Title: The Conway-Kochen-Specker Theorems

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Abstract: TBA

The Kochen-Specker Theorem

Klaas Landsman (with Chris Heunen and Bas Spitters)
Radboud Universiteit Nijmegen

Perimeter Institute,
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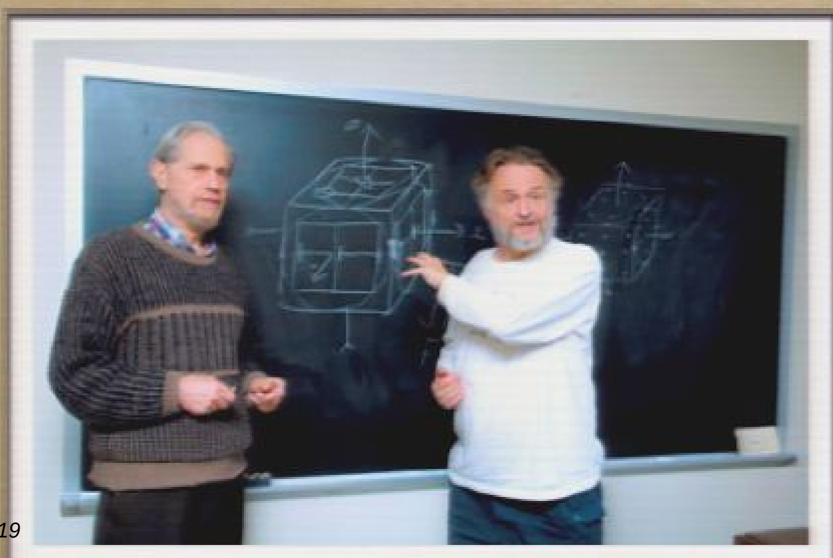

Condemned to indeterminism?

- **Borel space** $(X, \Sigma(X) \subseteq P(X))$; probability measure $\mu: \Sigma(X) \to [0,1]$

  μ is "σ-Boolean map" (morphism of Boolean algebras, countable sups)

  $x \in X$ defines point measure $\delta_x: \Sigma(X) \to \{0, 1\}$, $U \longmapsto 1$ iff $x \in U$

- **Hilbert space** H; "quantum analogue" of **Boolean lattice** $\Sigma(X)$ is **orthomodular lattice** $\mathbf{P}(H)$ of closed linear subspaces of H, $\leq = \subseteq$

  $\mathbf{P}(H) \cong$ lattice of projections $p: H \to H$, $p^* = p^2 = p$, $p \leq q$ iff $pq = p$

- Unit vector $\psi \in H$ defines map $\underline{\psi}: \mathbf{P}(H) \to [0, 1]$, $p \longmapsto (\psi, p\psi)$

  Extension: $p \longmapsto Tr(\varrho p) = \sum_i \lambda_i (\psi_i, p\psi_i)$, with $0 \leq \lambda_i \leq 1, \sum_i \lambda_i = 1$

  is "**locally** σ-**Boolean**" (σ- Boolean on each Boolean part of $\mathbf{P}(H)$)

- **Are there any locally σ-Boolean maps $\mathbf{P}(H) \to \{0, 1\}$?** **(Hidden variables)**

Gleason & Kochen-Specker

- **Gleason (1957):** *If dim(H) > 2, then each locally σ-Boolean map* $\mathbf{P}(H) \to [0, 1]$ *is of the form* $p \longmapsto Tr(\varrho p)$ *for some density matrix* ϱ

  Corollary: *there is no locally (σ-) Boolean map V:* $\mathbf{P}(H) \to \{0, 1\}$

- ***Kochen-Specker Theorem (1967)*** *= this corollary (with direct proof)*

  P*(H) has no local points/models, QM has no non-contextual hidden variables*

- ***Proof*** *for* $H = \mathbb{R}^3$ *(implies result for all complex Hilbert spaces H):*

  If p, q, r orthogonal 1-dimensional projections with $p + q + r = 1$*, then* $(V(p),V(q),V(r)) = (1, 0, 0)$ *or* $(0, 1, 0)$ *or* $(0, 0, 1)$

  This leads to contradiction for specific choice of 33 frames built from 16 different projections (Kochen-Specker, Penrose, Peres, ...)

Enter topos theory

P*(H) = lattice of projections on H (≅ closed linear subspaces of H)*

C*(H) = poset of Boolean sublattices of* ***P****(H)*

Can add conditions to relate to operator algebras: W, AW*, Rickart, spectral ..*

W*(H) = topos of* ***contravariant*** *functors* ***C****(H)* → ***Sets***

Pt*:* ***C****(H)* → ***Sets*** *("dual presheaf"),* ***Pt****(B) = Hom(B, {0,1})*

- *Isham-Butterfield's Kochen-Specker Theorem (1998):*

D has no "global" points, i.e. there is no arrow 1 → Pt in W(H)

- *Follow-up by Hamilton-Isham-Butterfield (2000), Döring (2005)*

General framework for "topos physics": Döring-Isham (2007)

Logic behind Kochen-Specker

Goal: ***clarify role of (intuitionistic) logic in topos quantum theory***

- *Will construct **Heyting algebra** $\Sigma(H)$ in topos $Sh(\mathbf{C}(H))$ so that*

  Kochen-Specker Theorem $\Leftrightarrow$ ***$\Sigma(H)$ has no standard models in*** $Sh(\mathbf{C}(H))$

- ***Complication**: $\Sigma(H)$ and its models live in topos $Sh(\mathbf{C}(H))$ - not in **Sets***

  ***Remedy**: "external description" of $\Sigma(H)$ in **Sets**: Heyting algebra $\Sigma(H)$*

  Kochen-Specker Theorem $\Leftrightarrow$ ***$\Sigma(H)$ has no Kripke models on $\mathbf{C}(H)$***

- *Kochen-Specker already excluded true/false semantics of $\mathbf{P}(H)$*

  Reformulation also excludes "natural" possible world semantics

- *Positive turn: road open for other types of models of quantum logic*

Heyting algebras

- ***Heyting algebra***: *Distributive lattice* Σ *(i.e. with top* $\top$*, bottom* $\bot$*)*

  with map $\Rightarrow: \Sigma \to \Sigma$ *such that* $x \leq (y \Rightarrow z)$ *iff* $(x \wedge y) \leq z$

$\leftrightarrow$ ***Intuitionistic propositional logic*** *with negation* $\neg x := (x \Rightarrow \bot)$

Examples: 1) *Boolean algebras with* $(x \Rightarrow y) = \neg x \vee y$ *("classical")*

2) *Poset P (Kripke frame);* $\Sigma = \{\text{upper sets in } P\}, \leq\ = \subseteq$

3) *Topology* $\Sigma = O(X)$ *with* $\leq\ = \subseteq$ *(Tarski)*

4) Locale **=** *complete distributive lattice where* $x \wedge \bigvee_i \{y_i\} = \bigvee_i \{x \wedge y_i\}$

$\leftrightarrow$ ***Complete Heyting algebra*** *with implication* $(y \Rightarrow z) = \bigvee \{x \mid (x \wedge y) \leq z\}$

5) *Heyting algebra of "intuitionistic quantum logic" (pointwise ordering):*

$\Sigma(H) = \{S: \mathbf{C}(H) \to \mathbf{P}(H) \mid S(B) \in B, S(C) \leq S(D) \text{ if } C \subseteq D$

Models of locales (in Sets)

Locales are Lindenbaum algebras of "geometric" propositional theories (signature Σ; $\top, \wedge, \vee$; axioms $\psi \rightarrow \phi$); "Algebraic" models $\leftrightarrow$ "logical" models

- ***Standard model*** *of locale Σ is $(\wedge, \vee)$-map $\Sigma \rightarrow \{0, 1\} = O(pt)$*

  Topology $O(X)$ has standard models $\delta_x: O(X) \rightarrow \{0, 1\}, x \in X$

- ***Kripke model*** *on poset ("frame") P is $(\wedge, \vee)$- map $\Sigma \rightarrow O_A(P)$*

  locale $O_A(P)$ = ***Alexandrov topology*** *on P = {upper sets in P}*

- $\Sigma(H) = \{S: \mathbf{C}(H) \rightarrow \mathbf{P}(H) \mid S(B) \in B, S(C) \leq S(D) \text{ if } C \subseteq D\}$

  would ***like*** *to have Kripke models δ_ψ on frame $\mathbf{C}(H)$ for $\psi \in H_1$*

  $\delta_\psi: S \mapsto \{B \in \mathbf{C}(H) \mid (\psi, S(B)\, \psi) = 1\}$ = {worlds B in which $S(B)$ is true}

Models of locales in topoi

Can define Heyting algebras, locales and their models in any topos

- $Sh(\mathbf{C}(H)) \simeq \mathbf{Sets}^{\mathbf{C}(H)}$ = topos of **covariant** functors $\mathbf{C}(H) \to \mathbf{Sets}$

  Functor $\underline{\mathbf{P}}: B \mapsto B$ is internal **Boolean** lattice in $Sh(\mathbf{C}(H))$

  Stone spectrum $Pt(B)$ of Boolean lattice B in **Sets**; $B \hookrightarrow O(Pt(B))$, $U \mapsto \{p: B \to \{0,1\} \mid p(U) = 1\}$

  $B \hookrightarrow O(Pt(B))$ isomorphic to $B \hookrightarrow Idl(B) = \{I \subseteq B \mid x, y \in I \Rightarrow x \vee y \in I, x \leq y \in I \Rightarrow x \in I\}$

  "Stone spectrum" of $\underline{\mathbf{P}}$ in $Sh(\mathbf{C}(H))$ is functor $\underline{\mathbf{Idl}}(\underline{\mathbf{P}})$: $B \mapsto \Sigma(H)_{\mid \uparrow B}$

  $\Sigma(H) = \{S: \mathbf{C}(H) \to \mathbf{P}(H) \mid S(B) \in B, S(C) \leq S(D) \text{ if } C \subseteq D\}$ $(\dim(H) < \infty)$

- $\underline{\mathbf{Idl}}(\underline{\mathbf{P}})$ is internal locale/complete Heyting algebra in $Sh(\mathbf{C}(H))$

  Standard models of $\underline{\mathbf{Idl}}(\underline{\mathbf{P}})$ in $Sh(\mathbf{C}(H))$ are $(\wedge, \vee)$- maps $\underline{\mathbf{Idl}}(\underline{\mathbf{P}}) \to \underline{\Omega}$

  locale $\underline{\Omega} = O_A(\mathbf{C}(H))$ is subobject classifier/ "truth object" in $Sh(\mathbf{C}(H))$; cf. $\{0, 1\}$ in **Sets**

From models to models

***Theorem**: There are bijective correspondences between:*

1. ***Standard models*** $\underline{\mathbf{Idl}}(\underline{\mathbf{P}}) \to \underline{\Omega}$ **in** $Sh(\mathbf{C}(H))$ *of Stone spectrum*
   $\underline{\mathbf{Idl}}(\underline{\mathbf{P}}): B \mapsto \Sigma(H)_{|\uparrow B}$ *of (internally) Boolean projection lattice* $\underline{\mathbf{P}}: B \mapsto B$

2. ***Kripke models*** $\Sigma(H) \to O_A(\mathbf{C}(H))$ *of "quantum logic" Heyting algebra*
   $\Sigma(H) = \{S: \mathbf{C}(H) \to \mathbf{P}(H) \mid S(B) \in B, S(C) \le S(D) \text{ if } C \subseteq D\}$ ***in Sets***

3. ***Locally Boolean maps*** $P(H) \to \{0, 1\}$

Idea of proof: *1 ⇔ 3:* $\underline{\mathbf{P}}$ *is basis of "clopens" for locale* $\underline{\mathbf{Idl}}(\underline{\mathbf{P}})$*; map* $\underline{\mathbf{P}} \hookrightarrow \underline{\Sigma}(\underline{\mathbf{P}})$ *composes with* $\underline{\Sigma}(\underline{\mathbf{P}}) \to \underline{\Omega}$ *to natural transformation* $\underline{\mathbf{P}} \to \underline{\Omega}$*; components yield locally Boolean* $\mathbf{P}(H) \to \{0,1\}$

*1 ⇔ 2: Maps(X,Y) = Geom(Sh(Y), Sh(X)); interpret this first in **Sets**, then in* $Sh(\mathbf{C}(H))$*, and use equivalences* $Sh(\mathbf{C}(H)) \simeq \underline{Sh}(\underline{\Omega})$ *and* $Sh(\Sigma(H)) \simeq \underline{Sh}(\underline{\mathbf{Idl}}(\underline{\mathbf{P}}))$ *(Joyal-Tierney, Moerdijk, Johnstone):*

$$Maps(\Sigma(H), \mathbf{C}(H)) \simeq Geom(Sh(\mathbf{C}(H)), Sh(\Sigma(H))) \simeq Geom(\underline{Sh}(\underline{\Omega}), \underline{Sh}(\underline{\mathbf{Idl}}(\underline{\mathbf{P}}))) \simeq Maps(\underline{\mathbf{Idl}}(\underline{\mathbf{P}}), \underline{\Omega})$$

Kochen & Specker strike back

1. *Locally Boolean maps* **P**$(H) \rightarrow \{0, 1\}$ *do not exist (Kochen-Specker)*
2. *Stone spectrum* **Idl**(**P**) *of* **P** *has no standard models in Sh(***C***(H))*
3. *Heyting algebra* $\Sigma(H)$ *has no Kripke models on frame* **C***(H) in* **Sets**

Locales $\Sigma(H)$ *and* **Idl**(**P**) *may have other models (also in other topoi)*

Interpretation*: such models would correspond to unusual hidden variables constructed by forcing, as in unusual models of set theory in which continuum hypothesis holds/fails (cf. Boos, 1996; Van Wesep, 2006)*

Have we just proved $\varnothing \simeq \varnothing$?

Replace Hilbert space H [really: B(H)] by unital C-algebra A*

Replace $\mathbf{C}(H)$ *= poset of Boolean sublattices of* $\mathbf{P}(H)$

by $\mathbf{C}(A)$ *= poset of unital commutative C*-subalgebras of A*

for $\dim(H) < \infty$, $\mathbf{C}(B(H)) \cong \mathbf{C}(H)$, *for general H need special classes of C*-(sub)algebras*

- *Replace* $Sh(\mathbf{C}(H))$ *by* $Sh(\mathbf{C}(A)) \simeq$ *topos of covariant functors* $\mathbf{C}(A) \to \mathbf{Sets}$
- *Replace internal **Boolean** lattice* $\underline{\mathbf{P}}: B \mapsto B$ *in* $Sh(\mathbf{C}(H))$ *by*

*internal **commutative** C*-algebra* $\underline{\mathbf{A}}: C \mapsto C$ *in* $Sh(\mathbf{C}(A))$

*Replace **Stone** spectrum* $\underline{\mathbf{Idl}}(\underline{\mathbf{P}})$ *of* $\underline{\mathbf{P}}$ *in* $Sh(\mathbf{C}(H))$ *by*

*(localic) **Gelfand** spectrum* $\underline{\Sigma}(\underline{\mathbf{A}})$ *of* $\underline{\mathbf{A}}$ *in* $Sh(\mathbf{C}(A))$

$\dim(H) < \infty$: $\underline{\Sigma}(B(H)) \cong \underline{\mathbf{Idl}}(\underline{\mathbf{P}})$

No!

***Theorem*: There are bijective correspondences between:**

1. ***Standard models*** $\underline{\Sigma}(\underline{A}) \to \underline{\Omega}$ ***in*** $Sh(\mathbf{C}(A))$

2. ***Kripke models*** $\Sigma(A) \to O_A(\mathbf{C}(A))$ *of "external description"*

   $\Sigma(A) = \underline{\Sigma}(\underline{A})(\mathbf{C}(A))$ *of* $\underline{\Sigma}(\underline{A})$ *in* ***Sets***

3. ***Valuations*** $A_{sa} \to \mathbb{R}$ *i.e. maps that are linear and multiplicative on commutative C*-subalgebras of A*

- *Kochen-Specker situation recovered for* $A = B(H)$
- *New phenomena for general C*-algebras A, for example:*

$\Sigma(C(X)) \cong O(X)$ *and hence* $A \mapsto Sh(\Sigma(A))$ *is noncommutative extension of sheaf functor* $X \mapsto Sh(X)$ [*since* $Sh(C(X)) \cong Sh(X)$]

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