Title: Quantum analogues of Bayes' theorem, sufficient statistics and the pooling problem

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Abstract: The notion of a conditional probability is critical for Bayesian reasoning. Bayes’ theorem, the engine of inference, concerns the inversion of conditional probabilities. Also critical are the concepts of conditional independence and sufficient statistics. The conditional density operator introduced by Leifer is a natural generalization of conditional probability to quantum theory. This talk will pursue this generalization to define quantum analogues of Bayes' theorem, conditional independence and sufficient statistics. These can be used to provide simple proofs of certain well-known results in quantum information theory, such as the isomorphism between POVMs and convex decompositions of a mixed state and the remote collapse postulate, and to prove some novel results on how to pool quantum states. This is joint work with Matt Leifer. I will also briefly discuss the possibility of a diagrammatic calculus for classical and quantum Bayesian inference (joint work with Bob Coecke).

Quantum analogues of Bayes' theorem, conditional independence and Markov chains

Robert Spekkens
Perimeter Institute for Theoretical Physics

Including joint work with Matt Leifer and Bob Coecke

Industry Canada Industrie Canada

Canada

June 3, 2009
Categories, Quanta and Concepts

MINISTRY OF RESEARCH & INNOVATION

or

Evidence for quantum theory being a noncommutative generalization of classical probability theory

Classical

A, B : random variables

$P(A) \equiv \{P(A=a) \mid a \in \mathcal{A}\}$

+ve fn' on $\mathcal{A}$

$$\sum_A P(A) \equiv \sum_a P(A=a) = 1$$

joint $P(A,B) \equiv \{P(A=a, B=b) \mid a \in \mathcal{A}, b \in \mathcal{B}\}$

$$\sum_{A,B} P(A,B) = 1$$

marginal $P(B) = \sum_A P(A,B)$

conditional probability $P(A|B) = \dfrac{P(A,B)}{P(B)}$

+ve function on $\mathcal{A} \times \mathcal{B}$

defined for values of B s.t. $P(B=b) \neq 0$

$$\sum_A P(A|B) = 1$$

Quantum

$\mathcal{H}_A, \mathcal{H}_B$: Hilbert spaces

$\rho_A \in \mathcal{B}(\mathcal{H}_A)$

+ve operator on $\mathcal{H}_A$

$$\text{Tr}_A(\rho_A) = 1$$

joint $\rho_{AB} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$

$$\text{Tr}_{AB}(\rho_{AB}) = 1$$

marginal $\rho_B = \text{Tr}_A(\rho_{AB})$

?

Classical

conditional probability

$$P(A|B) = \frac{P(A,B)}{P(B)}$$

+ve function on $A \times B$

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Quantum

conditional density operator

$$\rho_{A|B} \equiv \rho_{AB}(I_A \otimes \rho_B^{-1}) ?$$

Problem: ρ_B need not have an inverse on $\mathcal{H}_B$

Solution: Let ρ_B^{-1} be inverse on $\mathrm{supp}(\rho_B)$

if $\rho_B = \sum_k p_k |\psi_k\rangle\langle\psi_k|$

$$\rho_B^{-1} = \sum_{k|p_k \neq 0} p_k^{-1} |\psi_k\rangle\langle\psi_k|$$

$$\rho_B^{-1}\rho_B = \sum_{k|p_k \neq 0} |\psi_k\rangle\langle\psi_k|$$

$$= I_{\mathrm{supp}(\rho_B)}$$

Classical

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defined for values of B s.t. $P(B=b) \neq 0$

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$\sum_A P(A|B) = 1$

Quantum

conditional density operator $\rho_{A|B} \equiv \rho_{AB}(I_A \otimes \rho_B^{-1})$?

Problem: $[\rho_{AB}, I_A \otimes \rho_B^{-1}] \neq 0$ in general

$\therefore \rho_{A|B}$ is not +ve in general

Solution: use a symmetric expression

$$\rho_{A|B} \equiv (I_A \otimes \rho_B^{-1/2})\, \rho_{AB}\, (I_A \otimes \rho_B^{-1/2})$$

Notational conventions:
- drop "I"s
- drop "$\otimes$"s

$$\rho_{A|B} \equiv \rho_B^{-1/2} \rho_{AB} \rho_B^{-1/2}$$

Note: $\mathrm{Tr}_A \rho_{A|B} = \rho_B^{-1/2} \rho_B \rho_B^{-1/2} = I_{supp(\rho_B)}$

see: M.S. Leifer Phys Rev A 74 042310 (2006)

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+ve operator on $\mathcal{H}_A \otimes \mathcal{H}_B$

nonzero only on $\mathrm{supp}(\rho_B)$

$$\mathrm{Tr}_A(\rho_{A|B}) = I_{\mathrm{supp}(\rho_B)}$$

Example: Pure states

$$|\Psi\rangle_{AB} = \sum_k \sqrt{p_k}\, |u_k\rangle_A |v_k\rangle_B$$

$$\rho_{AB} = \sum_{k,k'} \sqrt{p_k p_{k'}}\, |u_k)_A\langle u_{k'}| \otimes |v_k)_A\langle v_{k'}|$$

$$\rho_B = \sum_k p_k\, |v_k)_A\langle v_k|$$

$$\rho_B^{-1/2} = \sum_k \frac{1}{\sqrt{p_k}}\, |v_k)_A\langle v_k|$$

$$\therefore \quad \rho_{A|B} = \rho_B^{-1/2}\, \rho_{AB}\, \rho_B^{-1/2}$$

$$= \sum_{k,k'} |u_k)_A\langle u_{k'}| \otimes |v_k)_A\langle v_{k'}|$$

$$= |\Psi\rangle_{A|B}\langle\Psi|$$

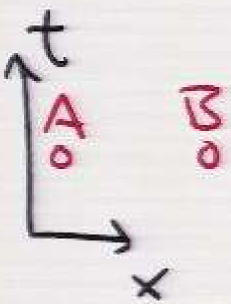
where $|\Psi\rangle_{A|B} \equiv \sum_k |u_k)_A |v_k\rangle_B$

$\rho_{A|B}$ encodes relation between Schmidt bases of A & B

ρ_B encodes Schmidt basis of B & relative amplitudes

Correlations & Causal structure

Space-like correlations

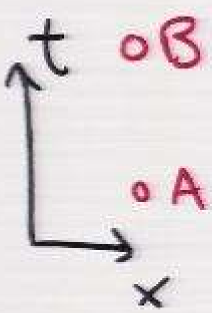


know $p(B|A)$

learn about A: $p(A)$

$$p(B) = \sum_A p(B|A)\,p(A)$$

Time-like correlations



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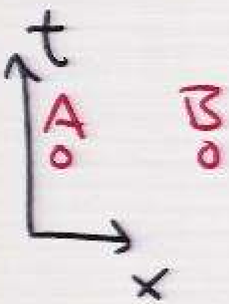

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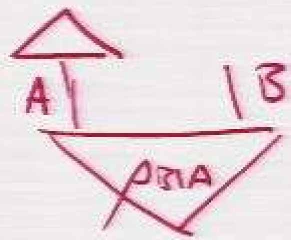

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$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\,\rho_A)$$

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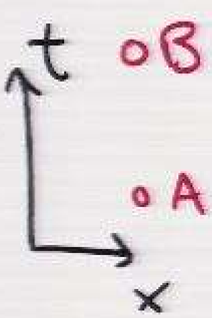

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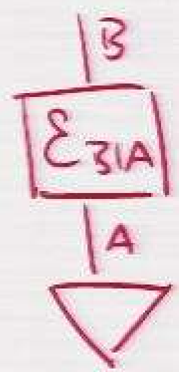

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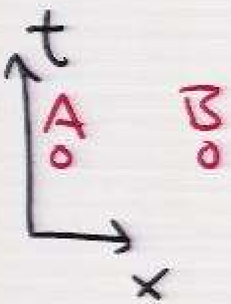

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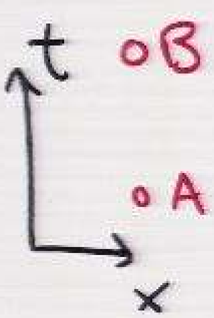

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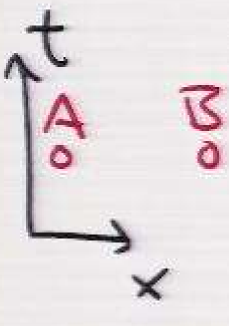
Correlations & Causal structure

Space-like correlations



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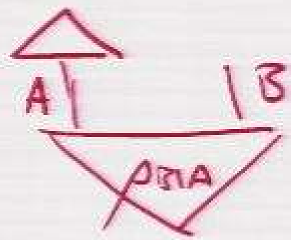

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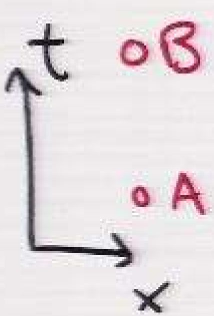

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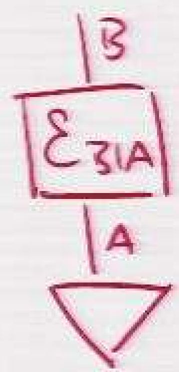

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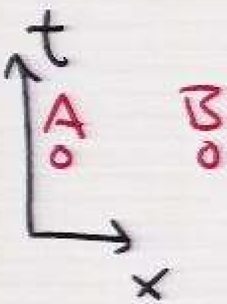

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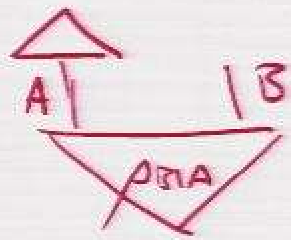

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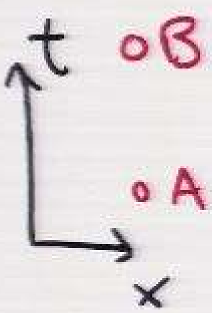

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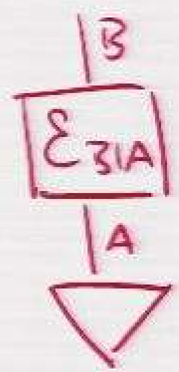

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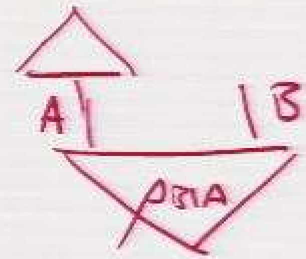

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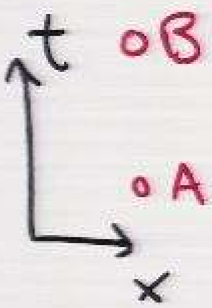
$$\rho_B=\mathrm{Tr}_A(\rho_{B|A}\,\rho_A)$$

$$=\mathcal{E}_{B|A}(\rho_A^{T_A})$$

Time-like correlations



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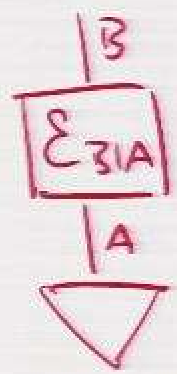

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learn about A: ρ_A

$$\rho_B=\mathrm{Tr}_A(\rho_{B|A}^{T_A}\,\rho_A)$$

$$=\mathcal{E}_{B|A}(\rho_A)$$

We shall prove:

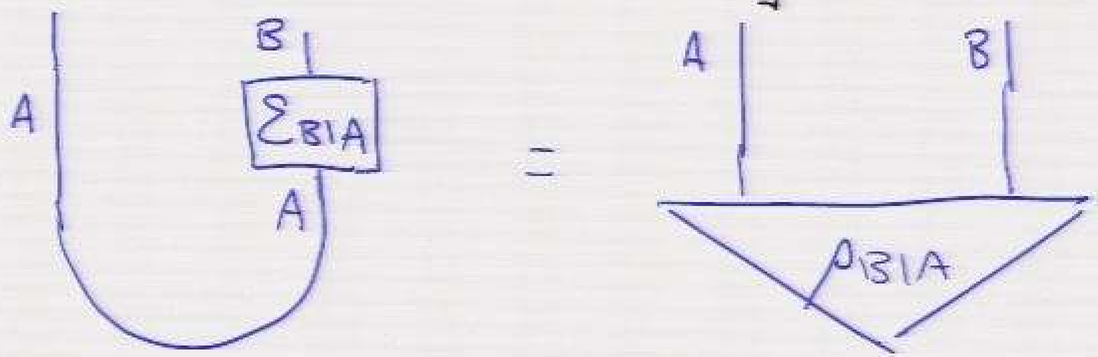
$$\text{Thm:} \quad \mathcal{E}_{B|A}(\rho_A) = \mathrm{Tr}_A\left(\rho_{B|A}^{T_A}\,\rho_A\right)$$

Choi-Jamiolkowski isomorphism (nonstandard form)

$\mathcal{E}_{B|A}$ $\longleftrightarrow$ $\rho_{B|A}$

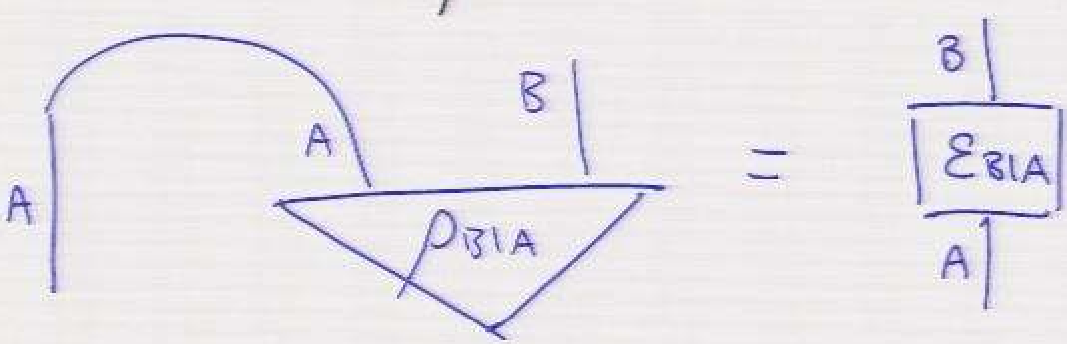
trace-preserving CP maps from $\mathcal{B}(\mathcal{H}_A)$ to $\mathcal{B}(\mathcal{H}_B)$ — conditional density operator: state on $\mathcal{H}_A \otimes \mathcal{H}_B$ s.t. $\mathrm{Tr}_B(\rho_{B|A}) = \mathbb{I}_A$

$$\rho_{B|A} = d_A \, \mathcal{I}_A \otimes \mathcal{E}_{B|A'}\left[|\Phi^+\rangle_{AA'}\langle\Phi^+| \right]$$



(modulo a scalar)

$$\mathcal{E}_{B|A}(\sigma_A) = d_A \langle\Phi^+|_{AA'} \, \rho_{B|A'} \, \sigma_A \, |\Phi^+\rangle_{AA'}$$



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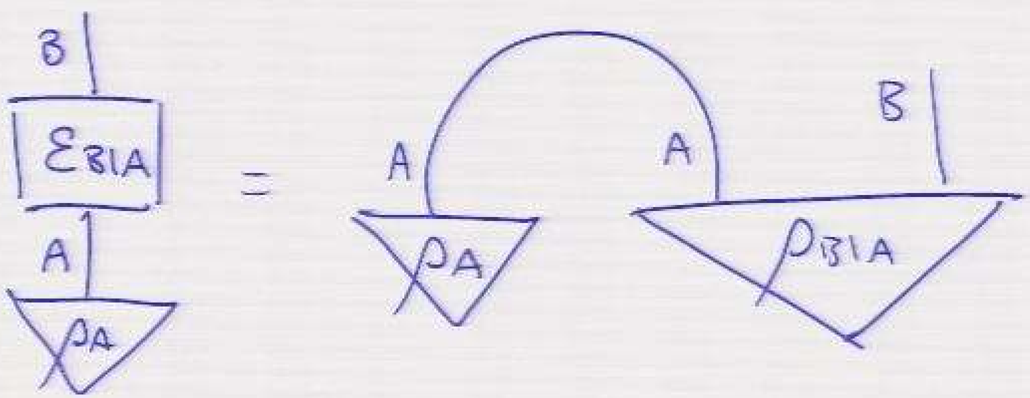
$$\text{Thm: } \mathcal{E}_{B|A}(\rho_A) = \mathrm{Tr}_A\left(\rho_{B|A}^{T_A}\,\rho_A\right)$$

proof:

$$\begin{aligned}
\mathcal{E}_{B|A}(\rho_A) &= d^2 \langle \Phi^+|_{AA'}\, \rho_{BA'}\, \rho_A \,|\Phi^+\rangle_{AA'} \\
&= \langle \Phi^+|_{A'|A}\, \rho_{B|A'}\, \rho_A \,|\Phi^+\rangle_{A'|A} \\
&= \sum_j \langle j|_{A'} \langle j|_A\, \rho_{B|A'}\, \rho_A \sum_{j'} |j'\rangle_{A'} |j'\rangle_A \\
&= \sum_{j,j'} \langle j|_A \rho_A |j'\rangle_A \,\langle j'|_{A'}\, \rho_{B|A'}^{T_{A'}} |j\rangle_{A'} \\
&= \mathrm{Tr}_A\left(\rho_{B|A}^{T_A}\,\rho_A\right)
\end{aligned}$$



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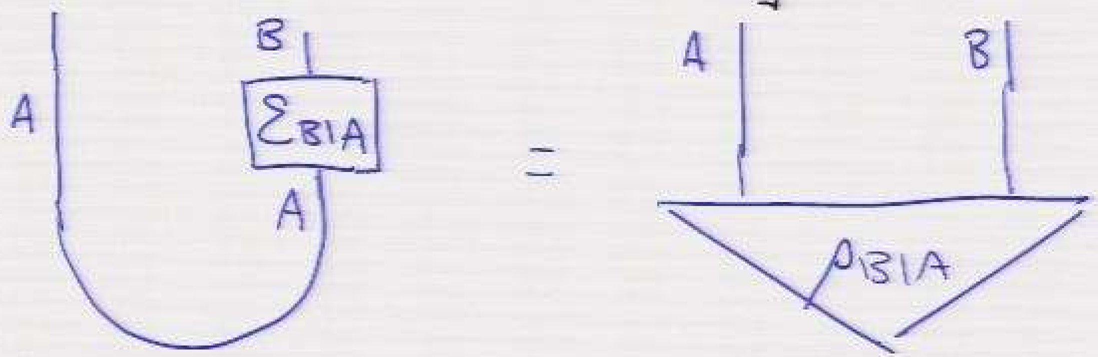

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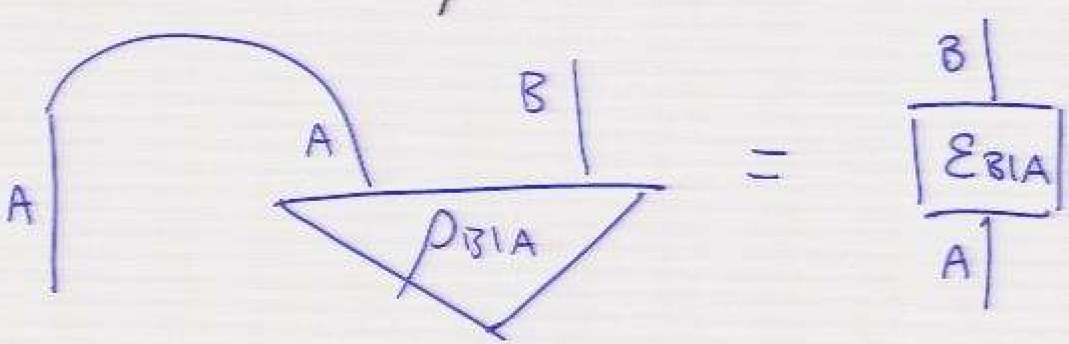

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$$\text{Thm: } \mathcal{E}_{B|A}(\rho_A) = \mathrm{Tr}_A\left(\rho_{B|A}^{T_A}\rho_A\right)$$

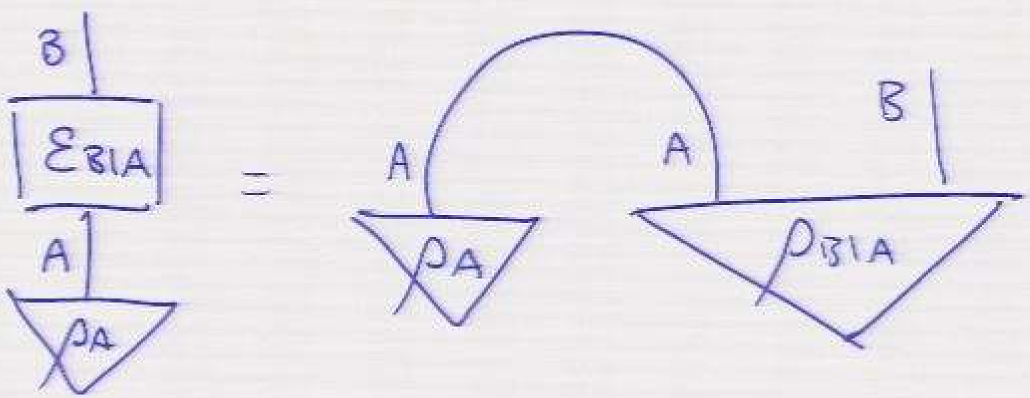
$$\text{proof: } \mathcal{E}_{B|A}(\rho_A) = d^2 \langle\Phi^+|_{AA'}\, \rho_{BA'}\rho_A\, |\Phi^+\rangle_{AA'}$$

$$= \langle\Phi^+|_{A'|A}\, \rho_{B|A'}\rho_A\, |\Phi^+\rangle_{A'|A}$$

$$= \sum_j \langle j|_{A'}\langle j|_A\, \rho_{B|A'}\rho_A \sum_{j'} |j'\rangle_{A'}|j'\rangle_A$$

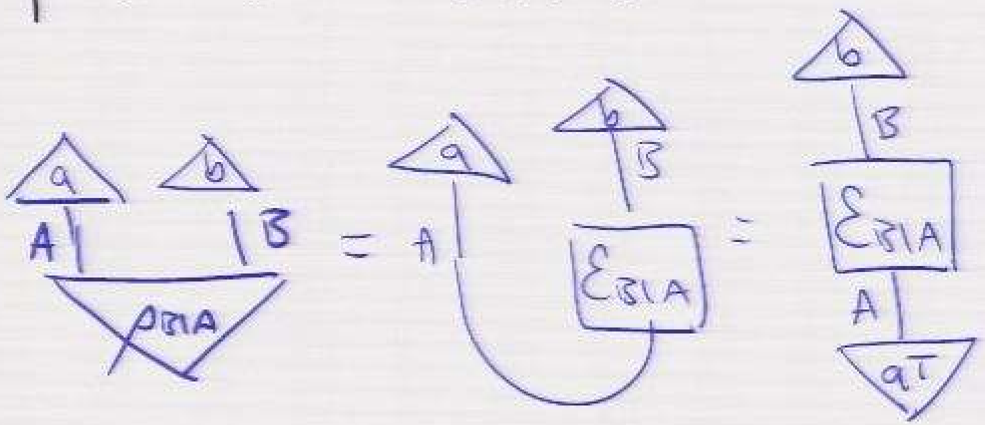
$$= \sum_{j,j'} \langle j|_A \rho_A |j'\rangle_A \langle j'|_{A'}\rho_{B|A'}^{T_{A'}}|j\rangle_{A'}$$

$$= \mathrm{Tr}_A\left(\rho_{B|A}^{T_A}\rho_A\right)$$



Correlations & Causal structure

Space-like correlations



Time-like correlations

Note: classical prob. distributions are diagonal ∴ invariant under transpose

Thm: $\mathcal{E}_{B|A}(\rho_A) = \mathrm{Tr}_A\left(\rho_{B|A}^{T_A}\,\rho_A\right)$

proof:

$$\mathcal{E}_{B|A}(\rho_A) = d^2 \langle \Phi^+|_{AA'}\, \rho_{BA'}\, \rho_A\, |\Phi^+\rangle_{AA'}$$

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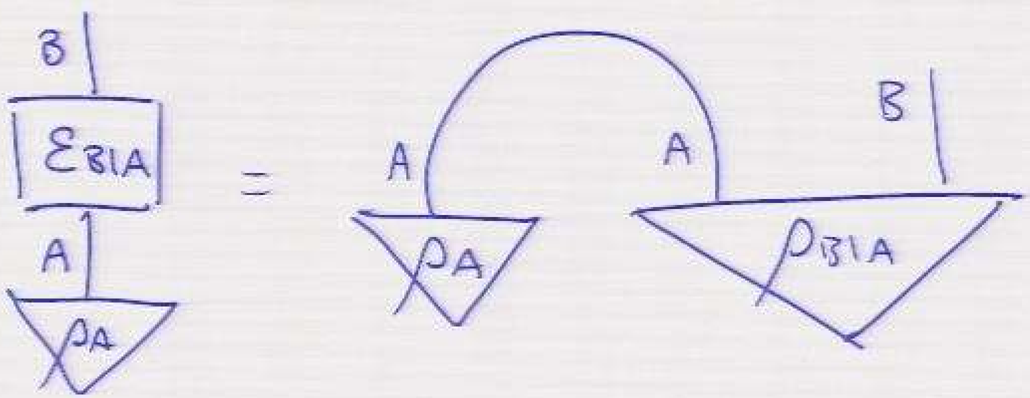
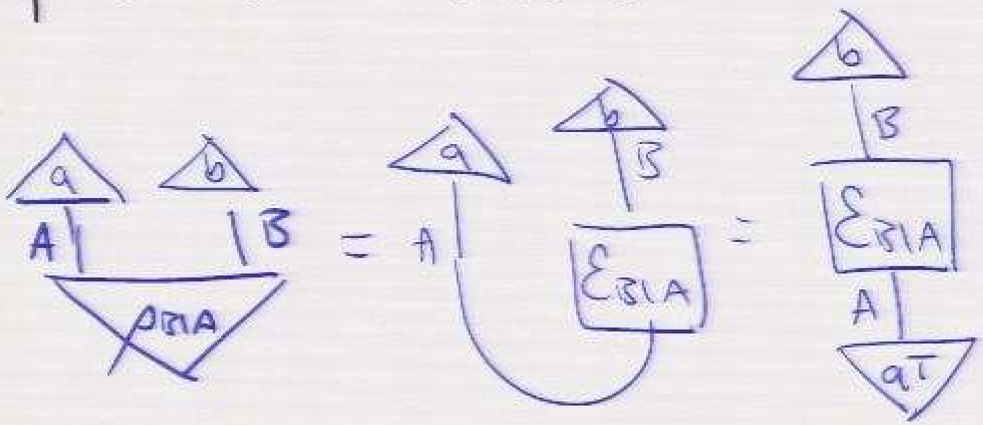


Correlations & Causal structure

Space-like correlations



Time-like correlations

Note: classical prob. distributions are diagonal
∴ invariant under transpose

Bayesian Inversion

Bayes' theorem

$$p(B|A) = \frac{p(A|B)\,p(B)}{p(A)}$$

Proof: $p(A,B) = p(A|B)\,p(B)$

$p(A,B) = p(B|A)\,p(A)$

A quantum analogue of Bayes' theorem

$$\rho_{B|A} = \rho_A^{-1/2}\,\rho_B^{1/2}\,\rho_{A|B}\,\rho_B^{1/2}\,\rho_A^{-1/2}$$

Proof: $\rho_{AB} = \rho_B^{1/2}\,\rho_{A|B}\,\rho_B^{1/2}$

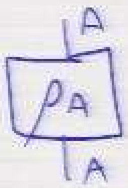
$\rho_{AB} = \rho_A^{1/2}\,\rho_{B|A}\,\rho_A^{1/2}$

Note: $[\rho_A, \rho_B] = 0$

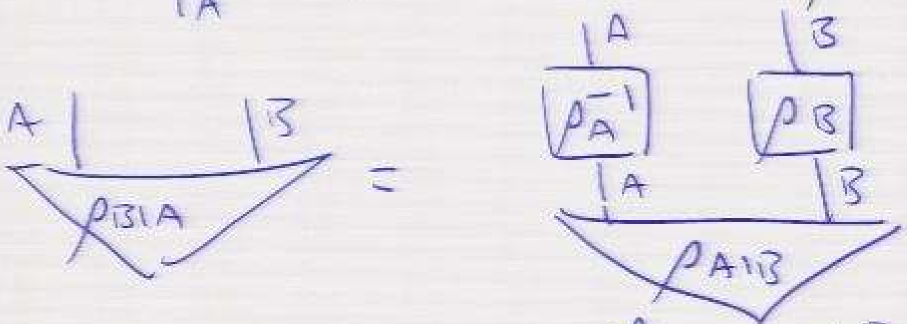
∴ order is not important

Bayesian Inversion

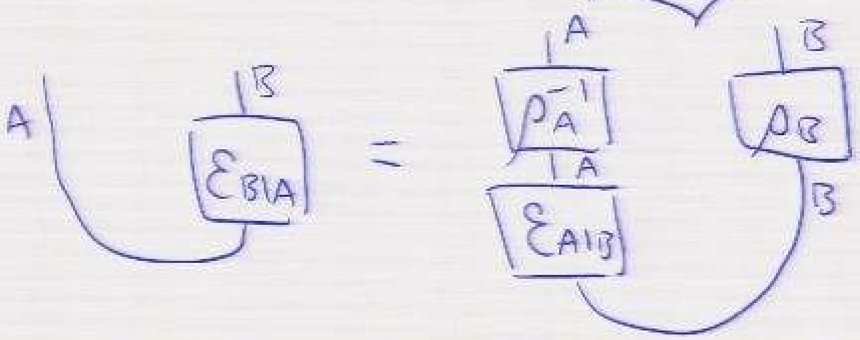
$$\rho_{B|A} = \rho_A^{-1/2} \rho_B^{1/2} \rho_{A|B} \rho_B^{1/2} \rho_A^{-1/2}$$

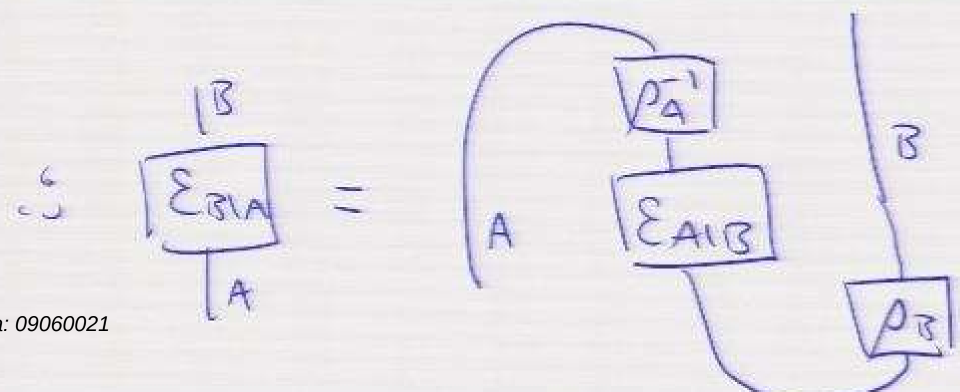
let 

represent $\rho_A^{1/2}(\cdot)\rho_A^{1/2}$



← commutation is apparent





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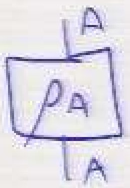
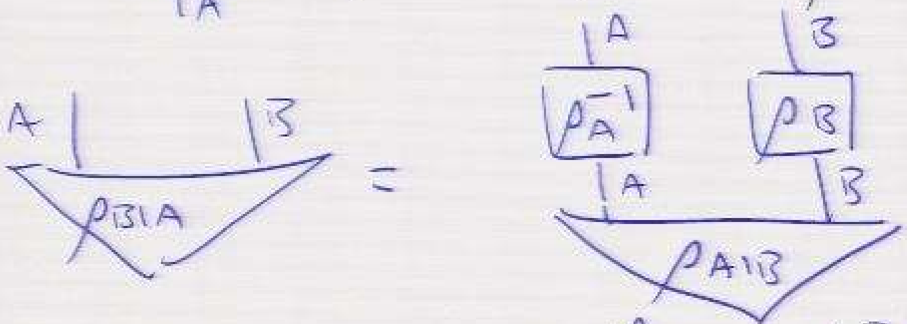
let  represent $\rho_A^{1/2}(\,\cdot\,)\,\rho_A^{1/2}$



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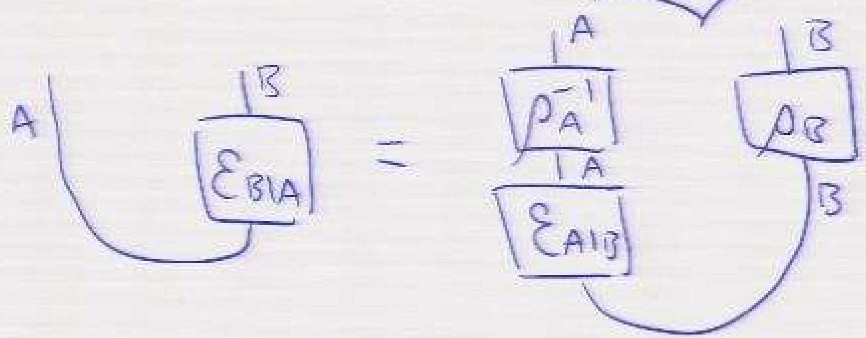

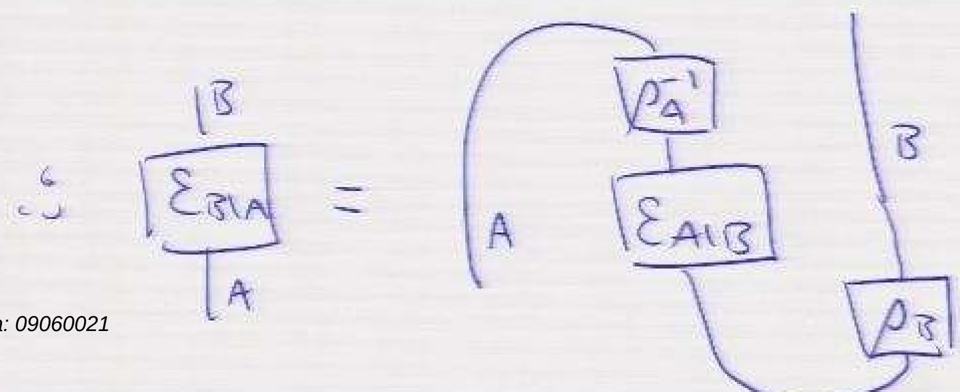

Conditional Independence

A & C are conditionally independent given B

Classical

(i) $p(AC|B) = p(A|B)\,p(C|B)$

(ii) $p(A|BC) = p(A|B)$

(iii) $p(C|BA) = p(C|B)$

Quantum

(i) $\rho_{AC|B} = \rho_{A|B}\,\rho_{C|B}$

(ii) $\rho_{A|BC} = \rho_{A|B} \otimes 1_{supp(\rho_C)}$

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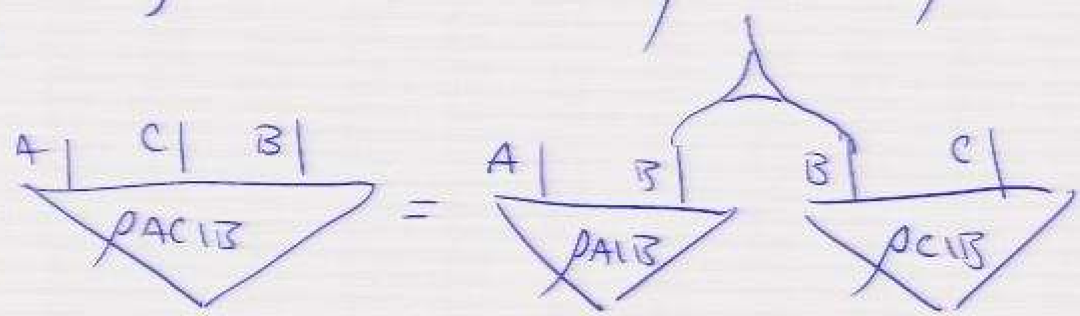

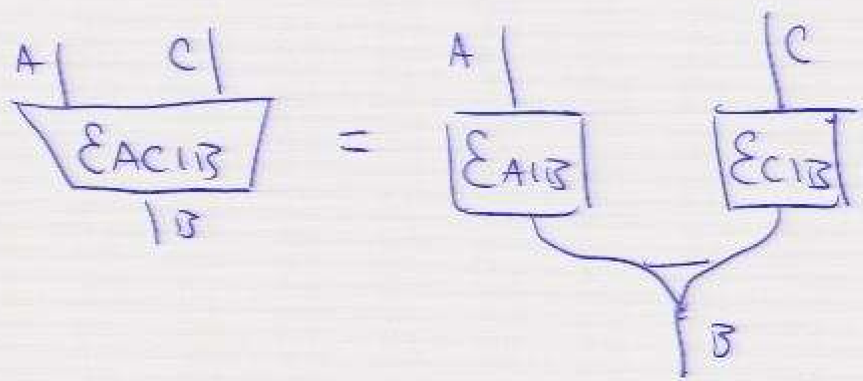

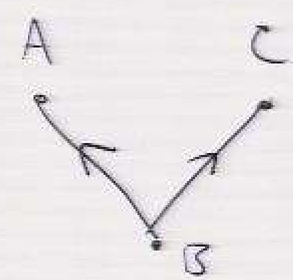

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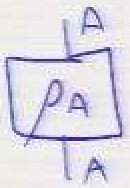
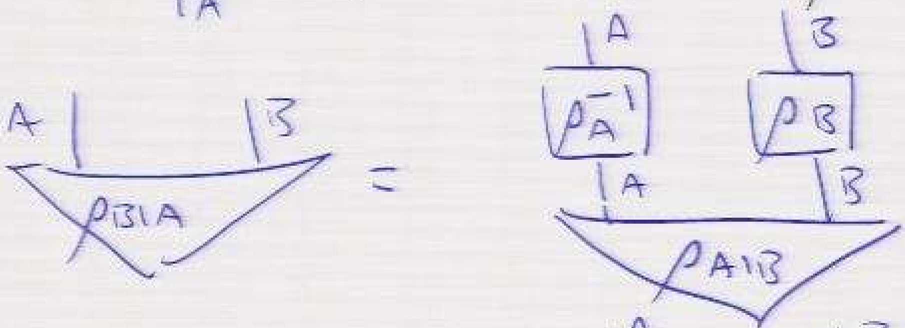
$$\rho_{B|A} = \rho_A^{-1/2}\,\rho_B^{1/2}\,\rho_{A|B}\,\rho_B^{1/2}\,\rho_A^{-1/2}$$

let  represent $\rho_A^{1/2}(\cdot)\,\rho_A^{1/2}$



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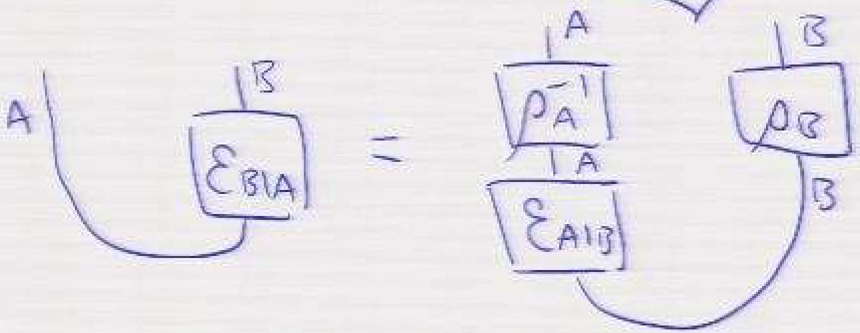

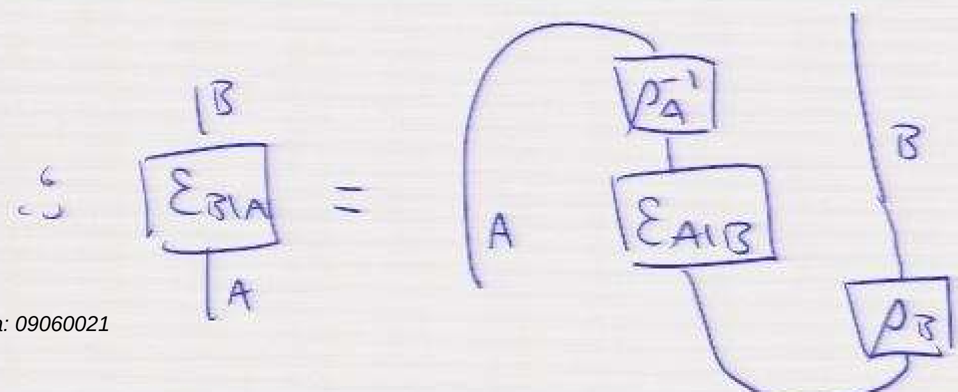

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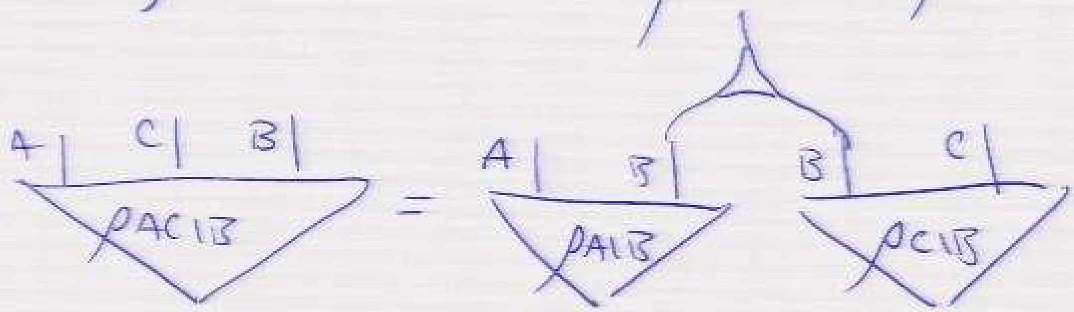

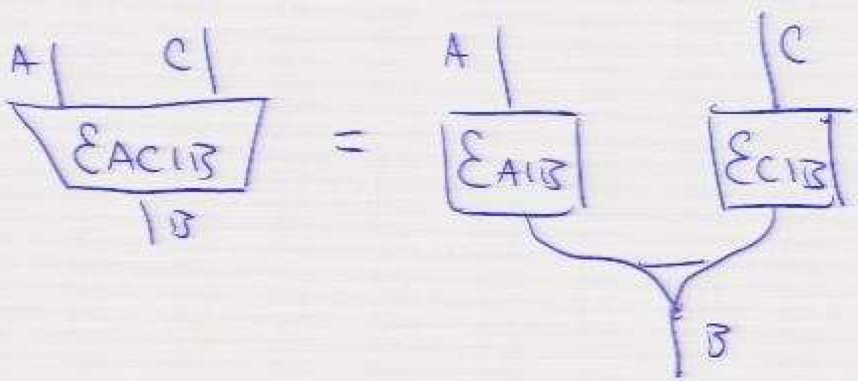

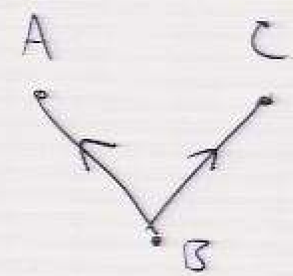

Classical Markov chains

Given $p(A,B,C)$
find $p(A|C)$

$$p(A|C) = \sum_B p(AB|C)$$

$$= \sum_B \frac{p(ABC)}{p(C)}$$

$$p(ABC) = p(A|BC)\,p(B|C)\,p(C)$$

$$\therefore\ p(A|C) = \sum_B p(A|BC)\,p(B|C)$$

But if A & C are conditionally independent given B

$$p(A|BC) = p(A|B)$$

Then $p(A|C) = \sum_B p(A|B)\,p(B|C)$

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But if A & C are conditionally independent given B

$$p(A|BC) = p(A|B)$$

Then $p(A|C) = \sum_B p(A|B)\,p(B|C)$

Quantum Markov chains

Given ρ_{ABC}
find $\rho_{A|C}$

$$\rho_{A|C} = \mathrm{Tr}_B\, \rho_{AB|C}$$
$$= \mathrm{Tr}_B\left(\rho_C^{-1/2}\rho_{ABC}\,\rho_C^{-1/2}\right)$$

$$\rho_{ABC} = \rho_C^{1/2}\rho_{B|C}^{1/2}\,\rho_{A|BC}\,\rho_{B|C}^{1/2}\rho_C^{1/2}$$

$$\therefore\ \rho_{A|C} = \mathrm{Tr}_B\left(\rho_{B|C}^{1/2}\,\rho_{A|BC}\,\rho_{B|C}^{1/2}\right)$$

But if A & C are conditionally independent given B

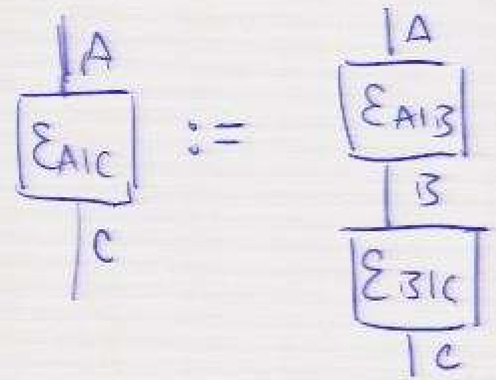
$$\rho_{A|BC} = \rho_{A|B} \otimes I_{\mathrm{supp}(\rho_C)}$$

Then $\rho_{A|C} = \mathrm{Tr}_B\left(\rho_{A|B}\,\rho_{B|C}\right)$

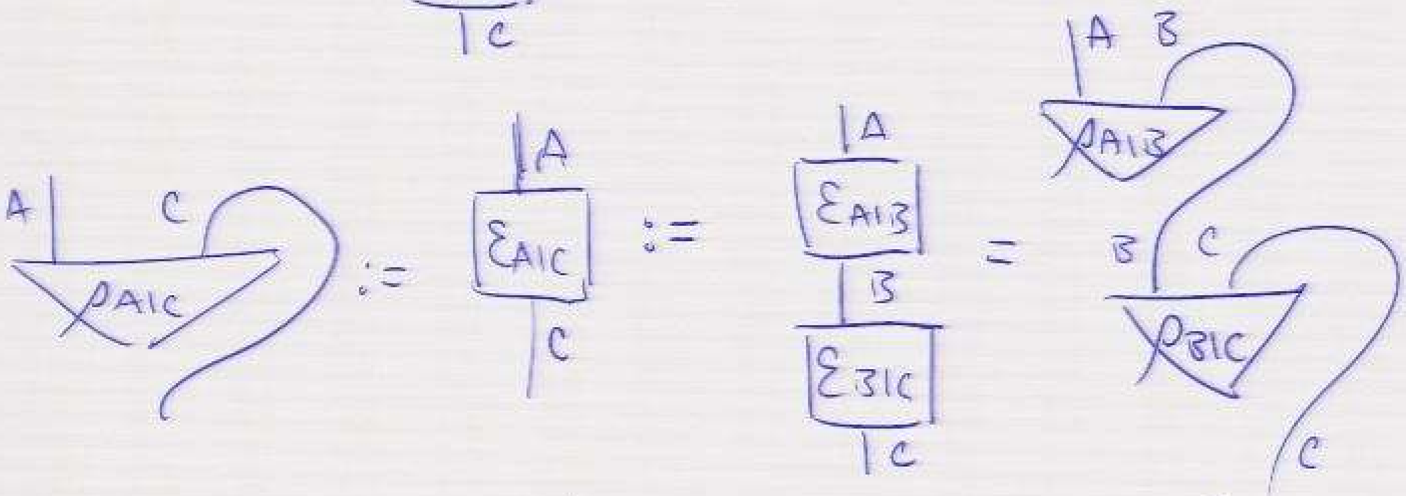
Composition of quantum operations

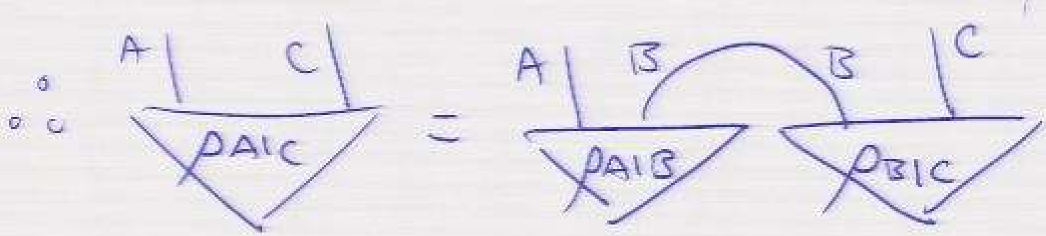
$\mathcal{E}_{A|C} := \mathcal{E}_{A|B} \circ \mathcal{E}_{B|C}$

Note: A & C are cond. indep. given B



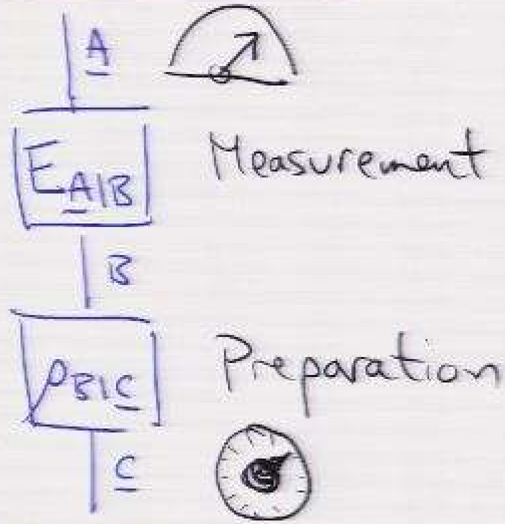
Find $\rho_{A|C}$





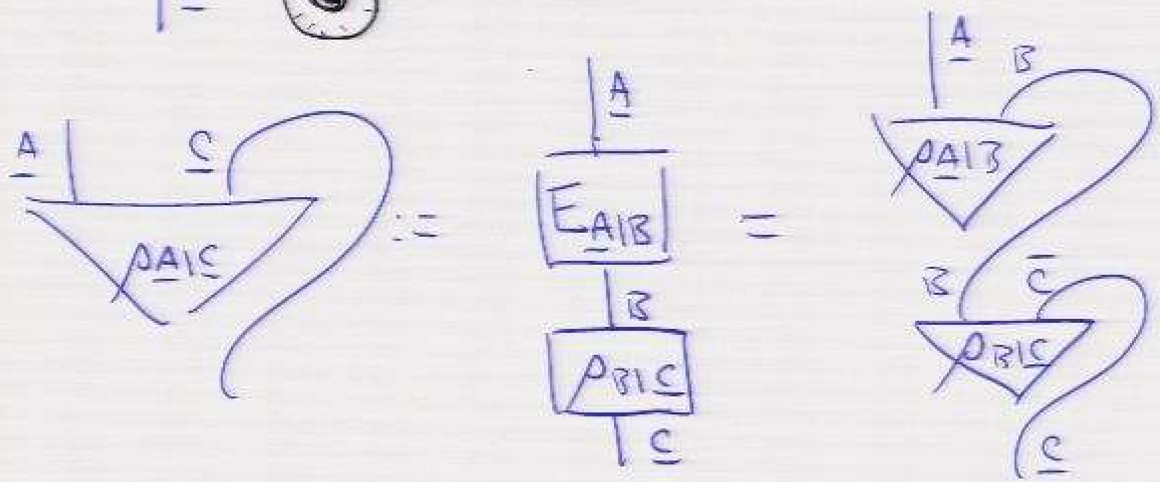
$$\therefore\ \rho_{A|C} = \mathrm{Tr}_B\left(\rho_{A|B}^{T_B}\,\rho_{B|C}\right)$$

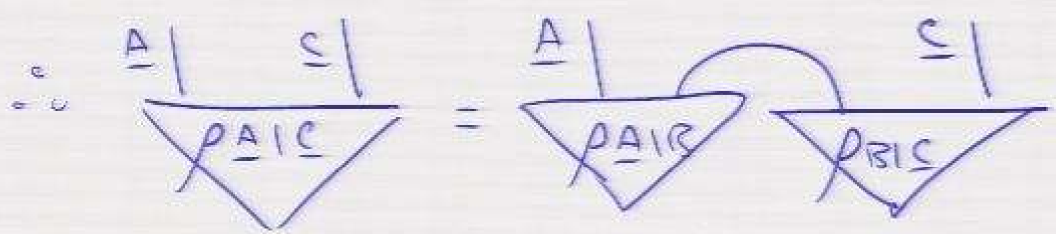
The Born rule



Note: $\underline{A}$ & $\underline{C}$ are cond. indep. given $\underline{B}$

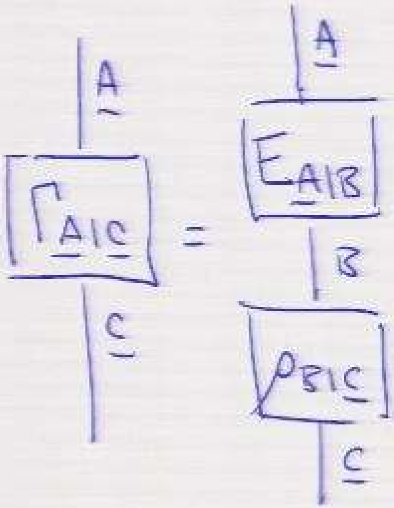
Find $\rho_{\underline{A}|\underline{C}}$





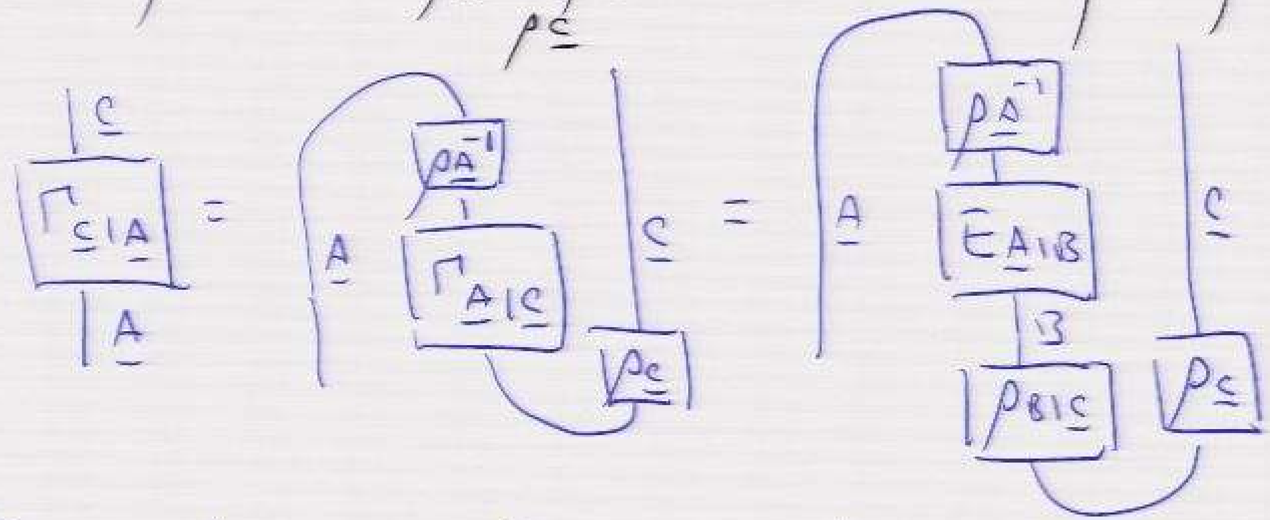
$$\therefore \rho_{\underline{A}|\underline{C}} = \mathrm{Tr}_{\underline{B}}\left(\rho_{\underline{A}|\underline{B}}^{T_{\underline{B}}}\, \rho_{\underline{B}|\underline{C}}\right)$$

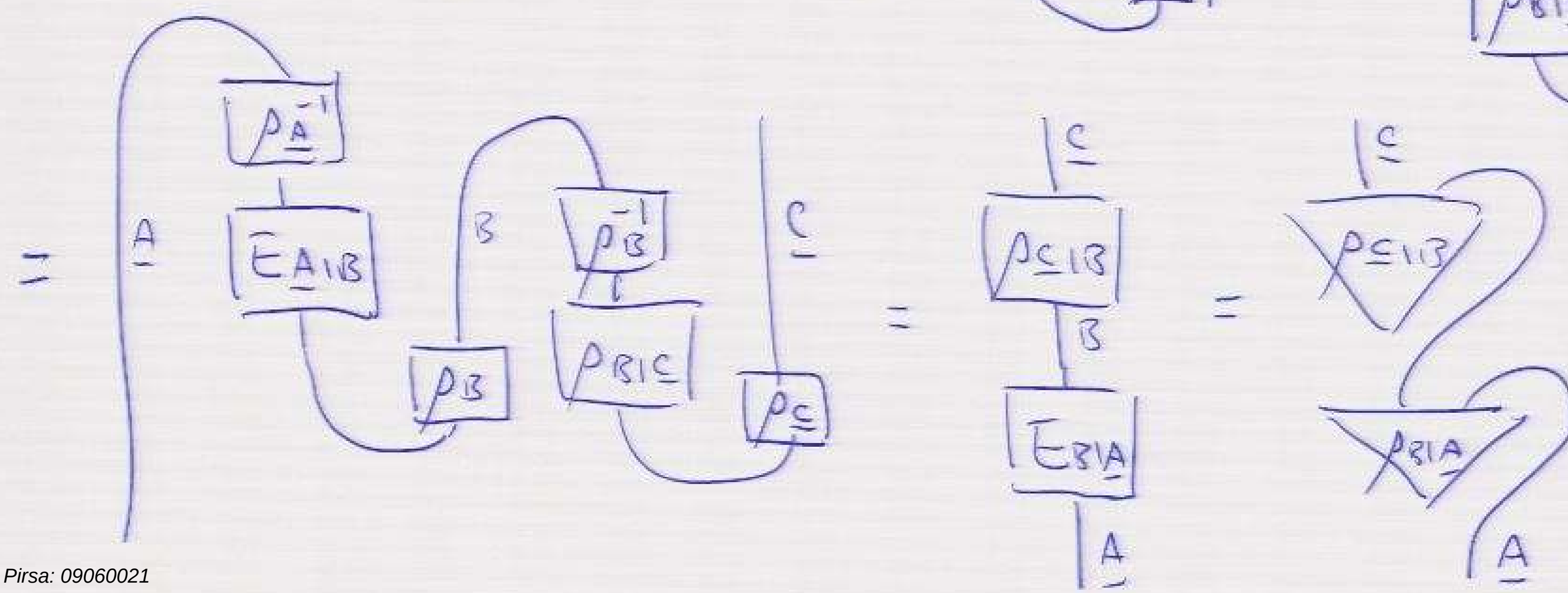
Retrodictive Born rule



Recall $\rho_{A|C} = \mathrm{Tr}_B\left(\rho_{A|B}^{T_B}\,\rho_{B|C}\right)$

Find $\rho_{C|A} = \dfrac{\rho_{C|A}\,\rho_A}{\rho_C} = \ldots = \mathrm{Tr}_B\left(\rho_{C|B}^{T_B}\,\rho_{B|A}\right)$





Tripartite state of Markov form

$$\rho_{ABC} = \sum_k p_k \, \rho_{AB_k^{(1)}} \otimes \rho_{B_k^{(2)}C}$$

where $B = \bigoplus_k B_k^{(1)} \otimes B_k^{(2)}$

For such a state, A & C are cond. indep. given B

i.e. $\boxed{\rho_{AC|B} = \rho_{A|B}\,\rho_{C|B}}$

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For such a state, A & C are cond. indep. given B

i.e. $\rho_{AC|B} = \rho_{A|B}\, \rho_{C|B}$

Proof: $\rho_B = \mathrm{Tr}_{AC}\, \rho_{ABC} = \sum_k p_k \, \rho_{B_k^{(1)}} \otimes \rho_{B_k^{(2)}}$

$$\rho_{AC|B} = \rho_B^{-1/2} \rho_{ACB}\, \rho_B^{-1/2}$$

$$= \sum_k \rho_{B_k^{(1)}}^{-1/2} \rho_{AB_k^{(1)}} \rho_{B_k^{(1)}}^{-1/2} \otimes \rho_{B_k^{(2)}}^{-1/2} \rho_{B_k^{(2)}C}\, \rho_{B_k^{(2)}}^{-1/2}$$

$$= \sum_k \rho_{A|B_k^{(1)}} \otimes \rho_{C|B_k^{(2)}}$$

$$\rho_{A|B} = \mathrm{Tr}_C\, \rho_{AC|B} = \sum_k \rho_{A|B_k^{(1)}} \otimes \mathbb{1}_{B_k^{(2)}}$$

$$\rho_{C|B} = \mathrm{Tr}_A\, \rho_{AC|B} = \sum_k \mathbb{1}_{B_k^{(1)}} \otimes \rho_{A|B_k^{(2)}}$$

$\therefore \rho_{AC|B} = \rho_{A|B}\, \rho_{C|B}$ Q.E.D.

Graphical proof?

Tripartite state of Markov form

$$\rho_{ABC} = \sum_k p_k \, \rho_{AB_k^{(1)}} \otimes \rho_{B_k^{(2)}C}$$

where $B = \bigoplus_k B_k^{(1)} \otimes B_k^{(2)}$

For such a state, A & C are cond. indep. given B

i.e.

$$\rho_{AC|B} = \rho_{A|B}\,\rho_{C|B}$$

or $$\rho_{A|BC} = \rho_{A|B} \otimes \mathbb{1}_{supp(\rho_C)}$$

Find $\rho_{A|C}$

$$\rho_{A|C} = \mathrm{Tr}_B\, \rho_{AB|C}$$

$$= \mathrm{Tr}_B\left(\rho_{B|C}^{1/2}\, \rho_{A|BC}\, \rho_{B|C}^{1/2}\right)$$

$$= \mathrm{Tr}_B\left(\rho_{B|C}^{1/2}\, \rho_{A|B} \otimes \mathbb{1}_{supp(\rho_C)}\, \rho_{B|C}^{1/2}\right)$$

$$= \mathrm{Tr}_B\left(\rho_{A|B}\, \rho_{B|C}\right)$$

Ongoing work

- Criterion for applicability of pooling
  → conditional independence of jointly sufficient statistics
- Pre & post selection
- Joint measurability of POVMs
- Develop a graphical calculus for Bayesian inference & explore differences between classical & quantum
- Connection to graphical models of Bayesian networks