Title: Towards an abstract description of tensor product

Date: Jun 04, 2009 11:30 AM

URL: http://pirsa.org/09060025

Abstract: Tensor product is described in a family of categories that includes Set and Hilbert spaces. Such categories admit a "scalar" object which enables a definition of bi-arrows with two domains, generalizing functions of two variables. The tensor product is characterized by the expected universal property relating bi-arrows to arrows.

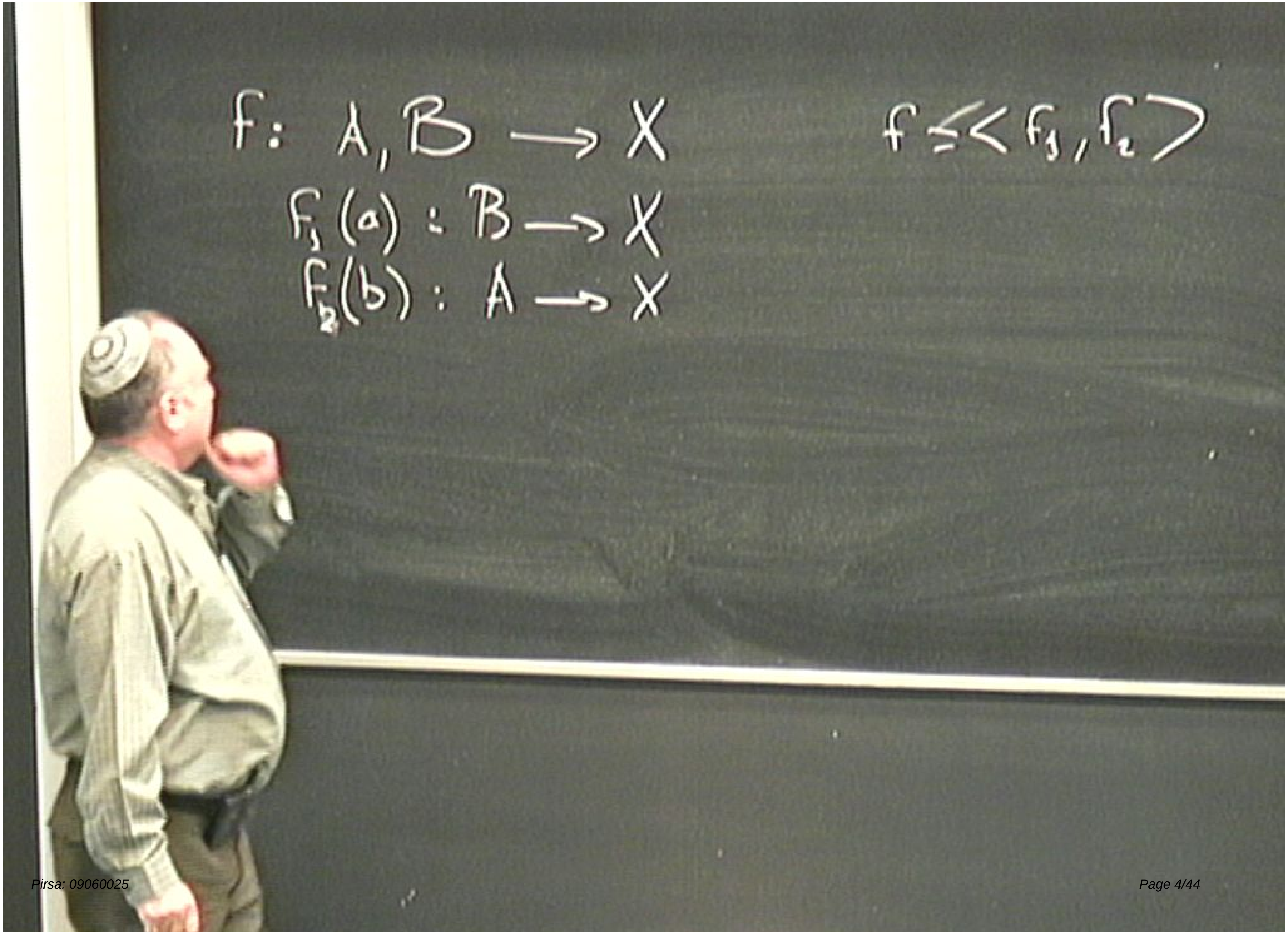
Towards an abstract definition of tensor product

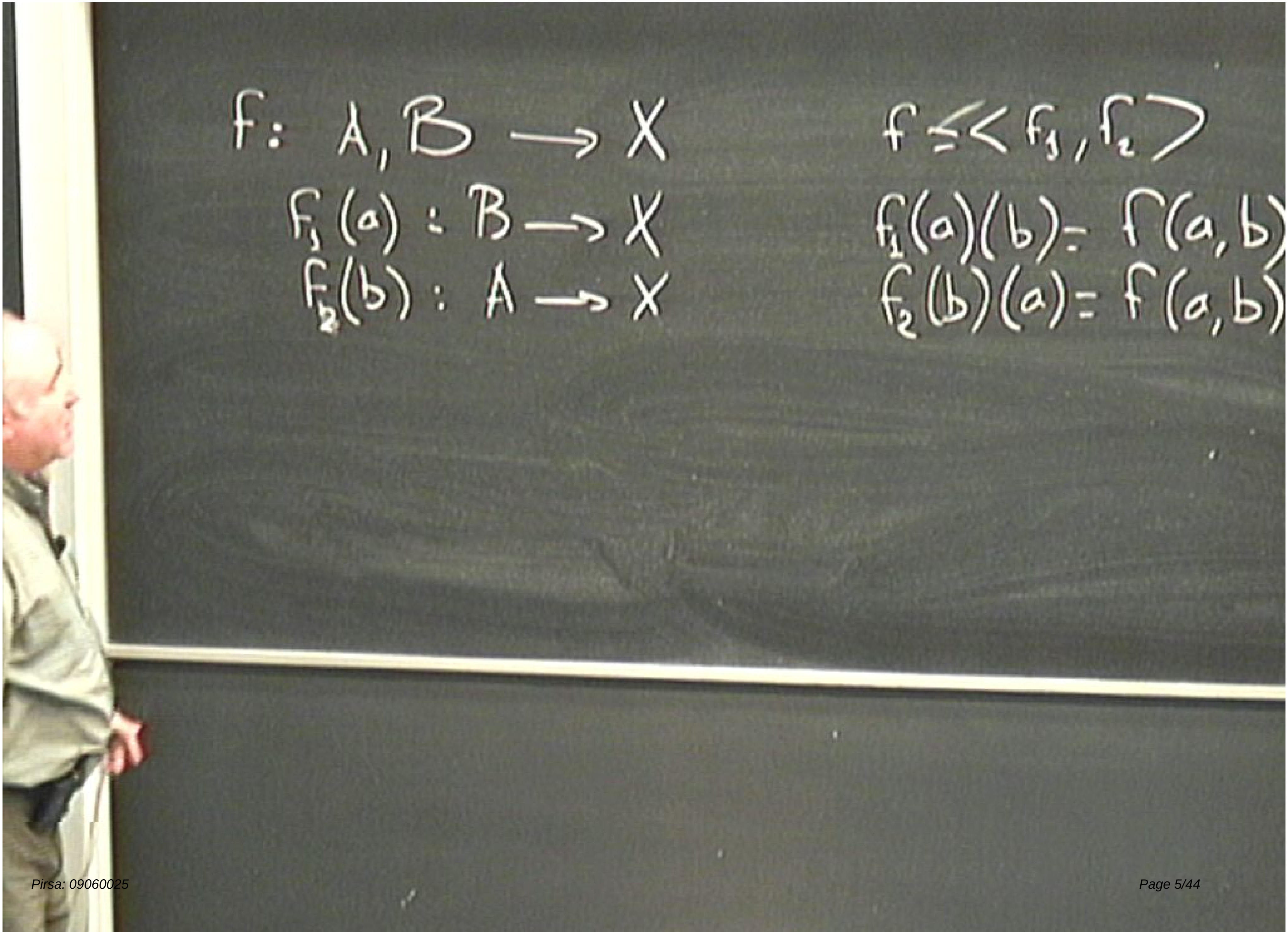
Daniel Lehmann

Towards an abstract definition of tensor product

Daniel Lehmann

$$
\begin{tikzcd}
A \arrow[dr] \arrow[drr, bend right] & & \\
& A \otimes B \arrow[r, dashed, "\text{linear}"] & X \\
B \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}
$$


$f : A, B \longrightarrow X$
$f_1(a) : B \longrightarrow X$
$f_2(b) : A \longrightarrow X$
$f \leq \langle f_1, f_2 \rangle$


$f : A, B \longrightarrow X$
$f_1(a) : B \longrightarrow X$
$f_2(b) : A \longrightarrow X$
$f \leq \langle f_1, f_2 \rangle$
$f_1(a)(b) = f(a,b)$
$f_2(b)(a) = f(a,b)$

$$f : A, B \longrightarrow X \qquad f \simeq \langle f_1, f_2 \rangle$$

$$f_1(a) : B \longrightarrow X \qquad f_1(a)(b) = f(a,b)$$

$$f_2(b) : A \longrightarrow X \qquad f_2(b)(a) = f(a,b)$$

$$a : I \longrightarrow A$$

Def. of a bi-morphism (resp. to I): $A, B \longrightarrow X$

$\langle f_1, f_2 \rangle$

$$f_1 : \underset{I \to A}{a} \longmapsto f_1(a) : B \longrightarrow X$$

$$f_2 : \underset{I \to B}{b} \longmapsto f_2(b) : A \longrightarrow X$$

s.t. $f_1(a) \circ b = f_2(b) \circ a$

$$f : A, B \longrightarrow X$$

$$f_1(a) : B \longrightarrow X$$

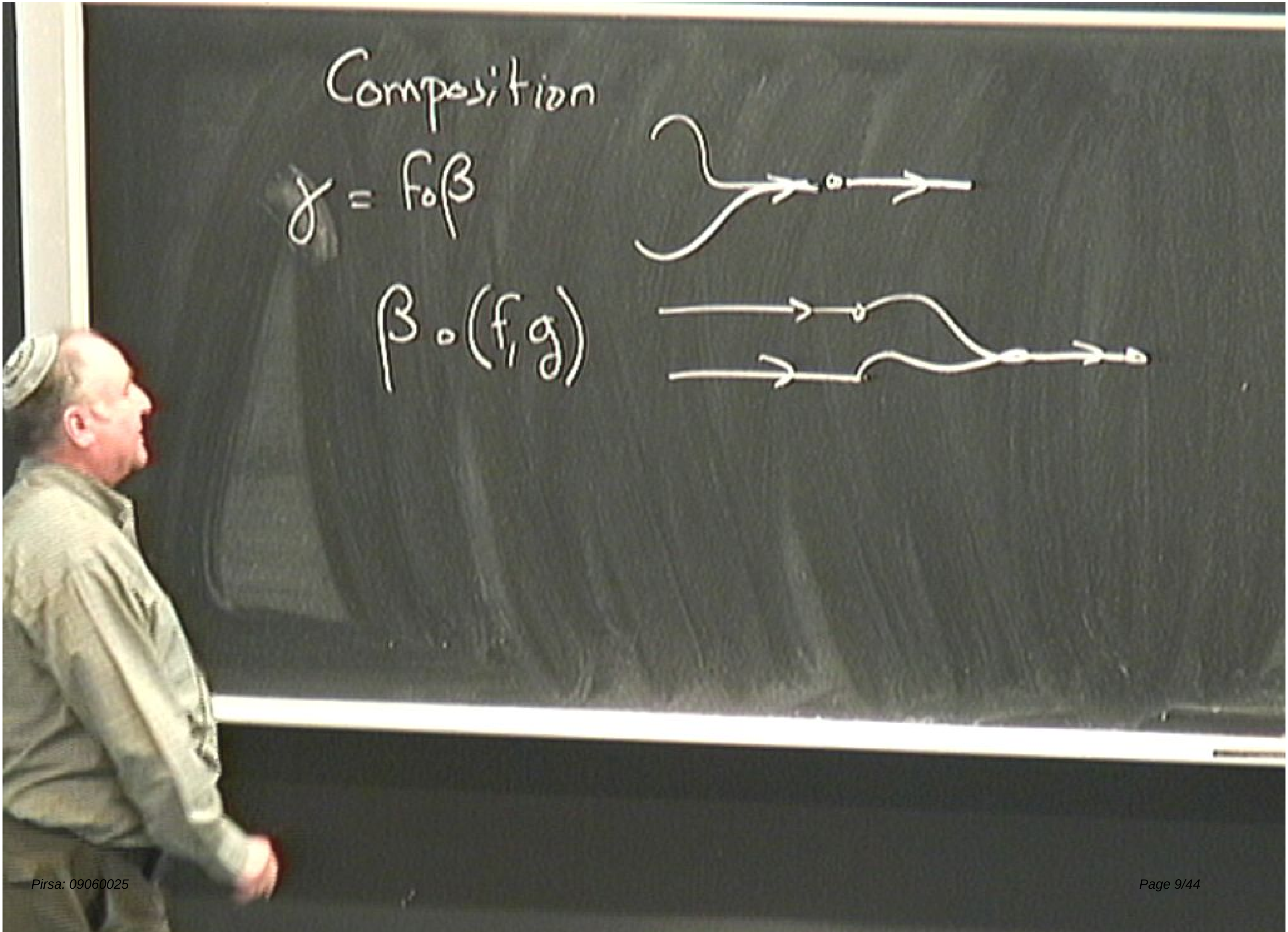
$$f_2(b) : A \longrightarrow X$$

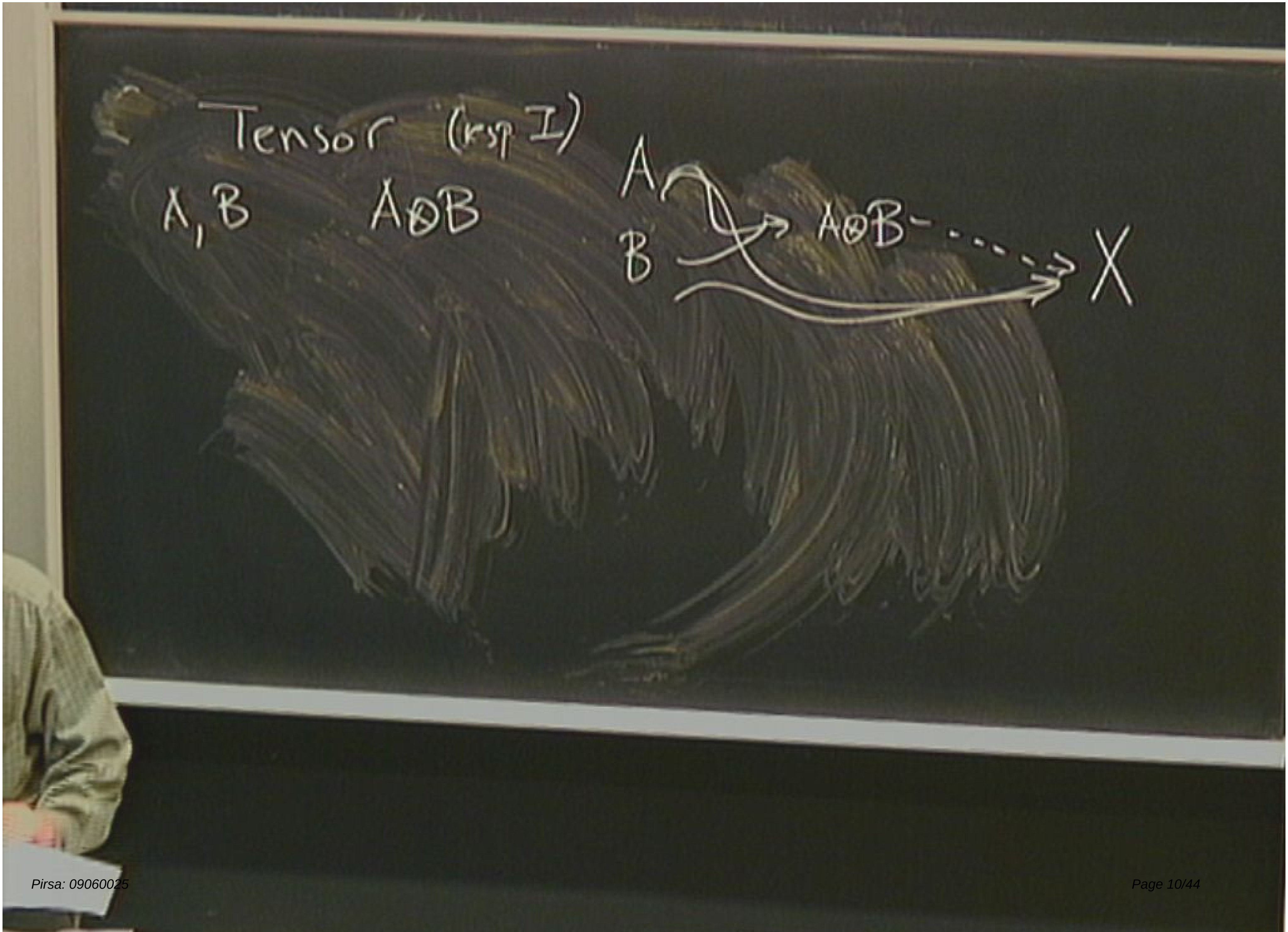
$$f \leq \langle f_1, f_2 \rangle$$

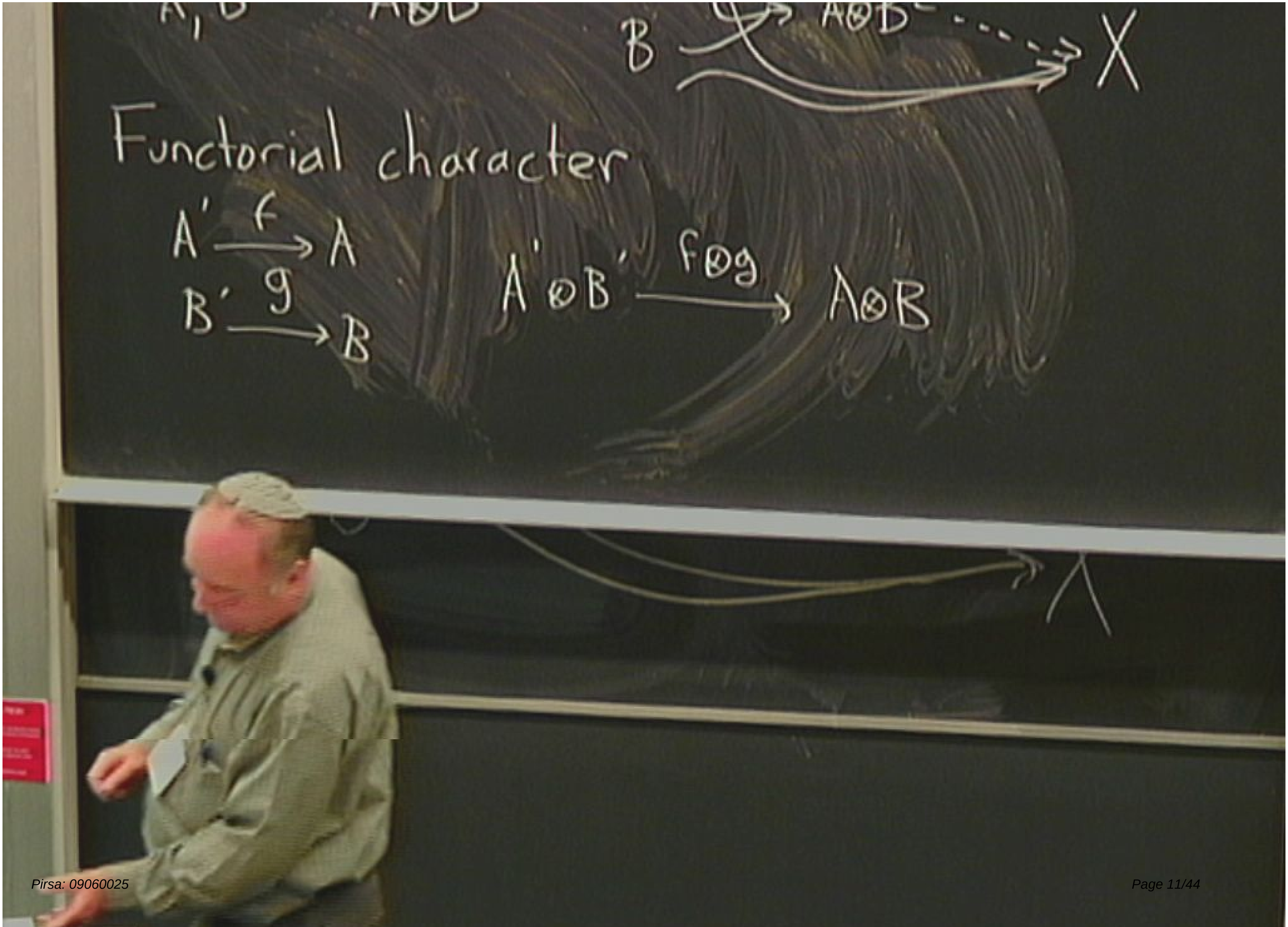
$$f_1(a)(b) = f(a, b)$$

$$f_2(b)(a) = f(a, b)$$

$$a : I \longrightarrow A$$


Composition
$\gamma = f \circ \beta$
$\beta \circ (f, g)$


Tensor (rsp I)
A, B
A⊗B
A
B
A⊗B
X

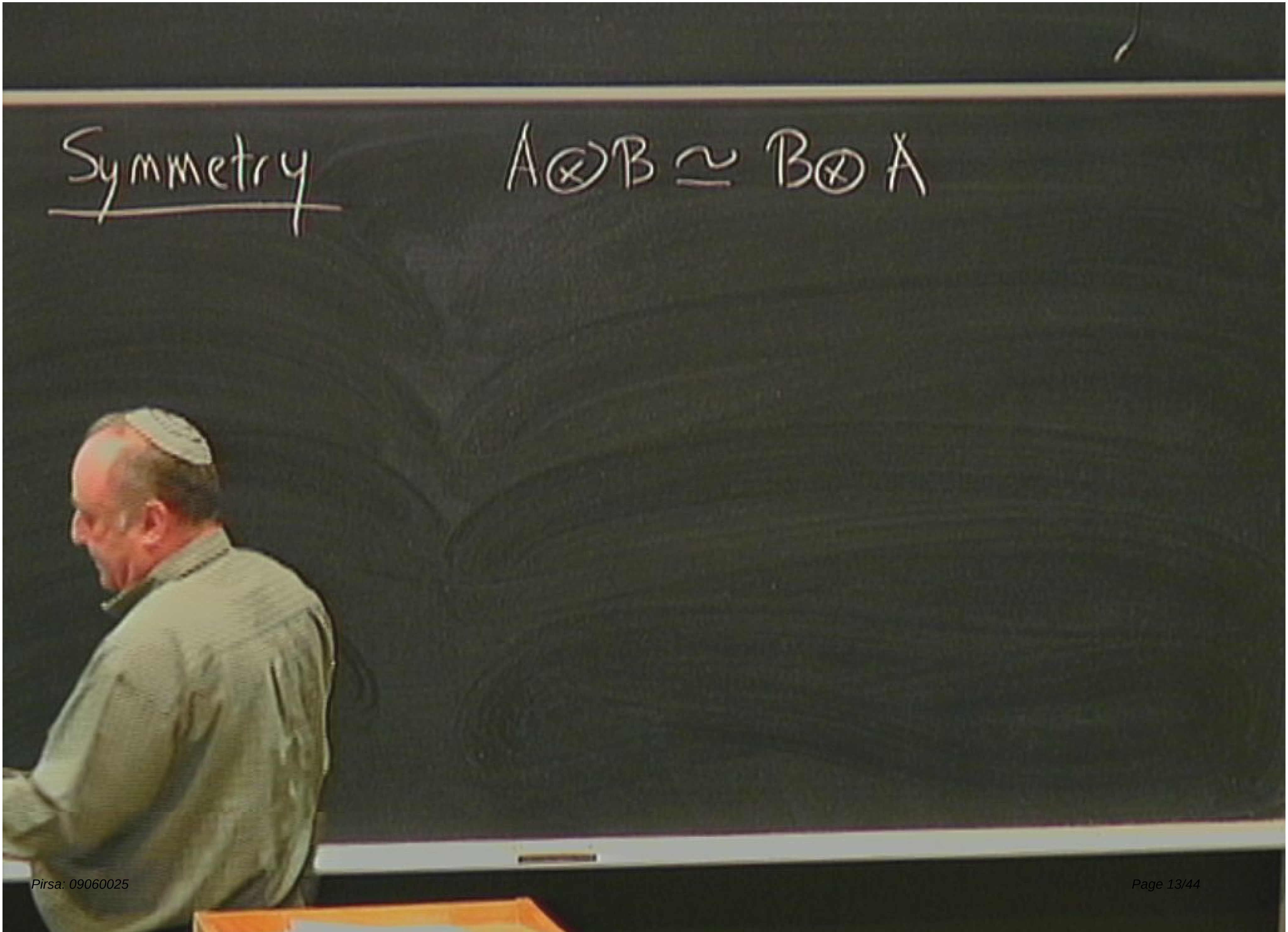

Functorial character
A' f A
B' g B
A'⊗B' f⊗g A⊗B
B
X

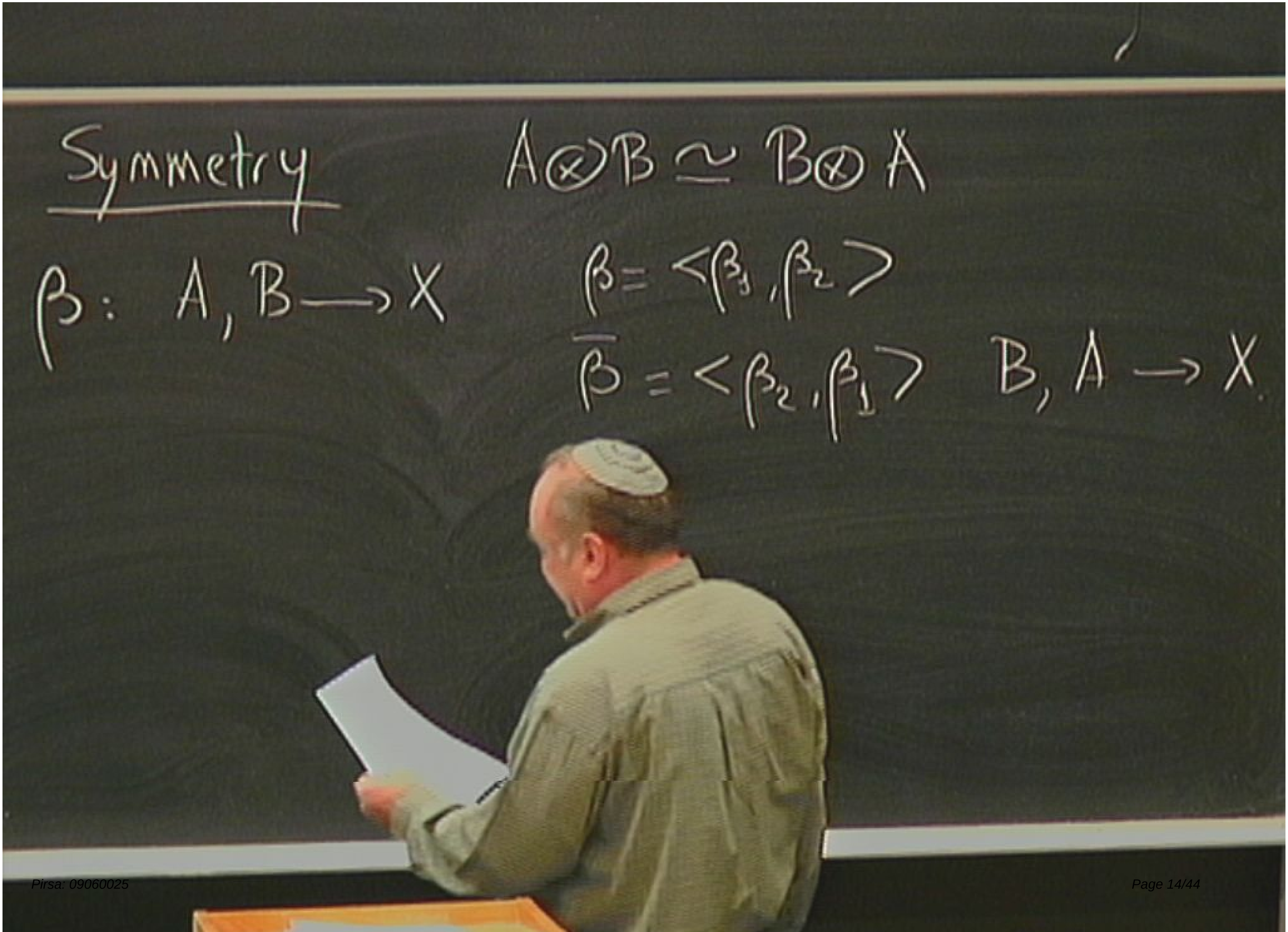
Functorial character

$$A' \xrightarrow{f} A \qquad A' \otimes B' \xrightarrow{f \otimes g} A \otimes B$$

$$B' \xrightarrow{g} B$$

$$A',\ B' \rightsquigarrow A \otimes B = (f, g) \circ \alpha(A, B)$$


Symmetry
$A \otimes B \simeq B \otimes A$


Symmetry
$A \otimes B \simeq B \otimes A$
$\beta : A, B \longrightarrow X$
$\beta = \langle \beta_1, \beta_2 \rangle$
$\bar{\beta} = \langle \beta_2, \beta_1 \rangle \quad B, A \longrightarrow X$

Symmetry

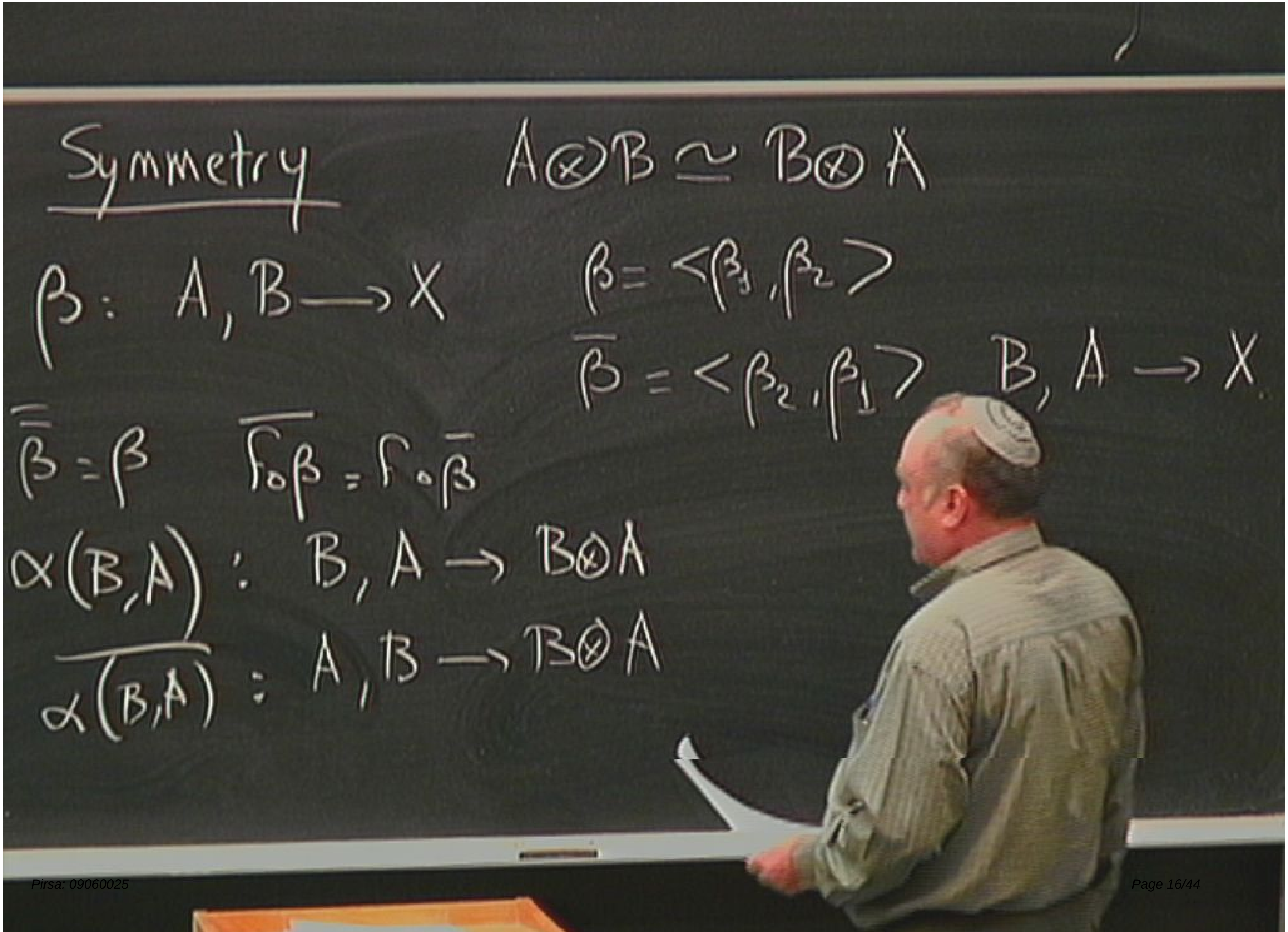
$$A \otimes B \simeq B \otimes A$$

$$\beta : A, B \longrightarrow X$$

$$\beta = \langle \beta_1, \beta_2 \rangle$$

$$\bar{\beta} = \langle \beta_2, \beta_1 \rangle \quad B, A \longrightarrow X$$

$$\bar{\bar{\beta}} = \beta \qquad \overline{f \circ \beta} = f \circ \bar{\beta}$$


Symmetry
$A \otimes B \simeq B \otimes A$
$\beta : A, B \longrightarrow X$
$\beta = \langle \beta_1, \beta_2 \rangle$
$\overline{\beta} = \langle \beta_2, \beta_1 \rangle \quad B, A \longrightarrow X$
$\overline{\overline{\beta}} = \beta \qquad \overline{f \circ \beta} = f \circ \overline{\beta}$
$\alpha(B, A) : B, A \longrightarrow B \otimes A$
$\overline{\alpha(B, A)} : A, B \longrightarrow B \otimes A$

Symmetry

$$A \otimes B \simeq B \otimes A$$

$$\beta : A, B \longrightarrow X \qquad \beta = \langle \beta_1, \beta_2 \rangle$$

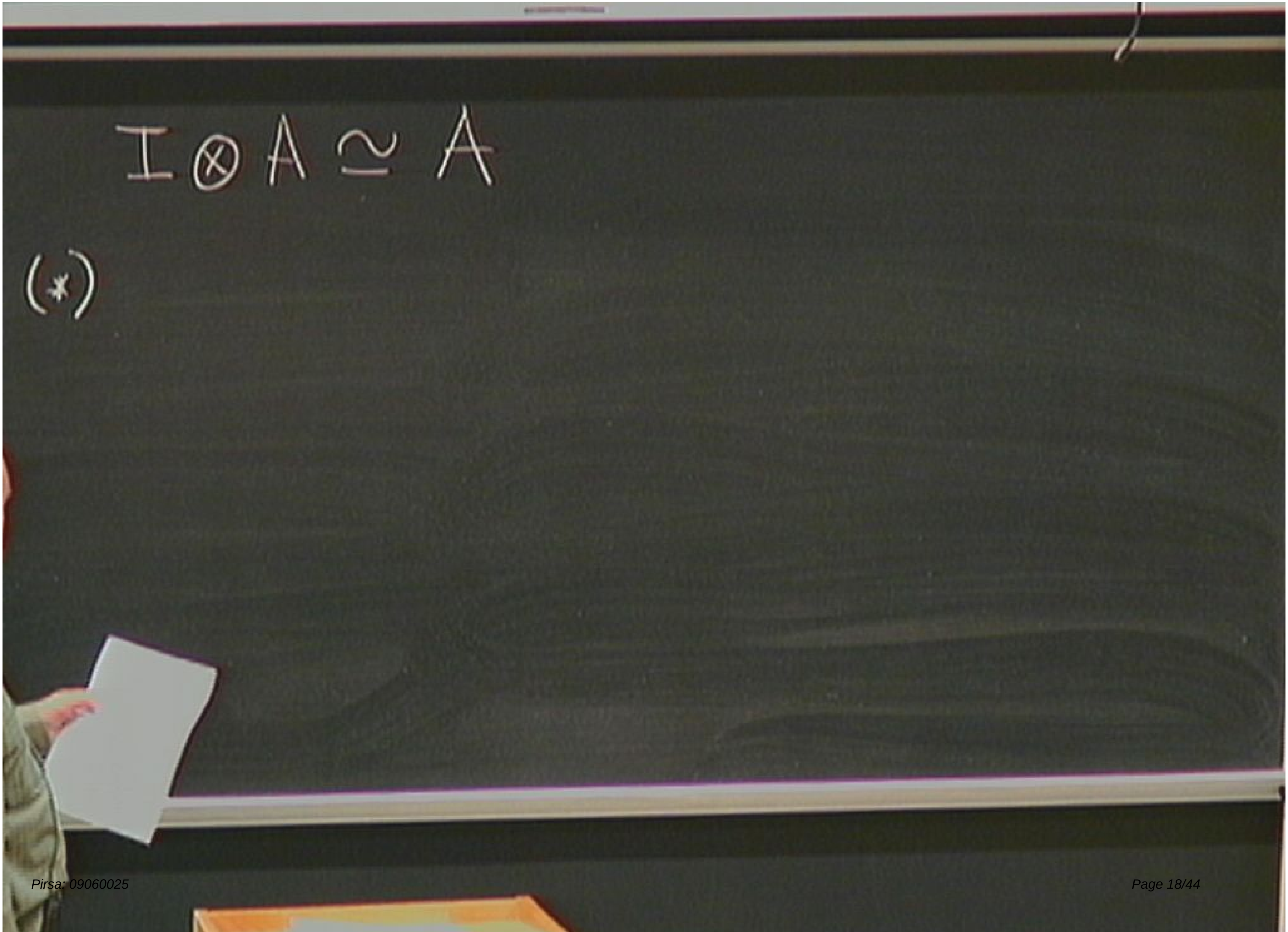
$$\bar{\beta} = \langle \beta_2, \beta_1 \rangle \quad B, A \longrightarrow X$$

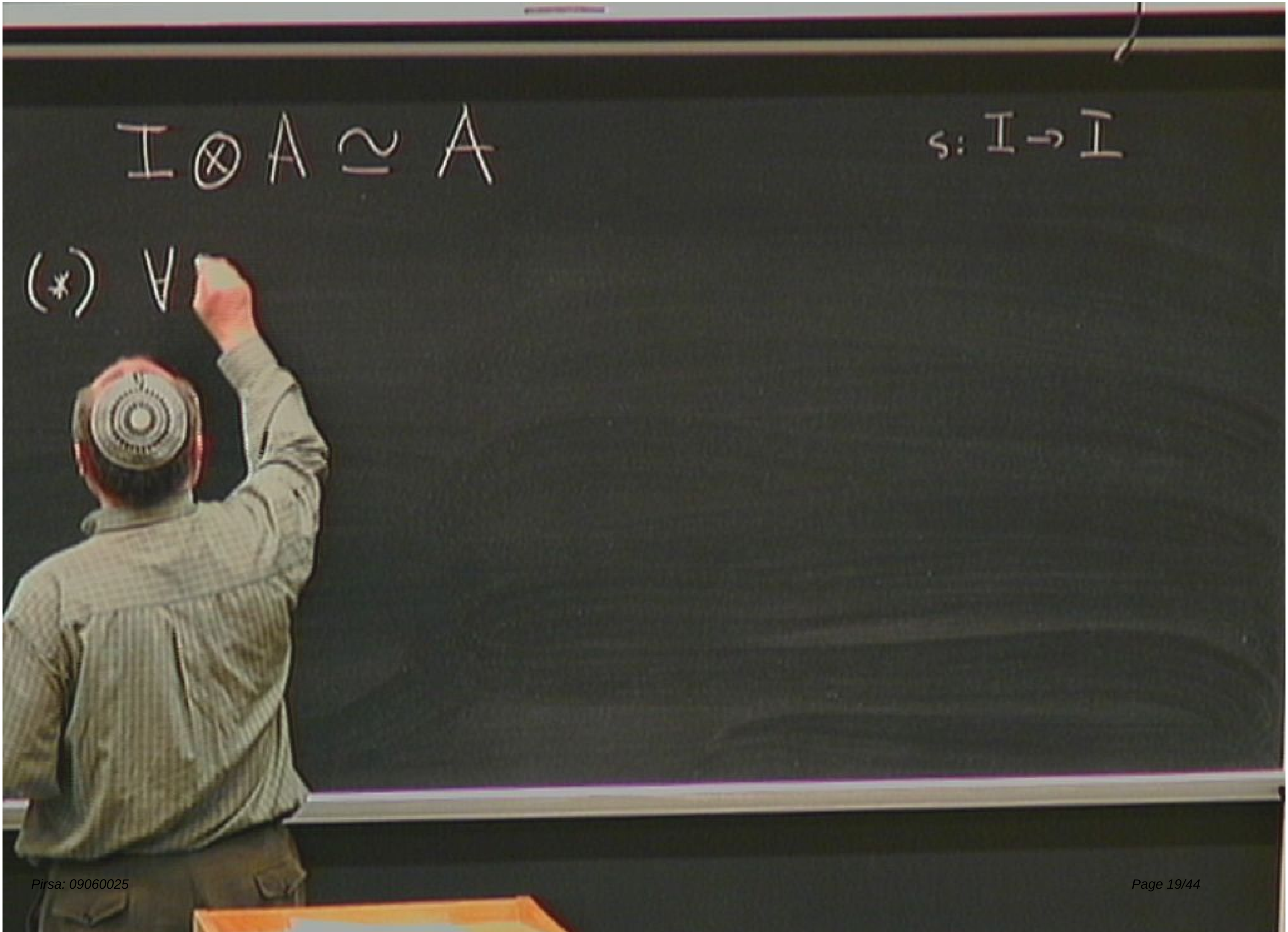
$$\bar{\bar{\beta}} = \beta \qquad \overline{f \circ \beta} = f \circ \bar{\beta}$$

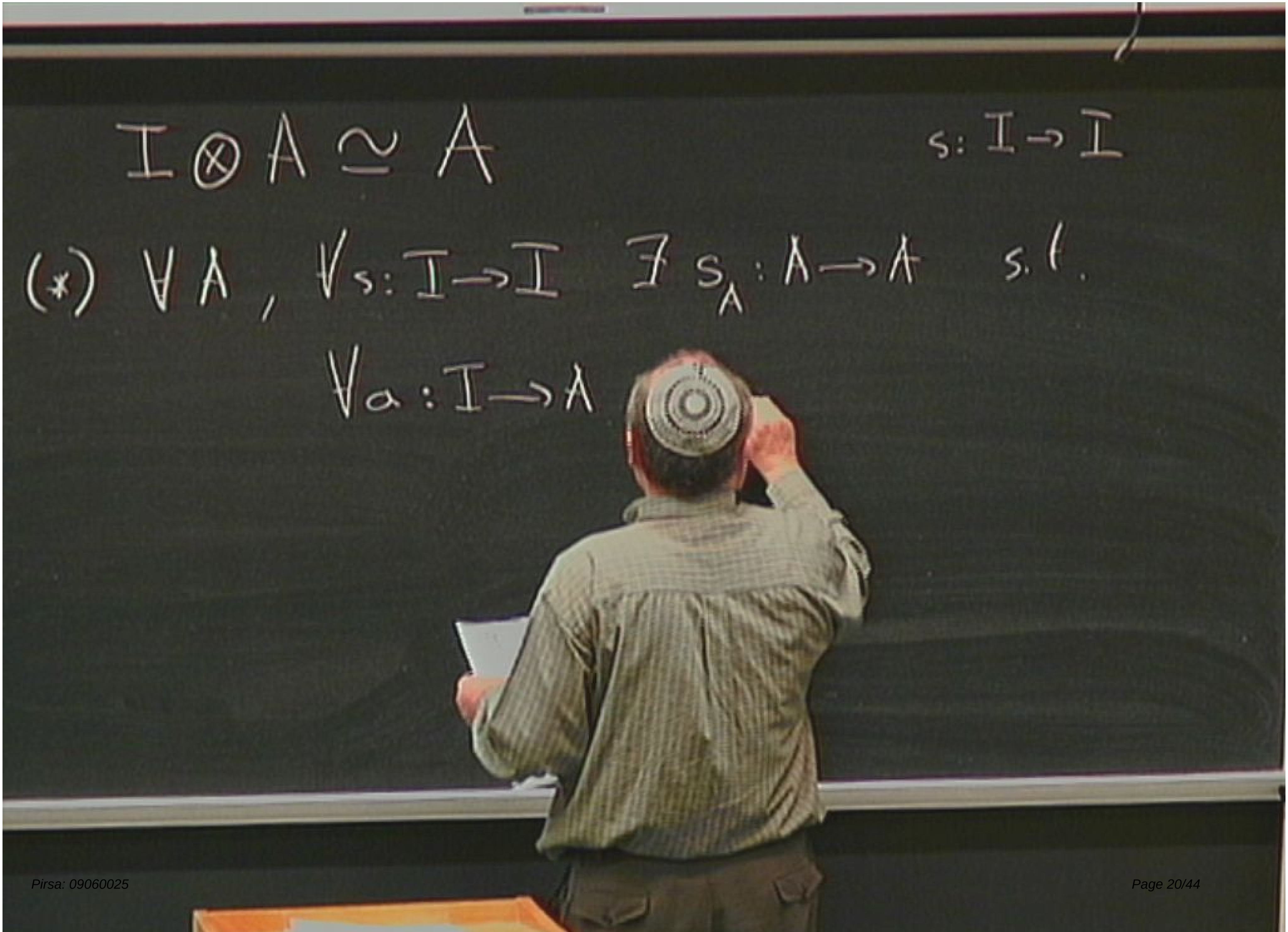
$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[rr, bend right] & A \otimes B \arrow[r, dotted] & B \otimes A \\
B \arrow[u] \arrow[urr, bend right, "\bar{\alpha}"'] & &
\end{tikzcd}$$

$$\alpha(B, A) : B, A \longrightarrow B \otimes A$$

$$\overline{\alpha(B, A)} : A, B \longrightarrow B \otimes A$$


$I \otimes A \cong A$
(*)


I ⊗ A ≅ A
s: I → I
(*) ∀


$I \otimes A \cong A$
$s : I \to I$
$(*)$ $\forall A,\ \forall s : I \to I$ $\exists\, s_A : A \to A$ s.t.
$\forall a : I \to A$

$$I \otimes A \cong A \qquad s : I \to I$$

$(*)$ $\forall A$, $\forall s : I \to I$ $\exists s_A : A \to A$ s.t.

$\forall a : I \to A$ $\quad a \circ s = s_A \circ a$

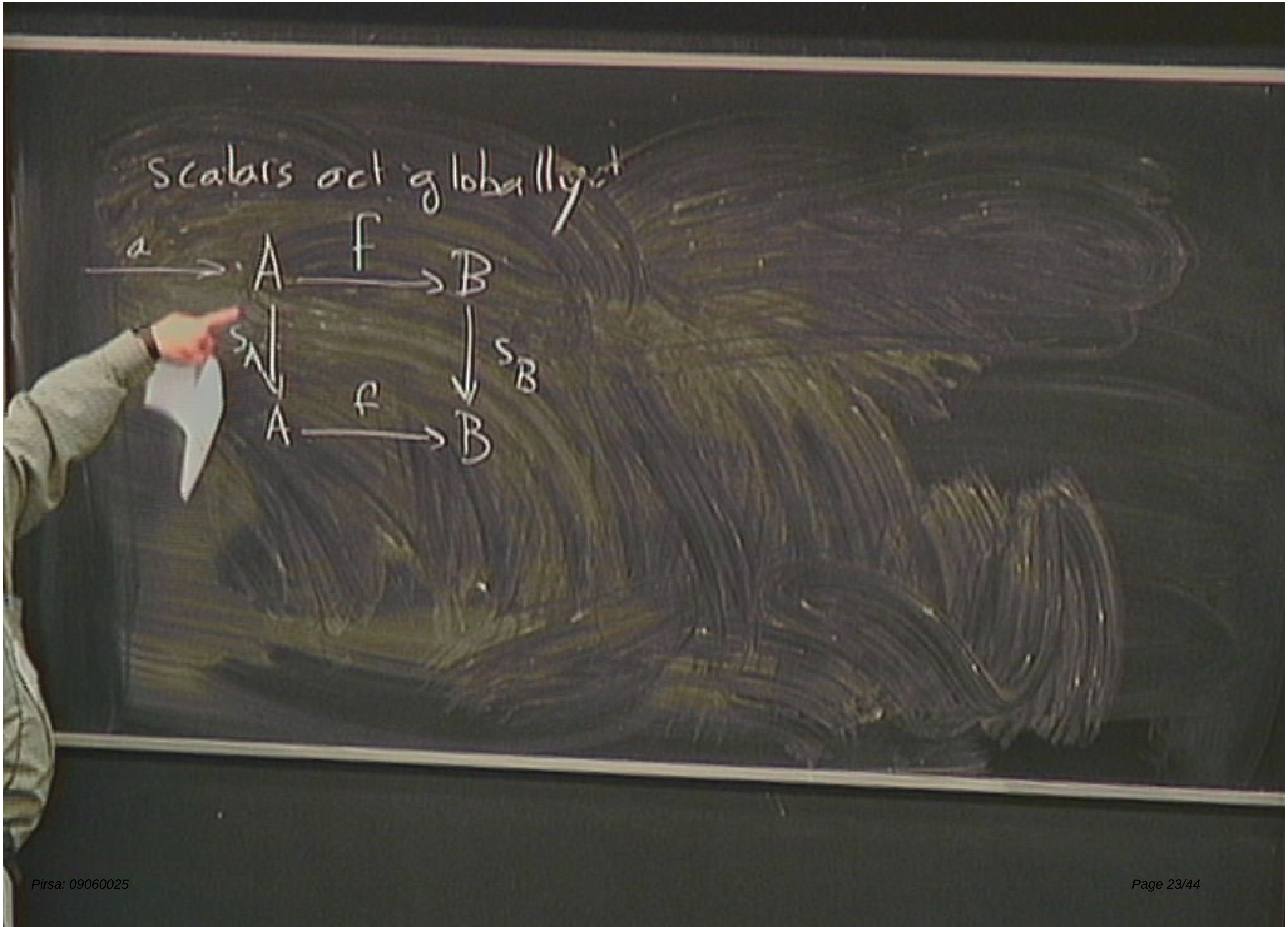
$$I \otimes A \simeq A \qquad s : I \to I$$

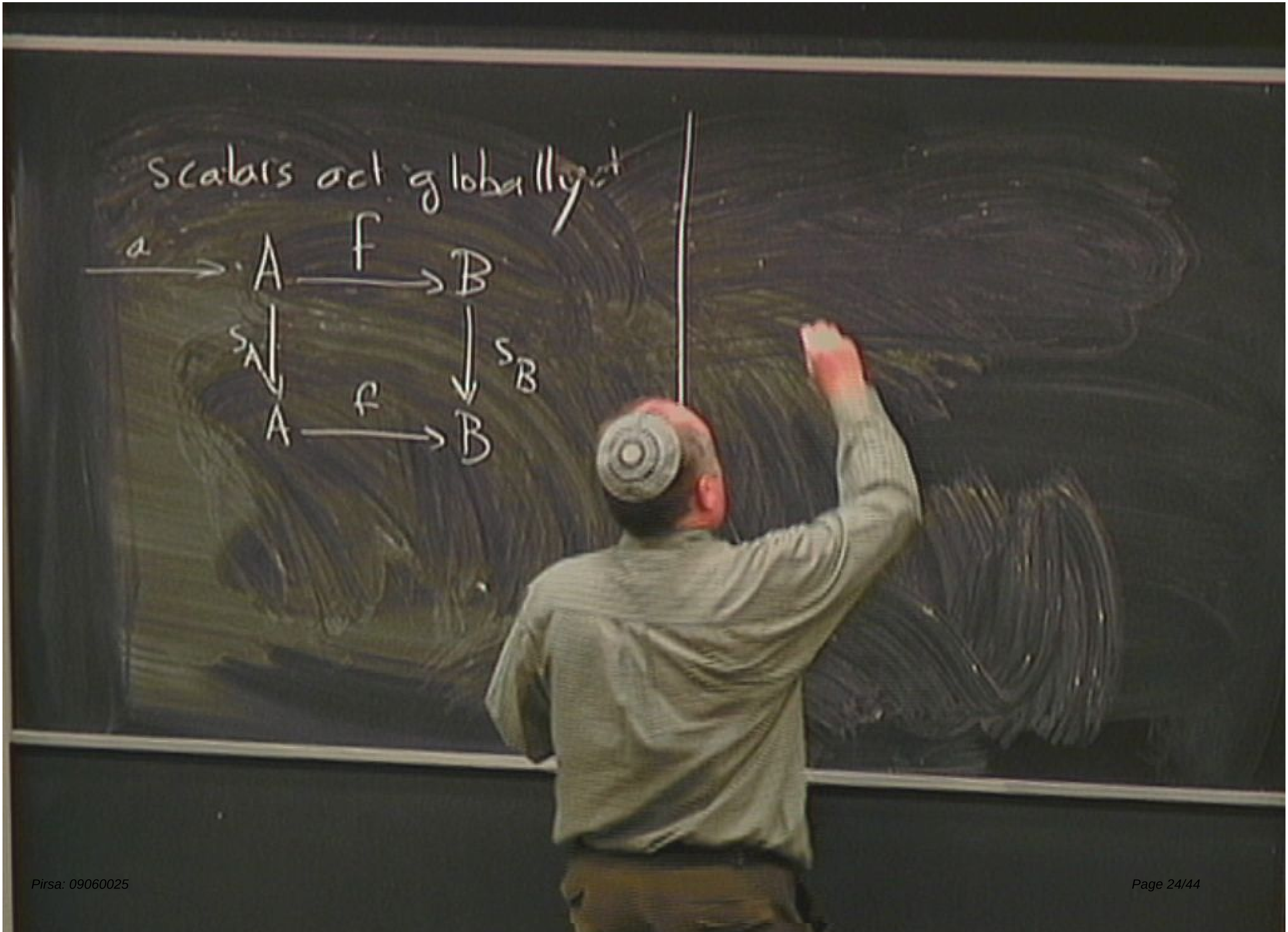
(*) $\forall A$, $\forall s : I \to I$ $\exists s_A : A \to A$ s.t.

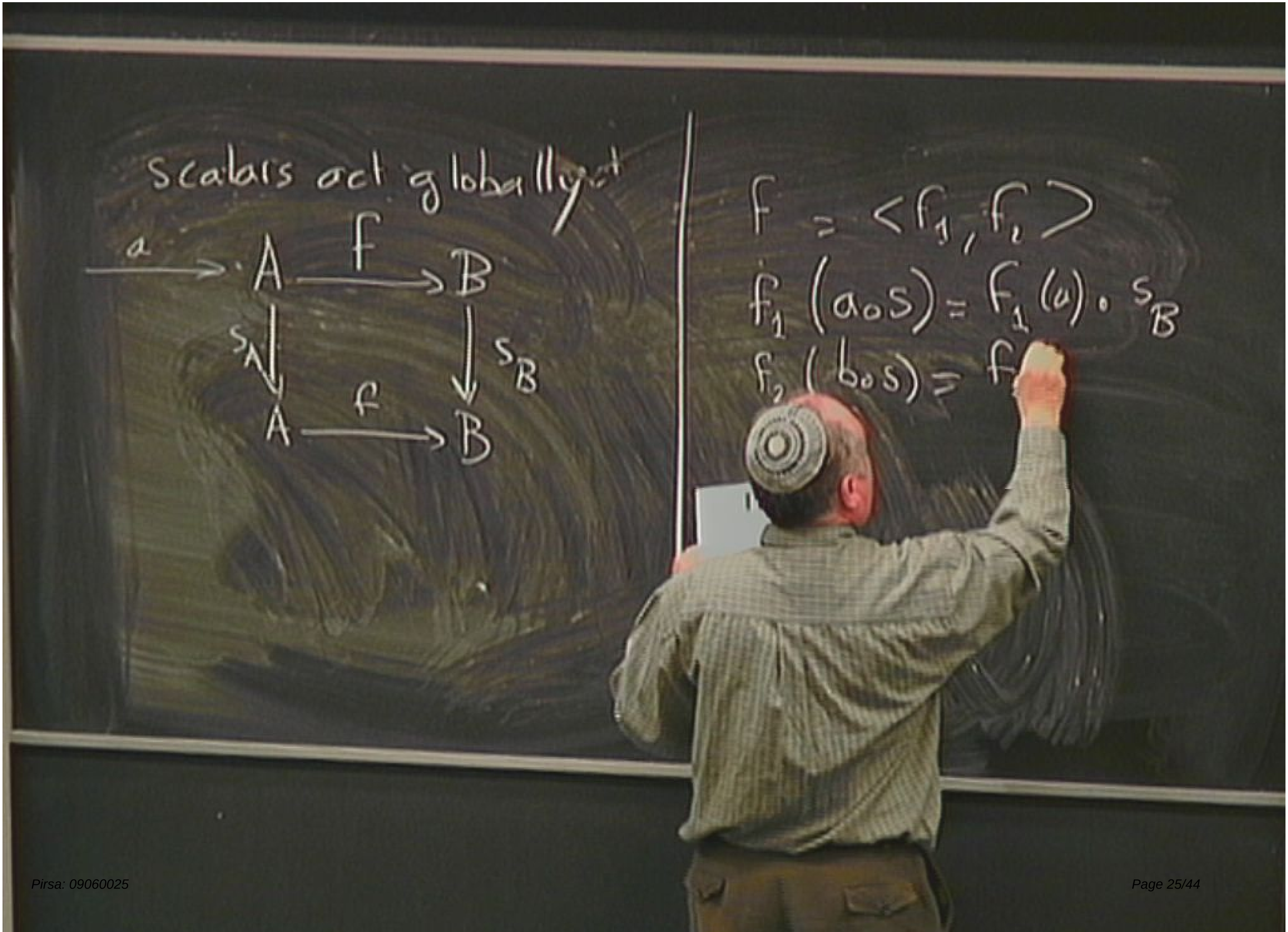
$\forall a : I \to A$ $\quad a \circ s = s_A \circ a$

(**) $\forall A, B$ $\quad f, g : A \to B$ if $f \neq g$ $\exists a : I \to A$

s.t. $\quad f \circ a \neq g \circ a$


Scalars act globally
a
A
f
B
s_A
s_B
A
f
B


Scalars act globally
a
A
f
B
s_A
s_B
A
f
B


Scalars act globally
f = <f1, f2>
f1 (a∘s) = f1 (a)∘ sB
f2 (b∘s) = f

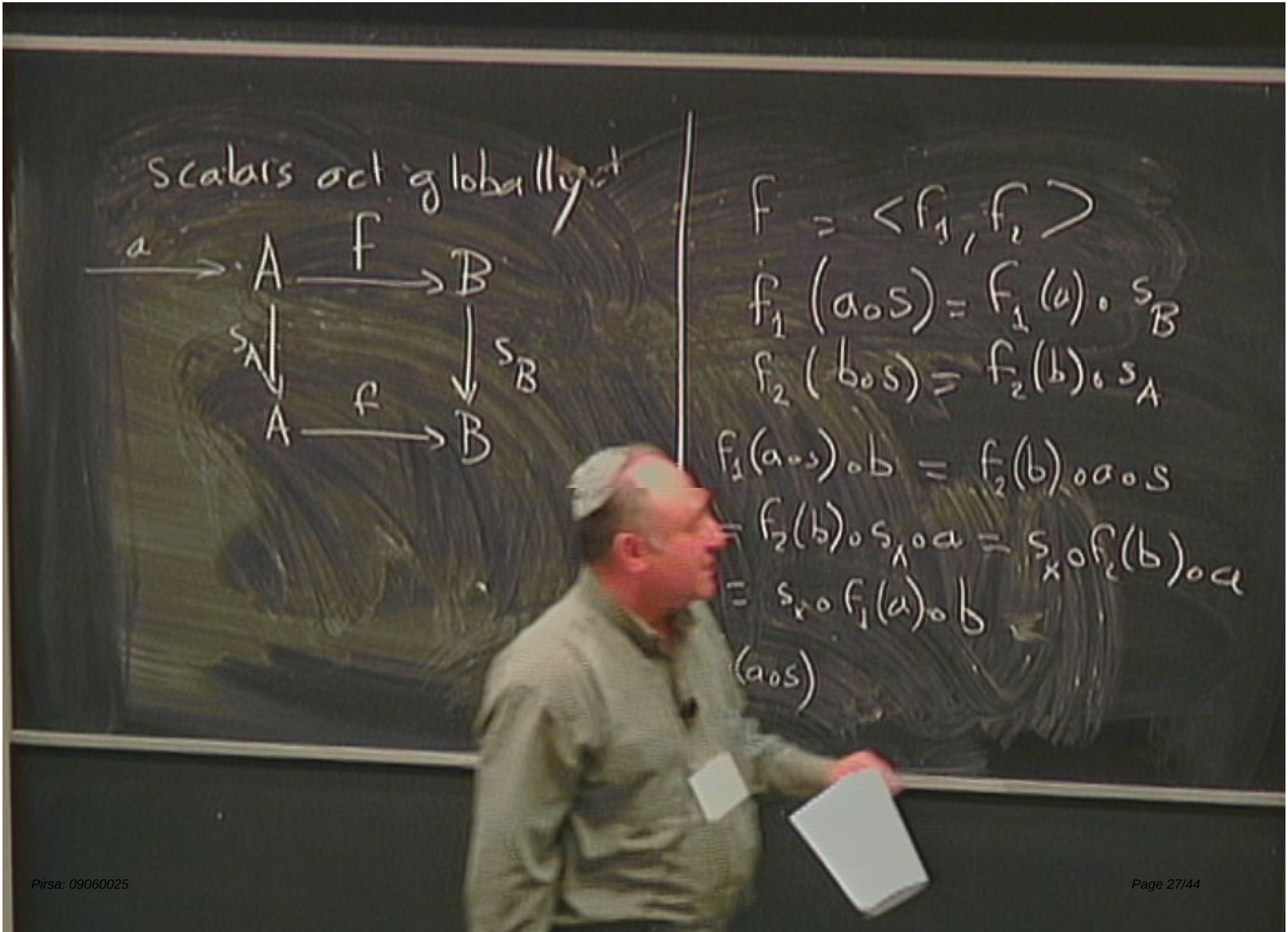
Scalars act globally

$$\begin{tikzcd} {} \arrow[r, "a"] & A \arrow[r, "f"] \arrow[d, "s_A"'] & B \arrow[d, "s_B"] \\ & A \arrow[r, "f"] & B \end{tikzcd}$$

$$f = \langle f_1, f_2 \rangle$$

$$f_1(a \circ s) = f_1(a) \circ s_B$$

$$f_2(b \circ s) = f_2(b) \circ s_A$$



Scalars act globally
f = <f1, f2>
f1 (a∘s) = f1(a)∘sB
f2 (b∘s) = f2(b)∘sA
f1(a∘s)∘b = f2(b)∘a∘s
= f2(b)∘sA∘a = sx∘f2(b)∘a
= sx∘f1(a)∘b
(a∘s)

Scalars act globally

$$\begin{tikzcd} \arrow[r, "a"] & A \arrow[r, "f"] \arrow[d, "s_A"'] & B \arrow[d, "s_B"] \\ & A \arrow[r, "f"] & B \end{tikzcd}$$

$$f = \langle f_1, f_2 \rangle$$

$$f_1(a \circ s) = f_1(a) \circ s_B$$

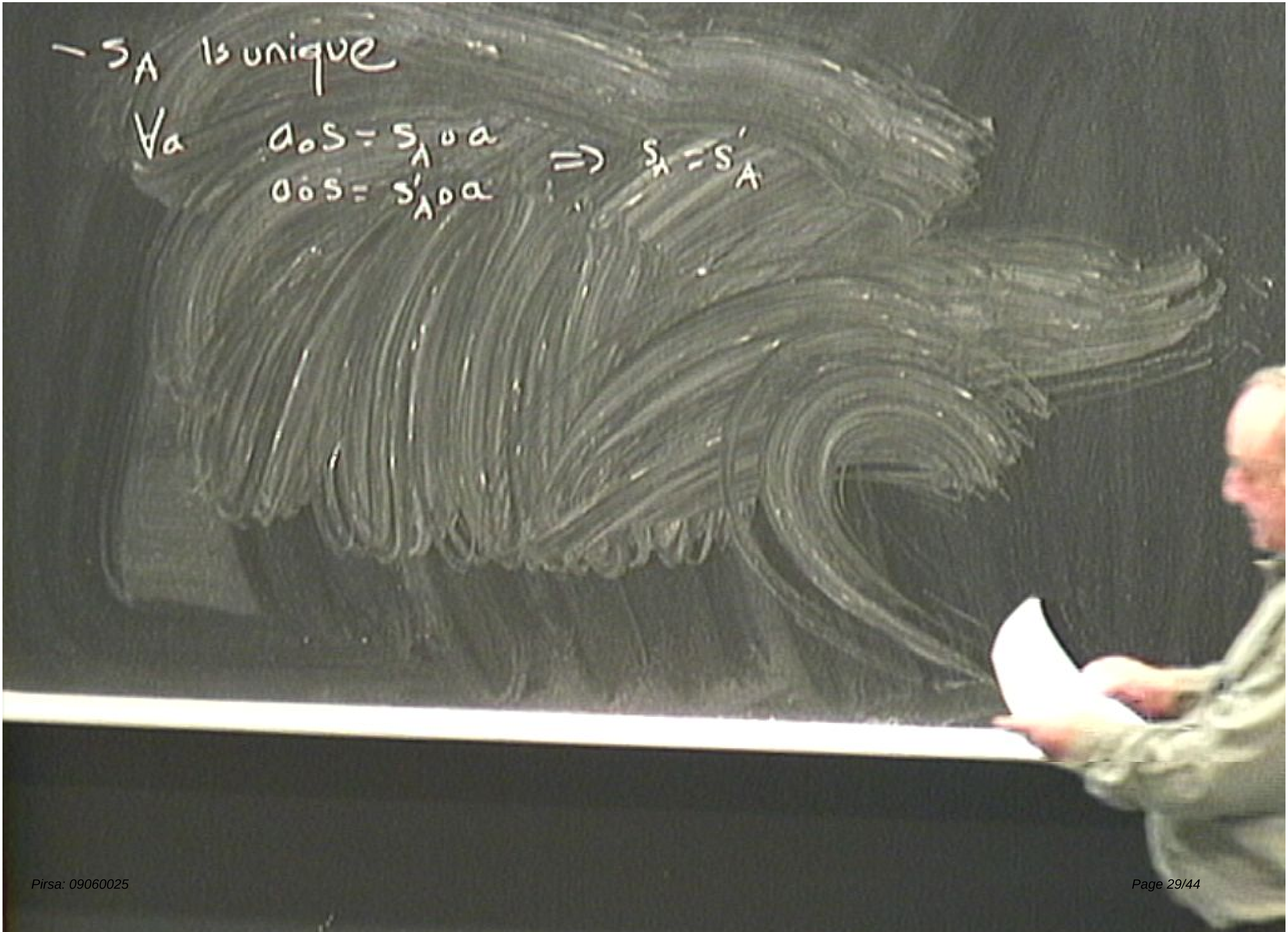
$$f_2(b \circ s) = f_2(b) \circ s_A$$

$$f_1(a \circ s) \circ b = f_2(b) \circ a \circ s$$

$$= f_2(b) \circ s_A \circ a = s_x \circ f_2(b) \circ a$$

$$= s_x \circ f_1(a) \circ b$$

$$f_1(a \circ s) = s_x \circ f_1(a)$$

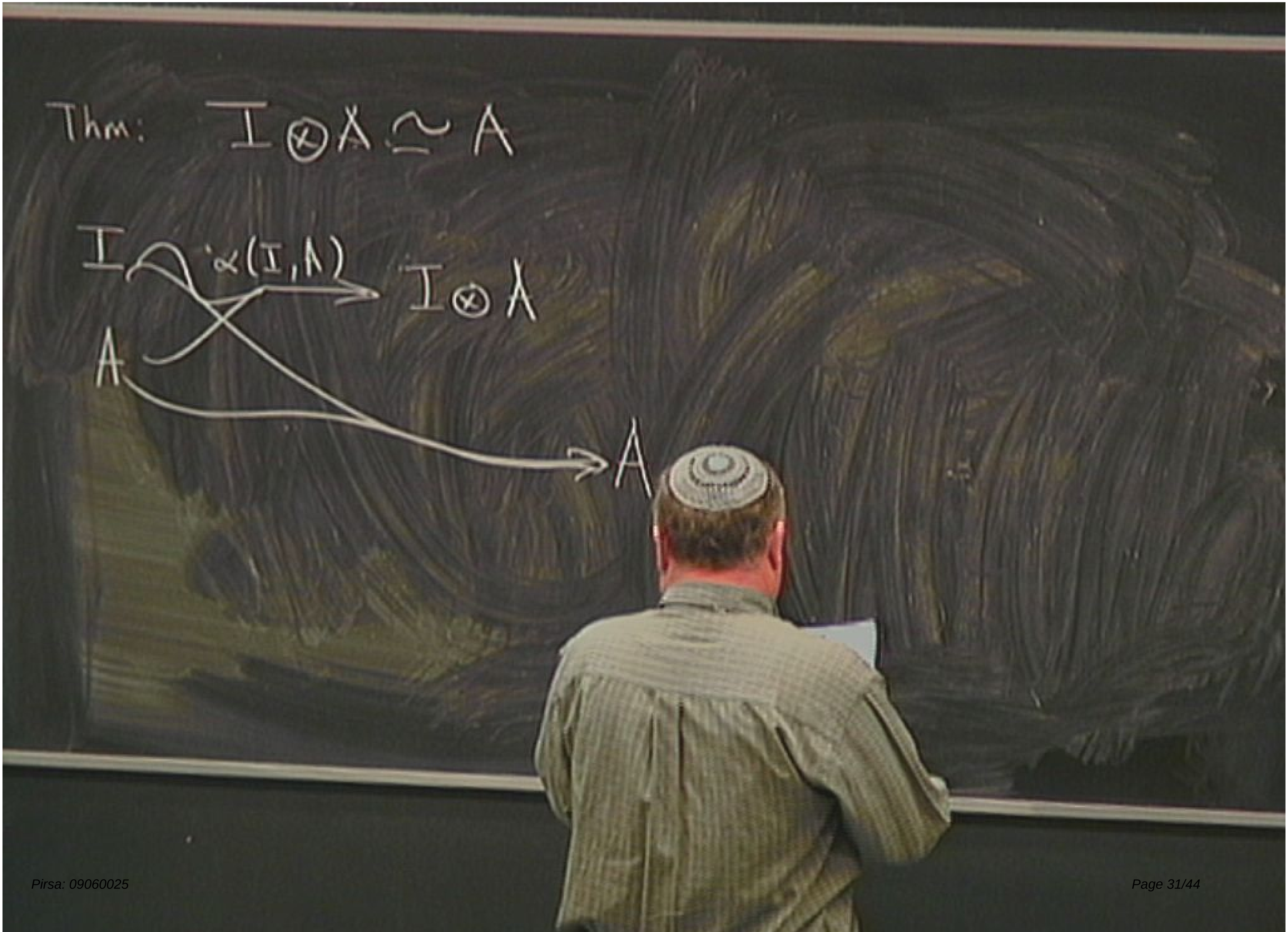


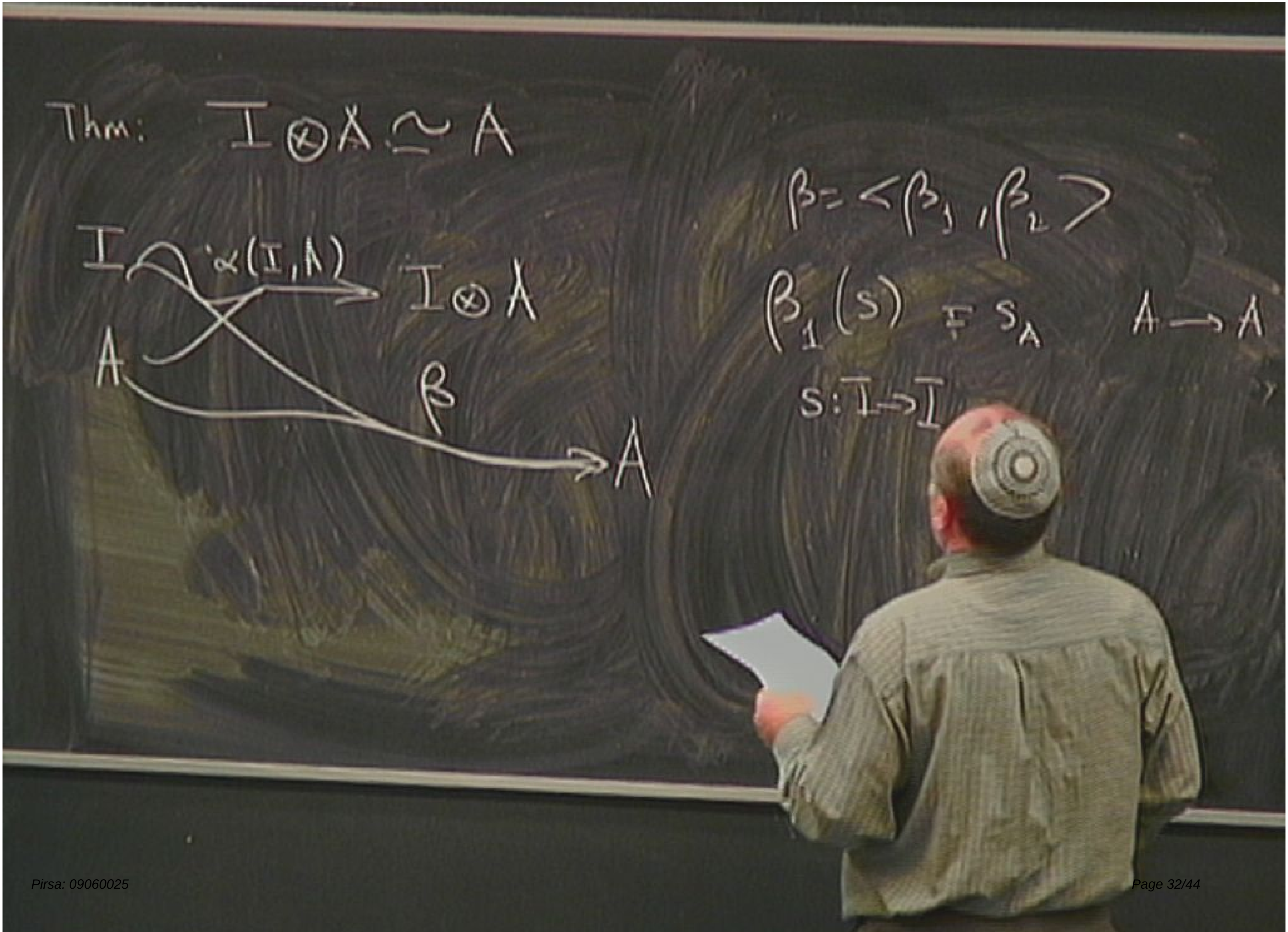
$- s_A$ is unique
$\forall a \quad a \circ s = s_A \circ a \quad \Rightarrow s_A = s'_A$
$a \circ s = s'_A \circ a$

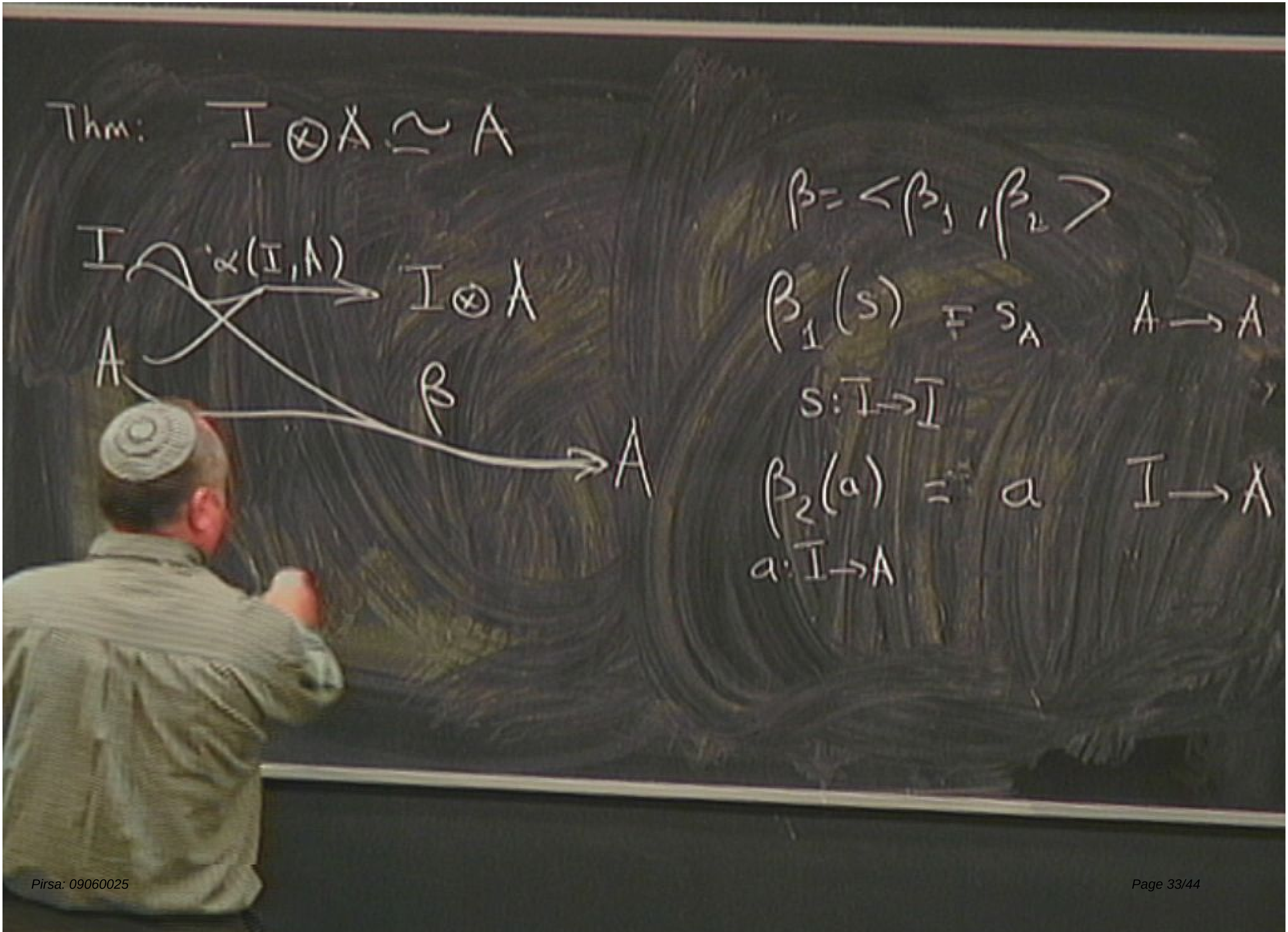
- s_A is unique

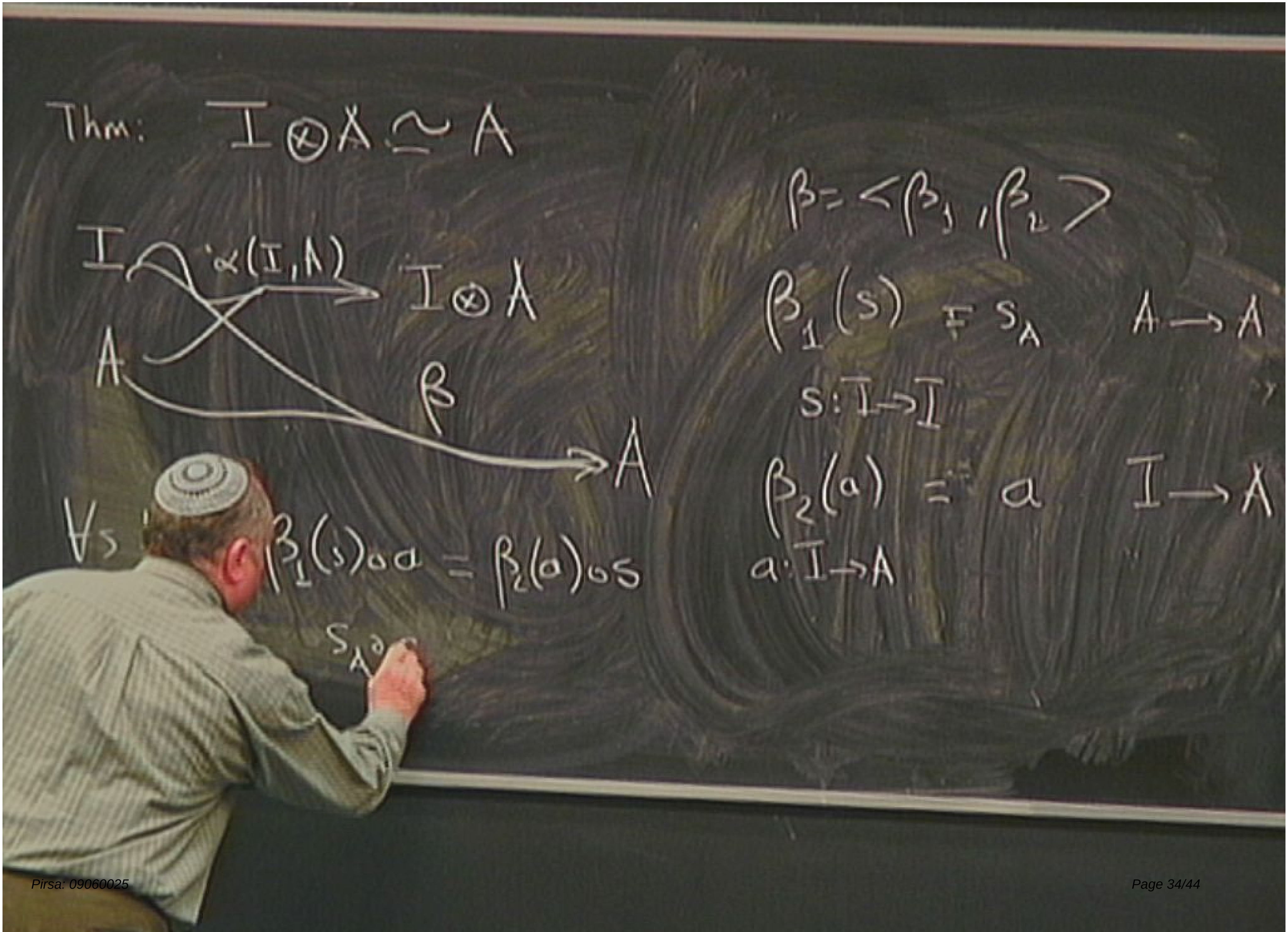
$\forall a \quad a \circ s = s_A \circ a$, $a \circ s = s'_A \circ a \Rightarrow s_A = s'_A$

- $(id_I)_A = id_A$
- $(s \circ t)_A = s_A \circ t_A$
- $s = s_I$
- $Hom(I, I)$ is commutative


Thm:


Thm: $I \otimes A \simeq A$
I
A
$\alpha(I, A)$
$I \otimes A$
β
A
$\beta = \langle \beta_1, \beta_2 \rangle$
$\beta_1(s) \otimes s_A$
$A \to A$
$s: I \to I$


Thm: $I \otimes A \simeq A$
I
A
$\alpha(I,A)$
$I \otimes A$
β
A
$\beta = \langle \beta_1, \beta_2 \rangle$
$\beta_1(s) = s_A$
$A \to A$
$s : I \to I$
$\beta_2(a) = a$
$I \to A$
$a : I \to A$


Thm: $I \otimes A \simeq A$
$\beta = \langle \beta_1, \beta_2 \rangle$
$\alpha(I, A)$
$I \otimes A$
$\beta_1(s) = s_A$
$A \to A$
$s : I \to I$
β
A
$\beta_2(a) = a$
$I \to A$
$\forall s$
$\beta_1(s) \circ a = \beta_2(a) \circ s$
$a : I \to A$
$s_A \circ$

Thm: $I \otimes A \simeq A$

$$\begin{tikzcd} I \arrow[r, "\alpha(I,A)"] \arrow[rd, bend left] & I \otimes A \arrow[d, dashed, "m"] \\ A \arrow[ru] \arrow[r, bend right, "\beta"'] & A \end{tikzcd}$$

$\beta = \langle \beta_1, \beta_2 \rangle$

$\beta_1(s) = s_A \qquad A \to A$

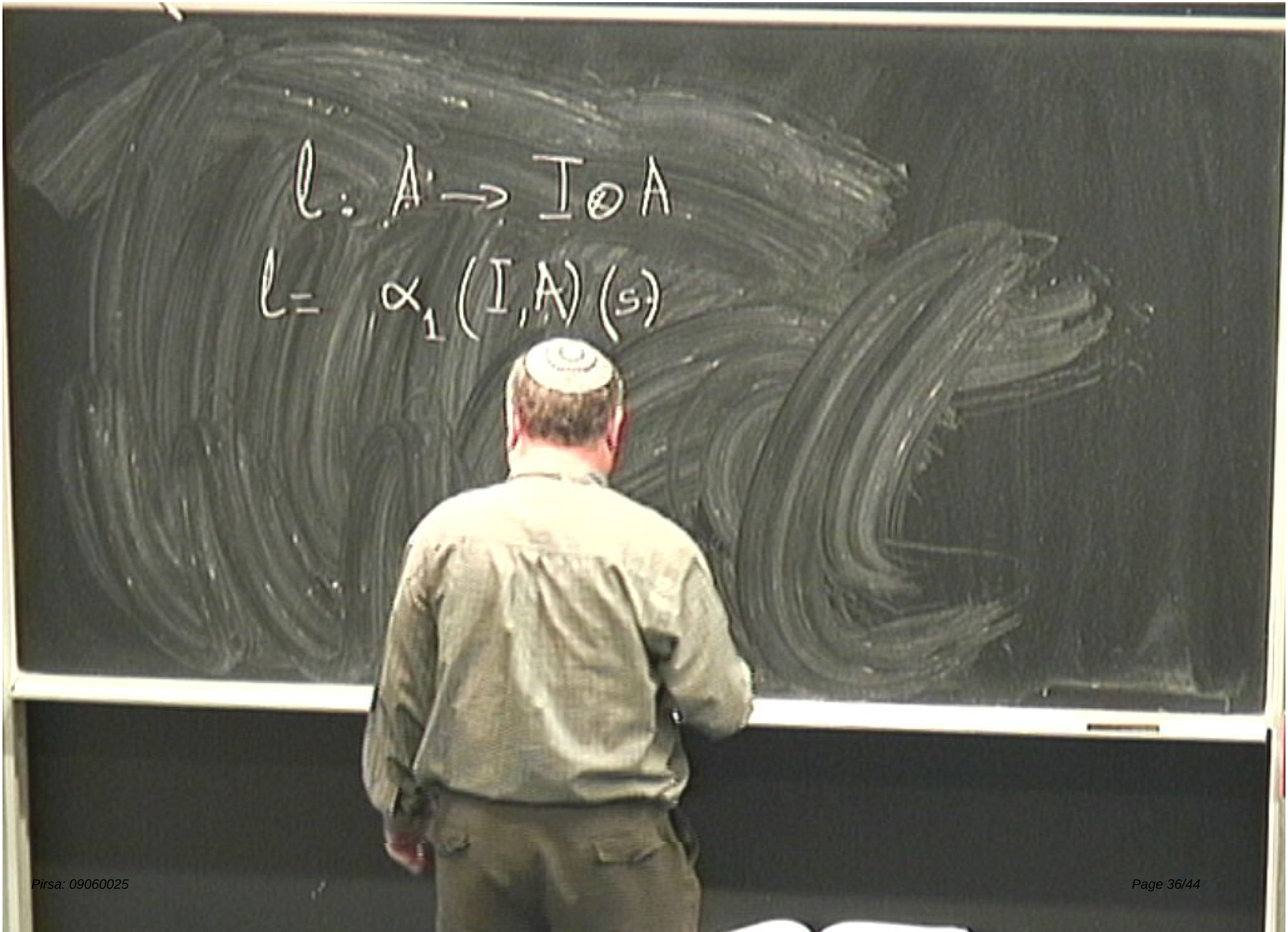
$s : I \to I$

$\beta_2(a) = a \qquad I \to A$

$a : I \to A$

$\forall s\ \forall a \quad \beta_1(s) \circ a = \beta_2(a) \circ s$

$s_A \circ a = a \circ s$

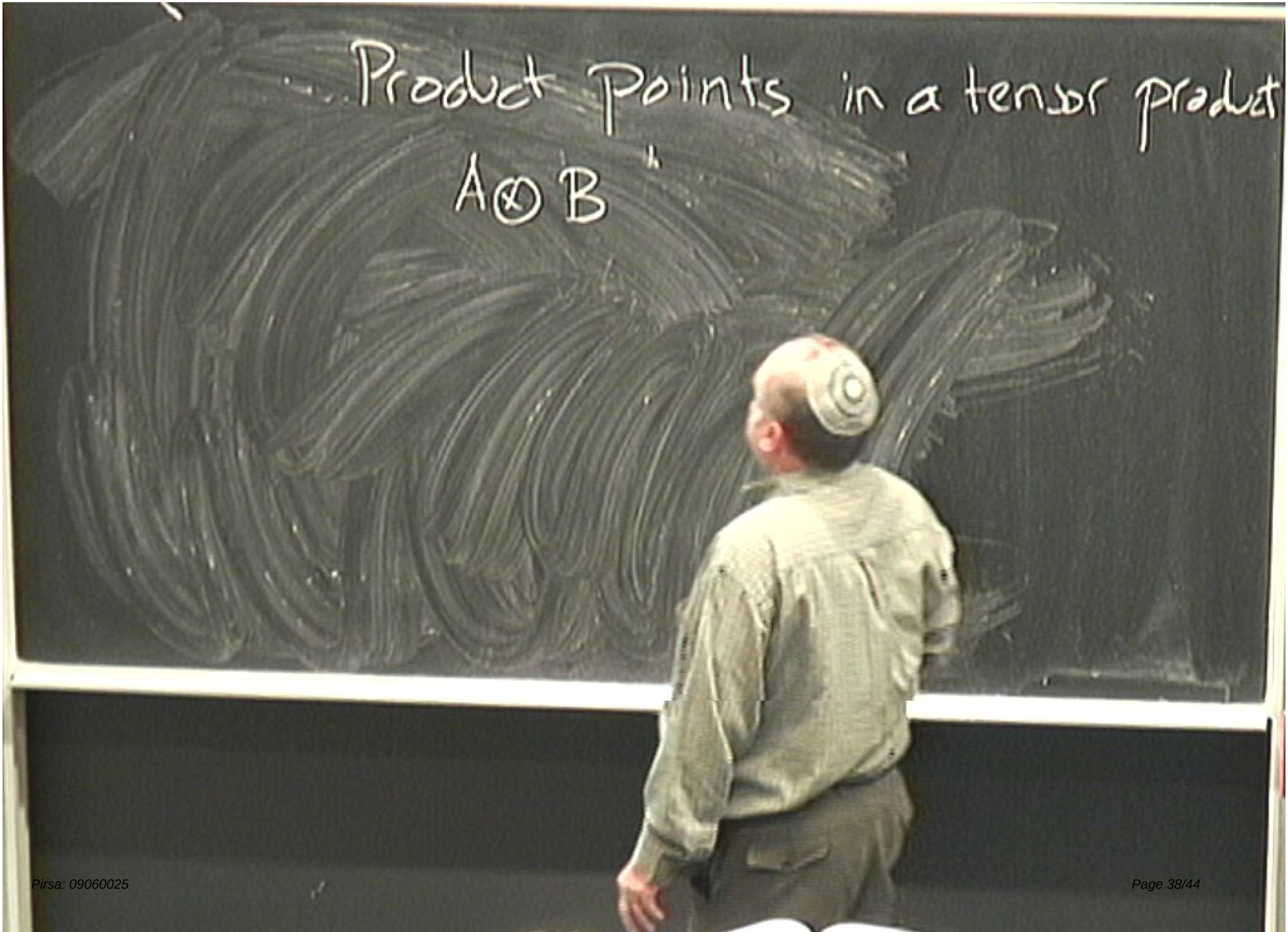

l: A → I ⊗ A
l = α1(I,A)(s)

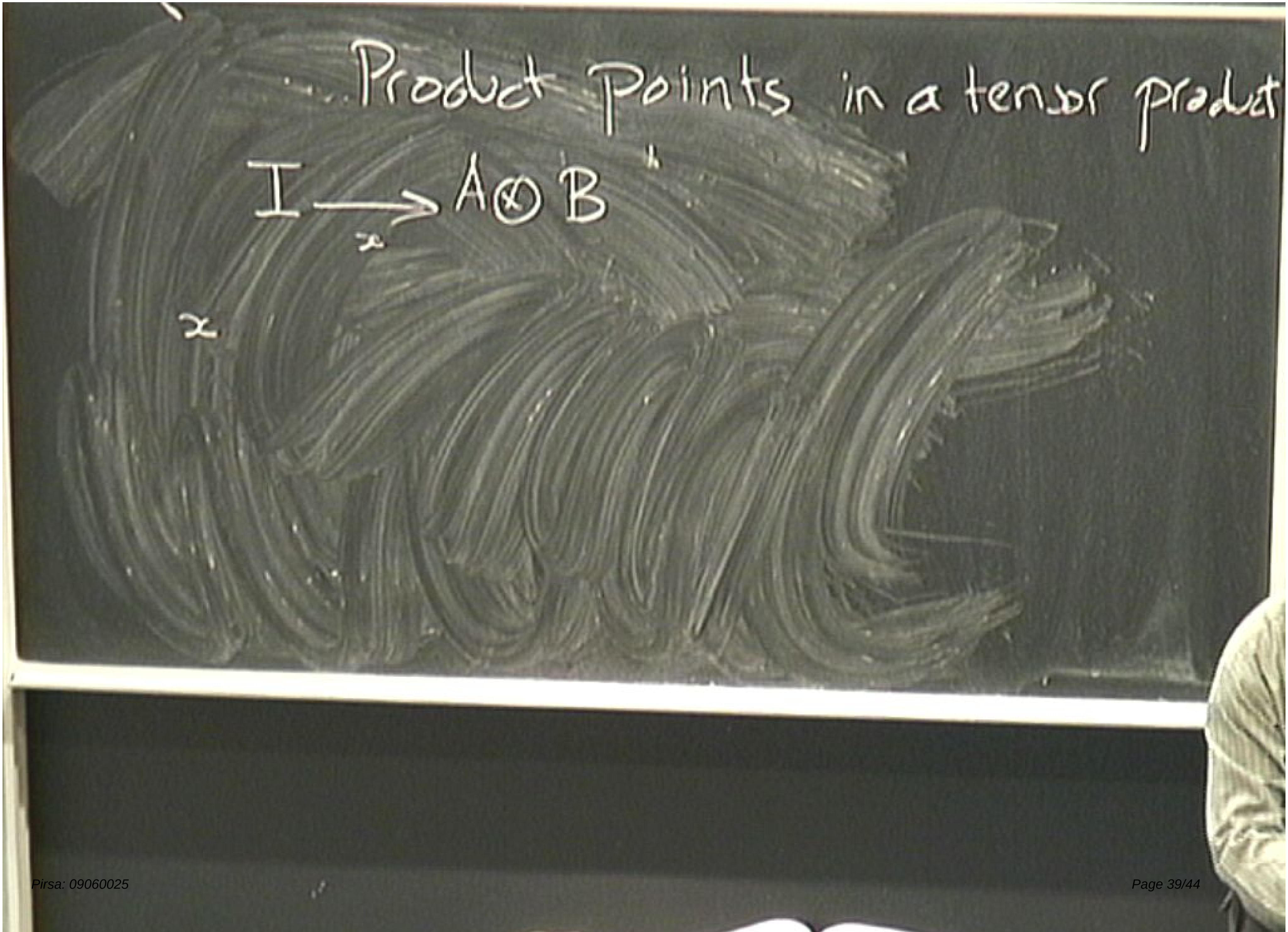
$l: A \to I \otimes A$

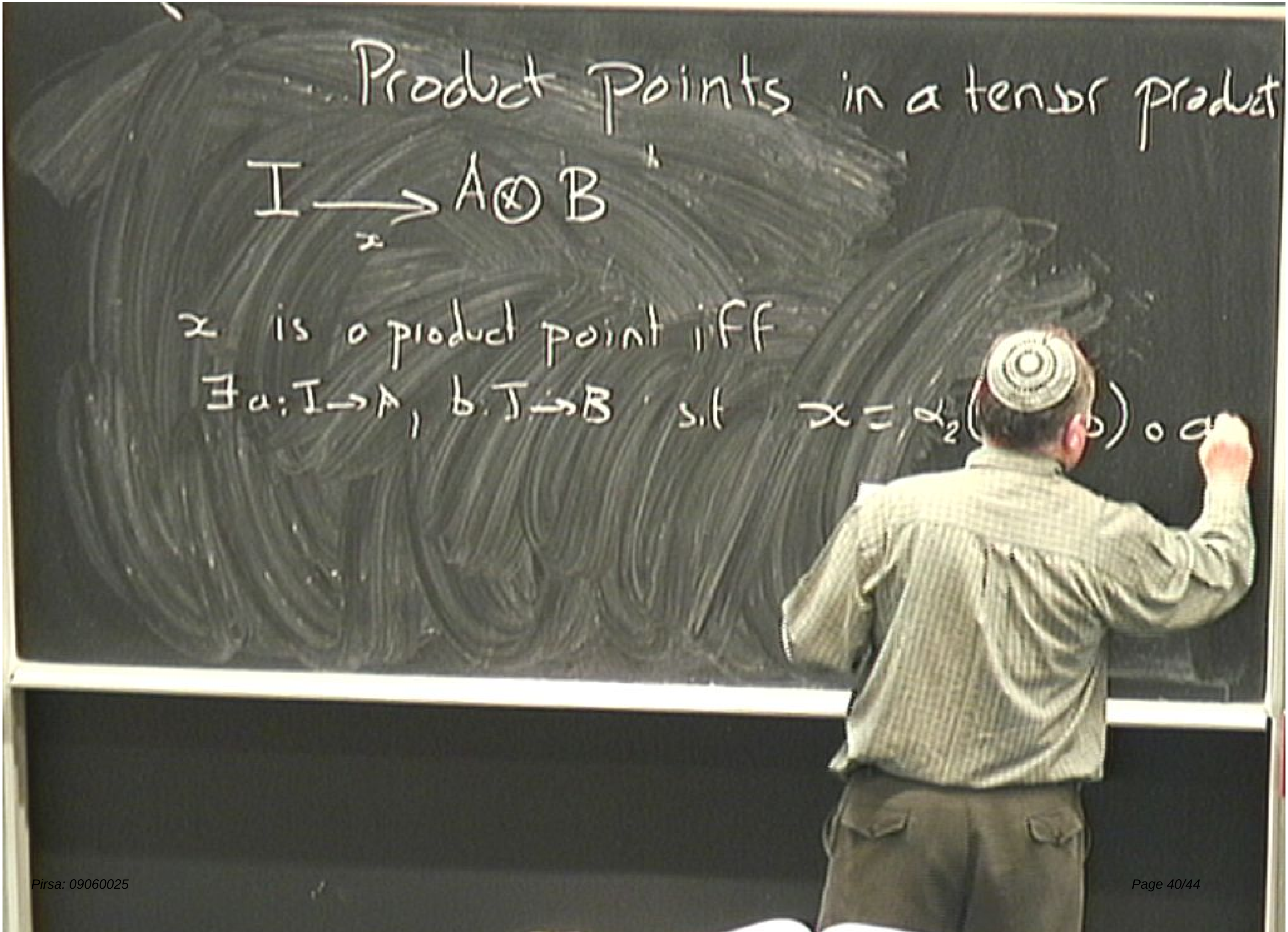
$l = \alpha_1(I, A)(\text{I}_{\otimes \text{I}})$

$m \circ l = id_A$

$l \circ m = id_{I \otimes A}$


Product Points in a tensor product
$A \otimes B$


Product Points in a tensor product
$I \xrightarrow{x} A \otimes B$
x

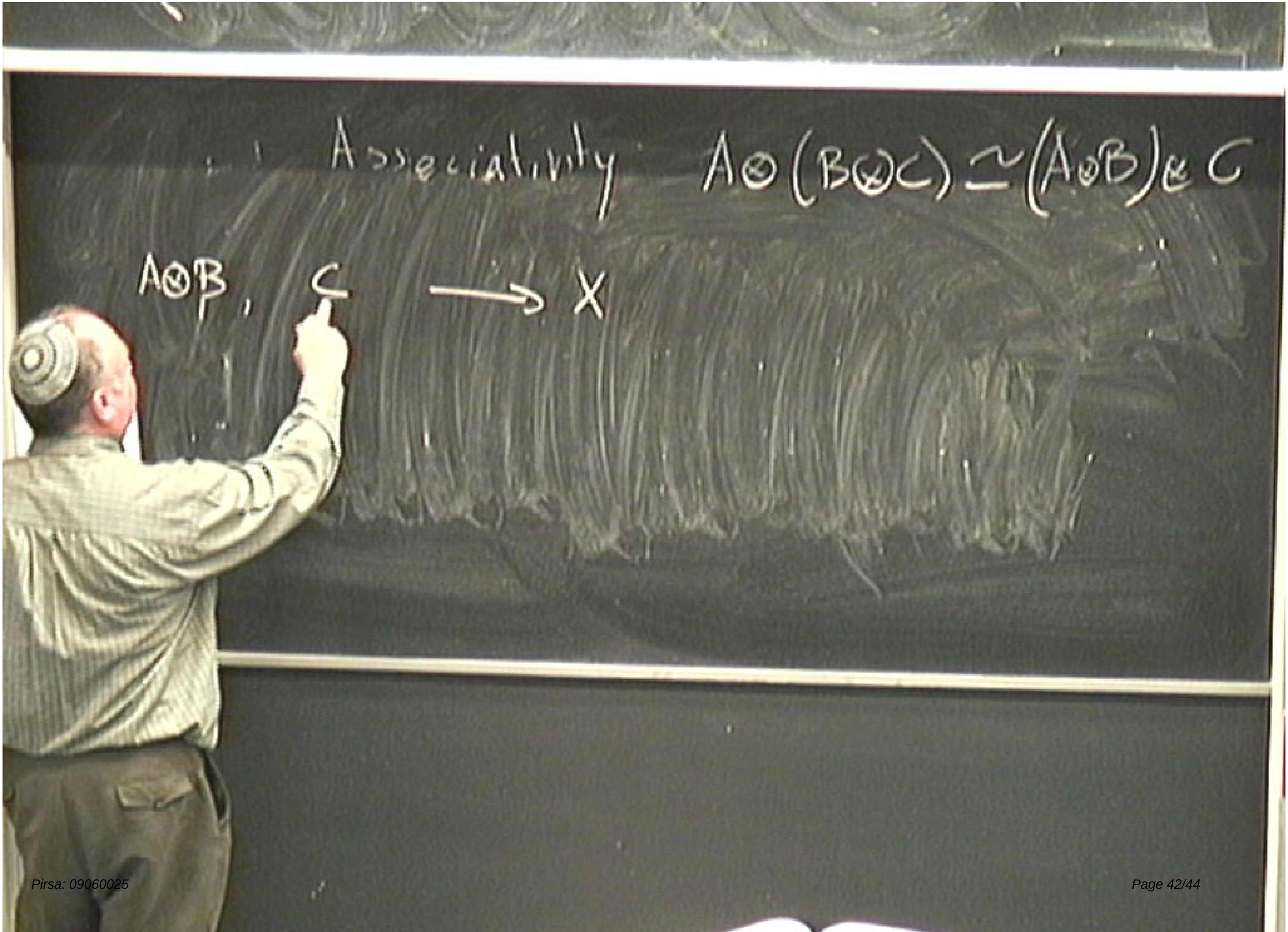

Product Points in a tensor product
$I \xrightarrow[x]{} A \otimes B$
x is a product point iff
$\exists a : I \to A,\ b : J \to B$ s.t. $x = \alpha_2(\quad b) \circ a$

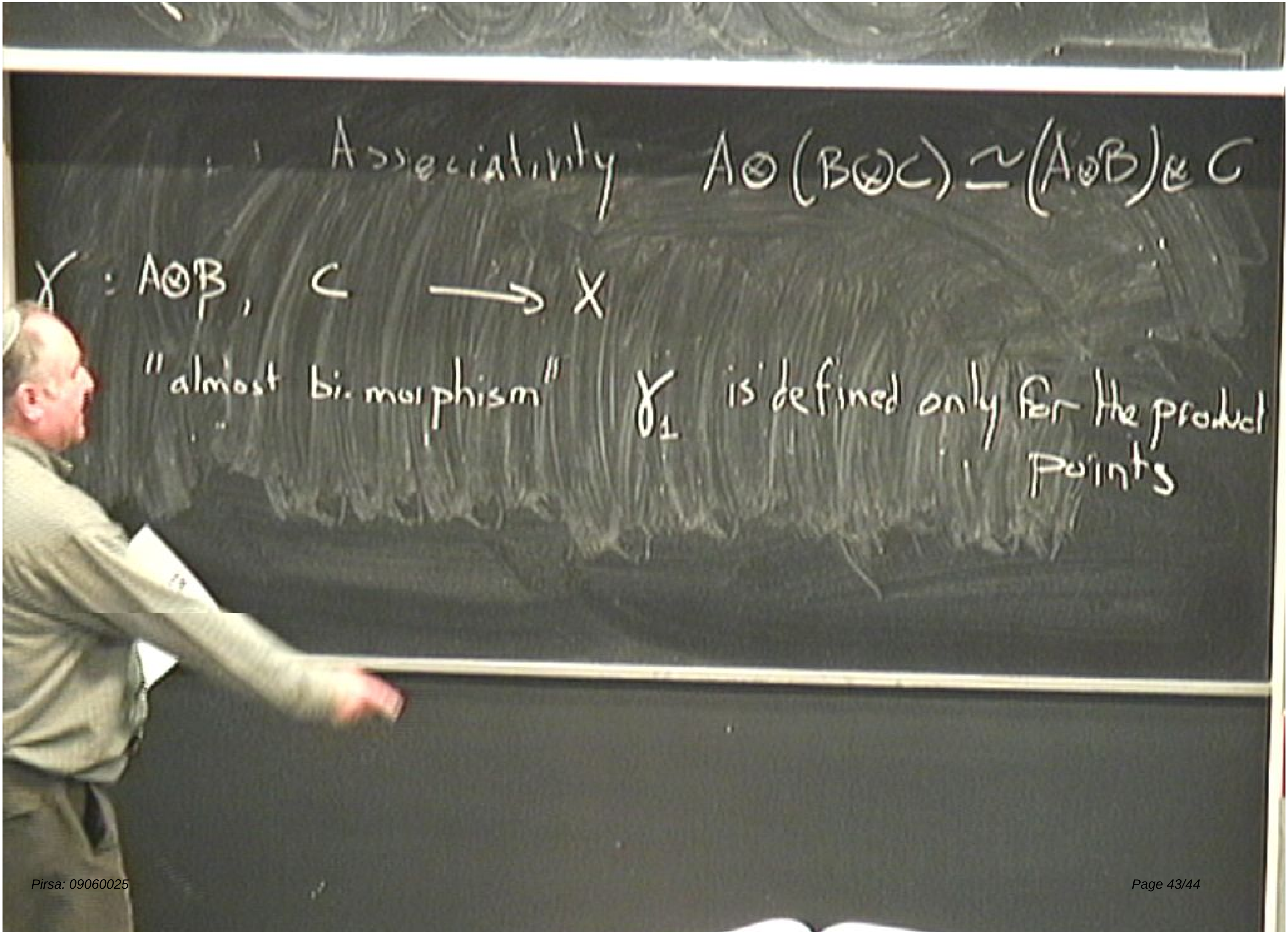
Products are dense in $A \otimes B$:

$$A \otimes B \xrightarrow{f,g} X$$

$\forall$ product point $x : I \to A \otimes B$

we have $f \circ x = g \circ x$ then $f = g$


Associativity A⊗(B⊗C) ≃ (A⊗B)⊗C
A⊗B, C ⟶ X


Associativity $A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$
$\gamma : A \otimes B,\ C \longrightarrow X$
"almost bi-morphism"
γ_1 is defined only for the product points

Associativity $A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$

$\gamma : A \otimes B, \quad C \longrightarrow X$

"almost bi-morphism" γ_1 is defined only for the product points