Title: Phase Groups and Complementarity

Date: Jun 01, 2009 10:15 AM

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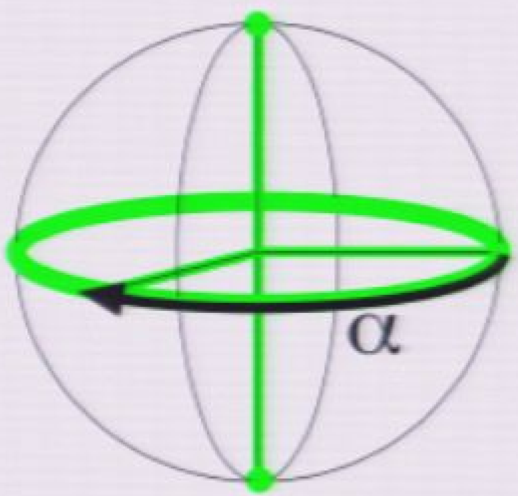
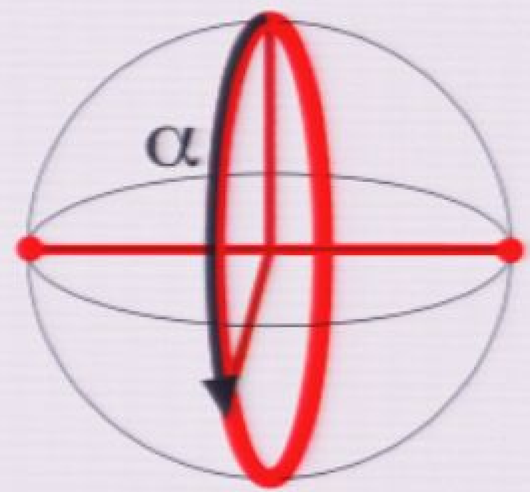
Abstract: TBA

Phase Groups and Complementarity





Ross Duncan
Oxford University Computing Laboratory

Motivations

$$A \Rightarrow B$$

Motivations

$$A \Rightarrow B \qquad\qquad !A \multimap B$$

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Hilbert space,
unitary transforms,
self-adjoint
operators....

?

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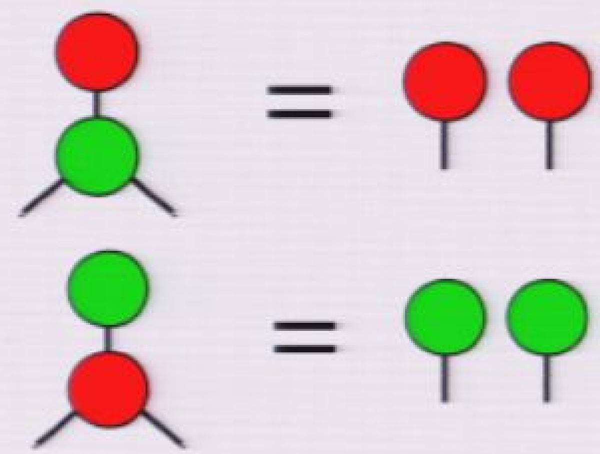

What we did:

We reformulated (a large part of) quantum mechanics:

- high level structural approach based on monoidal categories
- axiomatics expressive enough to cover universal quantum computation
  - and powerful enough to simulate algorithms and prove equivalence between quantum state and programs
- simple to understand and manipulate graphical calculus
  - which is being implemented in semi-automatic GUI based rewriting tool

In This Talk:

In This Talk:

Observables

In This Talk:

Observables ⟷ Special †-Frobenius Algebras

aka classical structures

aka basis structures

aka observable structures

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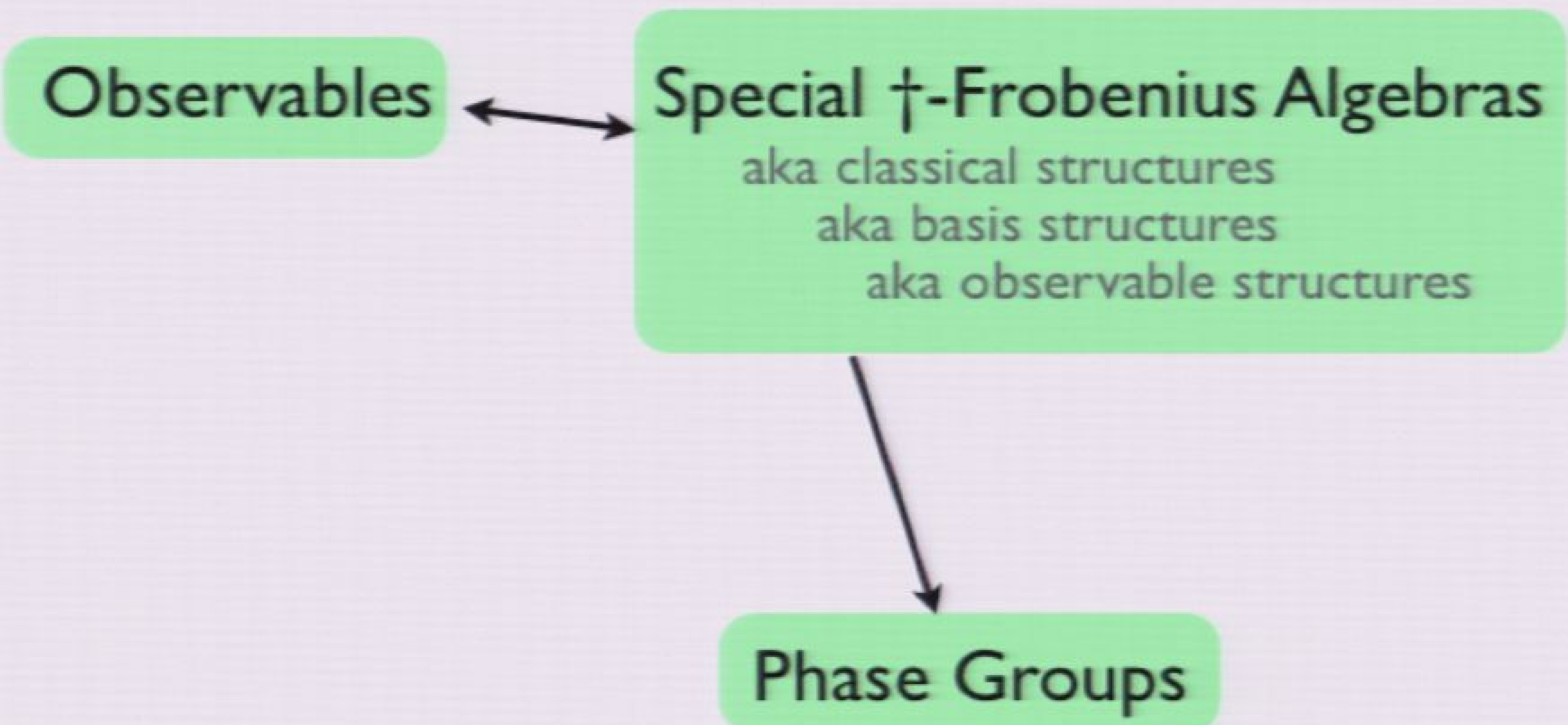
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Phase Groups

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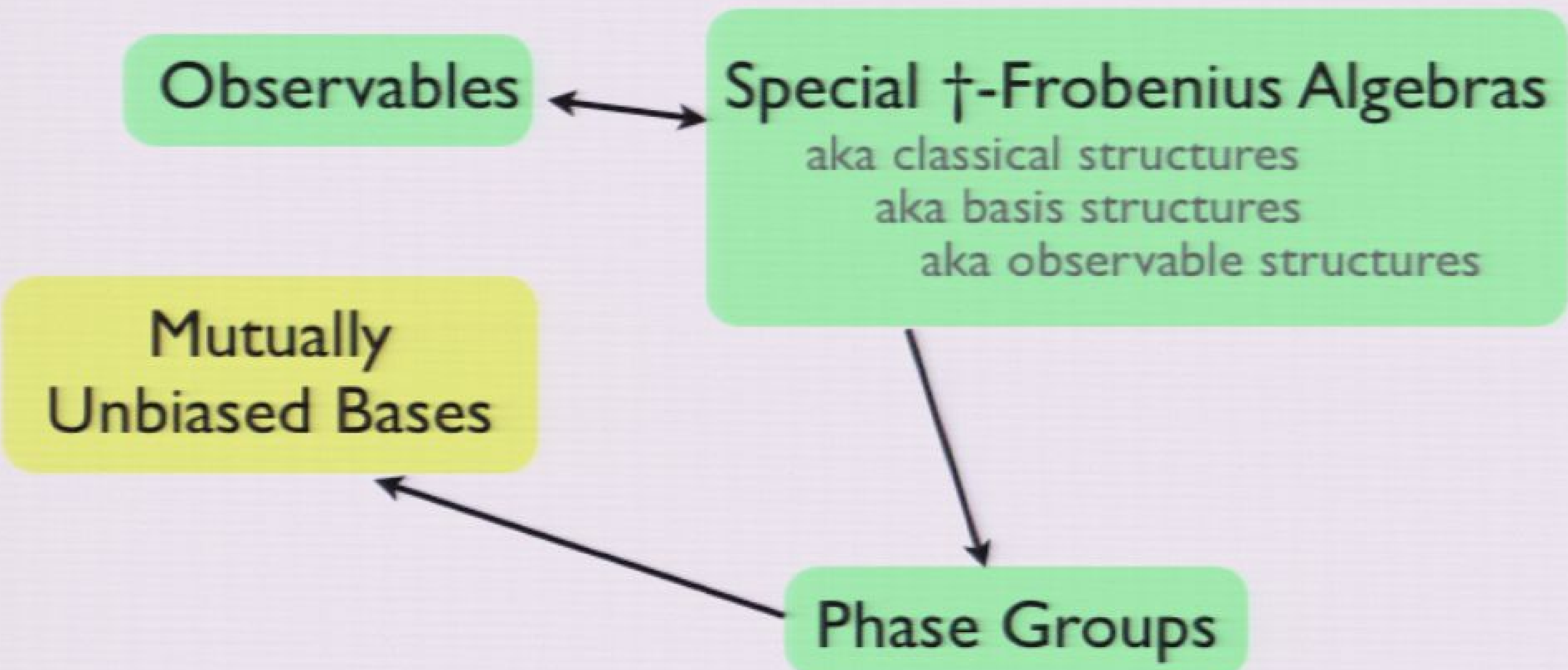
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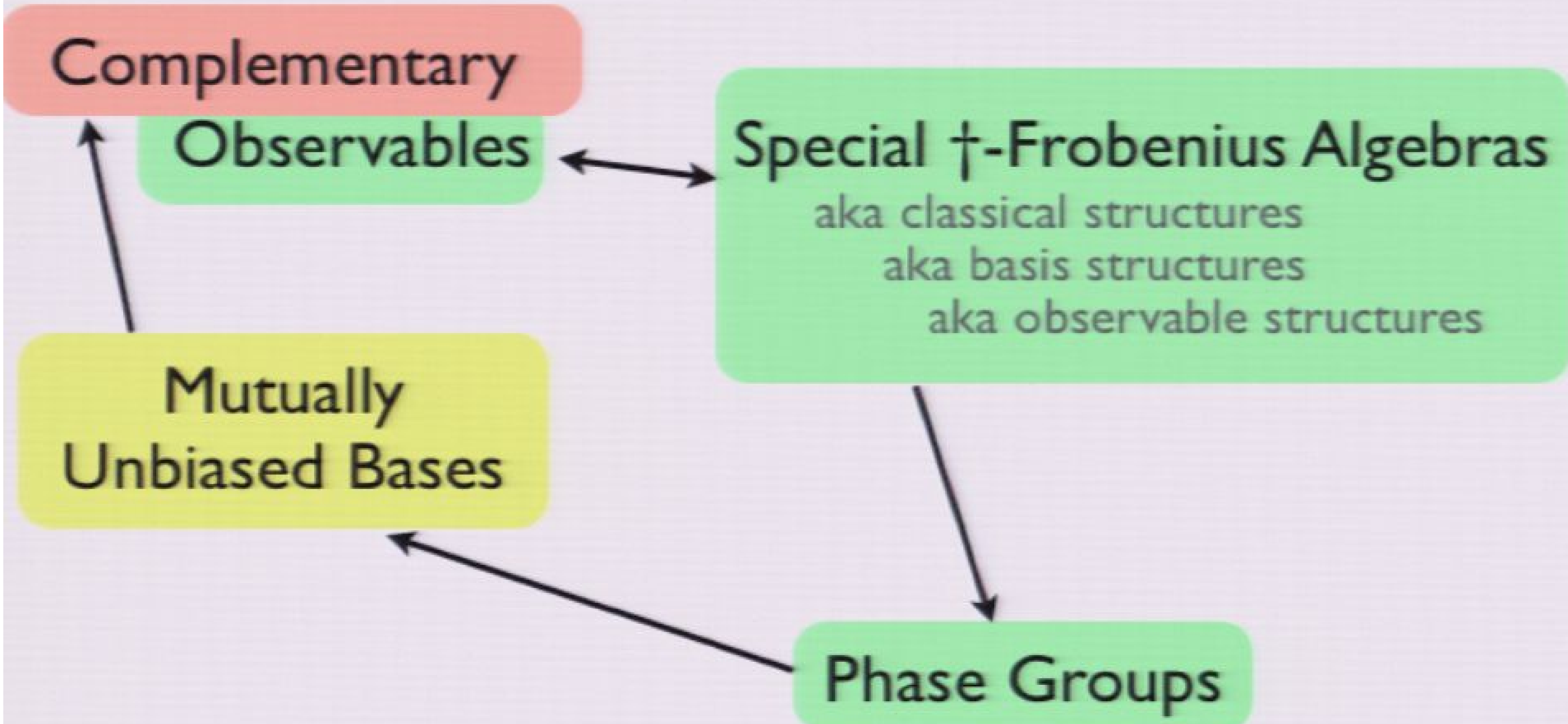
Complementary
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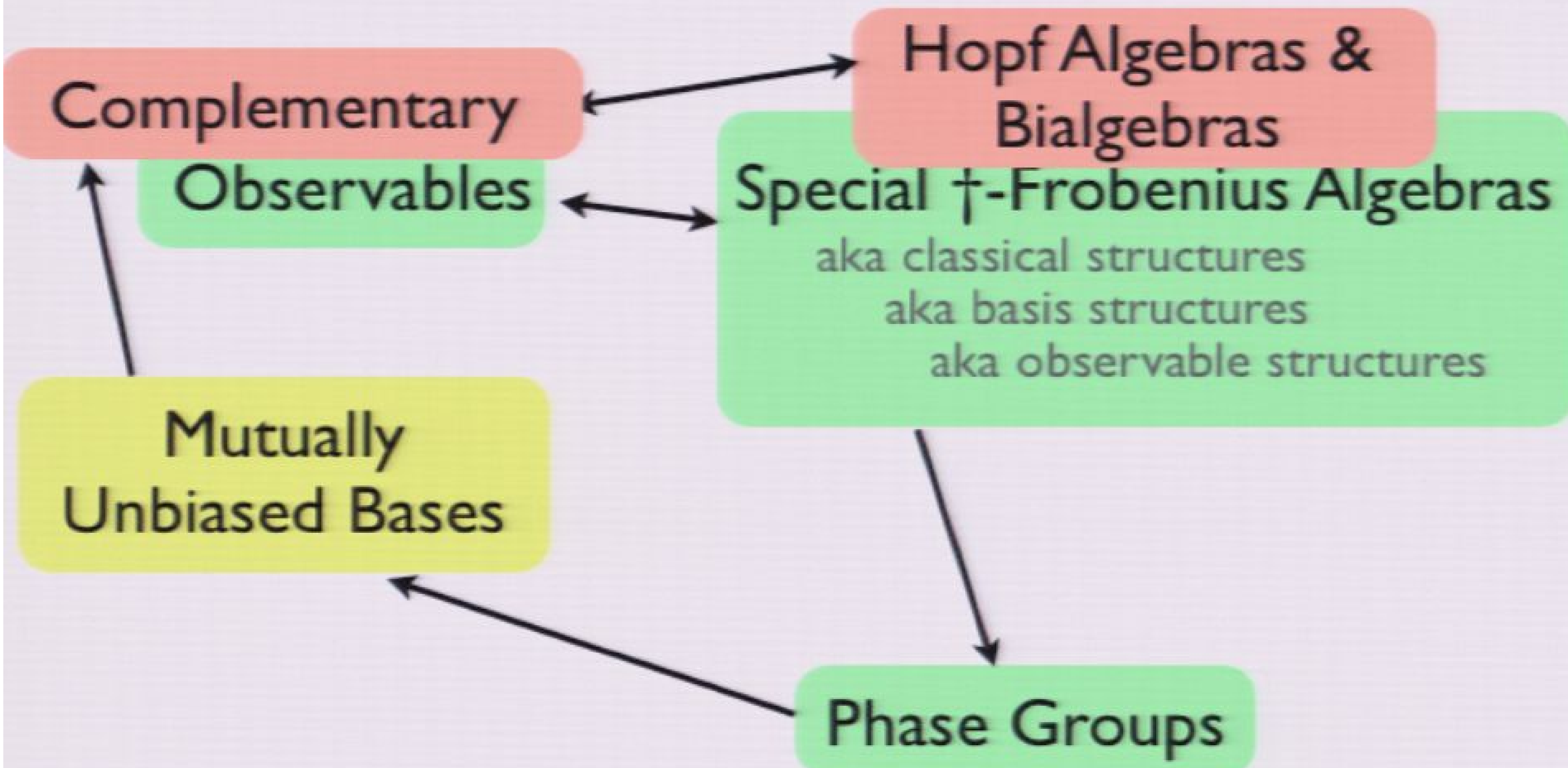
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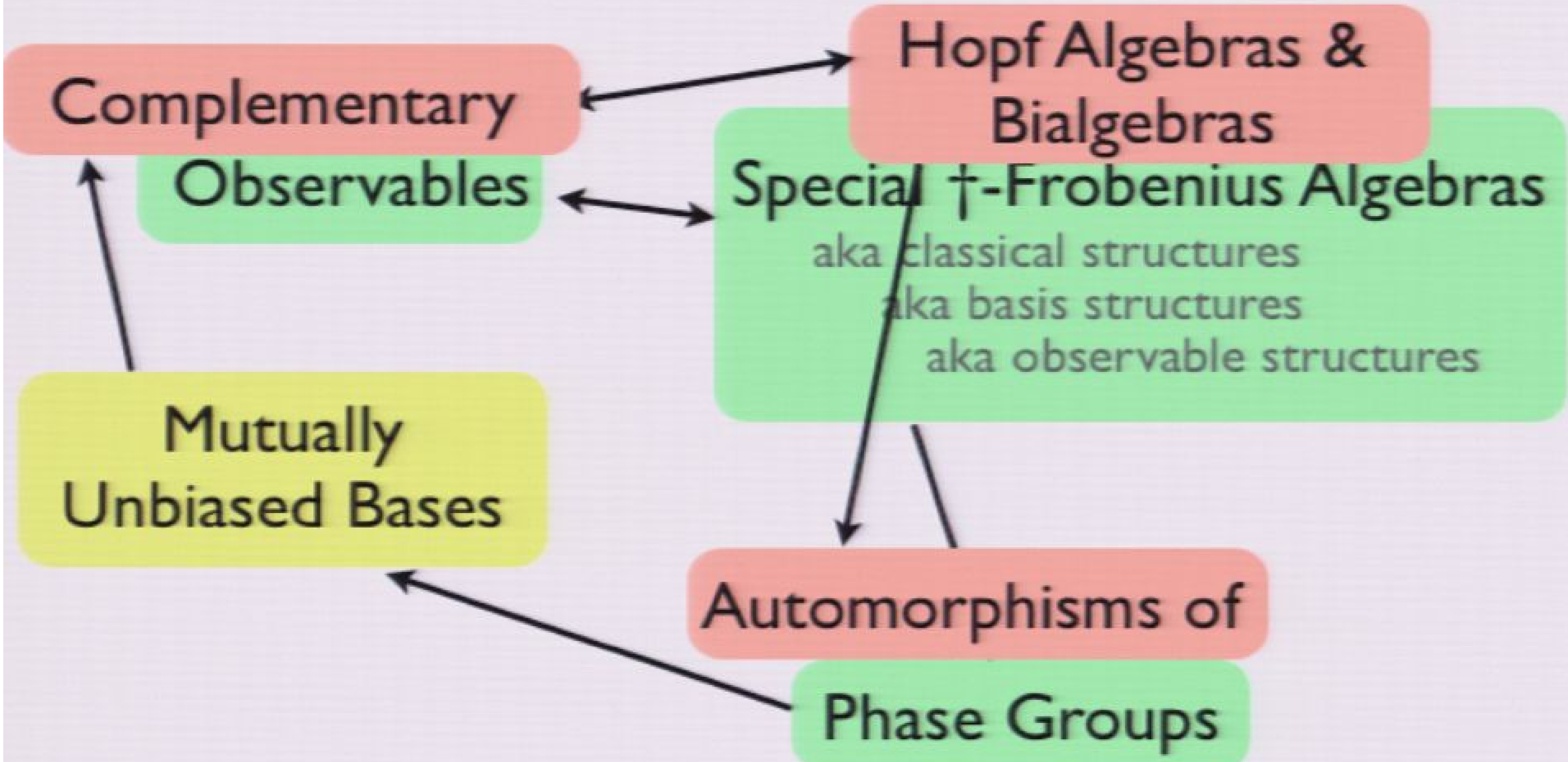
Complementary
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Automorphisms of
Phase Groups

What is a Phase Group?

- An abelian group of unitary maps on the state space

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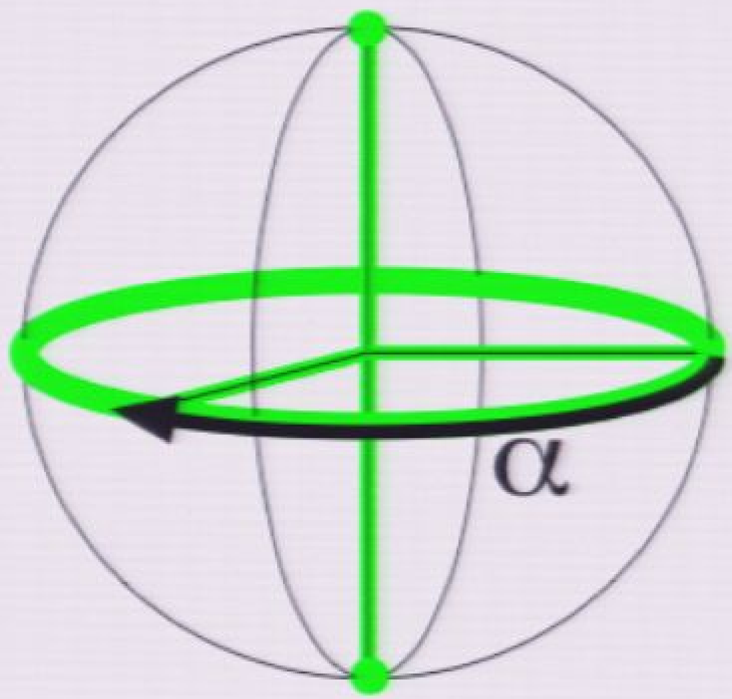

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Ingredients of the Theory

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To construct phase groups we need:

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  - *Symmetric monoidal categories*
- Unitarity:
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- Observables:
  - *Special †-Frobenius algebras*

This is a very general setting including much more than just quantum mechanics.

Why Bother?

What can phase groups say about quantum theory?

What can phase groups say about other theories?

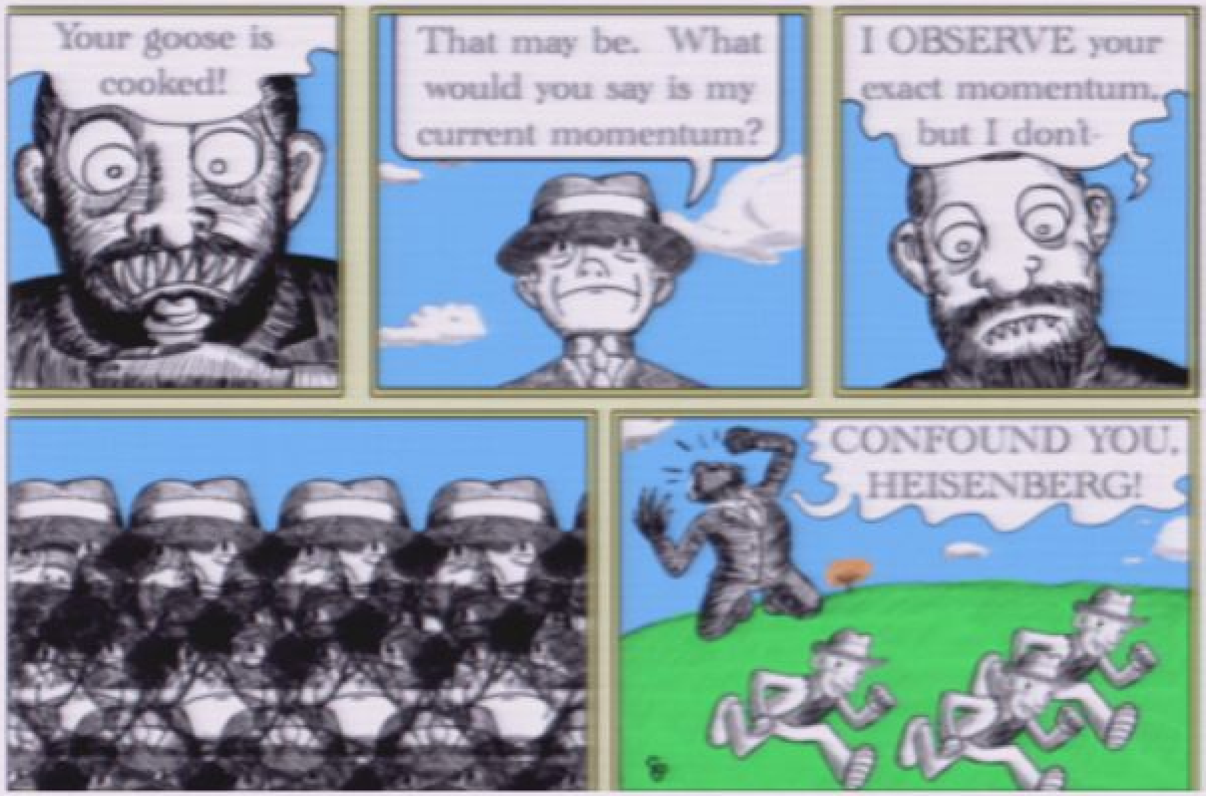
- Qubits, qutrits, etc
- Finite relations
- Toy models
- Convex operational theories
- Continuous variable QM
- **Stab** and **Spek**
- .....

DISCLAIMER

There are going to be a LOT of definitions in this talk.

sorry.

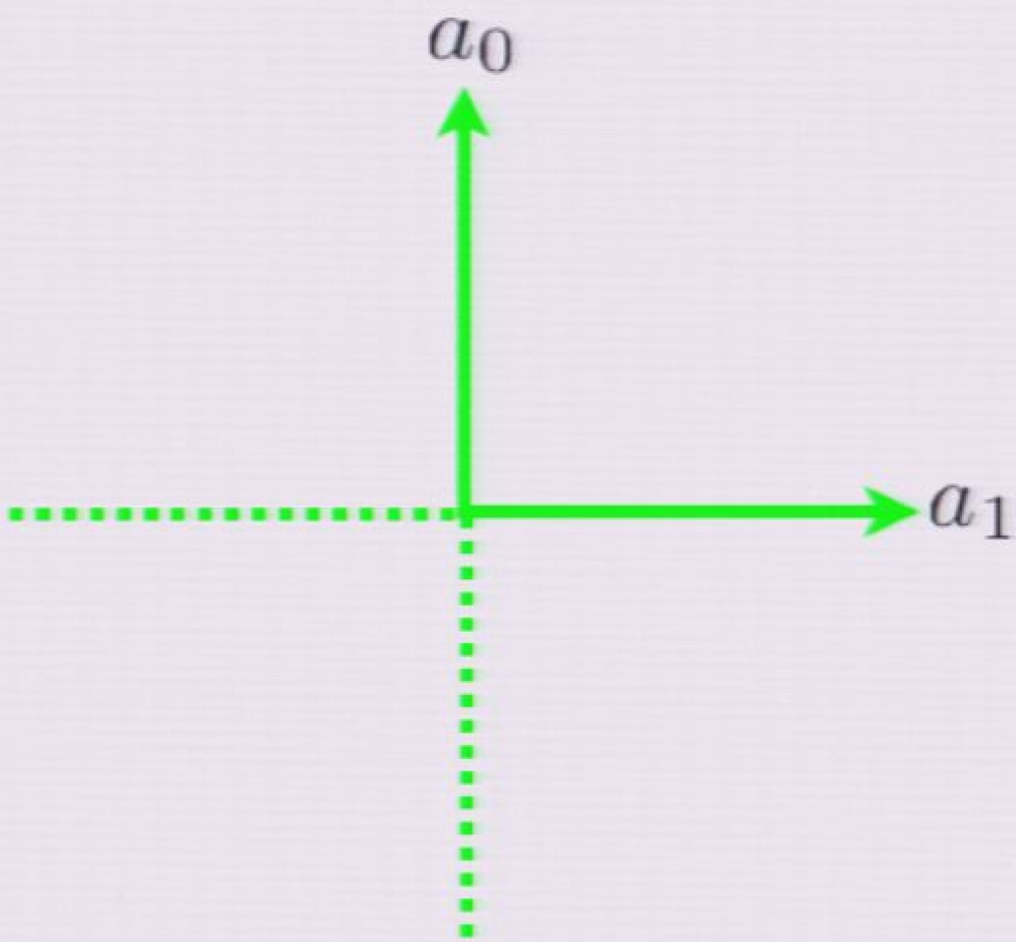
Quantum Observables



What is "classical" anyway?

http://dresdencodak.com

Complementary Observables



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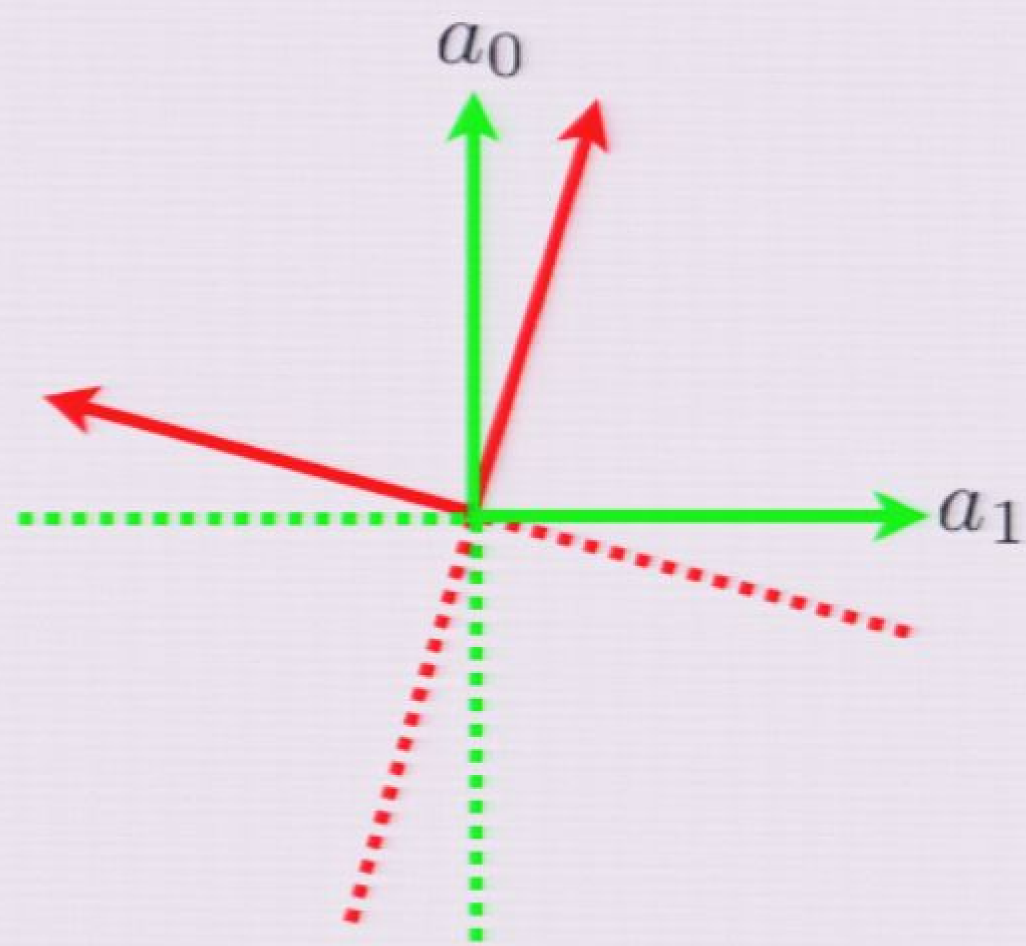

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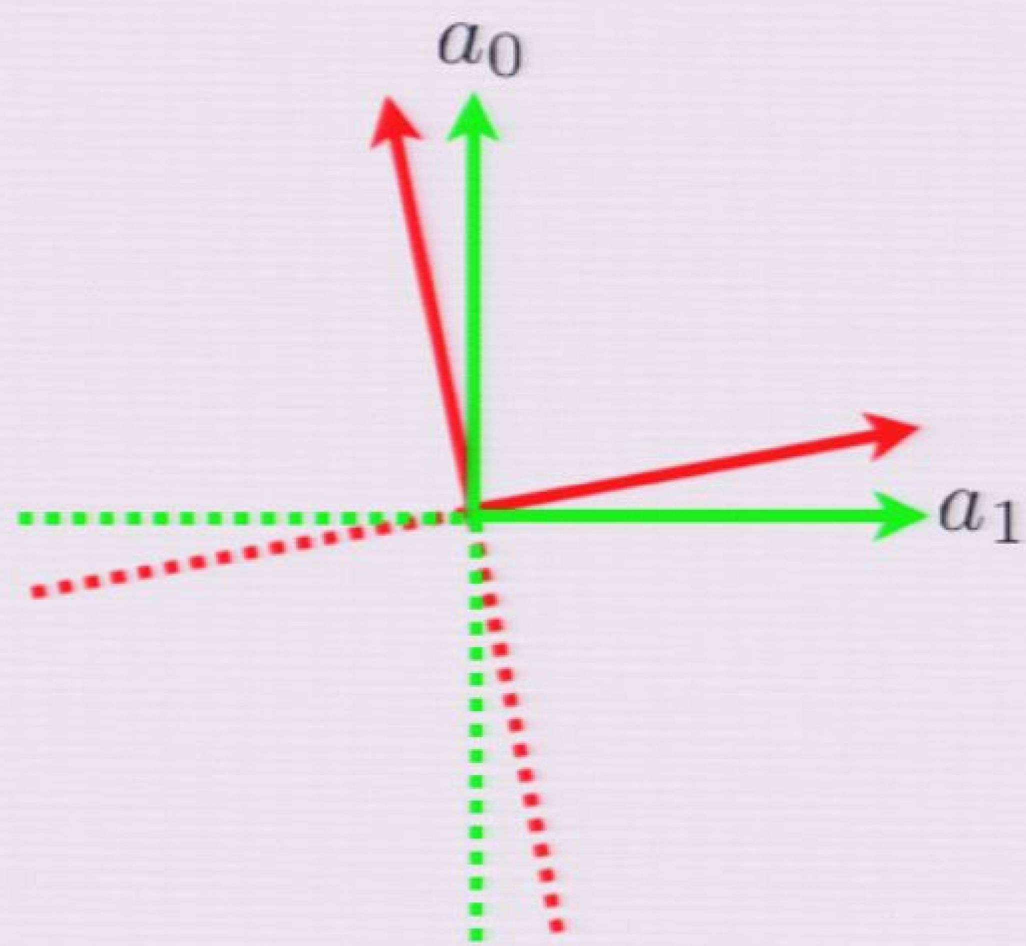

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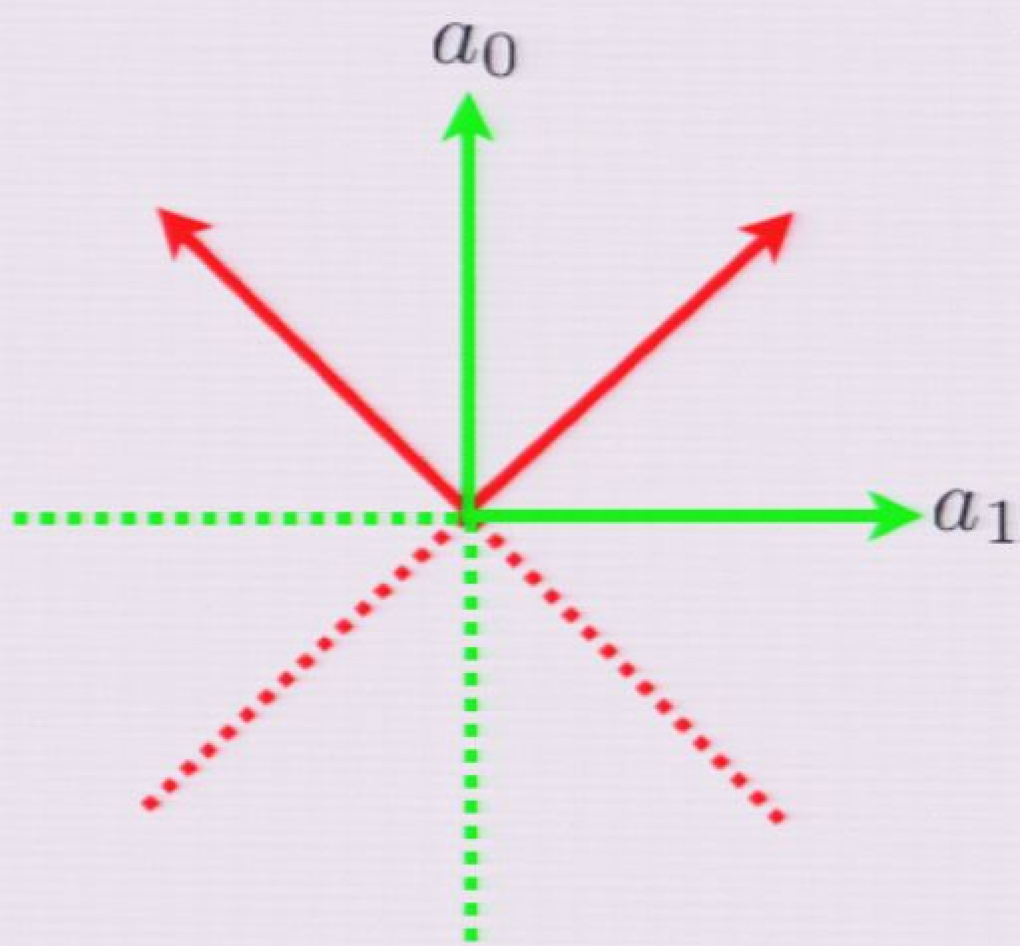

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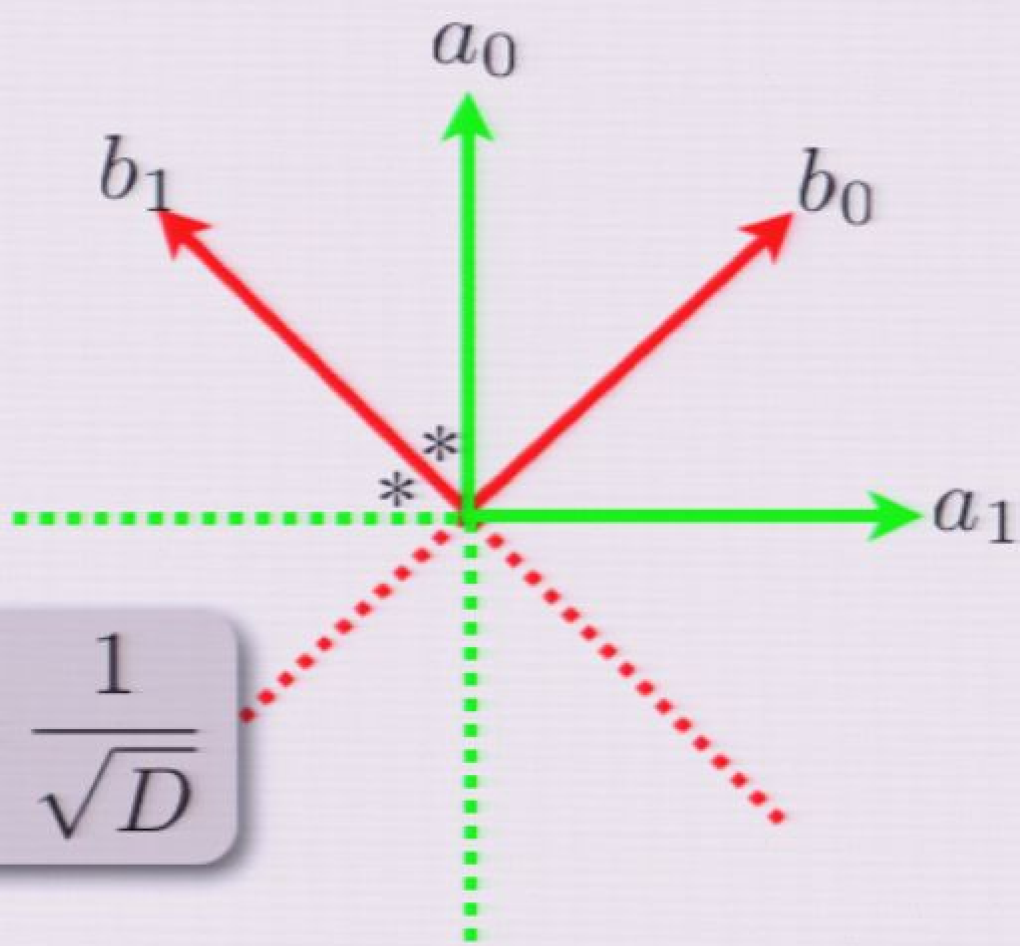

$$|\langle a_i \mid b_j \rangle| = \frac{1}{\sqrt{D}}$$

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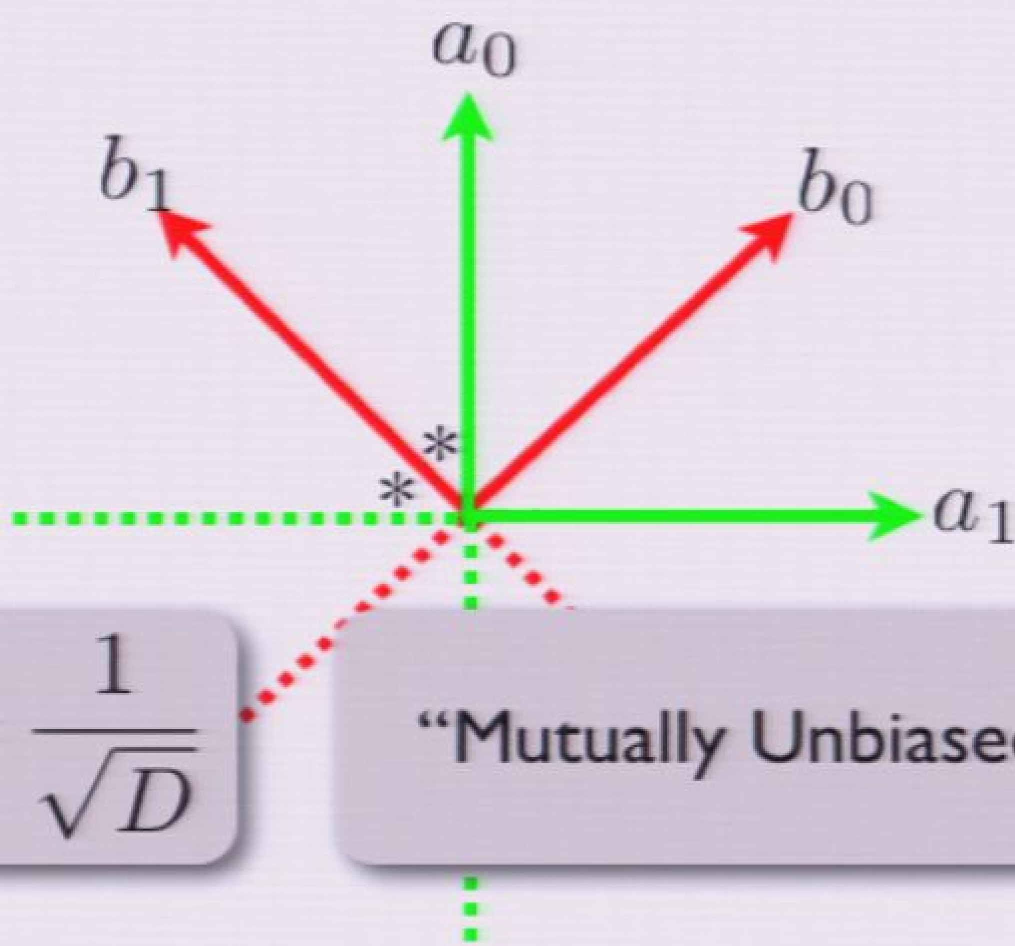


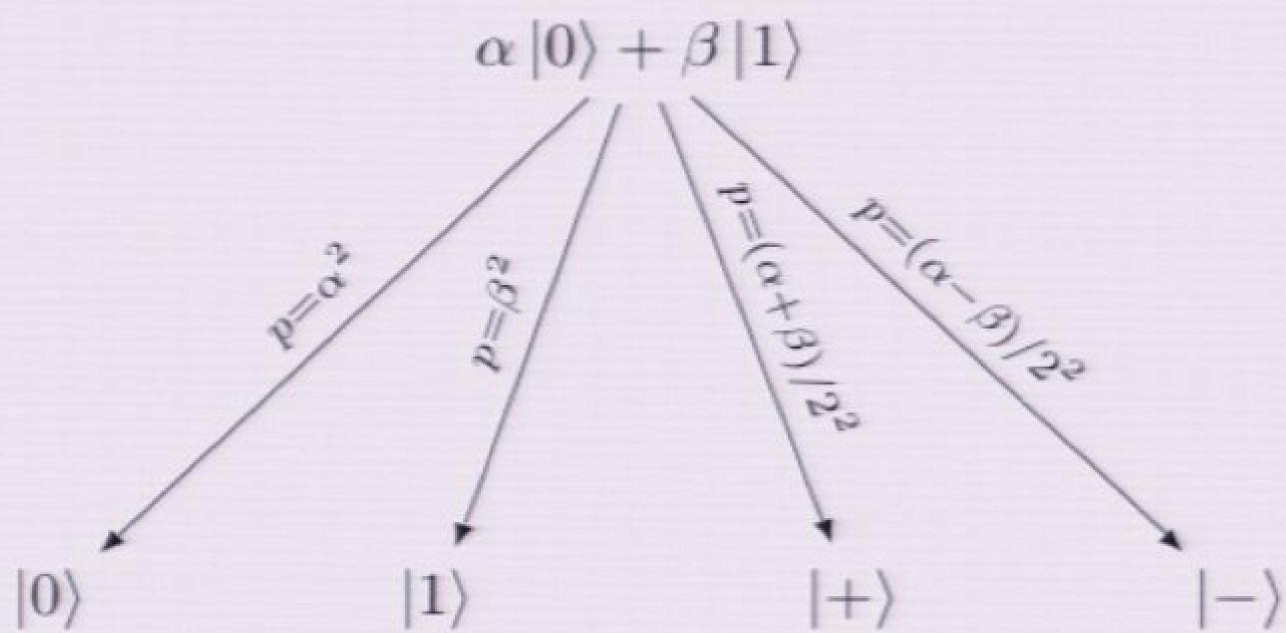
$$|\langle a_i \mid b_j \rangle| = \frac{1}{\sqrt{D}}$$

"Mutually Unbiased Bases"

X and Z Spins

We can measure the spin of qubit $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



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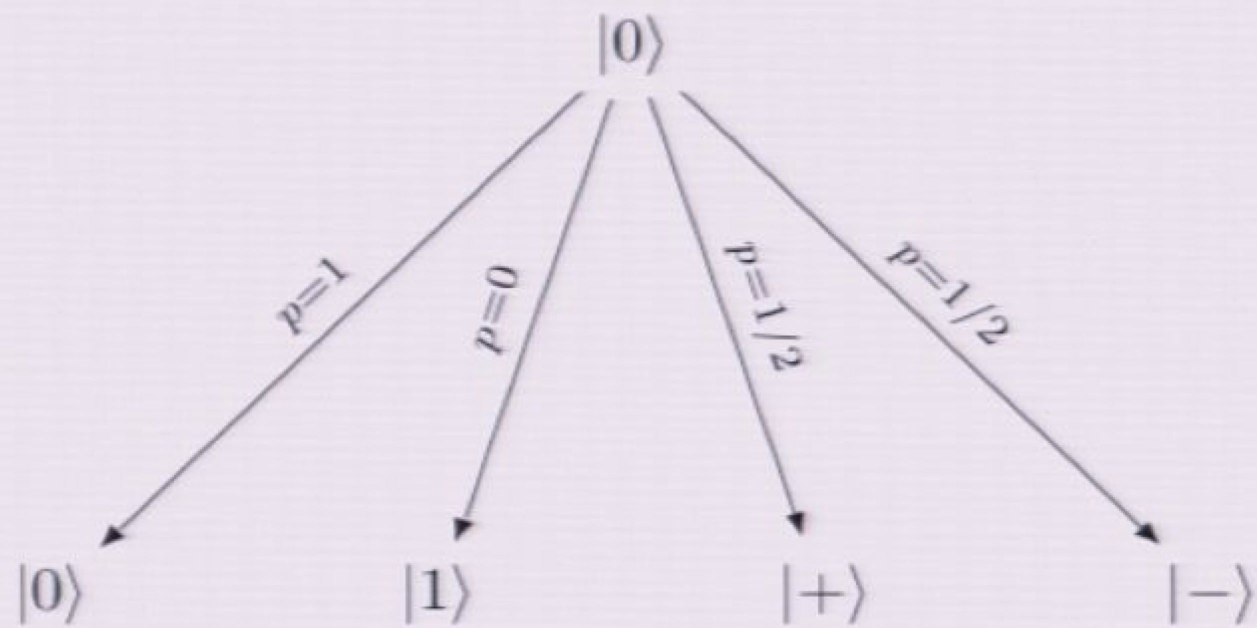

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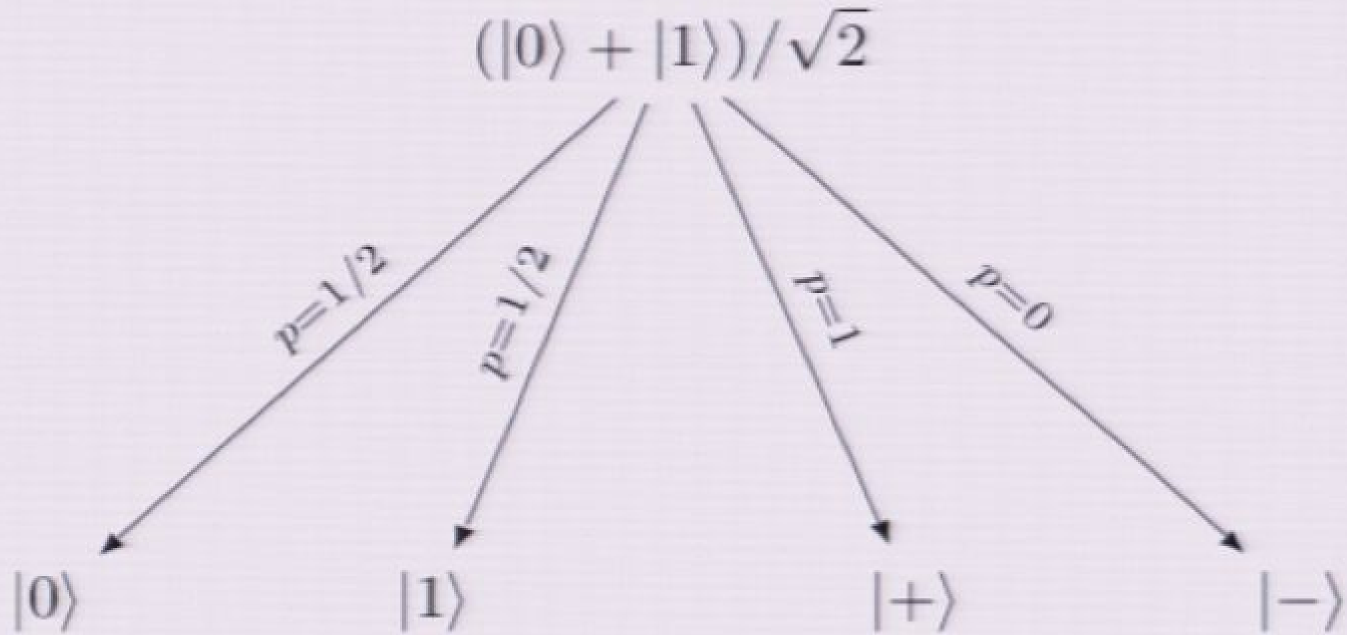


No-Cloning and No-Deleting

Theorem: There are no unitary operations D such that

$$D : |\psi\rangle \mapsto |\psi\rangle \otimes |\psi\rangle$$
$$D : |\phi\rangle \mapsto |\phi\rangle \otimes |\phi\rangle$$

unless $|\psi\rangle$ and $|\phi\rangle$ are orthogonal [Wooters & Zurek 1982]

Theorem: There are no unitary operations E such that

$$E : |\psi\rangle \mapsto |0\rangle$$
$$E : |\phi\rangle \mapsto |0\rangle$$

unless $|\psi\rangle$ and $|\phi\rangle$ are orthogonal [Pati & Braunstein 2000]

No-Cloning and No-Deleting, abstractly

Theorem: if a †-compact category has natural transformations

$$\delta : - \Rightarrow - \otimes -$$
$$\epsilon : - \Rightarrow I$$

then the category collapses [Abramsky 2007]

(Translation: in our abstract setting there are no universal cloning or deleting operations, just like in quantum mechanics)

“Classical” Quantum States

When can a quantum state be treated as if classical?

- no-go theorems allow copying and deleting of *orthogonal* states;

In other words:

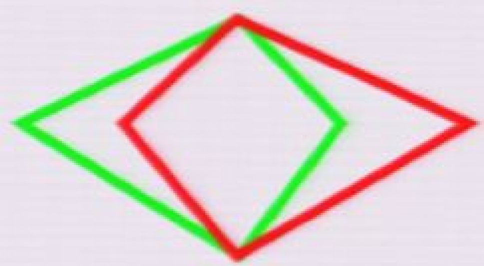
- A quantum state may be copied and deleted if it is an eigenstate of some *known observable.*

We’ll use this property to formalise *observables* in terms of *copying* and *deleting* operations.

Classical Properties

In general, quantum observables are incompatible - not defined at the same time : position and momentum; X and Z spin; etc.

Traditional quantum logic constructs a property lattice for each set of compatible observables; the incompatible properties are simply *incomparable* in the lattice.



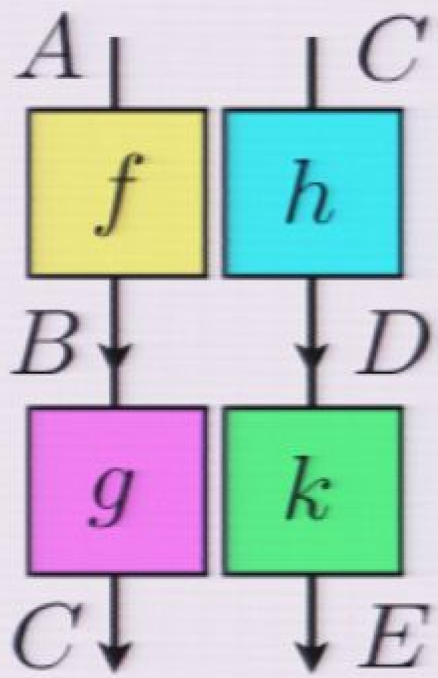
However if we want to compute with quantum mechanics we need to know how these observables relate to each other, i.e. how they *interfere*.

Our Approach

We aim to extract the *positive content* from the incompatibility of quantum observables:

- basis of monoidal categories : no-cloning, no-deleting;
- observable structures : axiomatised in terms of a copying operation
- Phase groups : constructed from the observable structures
- incompatible observables : how do classical operations which act on complementary states interact?

Monoidal Categories

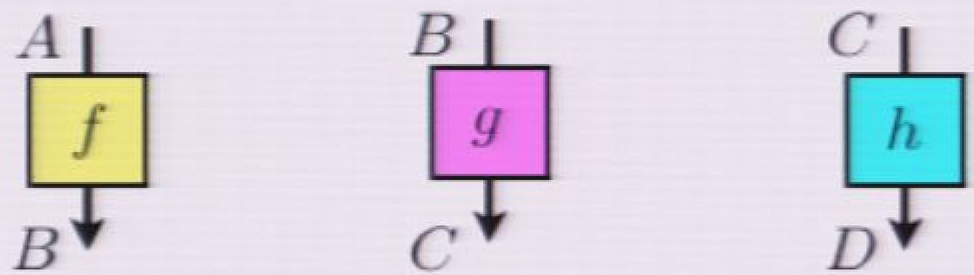


A theory of interacting processes

Categories

A *category* consists of *objects* A, B, C, etc, and *arrows* between them:

$$f : A \to B \qquad g : B \to C \qquad h : C \to D$$



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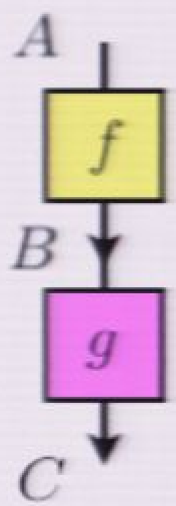
$$g \circ f : A \to C$$



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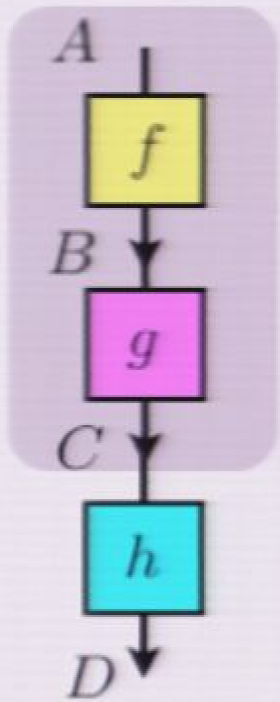

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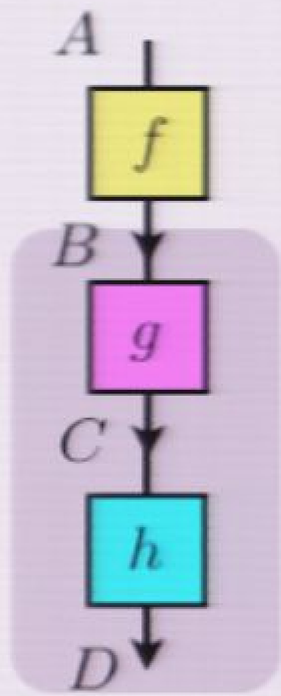

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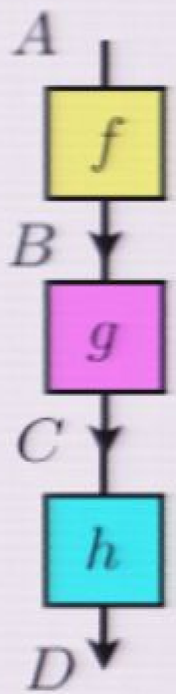
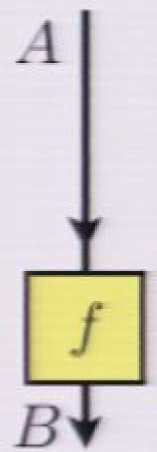


Categories

$$\mathrm{id}_A : A \to A$$

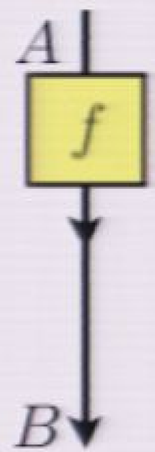
Categories

$$f \circ \mathrm{id}_A : A \to B$$



Categories

$$\mathrm{id}_B \circ f : A \to B$$



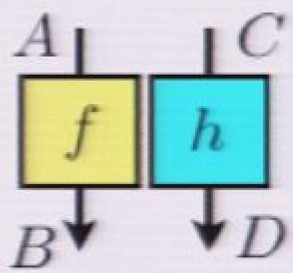
Categories

$$f : A \to B$$

Monoidal Categories

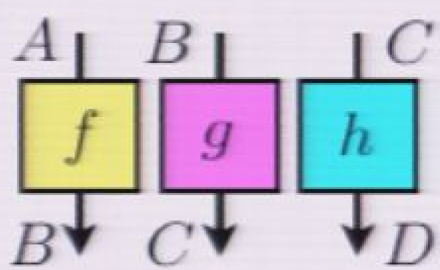
A *strict monoidal category* is a category equipped with a tensor product on both objects and arrows:

$$f \otimes h : A \otimes C \to B \otimes D$$



The tensor is associative:

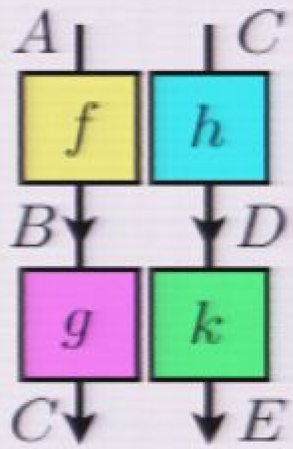
$$(f \otimes g) \otimes h = f \otimes (g \otimes h)$$



Monoidal Categories

The tensor product is *bifunctorial*, meaning that it preserves composition:

$$(g \circ f) \otimes (k \circ h) = (g \otimes k) \circ (f \otimes h)$$



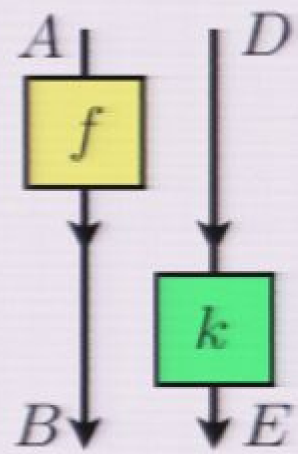
and identities:

$$\mathrm{id}_{A \otimes B} = \mathrm{id}_A \otimes \mathrm{id}_B$$

Monoidal Categories

In particular we have the following:

$$(\mathrm{id}_B \otimes k) \circ (f \otimes \mathrm{id}_D) = (f \otimes \mathrm{id}_E) \circ (\mathrm{id}_A \otimes k) = f \otimes k$$



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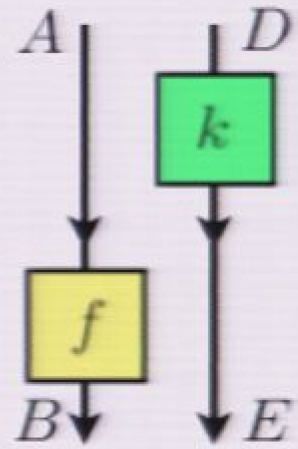

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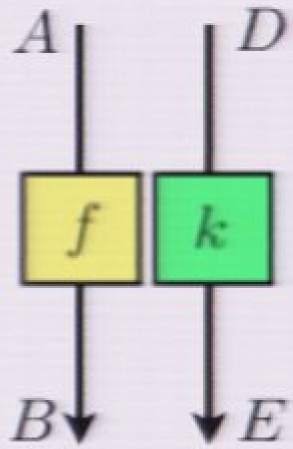
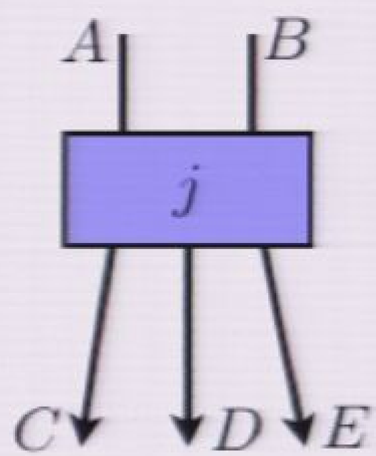


Monoidal Categories

Of course, it's quite possible to have arrows between tensors of objects which are not tensors themselves, e.g.

$$j : A \otimes B \rightarrow C \otimes D \otimes E$$

could be drawn like:



Monoidal Categories

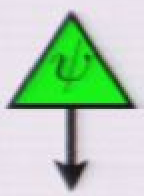
Monoidal categories have a special *unit* object called I which is a left and right identity for the tensor:

$$I \otimes A = A = A \otimes I$$

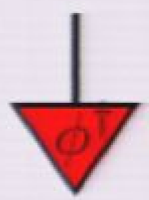
$$\mathrm{id}_I \otimes f = f = f \otimes \mathrm{id}_I$$

No lines are drawn for I in the graphical notation:

$\psi : I \to A$



$\phi^\dagger : A \to I$



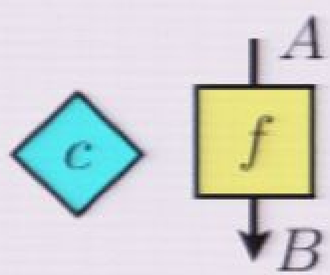
$\phi^\dagger \circ \psi : I \to I$

Monoidal Categories

The arrows $c : I \to I$ are called *scalars* and they enjoy some special properties:

$$\begin{aligned} c_1 \otimes c_2 &= c_1 \circ c_2 \\ c_1 \otimes c_2 &= c_2 \otimes c_1 \end{aligned}$$

Any arrow $f : A \to B$ can be multiplied by c using the tensor:



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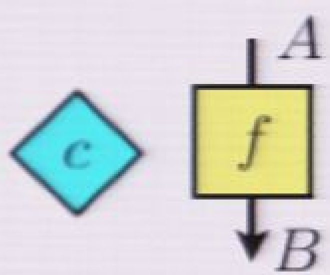
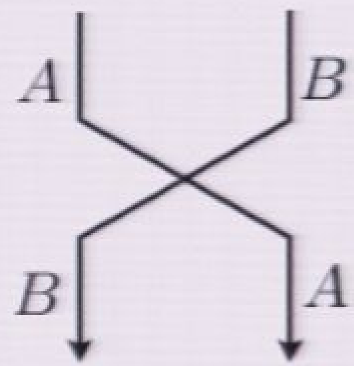


It doesn't matter where c is drawn.

Symmetric Monoidal Categories

$$\sigma_{A,B} : A \otimes B \to B \otimes A$$



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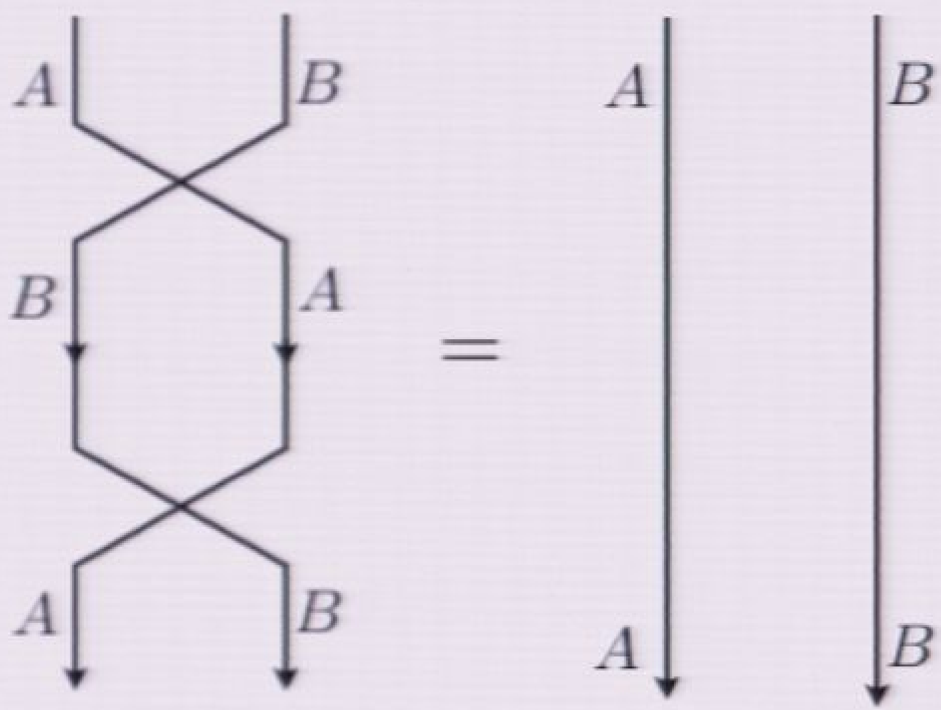

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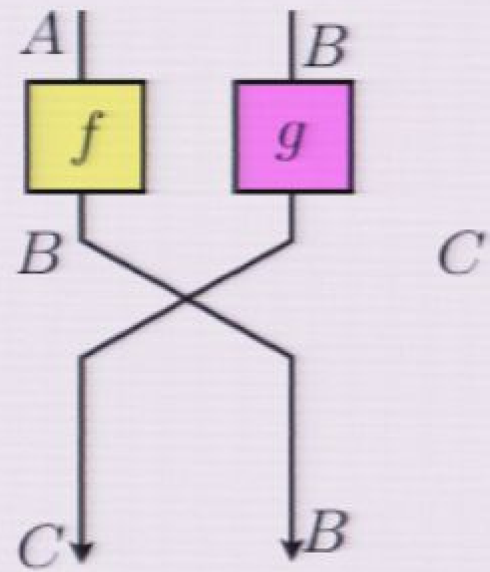

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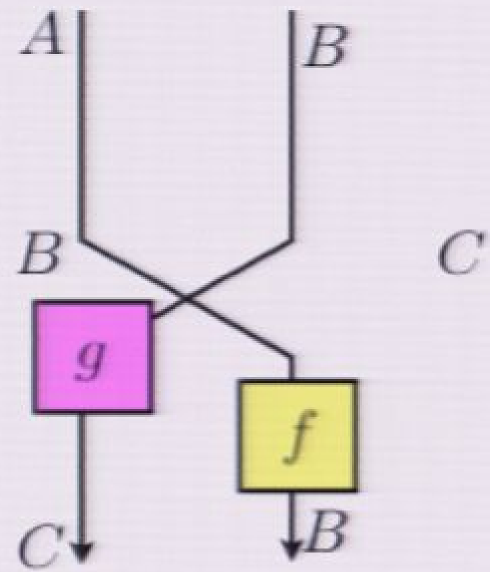

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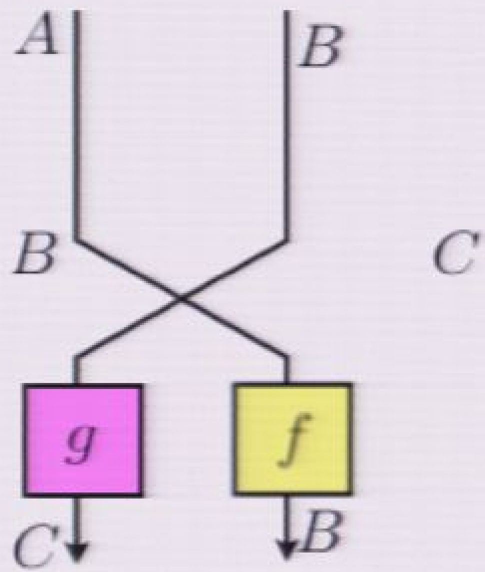


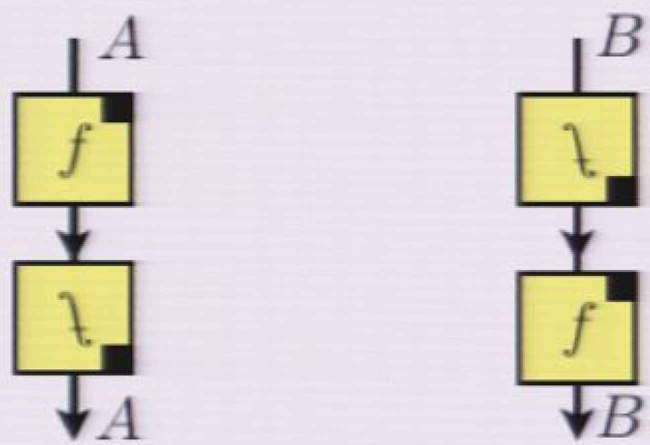
†-Monoidal Categories

A monoidal category is called *†-monoidal* if it is equipped with an involutive functor, $(\cdot)^\dagger$ which reverses the arrows while leaving the objects unchanged, which preserves the tensor structure.

$$f : A \to B \qquad\qquad f^\dagger : B \to A$$

†-Monoidal Categories

An arrow $f : A \to B$ is called *unitary* when:



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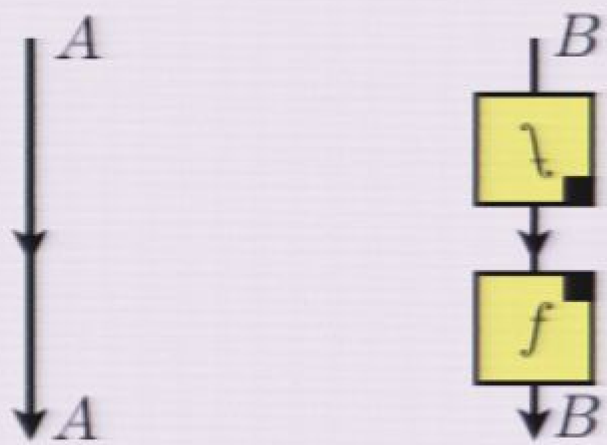

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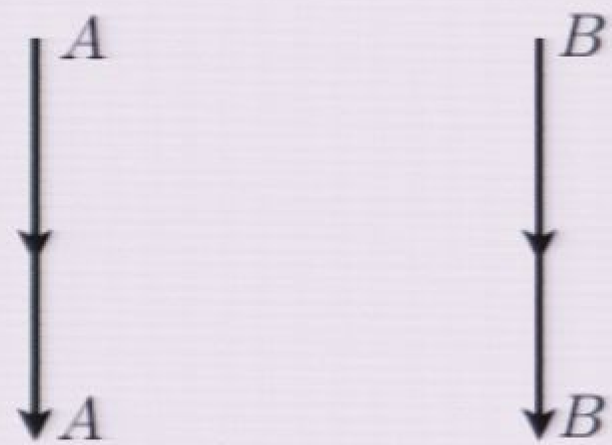


The Category **FDHilb**

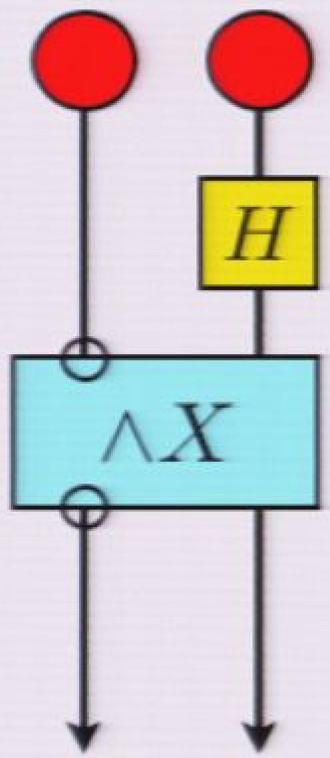
FDHilb is the category of finite dimensional complex Hilbert spaces. It is †-monoidal with the following structure.

- Objects: finite dimensional Hilbert spaces, A, B, C, etc
- Arrows: all linear maps
- Tensor: usual (Kronecker) tensor product; $I = \mathbb{C}$
- $f^\dagger$ is the usual adjoint (conjugate transpose)

A linear map $\psi : I \to A$ picks out exactly one vector. It is a ket and $\psi^\dagger : A \to I$ is the corresponding bra.

Hence $\psi^\dagger \circ \phi : I \to I$ is the inner product $\langle \psi \mid \phi \rangle$.

Example: Preparing a Bell state



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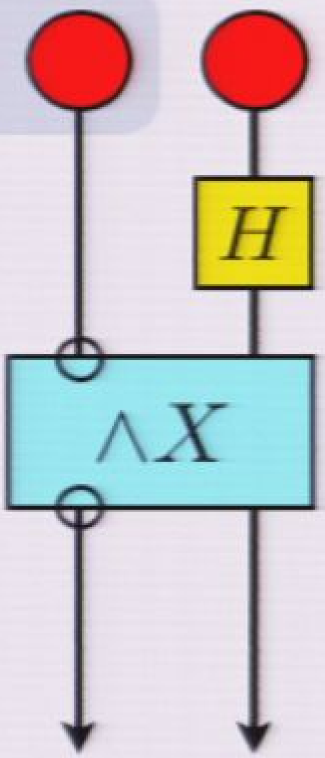
$|0\rangle : I \to Q$



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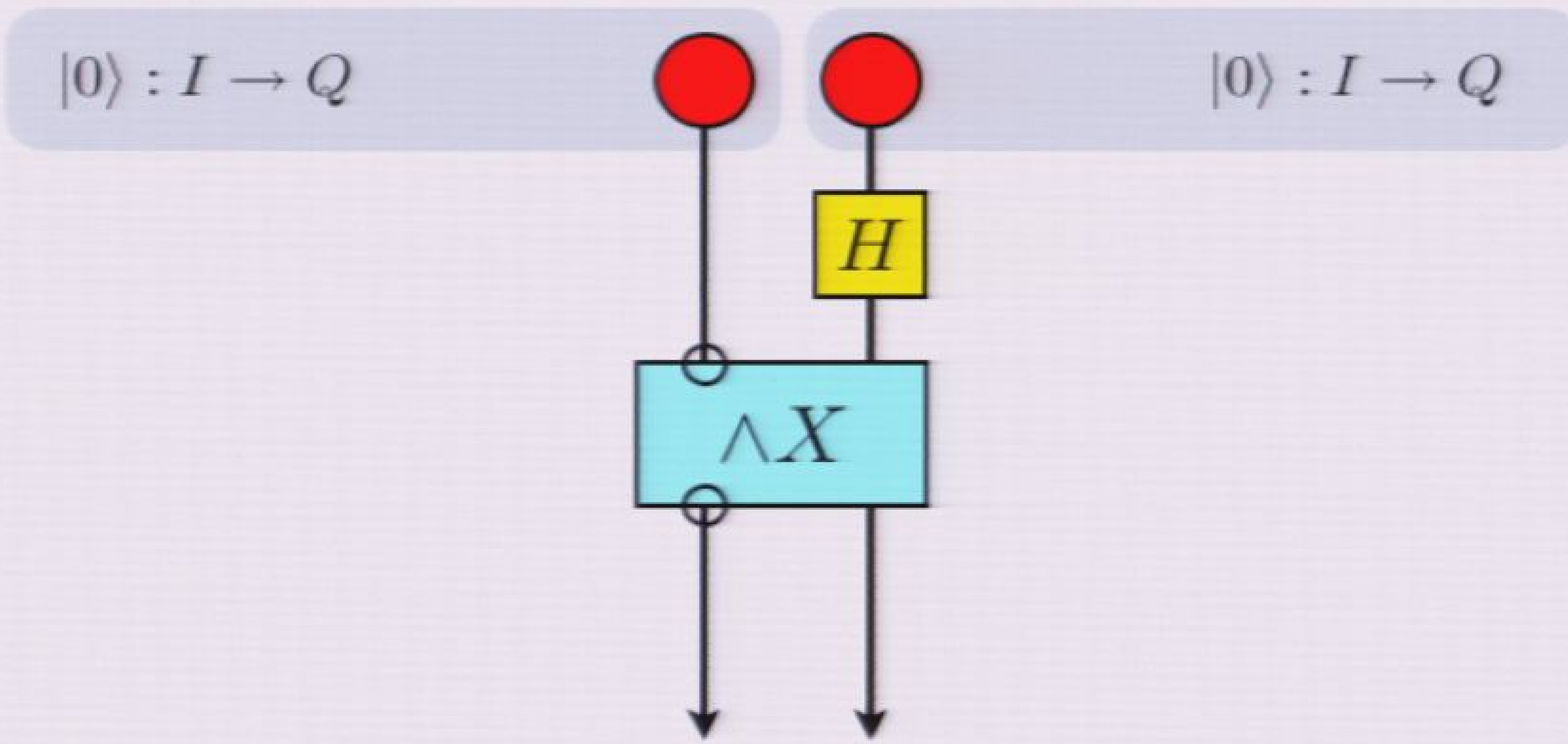

$$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$$

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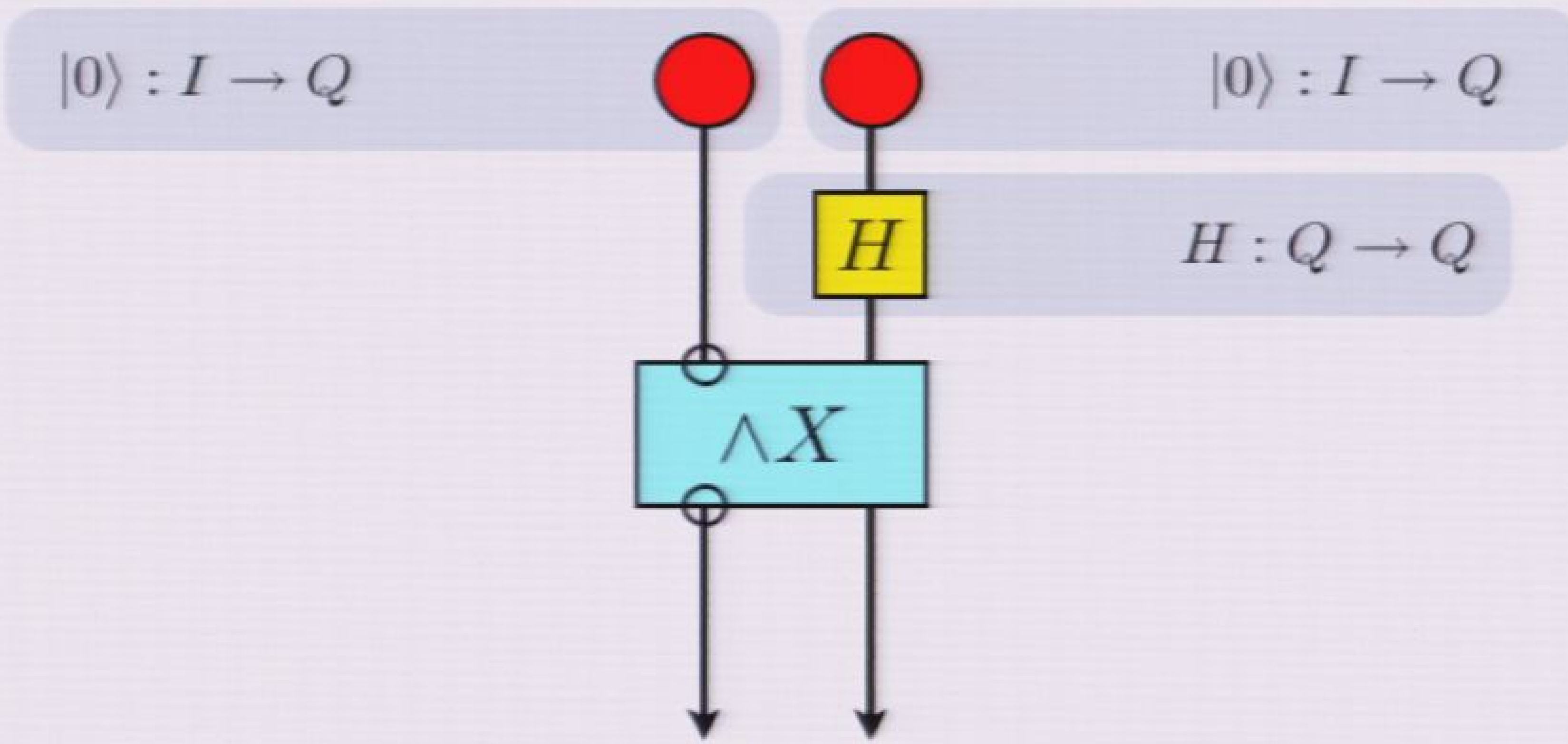

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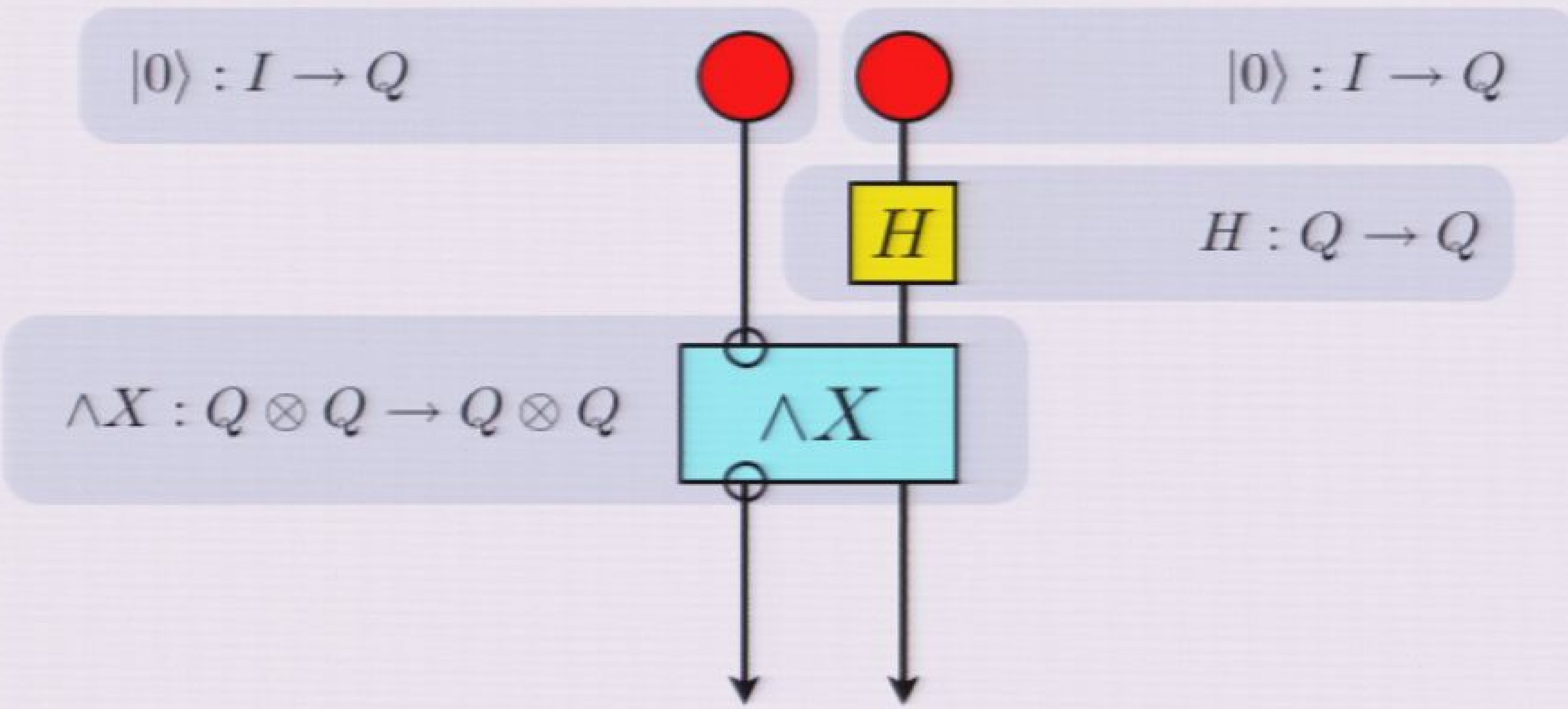

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The Category **FRel**

FDHilb is the category of finite relations. It is †-monoidal with the following structure.

- Objects: finite sets, $X, Y, Z,$ etc
- Arrows: Relations $R \subseteq X \times Y$
- Tensor: Cartesian product; $X \otimes Y := X \times Y, \;\; I = \{*\}$
- $f^\dagger$ is relational converse: $(x, y) \in f \Leftrightarrow (y, x) \in f^\dagger$

A relation $R \subseteq \{*\} \times X$ is simply a *subset* of X; its converse is also a subset.

Hence $S^\dagger \circ R : I \to I$ is non-empty iff $R \cap S \neq \emptyset$.

Compact Closure

$d : I \rightarrow A^* \otimes A$ $\qquad$ $e : A \otimes A^* \rightarrow I$

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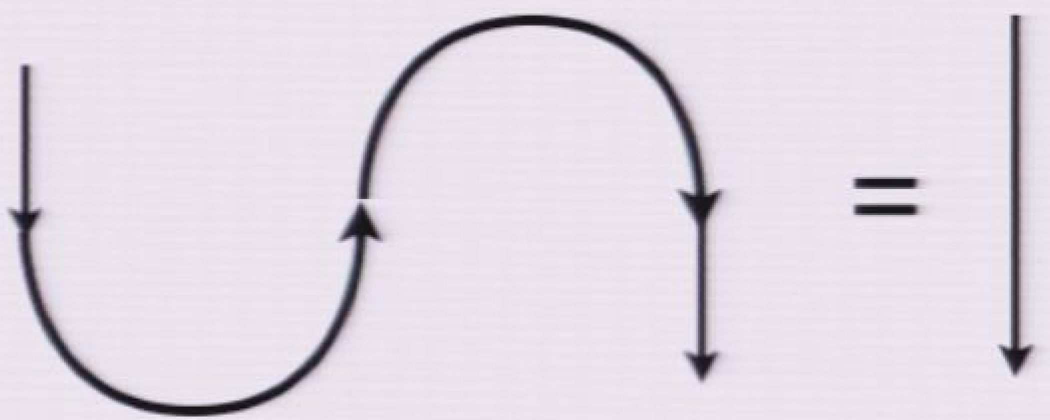

Compact Structure of **FDHilb**

In FDHilb the compact structure is given by the maps:

whenever $\{a_i\}_i$ is a basis for A and $\{\overline{a_i}\}_i$ is the corresponding basis for the dual space A^*

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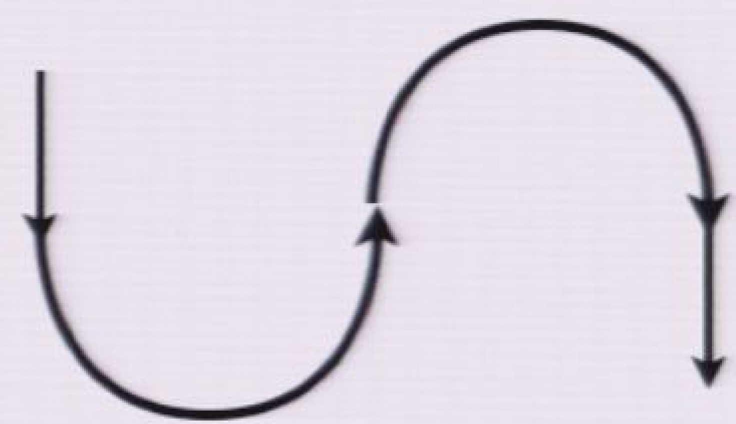

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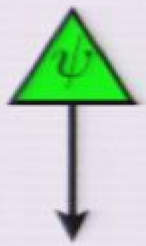
Example: Quantum Teleportation

Aleks



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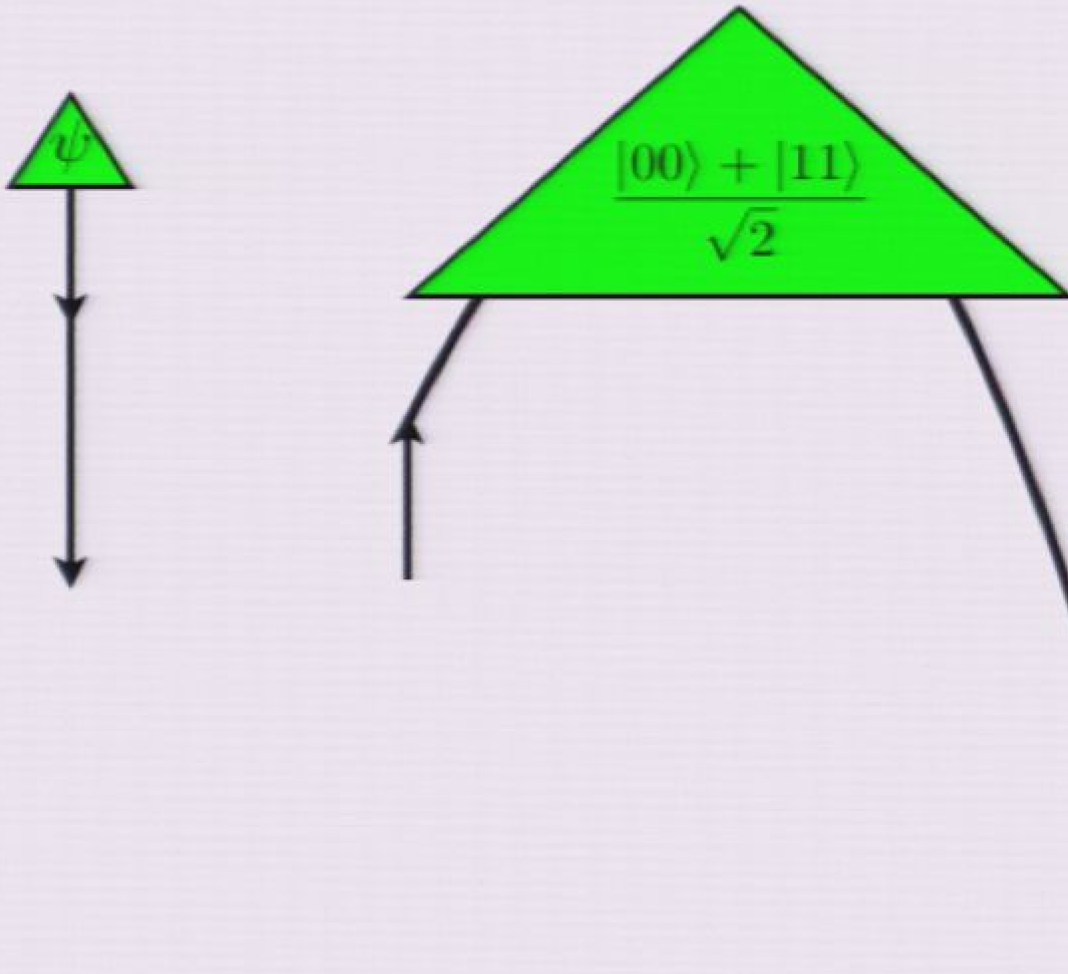

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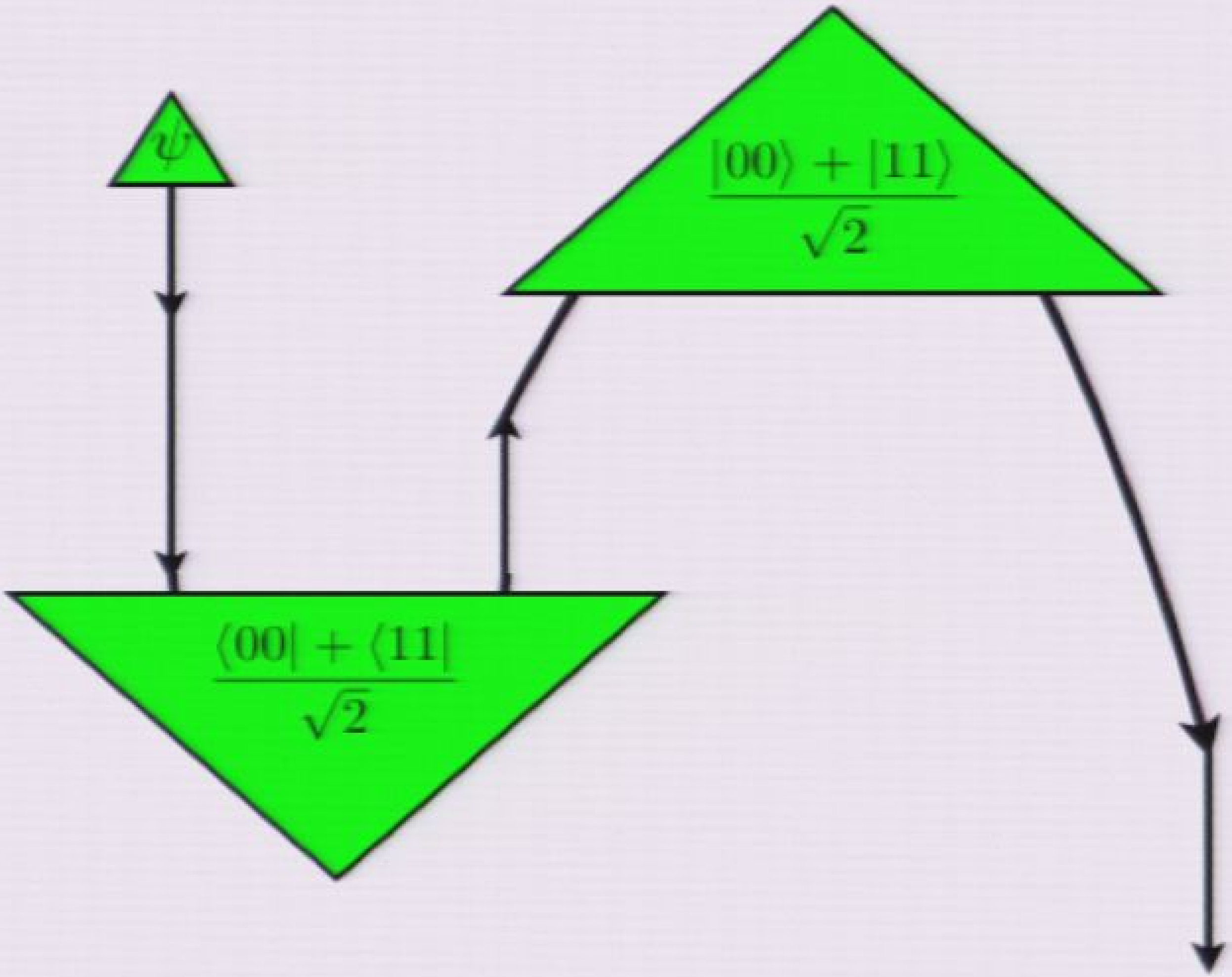

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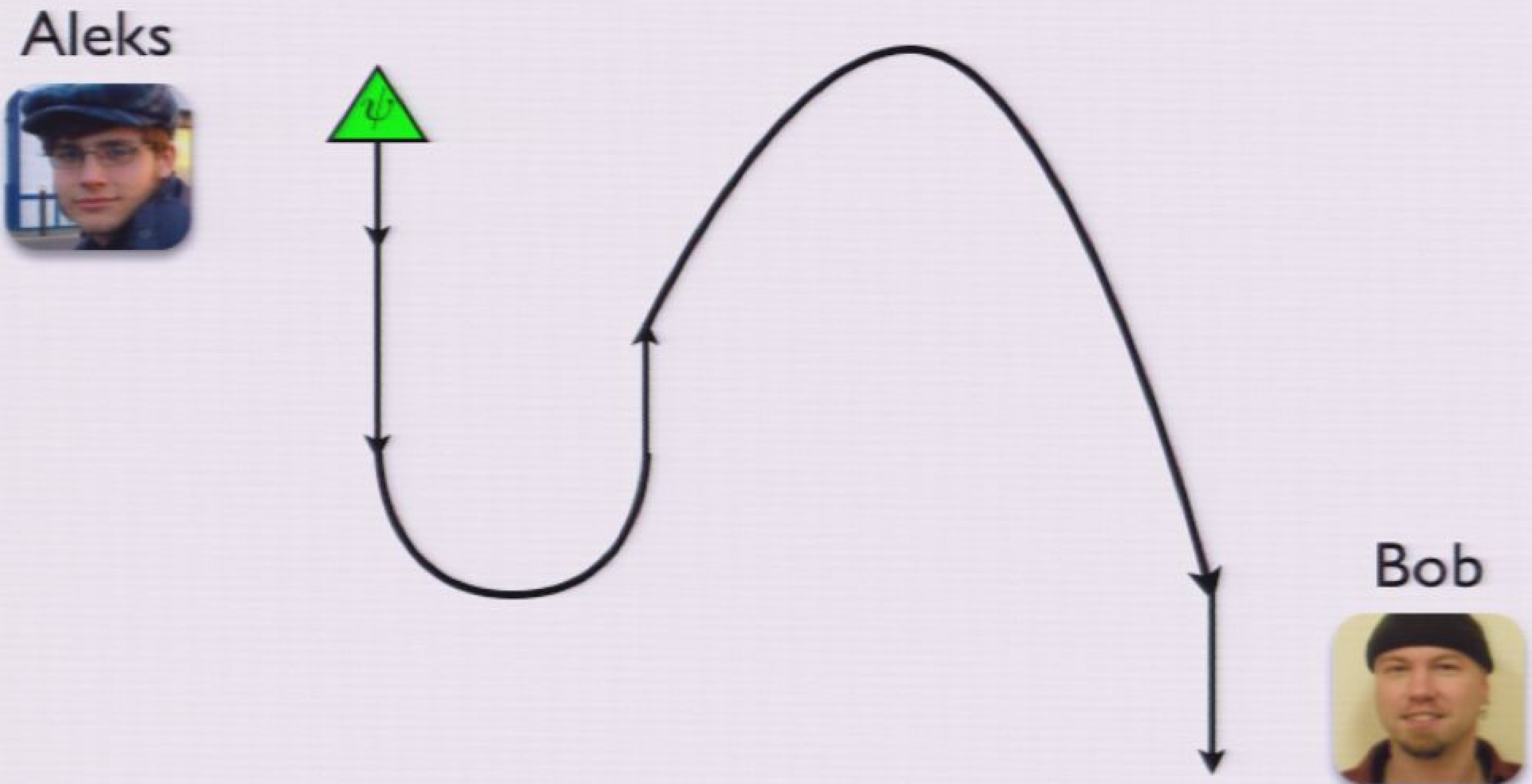

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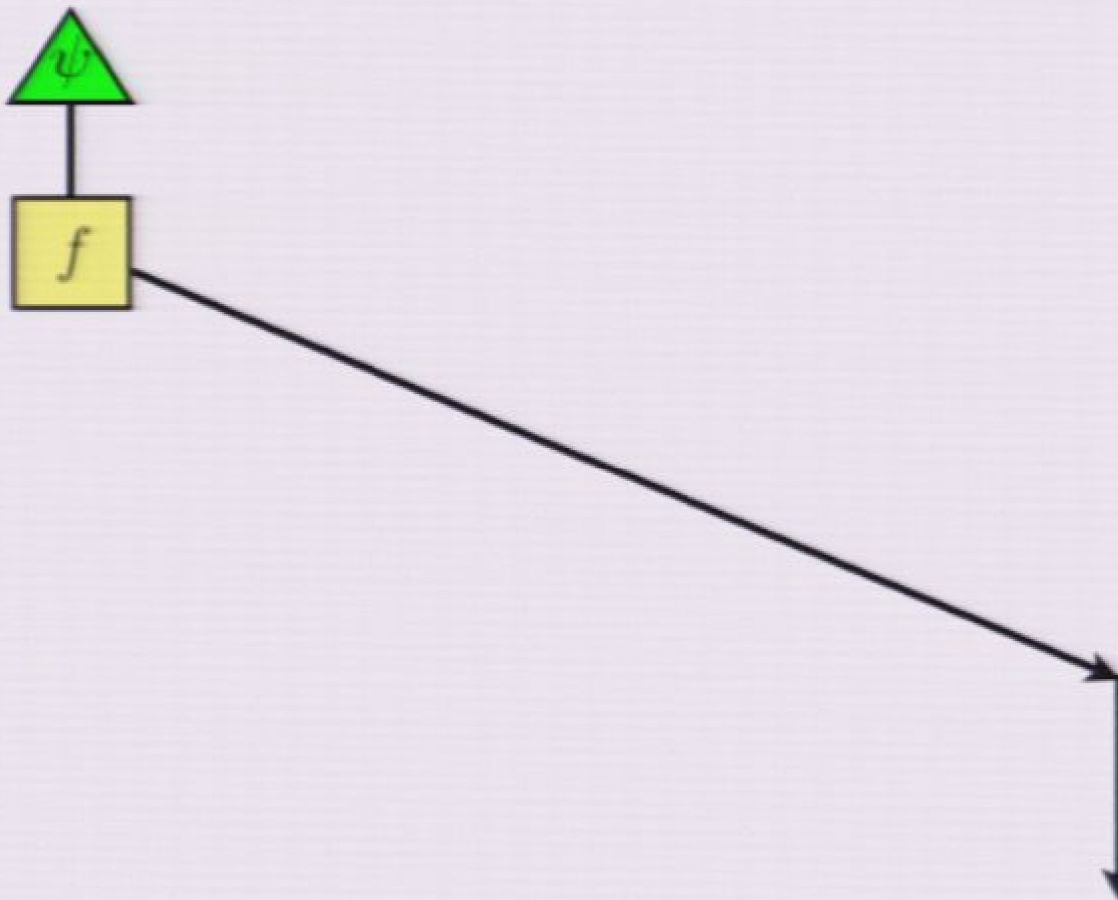

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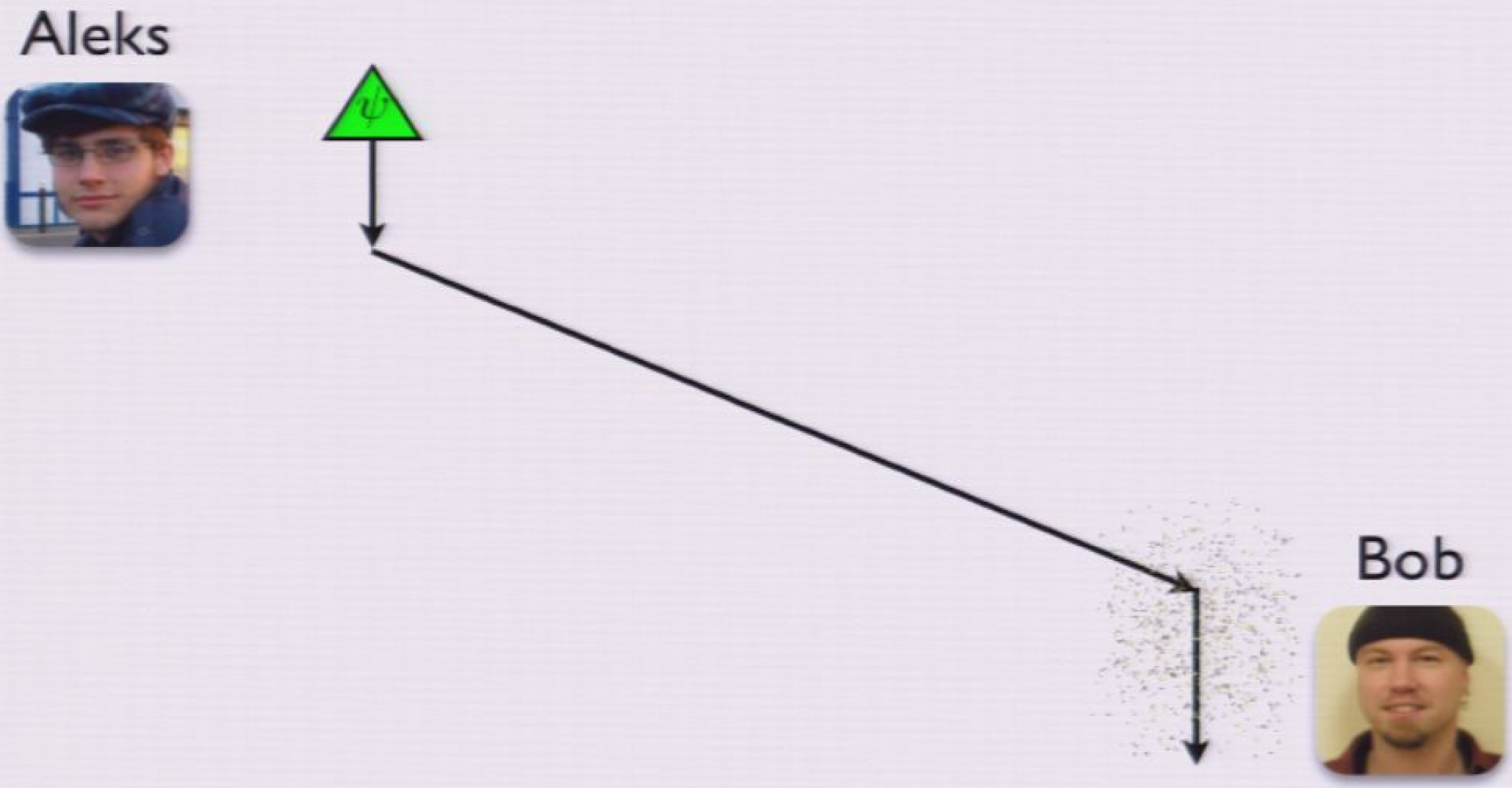

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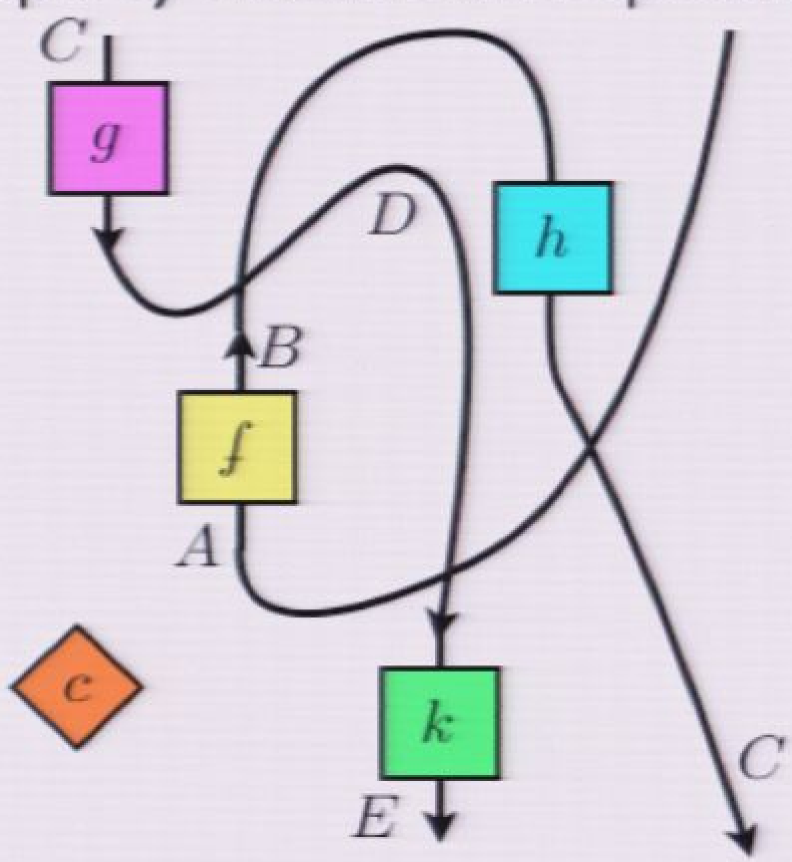
Thm: one diagram can be deformed to another if and only if their denotations are equal by the structural equations of the category.



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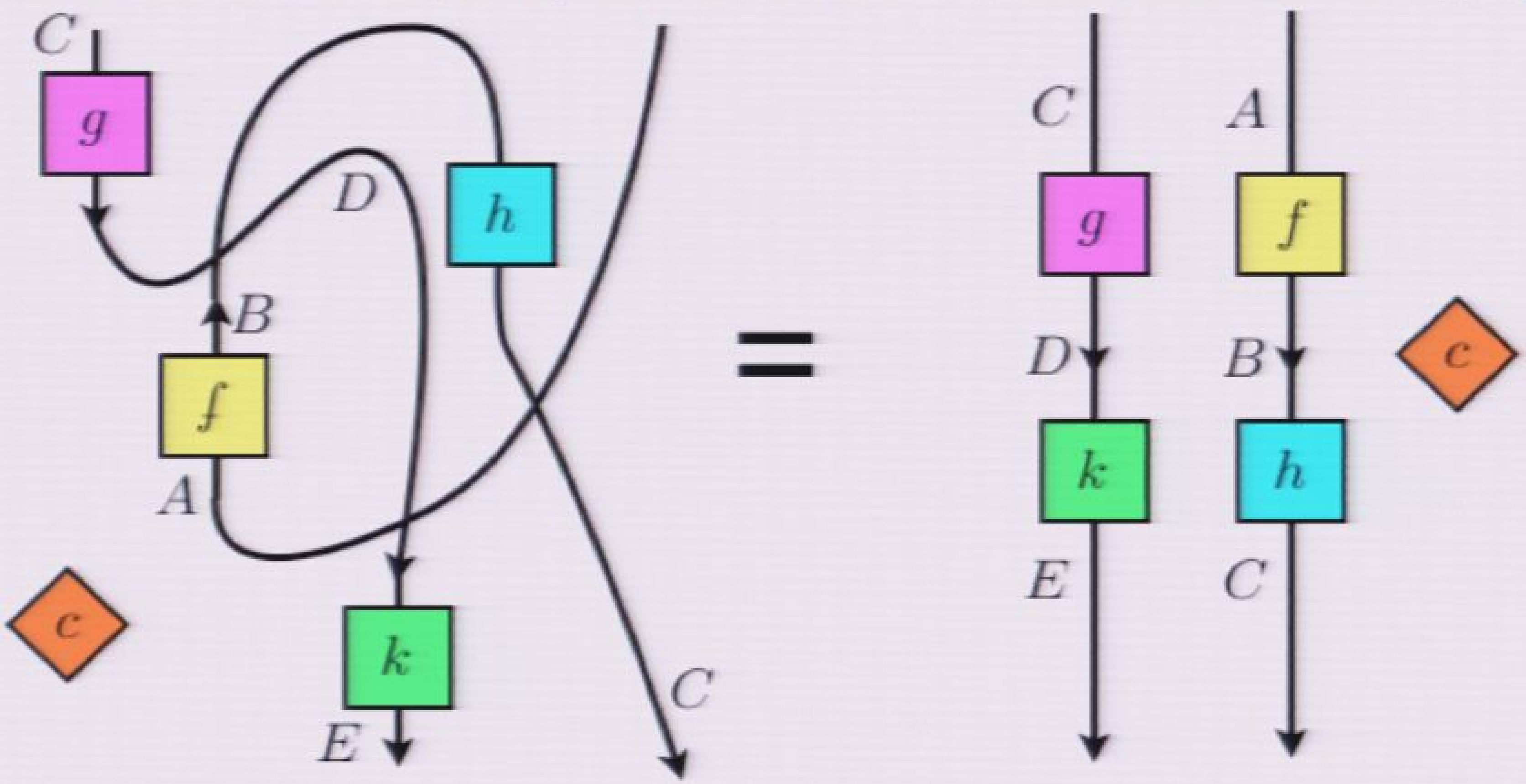
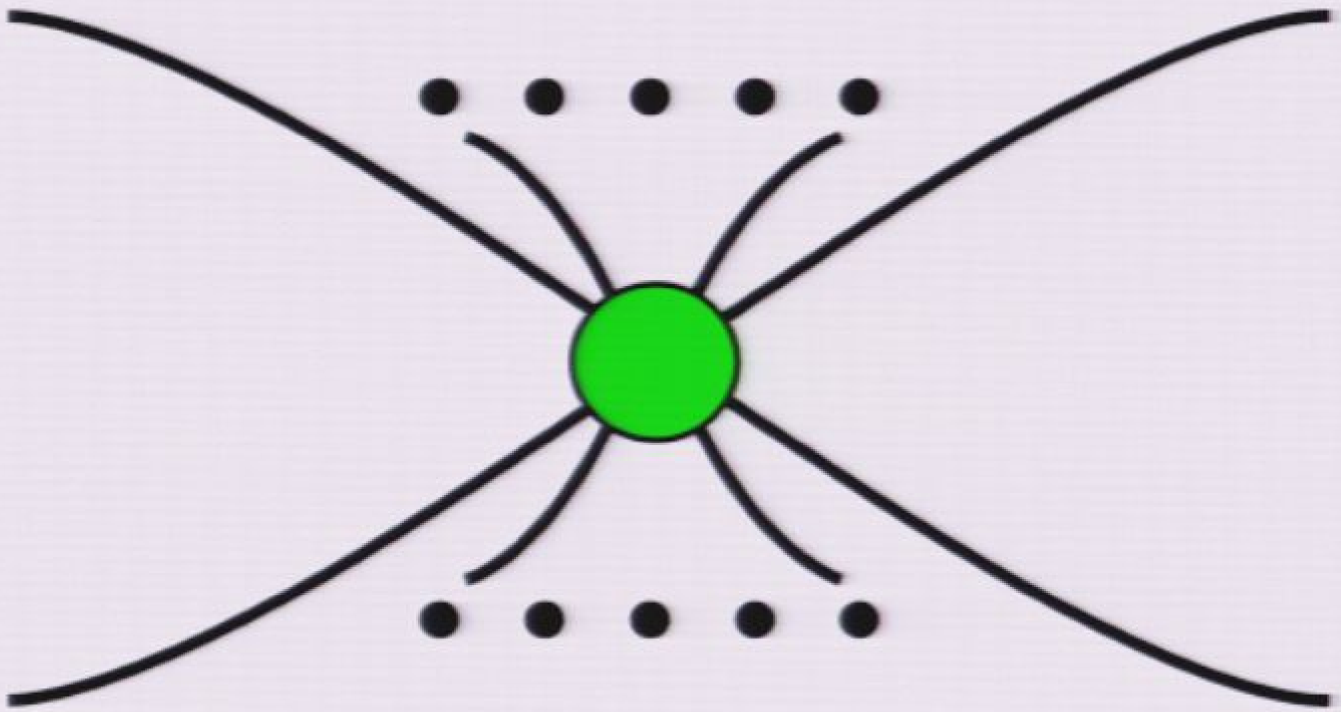


Observable Structures



Copying, deleting, and all that

Observable Structures

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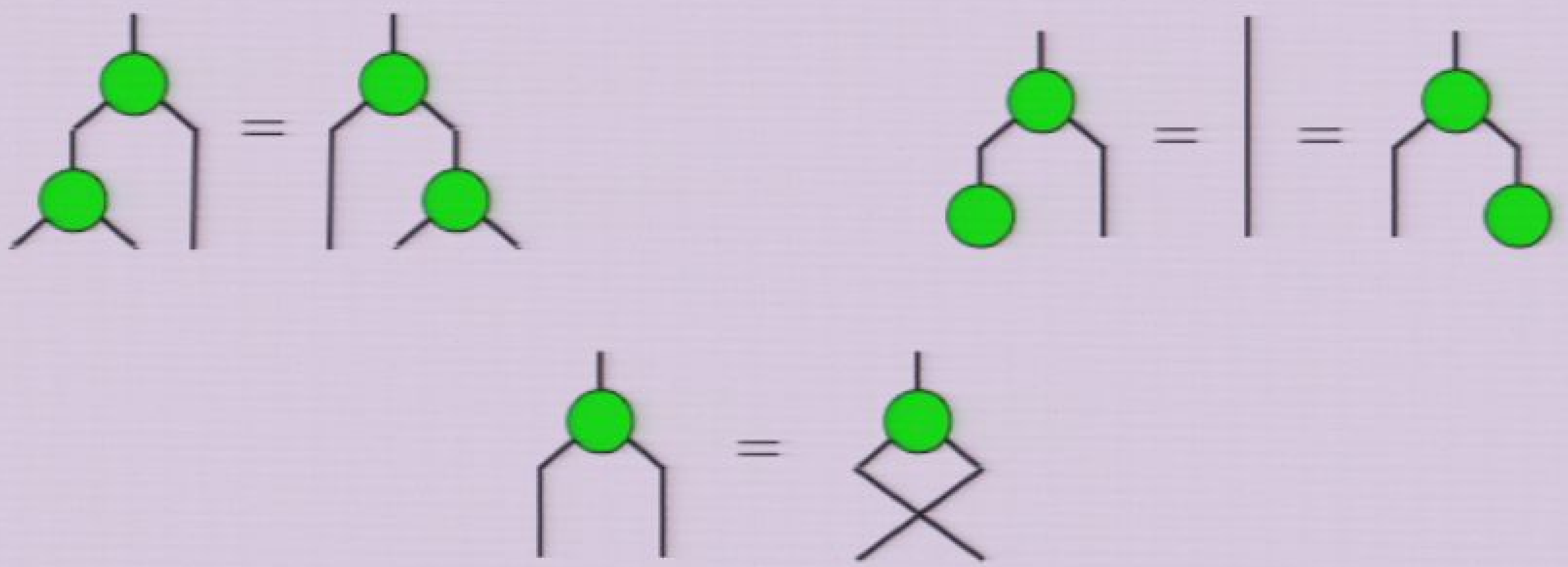

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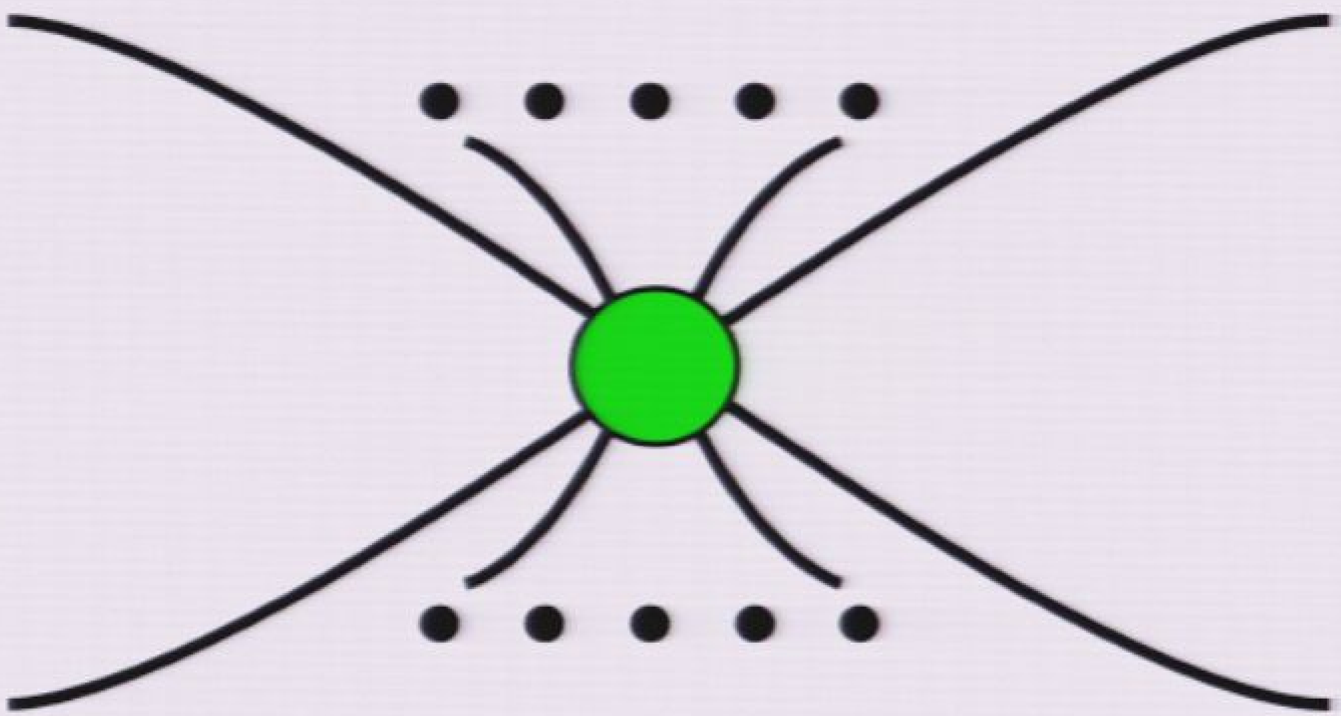

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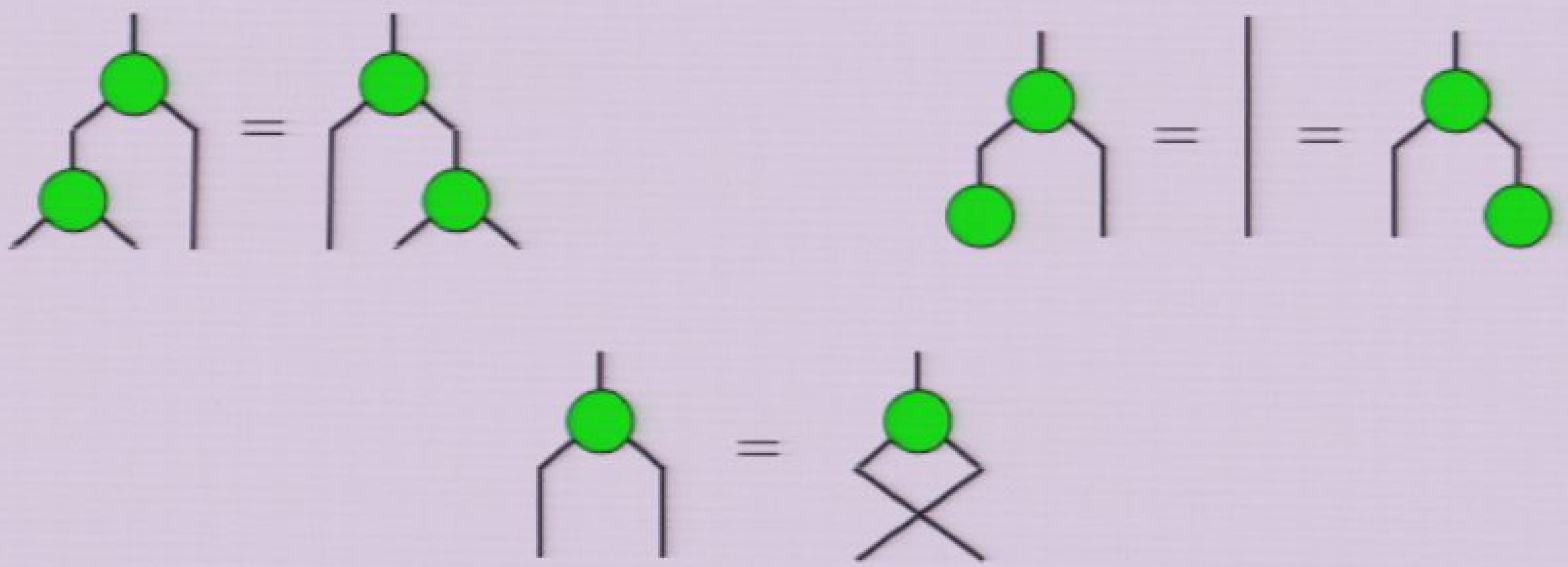

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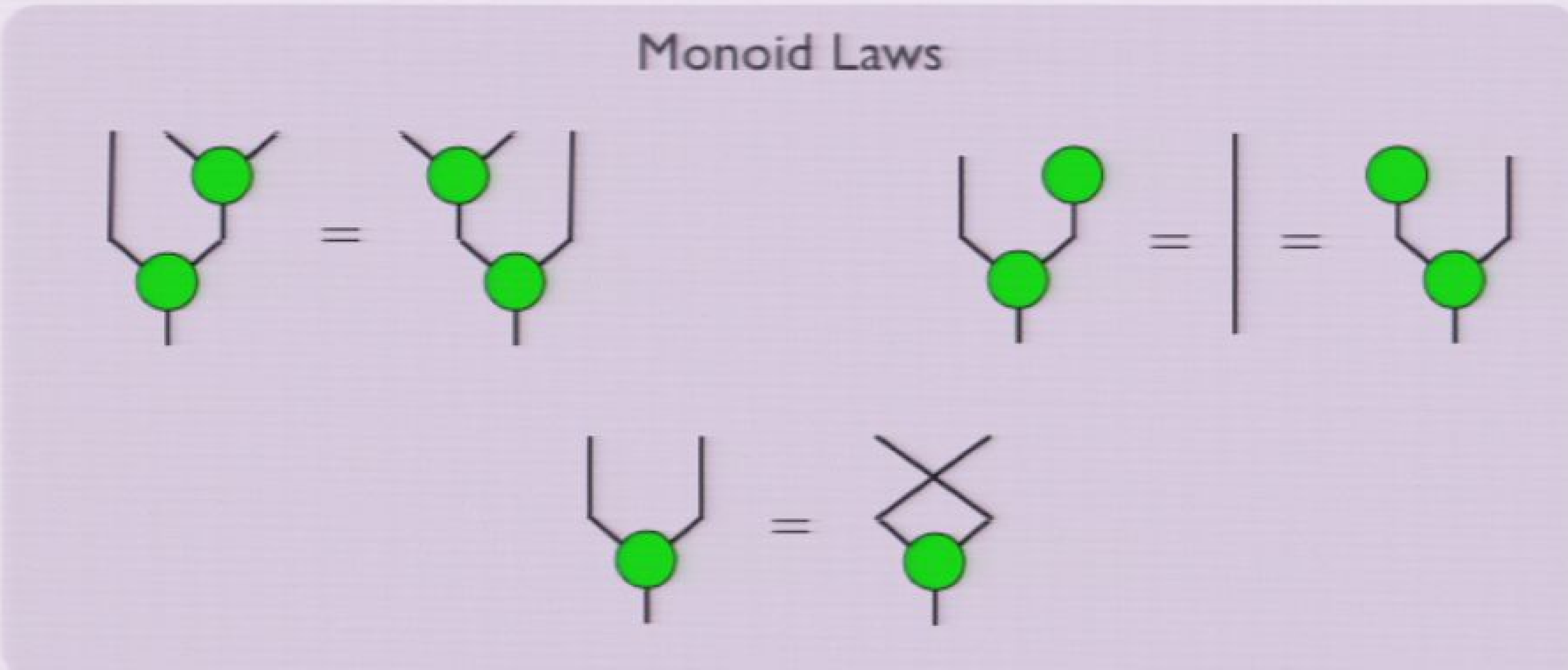

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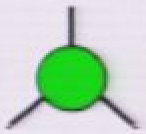
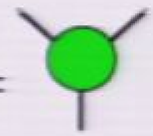
$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

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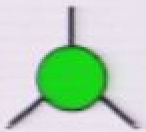
$\delta =$ 
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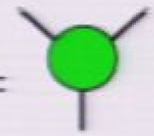
$\delta^\dagger =$ 
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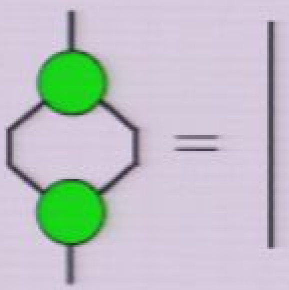
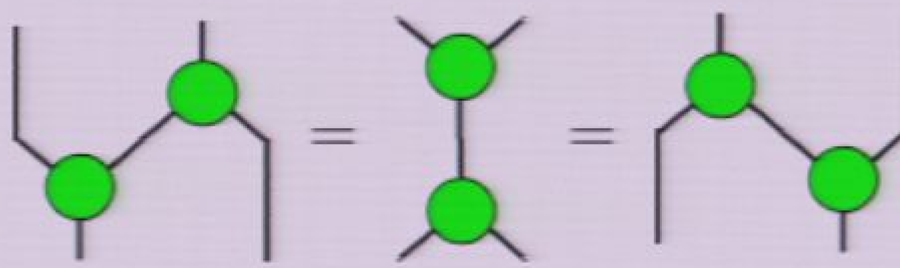


Frobenius Law



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Given any finite dimensional Hilbert space we can define an observable structure by

$$\delta : A \to A \otimes A :: a_i \mapsto a_i \otimes a_i$$
$$\epsilon : A \to I :: \sum_i a_i \mapsto 1$$

Example: $\delta : \begin{array}{l} |0\rangle \mapsto |00\rangle \\ |1\rangle \mapsto |11\rangle \end{array}$ $\qquad \epsilon : |0\rangle + |1\rangle \mapsto 1$

define a observable structure over qubits; the standard basis is copied and erased. Note however that:

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showing that not every state can be cloned.

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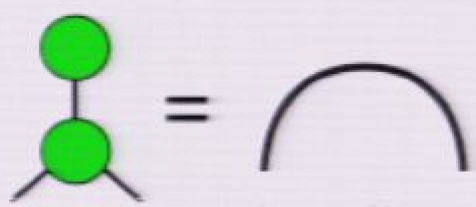

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<u>Theorem</u>: in **FDHilb**, observable structures are in bijective correspondence to bases. [Coecke, Pavlovic, Vicary]

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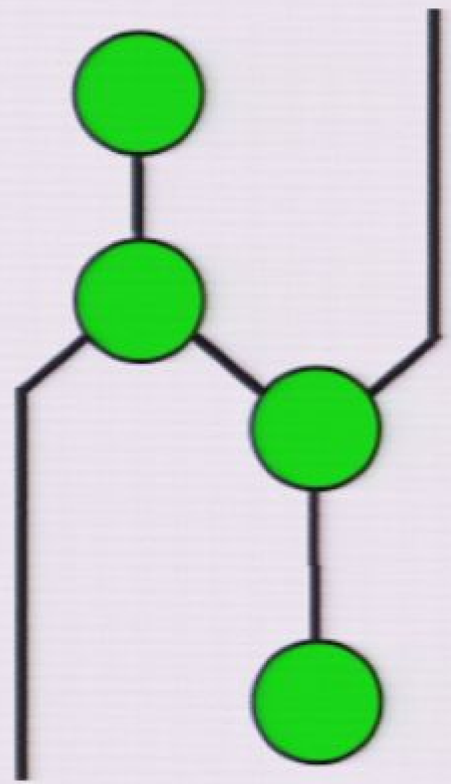
Each (well behaved) observable defines a basis, hence each observable defines an observable structure!

Classical Structure begets Compact Structure



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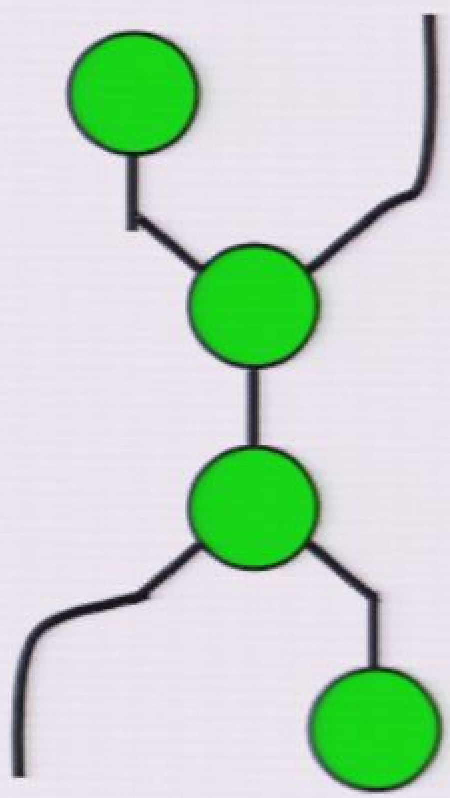

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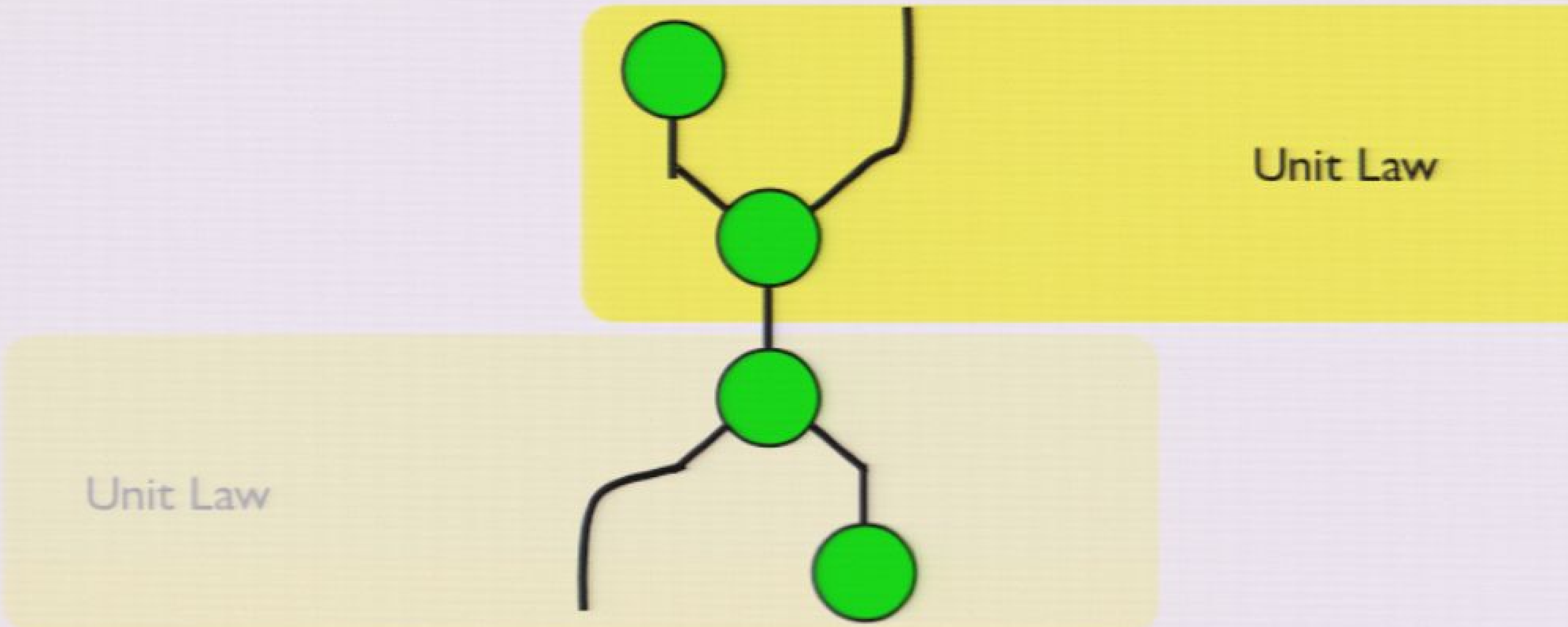

Classical Structure begets Compact Structure

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Each classical structure induces a self-dual compact structure.

Self duality is addressed in Coecke, Paquette, & Perdrix 2008

Spider Theorem

Theorem: any maps constructed from δ and ε , and their adjoints, whose graph is connected, is determined uniquely by the number of inputs and outputs.

Coecke & Paquette 2006

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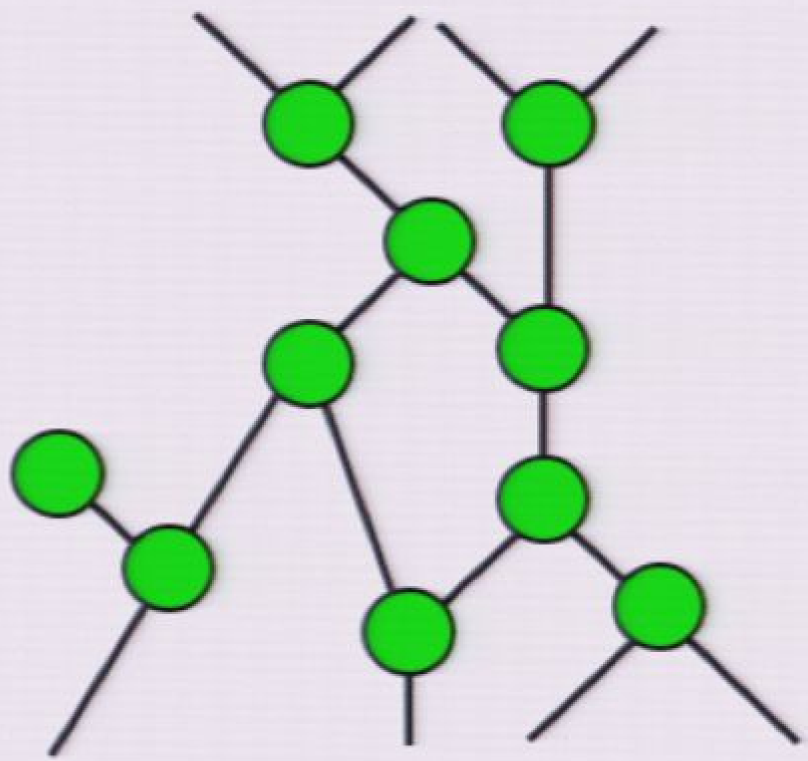

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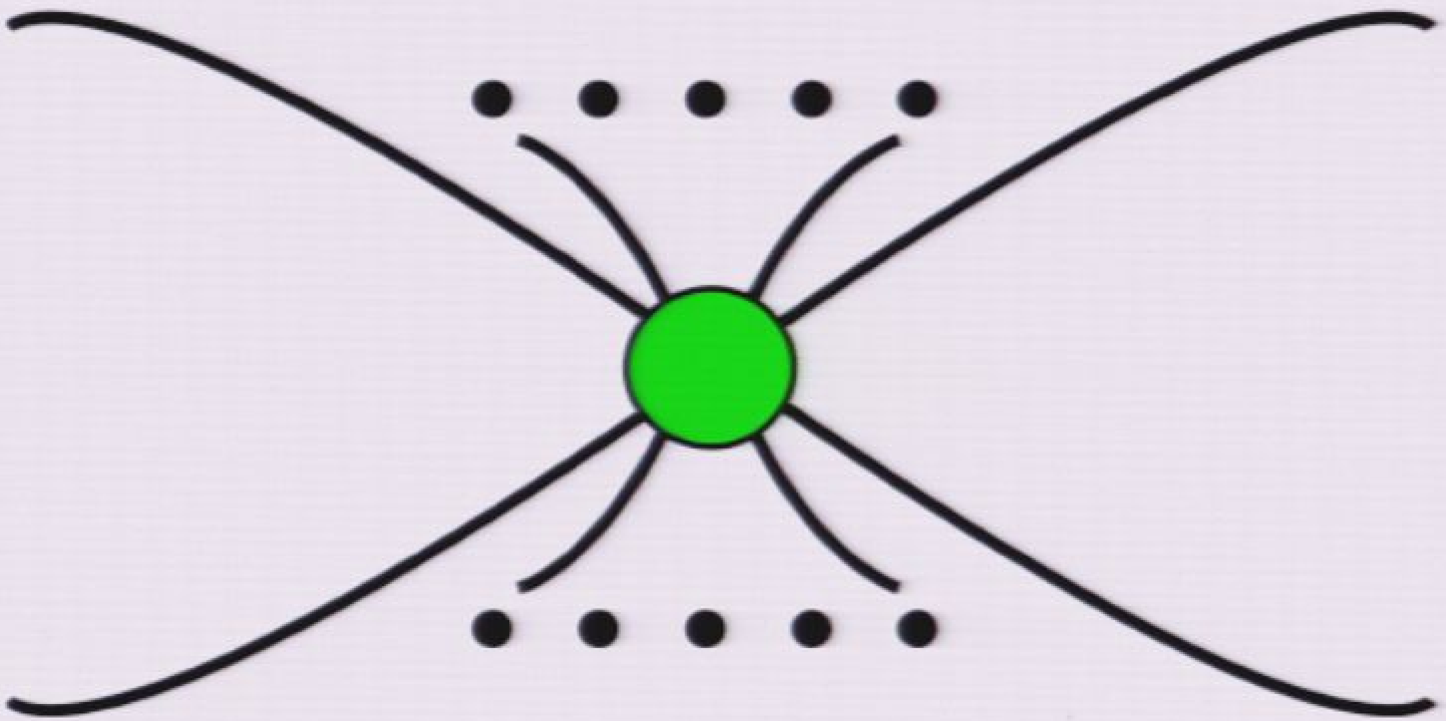

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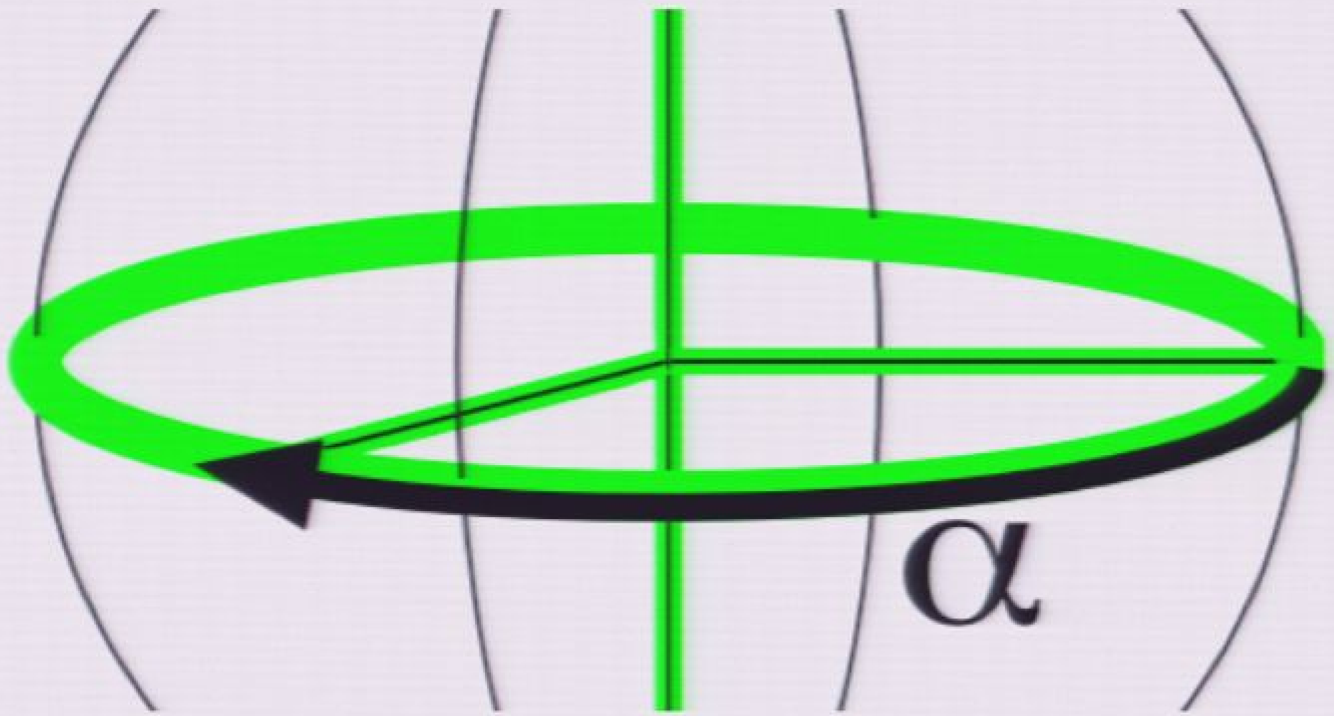
Phase Maps



Spinning around an observable

Using the monoid operation

Let $\psi, \phi : I \to A$ be points of A; we can combine them using the monoid operation $\delta^\dagger : A \otimes A \to A$

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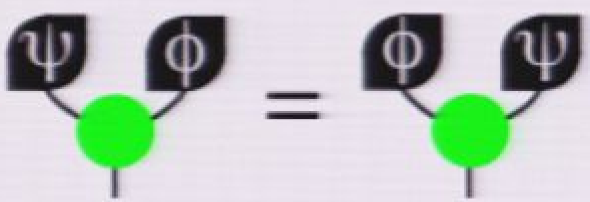

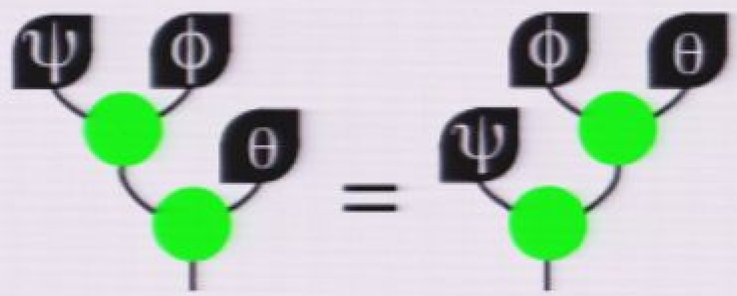

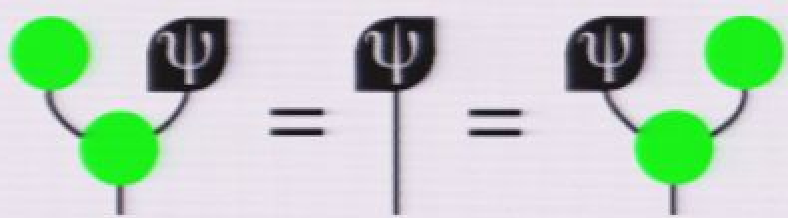

Example: qubits

$$\delta_Z^\dagger : \begin{array}{l} Q \to Q \otimes Q \\ |i\rangle \to |i\rangle \otimes |i\rangle \end{array} \qquad \begin{array}{l} |\phi\rangle : I \to Q \\ |\psi\rangle : I \to Q \end{array}$$

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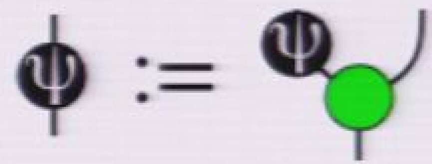
$$|\psi\rangle \odot |\phi\rangle := \delta_Z^\dagger \circ (|\psi\rangle \otimes |\phi\rangle) = \begin{pmatrix} \psi_1\phi_1 \\ \psi_2\phi_2 \end{pmatrix}$$

Using the monoid operation

Moreover, each point $\psi : I \to A$ can be lifted to an endomorphism $\Lambda(\psi) : A \to A$

$$\Lambda(\psi) := \delta^{\dagger} \circ (\psi \otimes \mathrm{id}_A)$$



This yields a homomorphism of monoids so we have:

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$$\delta_Z^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad |\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

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Generalised Spider Theorem

Theorem: any maps constructed from δ, ε, some points $\psi_i : I \to A$ and their adjoints, whose graph is connected, is determined by the number of inputs and outputs and the product $\bigodot_i \psi_i$.

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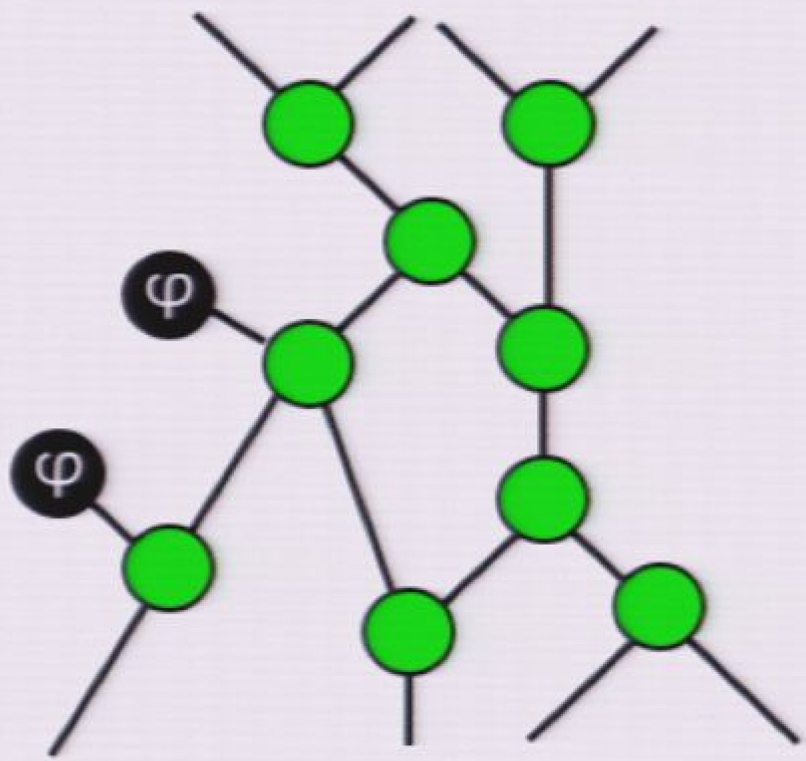

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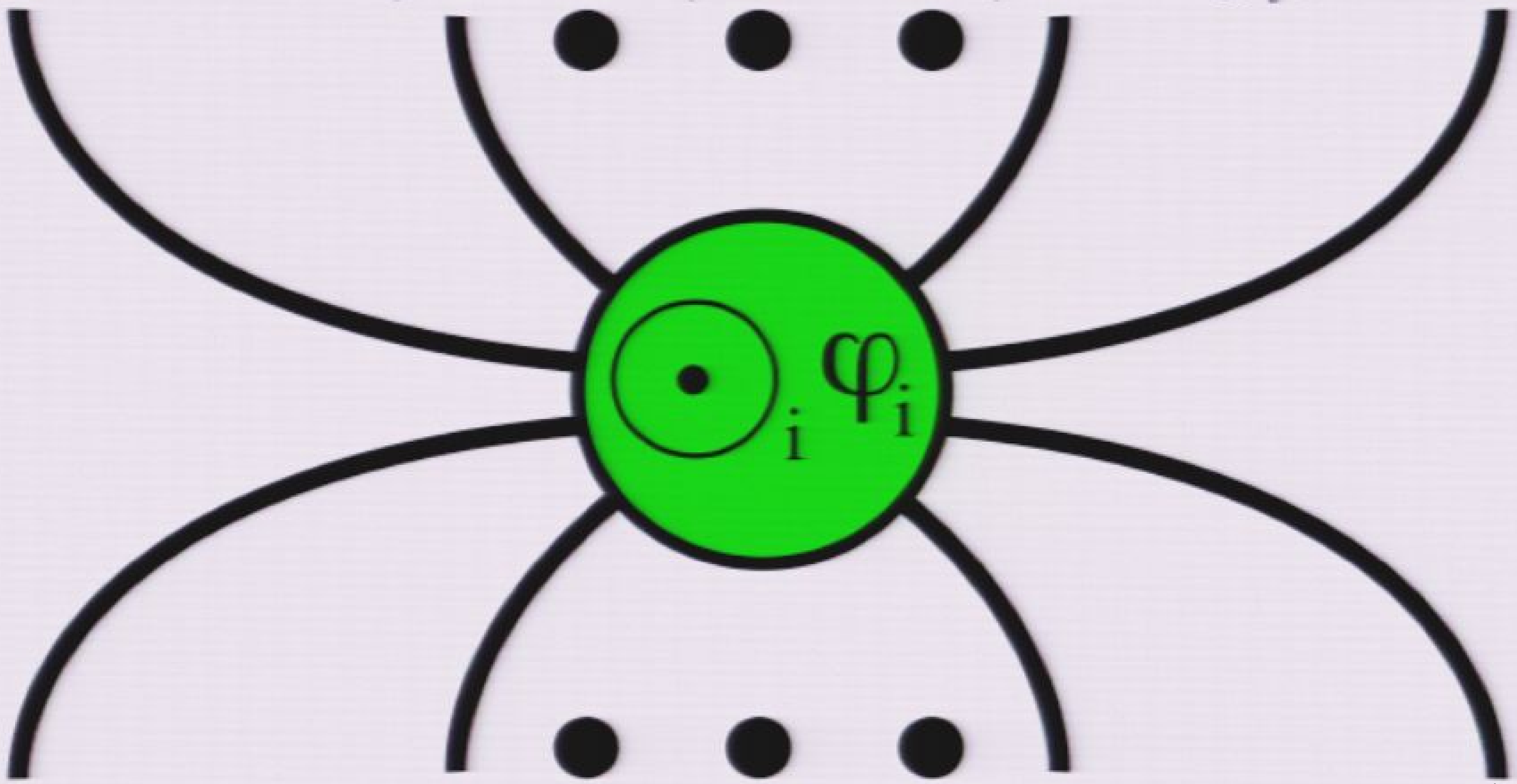

Unbiased Points

Q: When is  unitary?

A: In Hilbert spaces, $\Lambda(\psi)$ is unitary iff $|\psi\rangle$ is unbiased w.r.t. the basis copied by δ.

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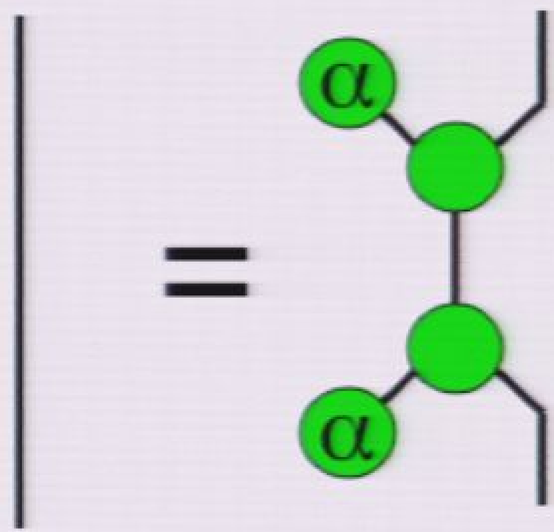

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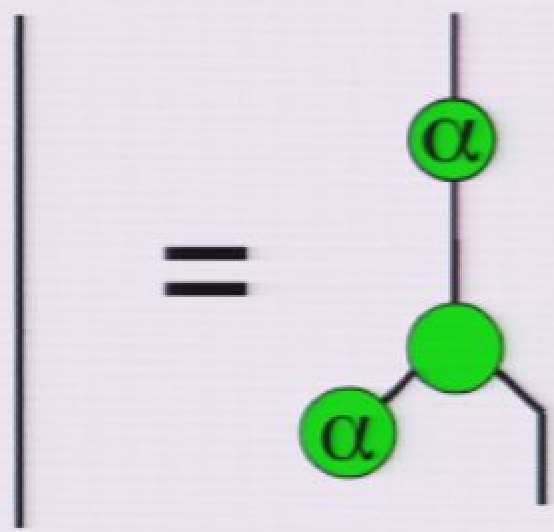

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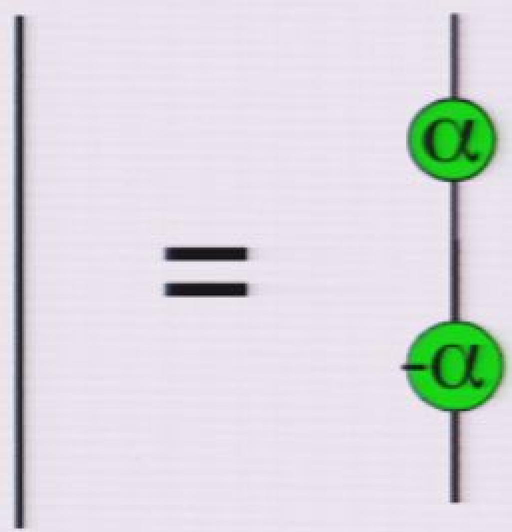

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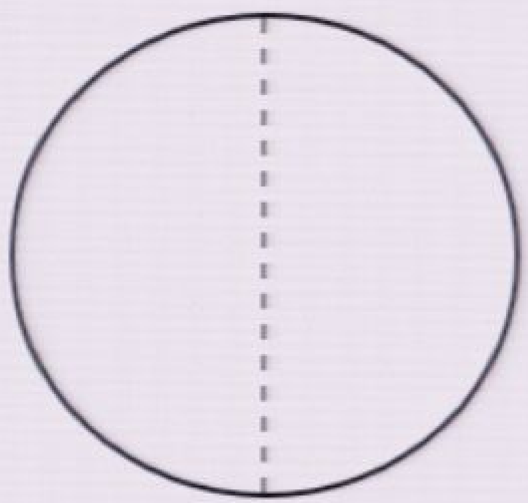
Prop:
1. the unbiased points for (δ, ϵ) form an abelian group w.r.t. to $\odot$;
2. the arrows generated by the unbiased points form an abelian group w.r.t. composition.

Example: qubits



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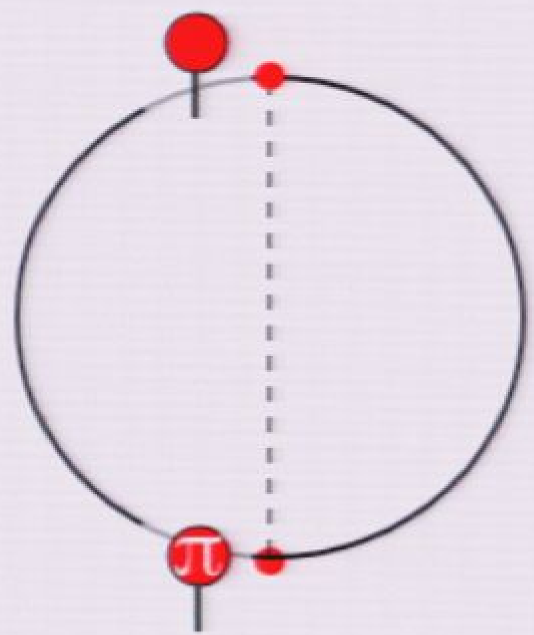

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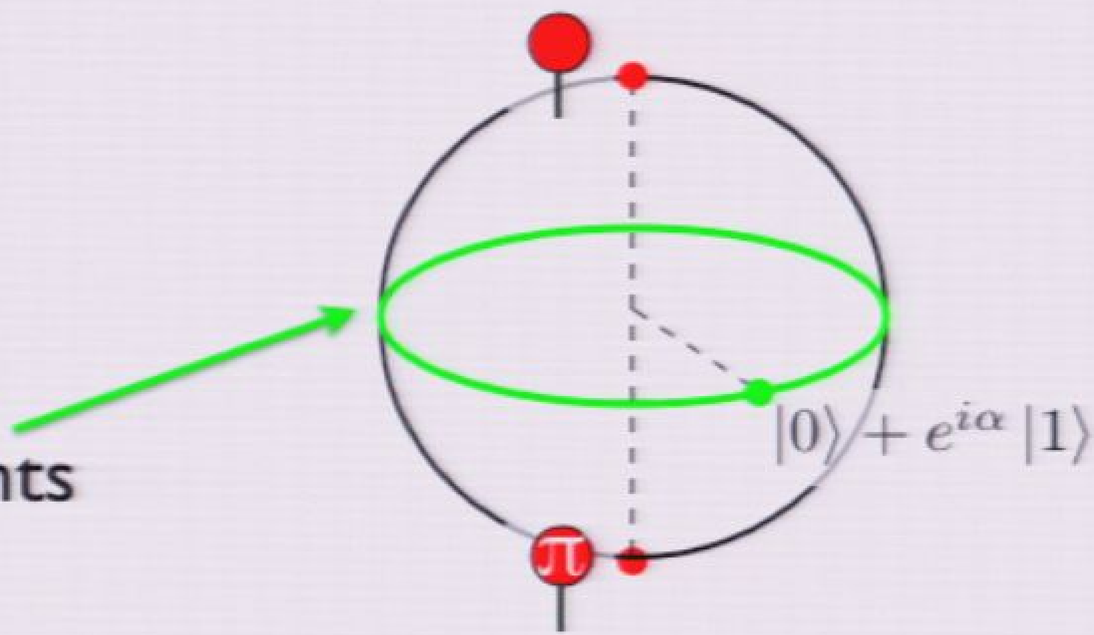
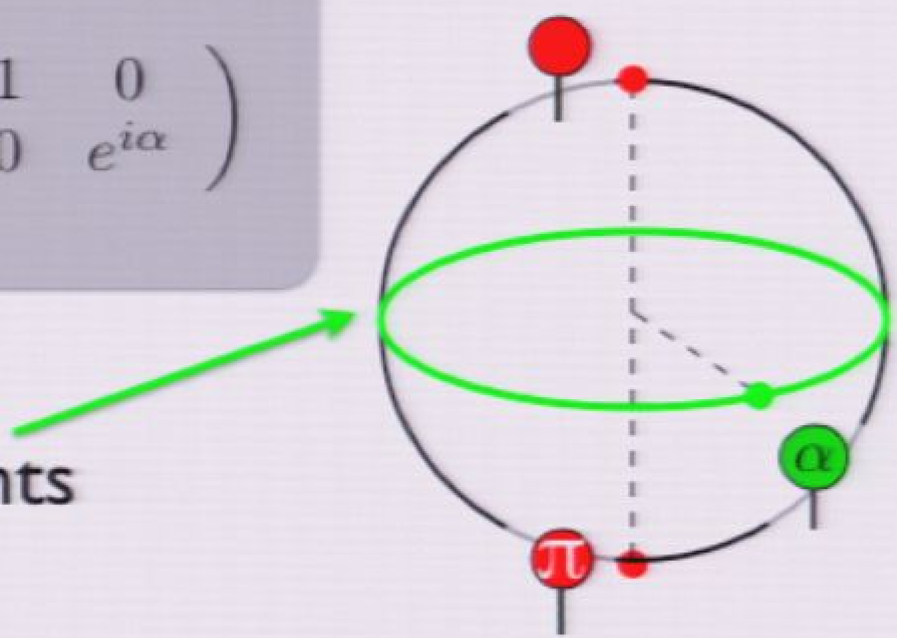


Unbiased points

Example: qubits

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

Unbiased points



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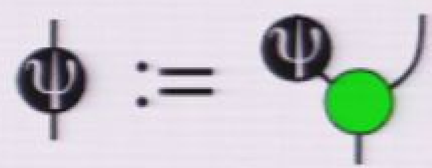

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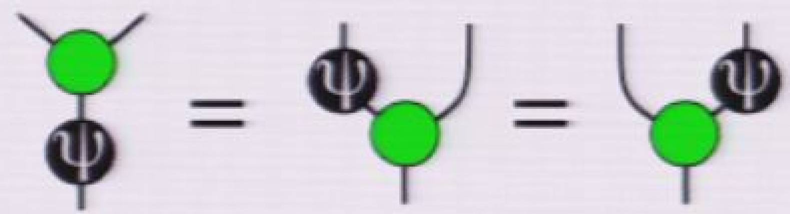

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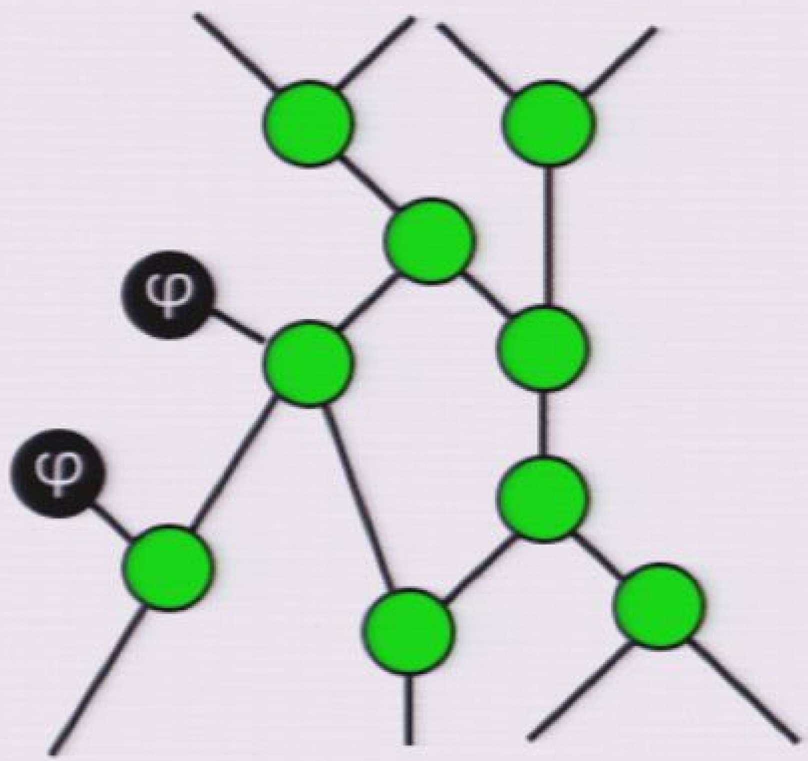

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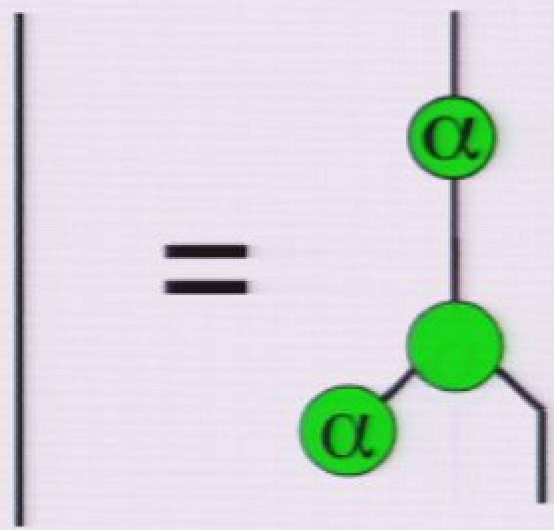

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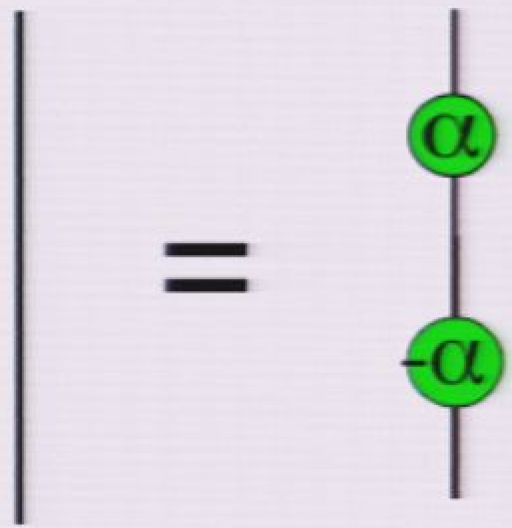

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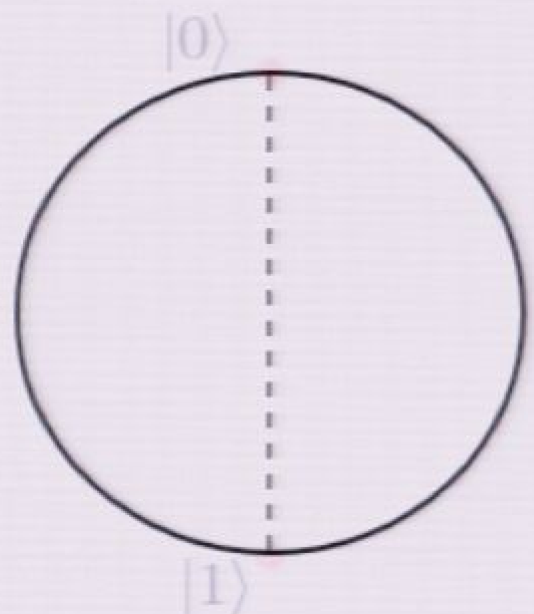

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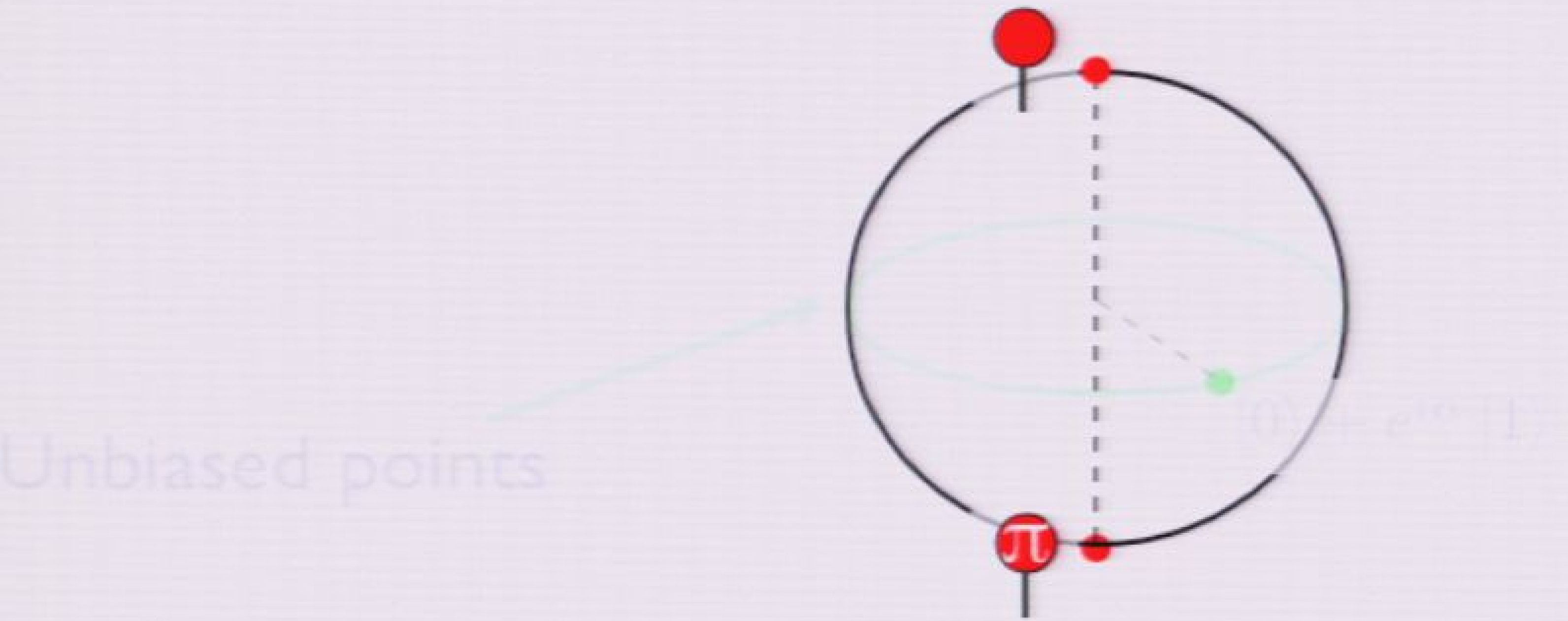
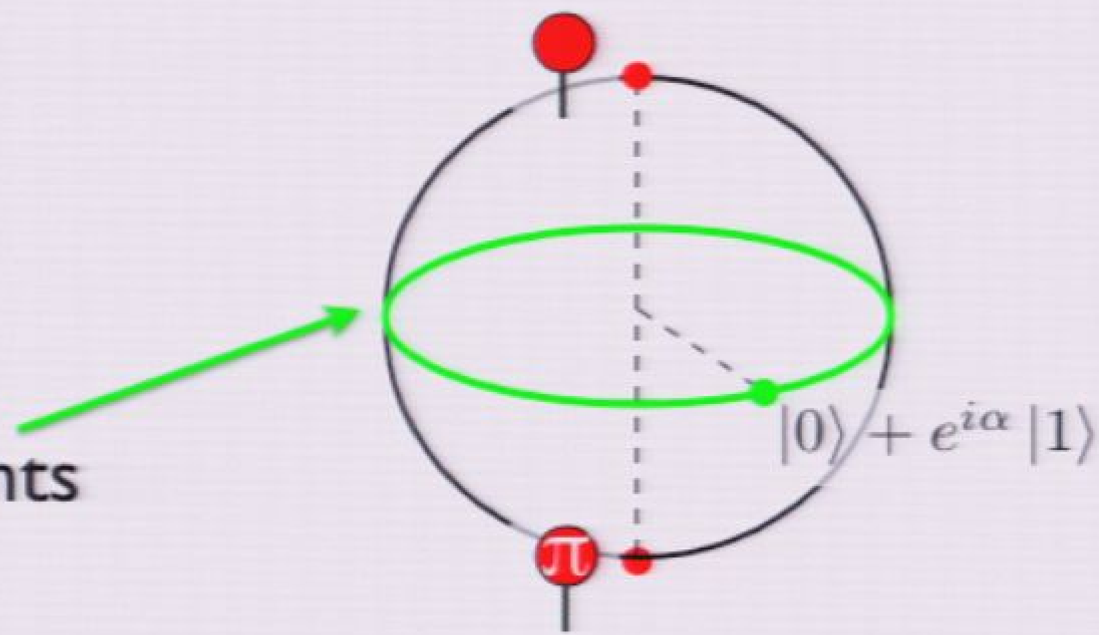


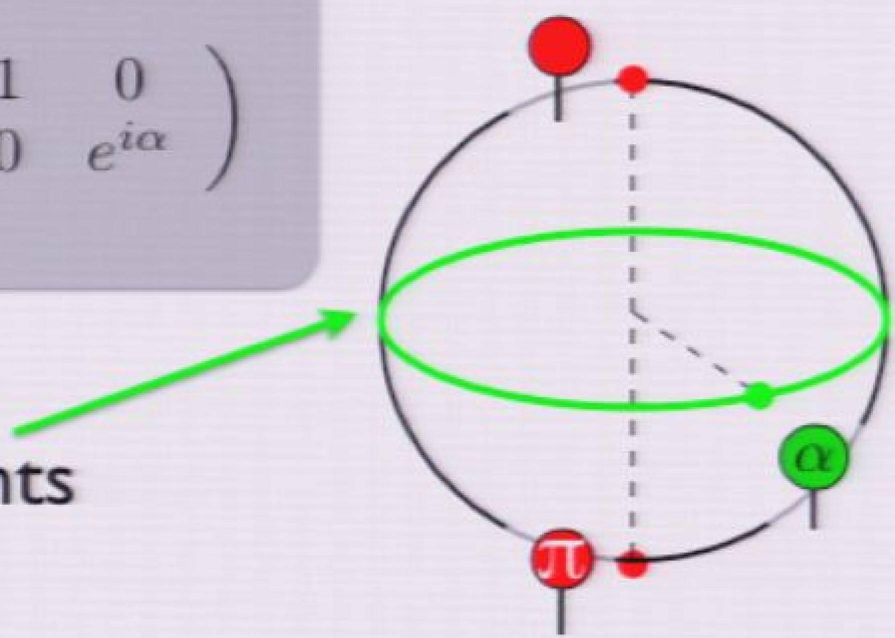
Example: qubits

Unbiased points



Example: qubits

$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$



Unbiased points

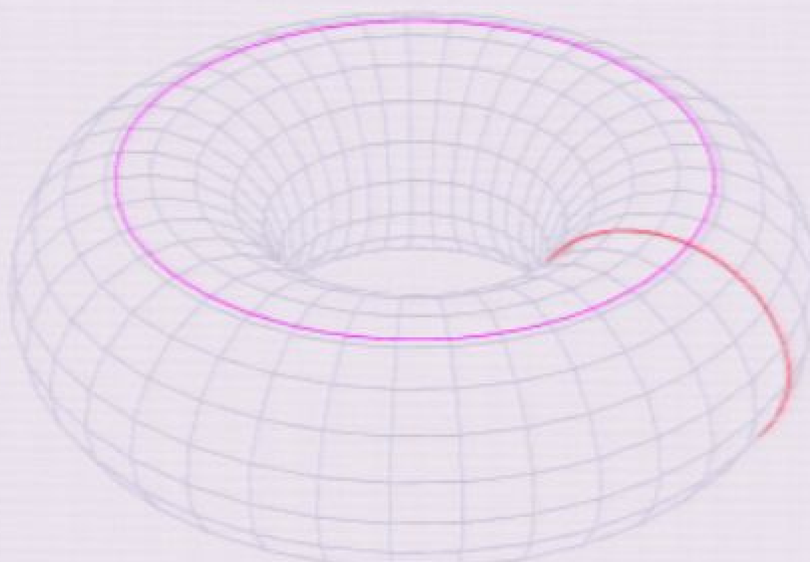
Example: qutrits

$$\Delta_Z : \begin{array}{ccc} |0\rangle & \mapsto & |00\rangle \\ |1\rangle & \mapsto & |11\rangle \\ |2\rangle & \mapsto & |22\rangle \end{array}$$

Unbiased points

$$|0\rangle + e^{i\alpha}|1\rangle + e^{i\beta}|2\rangle$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix}$$



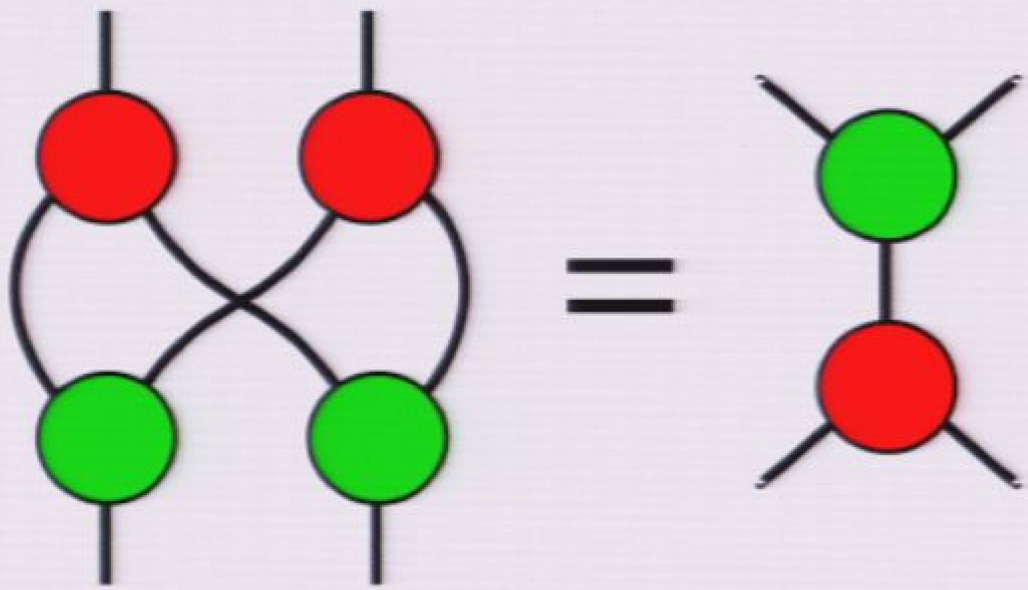
Example: FRel

$$D^\dagger : (x, x) \sim x, \forall x \in X$$

$D^\dagger \circ (\mathrm{id} \otimes \psi)$ is unitary iff $\psi = X$

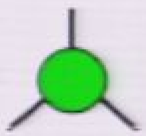
Hence the phase group is trivial.

Complementary Observables



A very general theory of interference

Two kinds of points

$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

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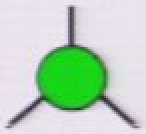
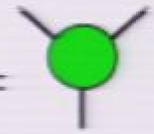
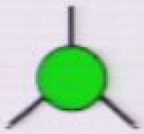
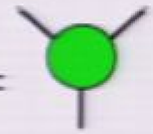
$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

Classical Points

Those points which can be copied by δ

Two kinds of points

$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

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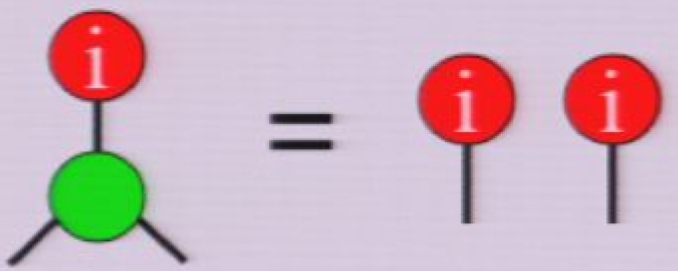
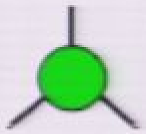
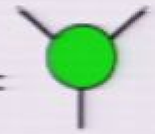


Those points which can be copied by δ

Two kinds of points

$\delta =$  $\epsilon =$  $\delta^\dagger =$  $\epsilon^\dagger =$ 

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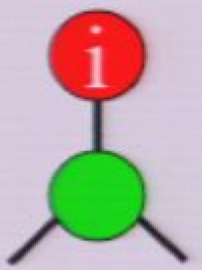

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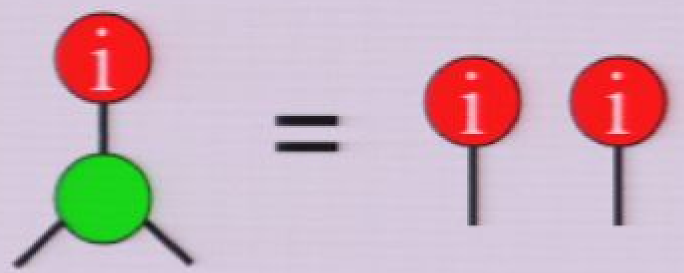
Those points which can be copied by δ

Unbiased Points

Two kinds of points

Classical Points



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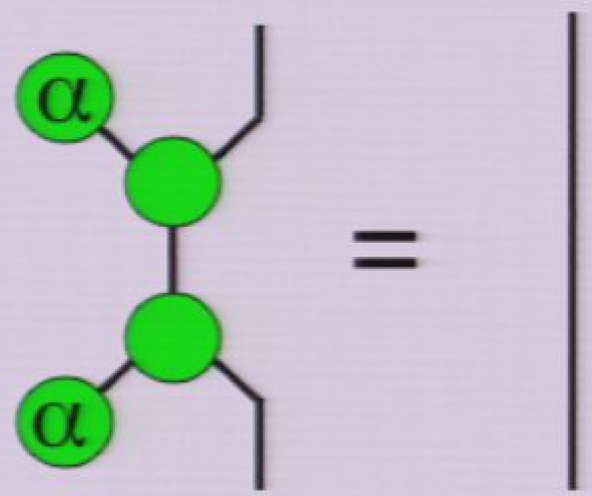
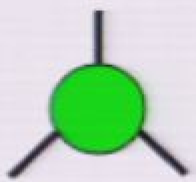
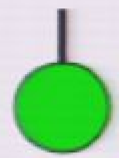
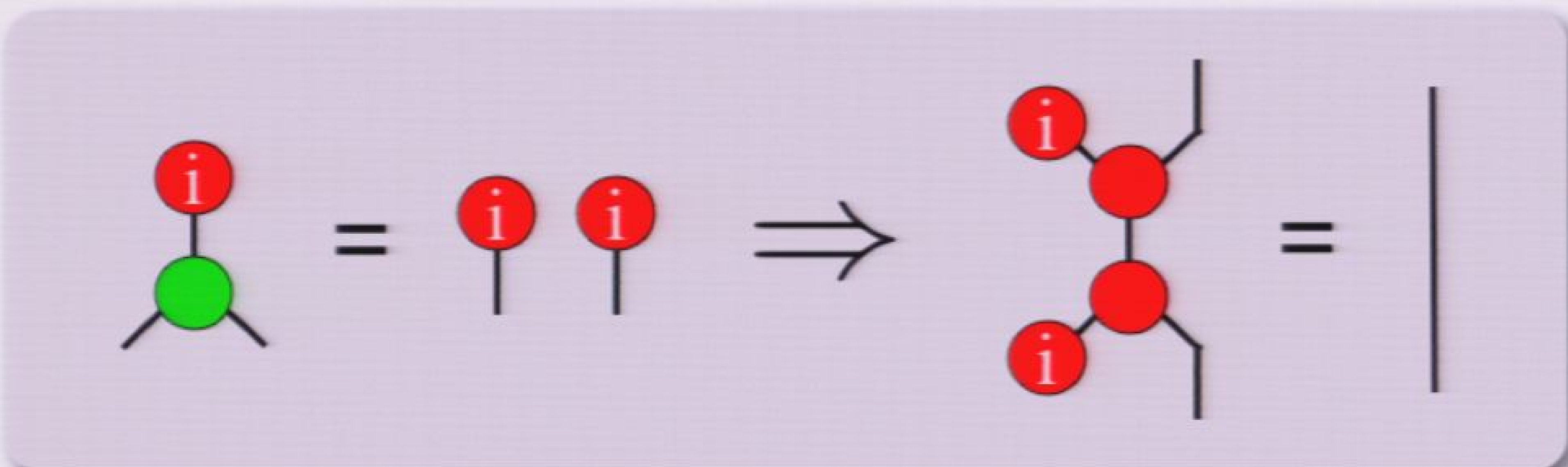


Complementary Classical Structures

$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 

$|i\rangle \mapsto |ii\rangle$ $\quad |+\rangle \mapsto 1$ $\quad |\pm\rangle \mapsto |\pm\pm\rangle$ $\quad |0\rangle \mapsto 1$

Complementary Classical Structures



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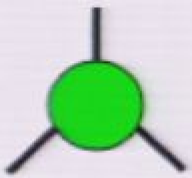
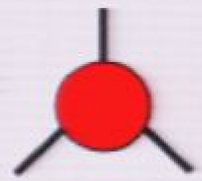
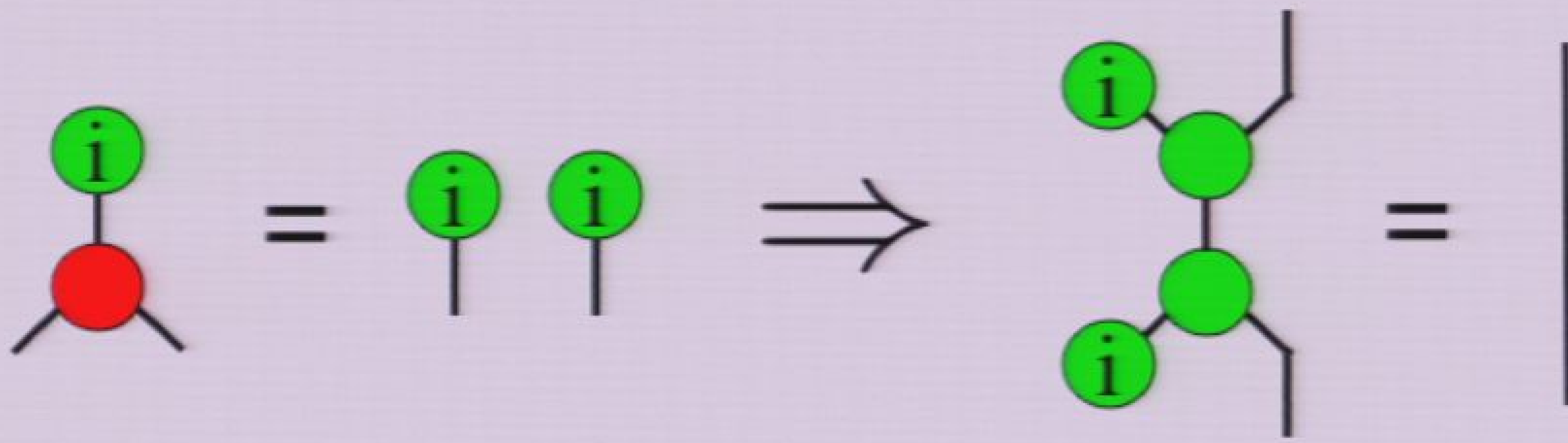
$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 



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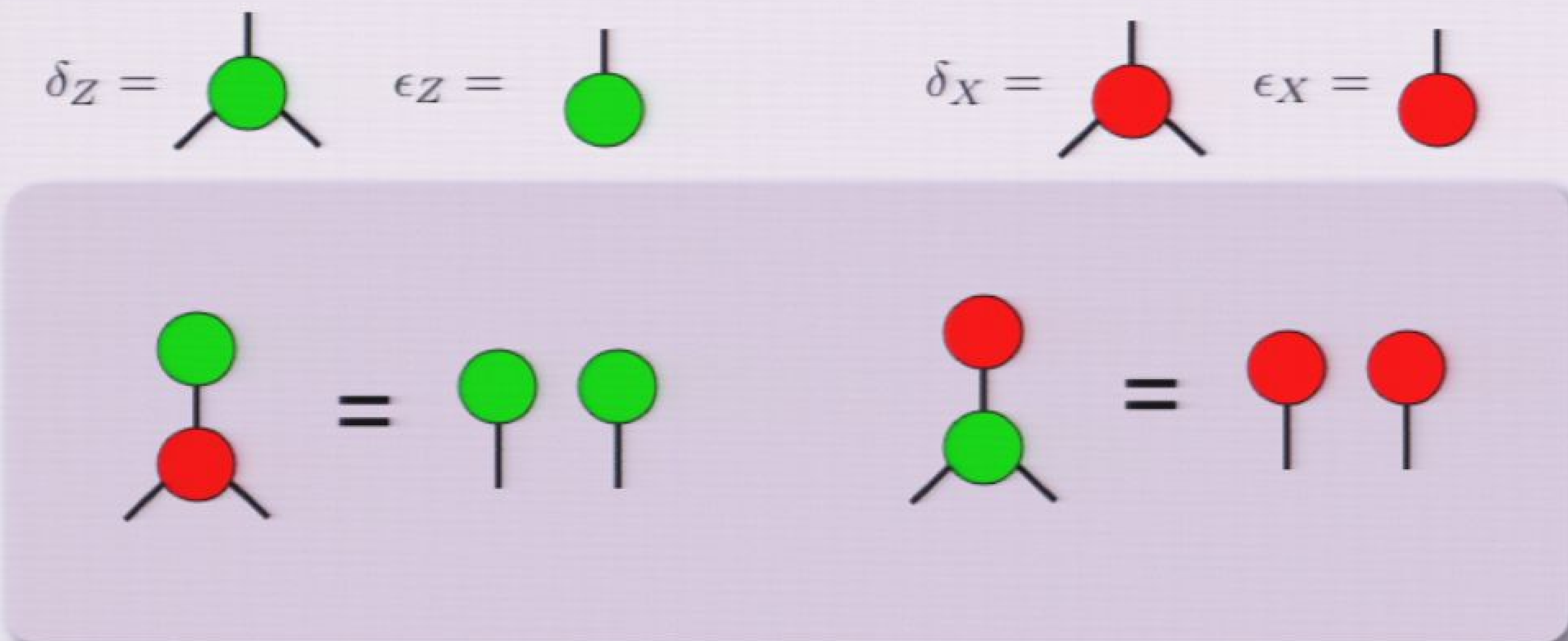

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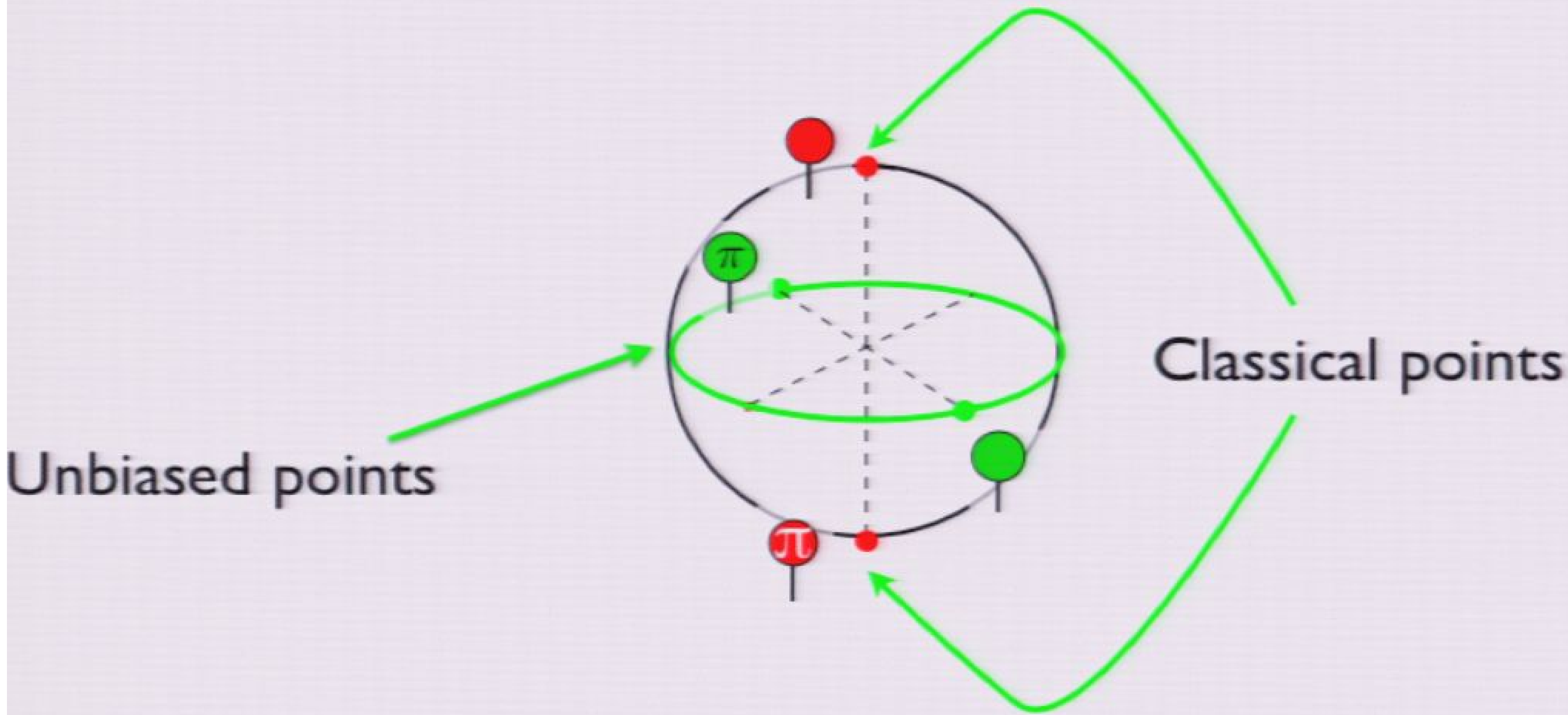

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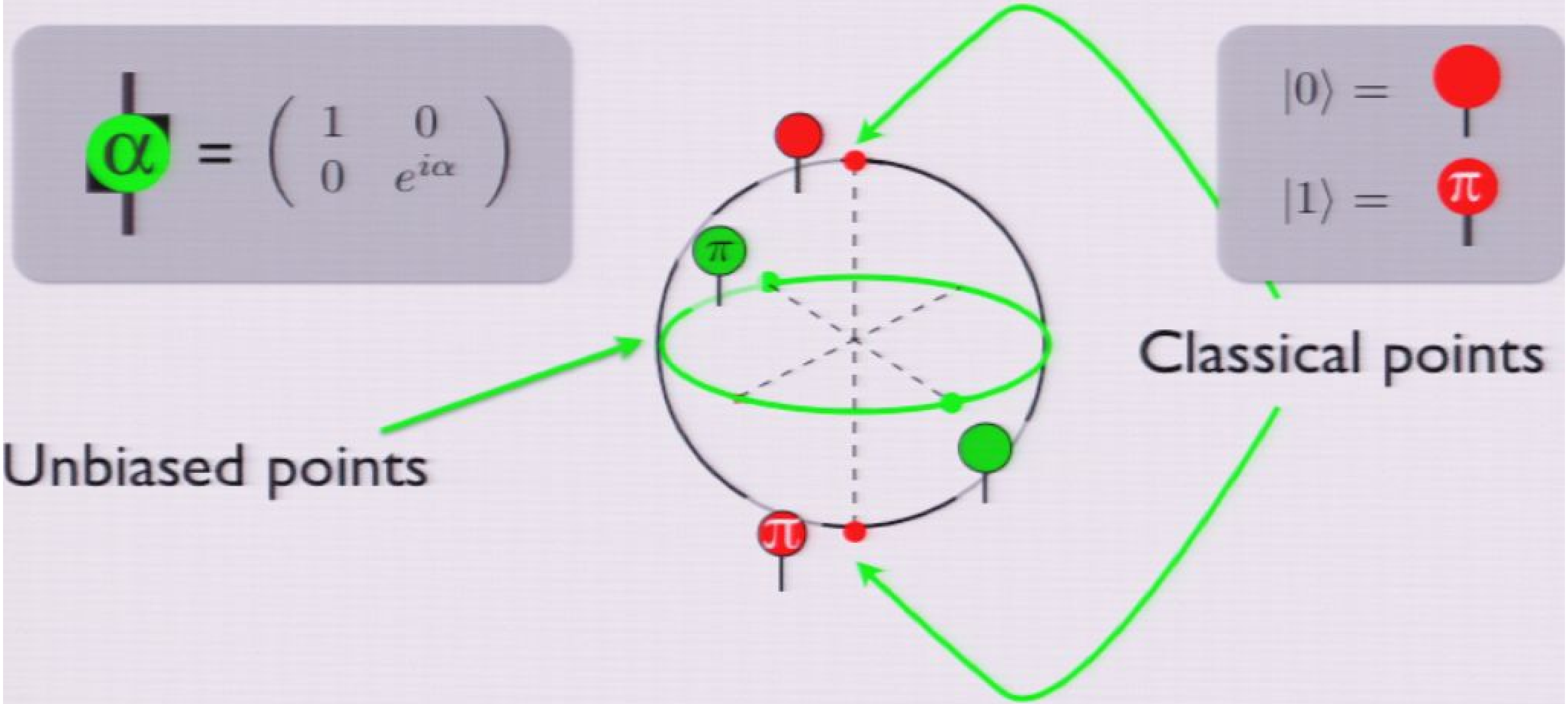

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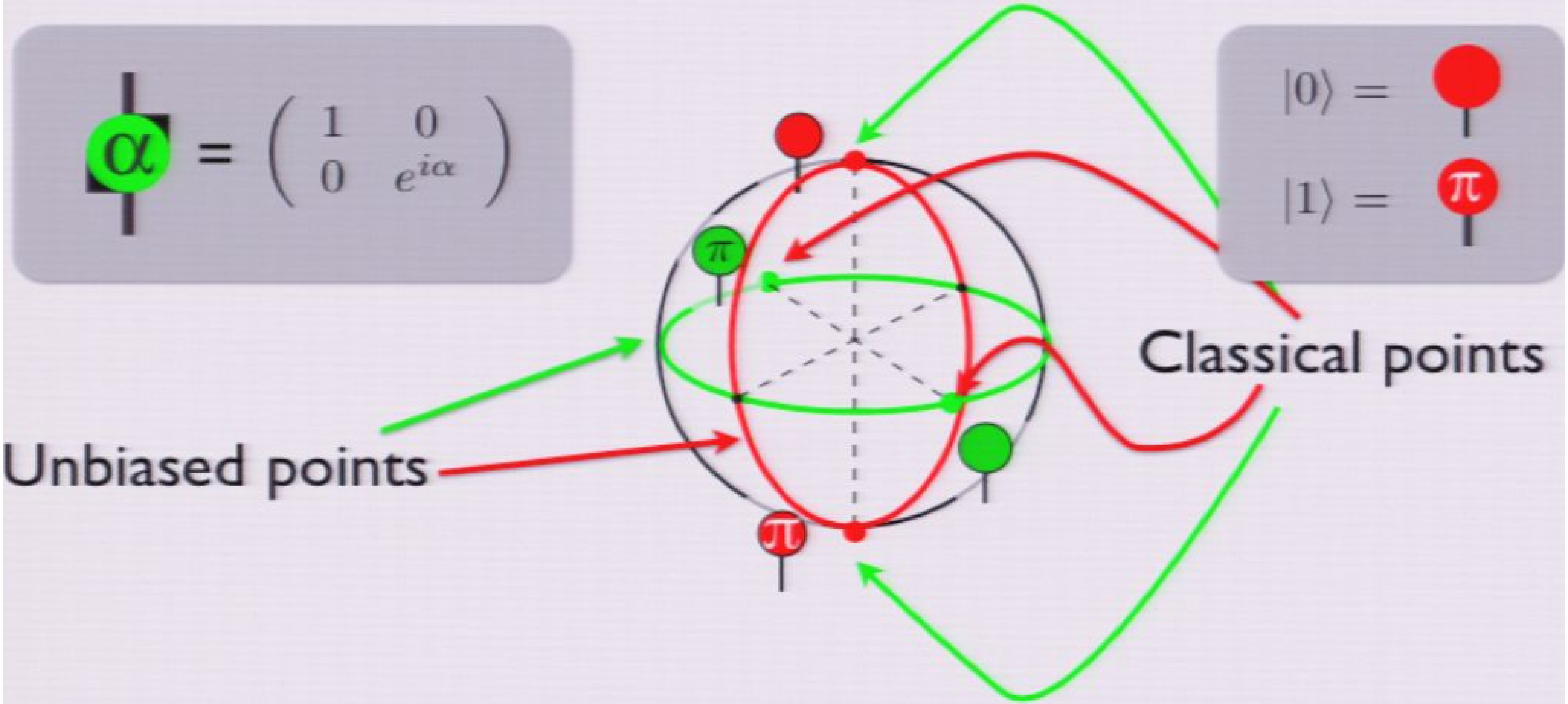
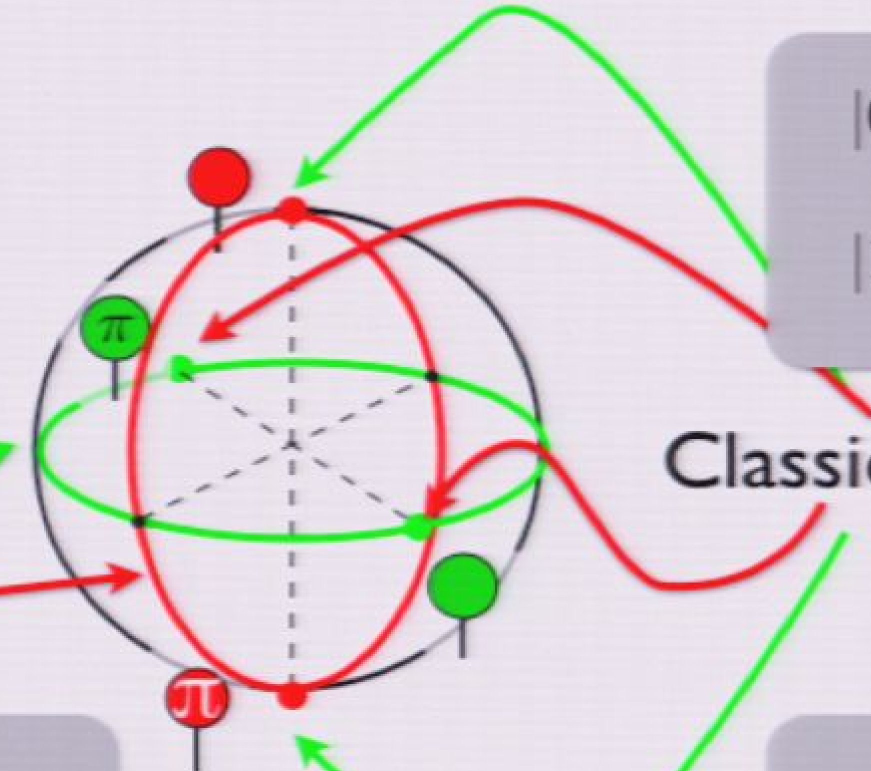


Complementary Classical Structures

(green α) $= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$

(red α) $= \begin{pmatrix} \cos\frac{\alpha}{2} & i\sin\frac{\alpha}{2} \\ i\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$



$|0\rangle =$ (red point)

$|1\rangle =$ (red π)

$|+\rangle =$ (green point)

$|-\rangle =$ (green π)

Example: qutrits

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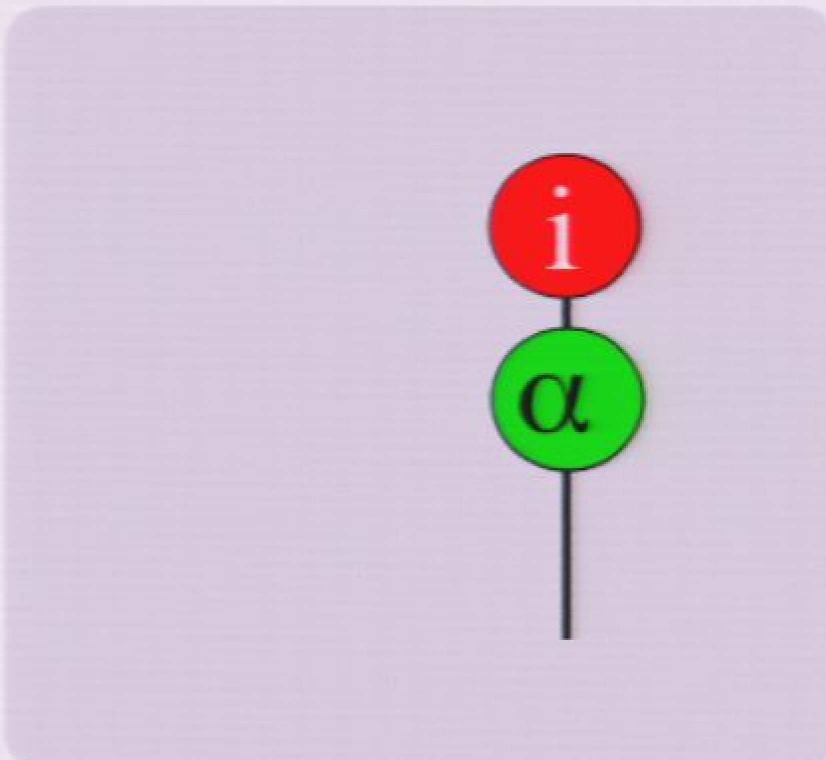
$$\Delta_X : \begin{array}{ccc} |+\rangle & \mapsto & |++\rangle \\ |\omega\rangle & \mapsto & |\omega\omega\rangle \\ |\overline{\omega}\rangle & \mapsto & |\overline{\omega\omega}\rangle \end{array} \qquad \begin{pmatrix} 1+e^{i\alpha}+e^{i\beta} & 1+\overline{\omega}e^{i\alpha}+\omega e^{i\beta} & 1+\omega e^{i\alpha}+\overline{\omega}e^{i\beta} \\ 1+\omega e^{i\alpha}+\overline{\omega}e^{i\beta} & 1+e^{i\alpha}+e^{i\beta} & 1+\overline{\omega}e^{i\alpha}+\omega e^{i\beta} \\ 1+\overline{\omega}e^{i\alpha}+\omega e^{i\beta} & 1+\omega e^{i\alpha}+\overline{\omega}e^{i\beta} & 1+e^{i\alpha}+e^{i\beta} \end{pmatrix}$$

Classical points are eigenvectors



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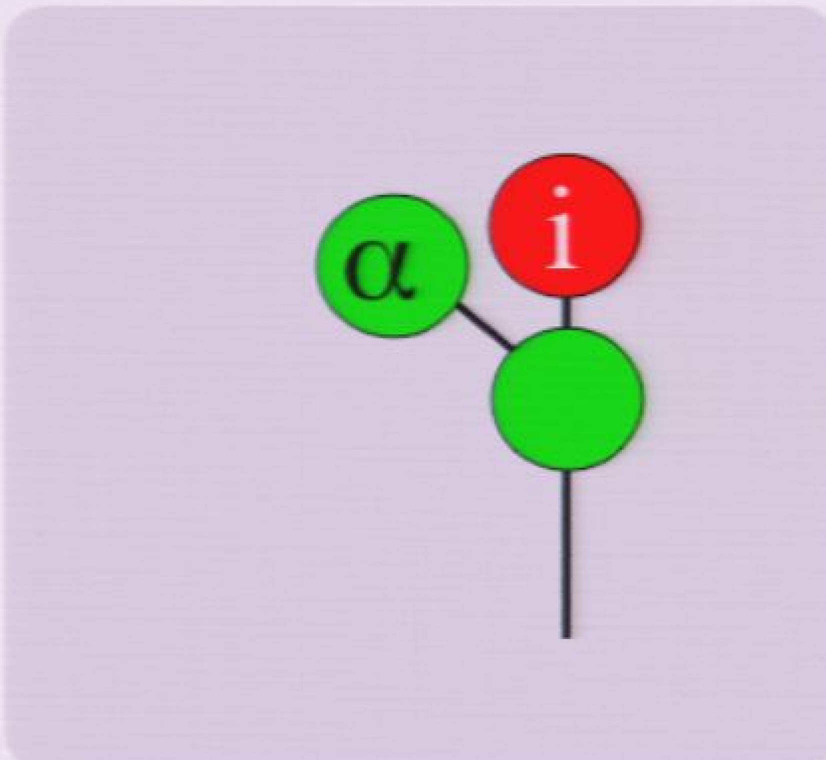

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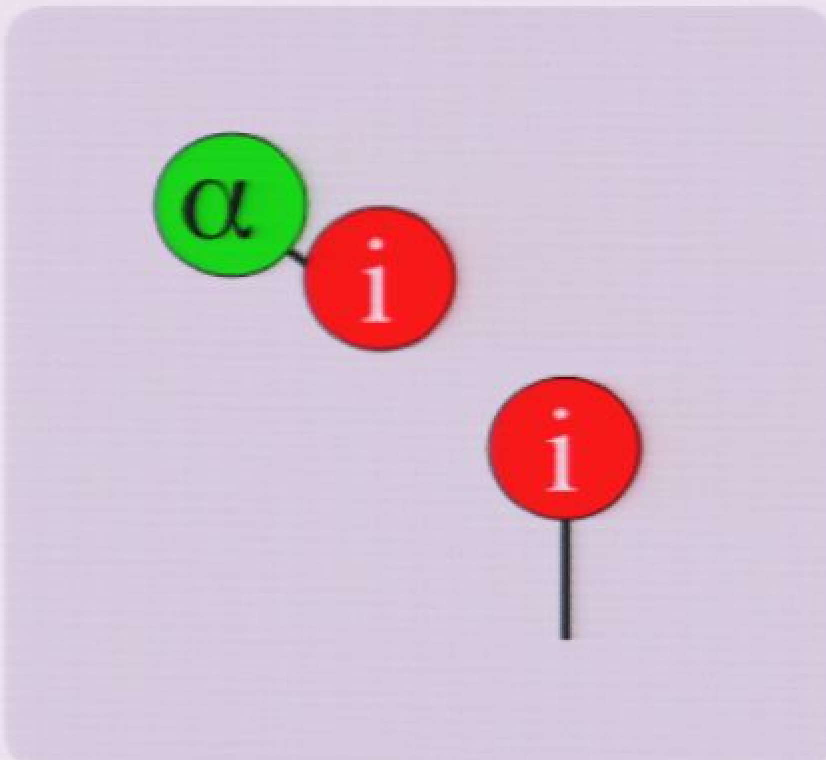
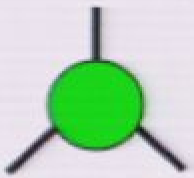
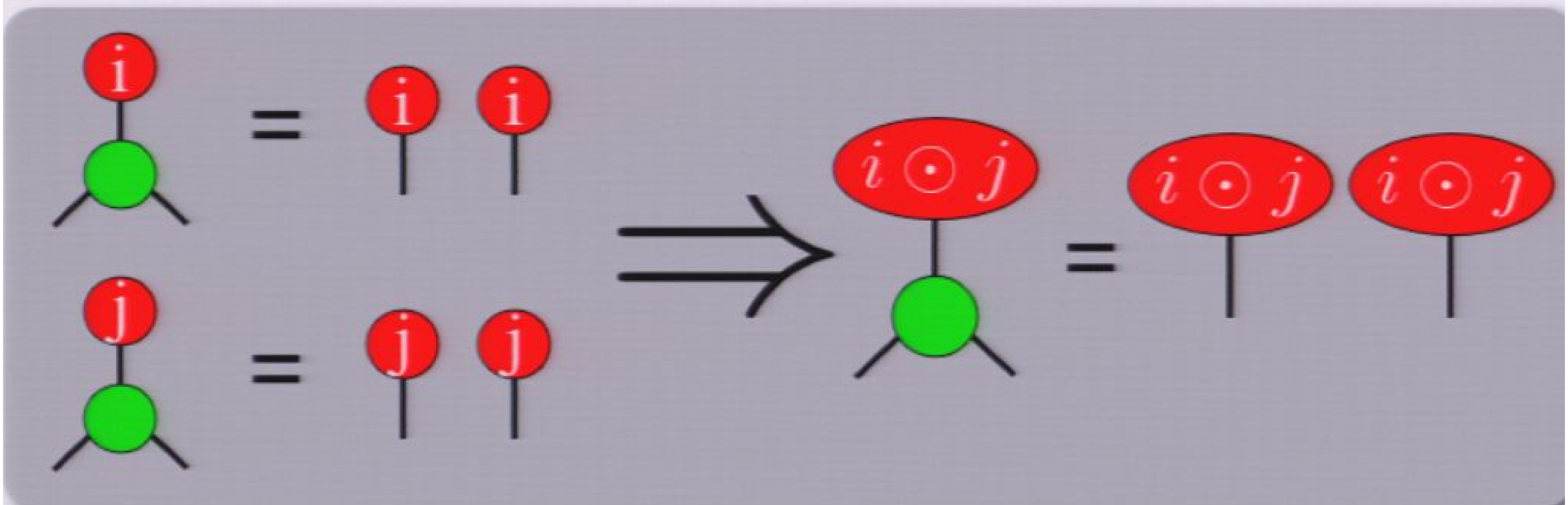


Closedness Property

$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 

Defn: complentary classical structures are called *closed when...*

Closedness Property



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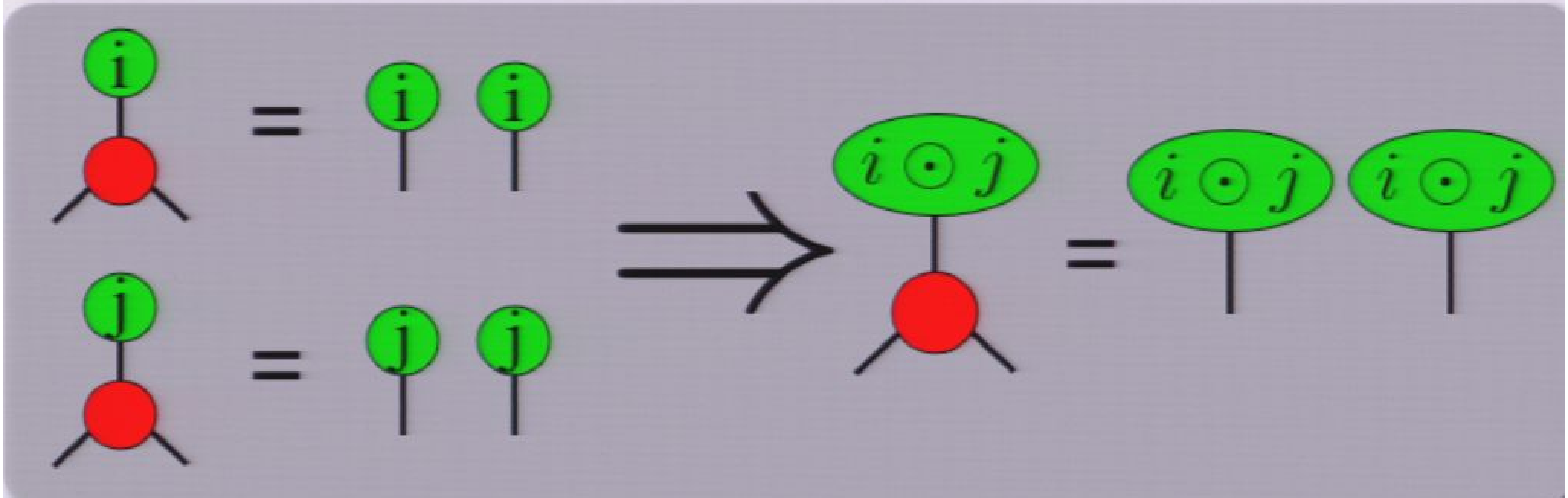
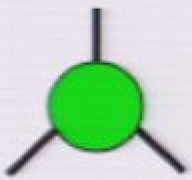
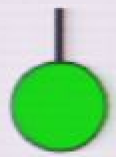


Closedness Property

$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 

In **fdHilb** it is always possible to construct a pair of mutually unbiased bases with this property...

... but in big enough dimension, it is possible to construct MUBs which are not closed.

Classical Points form a Subgroup

By the defn of complementarity, we have that:

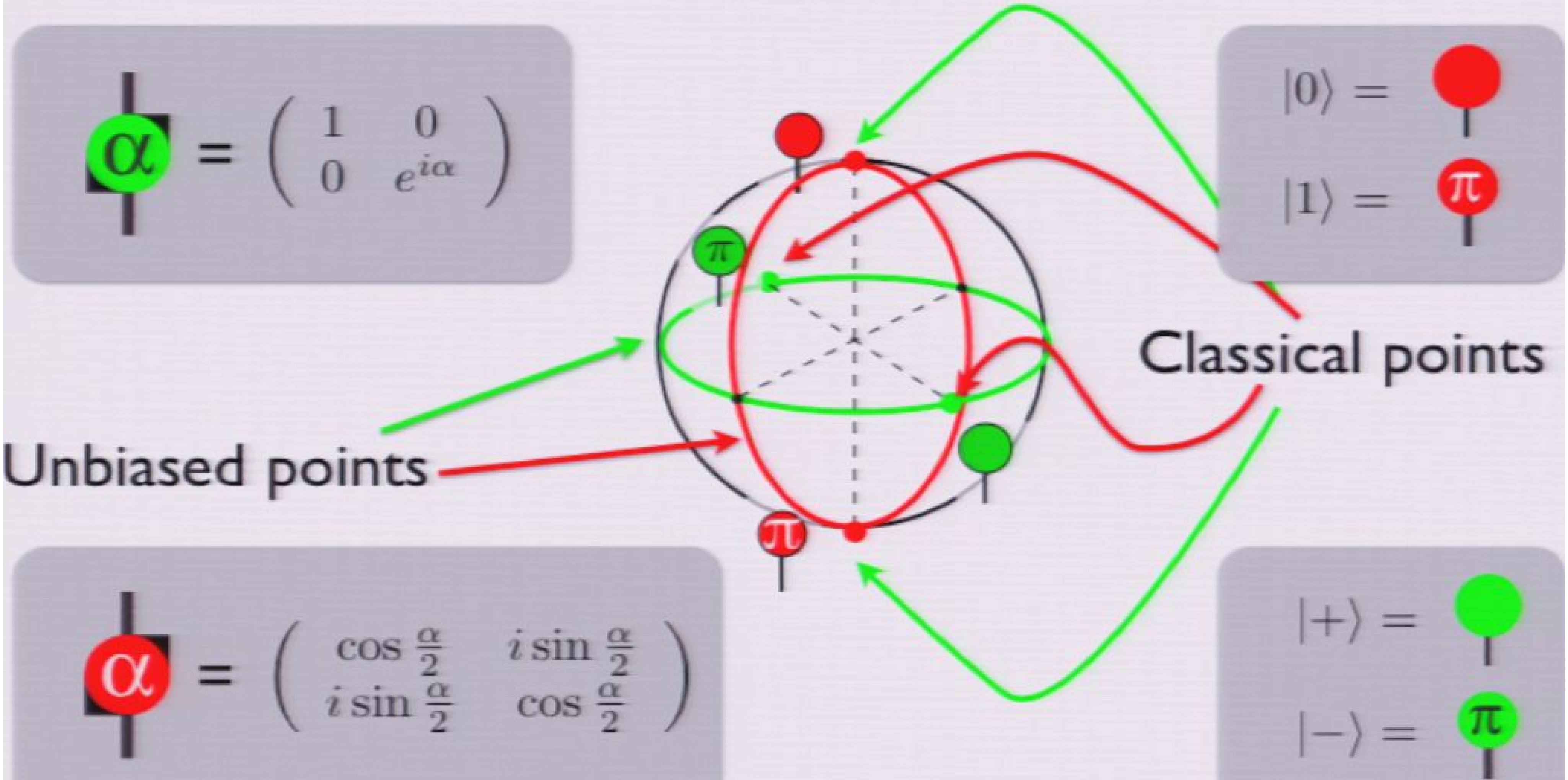
$$C_X \subseteq U_Z$$
$$C_Z \subseteq U_X$$

PROP: If the observable structure is closed, and there are finitely many classical points, then they form a subgroup of the unbiased points, i.e.

$$(C_Z, \odot_X) \leq (U_X, \odot_X)$$
$$(C_X, \odot_Z) \leq (U_Z, \odot_Z)$$

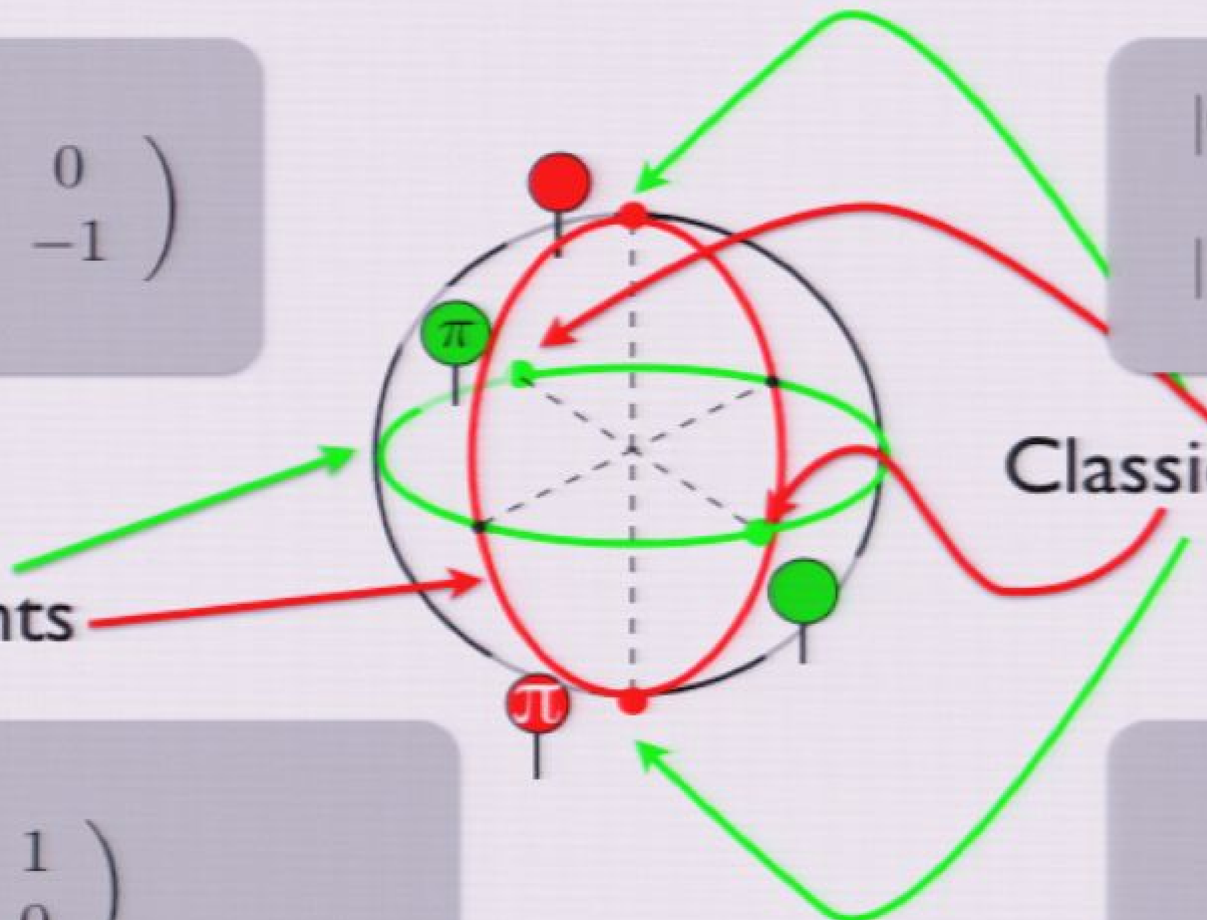
In particular, each $\Lambda^X(z_i)$ is a permutation on C_Z.

Complementary Classical Structures



Complementary Classical Structures

$$\pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



$$|0\rangle =$$

$$|1\rangle = \pi$$

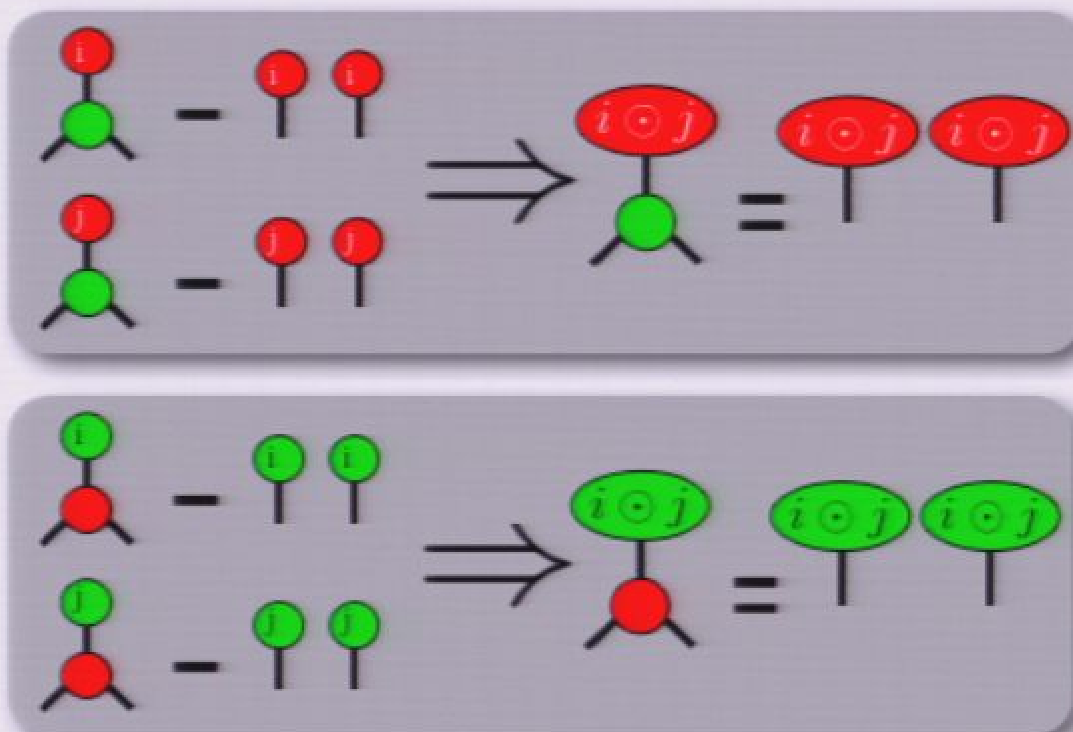
$$\pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$|+\rangle =$$

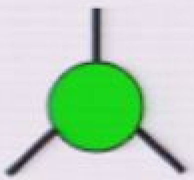
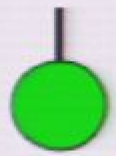
$$|-\rangle = \pi$$

The Following are Equivalent:

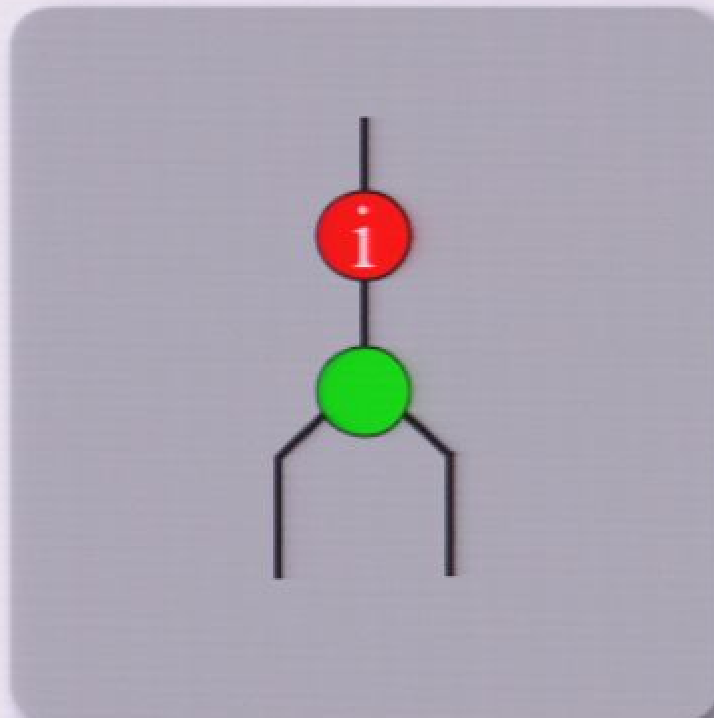
1. The classical structures are closed



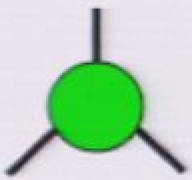
The Following are Equivalent:

$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 

2. The classical maps are comonoid homomorphisms



The Following are Equivalent:

$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 

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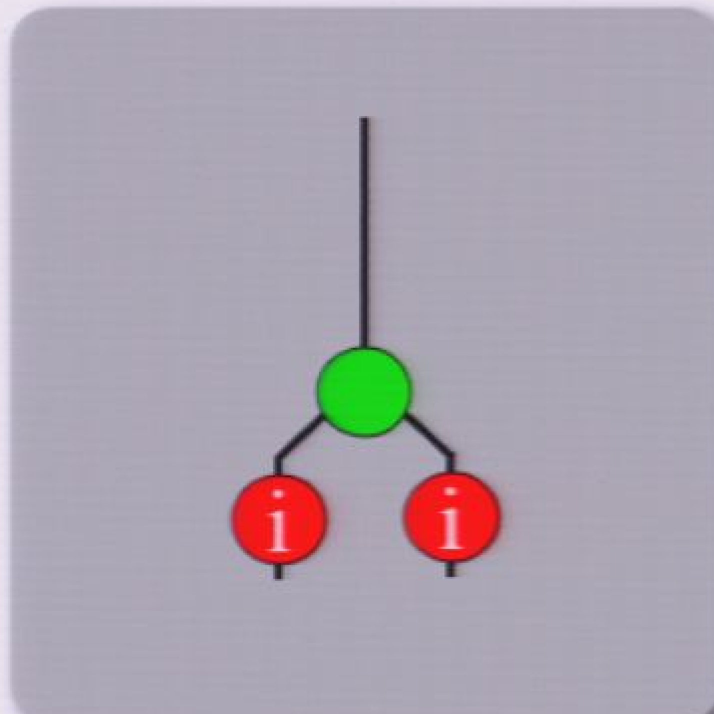
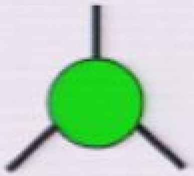
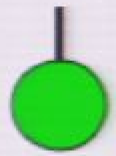
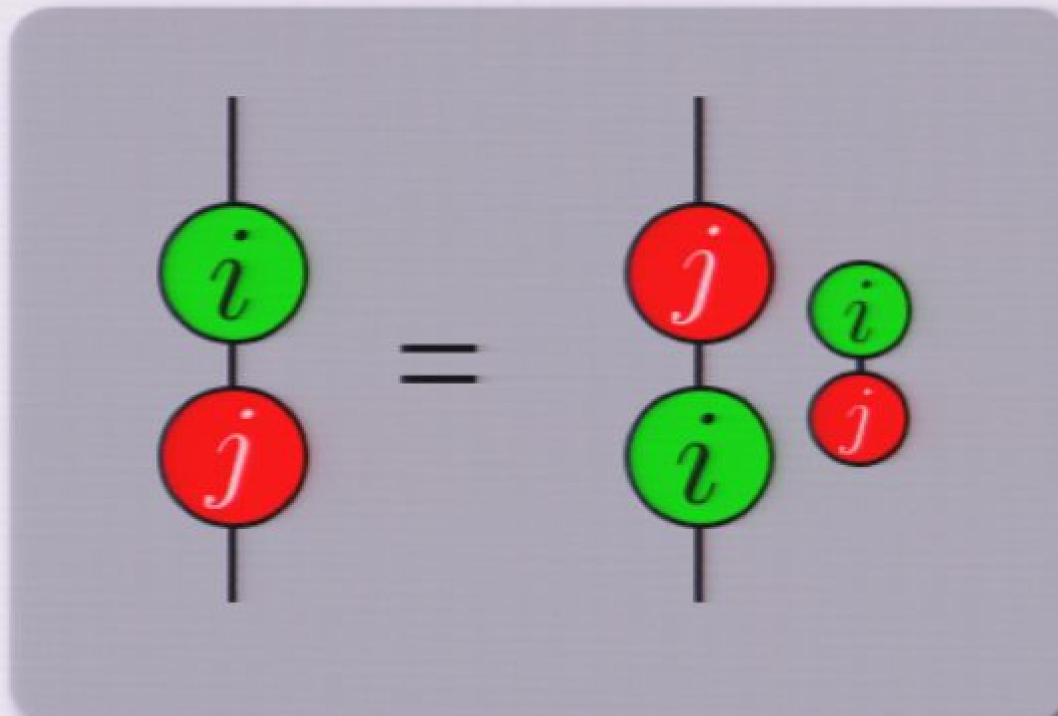


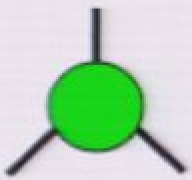
The Following are Equivalent:

$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 

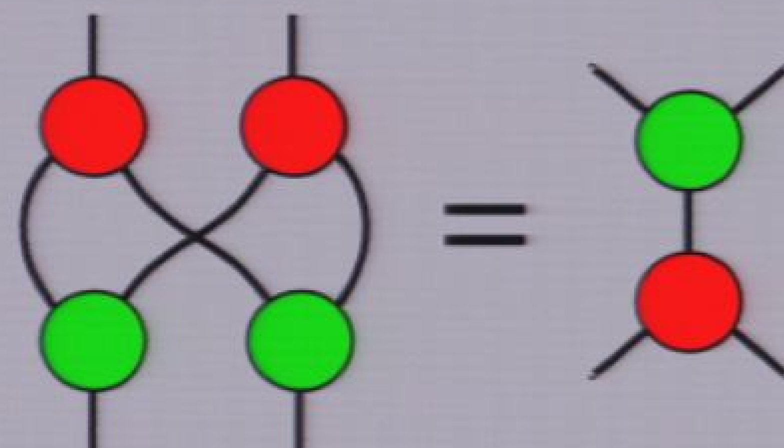
3. The classical maps satisfy canonical commutation relations



The Following are Equivalent:

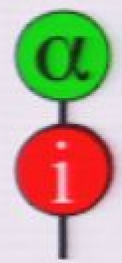
$\delta_Z =$  $\epsilon_Z =$  $\delta_X =$  $\epsilon_X =$ 

4. The classical structures form a bialgebra



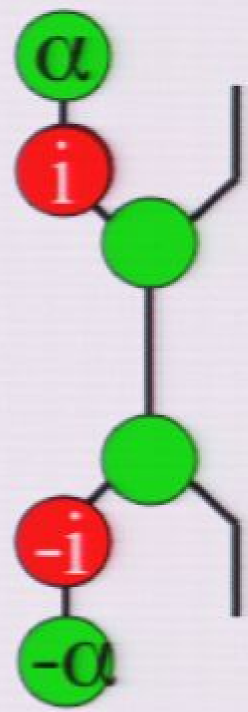
Automorphism Action

Theorem: The classical maps are group automorphisms of the unbiased points:



Automorphism Action

Theorem: The classical maps are group automorphisms of the unbiased points:



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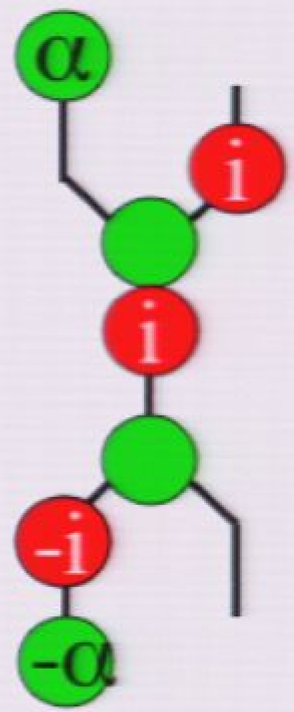
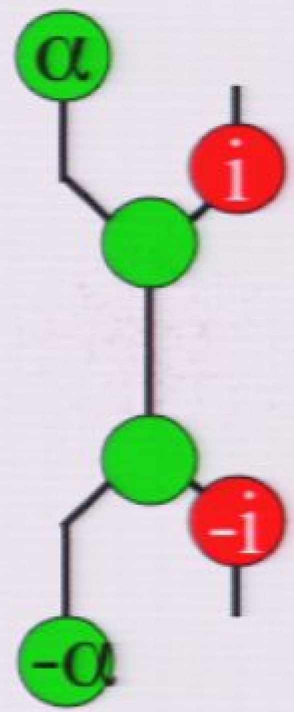
Theorem: The classical maps are group automorphisms of the unbiased points:



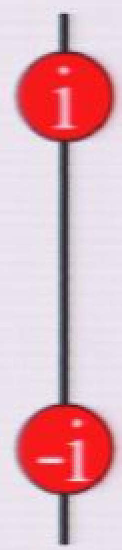
Automorphism Action

<u>Theorem</u>: The classical maps are group automorphisms of the unbiased points:



Automorphism Action

Theorem: The classical maps are group automorphisms of the unbiased points:



Automorphism Action

Theorem: The classical maps are group automorphisms of the unbiased points:

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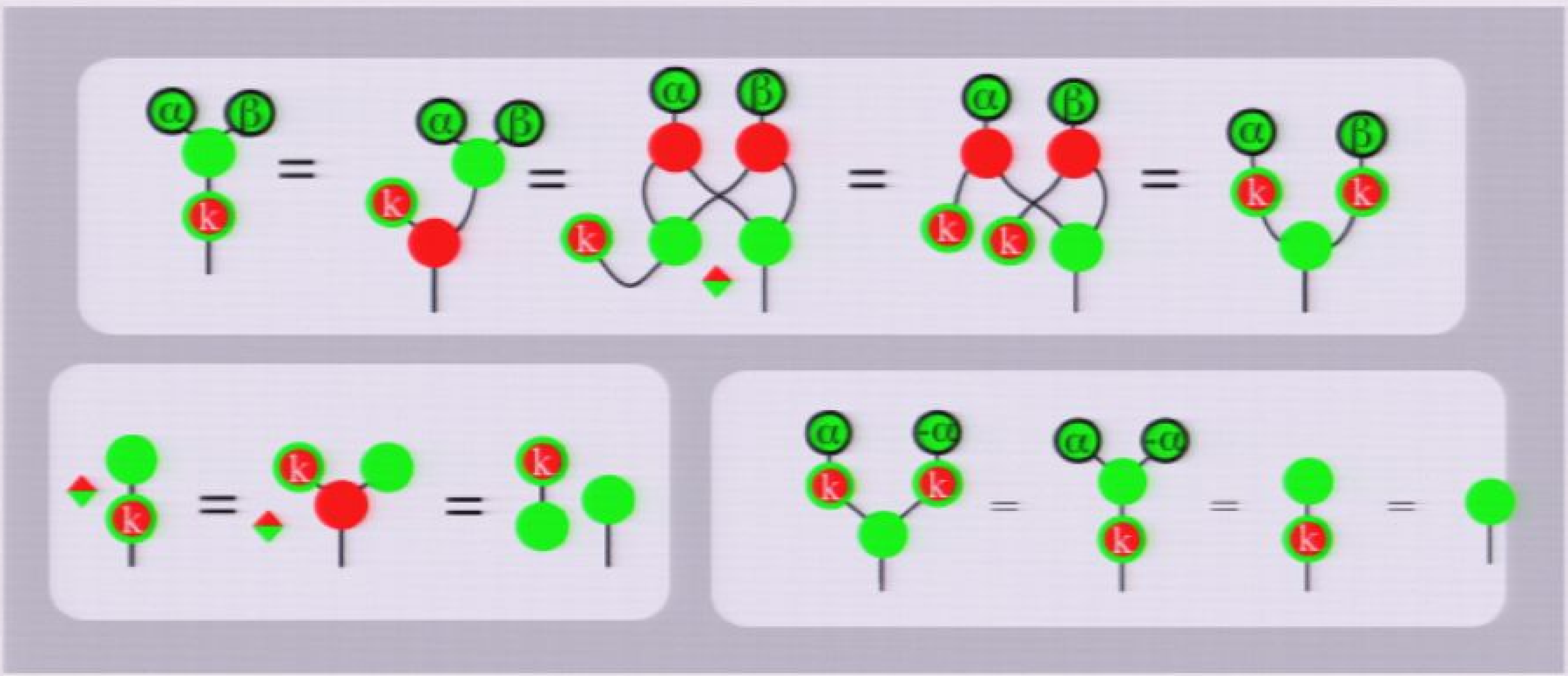

Automorphism Action

Theorem: The classical maps are group automorphisms of the unbiased points:

Classical points are symmetries of the phase group.

Example: qutrits

$$\Delta_Z : \begin{array}{ccc} |0\rangle & \mapsto & |00\rangle \\ |1\rangle & \mapsto & |11\rangle \\ |2\rangle & \mapsto & |22\rangle \end{array} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix}$$

$$\Delta_X : \begin{array}{ccc} |+\rangle & \mapsto & |++\rangle \\ |\omega\rangle & \mapsto & |\omega\omega\rangle \\ |\overline{\omega}\rangle & \mapsto & |\overline{\omega\omega}\rangle \end{array} \qquad \begin{pmatrix} 1+e^{i\alpha}+e^{i\beta} & 1+\overline{\omega}e^{i\alpha}+\omega e^{i\beta} & 1+\omega e^{i\alpha}+\overline{\omega}e^{i\beta} \\ 1+\omega e^{i\alpha}+\overline{\omega}e^{i\beta} & 1+e^{i\alpha}+e^{i\beta} & 1+\overline{\omega}e^{i\alpha}+\omega e^{i\beta} \\ 1+\overline{\omega}e^{i\alpha}+\omega e^{i\beta} & 1+\omega e^{i\alpha}+\overline{\omega}e^{i\beta} & 1+e^{i\alpha}+e^{i\beta} \end{pmatrix}$$

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Phase maps are classical when:

$$\alpha \in \{0, \frac{\pi}{3}, \frac{2\pi}{3}\} \qquad \beta = 2\pi - \alpha$$

$$\Delta_X : \begin{array}{ccc} |\omega\rangle & \mapsto & |\omega\omega\rangle \\ |\overline{\omega}\rangle & \mapsto & |\overline{\omega\omega}\rangle \end{array} \qquad \begin{pmatrix} \cdots & \cdots & \cdots \\ 1+\omega e^{i\alpha}+\overline{\omega}e^{i\beta} & 1+e^{i\alpha}+e^{i\beta} & 1+\overline{\omega}e^{i\alpha}+\omega e^{i\beta} \\ 1+\overline{\omega}e^{i\alpha}+\omega e^{i\beta} & 1+\omega e^{i\alpha}+\overline{\omega}e^{i\beta} & 1+e^{i\alpha}+e^{i\beta} \end{pmatrix}$$

Example: qutrits

$$C_X = \{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \overline{\omega} \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \overline{\omega} & 0 \\ 0 & 0 & \omega \end{pmatrix} \}$$

$$C_Z = \{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \}$$

Example: qutrits

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ e^{i\alpha} \\ e^{i\beta} \end{pmatrix} = \begin{pmatrix} e^{i\alpha} \\ e^{i\beta} \\ 1 \end{pmatrix} \simeq \begin{pmatrix} 1 \\ e^{i(\beta-\alpha)} \\ e^{-i\beta} \end{pmatrix}$$

Example: qutrits

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ e^{i\alpha} \\ e^{i\beta} \end{pmatrix} = \begin{pmatrix} e^{i\alpha} \\ e^{i\beta} \\ 1 \end{pmatrix} \simeq \begin{pmatrix} 1 \\ e^{i(\beta-\alpha)} \\ e^{-i\beta} \end{pmatrix}$$

$$(\alpha, \beta) \mapsto (\beta - \alpha, -\alpha)$$

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$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ e^{i\alpha} \\ e^{i\beta} \end{pmatrix} = \begin{pmatrix} e^{i\alpha} \\ e^{i\beta} \\ 1 \end{pmatrix} \simeq \begin{pmatrix} 1 \\ e^{i(\beta-\alpha)} \\ e^{-i\beta} \end{pmatrix}$$

$$(\alpha, \beta) \mapsto (\beta - \alpha, -\alpha)$$

Conclusions

We formalised complementary quantum observables in the language of monoidal categories:

- each observable defines two groups: its classical points and its unbiased points
- Interference between pairs of complementary observables is characterised via a group of automorphisms

Ongoing work 1

Current research directions:

- Investigate connections with multipartite entanglement (with Bill Edwards)
- Algorithmic properties of MBQC (with Simon Perdrix)
- Study toy models of QM based on the automorphism approach.
- Use phase groups to construct MUBs

Ongoing work 2

Collaborating with Lucas Dixon (Edinburgh) and Aleks Kissinger (Oxford) to automate this calculus using a graphical rewriting system / interactive theorem prover.

http://dream.inf.ed.ac.uk/projects/quantomatic/

- Rewriting properties e.g. normal forms?
- Pattern languages: ellipses and indexed containers?

I cited results by these people:

Bob Coecke

Samson Abramsky

Eric Paquette

Dusko Pavlovic

Peter Selinger

Simon Perdrix

... and Jamie Vicary

Use it: http://arxiv.or

Reference

B. Coecke and R. Duncan, *Interacting Quantum Observables*, Proceedings of ICALP, 2008

(Long version going onto arxiv.org as soon as we can get the 100s of diagrams past their TeX robot....maybe later today)

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Thanks!