Title: Convex and Categorial Frameworks for Information Processing and Physics

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Abstract: TBA

Convex and Categorial Frameworks for Information Processing and Physics

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Rough overview of convex operational formalism

- Systems $A, B, C...$
- Convex set $\Omega_A, \Omega_B...$ of states (for each system)
- Convex sets of measurement outcomes
- Bilinear map: states $\times$ outcomes $\rightarrow$ *probabilities*.
- Convex set of allowable dynamics taking states to states
- Way(s) of making "composite" systems, or of recognizing compositeness: $C = A \otimes B$

A categorial (process-based) view

Basic idea: processes have *probabilities*
sets of processes *convex*

- States $\leftrightarrow$ preparation processes
- Outcomes $\leftrightarrow$ are processes that map to probabilities

Definition of category

Objects Class $\mathrm{Ob}\mathscr{C}$ of objects

Morphisms For each pair of objects A, B in $\mathrm{Ob}\mathscr{C}$, a set $\mathscr{C}(A, B)$ of *morphisms* (aka *arrows* or *maps*) "from A to B".

Notation: $f : A \to B$ means $f \in \mathscr{C}(A, B)$. We call A f's **domain**, B its **codomain**.

Identity For each object $A \in \mathrm{Ob}\mathscr{C}$, an *identity* morphism $\mathrm{id} : A \to A$.

Composition For each pair $\varphi : A \to B$, $\chi : B \to C$, a morphism $\chi \circ \varphi : A \to C$.

Axioms

$\varphi \circ \mathrm{id} = \varphi$, $\mathrm{id} \circ \varphi = \varphi$. Composition is associative.

Examples of categories

- Categories named after their objects (**Set** , **Grp**), or their morphisms (**Rel**), or both.
- Often, objects are sets-with-structure, morphisms structure-preserving functions.

Examples

Set Sets, functions.

Grp Groups, group homorphisms.

Vec Vector spaces, linear maps.

Rel Sets, relations.

FDOrdLin Finite dimensional ordered linear spaces, positive linear maps.

Poset categories Elements x, y of a fixed set, $\mathscr{C}(x, y)$ contains a single morphism if $x \geq y$.

Categories of Convex Operational Models

Convexity of state space and dynamics: instances of a

General principle

Whatever can happen or be done to a system, can happen or be done conditioned on the outcome of a coin toss.

Implement in categories $\mathscr{C}$ of processes acting on convex operational models $A, B, \ldots$ of state and effect spaces:

$\boxed{\text{Hom-sets } \mathscr{C}(A, B) \text{ are convex.}}$

We'll assume finite dimensionality for simplicity.

Convex operational models

Cone subset C of a real vector space, that is closed under addition and positive scalar multiplication.

Regular cone pointed, closed, generating cone.

An ordered linear space is regular if its positive cone is.

Dual cone to A_+ Set of linear functionals $f : A \to \mathbb{R}$ that are nonnegative on A_+.

Definition

A **Convex Operational Model** (COM) is a triple $A, A^\sharp, u_A$ with A a regular ordered linear space, $A^\sharp$ an ordered version of A^*, ordered by a regular cone $A^\sharp_+ \subseteq A^*_+$, u_A the *order unit*, a distinguished element in the interior of $A^\sharp$.

Concrete categories $\mathscr{C}$ of convex operational models and positive maps

- Objects: Convex operational models $A, A^\sharp, u_A$.
- Morphisms: $\mathscr{C}(A, B)$ are regular cones of positive (i.e. $f(A_+) \subseteq B_+$) linear maps $f : A \to B$ such that the *map* (not necessarily a morphism!) $f^* : B^* \to A^*$ satisfies $f^*(B^\sharp) \subseteq A^\sharp$. Composition and identity are inherited from **Vec**.

Caution: A^*, B^* not necessarily in $\mathscr{C}$. $\mathscr{C}(A, B)$ is of course closed under composition and contains the identity map!

Norms and operational interpretation

u_A defines a *base norm* on A, and an *order-unit norm* on $A^\sharp$, making them Banach spaces whose norms are bounded above by each others' dual norms.

- Operational interpretation: elements $a \in [0, u_A] \subseteq A^\sharp$ are effects, elements $\omega \in A$ with $u_A(\omega) = 1$ are normalized states; $a(\omega)$ is the probability of effect a given state ω.
- *Operationally meaningful* morphisms are those that are contractive with respect to the base norm.

Aside: base-norm generalizes the quantum mechanical "trace norm" $||X||_1 := \mathrm{Tr}\sqrt{X^\dagger X}$; base-norm distance $||\omega_1 - \omega_2||$ still gives best state discrimination probability.

Categories of contractive positive maps

Slight variation: *only* the operationally meaningful morphisms are morphisms.
Makes the extra structure categorial: objects are pairs of base-norm/order-unit Banach spaces satisfying the above conditions; morphisms are norm-contractive [hence automatically positive — check] maps $\varphi : A \to B$ such that φ^*, viewed as a map of order unit spaces, $B^\sharp \to A^\sharp$, is also norm-contractive.

Process-oriented variants (steps toward enrichment)

Put the convex structure entirely into the hom-sets?

- Equip $\mathscr{C}$ with *unit object* I: $\mathbb{R}$ ordered by $\mathbb{R}_+$. $I^\sharp$ the same, with u_I the identity functional $\mathbb{R} \to \mathbb{R}$.
- States as **preparation processes** in $\mathscr{C}(I, A)$. Effects as processes in $\mathscr{C}(A, I)$. Require $\mathscr{C}(I, A) \simeq A$, $\mathscr{C}(A, I) \simeq A^\sharp$ (naturally).
- Still require $\mathscr{C}(A, B)$ to be regular cones of linear maps, or (equipping each $\mathscr{C}(A, I)$ with a distinguished element determining norms) convex sets of norm-contractive ones.
- Allows us to dispense with explicit structure of objects if desired.
- Requirement that $\varphi^*(B^\sharp) \subseteq A^\sharp$ now automatic from the definition of category (composition) and the requirement that hom-sets be convex cones or convex sets of contractive maps.

[Diagram: Process $I \to A \to A \to I$]

Enriched categories

Formalize as a category *enriched* over the category of ordered linear spaces (say **FDOrdLin**), or that of Banach spaces and contractive maps ($\mathbf{BanSp}_2$).

Informally, a *category enriched over* $\mathscr{V}$, is a category whose sets $\mathscr{C}(A, B)$ of morphisms all have some additional structure: they "are objects of $\mathscr{V}$".

To make this systematic, $\mathscr{V}$ is taken to be a *closed category*.

Definition (Category enriched over $\mathscr{V}$)

Closed categories

Definition

A *formal closed category* is a category $\mathscr{C}$ equipped with:

1. A functor, called the *internal hom functor*, $[\,]:\mathscr{C}^{op}\times\mathscr{C}\to\mathscr{C}$
2. A distinguished object I, called the *unit object*, of $\mathscr{C}$.
3. A natural isomorphism $i_A : A \to [IA]$
4. A natural transformation $j_A : I \to [AA]$
5. A natural transformation $L_{ABC} : [BC] \to [[AB][AC]]$

such that $i_I = j_I$ and certain diagrams commute.
A formally closed category is *closed* if for all $A, B \in \mathrm{Ob}\mathscr{C}$ the map $f \mapsto [1_A, f_B]j : \mathscr{C}(A, B) \mapsto \mathscr{C}(I, [AB])$ is an isomorphism.

Monoidal closed categories

For every A, B, an MCC has an object $A \implies B$, the "internal hom". $\implies$ is a functor from $\mathscr{C}^{op} \times \mathscr{C}$ to $\mathscr{C}$. The functor "tensoring with A on the left" has a right adjoint "taking the internal hom from A".
Bijection between $\mathscr{C}(A \otimes B, C)$ and $\mathscr{C}(B, A \implies B)$, natural in B, C.
I.e. for each A, B, an object $A \implies B$ and a morphism
$e_{A,B} : A \otimes (A \implies B) \rightarrow B$ such that
$\forall f : A \otimes X \rightarrow B \exists ! h : X \rightarrow (A \implies B)$ such that
$f = e_{A,B} \circ (\mathrm{id}_A \otimes h)$.
FDOrdlin can be made monoidal closed, with $\otimes$ the minimal tensor product.

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Saturation

Let $\mathscr{C}$ be a category of convex operational models (CCOM).

Definition

An object for which $\mathscr{C}(A,I) \simeq \mathscr{C}(I,A)^*$ (i.e. $A^\sharp = A^*$) is called *saturated*. A category all of whose objects are saturated is *locally saturated*.

- Classical theory and quantum theory are locally saturated.
- A convexified categorical version of Rob Spekkens' toy theory would likely not be locally saturated.

Definition

We call a CCOM $\mathscr{C}$ *saturated* if there is no way to extend it by adding positive maps to some $\mathscr{C}(A,B)$.

Definition

A CCOM is *locally Hom-saturated at* (A, B) if the subcategory whose objects are A, B, I is saturated.

- Quantum theory is locally saturated, but neither locally Hom-saturated nor saturated.

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Compositeness in a convex approach

BBLW (for saturated models) and Barrett (for general models) definition of **a tensor product** of convex operational models A and B: state space $AB = A \otimes B$ ordered by $(A \otimes B)_+$ containing all product states, effect space $(A \otimes B)^*$ ordered by $(A \otimes B)^\sharp_+ \subseteq (A \otimes B)^*$ containing all products of effects, equipped with order unit $u_{AB} = u_A \otimes u_B$. This implies:

- No signalling (marginals are well defined)
- Local observability Expectations of products of local observables determine states.

NB: this definition doesn't determine $A \otimes B$, unless one of them is classical.

Composites II: Examples

(a) The *maximal tensor product*, $A \otimes_{\max} B$, consists of *all* states positive on product effects.

(b) The *minimal tensor product*, $A \otimes_{\min} B$, contains *only* convex combinations of product states.

(c) If $A = B = \mathscr{B}_h(\mathbf{H})$, then the positive cone on $\mathscr{B}_h(\mathbf{H} \otimes \mathbf{H})$, with its usual ordering, lies properly between the max. and min. cones.

Composite systems in categories: monoidality

Definition

The *product category* $\mathscr{C} \times \mathscr{D}$ has $\mathrm{Ob}(\mathscr{C} \times \mathscr{D}) = \mathrm{Ob}(\mathscr{C}) \times \mathrm{Ob}(\mathscr{D})$, $\mathscr{C} \times \mathscr{D}(C \times D, E \times F) = \mathscr{C}(C, E) \times \mathscr{D}(D, F)$, and the obvious composition $(\gamma, \delta) \circ (\alpha, \beta) = (\gamma \circ \alpha, \delta \circ \beta)$ and identities $\mathrm{id}_{(A,B)} := (\mathrm{id}_A, \mathrm{id}_B)$.

Some CCOM's may be equipped with the additional structure of a *monoidal tensor*: a bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$; the result is a *monoidal category*.

Bifunctoriality implies that $\mathscr{C}(A \otimes B, C \otimes D)$ contains $\mathscr{C}(A, C) \times \mathscr{C}(B, D)$. The morphisms $\alpha \otimes \beta$ are the usual tensored pairs of linear maps. Thus the space $\mathscr{C}(I, A \otimes B)$ contains all product states, and the effect space $\mathscr{C}(A \otimes B, I)$ contains all product effects. For saturated objects, these are two of the desiderata of the notion of *composite* used in BBLW. Local observability is not enforced. (Condition at end of Daniel Lehman's talk looked like l.o. to me)

Motivations for dropping local observability

1) Can be motivated by desire to preserve some other property, like self-duality. E.g., we can have a monoidal category of the mixed-state spaces of real FD Hilbert spaces.
PSD matrices over $\mathscr{H}_1 \otimes \mathscr{H}_2$ span a larger space than do the tensors of PSD matrices over $\mathscr{H}_1$ with those of $\mathscr{H}_2$, when $\mathscr{H}_i$ are *real*. But we can let $PSD(\mathscr{H}_1) \otimes_{\mathscr{C}} PSD(\mathscr{H}_2) := PSD(\mathscr{H}_1 \otimes \mathscr{H}_2)$.
2) Formalizing Smolin's "lockboxes" in convex categorial way (deals with "haecceity" objection of Bub and Halvorson); allows bit commitment and key distribution and to coexist. (So does ruling out entanglement in nonclassical theories even *with* local observability; Barnum, Dahlsten, Leifer, Toner Proc IEEE ITW 2008, Porto, May 2008.) (Might create an issue with functoriality of $\otimes$, though: I and **lockbox** look the same "locally".)

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