Title: Why Topos Theory in the Foundations of Physics

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Abstract: TBA

Why Topos Theory in the Foundations of Physics?

Perimeter Institute, Waterloo
CQC (Categories, Quanta, Concepts)
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"Metaphysics may be, after all, only the art of being sure of something that is not so, and logic only the art of going wrong with confidence."

Joseph Wood Krutch, The Modern Temper, 1929

What's the problem?

Why is quantum theory (QT) problematic conceptually?

- states do not assign values to all physical quantities, hence only probabilistic predictions
- interpretation needs classical structures: observers, preparations, measurements
- even with these classical structures: measurement problem
- quantum logic is too weak
- QT does not describe a system 'in itself', there is no picture of quantum reality arising
- ...

More problems...

Why is quantum gravity (QG) even more problematic?

- we cannot write down QG, it is extremely hard technically
- space and time would become quantum objects, but where and when would a measurement on them take place?
- we don't even have a good idea what QG is supposed to be (could be neither *quantum* nor *gravity*)

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What about quantum cosmology (QC)?

- can potentially arise from simplified QG, so simpler, but more generic
- clearly no external observer, no preparation, no measurements, no statistics
- not embedded into any classical structure, must be understood 'out of itself'

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One possible conclusion: it is not a good idea to search for a quantisation of GR using Hilbert space methods.

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- The odd one out seems to be quantum theory, not gravity. There are big open conceptual issues with the former, not so much the latter.
- In particular with an eye to QG and QC, we need a mathematical formalism that allows to describe physical systems 'in themselves', without the need for external observers and other classical structures.
- In other words, we need a framework that allows us to formulate physical theories in a realist way.

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- In particular with an eye to QG and QC, we need a mathematical formalism that allows to describe physical systems 'in themselves', without the need for external observers and other classical structures.
- In other words, we need a framework that allows us to formulate physical theories in a realist way.

This clearly is also interesting for QT, but can it be done? Problems: Hilbert space formalism is very rigid; Kochen-Specker theorem.

The prototype

The prototype of a realist theory is classical mechanics. Pure states form a state space $\mathcal{S}$ (symplectic manifold), physical quantities are real-valued functions f_A on $\mathcal{S}$. The evaluation

$$(s, f_A) \longmapsto f_A(s)$$

gives the value that the physical quantity A has in the state s. The subset $f_A^{-1}(\Delta)$ of $\mathcal{S}$ represents the proposition "$A \varepsilon \Delta$", that is, "the physical quantity A has a value in the (Borel) set Δ". All states $s \in \mathcal{S}$ that lie in $f_A^{-1}(\Delta)$ make the proposition true, all other states make it false.

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We observe an interplay between the geometric and the logical structure of the theory. This is made precise by Stone's theorem.

Abstraction

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From the KS theorem, we know that there is no naive state space picture of quantum theory, and that we cannot assign real values to all physical quantities at once.

The idea is to go beyond spaces, which are *sets*, to more general objects in a suitable *category*. This will also mean to go beyond Boolean logic. The collection of (representatives of) propositions of the form "$A \varepsilon \Delta$" will not form a Boolean algebra anymore, but we still want some interpretable logical structure.

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All this suggests the use of topos theory. We will reformulate (large parts of) QT in a suitable topos. QG and QC will follow ;-)

What kind of beast is a topos? Or: Which part of the elephant did we grab?

There are many facets of topos theory, and topoi show up in many mathematical situations.

For us, the central idea is that an *elementary topos* $\mathcal{E}$ is a generalisation of the category **Sets** of sets and functions (itself a topos, of course).

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The objects in such a topos can be seen as generalised sets, the arrows are generalised functions. Given two sets, we can form their cartesian product, disjoint union and the set of all functions from one set to the other. In a topos, there are analogous operations defined on all objects.

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Just as the topos **Sets** comes equipped with Boolean logic, each topos $\mathcal{E}$ has an *internal logic*, which is of *intuitionistic* type, i.e., the law of excluded middle does not hold in general. But, interestingly, topoi allow for a well-defined notion of *partial truth*.

The genesis of topos ideas in physics

In the beginning, there was Lawvere, who decided to reinvent mathematics in order to do physics properly (1960s). Generalising Grothendieck, he invented elementary topoi (together with Miles Tierney).

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In 1996, Chris Isham first suggested to use topos theory in foundations of quantum theory (in the consistent histories approach). IJTP 36 (1997), 785–814

A central idea was already there:

Consider the commutative/distributive parts of a non-commutative/ non-distributive structure and their relations and build presheaves over them. The latter form a topos, whose internal logic is then employed in the interpretation. The commutative/distributive parts serve as stages of truth.

The genesis of topos ideas in physics (2)

The idea to look at a quantum system from the collection of classical perspectives was then developed considerably in four papers (1998-2001) with Jeremy Butterfield, concerned with the KS theorem (and how to get around it). IJTP 37, 2669–2733 (1998), IJTP 38, 827–859 (1999), IJTP 39, 1413–1436 (2000), IJTP 41, 613–639 (2002)

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In particular, they introduced the set $\mathcal{V}(\mathcal{N})$ of abelian subalgebras of the non-abelian algebra $\mathcal{N}$ of physical quantities, partially ordered under inclusion. Each abelian subalgebra or *context* $V \in \mathcal{V}(\mathcal{N})$ gives a classical perspective on the quantum system.

Importantly, each context V has a spectrum $\underline{\Sigma}_V$ such that $V \simeq C(\underline{\Sigma}_V)$. Each $\underline{\Sigma}_V$ is a *local state space*. Butterfield and Isham introduced the *spectral presheaf* $\underline{\Sigma}$, which collects all these local state spaces into one large structure.

The spectral presheaf

The spectral presheaf $\underline{\Sigma}$ is the functor assigning to each $V \in \mathcal{V}(\mathcal{N})$ its Gel'fand spectrum, and to each inclusion $V' \subset V$ of contexts the restriction function

$$\begin{aligned} \underline{\Sigma}(i_{V'V}) : \underline{\Sigma}_V &\longrightarrow \underline{\Sigma}_{V'} \\ \lambda &\longmapsto \lambda|_{V'}. \end{aligned}$$

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Physically, the spectral presheaf $\underline{\Sigma}$ is the analogue of the state space $\mathcal{S}$ of a classical system.

$\underline{\Sigma}$ is an object in the topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{N})^{op}}$ of presheaves over the context category $\mathcal{V}(\mathcal{N})$. This is the topos associated to a quantum system (with $\mathcal{N}$ as algebra of physical quantities).

Contravariance

Some remarks:

- The fact that we are using *contravariant* functors incorporates the important concept of *coarse-graining*
- the idea of coarse-graining goes back to Kochen-Specker pairs of physical quantities

Daseinisation of projections

The spectral presheaf $\underline{\Sigma}$ is the analogue of state space, so propositions should be represented by subobjects of $\underline{\Sigma}$.

The relevant mapping is called *daseinisation of projections*. It maps every proposition "$A \varepsilon \Delta$" to a (clopen) subobject of $\underline{\Sigma}$, the analogue of a Borel subset $f_A^{-1}(\Delta)$ of state space. Daseinisation takes coarse-graining into account. The subobjects form a *Heyting algebra*.

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The resulting new form of quantum logic (Butterfield, Isham, D) is intuitionistic. This topos quantum logic has many sensible properties (different from standard quantum logic).

Daseinisation of self-adjoint operators

There is another mapping, called daseinisation of self-adjoint operators. Though mathematically related to the daseinisation of projections, it does something quite different: it 'translates' physical quantities A into arrows $\breve{\delta}(A)$ in our topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{N})^{op}}$.

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Specifically, $\breve{\delta}(A)$ is an arrow from the spectral presheaf to a certain presheaf of 'values', basically given by intervals of real numbers. The arrow $\breve{\delta}(A) : \underline{\Sigma} \to \underline{\mathbb{R}^{\leftrightarrow}}$ is the analogue of the function $f_A : \mathcal{S} \to \mathbb{R}$ in classical physics.

Pure states

In the topos formulation, a pure state ψ of a quantum system corresponds to a certain subobject $\mathfrak{w}^{\psi}$ of the spectral presheaf $\underline{\Sigma}$, called a *pseudo-state*. Since $\underline{\Sigma}$ has no points, $\mathfrak{w}^{\psi}$ must be 'bigger than a point', but it is minimal with respect to a certain natural condition.

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Using the internal logic of our topos of presheaves, we can assign a truth-value to every proposition "$A \varepsilon \Delta$" in any given state. In fact, the truth-value assignment is a direct generalisation of the truth-value assignment in classical physics.

Mixed states

A mixed state of a classical system is a probability measure μ on the state space $\mathcal{S}$. One can show easily that every state ρ of a quantum system gives a probability measure μ_ρ (in an appropriate sense) on its spectral presheaf $\underline{\Sigma}$.

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Probability measures on $\underline{\Sigma}$ can be characterised abstractly. Importantly, every such measure μ determines a unique state ρ_μ of the (von Neumann) algebra of physical quantities $\mathcal{N}$. This means that we fully capture the state space of the non-commutative algebra $\mathcal{N}$ in our formalism.

The table

Abstract	**Class. phys.**	**Standard QT**	**Topos QT**
state space	set $\mathcal{S}$	Hilb. sp. $\mathcal{H}$?	presheaf $\underline{\Sigma}$
propos. "$A \in \Delta$"	subset of $\mathcal{S}$	projection $\widehat{P}$	subobj. of $\underline{\Sigma}$
latt. of props.	Boolean algebra	$\mathcal{P}(\mathcal{N})$ (non-distrib.)	Heyting algebra
topos	**Sets**	**Sets**	$\mathbf{Sets}^{\mathcal{V}(\mathcal{N})^{op}}$
logic	Boolean	quantum log.	intuitionistic
deductive sys.	yes	no(?)	yes
truth values	true, false	? (probabilities)	elements of $\underline{\Omega}$
interpretation	realist	instrument.	neo-realist
quant.-val. obj.	$\mathbb{R}$	$\mathbb{R}$?	$\underline{\mathbb{R}^{\leftrightarrow}}$
phys. quant. A	$f_A : \mathcal{S} \to \mathbb{R}$	$\hat{A} : \mathcal{H} \to \mathcal{H}$	$\breve{\delta}(\hat{A}) : \underline{\Sigma} \to \underline{\mathbb{R}^{\leftrightarrow}}$
pure state	Dirac meas. δ_s	vector st. w_ψ	pseudo-st. $\mathfrak{w}^\psi$
general state	prob. m. μ on $\mathcal{S}$	state ρ of vNa	prob. m. μ on $\underline{\Sigma}$

Quantum theory is...

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All these aspects are interrelated. Topos theory delivers tools to generalise both Boolean logic and set-based mathematics/topology/geometry.

Related work

Recently, Caspers, Heunen, Landsman and Spitters used *covariant* functors over the context category to define a topos-internal commutative algebra $\overline{\mathcal{N}}$ from the external non-commutative algebra $\mathcal{N}$. The commutative algebra $\overline{\mathcal{N}}$ has a Gel'fand spectrum, which is a certain locale in the topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{N})}$ of functors. CMP *in print* (2009), FOP *in print* (2009), arXiv:0905.2275

They suggest to use opens in this locale as representatives of propositions. The resulting formalism, and in particular the intuitionistic quantum logic, are closely related to the earlier work in the contravariant approach.

It will be interesting to see the interpretational differences.

Flori has developed a topos reformulation of the consistent histories formalism. arXiv:0812.1290

References

A. Döring, C. J. Isham, "A Topos Foundation for Theories of Physics I-IV", J. Math. Phys. **49**, 5, 053515–18 (2008), arXiv:quant-ph/0703060, 62, 64 and 66

A. Döring, "Topos theory and 'neo-realist' quantum theory", in *Quantum Field Theory: Competitive models*, eds. B. Fauser, J. Tolksdorf, E. Zeidler (Birkhäuser 2009), arXiv:0712.4003

A. Döring, C. J. Isham, "'What is a thing?': Topos Theory in the Foundations of Physics", to appear in *New Structures in Physics*, ed. Bob Coecke (Springer 2008), arXiv:0803.0417

A. Döring, "Quantum States and Measures on the Spectral Presheaf", *Adv. Sci. Lett.* **2**, Number 2 (Special Issue on "Quantum Gravity, Cosmology and Black Holes", ed. M. Bojowald), 291–301 (2009), arXiv:0809.4847

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