Title: A Little Categorified Arithmetic from Entanglement

Date: Jun 03, 2009 10:15 AM

URL: http://pirsa.org/09060020

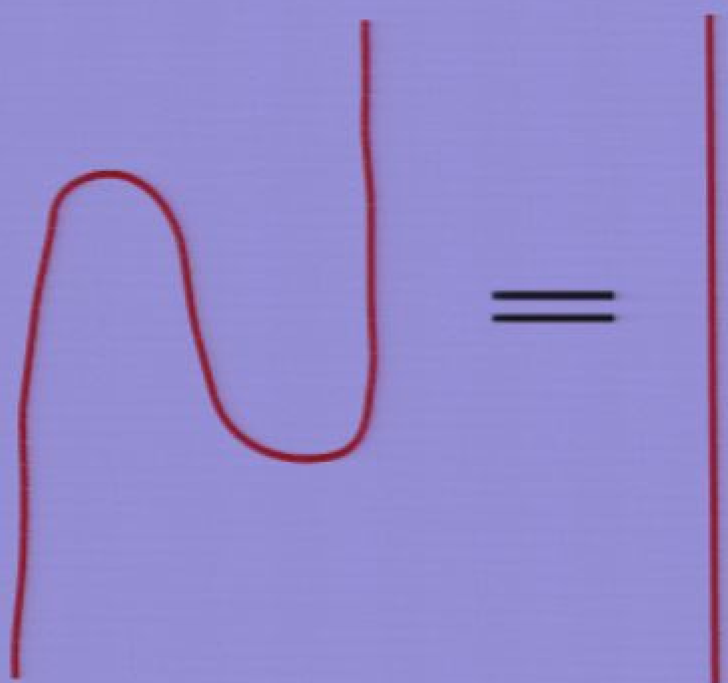
Abstract: TBA

a little categorified arithmetic

M. D. Sheppeard

Categories, Quanta, Concepts
Perimeter Institute, 1 - 5 Jun 2009

Entanglement



three qubits

$$\Psi = a_{000}|000\rangle + a_{100}|100\rangle + a_{010}|010\rangle + a_{001}|001\rangle$$
$$+a_{110}|110\rangle + a_{101}|101\rangle + a_{011}|011\rangle + a_{111}|111\rangle$$

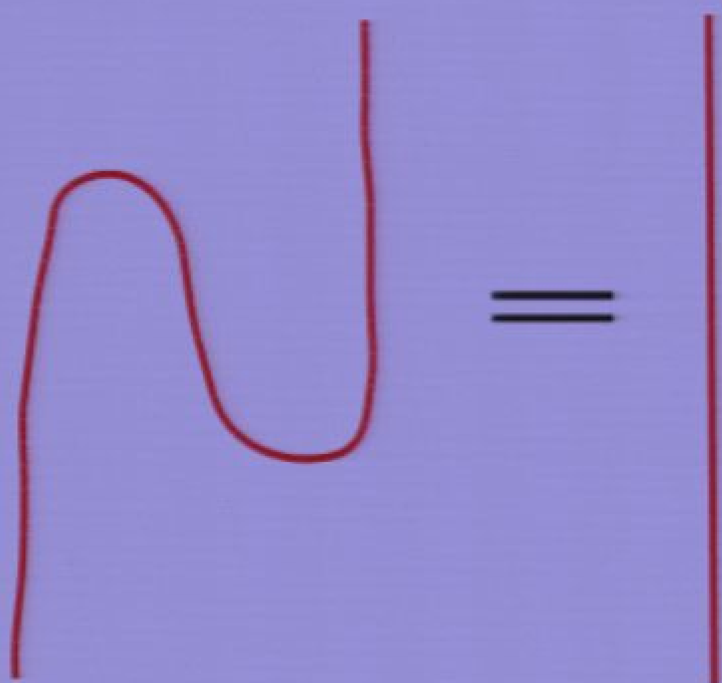
tripartite entanglement

$$\begin{aligned}\Delta(\Psi) = {} & a_{000}^2 a_{111}^2 + a_{100}^2 a_{011}^2 + a_{010}^2 a_{101}^2 + a_{001}^2 a_{110}^2 \\ & +4(a_{000}a_{011}a_{110}a_{101} + a_{111}a_{100}a_{010}a_{001}) \\ & -2(a_{000}a_{100}a_{011}a_{111} + a_{000}a_{010}a_{101}a_{111} + a_{000}a_{001}a_{110}a_{111} \\ & +a_{100}a_{010}a_{101}a_{011} + a_{010}a_{001}a_{110}a_{101} + a_{100}a_{001}a_{110}a_{011})\end{aligned}$$

dagger monoidal categories

compact closed structure

Entanglement



three qubits

$$\Psi = a_{000}|000\rangle + a_{100}|100\rangle + a_{010}|010\rangle + a_{001}|001\rangle$$
$$+a_{110}|110\rangle + a_{101}|101\rangle + a_{011}|011\rangle + a_{111}|111\rangle$$

tripartite entanglement

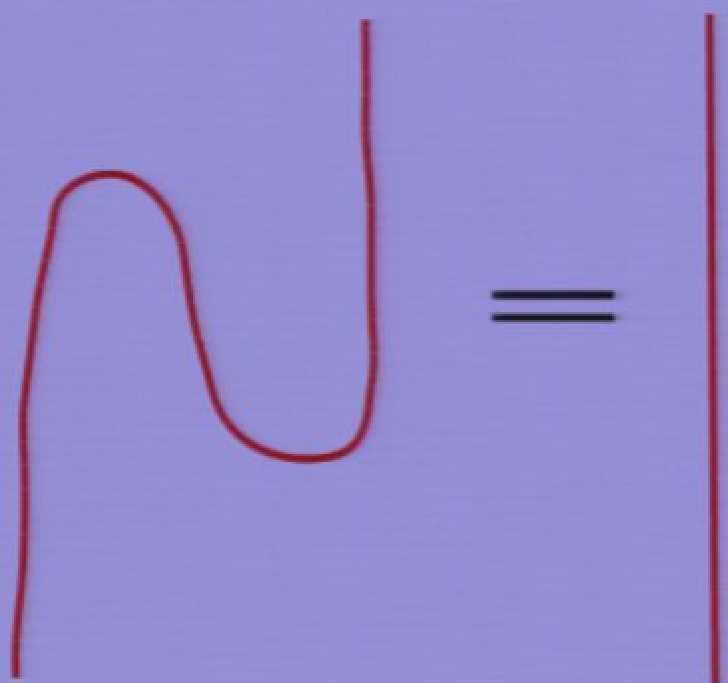
$$\Delta(\Psi) = a_{000}^2 a_{111}^2 + a_{100}^2 a_{011}^2 + a_{010}^2 a_{101}^2 + a_{001}^2 a_{110}^2$$
$$+4(a_{000} a_{011} a_{110} a_{101} + a_{111} a_{100} a_{010} a_{001})$$
$$-2(a_{000} a_{100} a_{011} a_{111} + a_{000} a_{010} a_{101} a_{111} + a_{000} a_{001} a_{110} a_{111}$$
$$+a_{100} a_{010} a_{101} a_{011} + a_{010} a_{001} a_{110} a_{101} + a_{100} a_{001} a_{110} a_{011})$$

see math.RA/0604374

dagger monoidal categories

compact closed structure

Entanglement



three qubits

$$\Psi = a_{000}|000\rangle + a_{100}|100\rangle + a_{010}|010\rangle + a_{001}|001\rangle$$
$$+a_{110}|110\rangle + a_{101}|101\rangle + a_{011}|011\rangle + a_{111}|111\rangle$$

tripartite entanglement

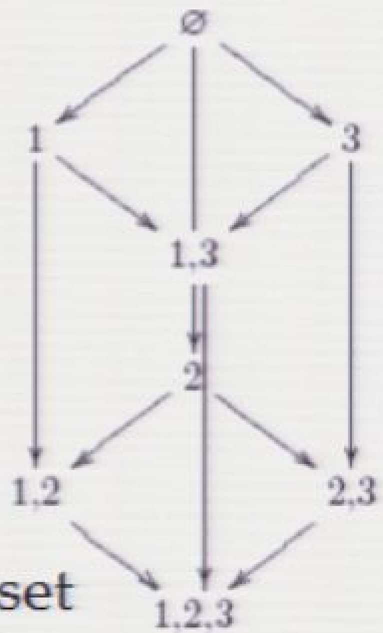
$$\Delta(\Psi) = a_{111}^2 + a_{100}^2 a_{011}^2 + a_{010}^2 a_{101}^2 + a_{001}^2 a_{110}^2$$
$$+4(a_{011}a_{110}a_{101} + a_{111}a_{100}a_{010}a_{001})$$
$$-2(a_{100}a_{011}a_{111} + a_{010}a_{101}a_{111} + a_{001}a_{110}a_{111}$$
$$+a_{100}a_{010}a_{101}a_{011} + a_{010}a_{001}a_{110}a_{101} + a_{100}a_{001}a_{110}a_{011})$$

see math.RA/0604374

dagger monoidal categories

compact closed structure

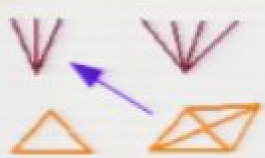
Entanglement



subset lattice

qudits

$2^n \rightarrow d^n$



van den Nest moves

W state

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \longrightarrow$$

three qubits

$$\Psi = a_{000}|000\rangle + a_{100}|100\rangle + a_{010}|010\rangle + a_{001}|001\rangle$$
$$+a_{110}|110\rangle + a_{101}|101\rangle + a_{011}|011\rangle + a_{111}|111\rangle$$

tripartite entanglement

$\Delta(\Psi) = 0$

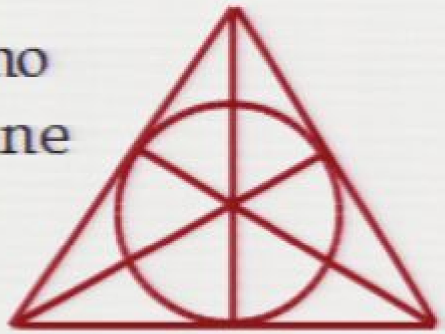
$$\begin{pmatrix} a_{100} & \sqrt{a_{100}a_{010} - a_{110}^2} & \sqrt{a_{100}a_{001} - a_{011}^2} \\ \sqrt{a_{100}a_{010} - a_{110}^2} & a_{010} & \sqrt{a_{010}a_{001} - a_{011}^2} \\ \sqrt{a_{100}a_{001} - a_{101}^2} & \sqrt{a_{010}a_{001} - a_{011}^2} & a_{001} \end{pmatrix}$$

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

complementary observables

Projective geometry of 3 qubits

Fano plane

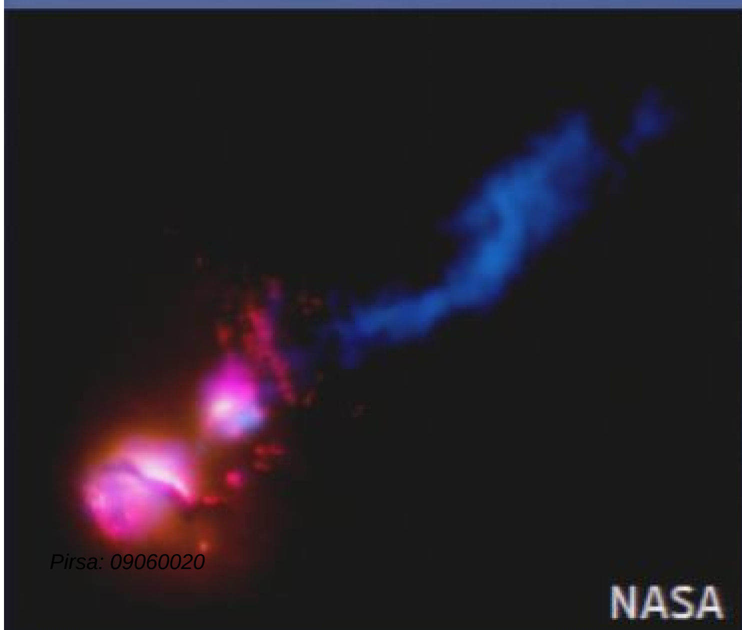


see Levay et al
arxiv 0808.3849

$\sigma_i \in \{X, Y, Z, I\}$ $\qquad$ $\sigma_i \otimes \sigma_j \otimes \sigma_k$

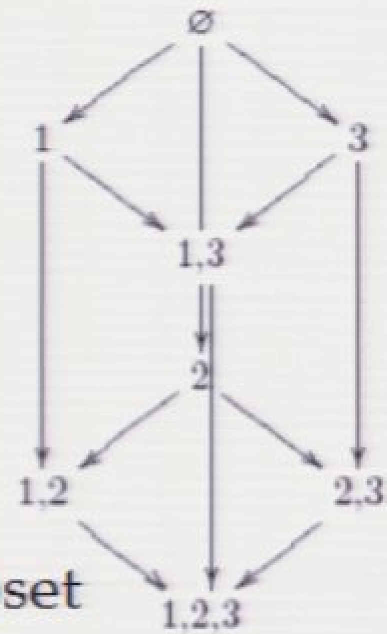
points	$ZZZ, IZZ, ZIZ, ZZI, IIZ, IZI, ZII$
lines	$XXX, IXX, XIX, XXI, IIX, IXI, XII$
flag	$ZZX, ZYY, IXZ, \cdots$
antiflag	$YZX, YIZ, \cdots$

F_2 swaps points and lines



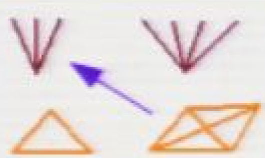
- local operations on qudits
- Fourier transform as duality
- entanglement – black hole correspondence arxiv 0809.4685
- invariants for QM?

Numbers and Decategorification

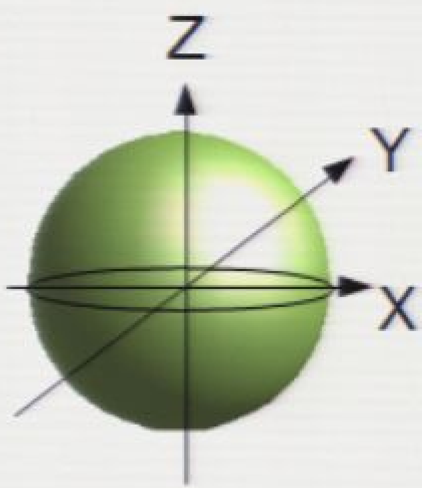


subset
lattice

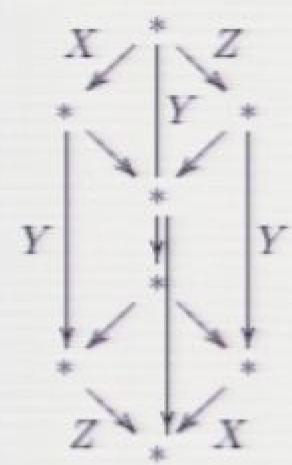
qudits
$2^n \rightarrow d^n$



van den Nest
moves



- Varied physical realisations of qudits
- Generalised cardinality
- n-category hierarchy
- Operads/PROPs
- Noncommutative Fourier transform
- Role of mutually unbiased bases



all paths

Mutually Unbiased Bases

- Construction for prime dimensions
- Prime power dimensions and finite fields
- Comments on *Decategorification*
- Comments on *Stone duality*

'It is a basic primitive question about the adelic line which we don't understand. It is a question about the way addition is fitting with multiplication.'

A. Connes

Mutually Unbiased Bases

bases $\{A_j\}_j$ and $\{B_k\}_k$

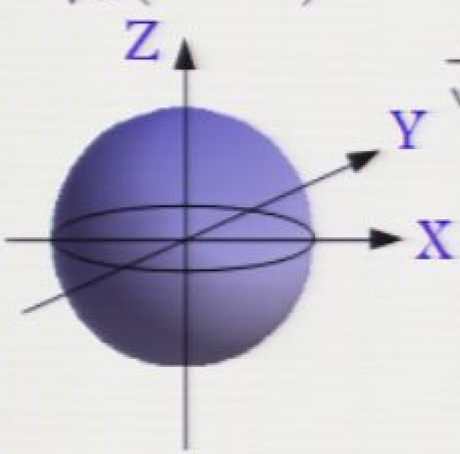
s.t. $\langle A_j | B_k \rangle = \frac{1}{\sqrt{d}}$

in prime power dimension $d = p^n$

there are $d + 1$ MUBs

qubits

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$F_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = R_2^8$$

qutrits

$$V_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad D_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \overline{\omega} \end{pmatrix}$$

$$M_3 = D_3 V_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \overline{\omega} & 0 & 0 \end{pmatrix} \qquad M_4 = D_3 V_3^2 = \begin{pmatrix} 0 & 0 & 1 \\ \omega & 0 & 0 \\ 0 & \overline{\omega} & 0 \end{pmatrix}$$

$$F_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{pmatrix} \qquad R_3 = \begin{pmatrix} 1 & \overline{\omega} & 1 \\ 1 & 1 & \overline{\omega} \\ \overline{\omega} & 1 & 1 \end{pmatrix}$$

circulant operators $R_{1k} = \omega^{-\frac{1}{2}k(k+1)}$

$R_{x+y} = R_x R_y$ for finite field index

see eg. Combescure arxiv 0710.5643

MUBs and finite fields

example in $\mathbb{F}_9$

two qutrits

$$\begin{matrix} 0, & 1, & 2 \\ t, & 1+t, & 2+t \\ 2t, & 1+2t, & 2+2t \end{matrix}$$

$$R_{2t} = \left(\begin{array}{ccc|ccc|ccc}
1 & 1 & 1 & 1 & \omega & \overline{\omega} & 1 & \overline{\omega} & \omega \\
1 & 1 & 1 & \overline{\omega} & 1 & \omega & \omega & 1 & \overline{\omega} \\
1 & 1 & 1 & \omega & \overline{\omega} & 1 & \overline{\omega} & \omega & 1 \\
\hline
1 & \overline{\omega} & \omega & 1 & 1 & 1 & 1 & \omega & \overline{\omega} \\
\omega & 1 & \overline{\omega} & 1 & 1 & 1 & \overline{\omega} & 1 & \omega \\
\overline{\omega} & \omega & 1 & 1 & 1 & 1 & \omega & \overline{\omega} & 1 \\
\hline
1 & \omega & \overline{\omega} & 1 & \overline{\omega} & \omega & 1 & 1 & 1 \\
\overline{\omega} & 1 & \omega & \omega & 1 & \overline{\omega} & 1 & 1 & 1 \\
\omega & \overline{\omega} & 1 & \overline{\omega} & \omega & 1 & 1 & 1 & 1
\end{array}\right)$$

$\rightarrow$ Qubit stabiliser formalism

$$\mathbb{F}_4 \rightarrow \mathbb{F}_8 \rightarrow \cdots \mathbb{F}_{2^n} \rightarrow \cdots$$

2-adic integers

prime qudits $\rightarrow$ profinite integers

$$\widehat{\mathbb{Z}} = \prod \mathbb{Z}_p$$

$\rightarrow$ integral adeles $\quad \mathbb{A} = \mathbb{R} \times \widehat{\mathbb{Z}}$

algebraic and metric closure?

Fourier transform diagonalises all circulants for all qudits and the R circulants act simply on cycle types

$$\begin{tikzcd}
* \arrow[r, "0"] \arrow[dr, "x"'] & \mathbf{N} \arrow[r, "s"] \arrow[d, "u"] & \mathbf{N} \arrow[d, "u"] \\
& A \arrow[r, "f"'] & A
\end{tikzcd}$$

natural number objects?

pretopos?

Galois Theory: Grothendieck, Lawvere, Tierney, Joyal, Street ...

MUBs in knotty mathematics

qubit spin model

$C_2 \equiv i R_2 F_2 R_2 \qquad F_2^2 = I$

$C_2 R_2 C_2 = R_2 C_2 R_2$

$$R_3 F_3 R_3 = \begin{pmatrix} 1-\omega & 0 & 0 \\ 0 & 0 & 1-\omega \\ 0 & \overline{\omega}-1 & 0 \end{pmatrix} \equiv C_3$$

qutrit plane swapping

$$R_3^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & x & -x \\ 0 & -x & x \end{pmatrix} R_3 = \begin{pmatrix} x & 0 & -x \\ 0 & 0 & 0 \\ -x & 0 & x \end{pmatrix}$$

pseudo Yang-Baxter rule

$$R_p C_p R_p = C_p R_p C_p \circ P$$

sets as vector spaces

$$\begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \omega_i & 0 \\ 0 & \overline{\omega}_i & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix}$$

field with 1 element

$\mathbb{F}_1$

modular

$(\sigma_1 \sigma_2 \sigma_1)^2 = 1$

$$\begin{pmatrix} 0 & 0 & \overline{\omega} \\ 0 & 1 & 0 \\ \omega & 0 & 0 \end{pmatrix} = \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$$

$M_q(2)$ for finite fields

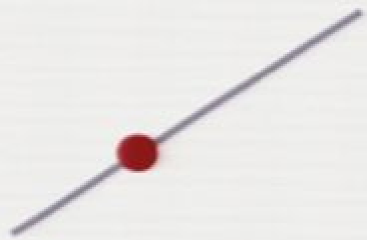
$$bc = cb, \quad ba = \omega ab, \quad ca = \omega ac$$

$$dc = \omega dc, \quad db = \omega bd$$

$$da = ad + (\omega - \overline{\omega}) bc$$

Decategorification

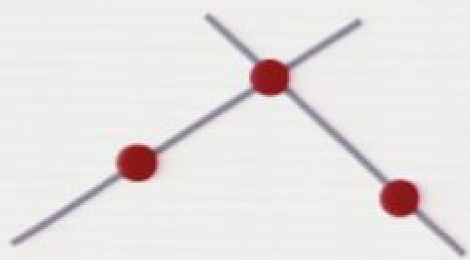
weak Yang-Baxter rule for *points* and *lines*



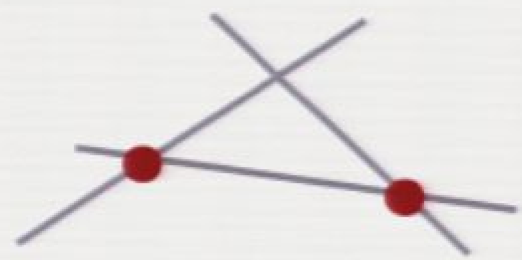
John Baez,
James Dolan et al
Groupoidification

Decategorification

weak Yang-Baxter rule for *points* and *lines*



PLP

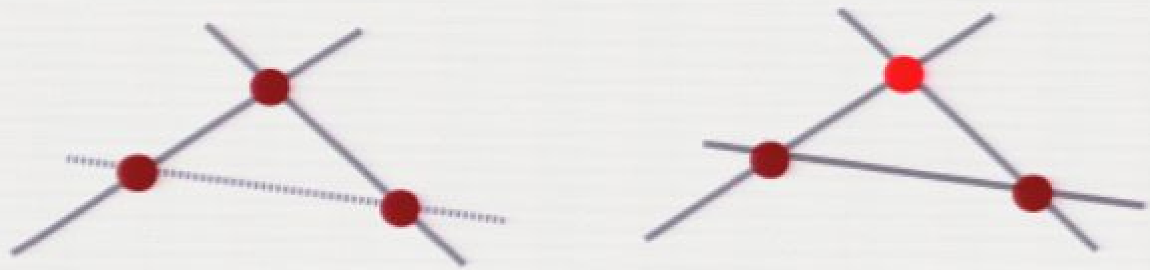


LPL

John Baez,
James Dolan et al
Groupoidification

Decategorification

weak Yang-Baxter rule for *points* and *lines*



PLP $\simeq$ LPL

eg. recall the three qubit Fano plane geometry:

points $ZZZ, IZZ, ZIZ, ZZI, IIZ, IZI, ZII$

lines $XXX, IXX, XIX, XXI, IIX, IXI, XII$

John Baez,
James Dolan et al
Groupoidification

Stone Duality

The abstract Fourier transform

Pontrjagin duality: $\widehat{\widehat{G}} = G$

eg. G = all p^nth roots of unity $\subset U(1)$

dual to the p-adic integers

$$\begin{tikzcd} \mathrm{BoolAlg}^{op} \arrow[rr, bend left, "\mathrm{Hom}(\_,2)"] & & \mathrm{StoneSpace} \arrow[ll, bend left, "\mathrm{Hom}(\_,2)"] \end{tikzcd}$$

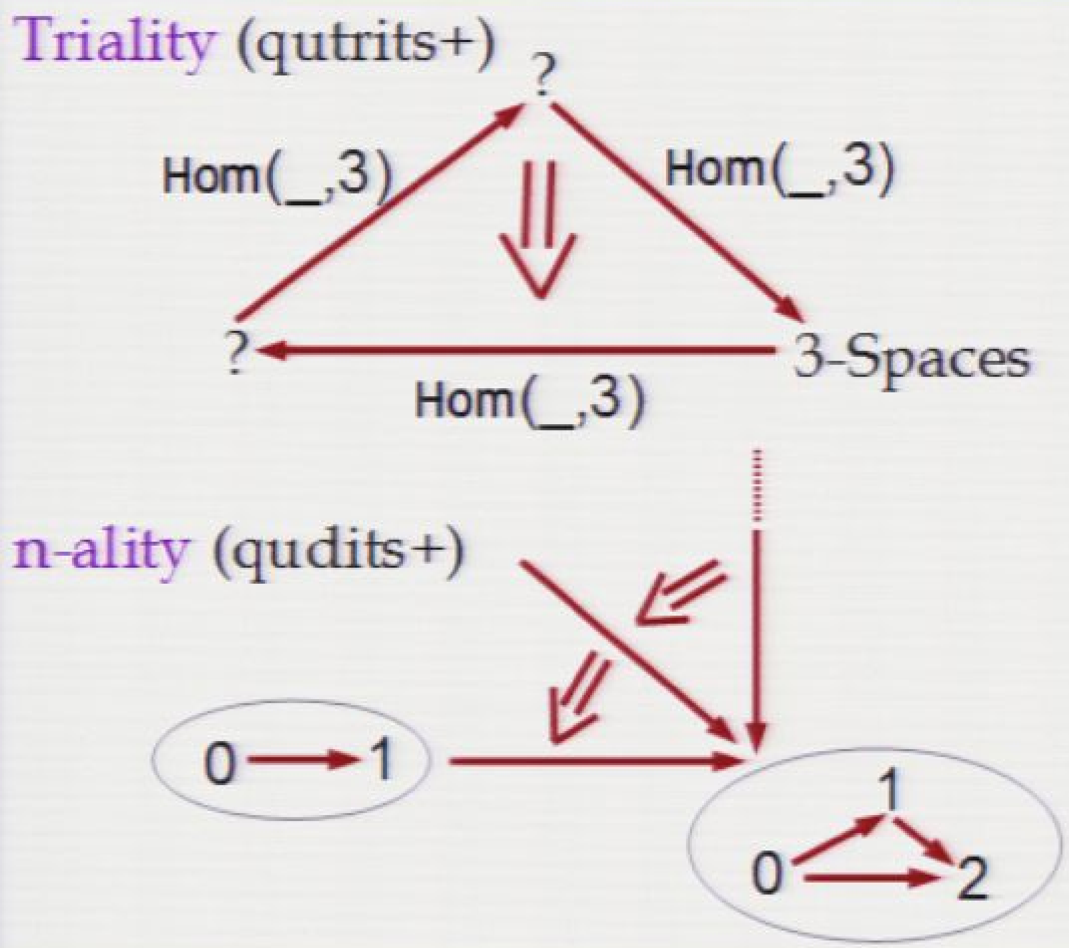
Qubits → Duality

P. T. Johnstone

Stone Spaces (CUP)

- locales - spaces
- phases - numbers
- topos foundations
- See, eg., Forssell's thesis on a Boolean coherent version

Stone n-ality?



Cosmoi etc — Set → Set_*

Ross Street — Mark Weber et al

Summary

- Axiomatic projective *geometry* in QM
- Connection between MUBs and finite fields
- Elements of higher categorical structures
- Real physical applications …

eg:

neutrino mixing

$TBM = F_3 F_2$

Koide matrices:

$$\begin{pmatrix} s & re^{i\theta} & re^{-i\theta} \\ re^{-i\theta} & s & re^{i\theta} \\ re^{i\theta} & re^{-i\theta} & s \end{pmatrix}$$

ref: Carl Brannen

http://carlbrannen.wordpress.com/

+ my blog