Title: Linear Logic for Local Algebras of Observables

Date: Jun 05, 2009 12:15 PM

URL: http://pirsa.org/09060030

Abstract: TBA

oo.pdf - Adobe Reader
View Document Tools Window Help
1 / 17 144%

Linear Logic for Algebras of Observables

I.T. Ivanov

"Categories, Quanta, Concepts", Waterloo 2009

.pdf - Adobe Reader
View Document Tools Window Help
2 / 17 144%

Comparision Theory for States

$\mathcal{M}$ von Neumann algebra, $\mathcal{M}_*^+$ its state space.

For $\phi \in \mathcal{M}_*^+$ let $s(\phi)$ be its support.

For $\phi, \psi \in \mathcal{M}_*^+$ we say that ϕ and ψ are *equivalent*, $\phi \sim \psi$ if there is a partial isometry $u \in \mathcal{M}$ such that $s(\phi) = u^*u$, $s(\psi) = uu*$ and $\phi(x) = \psi(uxu^*), x \in \mathcal{M}$.

Define $\mathcal{M}_\phi = \{x \in \mathcal{M} : \sigma_t^\phi(x) = x, t \in \mathbb{R}\}$ - centralizer of ϕ.

If there exists $e \in Proj(\mathcal{M}_\psi)$ such that $\phi \sim \psi_e$ write $\phi \preccurlyeq \psi$.

Cantor-Bernstein: If $\phi \preccurlyeq \psi$ and $\phi \succcurlyeq \psi$, then $\phi \sim \psi$.

pa.pdf - Adobe Reader
View Document Tools Window Help
3 / 17 144%

Group von Neumann Algebras

G - locally compact group

On $L^2(G)$ we have left/right regular representation

$$(\lambda(x)\xi)(y) = \xi(x^{-1}y)$$

$$(\rho(x)\xi)(y) = \delta^{1/2}(x)\xi(yx)$$

$x, y \in G$, $\xi \in L^2(G)$

$$\mathcal{L}(G) = [\lambda(x)]'', \qquad R(G) = [\rho(x)]''$$

Standard Form: antilinear isometry J on $L^2(G)$

$$J\mathcal{L}(G)J = R(G)$$

$\rightarrow$ $\forall \omega \in \mathcal{L}(G)_*$ is of the form $\omega_{\xi,\eta} = \langle \cdot\, \xi, \eta \rangle$

$\rightarrow$ $\forall \varphi \in \mathcal{L}(G)_*^+$ is $\varphi_\xi = \langle \cdot\, \xi, \xi \rangle$

pdf - Adobe Reader
View Document Tools Window Help
3 / 17 144%

On $L^2(G)$ we have left/right regular representation

$$(\lambda(x)\xi)(y) = \xi(x^{-1}y)$$

$$(\rho(x)\xi)(y) = \delta^{1/2}(x)\xi(yx)$$

$x, y \in G, \ \xi \in L^2(G)$

$$\mathcal{L}(G) = [\lambda(x)]'', \qquad R(G) = [\rho(x)]''$$

Standard Form: antilinear isometry J on $L^2(G)$

$$J\mathcal{L}(G)J = R(G)$$

$\rightarrow$ $\forall \omega \in \mathcal{L}(G)_*$ is of the form $\omega_{\xi,\eta} = \langle \cdot \xi, \eta \rangle$

$\rightarrow$ $\forall \varphi \in \mathcal{L}(G)_*^+$ is $\varphi_\xi = \langle \cdot \xi, \xi \rangle$

Ex $G = S_3$ $\quad \mathcal{L}(G) = M_2 \otimes 1_{\mathbb{C}^2} \oplus \mathcal{L}_{\mathbb{Z}_2}$

$$R(G) = 1_{\mathbb{C}^2} \otimes M_2 \oplus \mathcal{L}_{\mathbb{Z}_2}$$

Fourier Algebra

$f^{\vee}(g) = f(g^{-1})$

$\omega_{\xi,\eta}(\lambda(g)) = \int_G \xi(g^{-1}h)\,\overline{\eta(h)}\,dh = \bar{\eta} * \xi^{\vee}(g)$

$A(G) = \{\xi * \eta^{\vee} : \xi, \eta \in L^2(G)\} \subset C_0(G)$

W_G - unitary on $L^2(G\times G)$: $(W_G\xi)(g,h) = \xi(g, gh)$

$W_G^*(\lambda(g)\otimes 1)W_G = \lambda(g)\otimes\lambda(g)$

$\pi(x) = W_G^*(x\otimes 1)W_G$ is a map $\mathcal{L}(G) \to \mathcal{L}(G)\otimes\mathcal{L}(G)$

$\{\bar{\eta}_1 * \xi_1^{\vee}(g)\}\{\bar{\eta}_2 * \xi_2^{\vee}(g)\} = (\lambda(g)\otimes\lambda(g)(\xi_1\otimes\xi_2)\,|\,(\eta_1\otimes\eta_2)) =$

$= (W_G^*(\lambda(g)\otimes 1)W_G(\xi_1\otimes\xi_2)\,|\,\eta_1\otimes\eta_2) =$

pa.pdf - Adobe Reader
View Document Tools Window Help
4 / 17 144%

$$\omega_{\xi,\eta}(\lambda(g)) = \int_G \xi(g^{-1}h)\,\eta(h)\,dh = \eta * \xi^{\vee}(g)$$

$$A(G) = \{\xi * \eta^{\vee} : \xi, \eta \in L^2(G)\} \subset C_0(G)$$

W_G - unitary on $L^2(G\times G)$: $(W_G\xi)(g,h) = \xi(g, gh)$

$$W_G^*(\lambda(g)\otimes 1)W_G = \lambda(g)\otimes\lambda(g)$$

$\pi(x) = W_G^*(x\otimes 1)W_G$ is a map $\mathcal{L}(G) \to \mathcal{L}(G)\otimes\mathcal{L}(G)$

$$\{\bar{\eta}_1 * \xi_1^{\vee}(g)\}\{\bar{\eta}_2 * \xi_2^{\vee}(g)\} = (\lambda(g)\otimes\lambda(g)(\xi_1\otimes\xi_2)\,|\,(\eta_1\otimes\eta_2)) =$$

$$= (W_G^*(\lambda(g)\otimes 1)W_G(\xi_1\otimes\xi_2)\,|\,\eta_1\otimes\eta_2) =$$

$$= \langle \pi(\lambda(g)),\ \omega_{\xi_1,\eta_1}\otimes\omega_{\xi_2,\eta_2}\rangle = \bar{\eta} * \xi^{\vee}(g)$$

Theorem: $A(G) \ni \bar{\eta} * \xi^{\vee} \longleftrightarrow \omega_{\xi,\eta} \in \mathcal{L}(G)_*$ is an isomorphism of commutative Banach algebras.

pa.pdf - Adobe Reader
View Document Tools Window Help
4 / 17 144%

$$A(G) = \{ \xi * \eta^{\vee} : \xi, \eta \in L^2(G) \} \subset C_0(G)$$

W_G - unitary on $L^2(G \times G)$: $(W_G \xi)(g,h) = \xi(g, gh)$

$$W_G^* (\lambda(g) \otimes 1) W_G = \lambda(g) \otimes \lambda(g)$$

$\pi(x) = W_G^* (x \otimes 1) W_G$ is a map $\mathcal{L}(G) \to \mathcal{L}(G) \otimes \mathcal{L}(G)$

$$\{ \bar{\eta}_1 * \xi_1^{\vee}(g) \} \{ \bar{\eta}_2 * \xi_2^{\vee}(g) \} = (\lambda(g) \otimes \lambda(g) (\xi_1 \otimes \xi_2) | (\eta_1 \otimes \eta_2))$$

$$= (W_G^* (\lambda(g) \otimes 1) W_G (\xi_1 \otimes \xi_2) | \eta_1 \otimes \eta_2) =$$

$$= \langle \pi(\lambda(g)), \omega_{\xi_1, \eta_1} \otimes \omega_{\xi_2, \eta_2} \rangle = \bar{\eta} * \xi^{\vee}(g)$$

Theorem: $A(G) \ni \bar{\eta} * \xi^{\vee} \longleftrightarrow \omega_{\xi,\eta} \in \mathcal{L}(G)_*$ is an isomorphism of commutative Banach algebras.

$A(G) \cong \mathcal{L}_*(G)$ - Fourier algebra

Fourier Algebra

$\nabla(\lambda(g)) = \lambda(g) \otimes \lambda(g)$ extends to

$\nabla : \mathcal{L}(G) \to \mathcal{L}(G) \bar{\otimes} \mathcal{L}(G)$ Hopf-von Neumann alg.

The identification $(\mathcal{L}_*(G) \hat{\otimes} \mathcal{L}_*(G))^* \simeq \mathcal{L}(G) \bar{\otimes} \mathcal{L}(G)$

allows us to define

$$\Delta : \mathcal{L}_*(G) \hat{\otimes} \mathcal{L}_*(G) \to \mathcal{L}_*(G)$$

via $\langle \Delta(u), T \rangle = \langle u, \nabla(T) \rangle$

On generators : $(u \times v)(g,h) = u(g)v(h)$

$$[\Delta(u \times v)](g) = u(g)v(g)$$

$\mathcal{L}_*(G)$ - completely contractive Banach algebra

pa.pdf - Adobe Reader
View Document Tools Window Help
5 / 17 144%

$\nabla : \mathcal{L}(G) \to \mathcal{L}(G) \bar{\otimes} \mathcal{L}(G)$ Hopf-von Neumann alg.

The identification $(\mathcal{L}_*(G) \hat{\otimes} \mathcal{L}_*(G))^* \simeq \mathcal{L}(G) \bar{\otimes} \mathcal{L}(G)$

allows us to define

$$\Delta : \mathcal{L}_*(G) \hat{\otimes} \mathcal{L}_*(G) \to \mathcal{L}_*(G)$$

via $\langle \Delta(u), T \rangle = \langle u, \nabla(T) \rangle$

On generators : $(u \times v)(g,h) = u(g)v(h)$

$$[\Delta(u \times v)](g) = u(g)v(g)$$

$\mathcal{L}_*(G)$ - completely contractive Banach algebra

.pdf - Adobe Reader
View Document Tools Window Help
6 / 17 144%

Fourier Algebra (Groupoids)

$\{\mathcal{G}, \mu\}$ - measured groupoid

$\mathcal{L}(\mathcal{G})$ - the vN algebra generated by the left regular representation on $\{L^2(\mathcal{G}^u)\}_{u \in \mathcal{G}^0}$

$\mathcal{H} = L^\infty(\mathcal{G}^0, \mu)$ - commutative with trace μ.

$\nabla : \mathcal{L}(\mathcal{G}) *_\mu \mathcal{L}(\mathcal{G}) \longrightarrow \mathcal{L}(\mathcal{G})$ Hopf-von Neumann bimodule

$\omega, \varphi \in \mathcal{L}(\mathcal{G})_*$ we have $\omega \otimes_\mu \varphi$ densely defined.

Define: $\omega *_\mu \varphi = (\omega \otimes_\mu \varphi) \circ \nabla$

$\mathcal{L}(\mathcal{G})_*^\mu$ - the Fourier algebra of $\mathcal{G}$.

Local structure of the 1-particle space

$\mathcal{H} = L^2(H_m, d\Omega_m)$

Real-linear embedding of the space $S(M^4, \mathbb{R})$ of smooth rapidly decreasing functions (test functions) into $\mathcal{H}$, $E : S \to \mathcal{H}$

$\mathcal{O} \subset M^4$ - bounded open region

$\mathcal{K}(\mathcal{O}) = \{Ef : f \in S, \operatorname{supp}(f) \subset \mathcal{O}\}^-$ - real subspace

$\mathcal{O}' = \{x \in M^4 : (x-y)^2 < 0, \forall y \in \mathcal{O}\}$ - causal compliment

$\mathcal{K}' = \{h \in \mathcal{H} : \operatorname{Im}(h,k) = 0, k \in \mathcal{K}\}$

The map $\mathcal{O} \to \mathcal{K}(\mathcal{O})$ is an isomorphism of

pa.pdf - Adobe Reader
View Document Tools Window Help
7 / 17 144%

$\mathcal{H} = L^2(H_m, d\Omega_m)$

Real-linear embedding of the space $S(M^4, \mathbb{R})$ of smooth rapidly decreasing functions (test functions) into $\mathcal{H}$, $E : S \to \mathcal{H}$

$\mathcal{O} \subset M^4$ - bounded open region

$\mathcal{K}(\mathcal{O}) = \{Ef : f \in S, \operatorname{supp}(f) \subset \mathcal{O}\}^-$ - real subspace

$\mathcal{O}' = \{x \in M^4 : (x-y)^2 < 0, \forall y \in \mathcal{O}\}$ - causal compliment

$\mathcal{K}' = \{h \in \mathcal{H} : \operatorname{Im}(h,k) = 0, k \in \mathcal{K}\}$

The map $\mathcal{O} \to \mathcal{K}(\mathcal{O})$ is an isomorphism of complimented lattices, in particular

$\mathcal{K}(\mathcal{O}') = [\mathcal{K}(\mathcal{O})]'$.

...pdf - Adobe Reader
View Document Tools Window Help
7 / 17 144%

Local structure of the 1-particle space

$\mathcal{H} = L^2(H_m, d\Omega_m)$

Real-linear embedding of the space $S(M^4, \mathbb{R})$ of smooth rapidly decreasing functions (test functions) into $\mathcal{H}$, $E : S \to \mathcal{H}$

$\mathcal{O} \subset M^4$ - bounded open region

$\mathcal{K}(\mathcal{O}) = \{Ef : f \in S, \operatorname{supp}(f) \subset \mathcal{O}\}^-$ - real subspace

$\mathcal{O}' = \{x \in M^4 : (x-y)^2 < 0, \forall y \in \mathcal{O}\}$ - causal compliment

$\mathcal{K}' = \{h \in \mathcal{H} : \operatorname{Im}(h,k) = 0, k \in \mathcal{K}\}$

The map $\mathcal{O} \to \mathcal{K}(\mathcal{O})$ is an isomorphism of complimented lattices, in particular

bo.pdf - Adobe Reader
View Document Tools Window Help
7 / 17 144%

Real-linear embedding of the space $S(M^4, \mathbb{R})$ of smooth rapidly decreasing functions (test functions) into $\mathcal{H}$, $E : S \to \mathcal{H}$

$\mathcal{O} \subset M^4$ - bounded open region

$\mathcal{K}(\mathcal{O}) = \{Ef : f \in S, \operatorname{supp}(f) \subset \mathcal{O}\}^-$ - real subspace

$\mathcal{O}' = \{x \in M^4 : (x-y)^2 < 0, \forall y \in \mathcal{O}\}$ - causal compliment

$\mathcal{K}' = \{h \in \mathcal{H} : \operatorname{Im}(h,k) = 0, k \in \mathcal{K}\}$

The map $\mathcal{O} \to \mathcal{K}(\mathcal{O})$ is an isomorphism of complimented lattices, in particular

$\mathcal{K}(\mathcal{O}') = [\mathcal{K}(\mathcal{O})]'$.

Second Quantization Algebra Map

$\mathcal{H}$ - Hilbert space ; $\mathcal{K} \subseteq_{Re} \mathcal{H}$

Weyl unitaries : $W(h)\,W(k) = e^{i/2\,\mathrm{Im}(h,k)}\,W(h+k)$

$\mathcal{R}(\mathcal{K}) = \{ W(h) : h \in \mathcal{K} \}''$

Th: (Araki) The map $\mathcal{K} \to \mathcal{R}(\mathcal{K})$ is an isomorphism of complimented lattices:

i) $\mathcal{R}(\mathcal{K}_1) \subset \mathcal{R}(\mathcal{K}_2)$ iff $\mathcal{K}_1 \subset \mathcal{K}_2$

ii) $\mathcal{R}(\bigvee_\alpha \mathcal{K}_\alpha) = \bigvee_\alpha \mathcal{R}(\mathcal{K}_\alpha)$

iii) $\mathcal{R}(\bigwedge_\alpha \mathcal{K}_\alpha) = \bigwedge_\alpha \mathcal{R}(\mathcal{K}_\alpha)$

iv) $\mathcal{R}(\mathcal{K})' = \mathcal{R}(\mathcal{K}')$.

bo.pdf - Adobe Reader
View Document Tools Window Help
8 / 17 144%

Weyl unitaries: $W(h)\,W(k) = e^{i/2\,\mathrm{Im}(h,k)}\,W(h+k)$

$\mathcal{R}(\mathcal{K}) = \{ w(h) : h \in \mathcal{K} \}''$

Th: (Araki) The map $\mathcal{K} \to \mathcal{R}(\mathcal{K})$ is an isomorphism of complimented lattices:

i) $\mathcal{R}(\mathcal{K}_1) \subset \mathcal{R}(\mathcal{K}_2)$ iff $\mathcal{K}_1 \subset \mathcal{K}_2$

ii) $\mathcal{R}(\bigvee_\alpha \mathcal{K}_\alpha) = \bigvee_\alpha \mathcal{R}(\mathcal{K}_\alpha)$

iii) $\mathcal{R}(\bigwedge_\alpha \mathcal{K}_\alpha) = \bigwedge_\alpha \mathcal{R}(\mathcal{K}_\alpha)$

iv) $\mathcal{R}(\mathcal{K})' = \mathcal{R}(\mathcal{K}')$.

$\mathcal{R}(\mathcal{K}(\mathcal{O}))$ - algebras of local observables → net.

Type of $R(\mathcal{K})$

If $\mathcal{K} = \mathcal{K}' \oplus \mathcal{K}'$ Then $R(\mathcal{K}) \cong R(\mathcal{K}') \otimes R(\mathcal{K}'')$.

$\mathcal{H} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus \mathcal{H}^{(3)}$

$\mathcal{K} = \mathcal{N} \oplus \mathcal{H}^{(1)} \oplus \{0\} \oplus \mathbb{C}$

$\mathcal{K}' = \mathcal{N}' \oplus \{0\} \oplus \mathcal{H}^{(2)} \oplus \mathbb{C}$

$R(\mathcal{K}) = R \otimes B_{\mathcal{H}^{(1)}} \otimes 1_{\mathcal{H}^{(2)}} \otimes \mathcal{K}$

$R(\mathcal{K}') = R' \otimes 1_{\mathcal{H}^{(1)}} \otimes B_{\mathcal{H}^{(2)}} \otimes \mathcal{K}$

R - type III ; type III_1 is dense in the Grassmanian of real closed subspaces of $\mathcal{H}$

.pdf - Adobe Reader
View Document Tools Window Help
9 / 17 144%

Type of $R(\mathcal{K})$

If $\mathcal{K} = \mathcal{K}' \oplus \mathcal{K}'$ Then $R(\mathcal{K}) \cong R(\mathcal{K}') \otimes R(\mathcal{K}'')$.

$\mathcal{H} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus \mathcal{H}^{(3)}$

$\mathcal{K} = N \oplus \mathcal{H}^{(1)} \oplus \{0\} \oplus \mathbb{C}$

$\mathcal{K}' = N' \oplus \{0\} \oplus \mathcal{H}^{(2)} \oplus \mathbb{C}$

$R(\mathcal{K}) = R \otimes B_{\mathcal{H}^{(1)}} \otimes 1_{\mathcal{H}^{(2)}} \otimes \mathcal{F}$

$R(\mathcal{K}') = R' \otimes 1_{\mathcal{H}^{(1)}} \otimes B_{\mathcal{H}^{(2)}} \otimes \mathcal{F}$

R - type III ; type III_1 is dense in the Grassmanian of real closed subspaces of $\mathcal{H}$

$R(\mathcal{K})$ - in standard form.

bo.pdf - Adobe Reader
View Document Tools Window Help
10 / 17 144%

Linear Logic Reminder

$$\frac{}{\vdash A, A^{\perp}}\ \text{axiom} \qquad \frac{\Gamma_1 \vdash A, \Sigma_1 \quad \Gamma_2 \vdash A^{\perp}, \Sigma_2}{\Gamma_1, \Gamma_2 \vdash \Sigma_1, \Sigma_2}\ \text{cut}$$

$$\frac{\Gamma, A, B \vdash \Sigma}{\Gamma, A \otimes B \vdash \Sigma}\ \otimes_L \qquad \frac{\Gamma_1 \vdash A, \Sigma_1 \quad \Gamma_2 \vdash B, \Sigma_2}{\Gamma_1, \Gamma_2 \vdash A \otimes B, \Sigma_1, \Sigma_2}\ \otimes_R$$

multiplicative conjunction.

$$\frac{\Gamma, A \vdash \Sigma}{\Gamma, A \,\&\, B \vdash \Sigma}\ \&_L \qquad \frac{\Gamma \vdash A, \Sigma \quad \Gamma \vdash B, \Sigma}{\Gamma \vdash A \,\&\, B, \Sigma}\ \&_R$$

additive conjunction

$$\frac{\Gamma_1, A \vdash \Sigma_1 \quad \Gamma_2, B \vdash \Sigma_2}{\Gamma_1, \Gamma_2, A \parr B \vdash \Sigma_1, \Sigma_2}\ \parr_L \qquad \frac{\Gamma \vdash A, B, \Sigma}{\Gamma \vdash A \parr B, \Sigma}\ \parr_R$$

multiplicative disjunction

$$\frac{\Gamma, A \vdash \Sigma \quad \Gamma, B \vdash \Sigma}{\Gamma, A \oplus B \vdash \Sigma}\ \oplus_L \qquad \frac{\Gamma \vdash A, \Sigma}{\Gamma \vdash A \oplus B, \Sigma}\ \oplus_R$$

additive disjunction

o.pdf - Adobe Reader
View Document Tools Window Help
10 / 17 144%

$$\frac{\Gamma, A, B \vdash \Sigma}{\Gamma, A \otimes B \vdash \Sigma}\ \otimes_L \qquad \frac{\Gamma_1 \vdash A, \Sigma_1 \quad \Gamma_2 \vdash B, \Sigma_2}{\Gamma_1, \Gamma_2 \vdash A \otimes B, \Sigma_1, \Sigma_2}\ \otimes_R$$

multiplicative conjunction.

$$\frac{\Gamma, A \vdash \Sigma}{\Gamma, A \,\&\, B \vdash \Sigma}\ \&_L \qquad \frac{\Gamma \vdash A, \Sigma \quad \Gamma \vdash B, \Sigma}{\Gamma \vdash A \,\&\, B, \Sigma}\ \&_R$$

additive conjunction

$$\frac{\Gamma_1, A \vdash \Sigma_1 \quad \Gamma_2, B \vdash \Sigma_2}{\Gamma_1, \Gamma_2, A \parr B \vdash \Sigma_1, \Sigma_2}\ \parr_L \qquad \frac{\Gamma \vdash A, B, \Sigma}{\Gamma \vdash A \parr B, \Sigma}\ \parr_R$$

multiplicative disjunction

$$\frac{\Gamma, A \vdash \Sigma \quad \Gamma, B \vdash \Sigma}{\Gamma, A \oplus B \vdash \Sigma}\ \oplus_L \qquad \frac{\Gamma \vdash A, \Sigma}{\Gamma \vdash A \oplus B, \Sigma}\ \oplus_R$$

additive disjunction

$A \to A^\perp$; $\Gamma \to A, \Sigma \equiv \Gamma, A^\perp \vdash \Sigma$ involutive negation.

Interpretation of Axioms and Sequents

(1) Axiom $\vdash A, A^{\perp}$ → pair of von Neumann algebras A and its commutant A' acting on a separable Hilbert space in standard form $\vdash_{SH} A, A'$.

(2) Right-sided sequent $\vdash \Gamma \equiv \vdash A, B, \ldots$ → commuting von Neumann algebras $\vdash A, B, \ldots$

(3) Negation: If $\Gamma \vdash A, \Sigma$ is $\mathcal{P} \vdash A, \Sigma$ then $\Gamma, A^{\perp} \vdash \Sigma$ is $\Gamma, A' \vdash \Sigma$

(4) Proof of an axiom $\dfrac{}{\vdash A, A'}$ is $\vdash_{P} A, A'$ with $\varphi \in A_*$ i.e. $\varphi = \omega_{\xi,\eta}$

pa.pdf - Adobe Reader
View Document Tools Window Help
11 / 17 144%

(1) Axiom $\vdash A, A^{\perp}$ → pair of von Neumann algebras A and its commutant A' acting on a separable Hilbert space in standard form $\vdash_{\mathcal{H}} A, A'$.

(2) Right-sided sequent $\vdash \Gamma \equiv \vdash A, B, \ldots$ → commuting von Neumann algebras $\vdash A, B, \ldots$

(3) Negation: If $\Gamma \vdash A, \Sigma$ is $\mathcal{P} \vdash A, \Sigma$ then $\Gamma, A^{\perp} \vdash \Sigma$ is $\Gamma, A' \vdash \Sigma$

(4) Proof of an axiom $\dfrac{}{\vdash A, A'}$ is $\vdash_{\varphi} A, A'$ with $\varphi \in A_*$ i.e. $\varphi = \omega_{\xi,\eta}$

(5) Proof of a sequent $\dfrac{\vdots}{\Gamma \vdash \Sigma}$ is $\mathcal{P} \vdash_{\varphi} \Sigma$ with $\varphi = \omega_{\xi,\eta}$

pa.pdf - Adobe Reader
View Document Tools Window Help
12 / 17 144%

Multiplicative Connectives

$$\frac{\Sigma, R, S \vdash_\varphi}{\Sigma, R \wedge S \vdash_\varphi} \otimes_L \qquad \frac{\vdash_{\mathcal{H}_1, \varphi} A, \Gamma \quad \vdash_{\mathcal{H}_2, \psi} B, \Delta}{\vdash_{\mathcal{H}_1 \otimes \mathcal{H}_2, \varphi \otimes \psi} A \bar{\otimes} B, \Gamma \bar{\otimes} 1_{\mathcal{H}_2}, 1_{\mathcal{H}_1} \bar{\otimes} \Delta} \otimes_R$$

$$\frac{\Sigma, R \vdash_{\varphi, \mathcal{H}_1} \quad \Phi, S \vdash_{\psi, \mathcal{H}_2}}{\Sigma \bar{\otimes} \beta_{\mathcal{H}_2}, \beta_{\mathcal{H}_1} \bar{\otimes} \Phi, R \bar{\otimes} S \vdash_{\mathcal{H}_1 \otimes \mathcal{H}_2, \varphi \otimes \psi}} \wp_L \qquad \frac{\vdash_\varphi \Gamma, A, B}{\vdash_\varphi \Gamma, A \vee B} \wp_R$$

① Consistency with negation: $(A \vee B)' = (A \cup B)''' = A' \wedge B'$

$(\Gamma \bar{\otimes} 1_{\mathcal{H}})' = \Gamma' \bar{\otimes} \beta_{\mathcal{H}}$; $(A \bar{\otimes} B)' = A' \bar{\otimes} B'$

② Notice : $A \bar{\otimes} B = (A \bar{\otimes} \beta_{\mathcal{H}_2}) \wedge (\beta_{\mathcal{H}_1} \bar{\otimes} B)$ so both $\otimes_L$ and $\wp_i$ are intersections.

bo.pdf - Adobe Reader
View Document Tools Window Help
13 / 17 144%

Additive Connectives

$$\frac{R, \Sigma \vdash_{\mathcal{H},\varphi}}{R \oplus S, \Sigma \otimes \mathbb{B}_{\mathbb{C}^2} \vdash_{\mathcal{H}\oplus\mathcal{H},\varphi\oplus 0}} \&_L \qquad \frac{\vdash_{\mathcal{H},\varphi} A, \Gamma \quad \vdash_{\mathcal{H},\psi} B, \Gamma}{\vdash_{\mathcal{H}\oplus\mathcal{H},\varphi\oplus\psi} A \oplus B, \Gamma \bar{\otimes} 1_{\mathbb{C}^2}} \&_R$$

$$\frac{\Sigma, R \vdash_{\mathcal{H},\varphi} \quad \Sigma, S \vdash_{\mathcal{H},\psi}}{\Sigma \otimes \mathbb{B}_{\mathbb{C}^2}, R \oplus S \vdash_{\mathcal{H}\oplus\mathcal{H},\varphi\oplus\psi}} \oplus_L \qquad \frac{\vdash_{\mathcal{H},\varphi} A, \Gamma}{\vdash_{\mathcal{H}\oplus\mathcal{H},\varphi\oplus 0, A\oplus B, \Gamma \bar{\otimes} 1_{\mathbb{C}^2}}} \oplus_R$$

Consistency with negation:

$$(\Gamma \bar{\otimes} 1_{\mathbb{C}^2})' = \Gamma' \otimes \mathbb{B}_{\mathbb{C}^2} \; ; \; (A \oplus B)' = A' \oplus B'$$

Cut

$$\frac{\vdash_{\varphi} A, \Gamma \qquad A \vdash_{\psi} \Delta}{\vdash_{\varphi\,\psi} \Gamma, \Delta}\ \text{cut}$$

The most interesting case is cut between axioms (cut-elimination).

$$\frac{A \vdash_{\varphi} A \qquad A \vdash_{\psi} A}{A \vdash_{\varphi\,\psi} A}\ \text{cut}$$

So what is needed is commutative,

oo.pdf - Adobe Reader
View Document Tools Window Help
14 / 17 144%

Cut

$$\frac{\vdash_{\varphi} A, \Gamma \qquad A \vdash_{\psi} \Delta}{\vdash_{\varphi \llcorner \psi} \Gamma, \Delta}\ \text{cut}$$

The most interesting case is cut between axioms (cut-elimination).

$$\frac{A \vdash_{\varphi} A \qquad A \vdash_{\psi} A}{A \vdash_{\varphi \llcorner \psi} A}\ \text{cut}$$

So what is needed is commutative, associative composition in A_*.

Cut-elimination (non-principal formulas).

$$\dfrac{\dfrac{\vdash \Sigma, A \quad \vdash B, \Delta, C}{\vdash \Sigma, A \otimes B, \Delta, C}\otimes \quad C \vdash \Gamma}{\vdash \Sigma, A \otimes B, \Delta, \Gamma}\,cut \qquad \dfrac{\vdash \Sigma, A \quad \dfrac{\vdash B, \Delta, C \quad C \vdash \Gamma}{\vdash B, \Delta, \Gamma}\,cut}{\vdash \Sigma, A \otimes B, \Delta, \Gamma}\otimes$$

If $A = \mathcal{Y}(G_1)$; $B = \mathcal{Y}(G_2)$; $C = \mathcal{Y}(G_3)$;

$f(u) \in \mathcal{Y}(G_1)_*$; $g(x,y) \in \mathcal{Y}(G_2 \times G_3)_*$; $h(z) \in \mathcal{Y}(G_3)_*$

$$[f(u)\, g(x,y)]\, h(z) = f(u)\, [g(x,y)\, h(z)]$$

($y=z$ cut) ($y=z$ cut)

$$\dfrac{\vdash \Sigma, A, B, C}{\vdash \Sigma, A \parr B, C}\parr \quad C \vdash \Gamma \qquad \dfrac{\vdash \Sigma, A, B, C \quad C \vdash \Gamma}{\vdash \Sigma, A, B, \Gamma}\,cut$$

...pdf - Adobe Reader
View Document Tools Window Help
15 / 17 144%

If $A = \mathcal{L}(G_1)$; $B = \mathcal{L}(G_2)$; $C = \mathcal{L}(G_3)$,

$f(u) \in \mathcal{L}(G_1)_*$; $g(x,y) \in \mathcal{L}(G_2 \times G_3)_*$; $h(z) \in \mathcal{L}(G_3)_*$

$$[f(u)\, g(x,y)]\, h(z) = f(u)\, [g(x,y)\, h(z)]$$

($y=z$ cut) ($y=z$ cut)

$$\dfrac{\dfrac{\vdash \Sigma, A, B, C}{\vdash \Sigma, A \otimes B, C}\otimes \quad C \vdash \Gamma}{\vdash \Sigma, A \otimes B, \Gamma}\text{cut} \qquad \dfrac{\dfrac{\vdash \Sigma, A, B, C \quad C \vdash \Gamma}{\vdash \Sigma, A, B, \Gamma}\text{cut}}{\vdash \Sigma, A \otimes B, \Gamma}\otimes$$

$$f(x,y,z)\, g(u) = f(x,y,z)\, g(u)$$

(cut) (cut)

Cut-elimination (non-principal formulas)

$$\dfrac{\dfrac{\vdash \Sigma, A, C}{\vdash \Sigma, A \oplus B, C}\oplus \quad C \vdash \Gamma}{\vdash \Sigma, A \oplus B, \Gamma}\,cut \qquad\qquad \dfrac{\vdash \Sigma, A, C \quad C \vdash \Gamma}{\dfrac{\vdash \Sigma, A, \Gamma}{\vdash \Sigma, A \oplus B, \Gamma}\oplus}\,cut$$

$$[f(x,z) + O(y,z)]\, g(u) = f(x,z)\, g(u) + O(y,z)$$

$$\dfrac{\dfrac{\vdash \Sigma, A, C \quad \vdash \Sigma, B, C}{\vdash \Sigma, A \& B, C}\& \quad C \vdash \Gamma}{\vdash \Sigma, A \& B, \Gamma}\,cut \qquad\qquad \dfrac{\dfrac{\vdash \Sigma, A, C \quad C \vdash \Gamma}{\vdash \Sigma, A, \Gamma} \quad \dfrac{\vdash \Sigma, B, C \quad C \vdash \Gamma}{\vdash \Sigma, B, \Gamma}}{\vdash \Sigma, A \& B, \Gamma}\&$$

$$[f(x,z) + g(y,z)]\, h(u) = f(x,z)\, h(u) + f(y,z)\, h(u)$$

pa.pdf - Adobe Reader
View Document Tools Window Help
16 / 17 144%

Cut-elimination (non-principal formulas)

$$\dfrac{\vdash \Sigma, A, C}{\vdash \Sigma, A \oplus B, C}\oplus \qquad \dfrac{C \vdash \Gamma}{\vdash \Sigma, A \oplus B, \Gamma}\text{cut}$$

$$\dfrac{\vdash \Sigma, A, C \qquad C \vdash \Gamma}{\vdash \Sigma, A, \Gamma}\text{cut} \qquad \dfrac{\vdash \Sigma, A, \Gamma}{\vdash \Sigma, A \oplus B, \Gamma}\oplus$$

$$[f(x,z) + 0(y,z)]\, g(u) = f(x,z)\, g(u) + 0(y,z)$$

$$\dfrac{\vdash \Sigma, A, C \qquad \vdash \Sigma, B, C}{\vdash \Sigma, A \& B, C}\& \qquad \dfrac{C \vdash \Gamma}{\vdash \Sigma, A \& B, \Gamma}\text{cut}$$

$$\dfrac{\vdash \Sigma, A, C \qquad C \vdash \Gamma}{\vdash \Sigma, A, \Gamma}\text{cut} \qquad \dfrac{\vdash \Sigma, B, C \qquad C \vdash \Gamma}{\vdash \Sigma, B, \Gamma}\text{cut}$$

$$\dfrac{\vdash \Sigma, A, \Gamma \qquad \vdash \Sigma, B, \Gamma}{\vdash \Sigma, A \& B, \Gamma}\&$$

$$[f(x,z) + g(y,z)]\, h(u) = f(x,z)\, h(u) + f(y,z)\, h(u)$$

pa.pdf - Adobe Reader
View Document Tools Window Help
17 / 17 144%

Cut elimination (principal formulas)

$$\dfrac{\dfrac{\vdash \Sigma, A \quad \vdash B, \Delta}{\vdash \Sigma, A \otimes B, \Delta}\otimes_R \quad \dfrac{A, B \vdash \Gamma}{A \otimes B \vdash \Gamma}\otimes_L}{\vdash \Sigma, \Delta, \Gamma}\,cut \qquad\qquad \dfrac{\dfrac{\vdash B, \Delta \quad A, B \vdash \Gamma}{A, B \vdash \Delta, \Gamma}\,cut \quad \vdash \Sigma, A}{\vdash \Sigma, \Delta, \Gamma}\,cut$$

$$[f(x)\,g(y)]\,h(u,v) = f(x)\,[g(y)\,h(u,v)]$$

$$\dfrac{\dfrac{\vdash \Sigma, A \quad \vdash \Sigma, B}{\vdash \Sigma, A \& B}\&_R \quad \dfrac{A \vdash \Gamma}{A \& B \vdash \Gamma}\&_L}{\vdash \Sigma, \Gamma}\,cut \qquad\qquad \dfrac{\vdash \Sigma, A \quad A \vdash \Gamma}{\vdash \Sigma, \Gamma}\,cut$$

$$[f(x) + g(y)]\,[h(u) + 0(v)] \;?\; f(x)\,h(u)$$

.pdf - Adobe Reader
View Document Tools Window Help
17 / 17 144%

$$\dfrac{\dfrac{\vdash \Sigma, A \quad \vdash B, \Delta}{\vdash \Sigma, A \otimes B, \Delta}\otimes_R \quad \dfrac{A, B \vdash \Gamma}{A \otimes B \vdash \Gamma}\otimes_L}{\vdash \Sigma, \Delta, \Gamma}\,cut \qquad \dfrac{\dfrac{\vdash B, \Delta \quad A, B \vdash \Gamma}{\vdash \Sigma, A \quad A \vdash \Delta, \Gamma}\,cut}{\vdash \Sigma, \Delta, \Gamma}\,cut$$

$$[f(x)\,g(y)]\,h(u,v) = f(x)\,[g(y)\,h(u,v)]$$

$$\dfrac{\dfrac{\vdash \Sigma, A \quad \vdash \Sigma, B}{\vdash \Sigma, A \& B}\&_R \quad \dfrac{A \vdash \Gamma}{A \& B \vdash \Gamma}\&_L}{\vdash \Sigma, \Gamma}\,cut \qquad \dfrac{\vdash \Sigma, A \quad A \vdash \Gamma}{\vdash \Sigma, \Gamma}\,cut$$

$$[f(x)+g(y)]\,[h(u)+0(v)] \;?\; f(x)\,h(u)$$

pp.pdf - Adobe Reader
View Document Tools Window Help
17 / 17 144%

$$\dfrac{\dfrac{\vdash \Sigma, A \quad \vdash B, \Delta}{\vdash \Sigma, A \otimes B, \Delta}\otimes_R \quad \dfrac{A, B \vdash \Gamma}{A \otimes B \vdash \Gamma}\otimes_L}{\vdash \Sigma, \Delta, \Gamma}\,cut \qquad \dfrac{\dfrac{\vdash B, \Delta \quad A, B \vdash \Gamma}{\vdash \Sigma, A \quad A \vdash \Delta, \Gamma}\,cut}{\vdash \Sigma, \Delta, \Gamma}\,cut$$

$$[f(x)\,g(y)]\,h(u,v) = f(x)\,[g(y)\,h(u,v)]$$

$$\dfrac{\dfrac{\vdash \Sigma, A \quad \vdash \Sigma, B}{\vdash \Sigma, A \& B}\&_R \quad \dfrac{A \vdash \Gamma}{A \& B \vdash \Gamma}\&_L}{\vdash \Sigma, \Gamma}\,cut \qquad \dfrac{\vdash \Sigma, A \quad A \vdash \Gamma}{\vdash \Sigma, \Gamma}\,cut$$

$$[f(x)+g(y)]\,[h(u)+0(v)] \;?\; f(x)\,h(u)$$

$$\dfrac{\vdots}{\qquad}\;\text{axiom}$$

bo.pdf - Adobe Reader
View Document Tools Window Help
17 / 17 144%

$$[f(x)\,g(y)]\,h(u,v) = f(x)\,[g(y)\,h(u,v)]$$

$$\dfrac{\dfrac{\vdash \Sigma, A \quad \vdash \Sigma, B}{\vdash \Sigma, A \& B}\,\&_R \quad \dfrac{A \vdash \Pi}{A \& B \vdash \Pi}\,\&_L}{\vdash \Sigma, \Pi}\,cut \qquad \dfrac{\vdash \Sigma, A \quad A \vdash \Pi}{\vdash \Sigma, \Pi}\,cut$$

$$[f(x)+g(y)]\,[h(u)+0(v)] \;?\; f(x)\,h(u)$$

$$\dfrac{\dfrac{\vdots}{\vdash \Gamma, A} \quad \dfrac{}{A \vdash A}\,axiom}{\vdash \Gamma, A} \qquad \dfrac{\vdots}{\vdash \Gamma, A}$$

$$f(u)\,g(x) = f(u)\,g(u)$$

pdf - Adobe Reader
View Document Tools Window Help
17 / 17 144%

$$\dfrac{\dfrac{\vdash \Sigma, A \quad \vdash B, \Delta}{\vdash \Sigma, A \otimes B, \Delta}\otimes_R \quad \dfrac{A, B \vdash \Gamma}{A \otimes B \vdash \Gamma}\otimes_L}{\vdash \Sigma, \Delta, \Gamma}\,cut \qquad \dfrac{\dfrac{\vdash B, \Delta \quad A, B \vdash \Gamma}{\vdash \Sigma, A \quad A \vdash \Delta, \Gamma}\,cut}{\vdash \Sigma, \Delta, \Gamma}\,cut$$

$$[f(x)\,g(y)]\,h(u,v) = f(x)\,[g(y)\,h(u,v)]$$

$$\dfrac{\dfrac{\vdash \Sigma, A \quad \vdash \Sigma, B}{\vdash \Sigma, A \& B}\&_R \quad \dfrac{A \vdash \Gamma}{A \& B \vdash \Gamma}\&_L}{\vdash \Sigma, \Gamma}\,cut \qquad \dfrac{\vdash \Sigma, A \quad A \vdash \Gamma}{\vdash \Sigma, \Gamma}\,cut$$

$$[f(x) + g(y)]\,[h(u) + 0(v)] \;?\; f(x)\,h(u)$$

oo.pdf - Adobe Reader
View Document Tools Window Help
17 / 17 144%

$$\frac{\vdash \Sigma, A \& B \qquad A \& B \vdash \Pi}{\vdash \Sigma, \Pi}\ \text{cut}$$

$$\vdash \Sigma, \Pi$$

$$[f(x) + g(y)][h(u) + 0(v)] \ ? \ f(x)h(u)$$

$$\frac{\dfrac{\vdots}{\vdash \Gamma, A} \qquad \dfrac{}{A \vdash A}\ \text{axiom}}{\vdash \Gamma, A} \qquad \frac{\vdots}{\vdash \Gamma, A}$$

$$f(u)g(x) = f(u)g(u)$$