Title: Hamilton's diabolical singularity

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Abstract: The transformation of a narrow beam into a hollow cone when incident along the optic axis of a biaxial crystal, predicted by Hamilton in 1832, created a sensation when observed by Lloyd soon afterwards. It was the first application of his concept of phase space, and the prototype of the conical intersections and fermionic sign changes that now pervade physics and chemistry. But the fine structure of the bright cone contains many subtle features, slowly revealed by experiment, whose definitive explanation, involving new mathematical asymptotics, has been achieved only recently, along with definitive experimental test of the theory. Radically different phenomena arise when chirality and absorption are incorporated in addition to biaxiality.

Hamilton's diabolical singularity

Michael Berry
University of Bristol

http://www.phy.bris.ac.uk/people/berry_mv/index.html

Sir William Rowan Hamilton

1805-1865

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ook

direction

seriously

polarization optics

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basic crystal optics (Huygens, Young, Fresnel)

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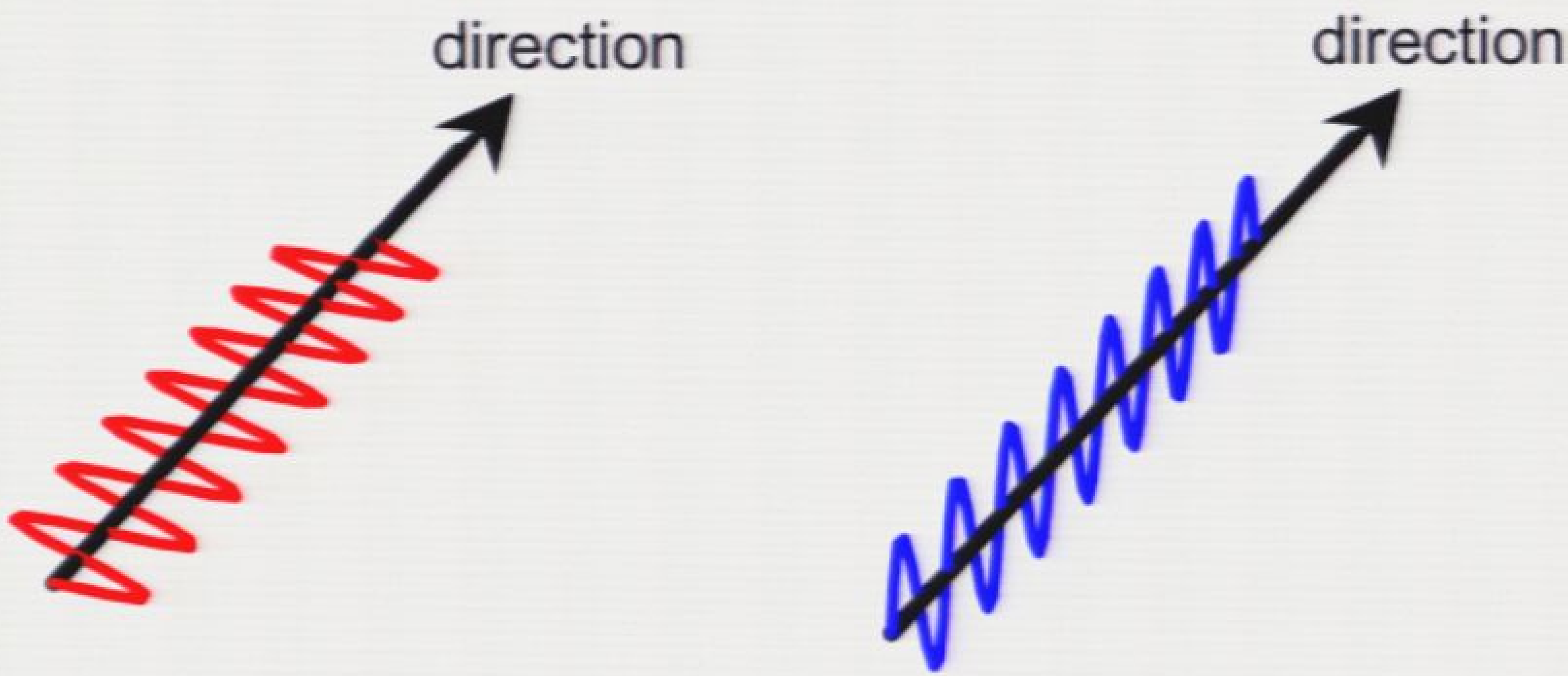

the wave surface: dependence of refractive indices on direction (polar plot)

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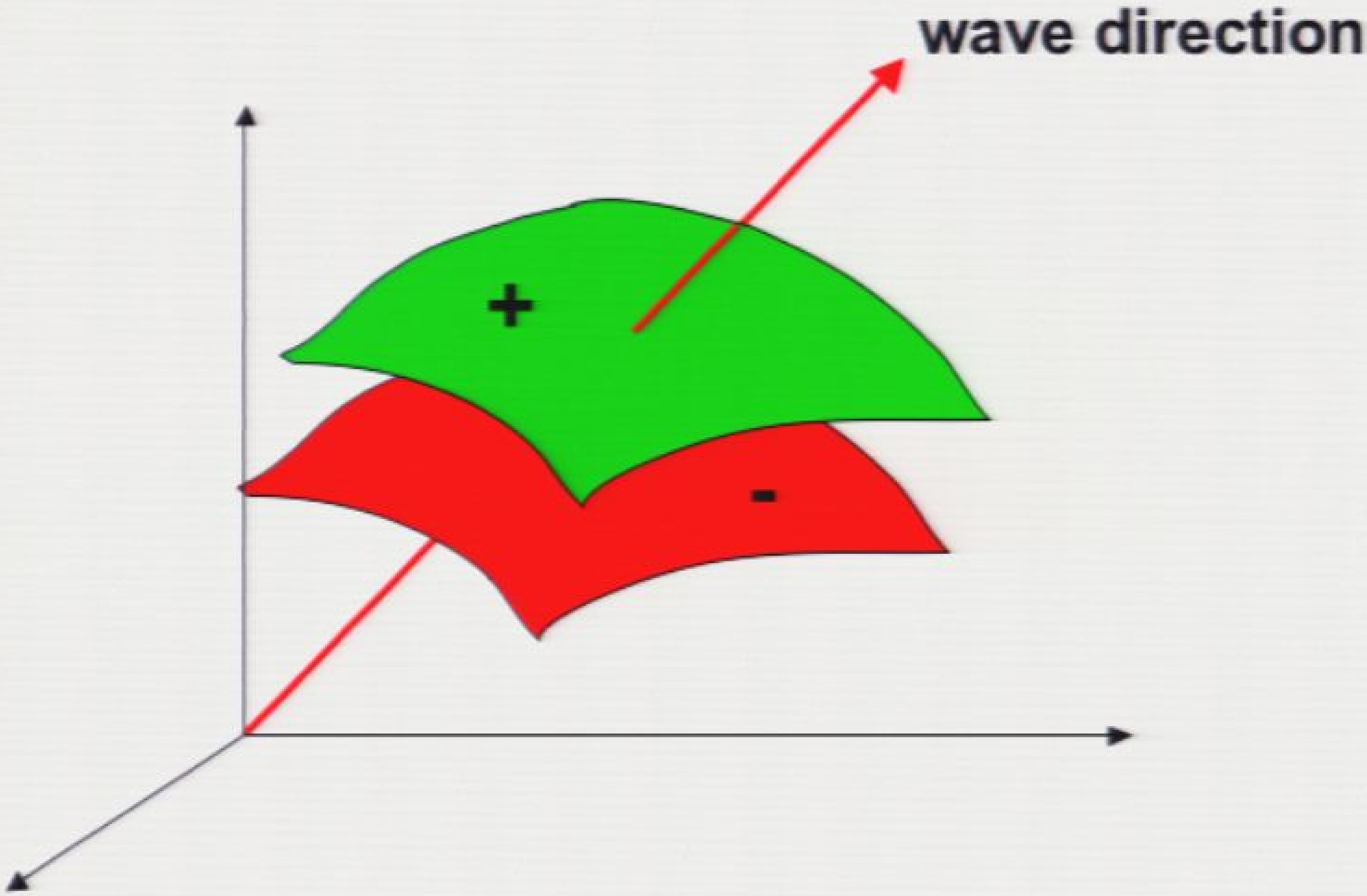
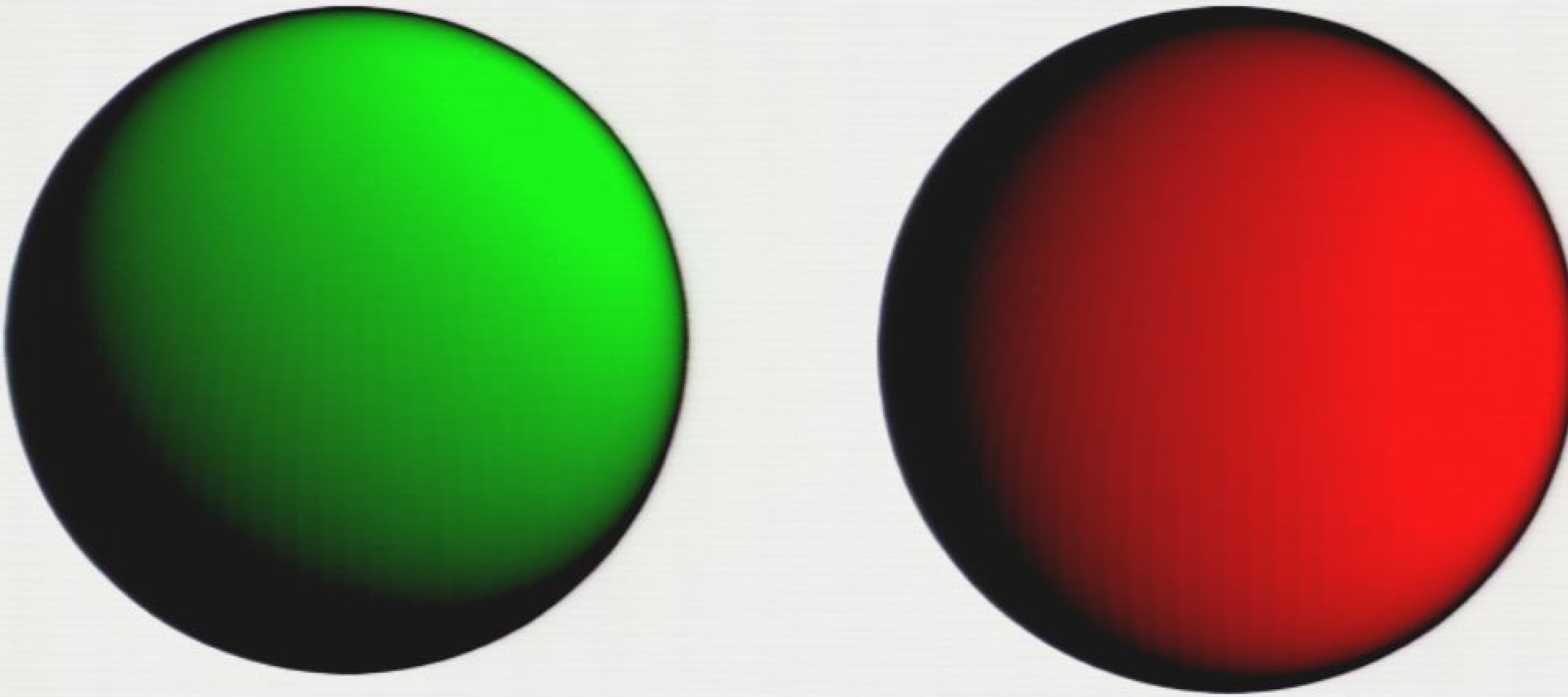
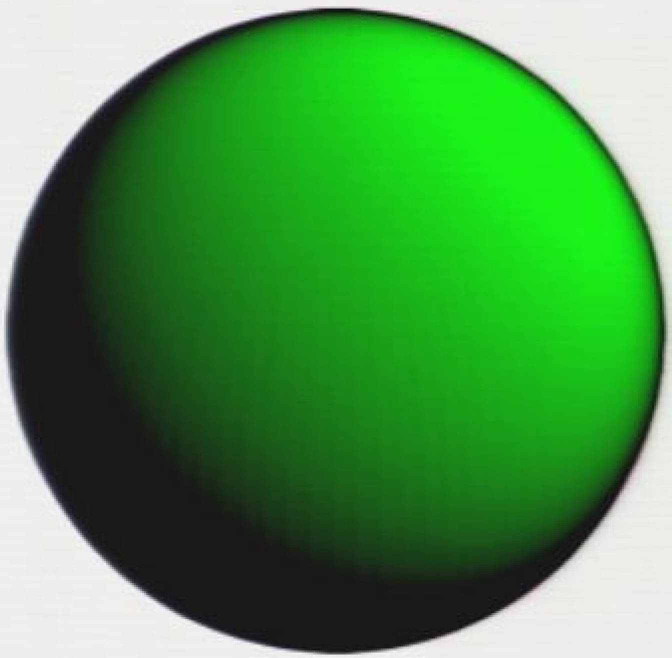


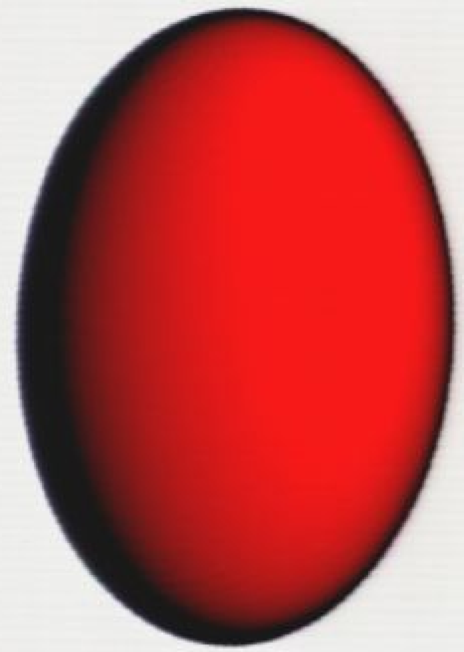
distance of each sheet (+ or -) from the origin equal to refractive index for isotropic materials (glass, water, air…) there is no direction-dependence, and the wave surfaces are identical spheres



for simple crystals (calcite (Iceland spar), cellophane…), where one direction is distinguished and the other two are the same, one surface is a sphere and the other is an ellipsoid

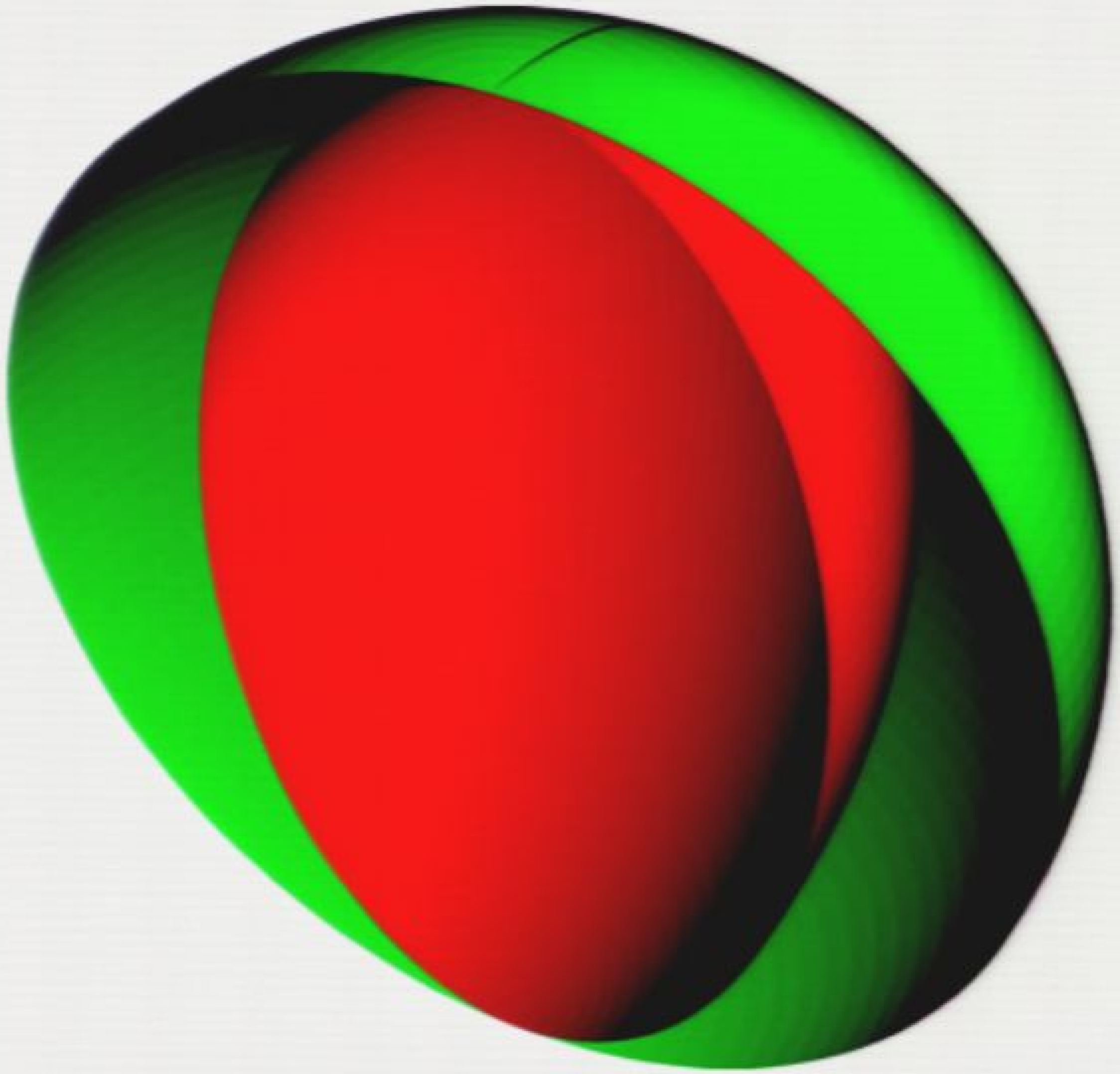


ordinary wave



extraordinary wave

the surfaces touch at two points (directions), on the *optic axis*



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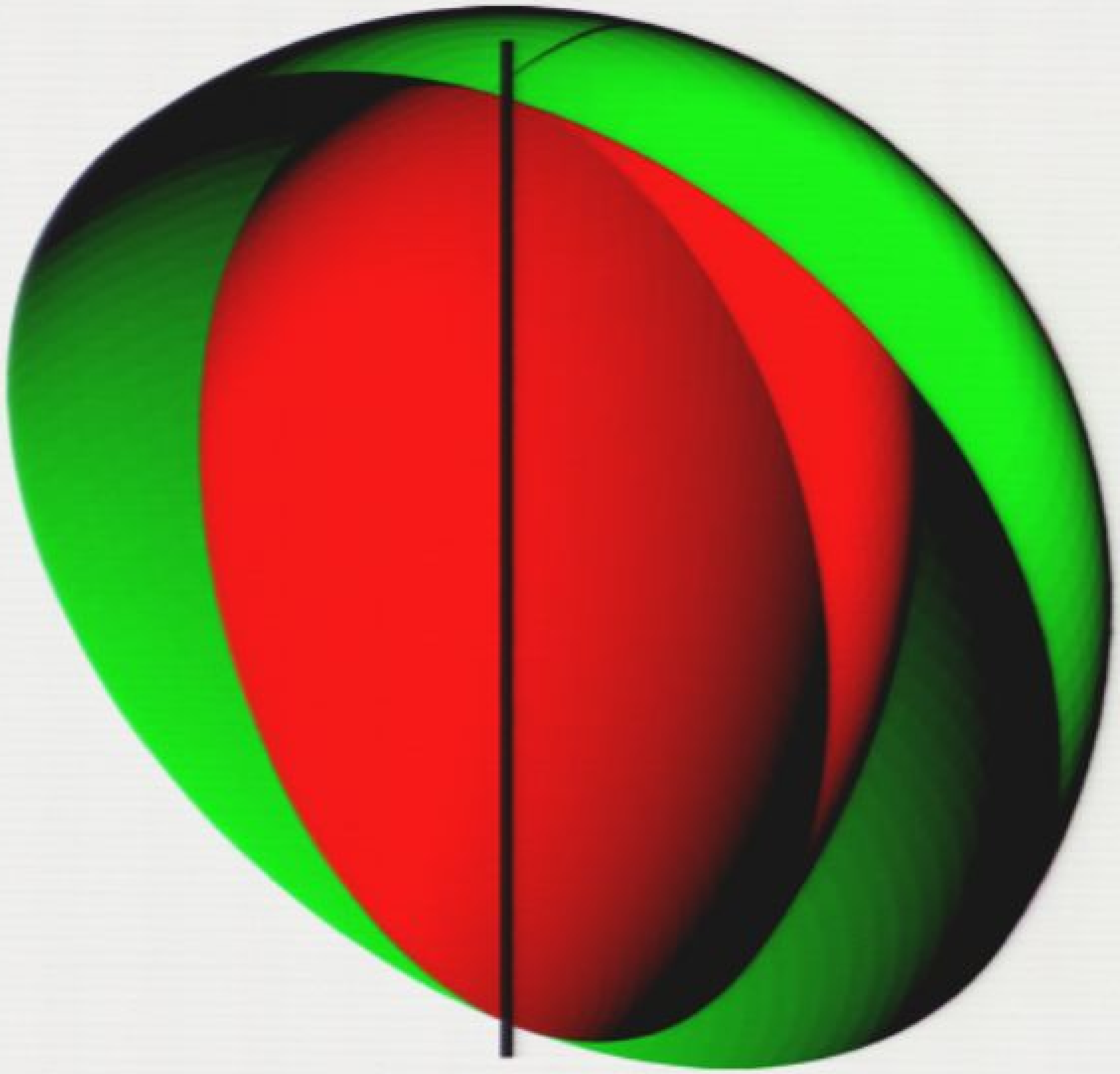

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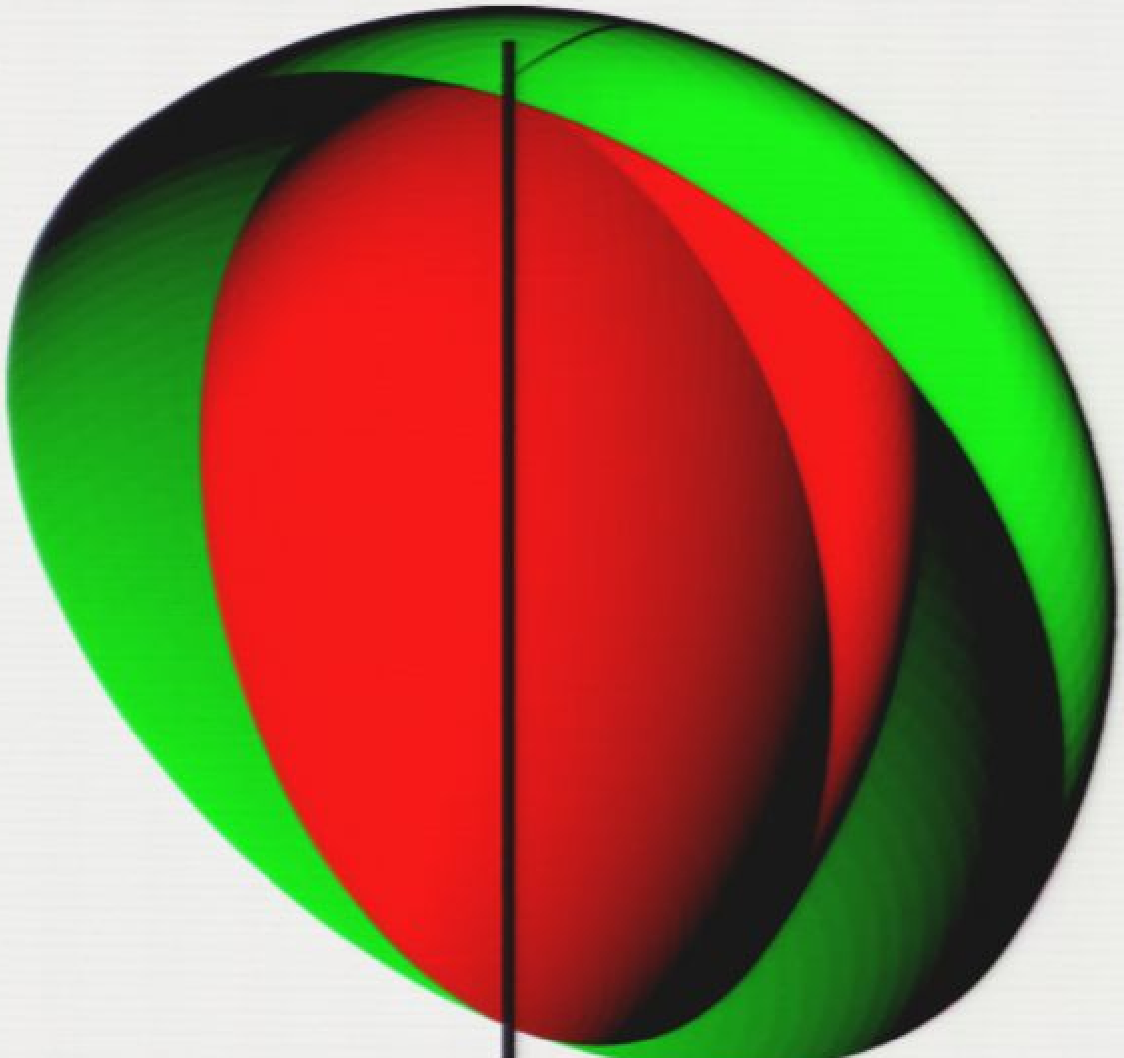

one axis: uniaxial crystal

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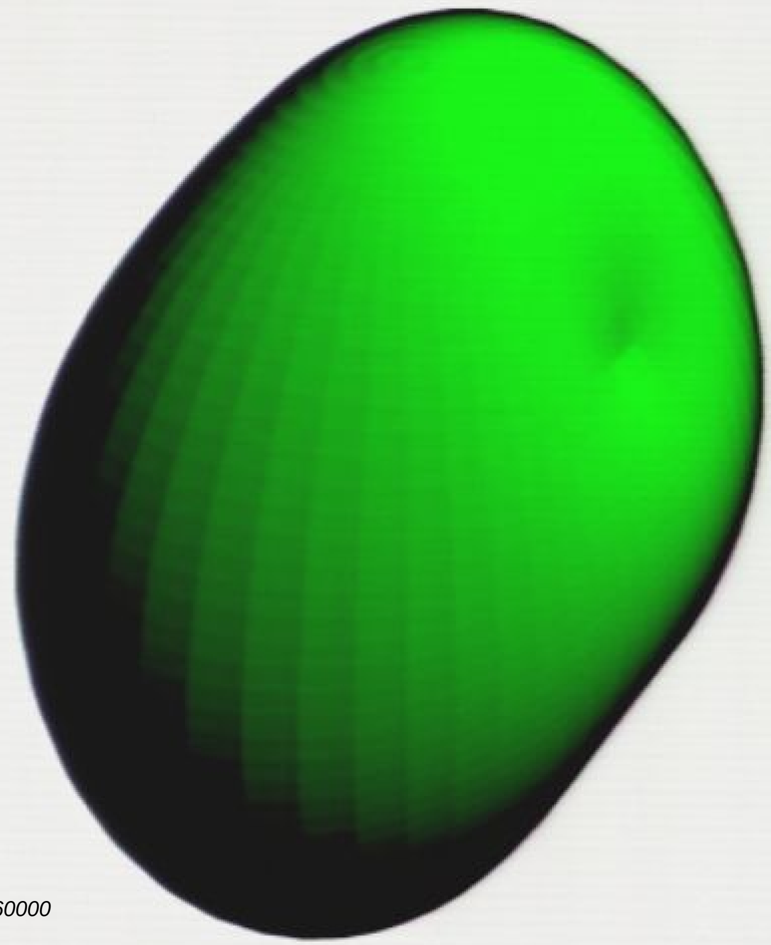
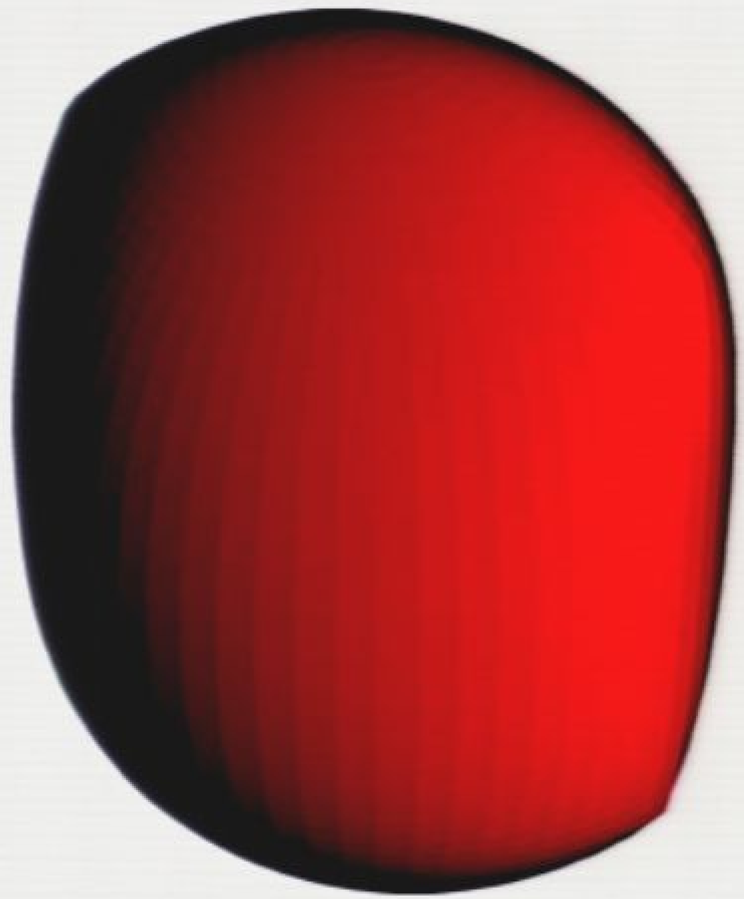
dimpled and bumpy wave surfaces





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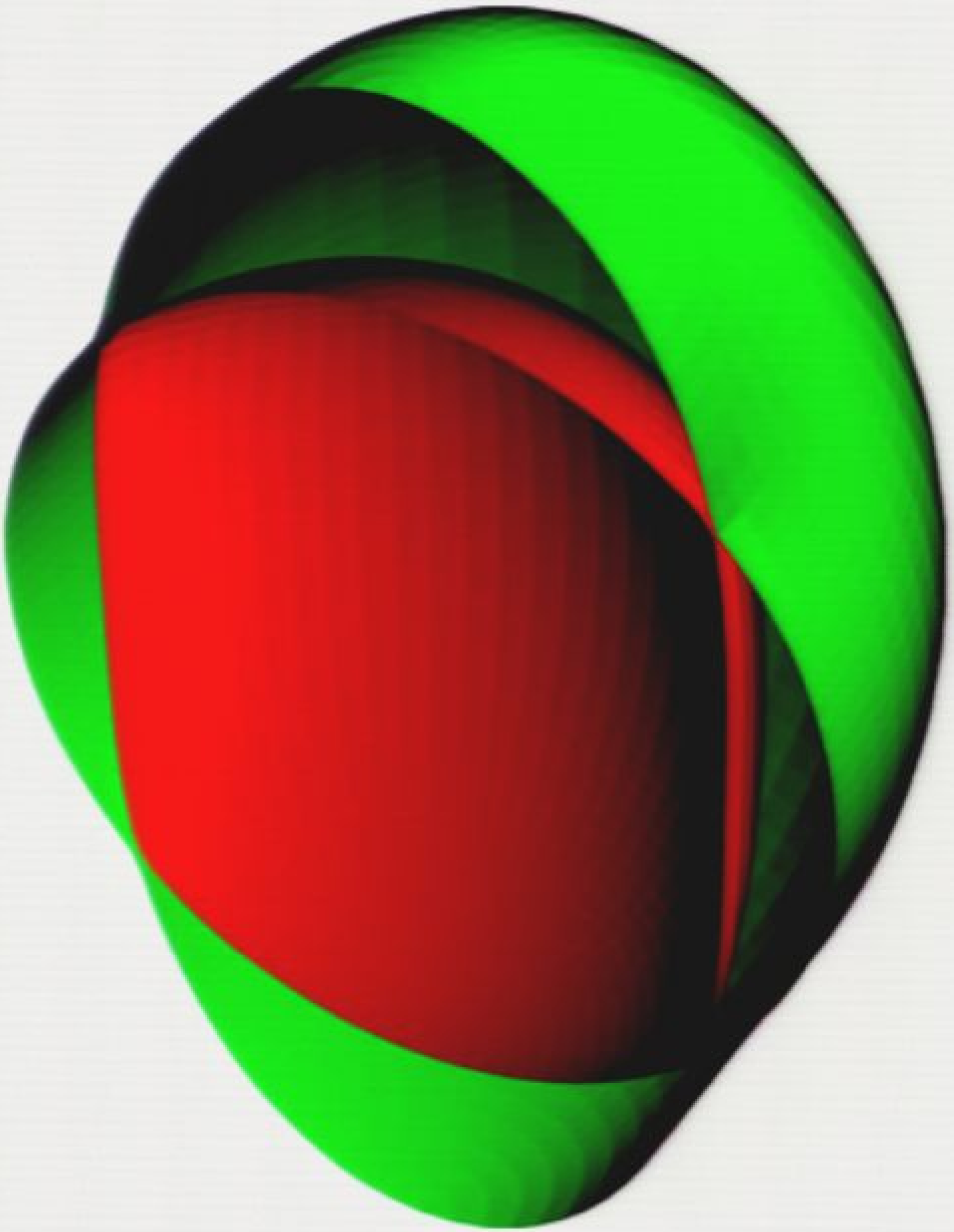

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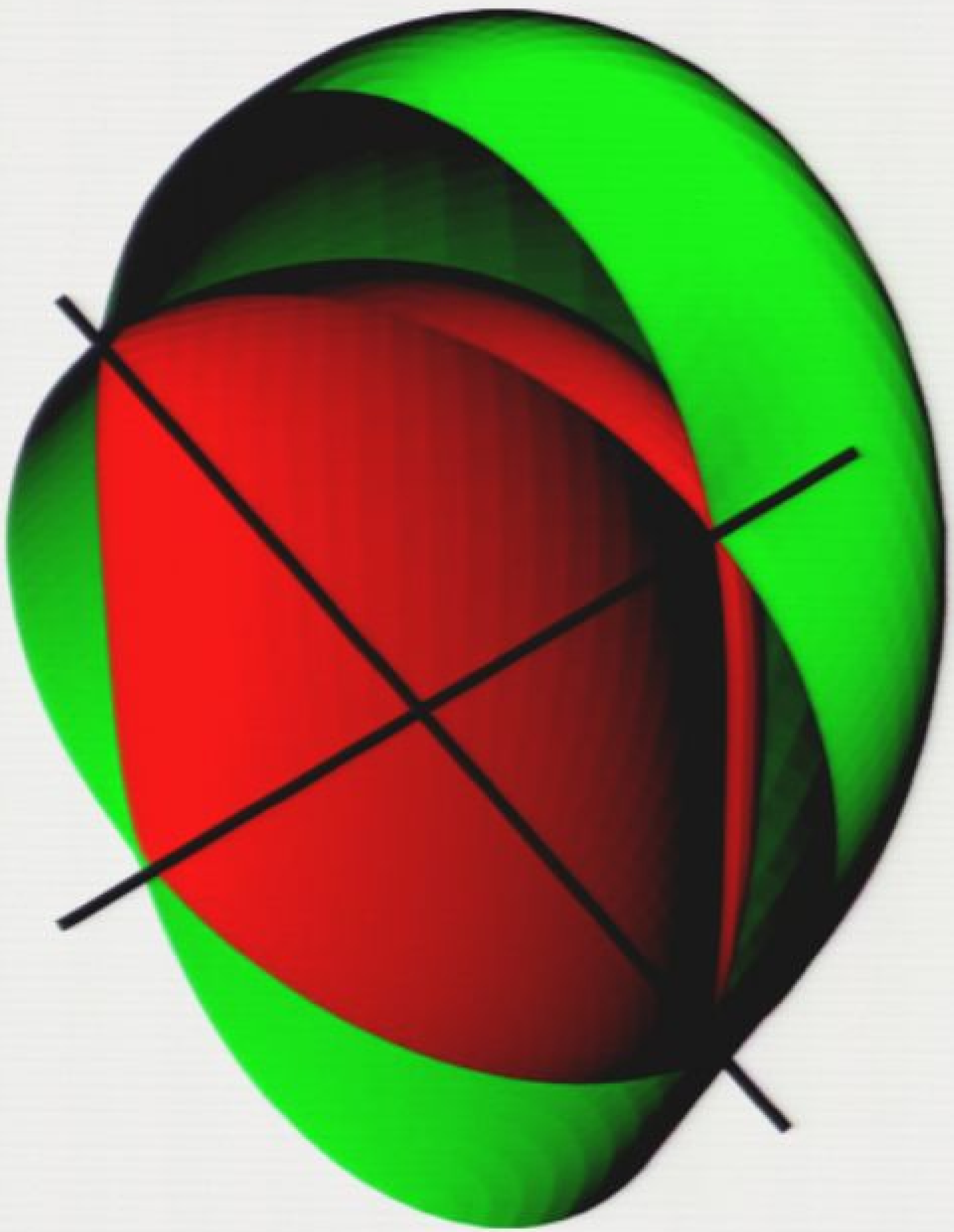
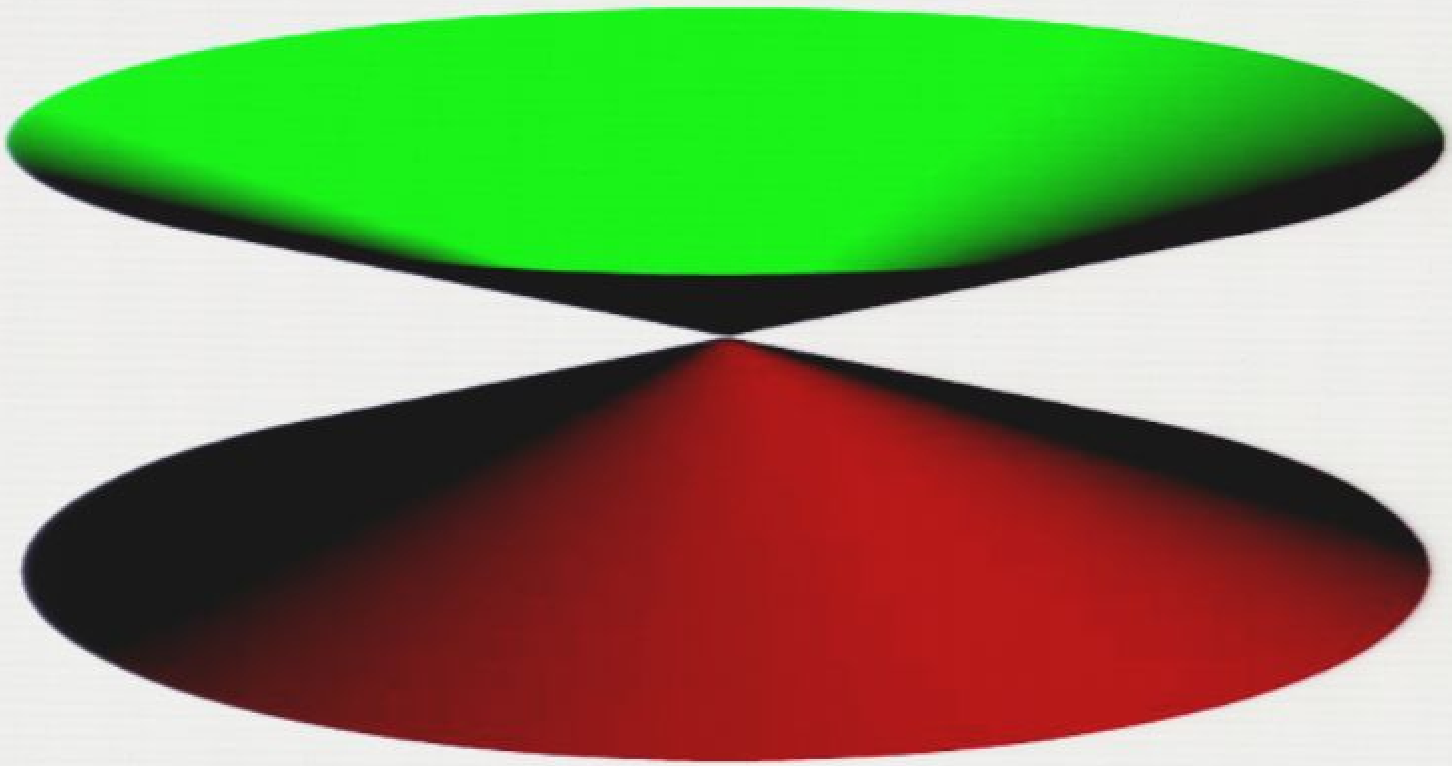


two axes:
biaxial crystal

each intersection is a ***double cone - a diabolo***



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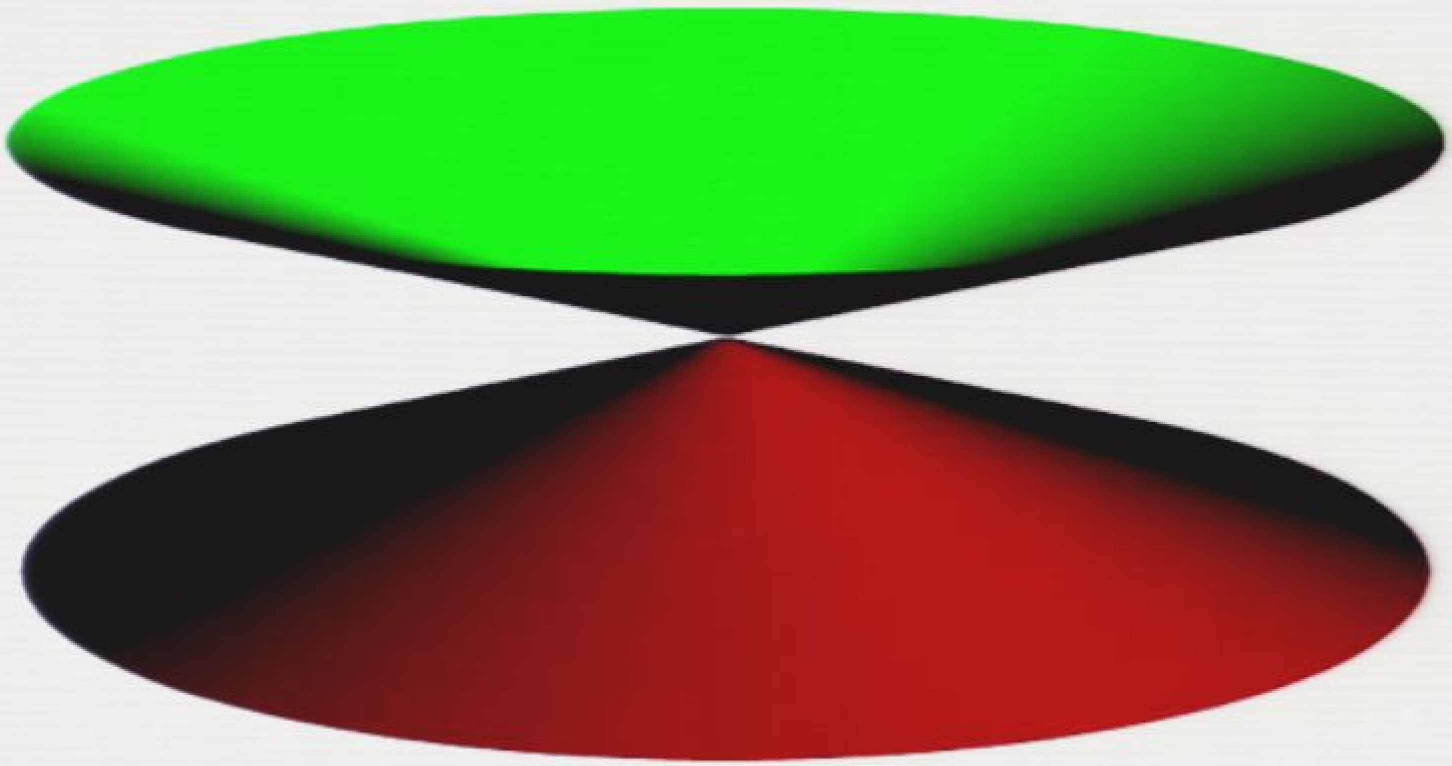

Hamilton's diabolical point

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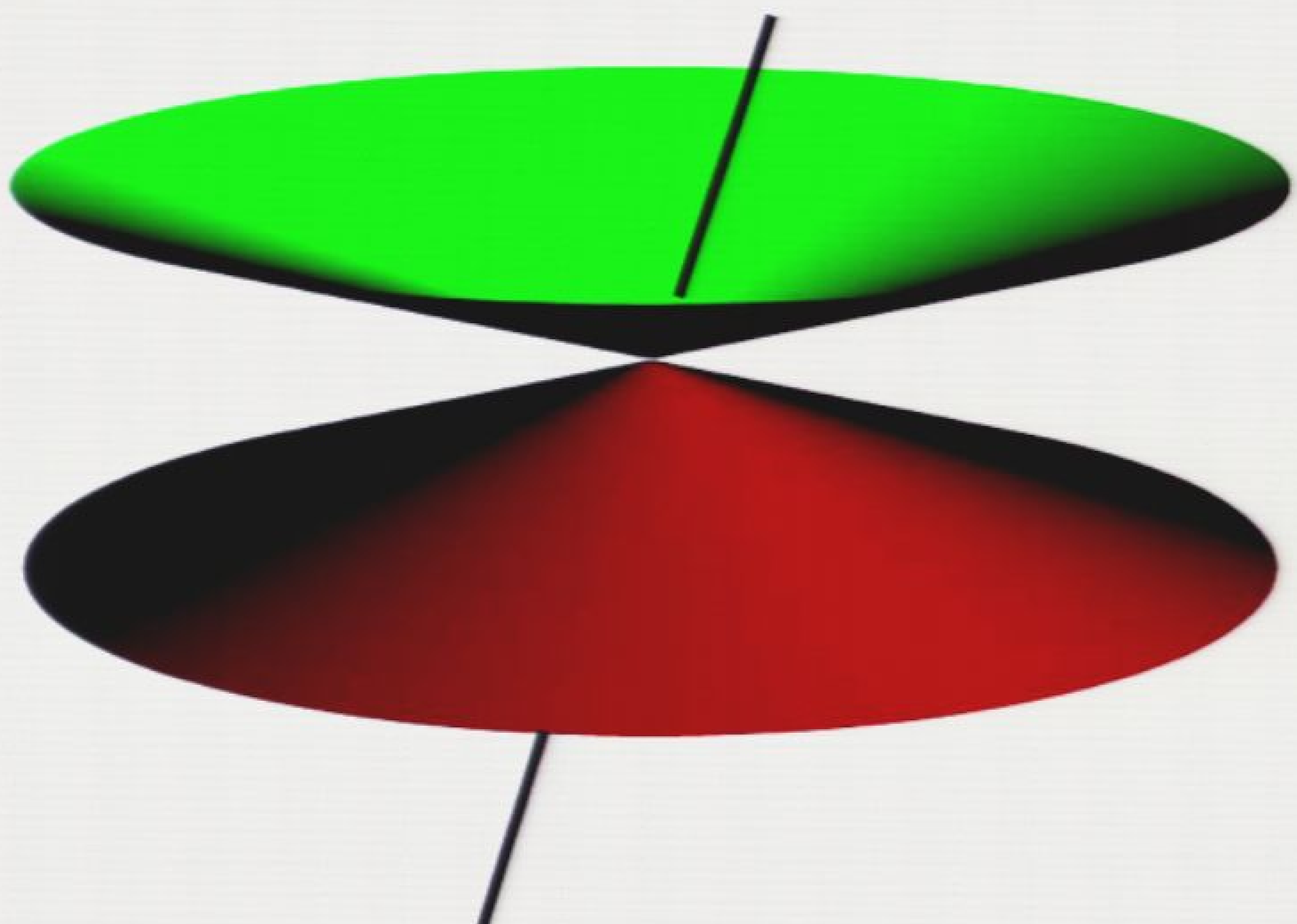

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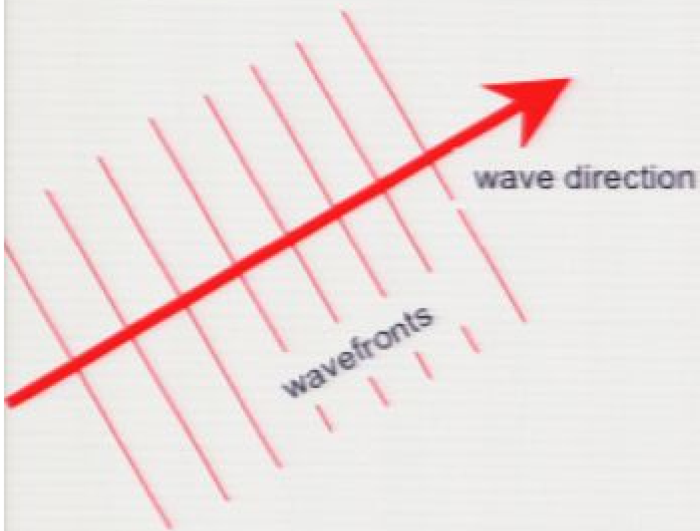

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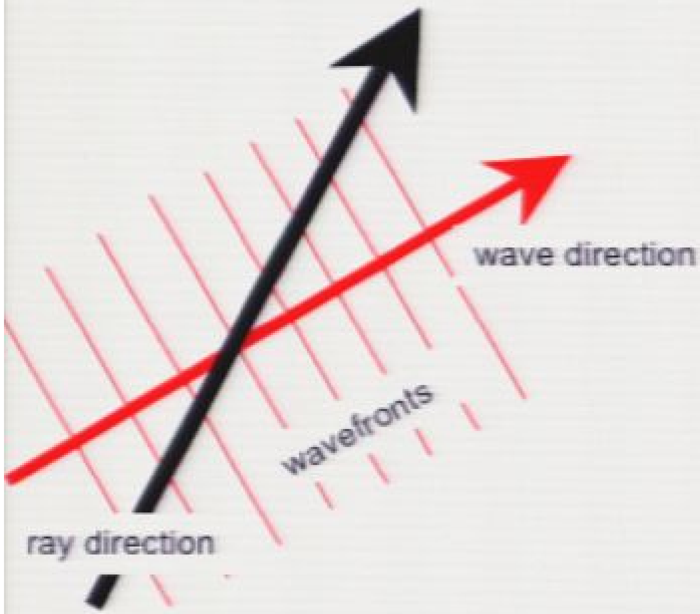

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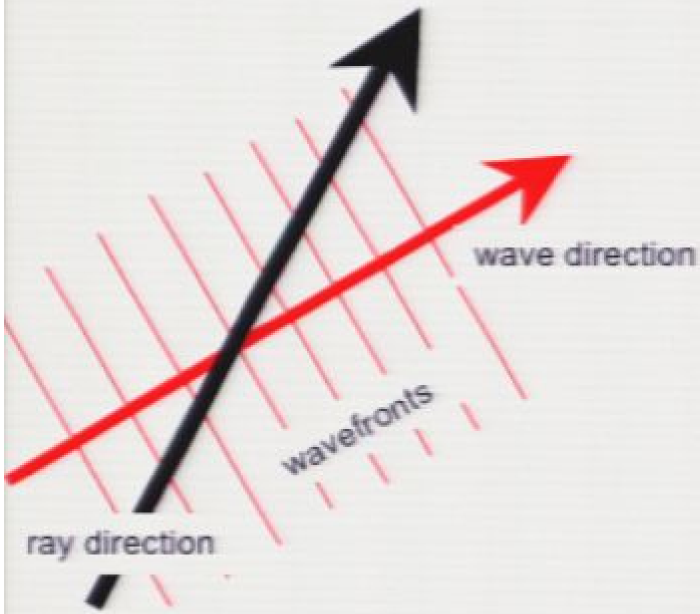

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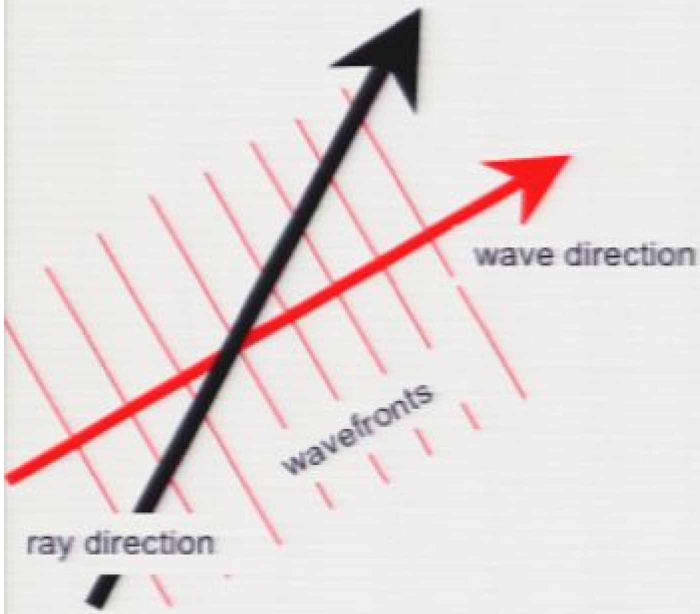

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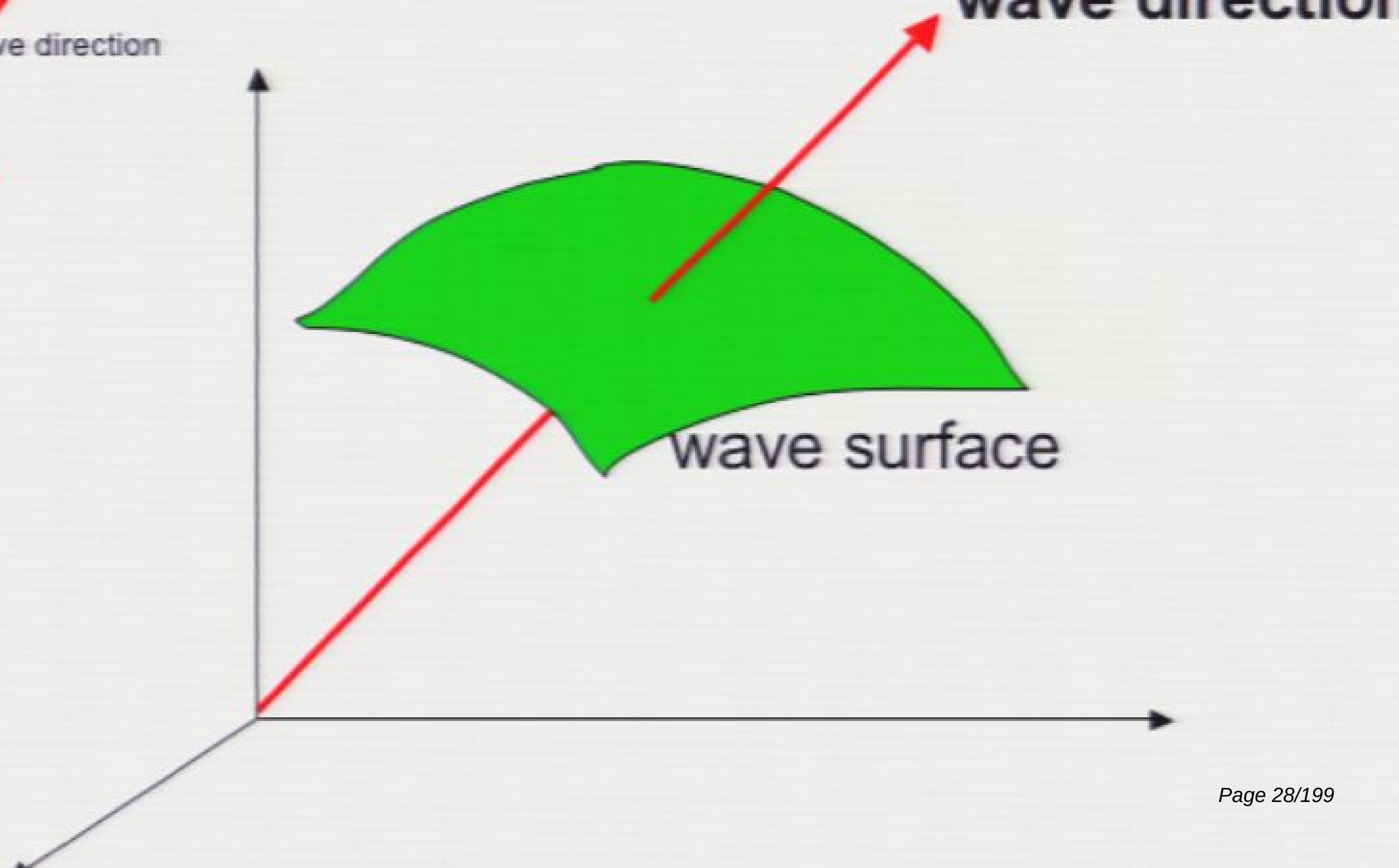

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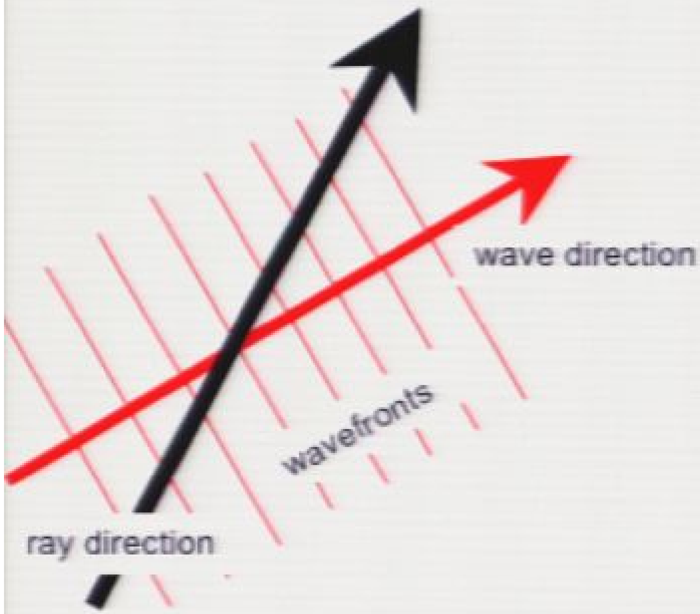

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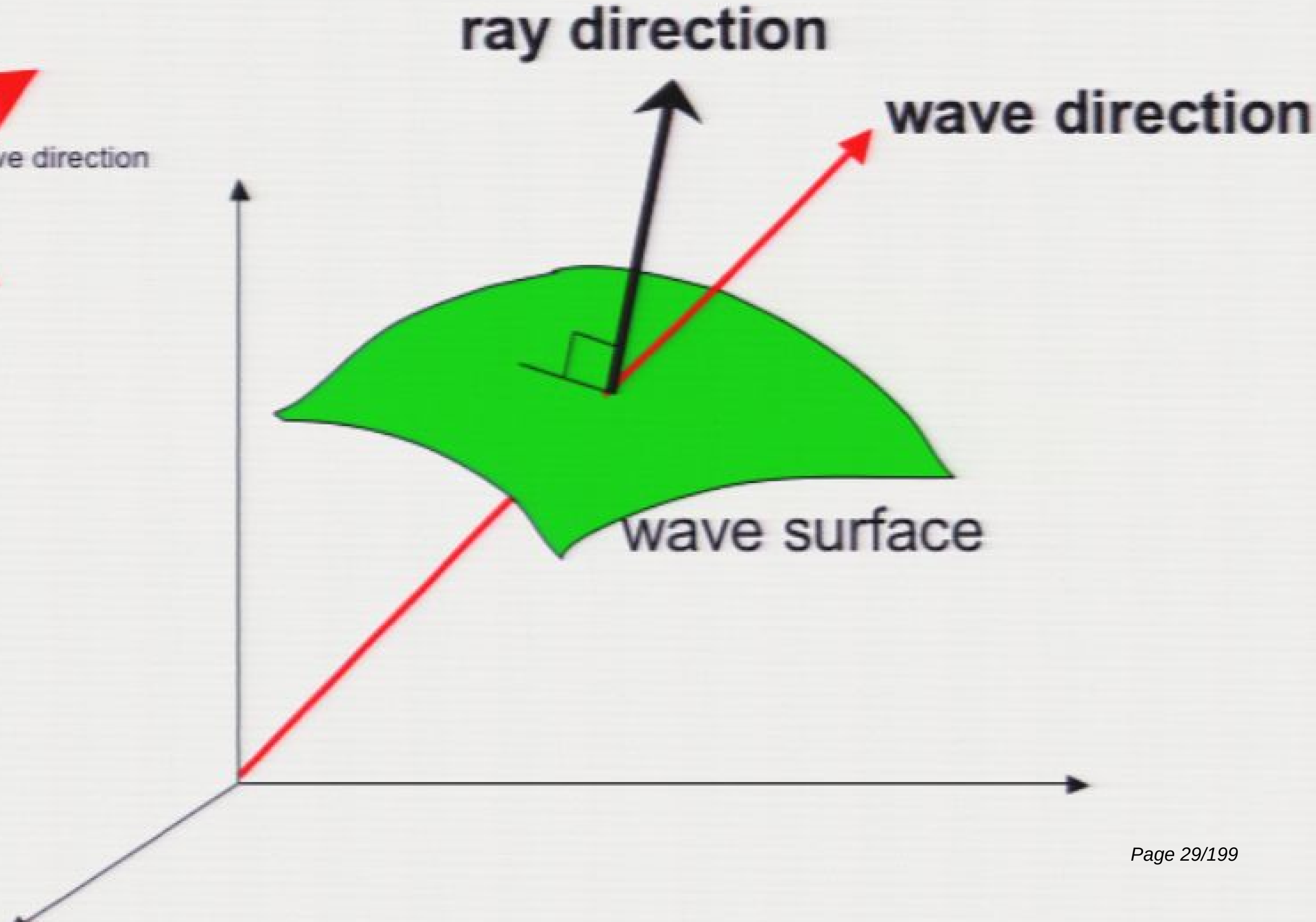

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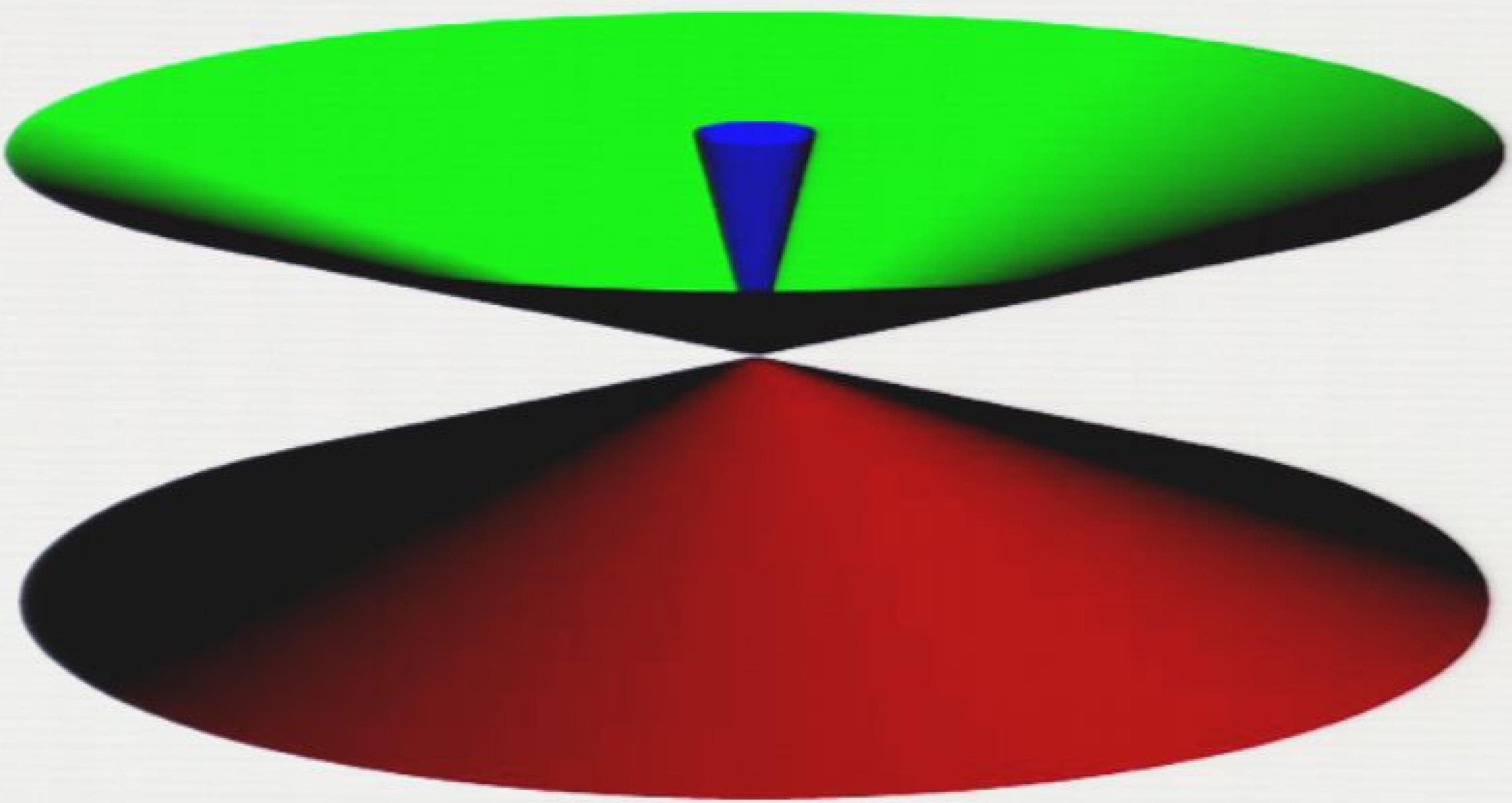

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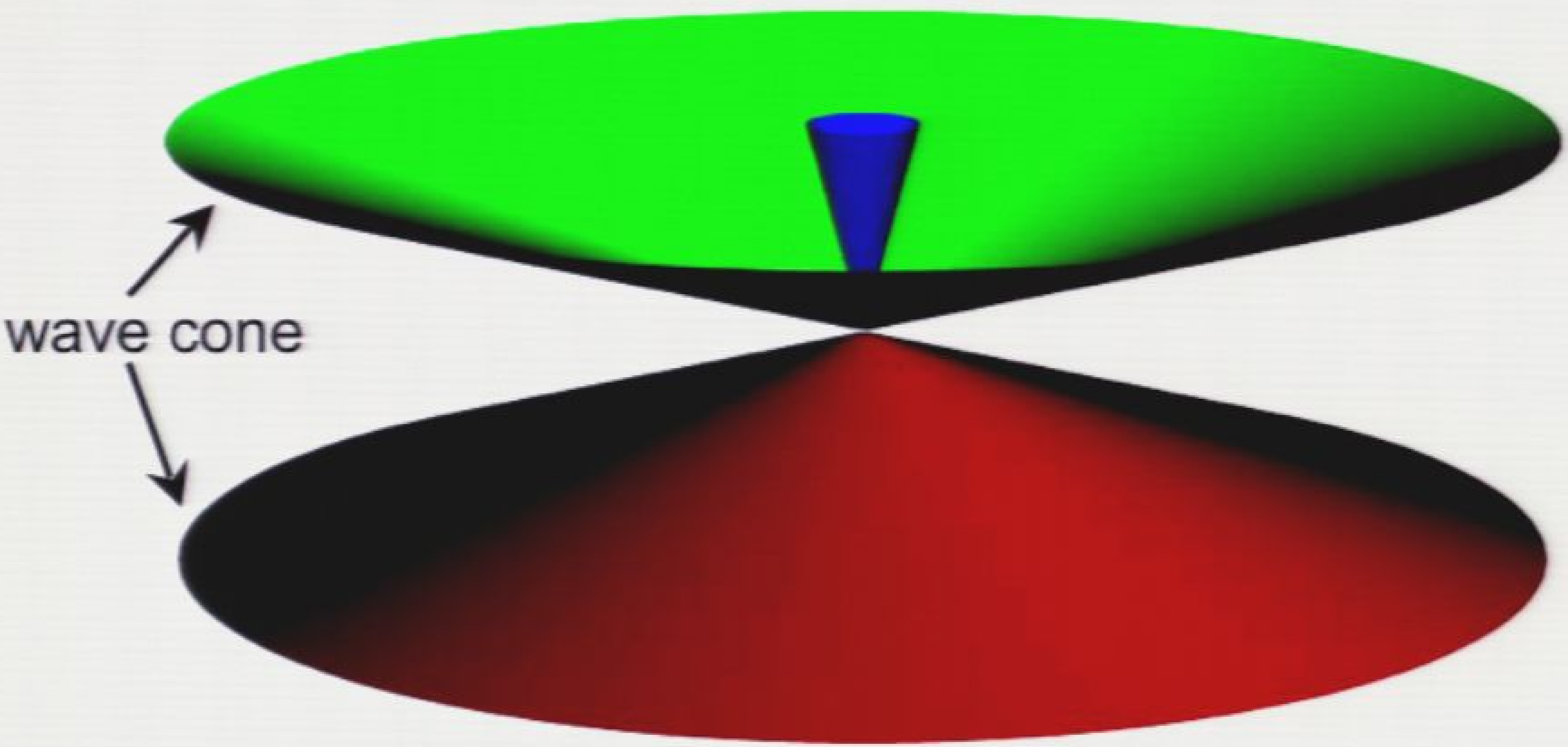

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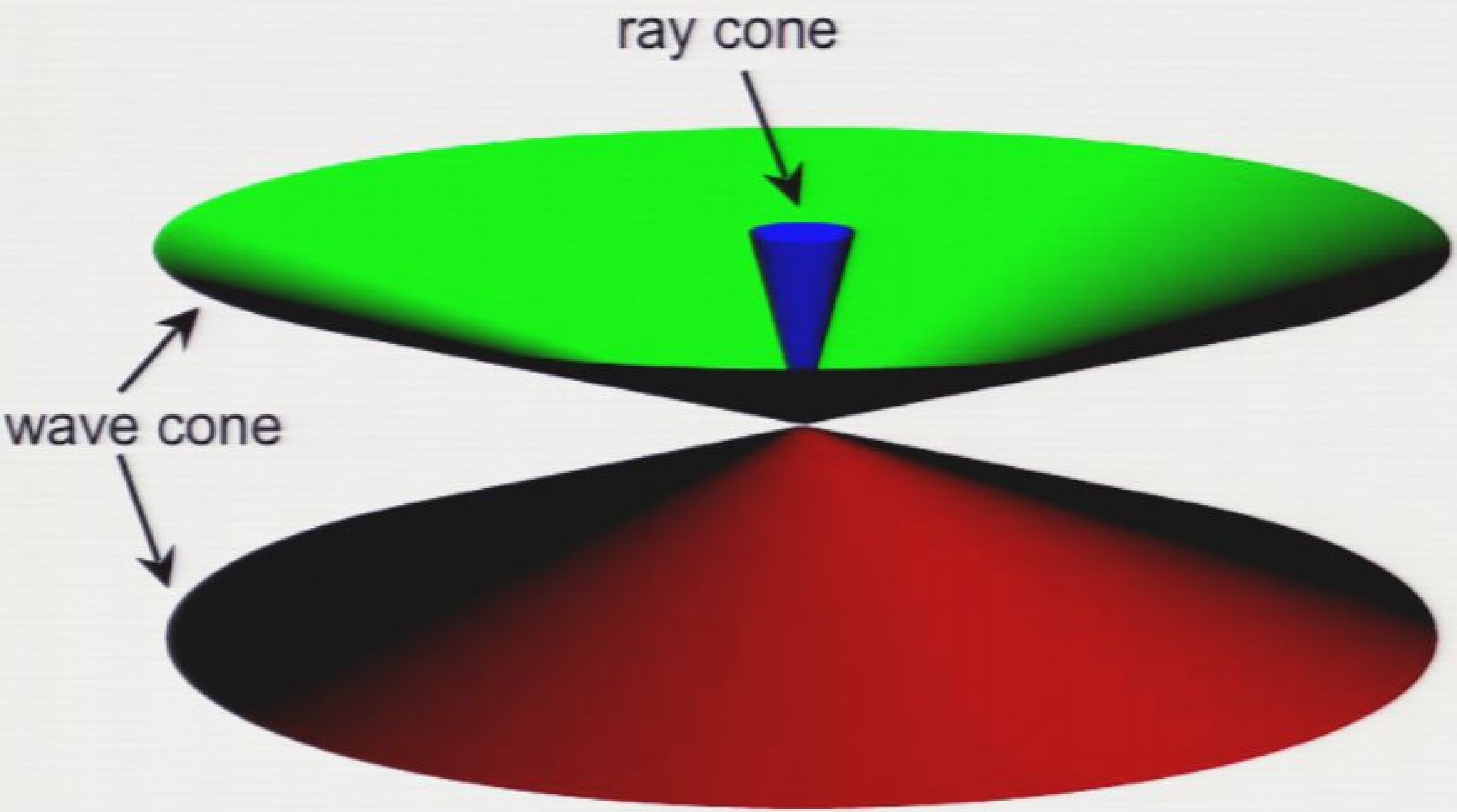

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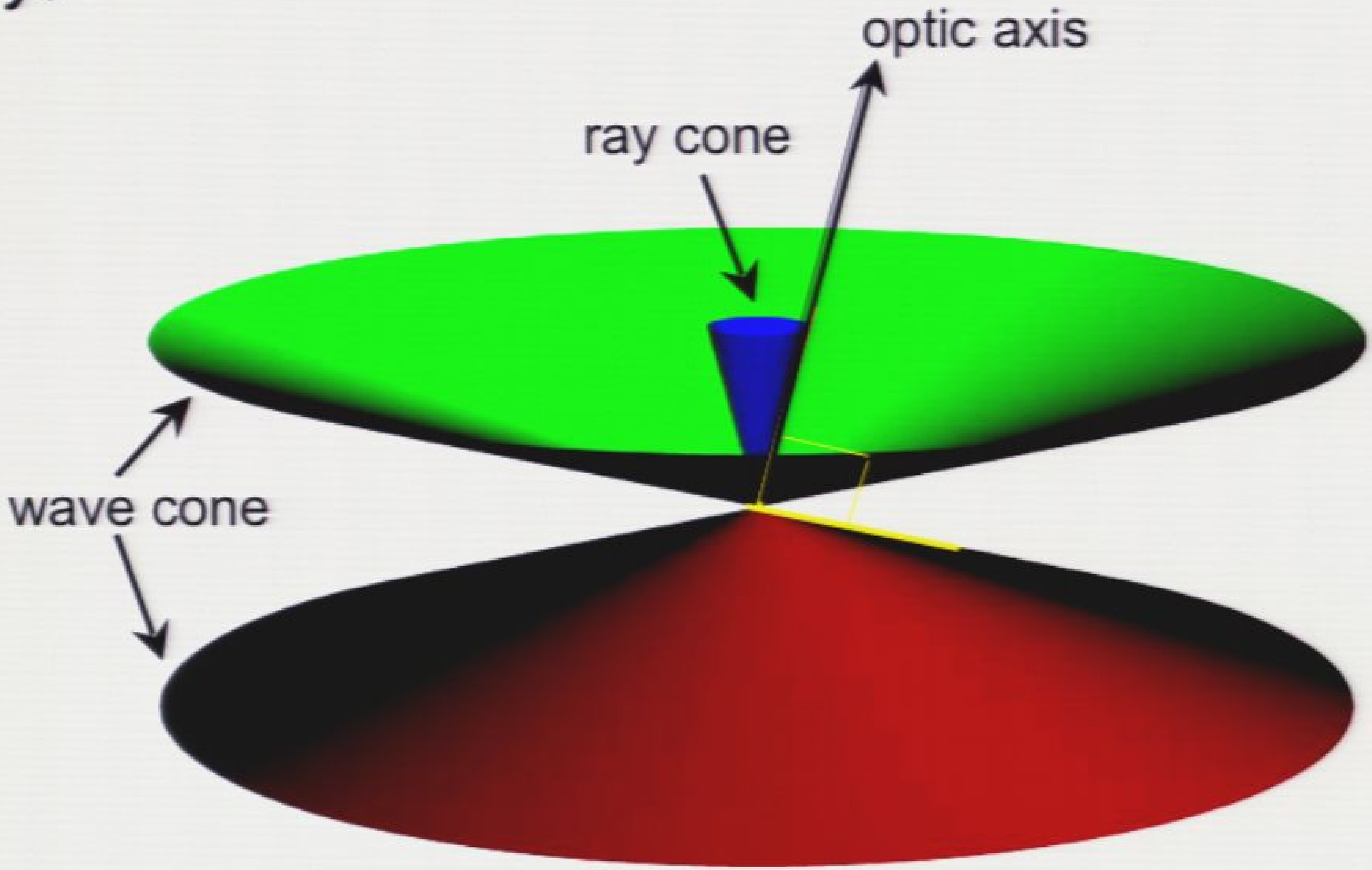

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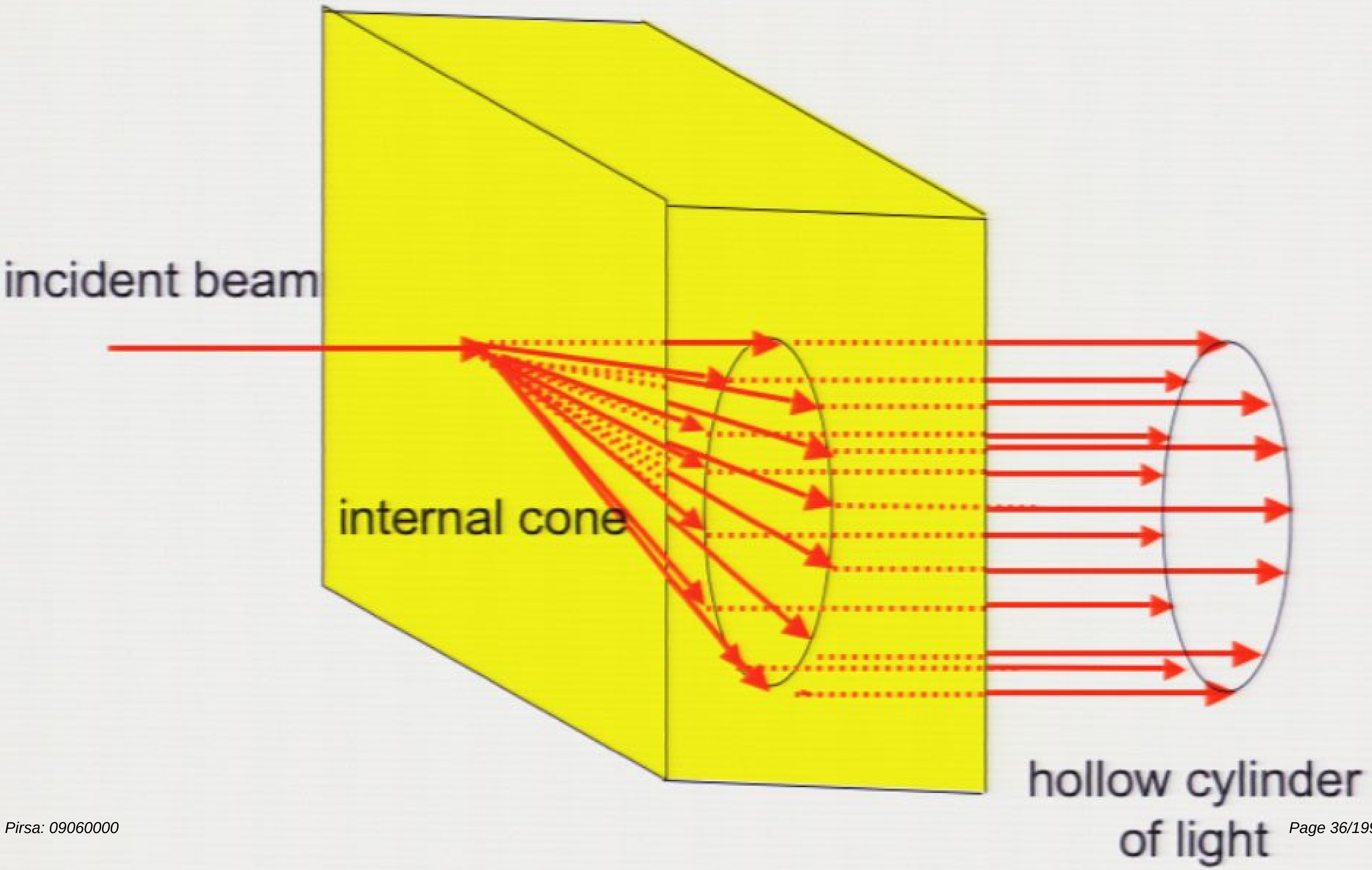

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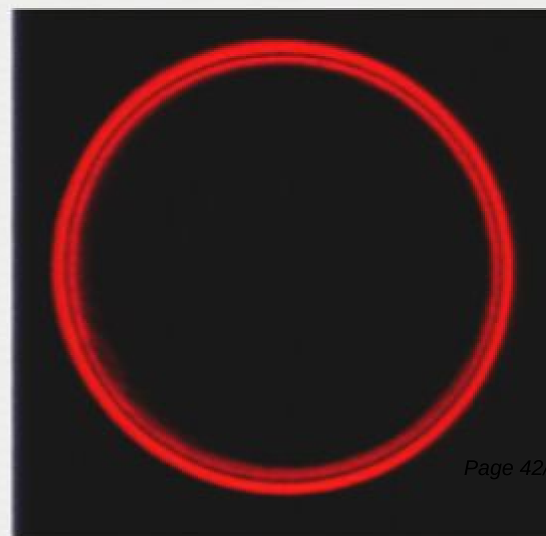

double refraction

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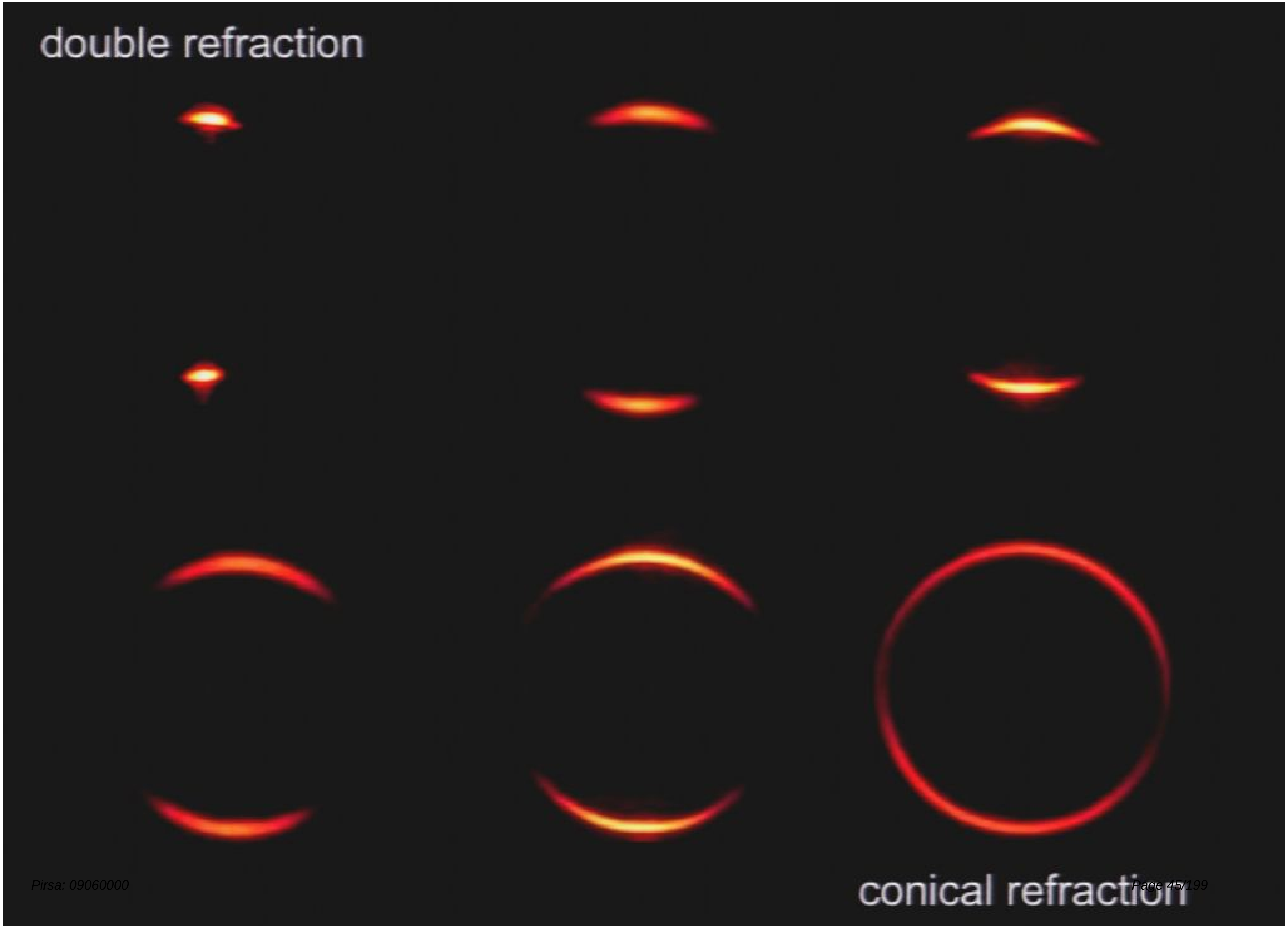
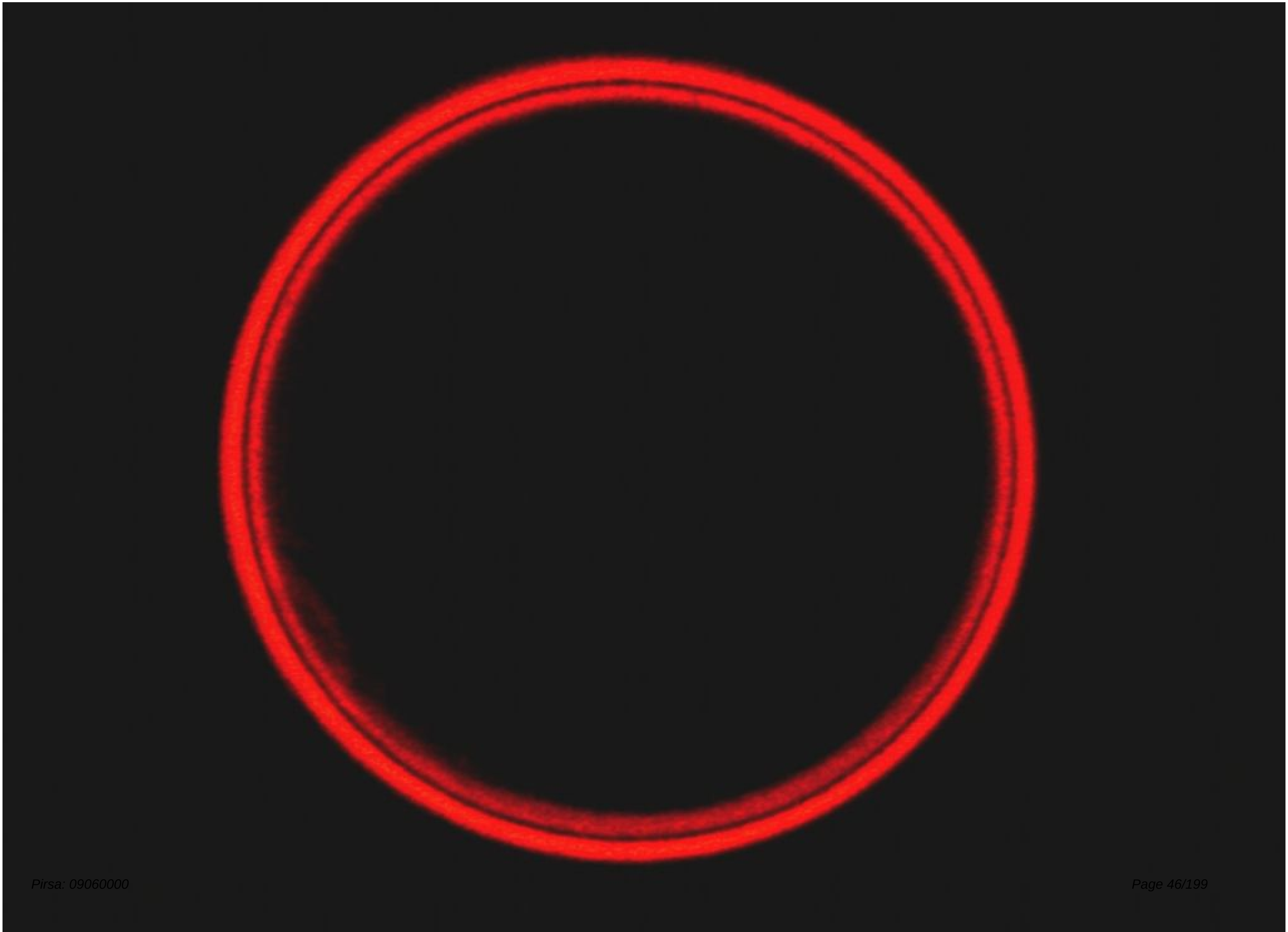

double refraction
conical refraction



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explanation 65 (!) years later, by Waldemar Voigt in 1905

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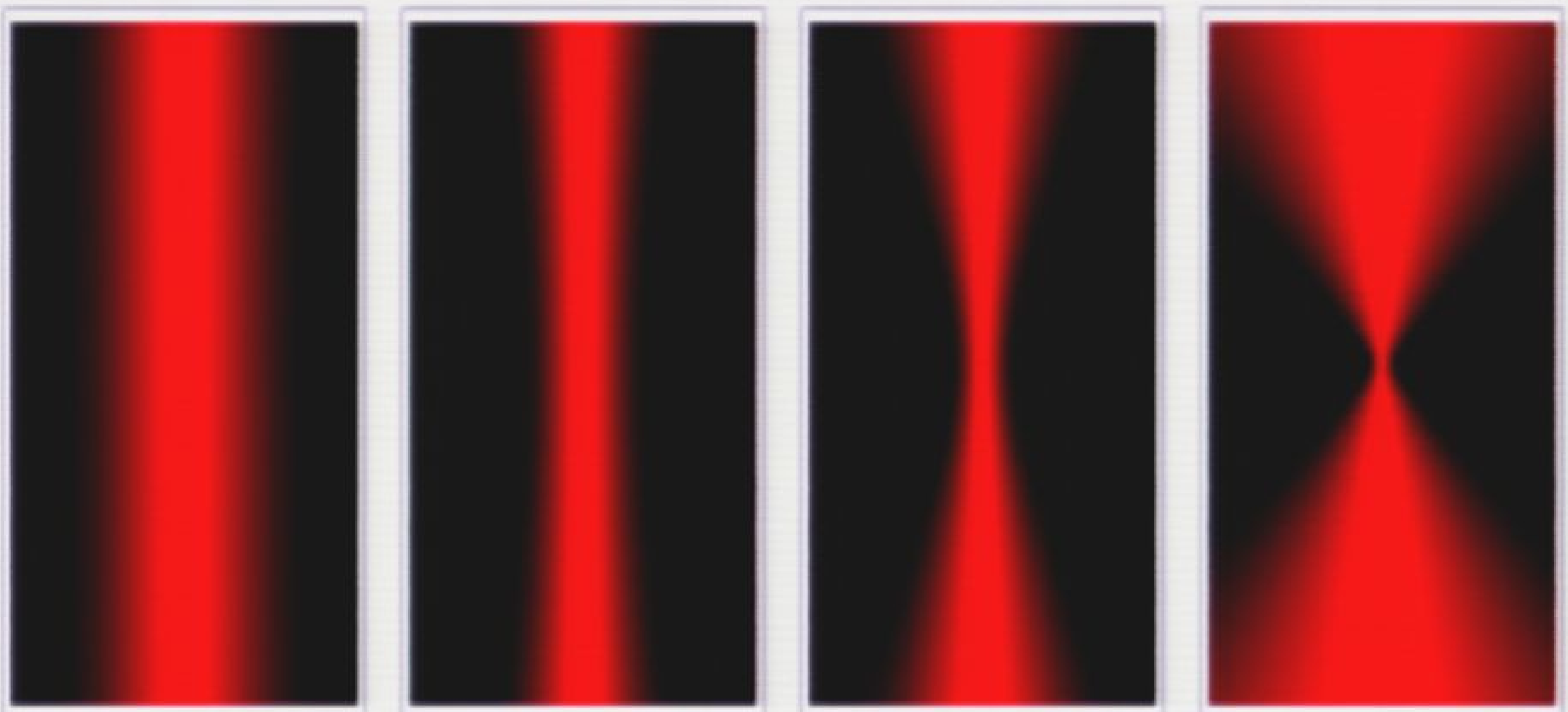

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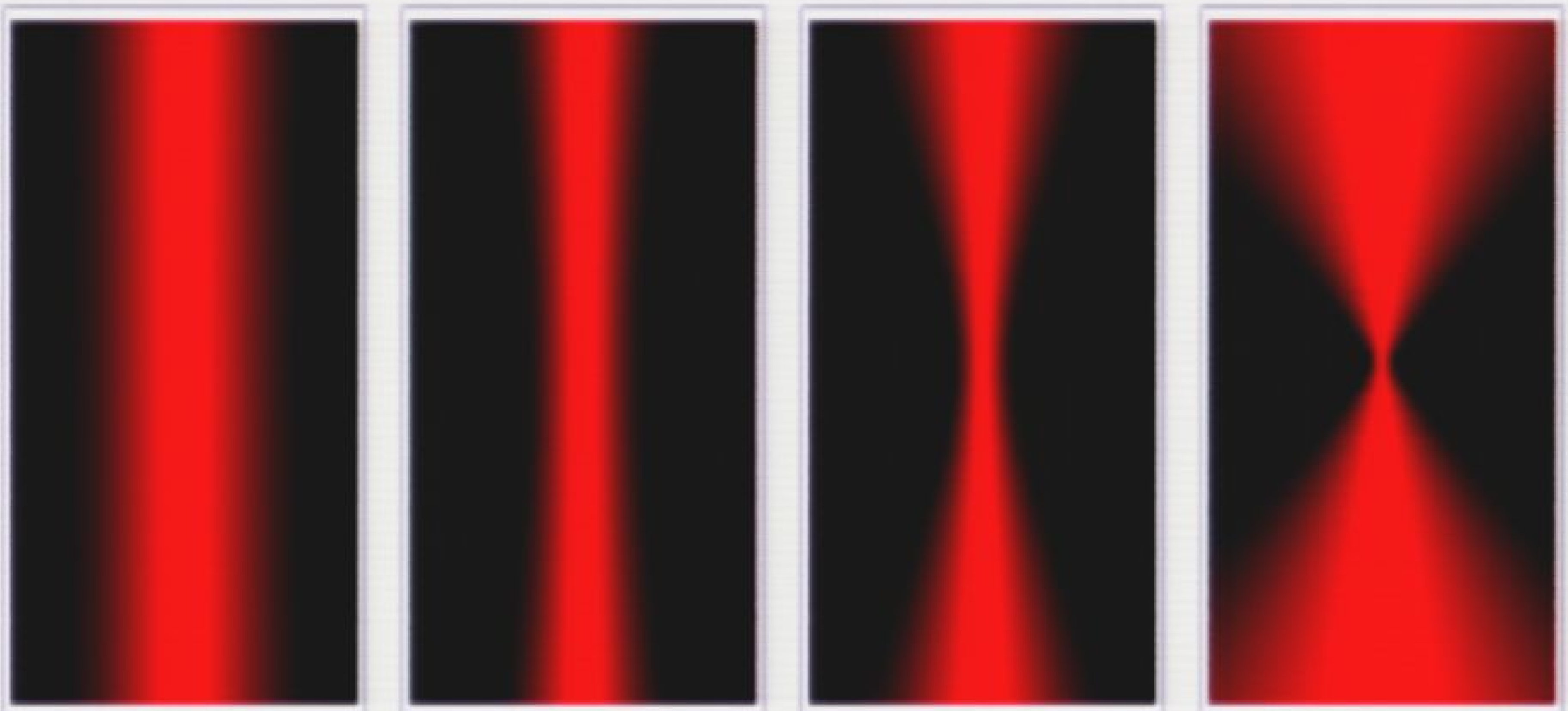

(cf: Heisenberg uncertainty principle)

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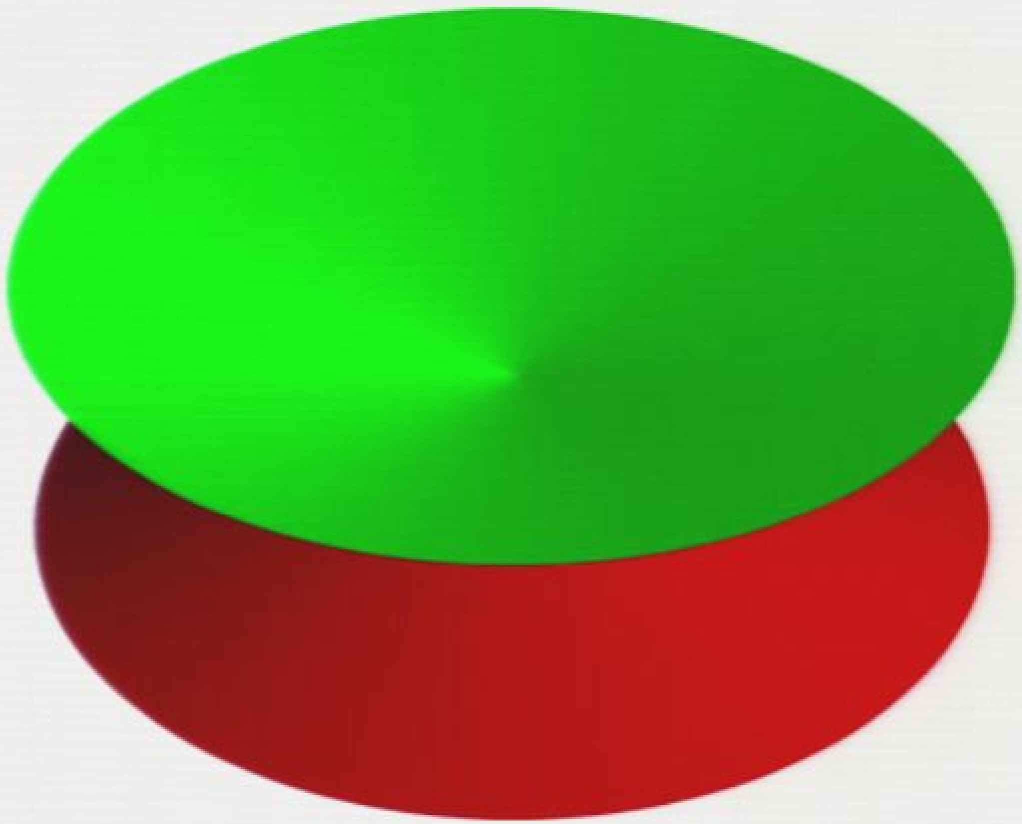

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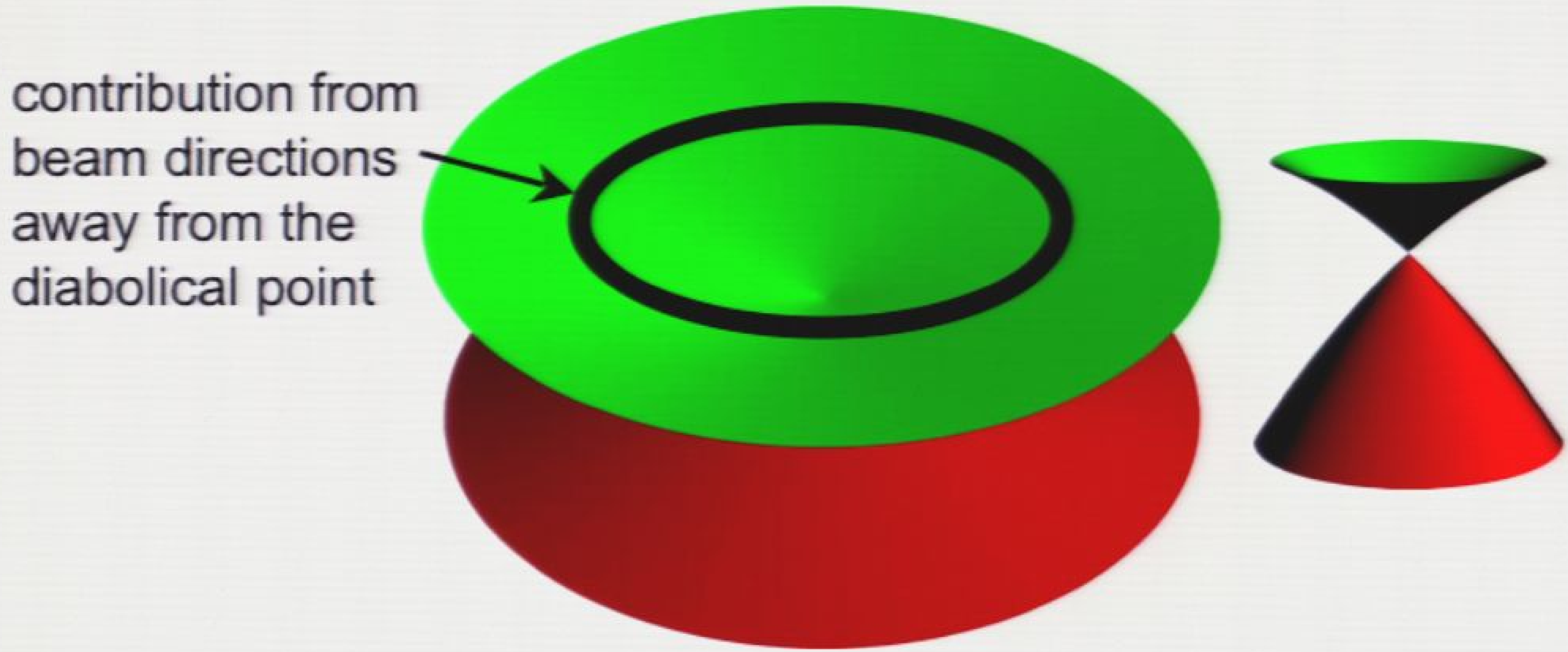

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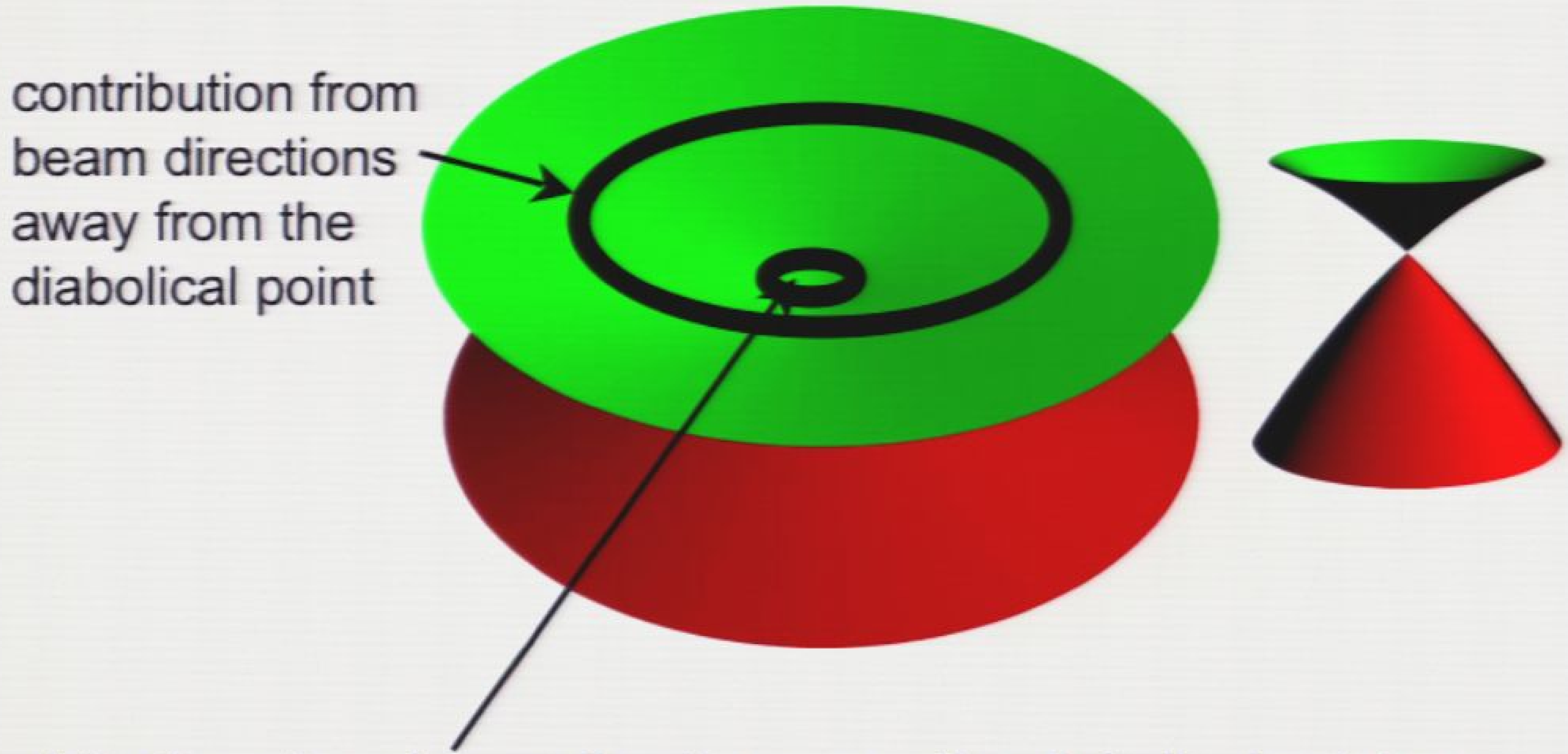

contributions from beam directions near the diabolical point are smaller, and vanish at the point itself - hence Poggendorff's dark ring at the cone direction

half-turn of polarization direction around the rings

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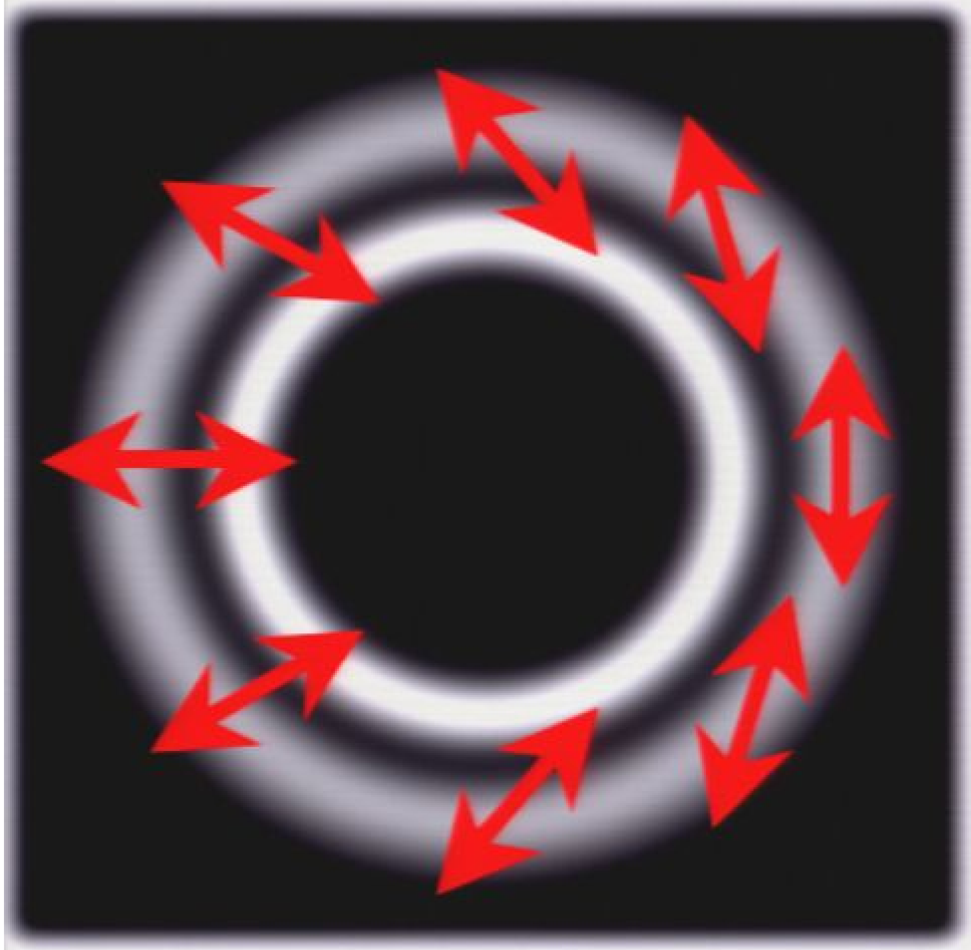

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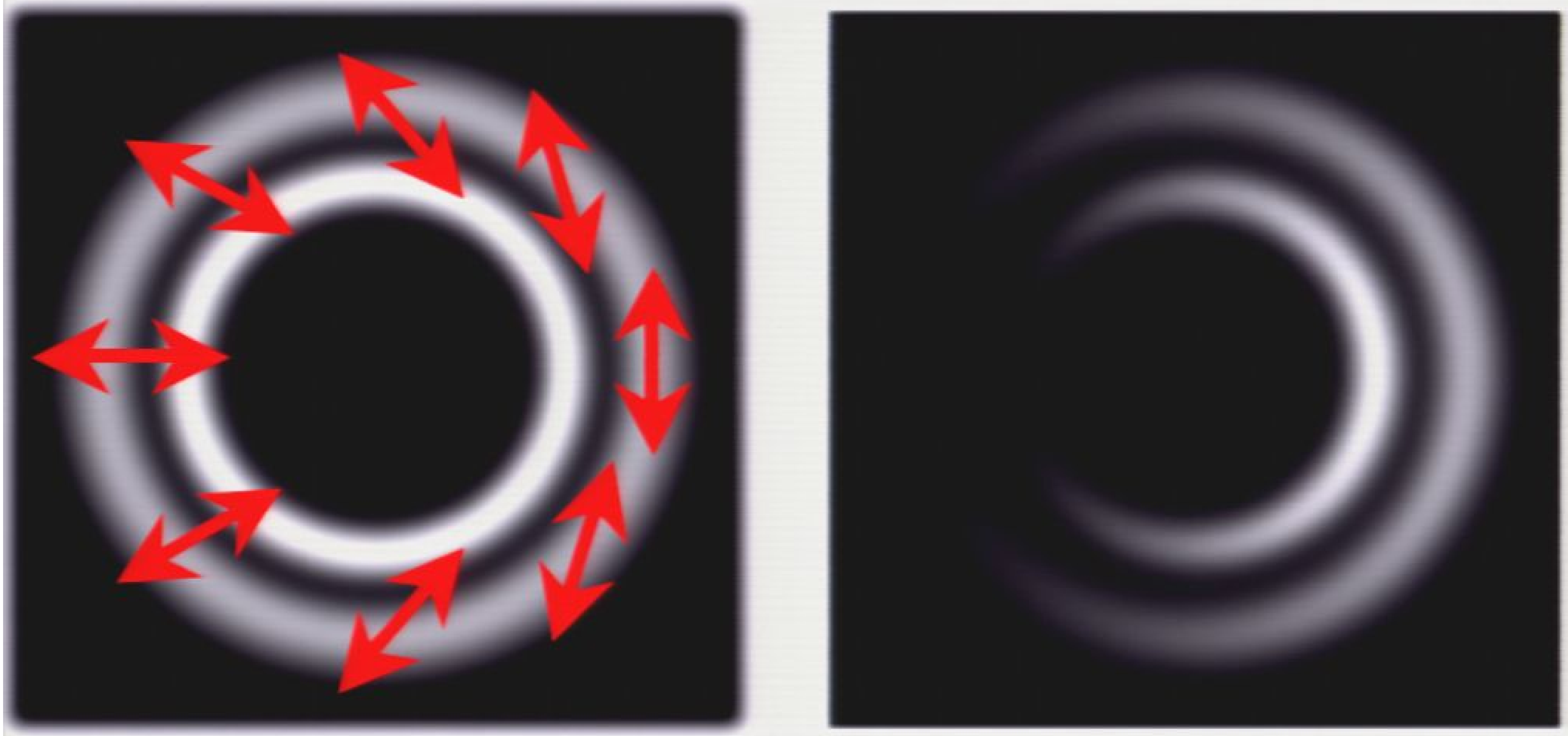

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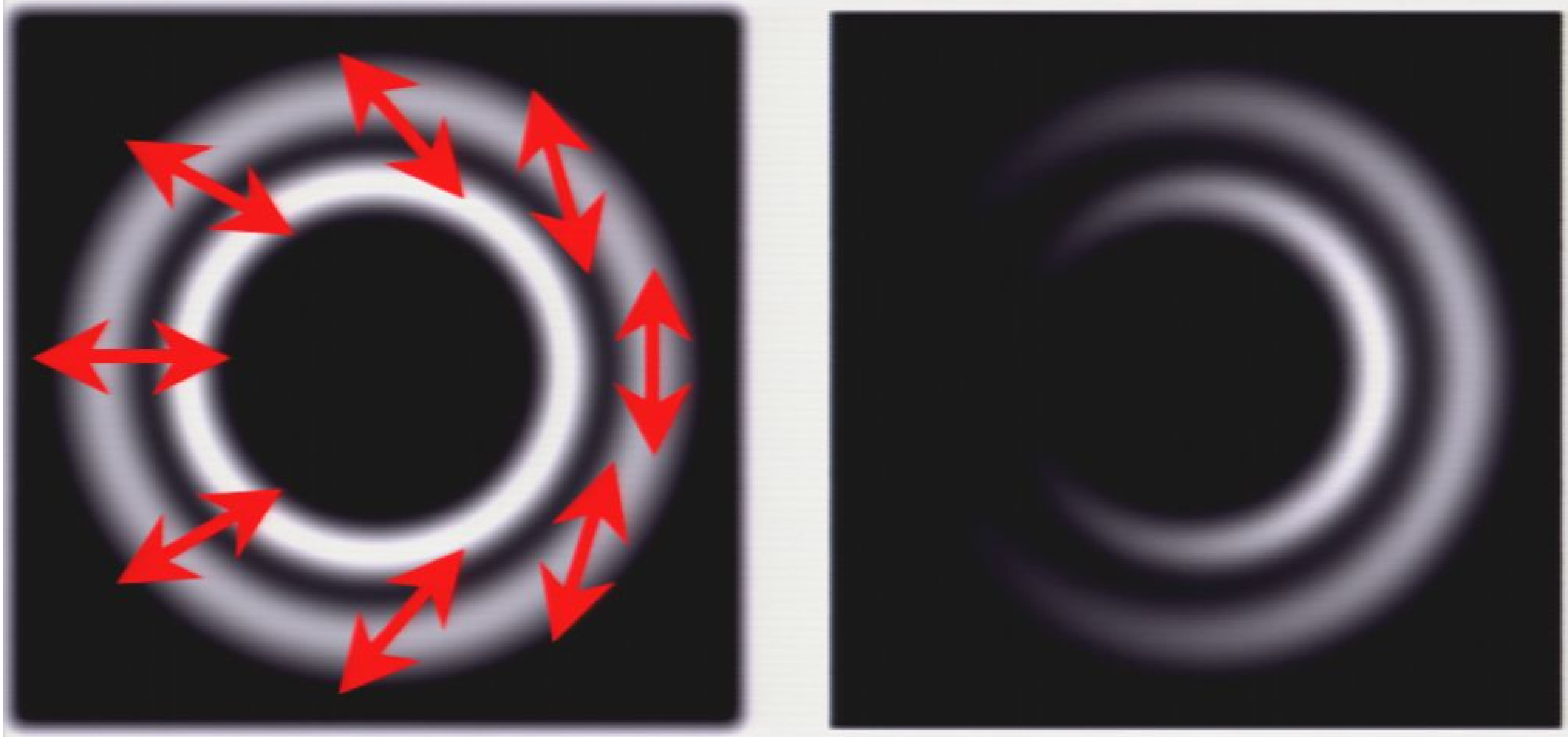
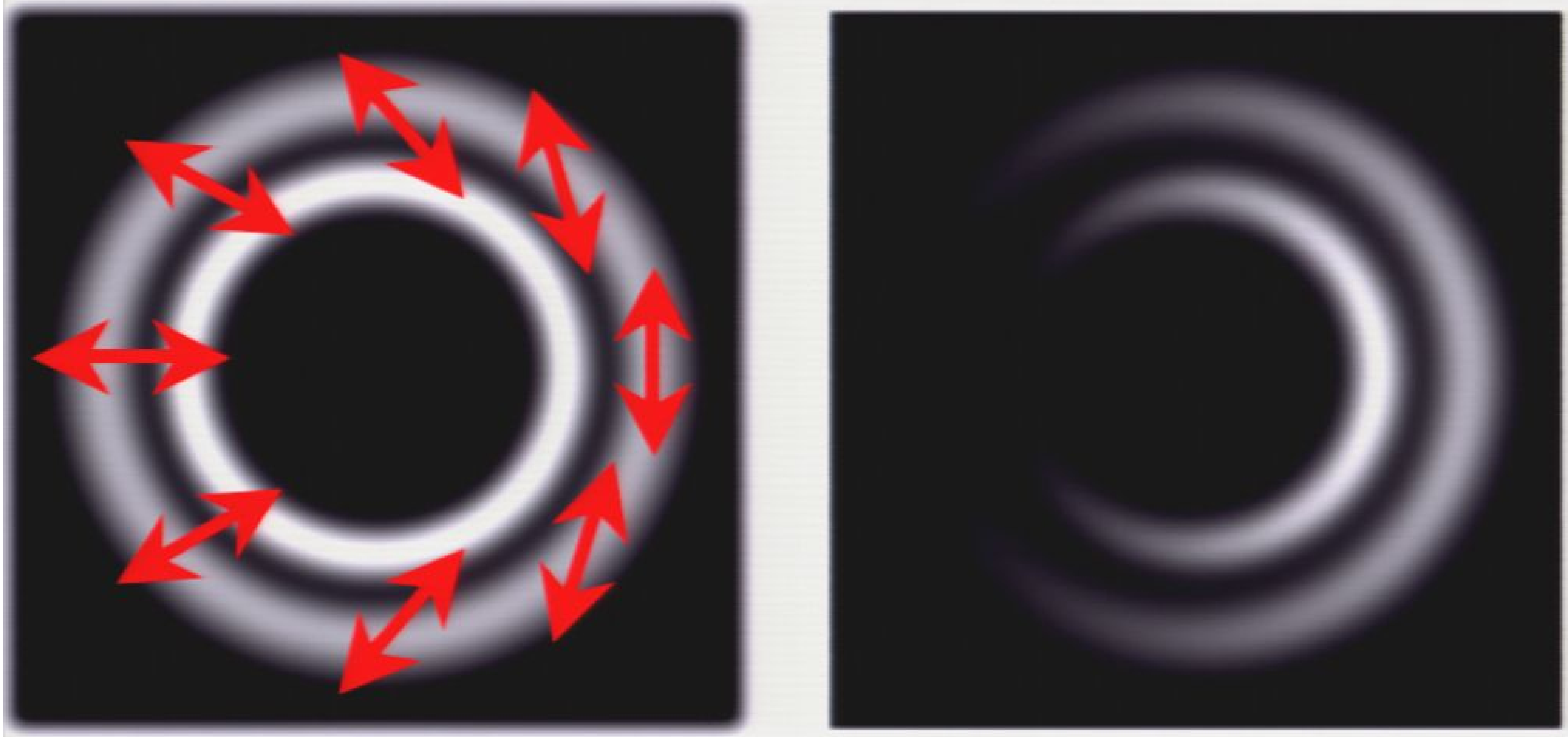


photon angular momentum transformed from pure spin (incident) to pure orbital (emergent) half-turn of polarization direction around the rings



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Berry, M V, Jeffrey, M, and Mansuripur, M R, (2005) 'Orbital and spin angular momentum in conical diffraction', *J.Optics.A: Pure Appl.Opt* 7, 685-690

enter C V Raman (1888-1975)

1941

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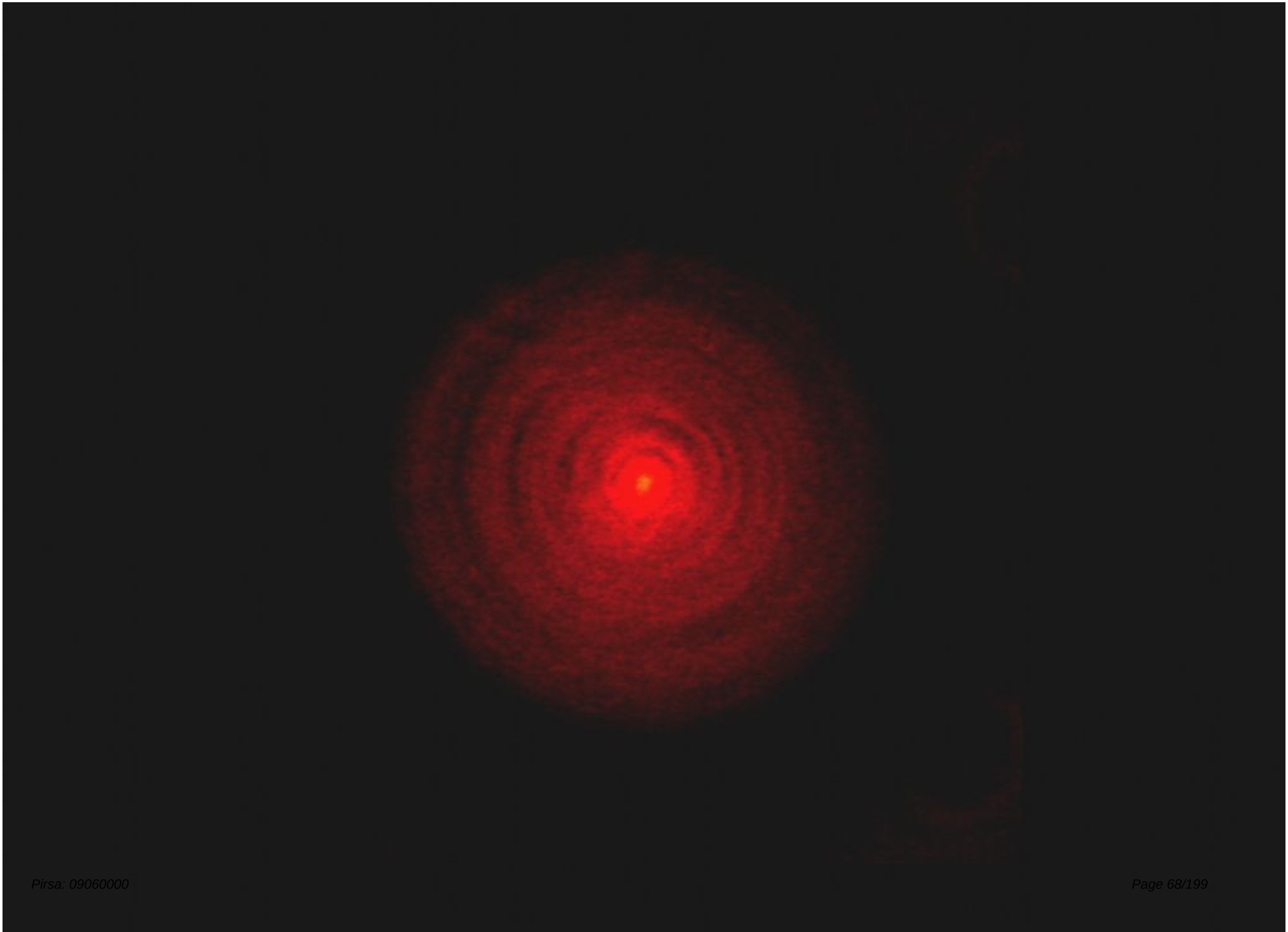
far from the crystal, rings fade and are replaced by a central spot



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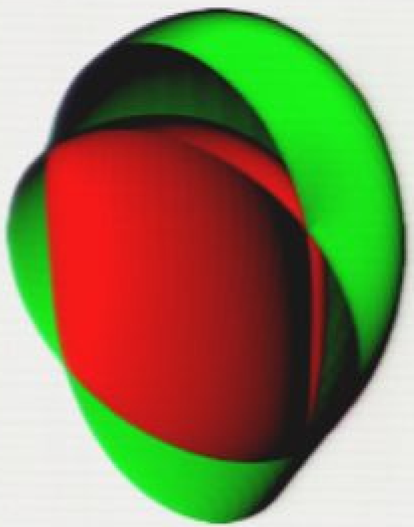

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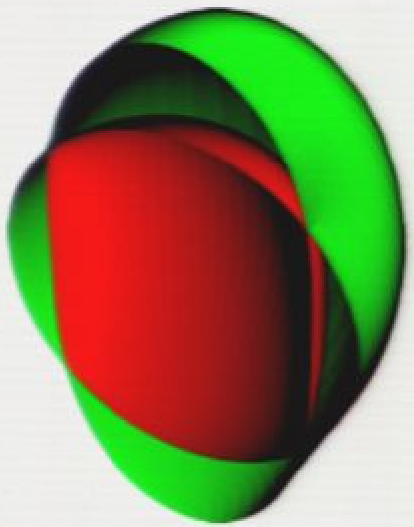

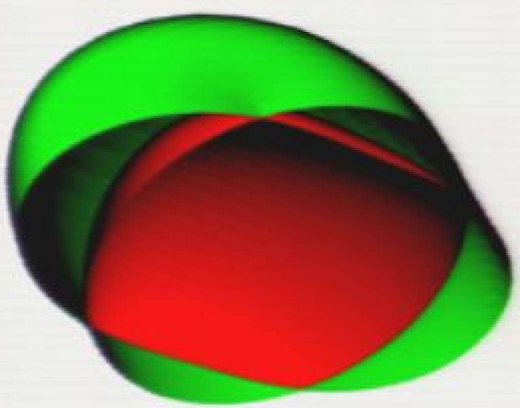

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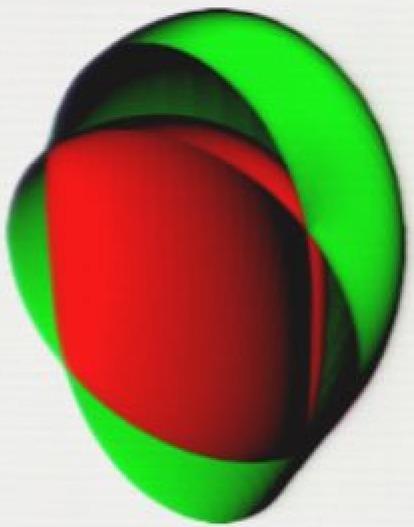

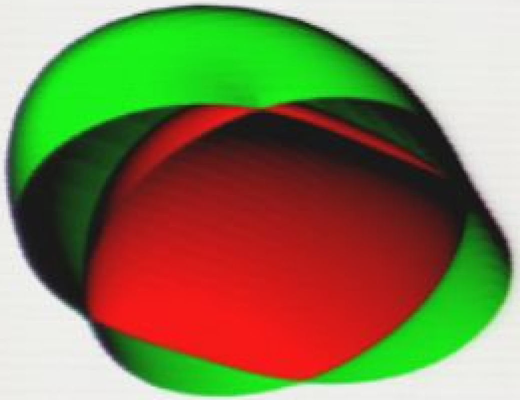

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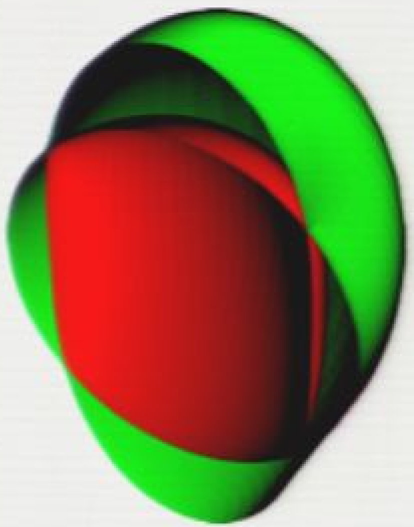

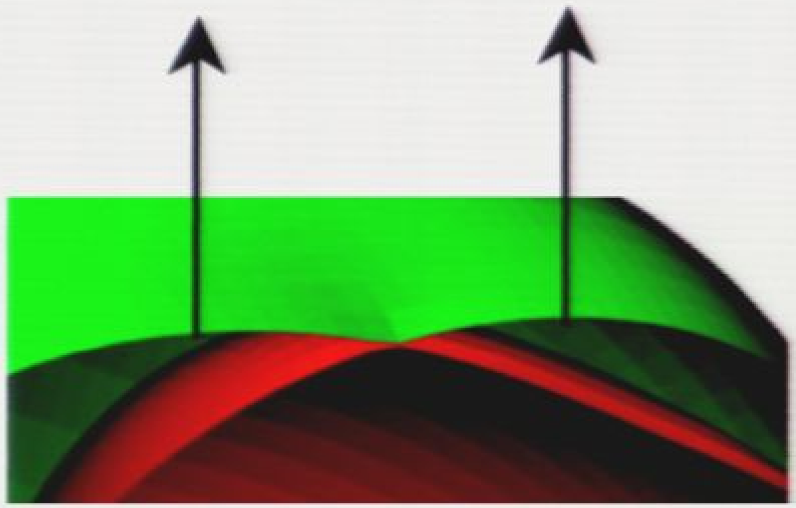

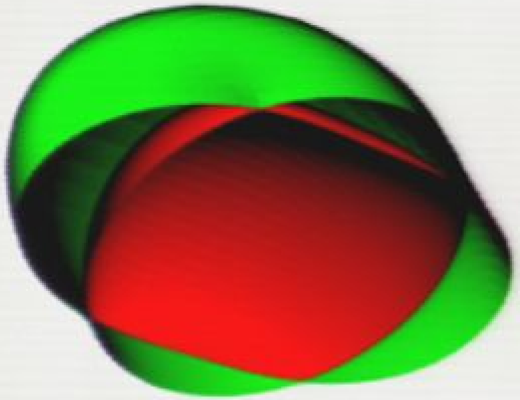

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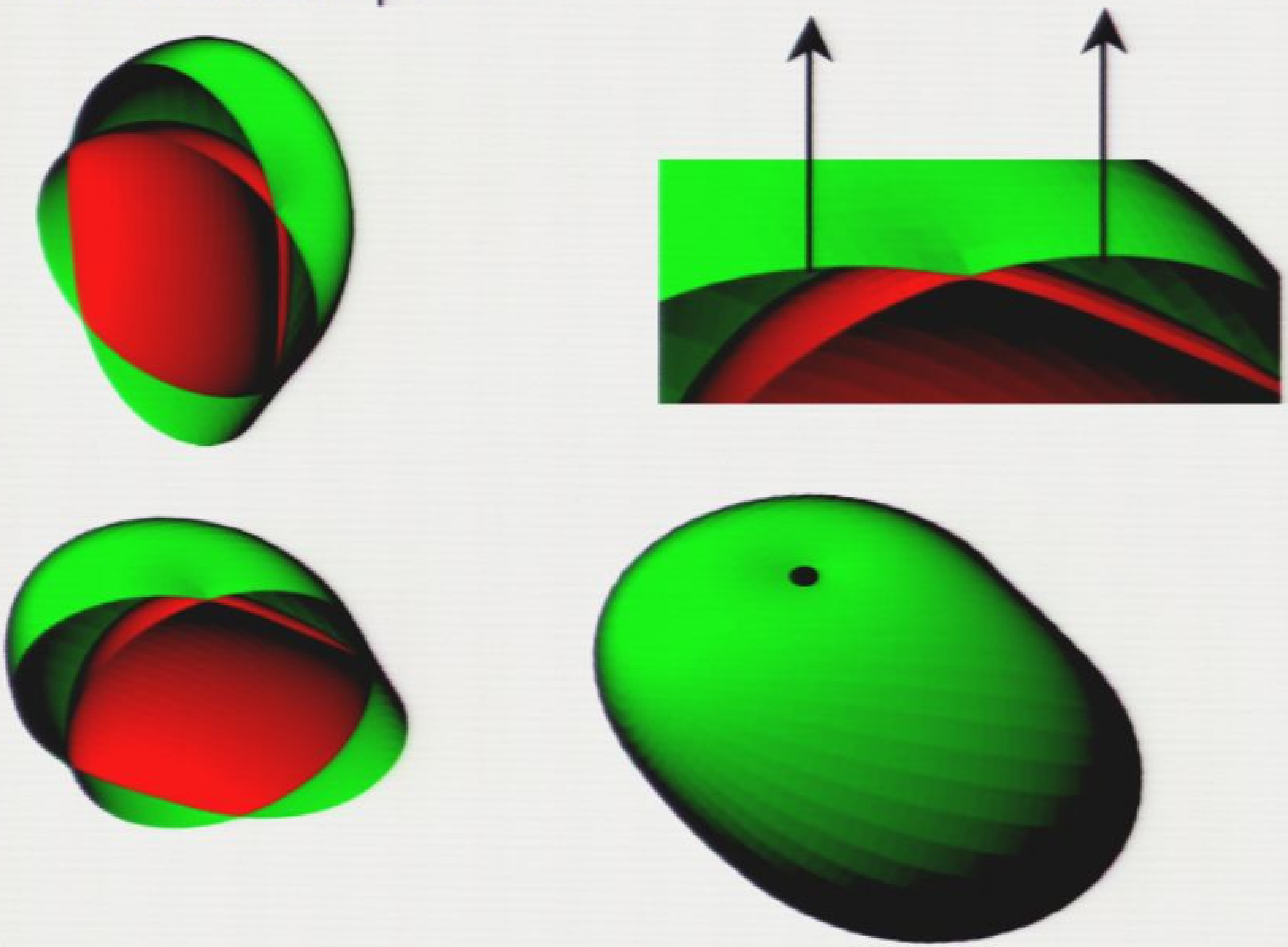

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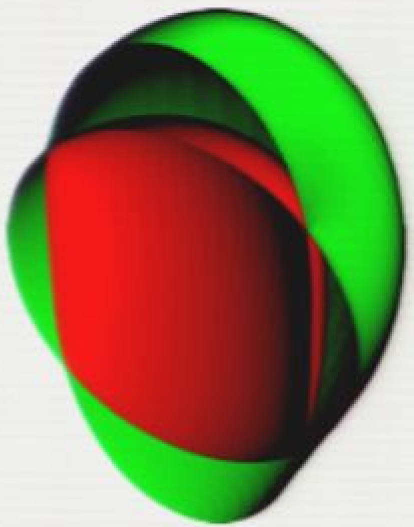

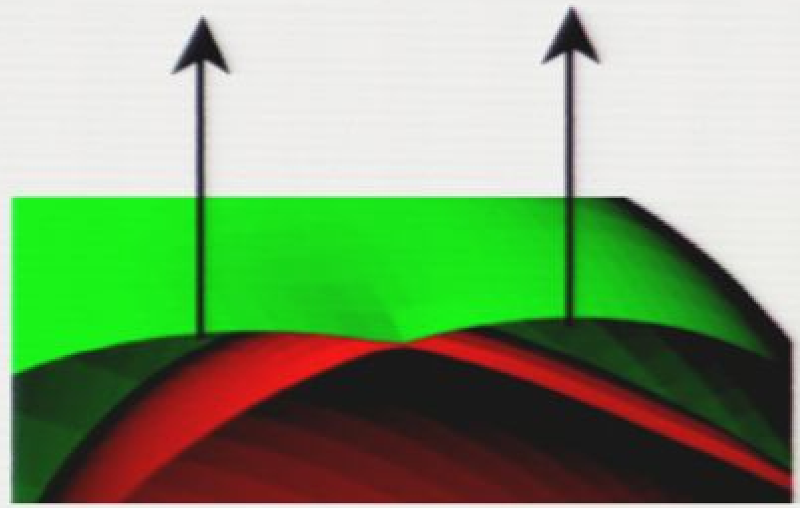

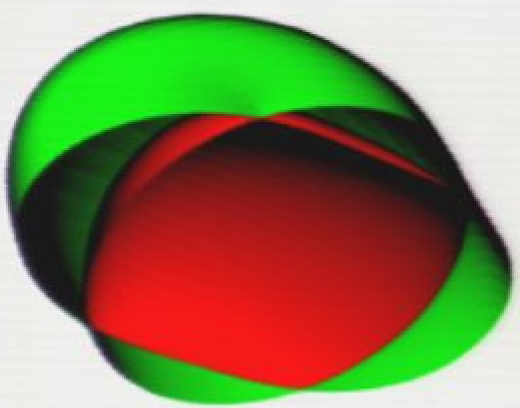

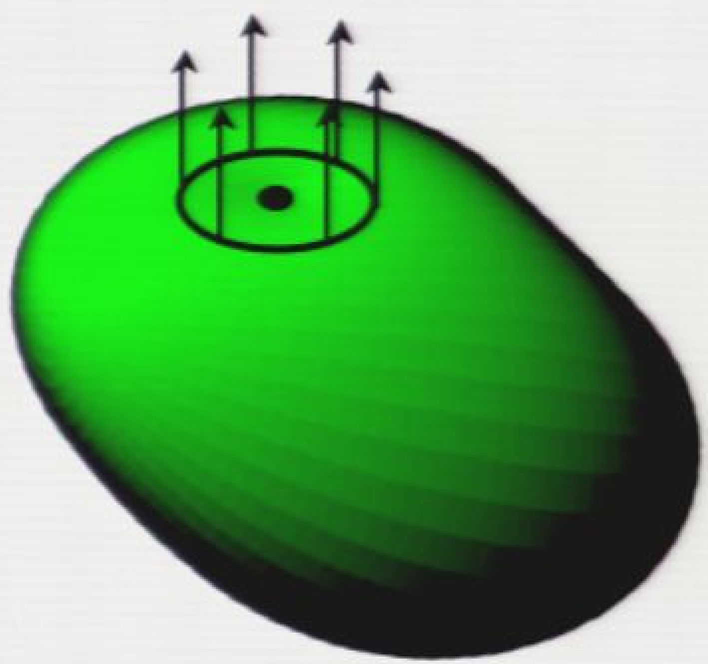

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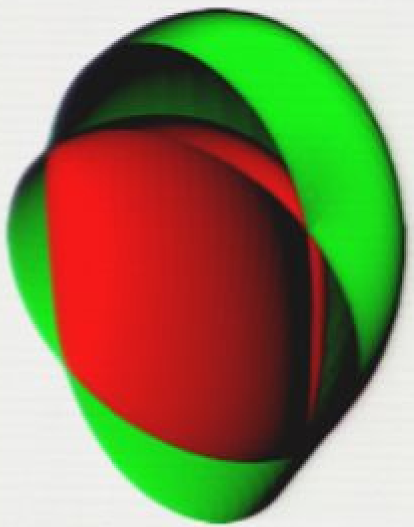

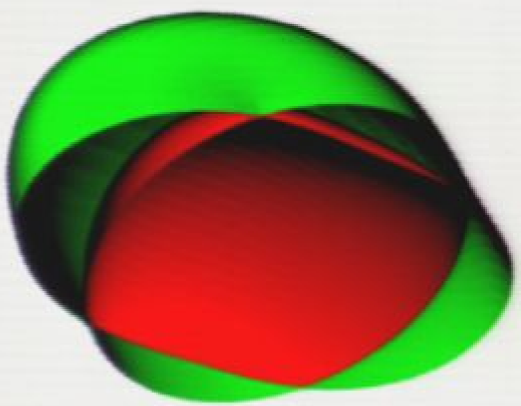

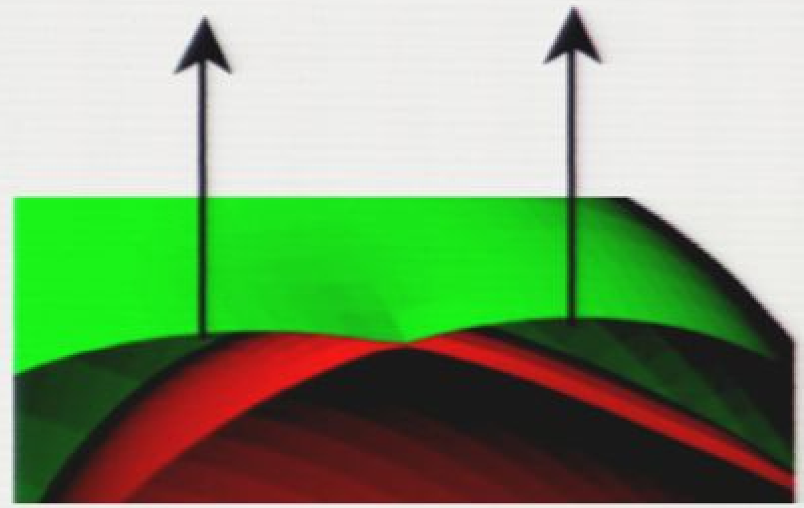

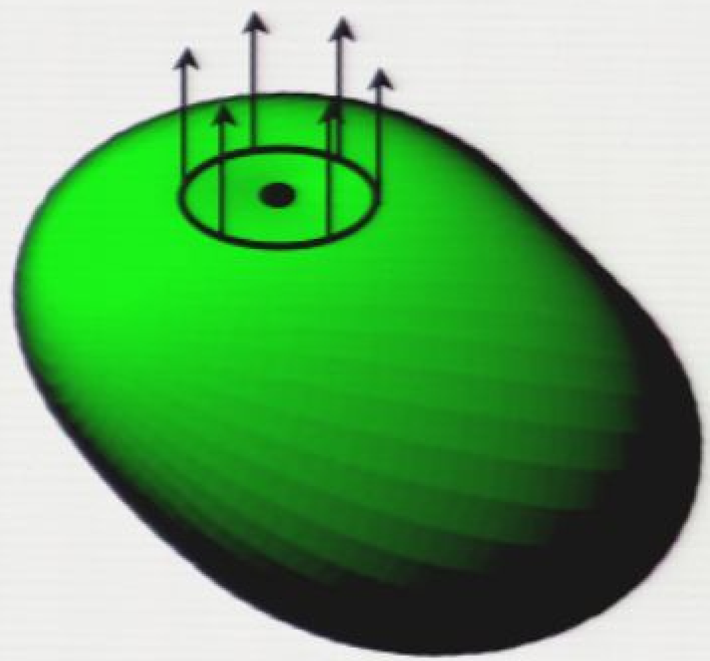

axial spike of
light rays,
focused
forwards
(line caustic)

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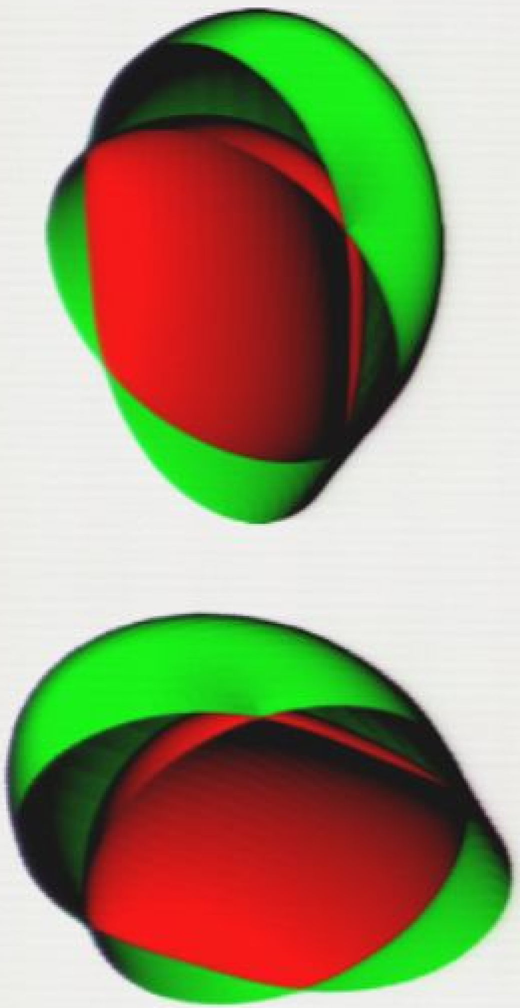

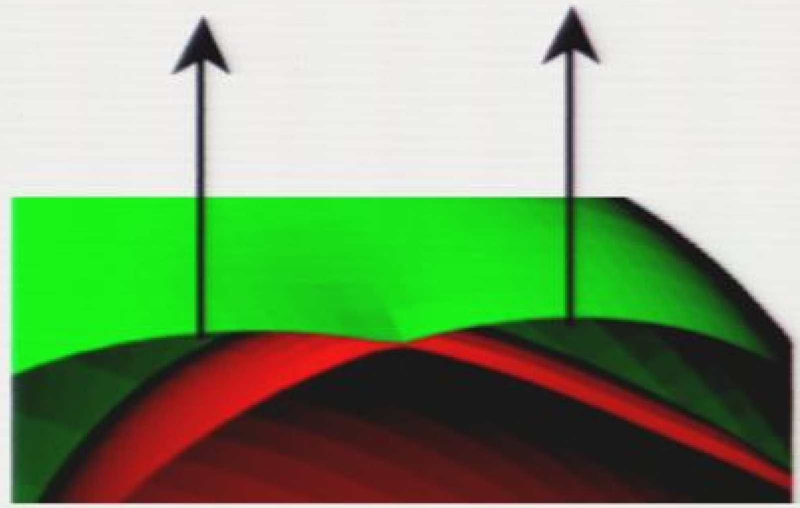

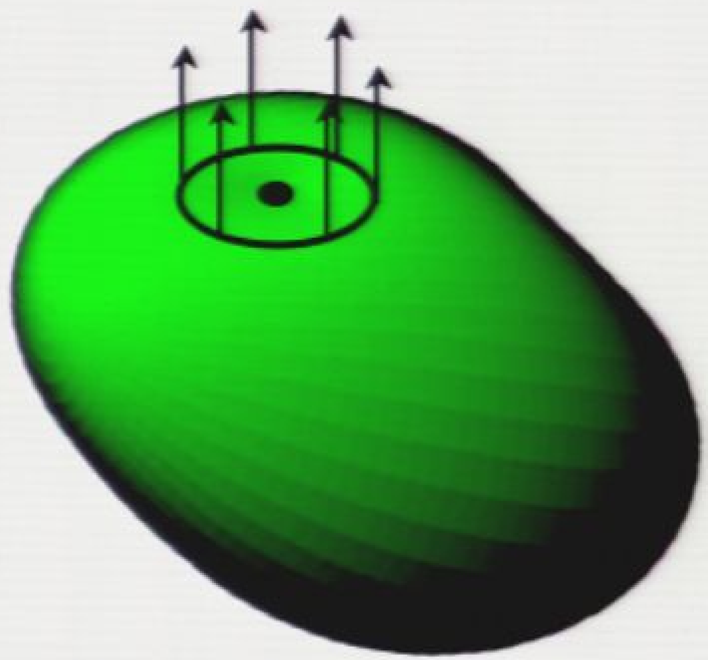

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"somewhat as a plum can be laid down on a table so as to touch and rest on the table in a whole circle of contact"

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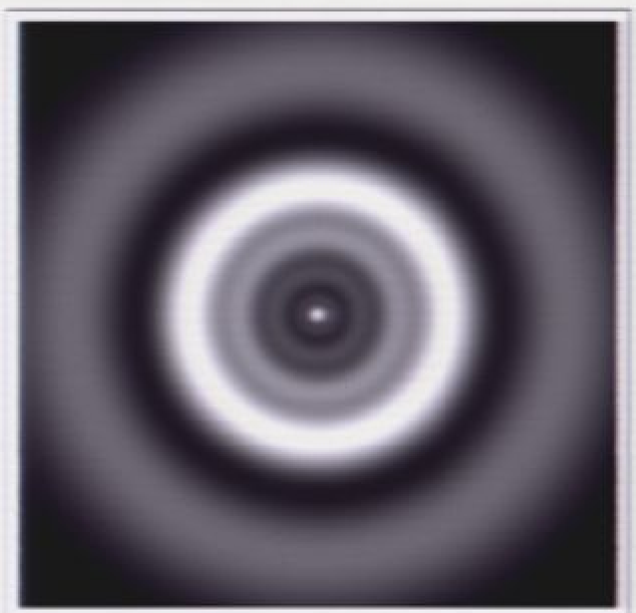

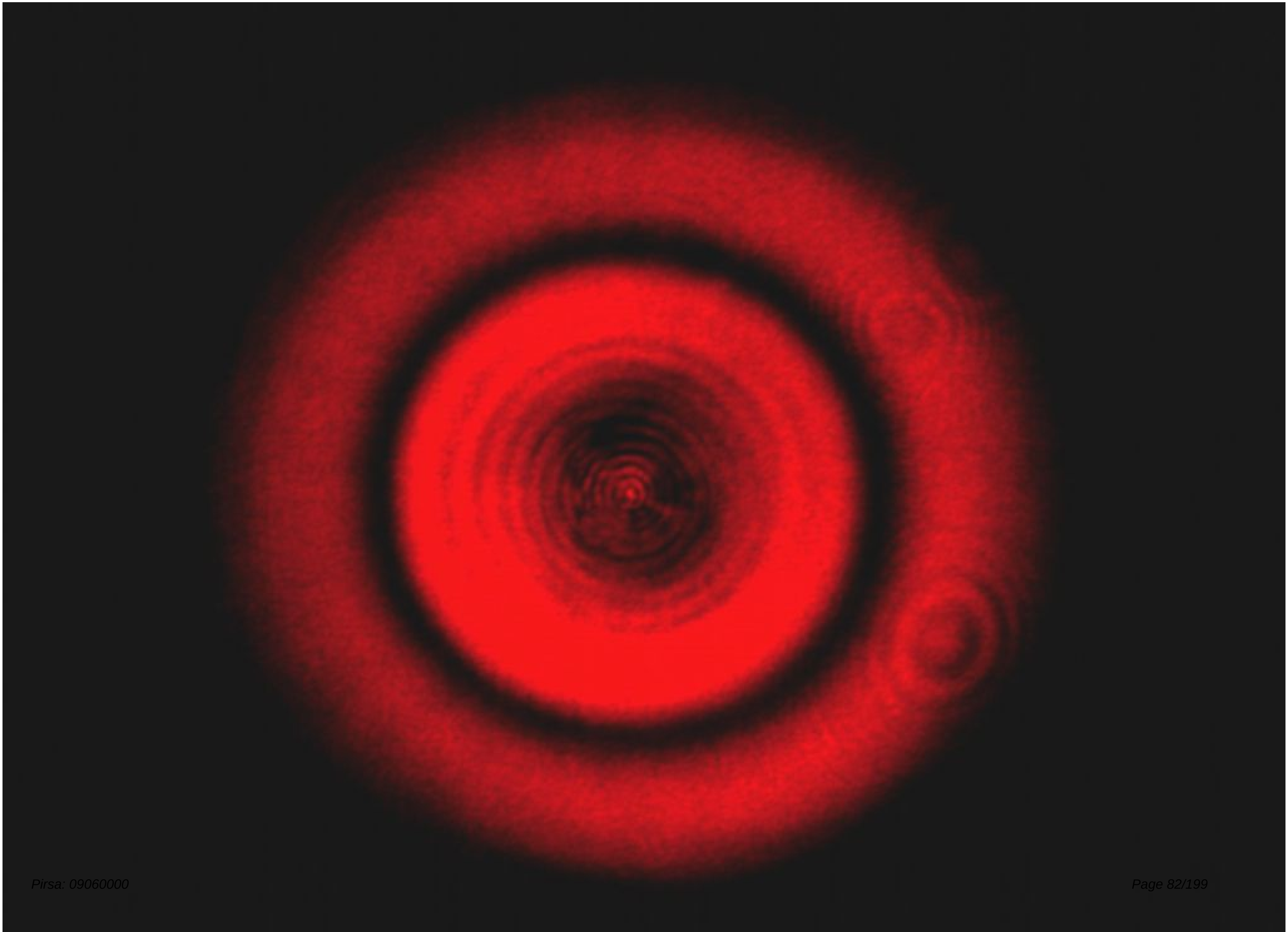

theory: the important simplifying ingredient: ***paraxiality***

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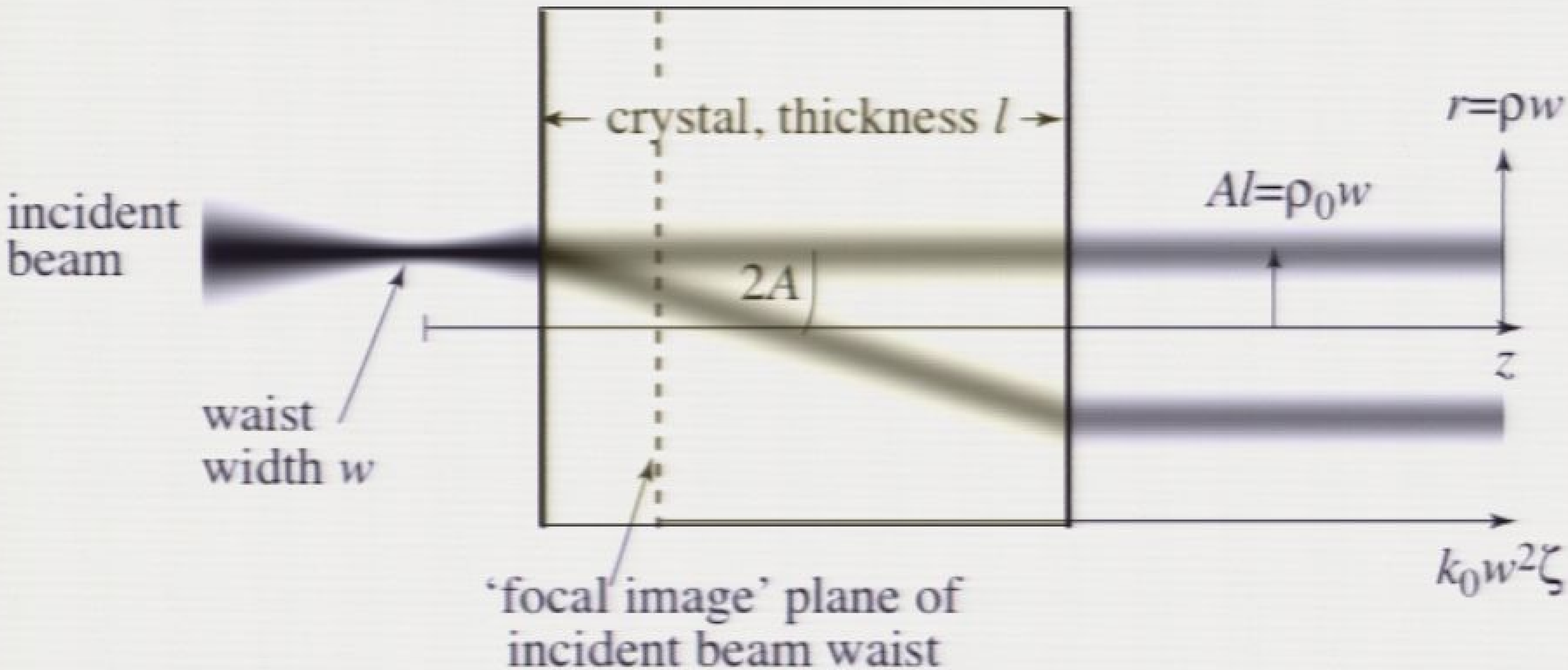

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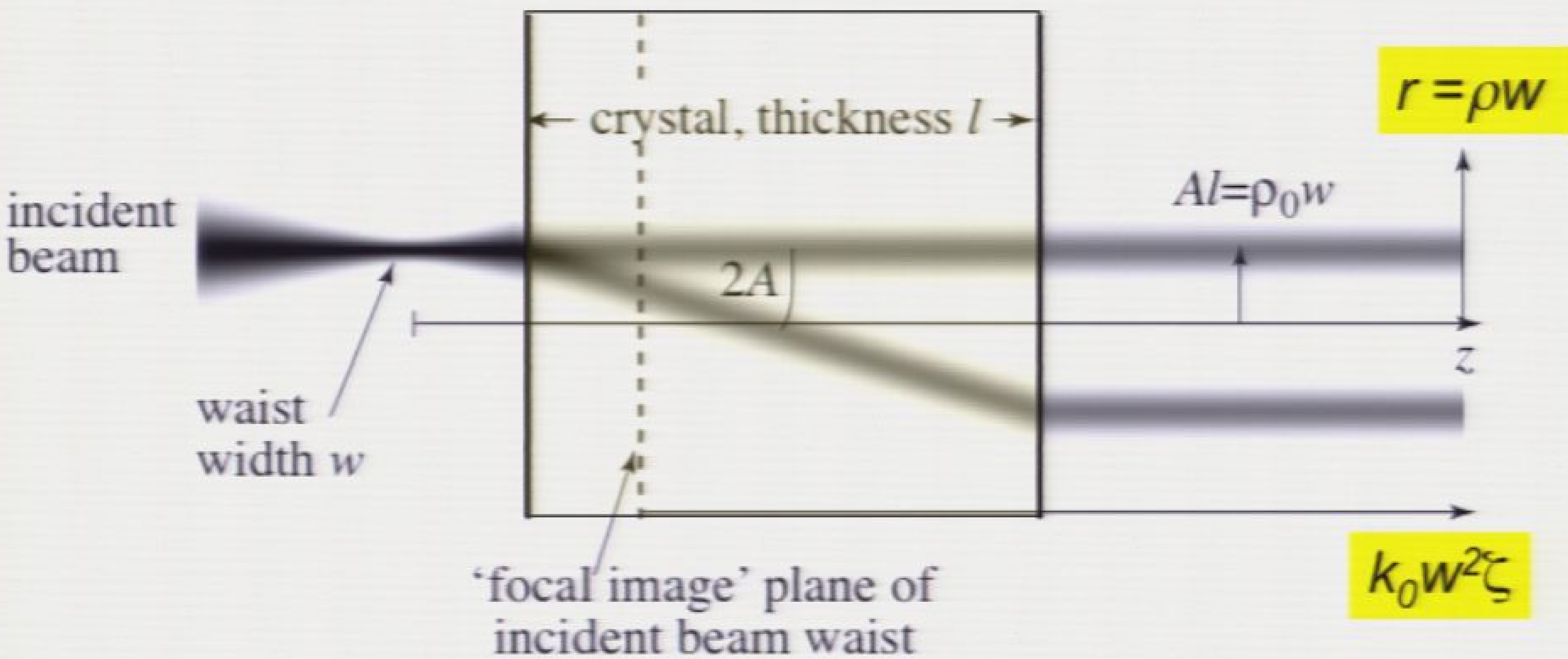

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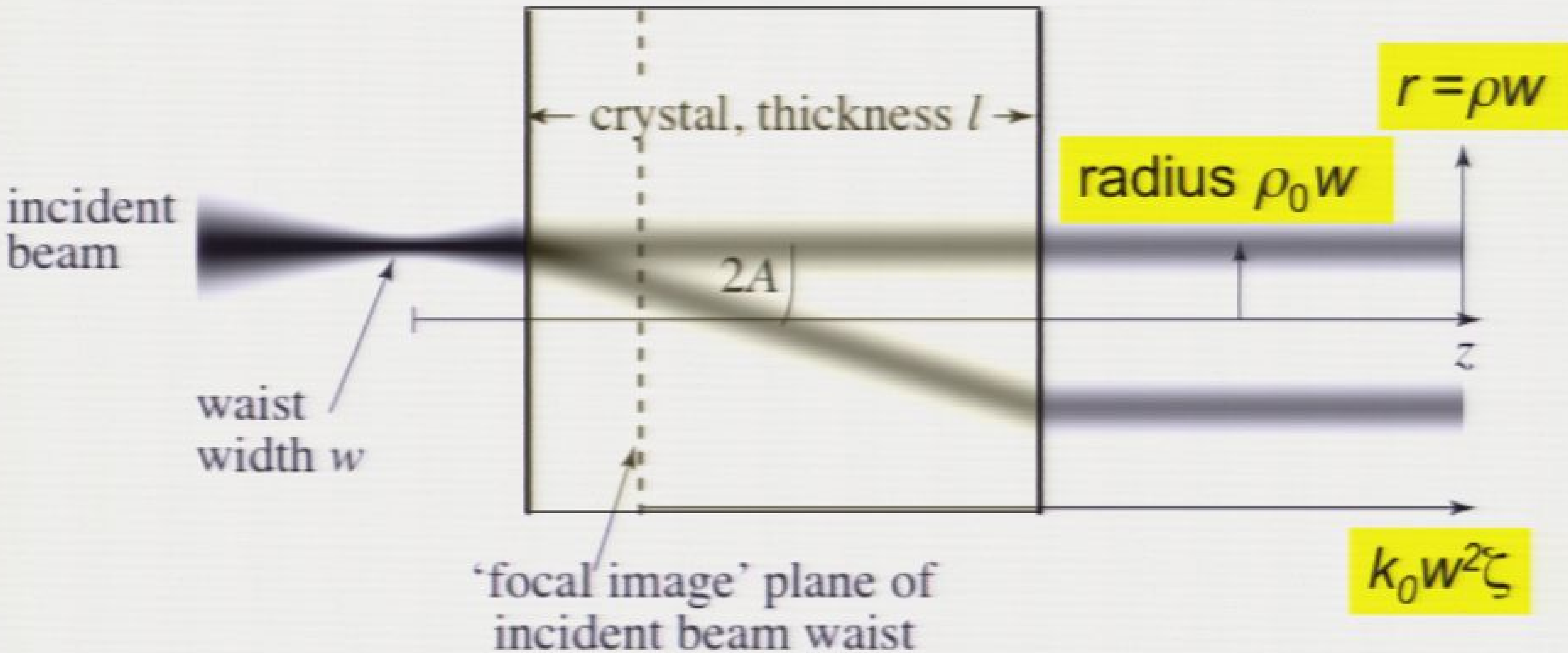

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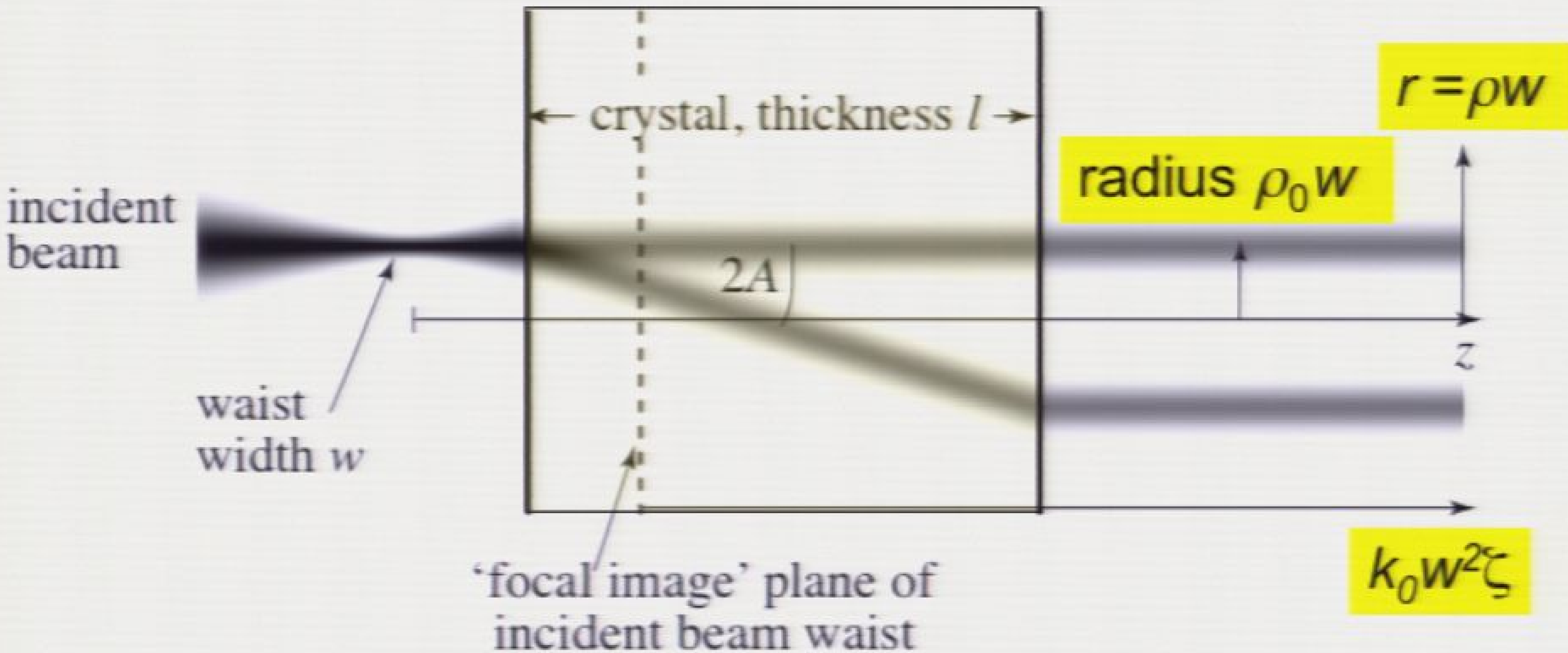

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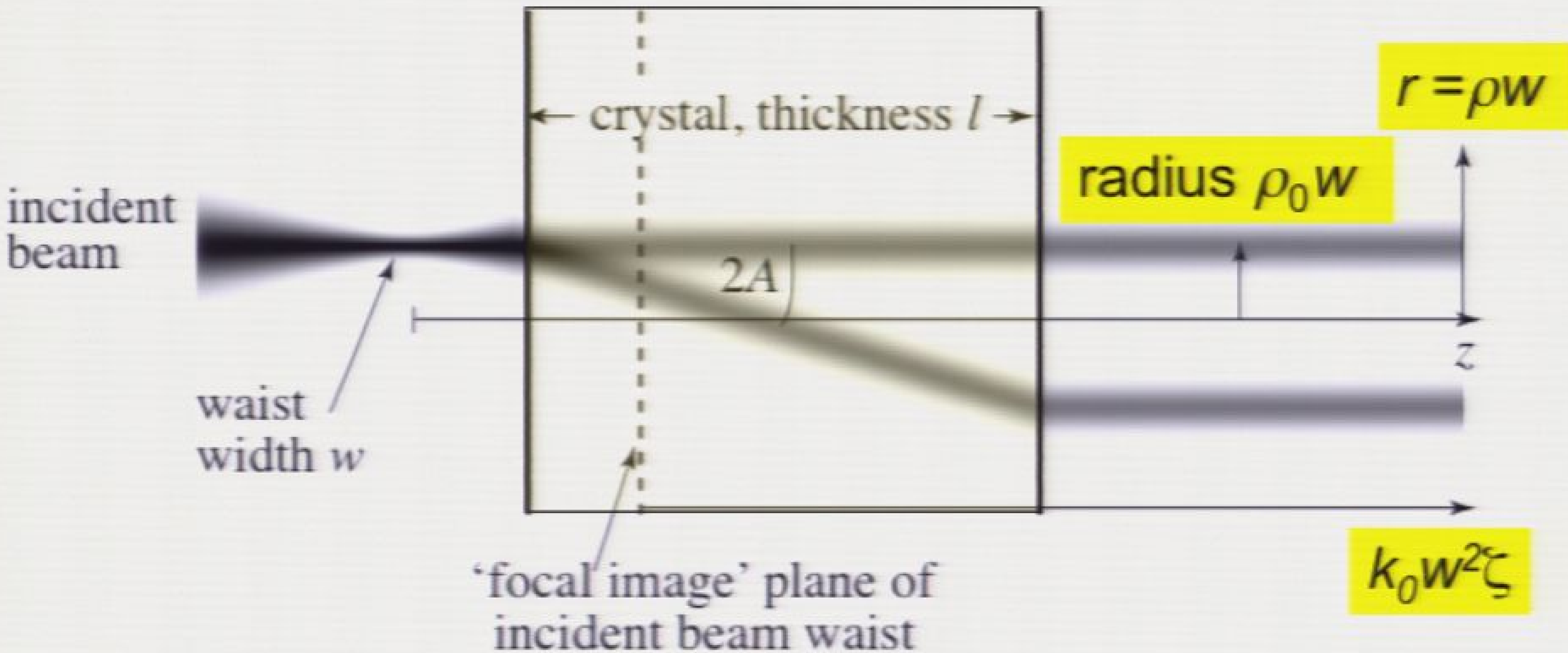

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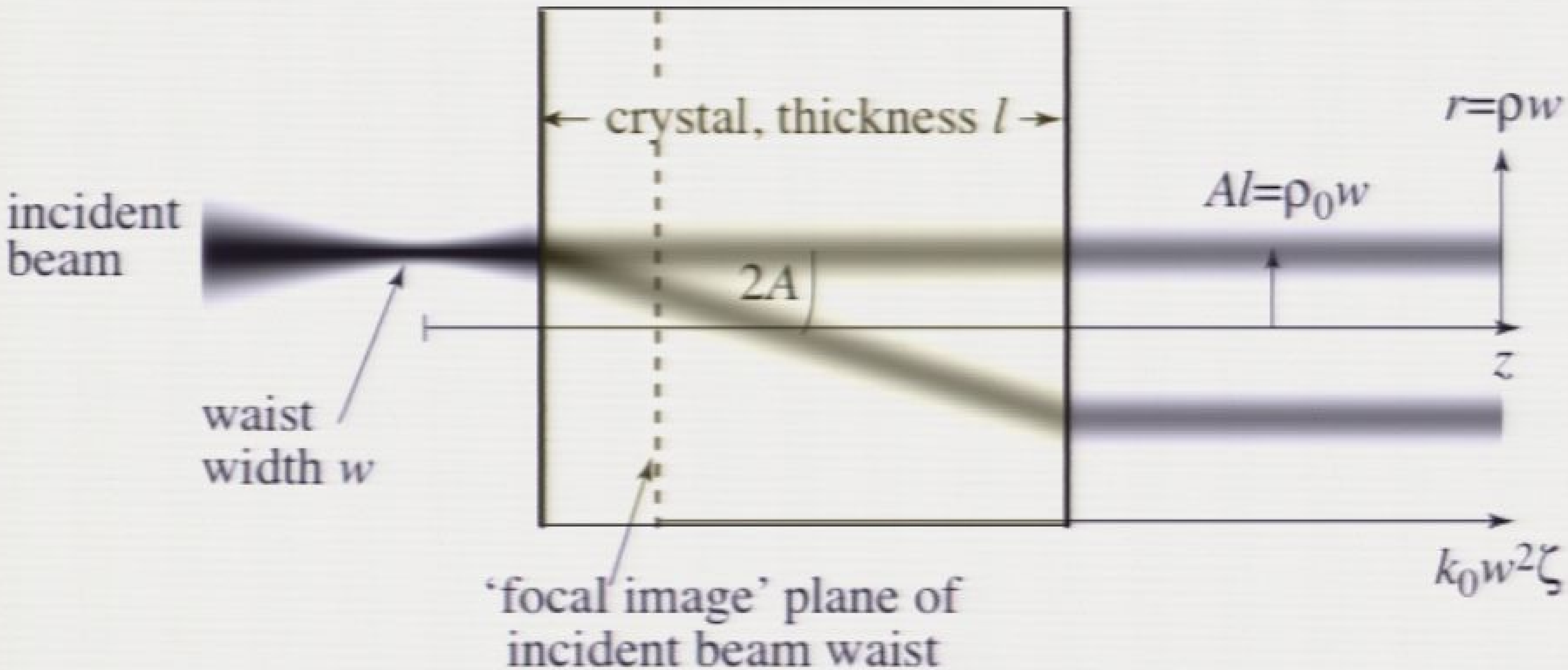

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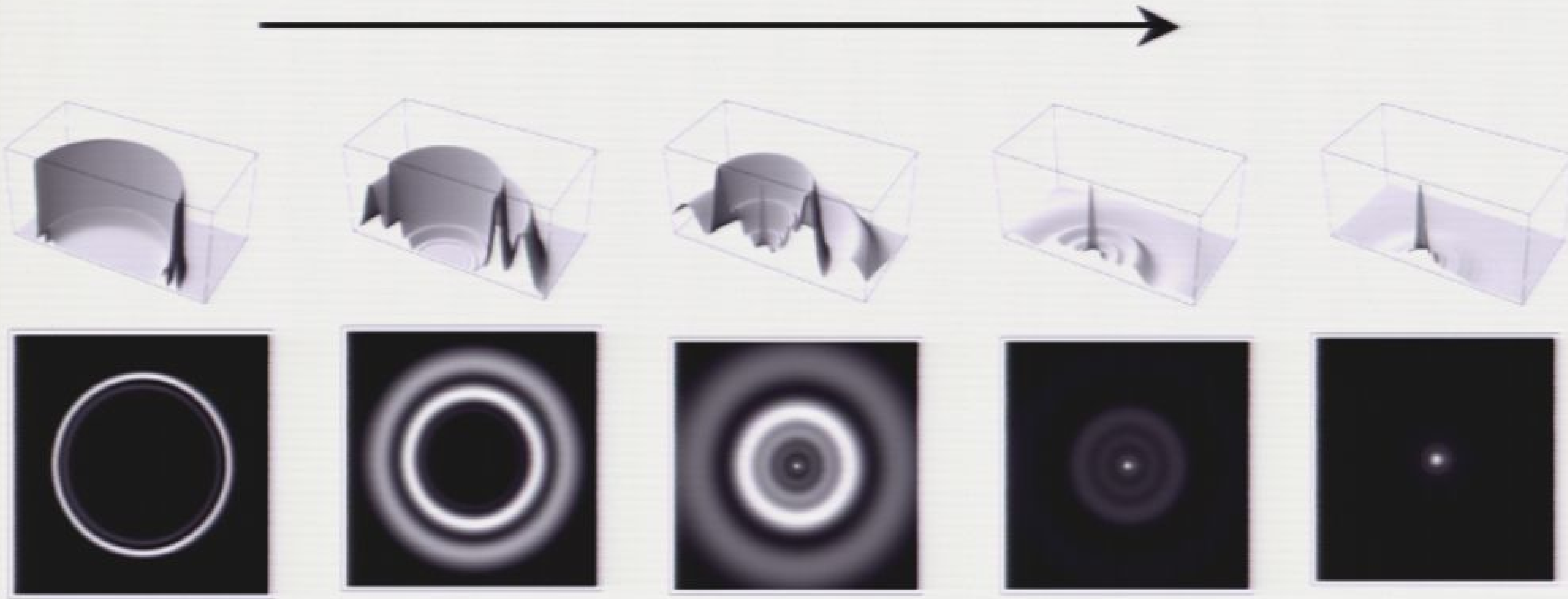
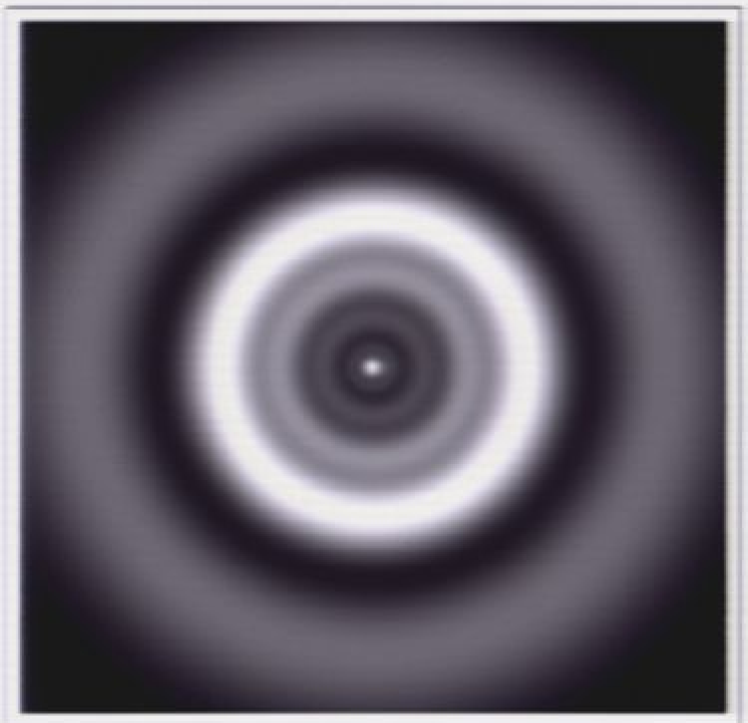
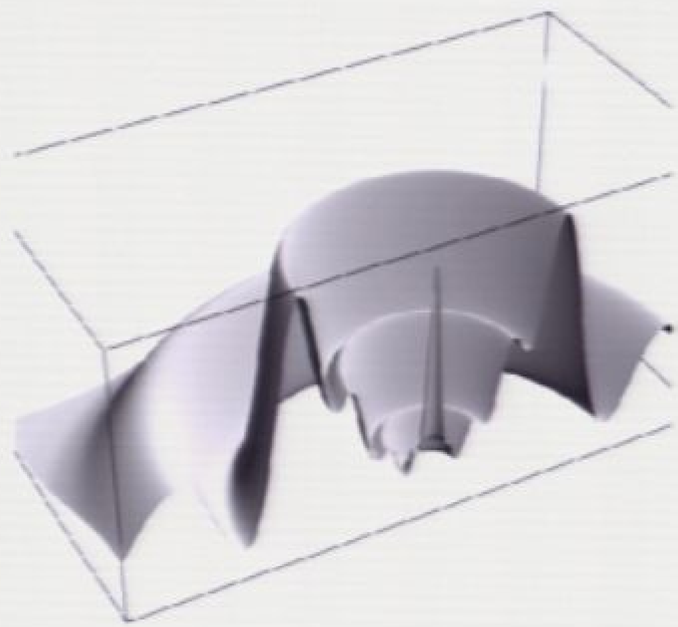
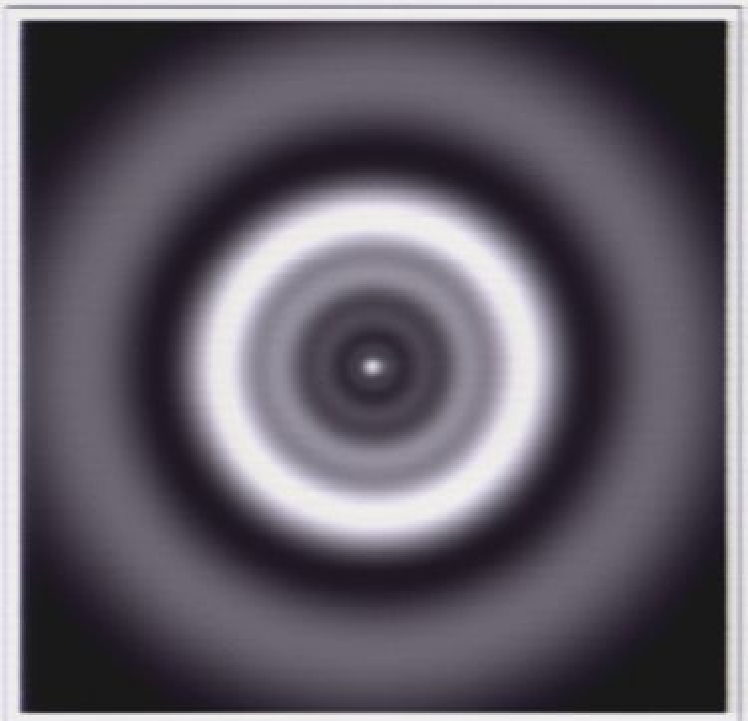
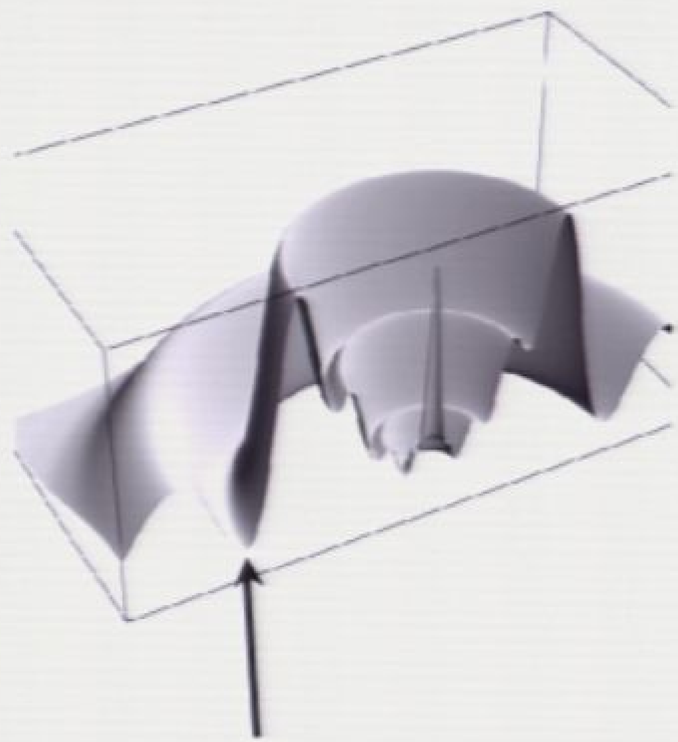
rings transforming into spot away from the crystal



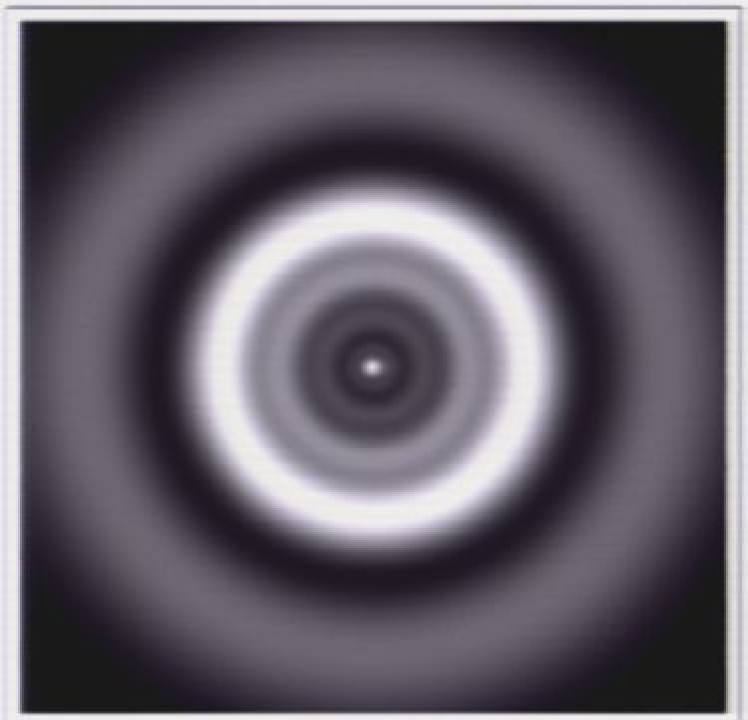


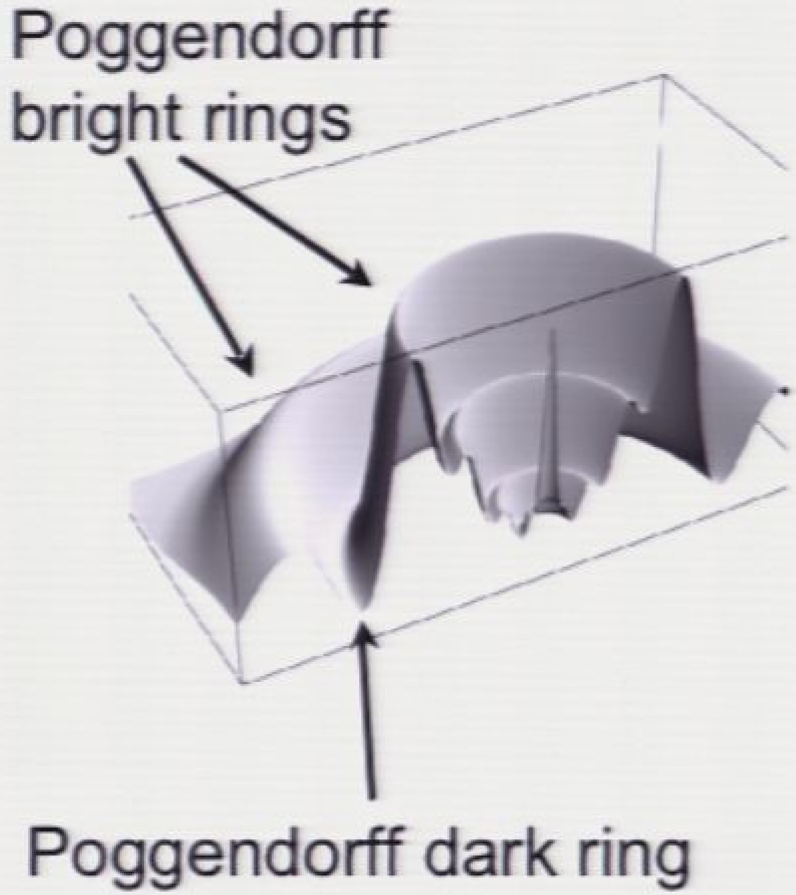






Poggendorff dark ring




Poggendorff
bright rings
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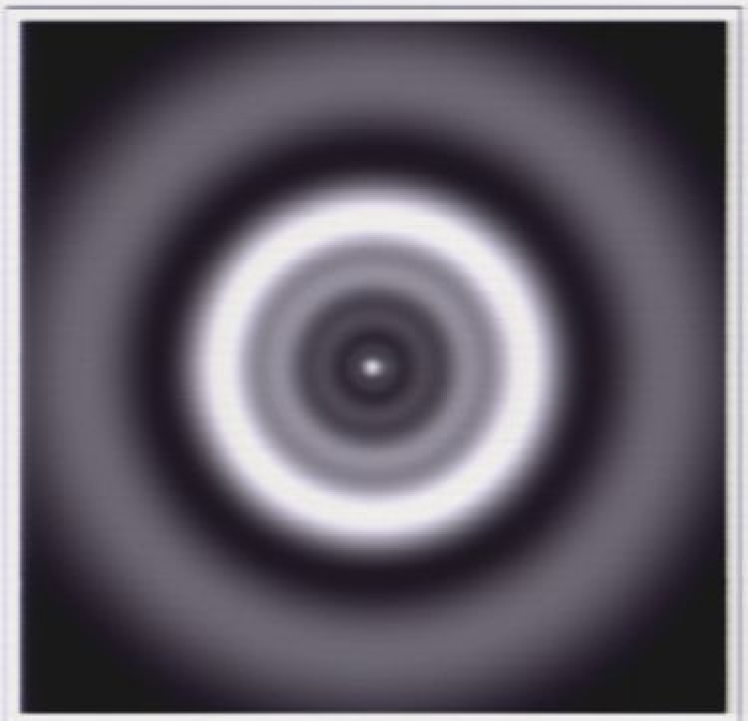

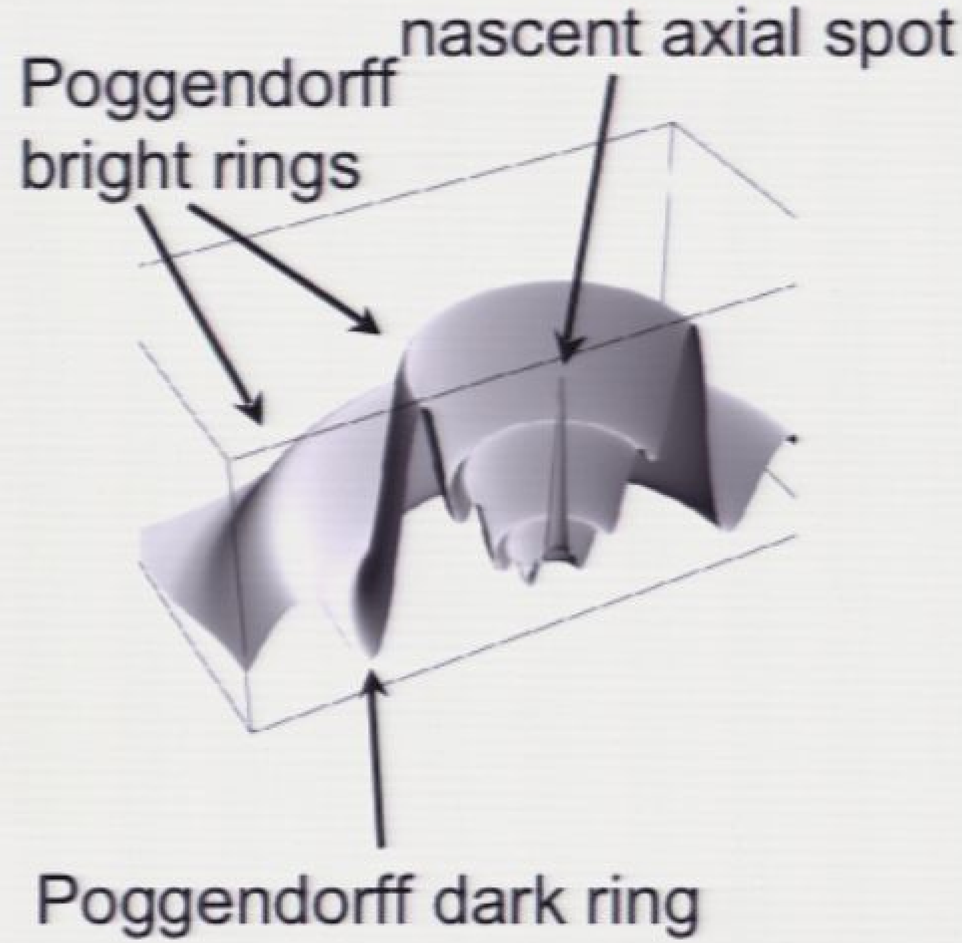
nascent axial spot
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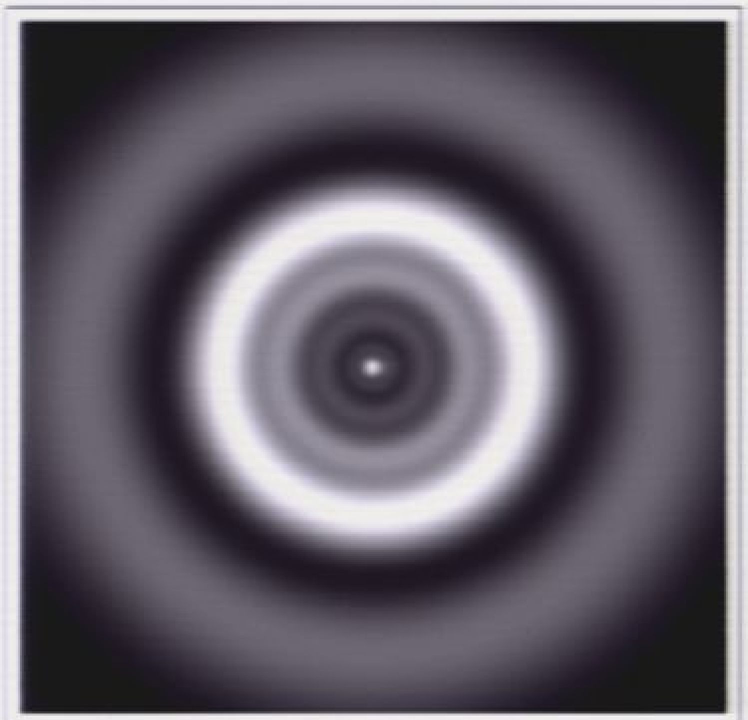

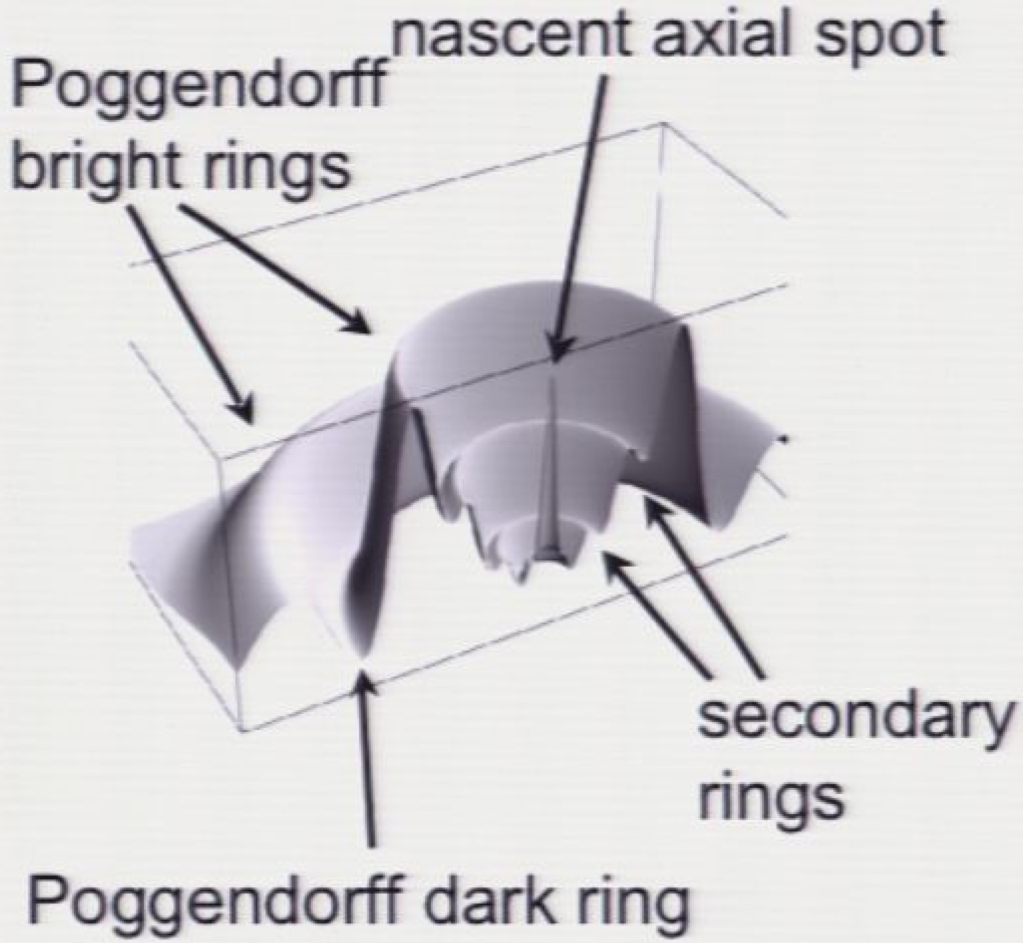
nascent axial spot
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geometrical optics, from stationary-phase approximation to integrals, or directly from Hamilton's principle

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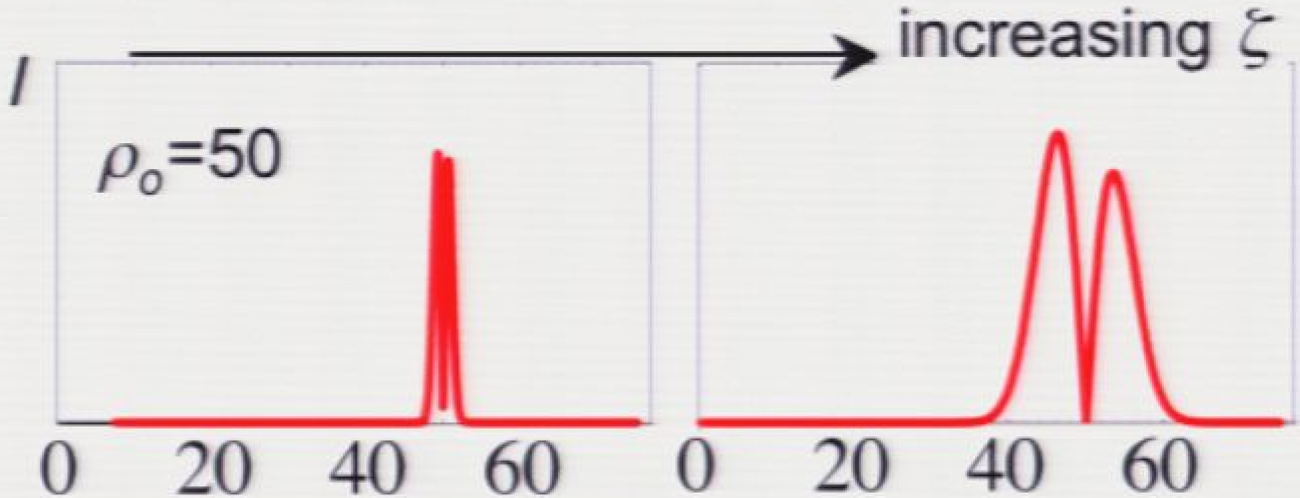

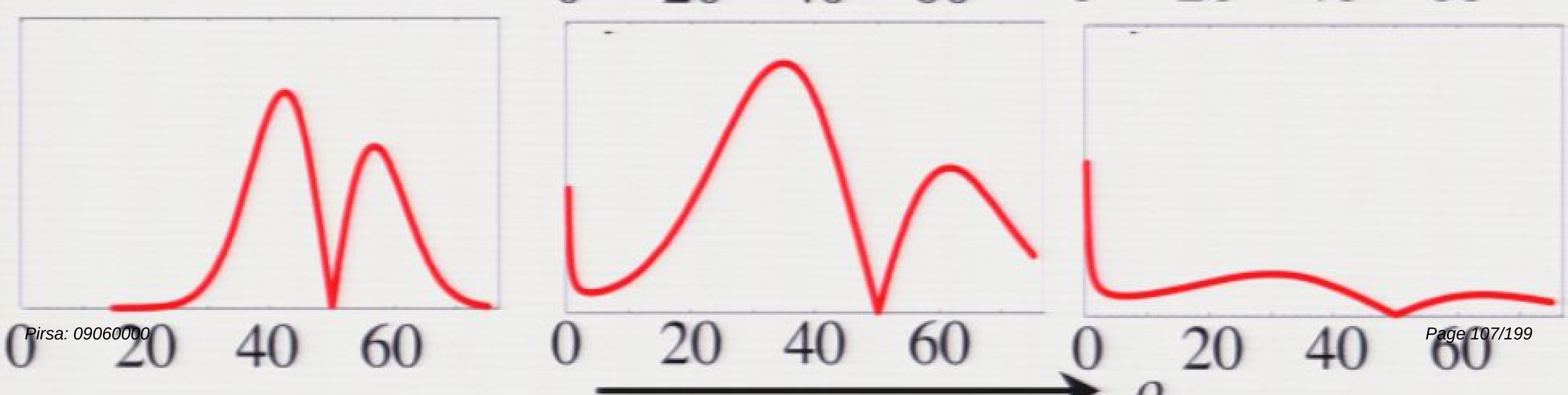

exact diffraction compared with geometrical optics

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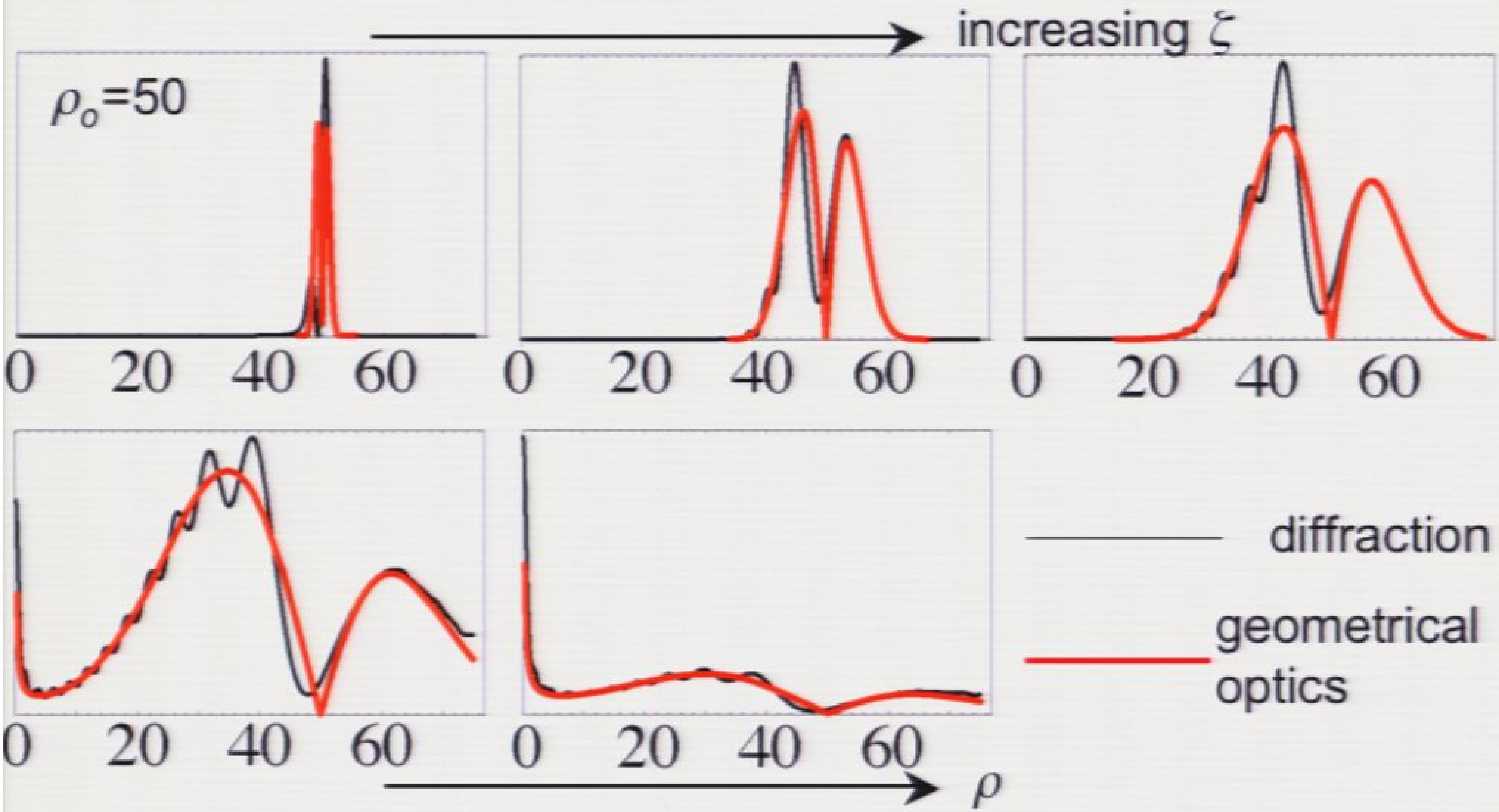

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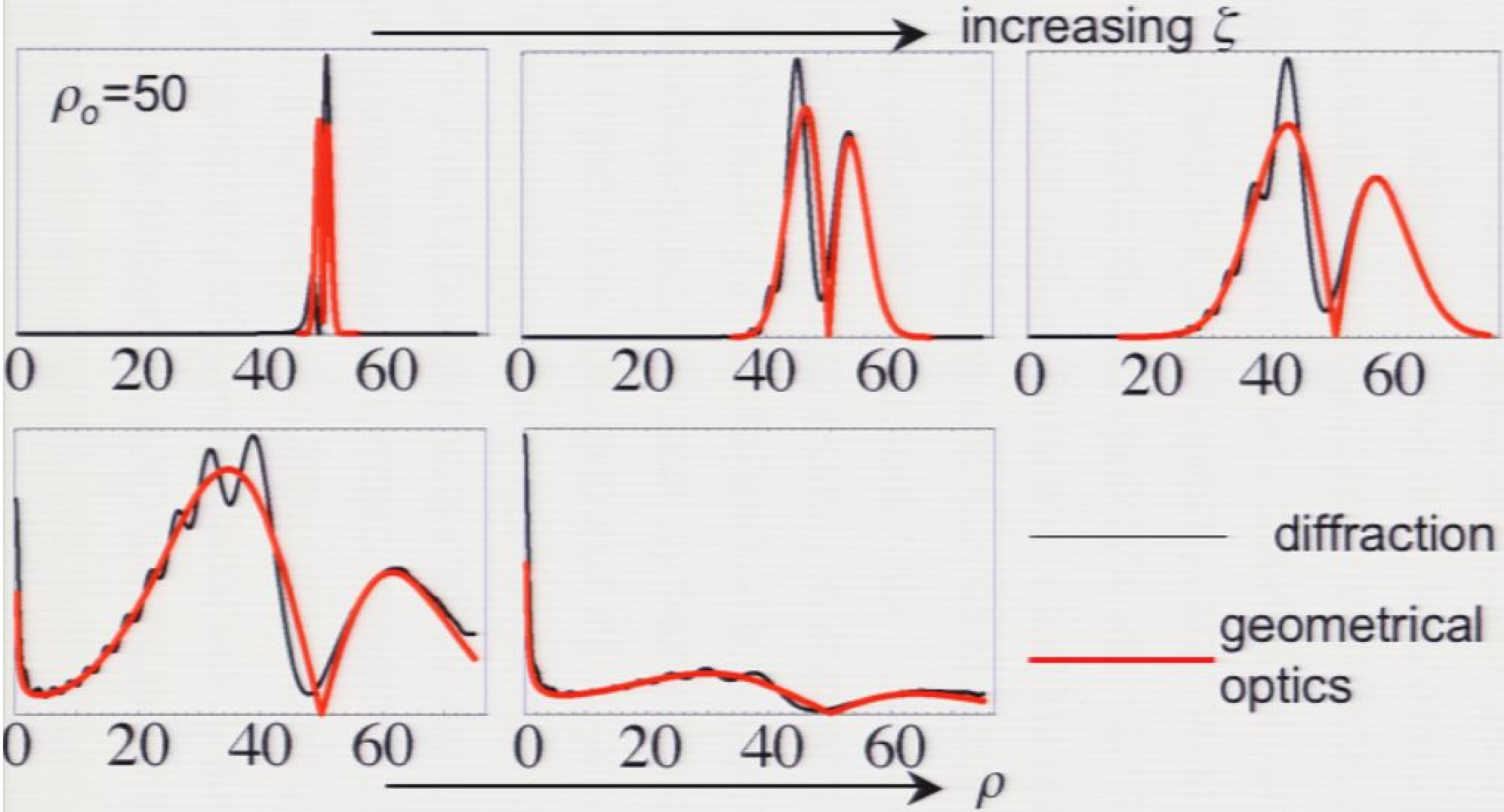

secondary rings result from interference between a geometrical ray and a 'diffracted ray' from the diabolical point full ring asymptotics for $\rho_0 \gg 1$, including diffraction full ring asymptotics for ρ_0>>1, including diffraction

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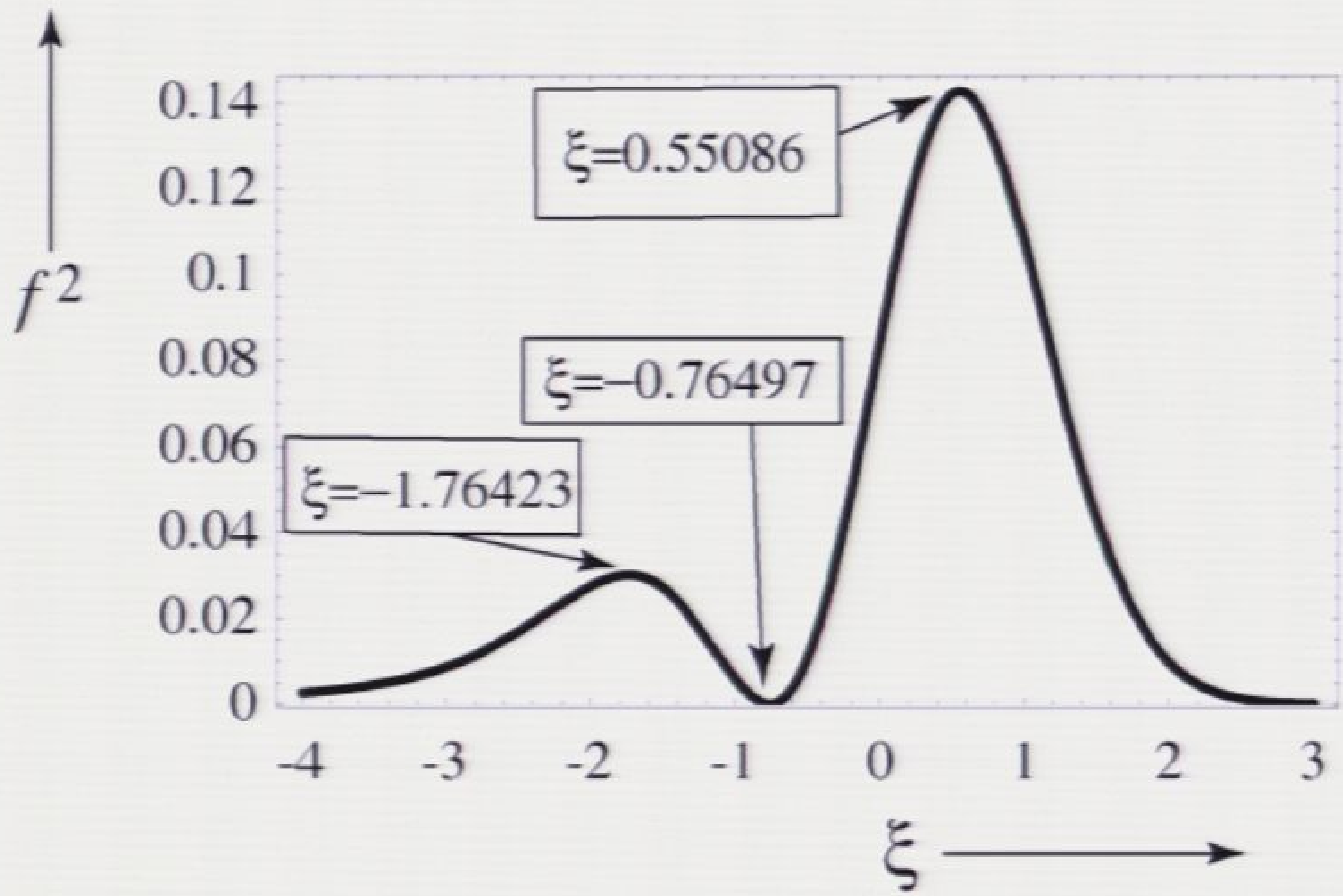
$$I_{\mathrm{rings}} = \frac{2}{\rho_0\left(1+\zeta^2\right)^{3/4}} \left| f\left(\frac{\rho-\rho_0}{\sqrt{1+\mathrm{i}\zeta}}\right)\right|^2$$

the sharpest rings, in the focal plane $\zeta=0$, as a function of $\xi=\rho-\rho_0$



Poggendorff rings in focal image depend on the nature of the incident beam

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$$f(\xi) = 0 \quad (\xi > 1) = \frac{1}{\pi}\left(-K\left(\frac{1-\xi}{2}\right) + 2E\left(\frac{1-\xi}{2}\right)\right) \quad (|\xi| < 1)$$

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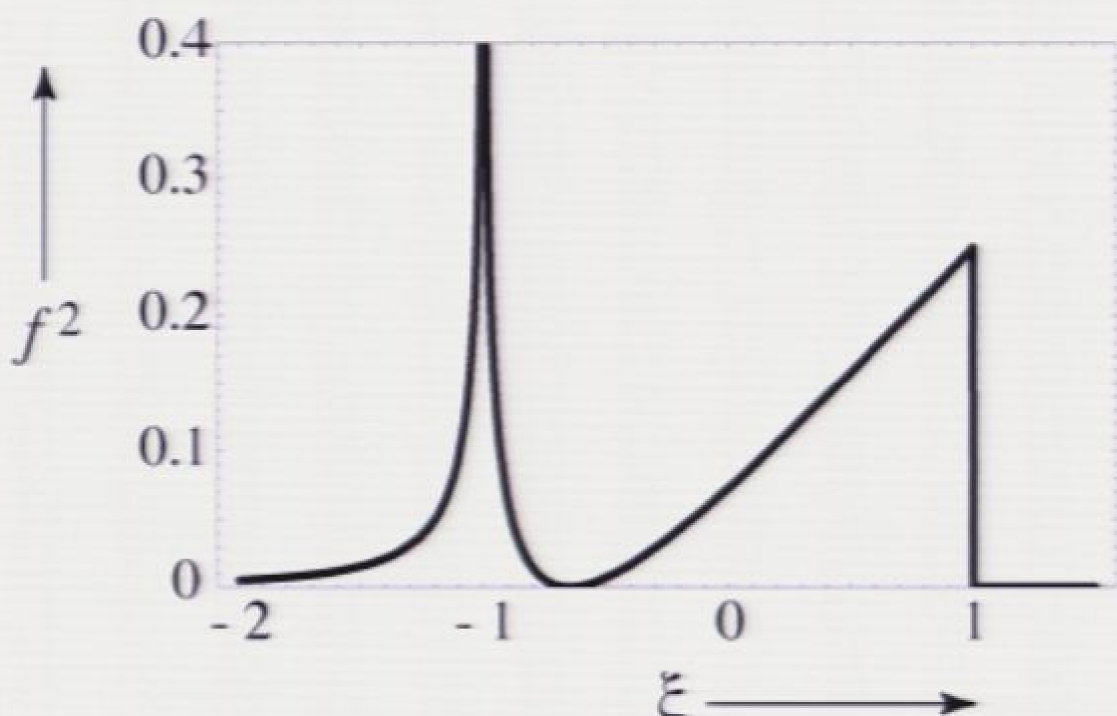

fringes surrounding central spot: weak interference between two geometrical rays, in post-geometrical approximation

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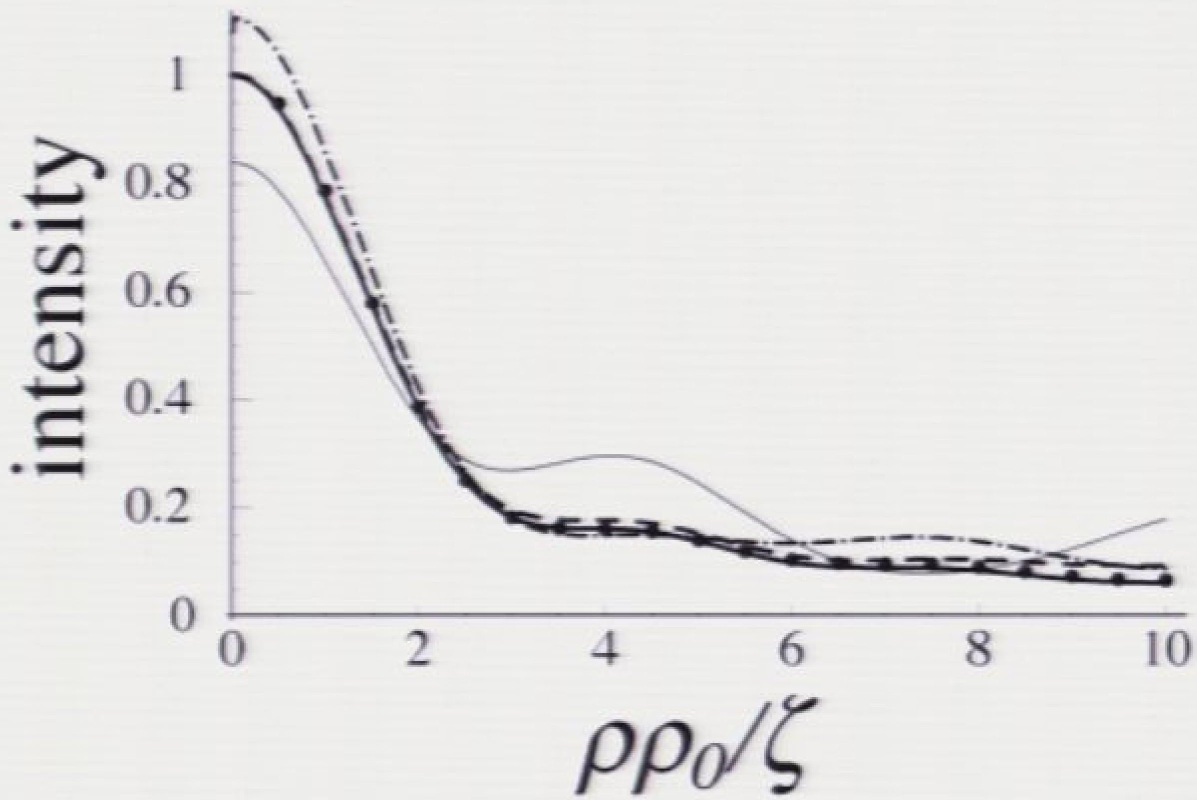

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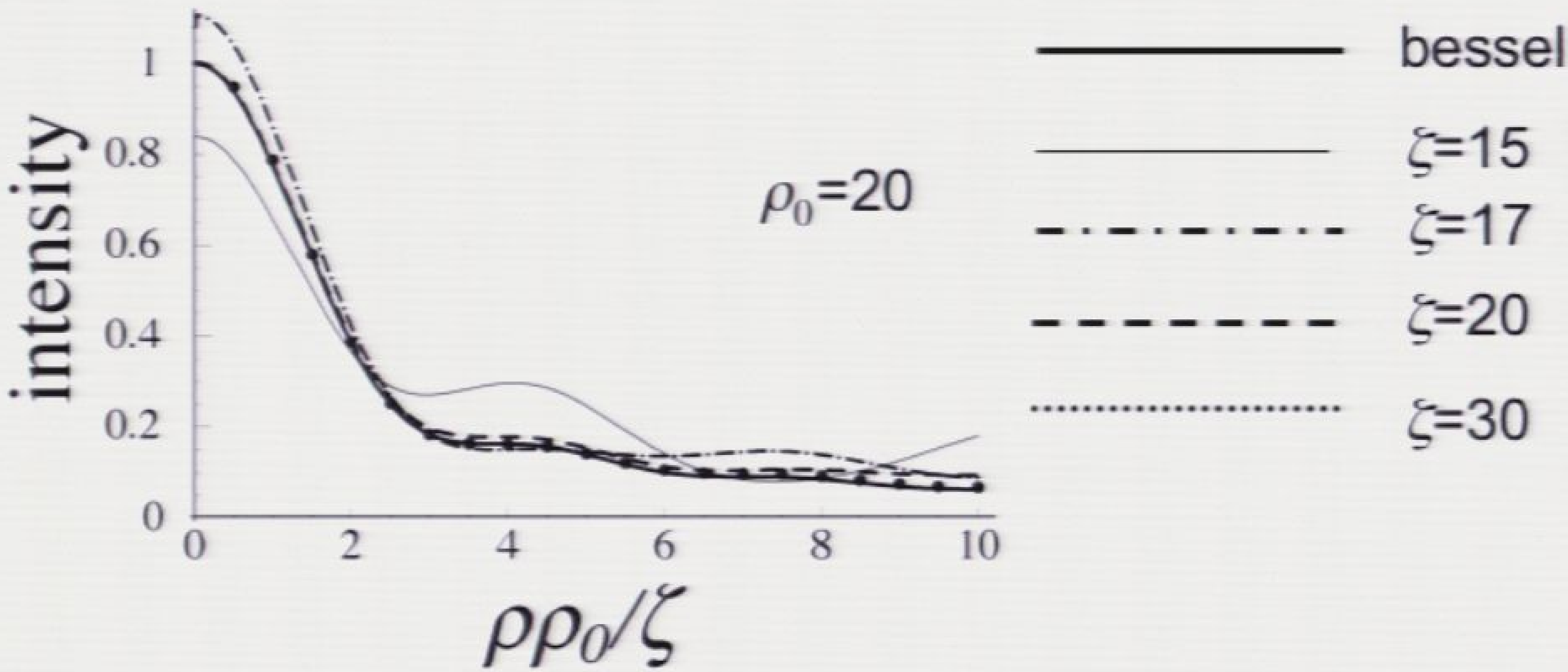

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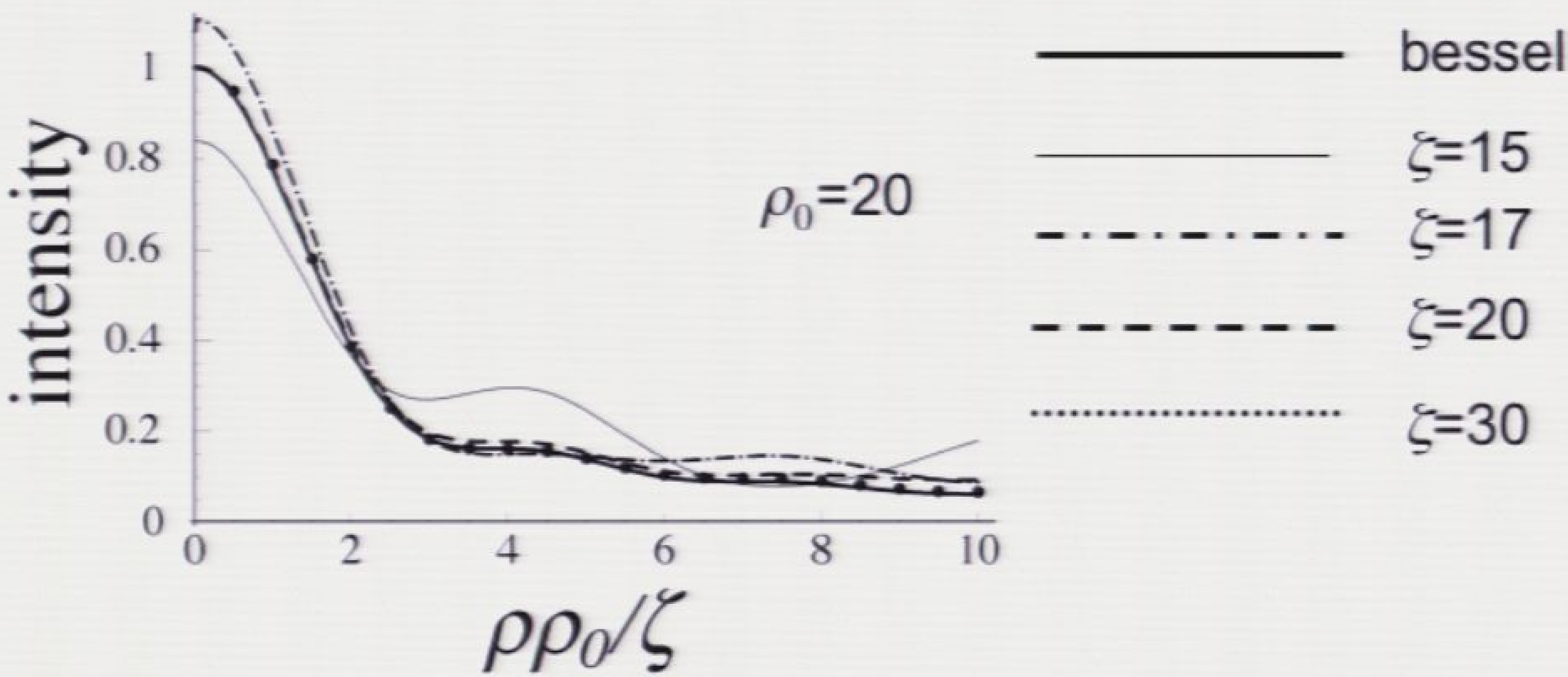

shoulders - flat inflections - at zeros of $J_1(\rho\rho_0/\zeta)$

experiment with $KGd(WO_4)_2$ (monoclinic double tungstate) with Mike Jeffrey and James Lunney

Berry, M V, Jeffrey, M R & Lunney, J G 2006 'Conical diffraction: observations and theory' *Proc R Soc A* 462 1629-1642

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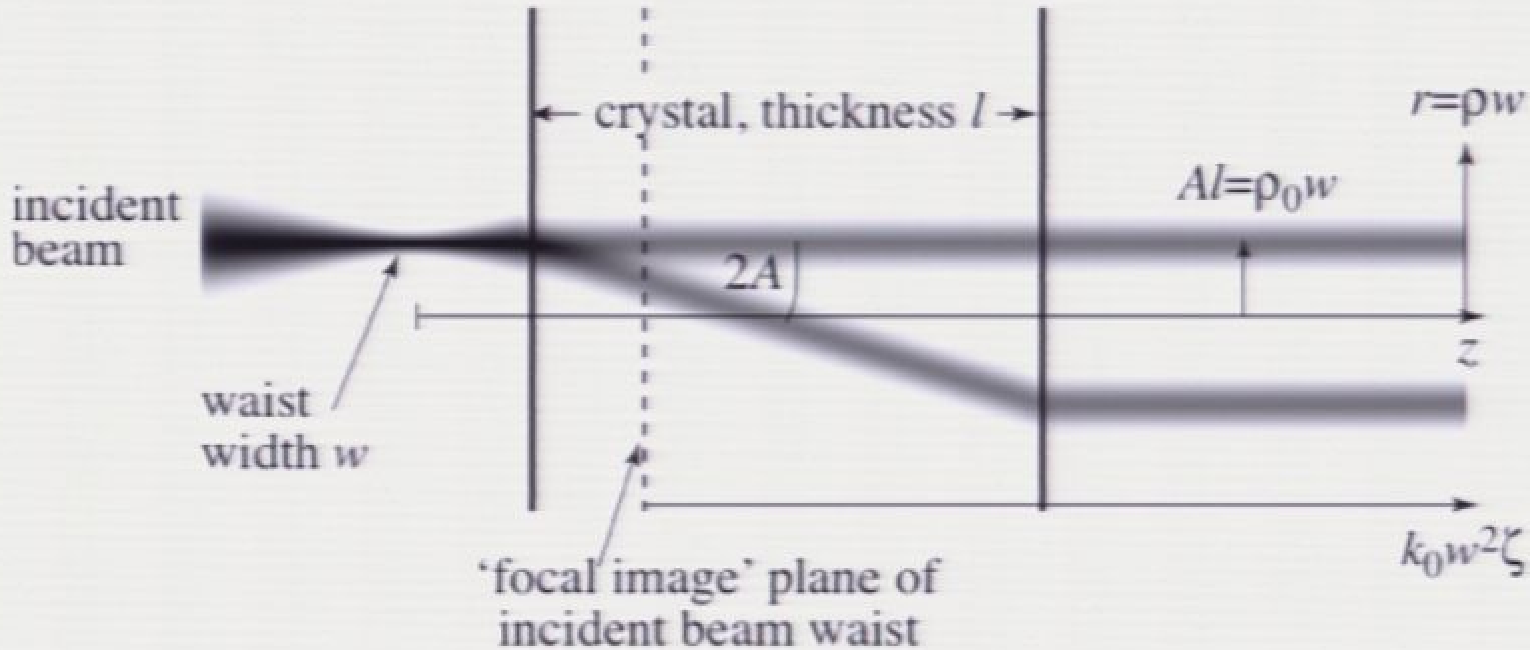

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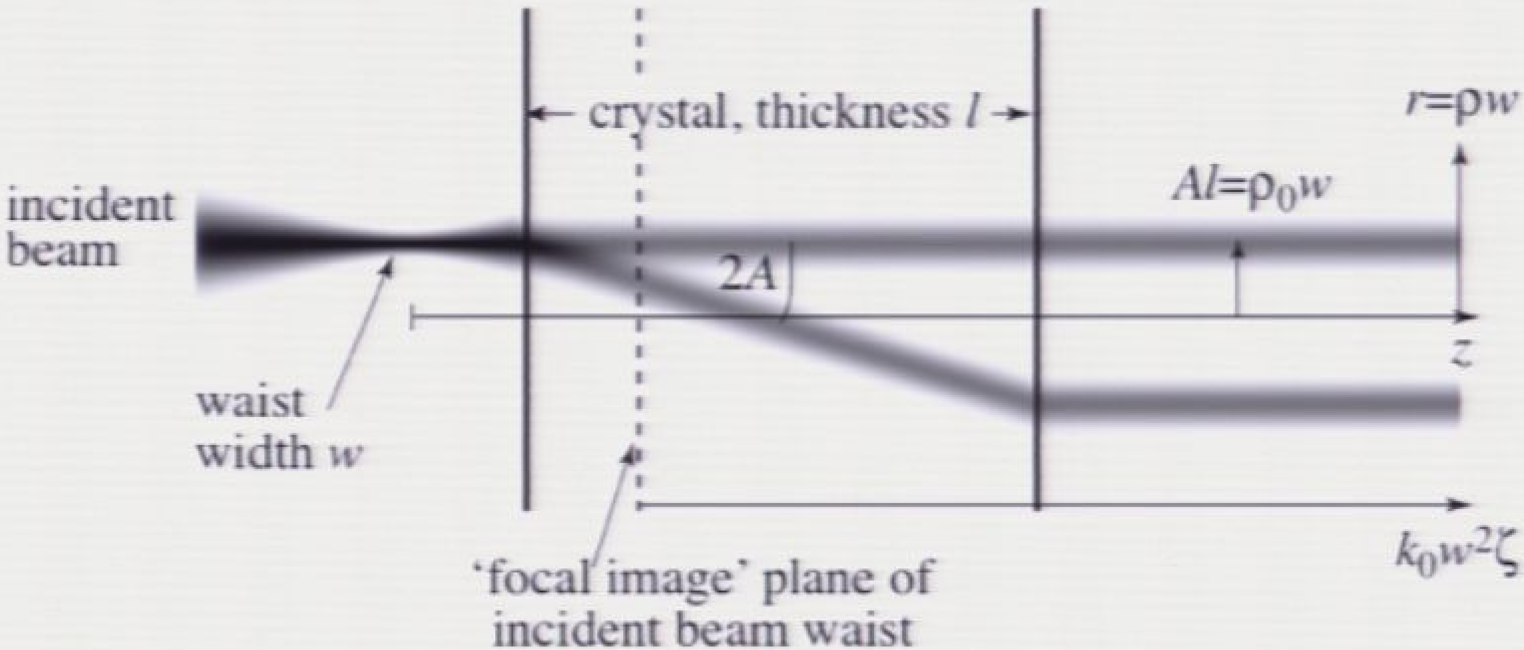

obtain ρ_0 by measuring ring radius *Al* (magnified on distant screen) and *w* (as expanded spot on distant screen)

Berry, M V, Jeffrey, M R & Lunney, J G 2006 'Conical diffraction: observations and theory' *Proc R Soc A* 462 1629-1642

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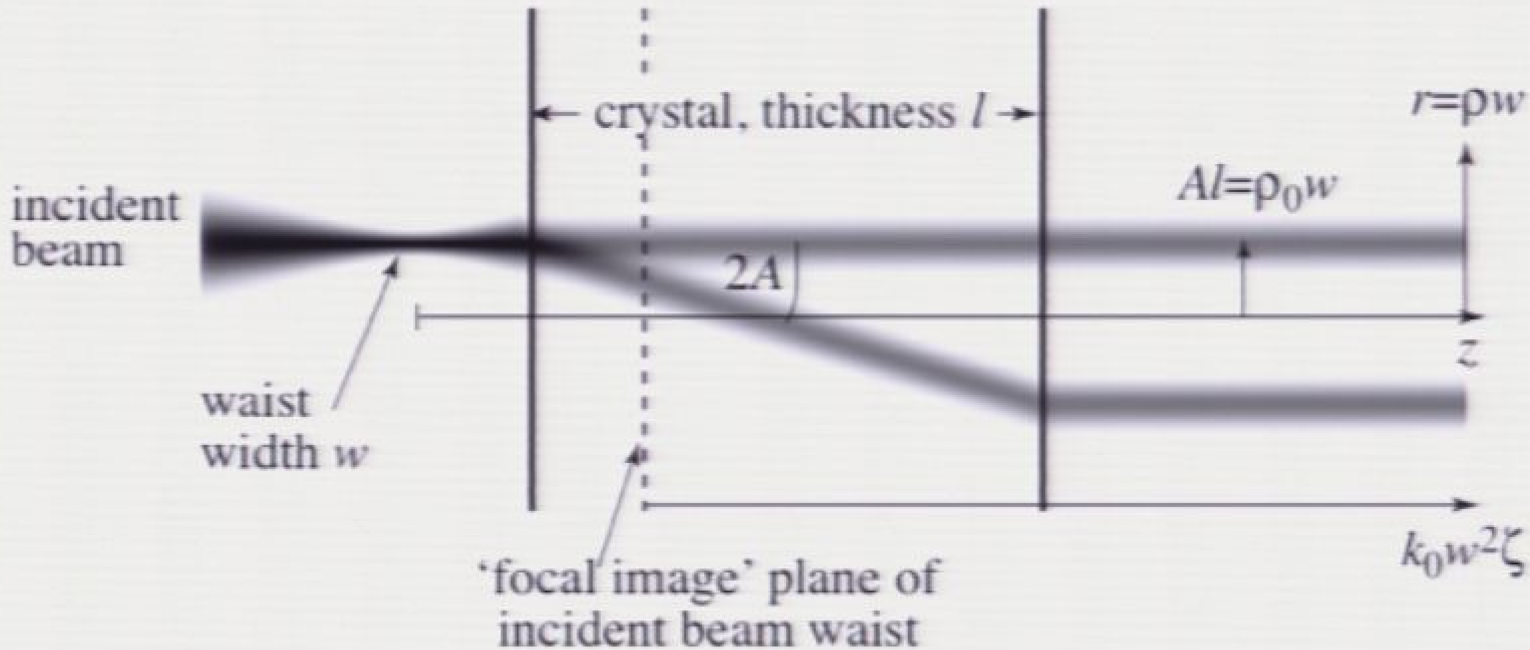
w=7.1±0.6μm, ρ_0=59±10



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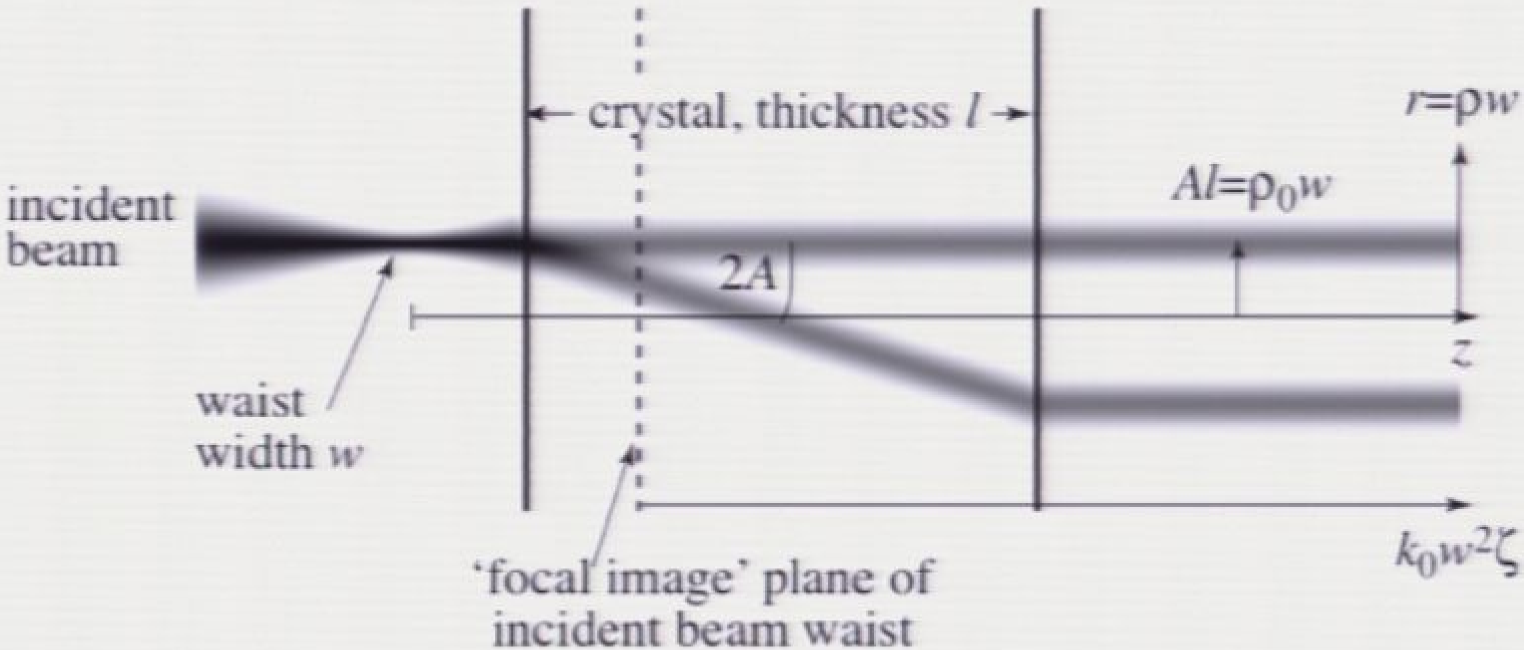

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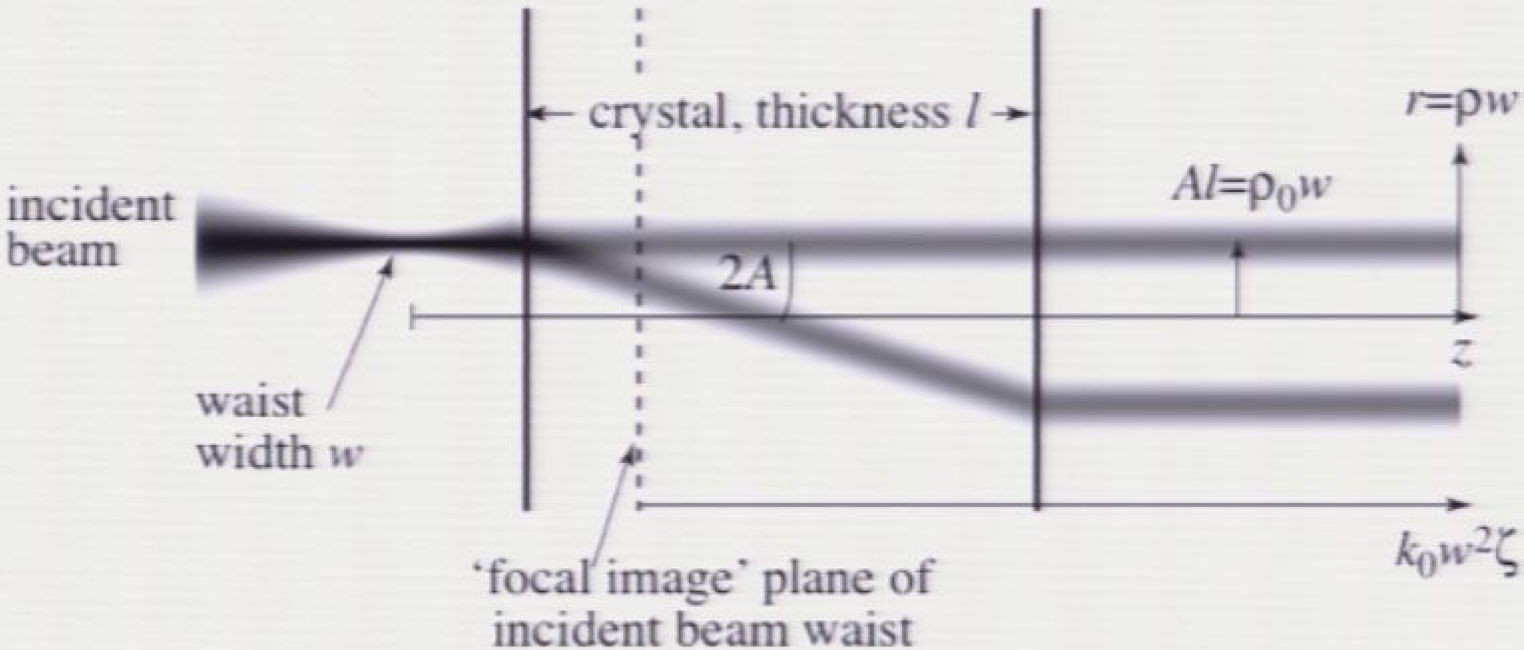

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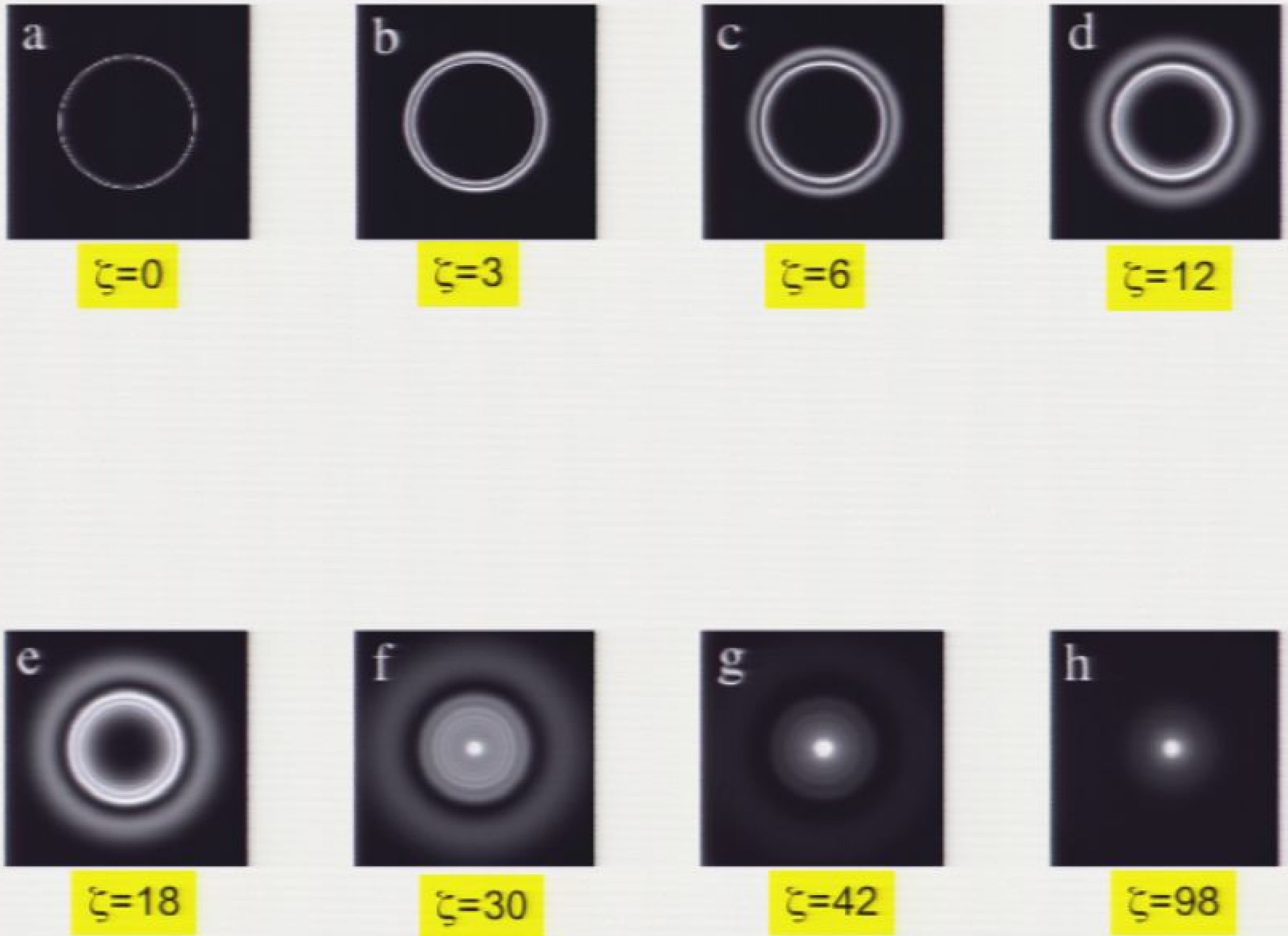
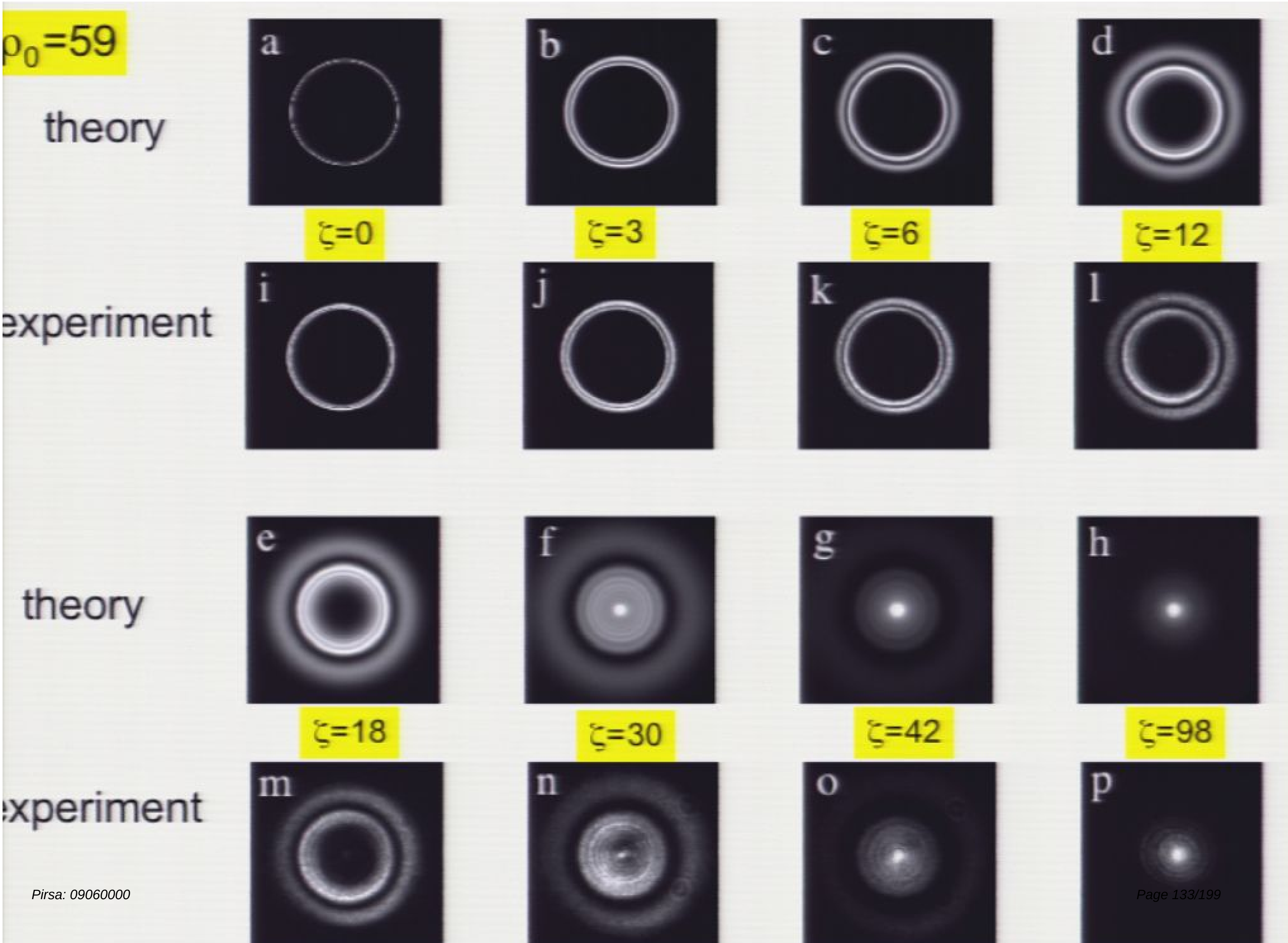
ρ_0=59

ρ_0=59

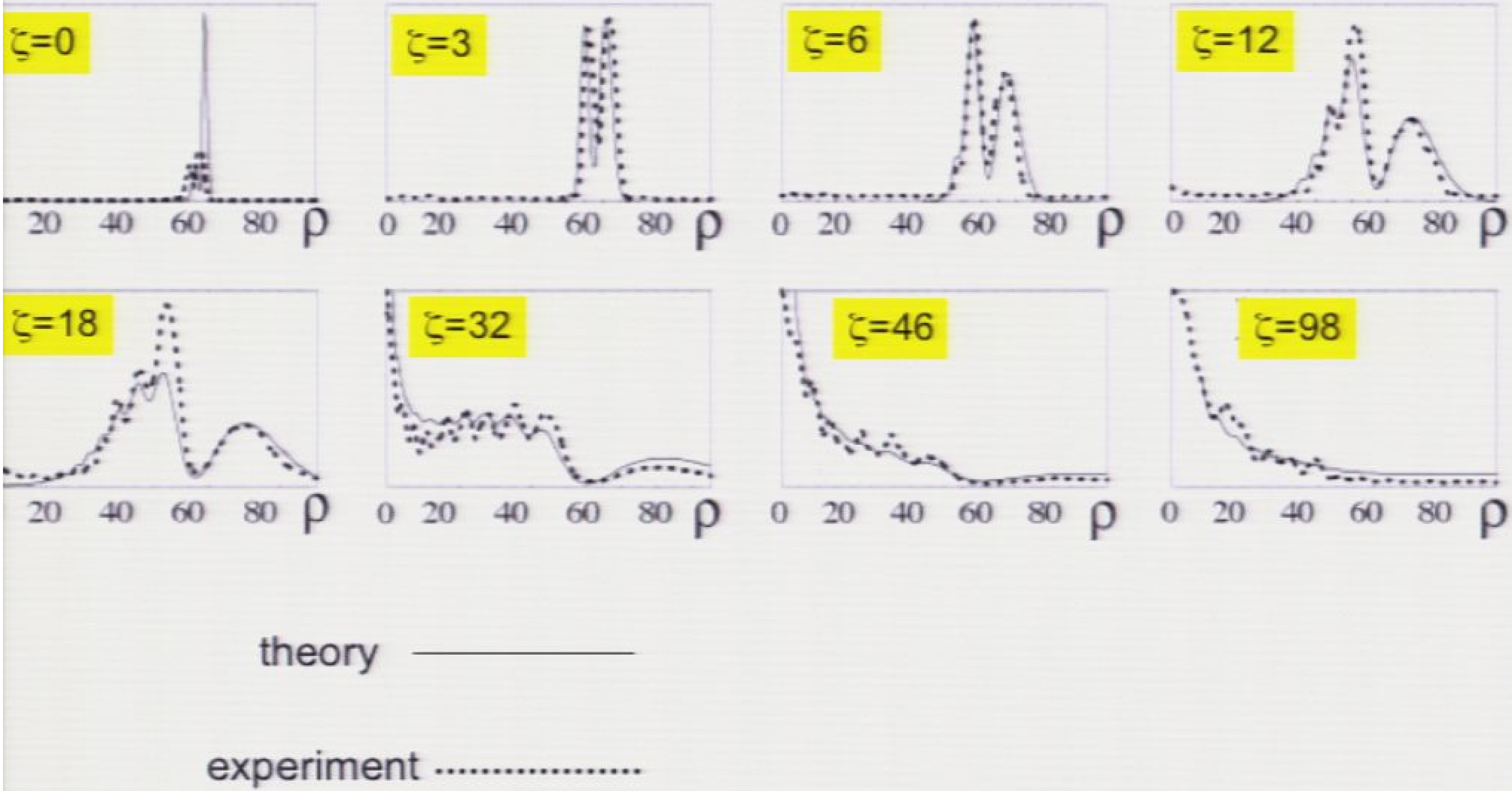
theory

theory



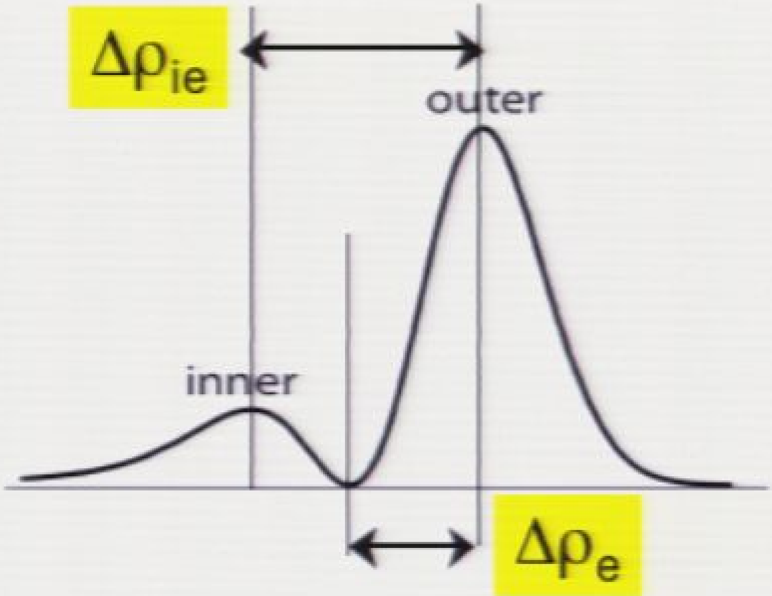


ρ0=59
theory
a
b
c
d
ζ=0
ζ=3
ζ=6
ζ=12
experiment
i
j
k
l
e
f
g
h
theory
ζ=18
ζ=30
ζ=42
ζ=98
experiment
m
n
o
p



ζ=0
ζ=3
ζ=6
ζ=12
ζ=18
ζ=32
ζ=46
ζ=98
20 40 60 80 ρ
0 20 40 60 80 ρ
theory
experiment

Poggendorff ring dimensions



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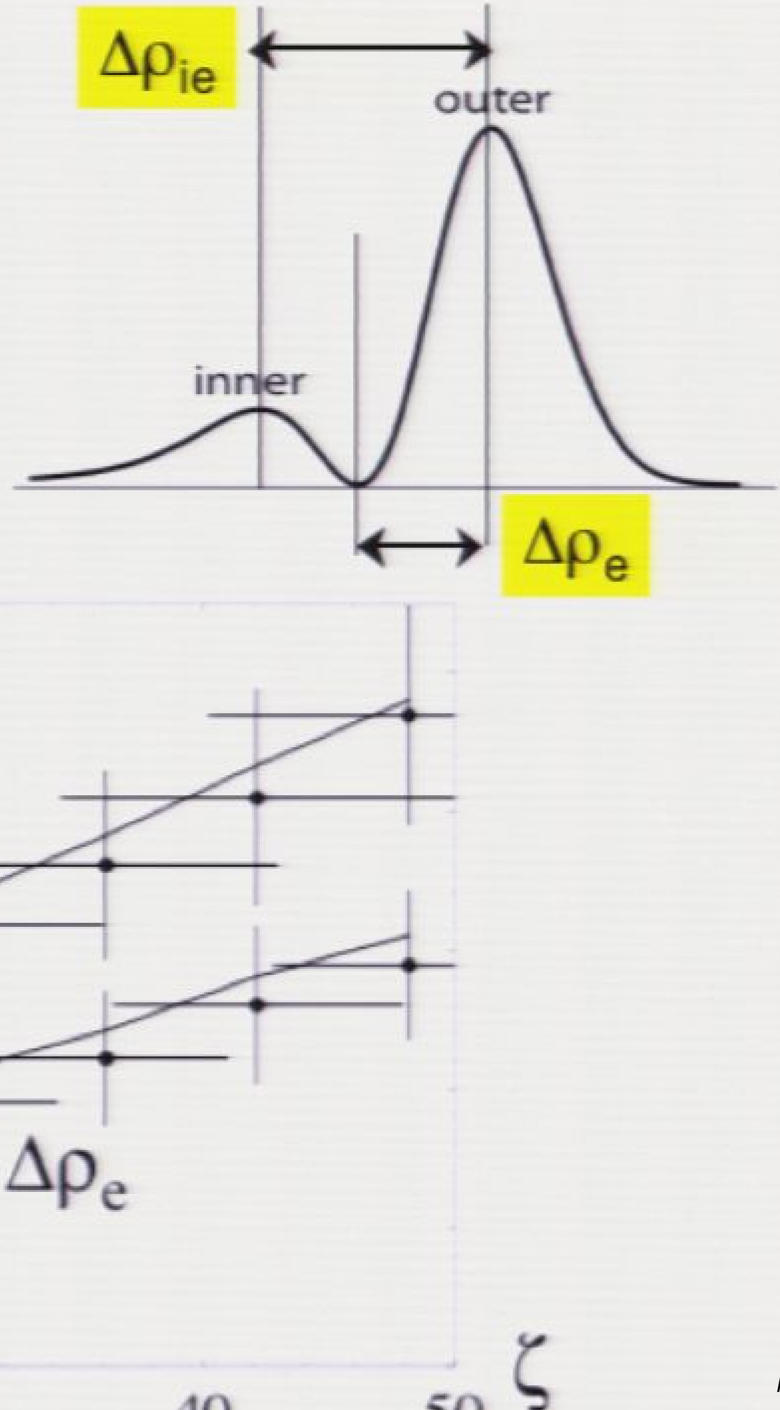
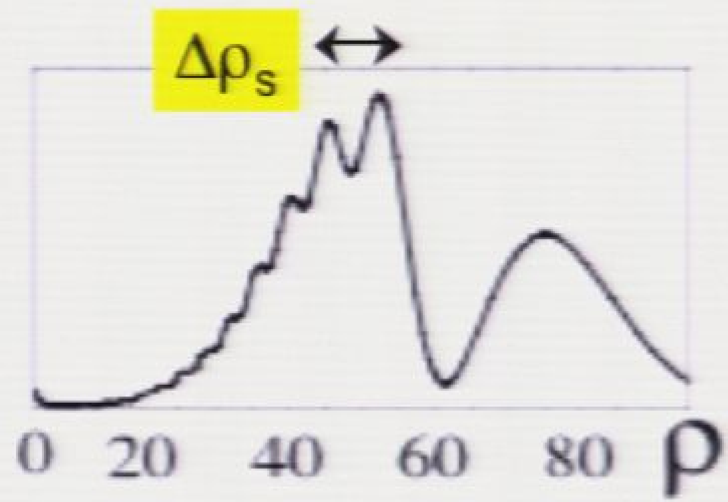


first secondary fringe width



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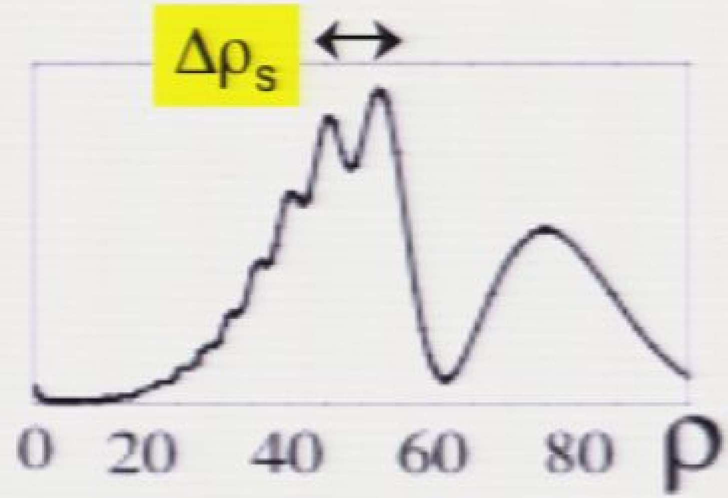
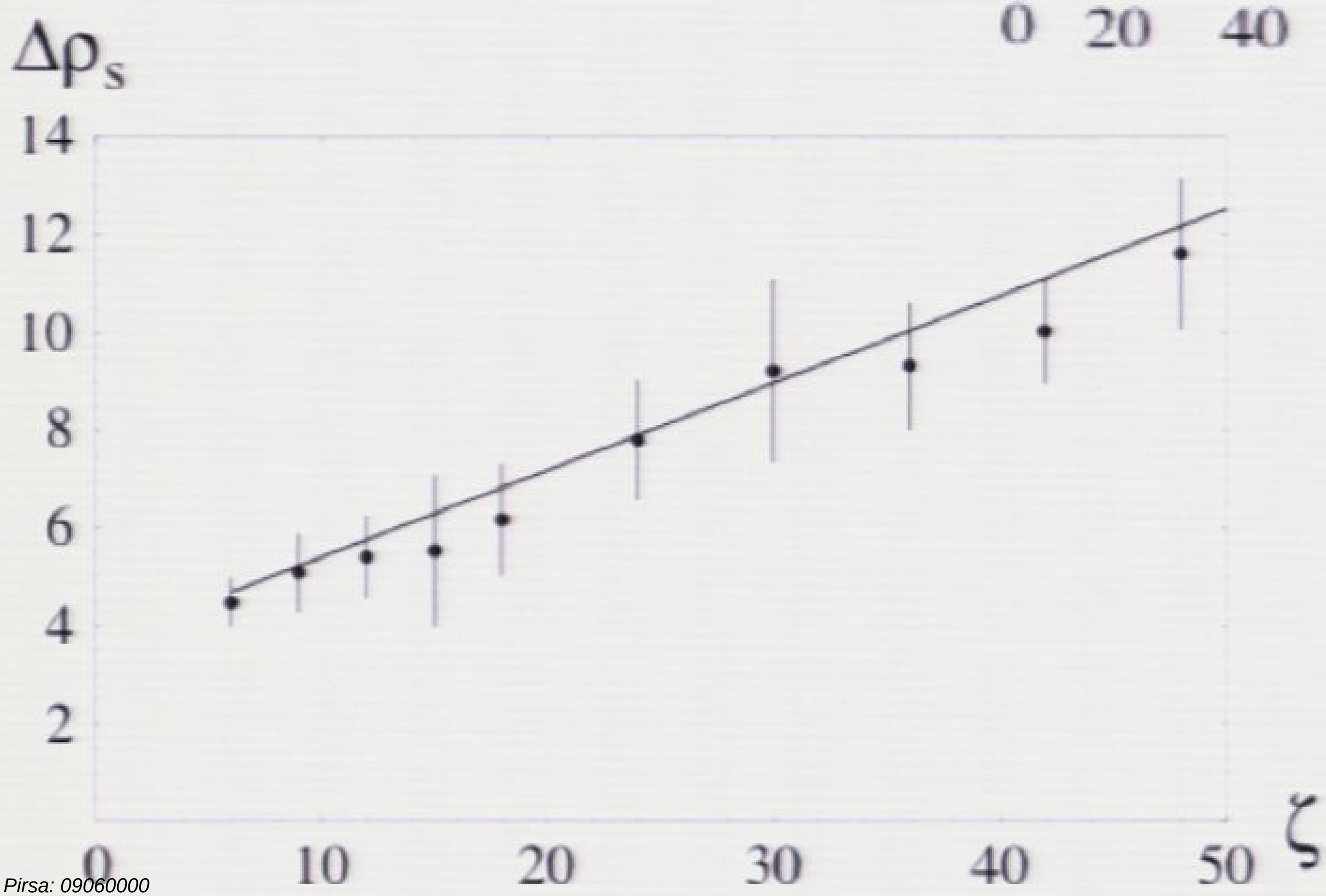
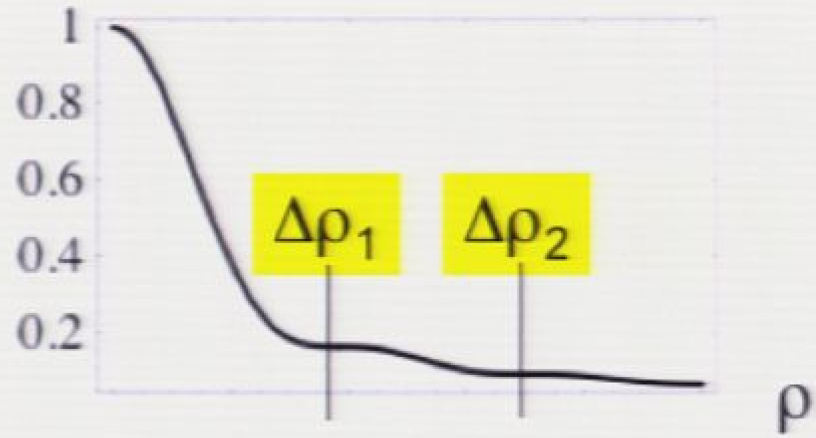




ringes decorating central spot



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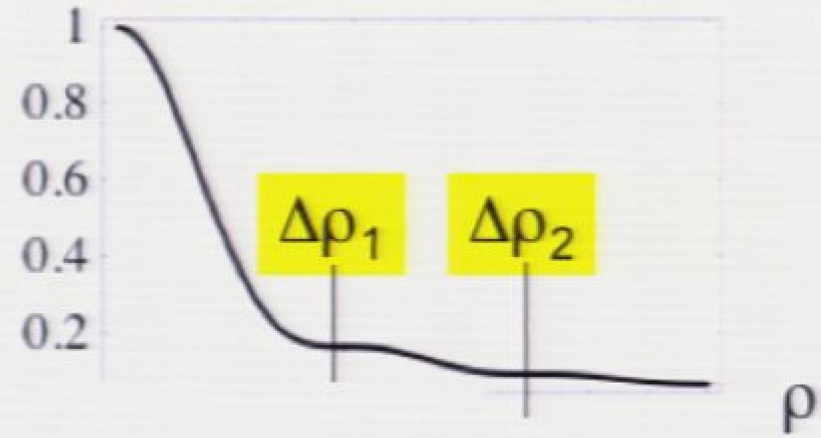

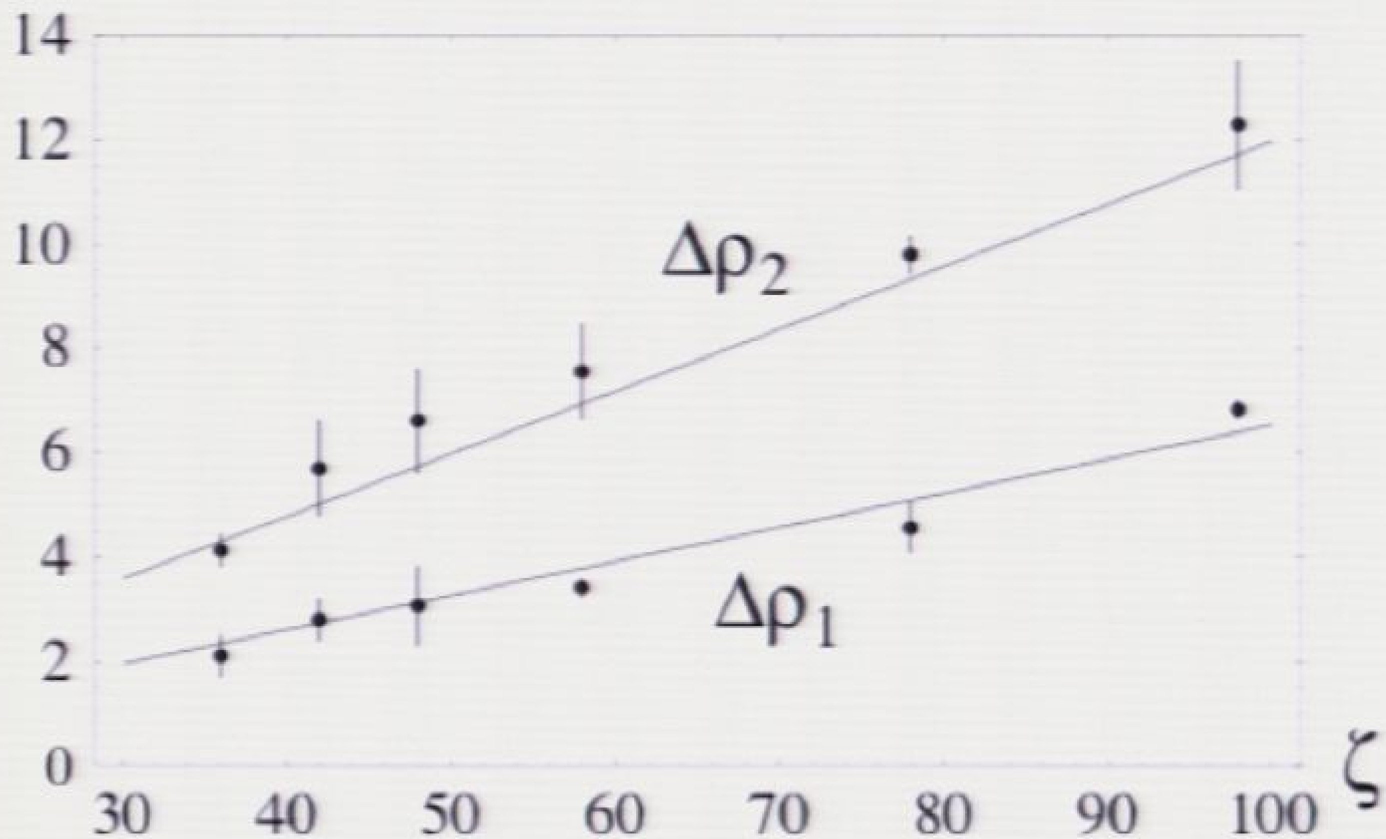

175 year multinational story:

Hamilton (Ireland 1832)

Lloyd (Ireland 1833)

Poggendorff (Germany 1839)

Voigt (Germany 1905)

Raman (India 1941)

Belskii-Khapalyuk (Belarus 1978)

Bloembergen-Schell (USA 1978)

Uhlmann (Chile 1982)

Warnick-Arnold (USA 1997)

Berry-Jeffrey (UK 2004)

Lunney (Ireland 2005)...

DIY conoscopy

travelling through the crystal, the two waves get out of step, and can be made to interfere with a ***black light sandwich***

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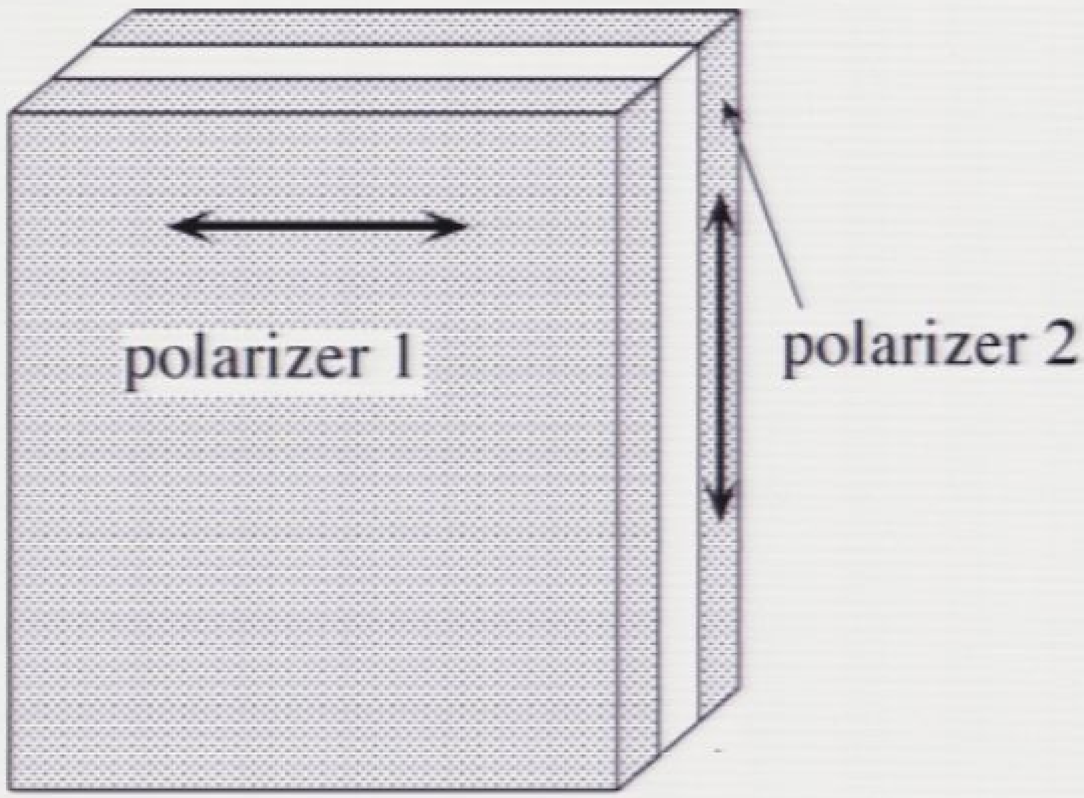
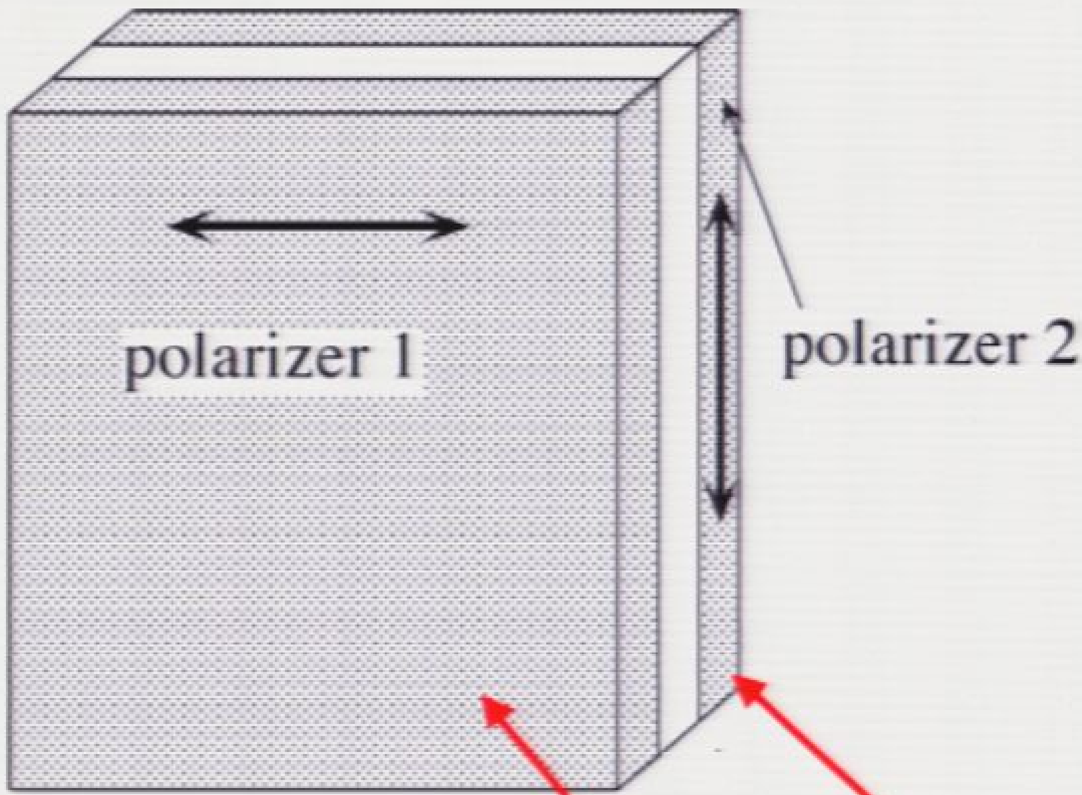


DIY conoscopy travelling through the crystal, the two waves get out of step, and can be made to interfere with a ***black light sandwich***



the bread: crossed polarizing sheets

DIY conoscopy

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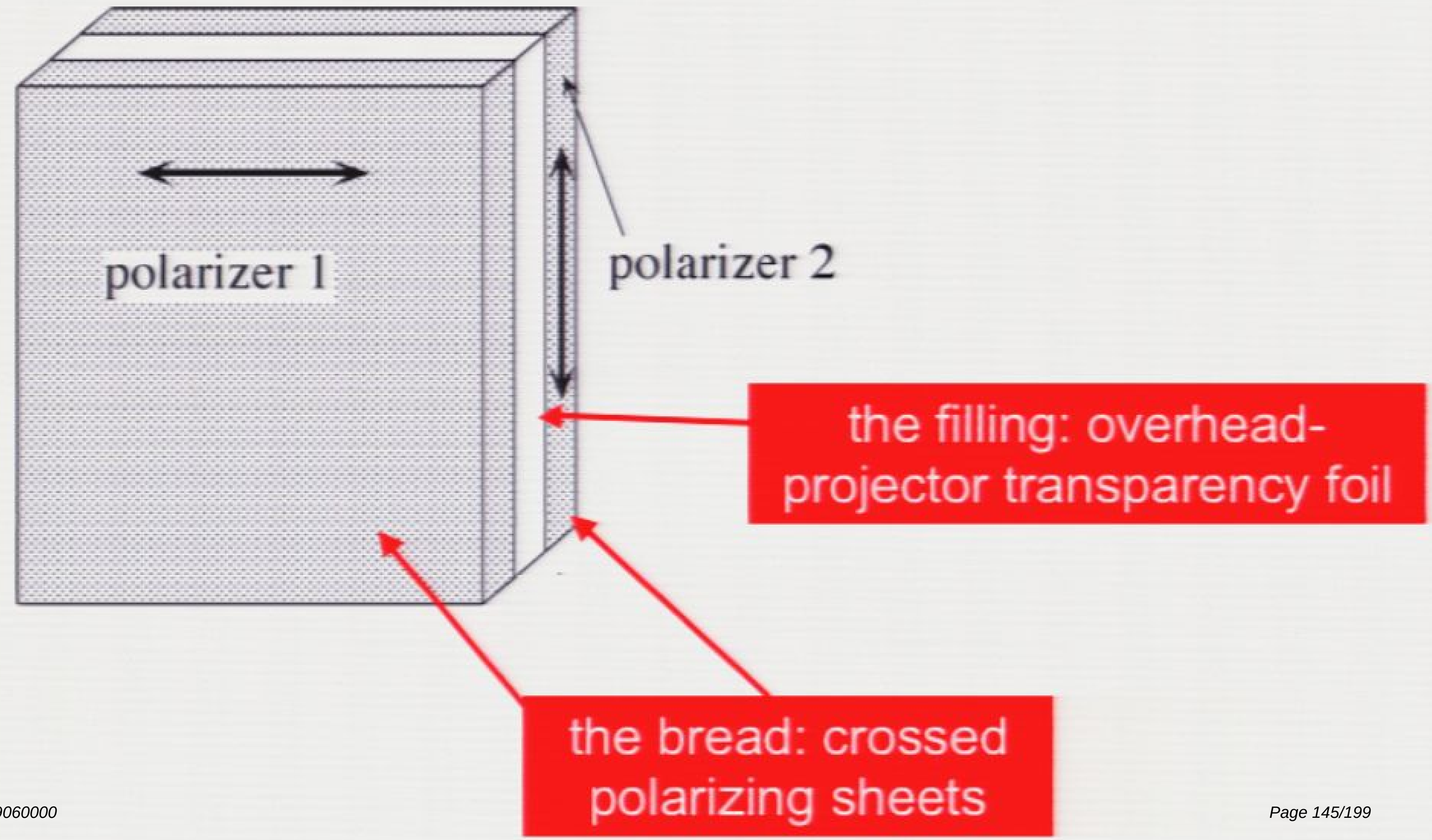
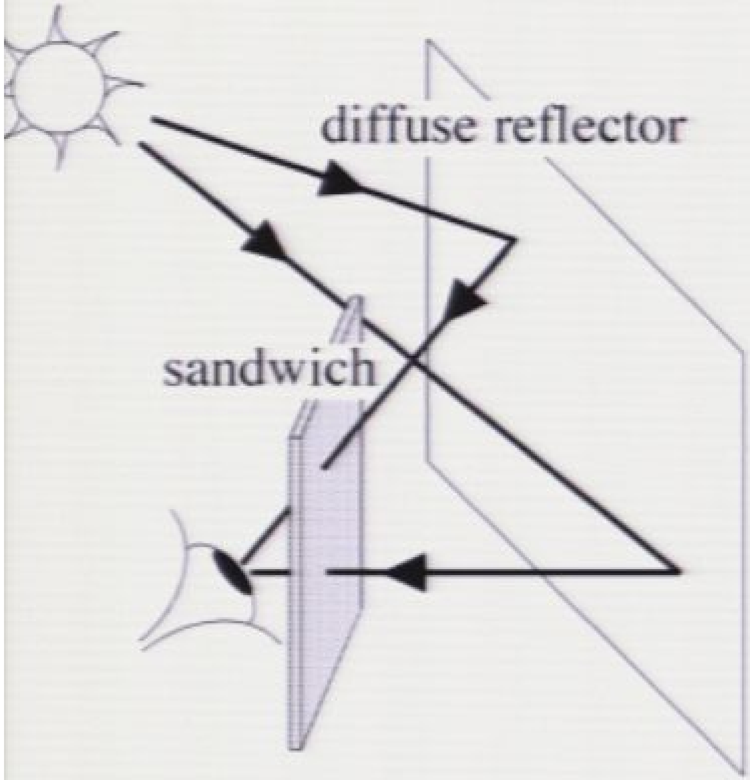
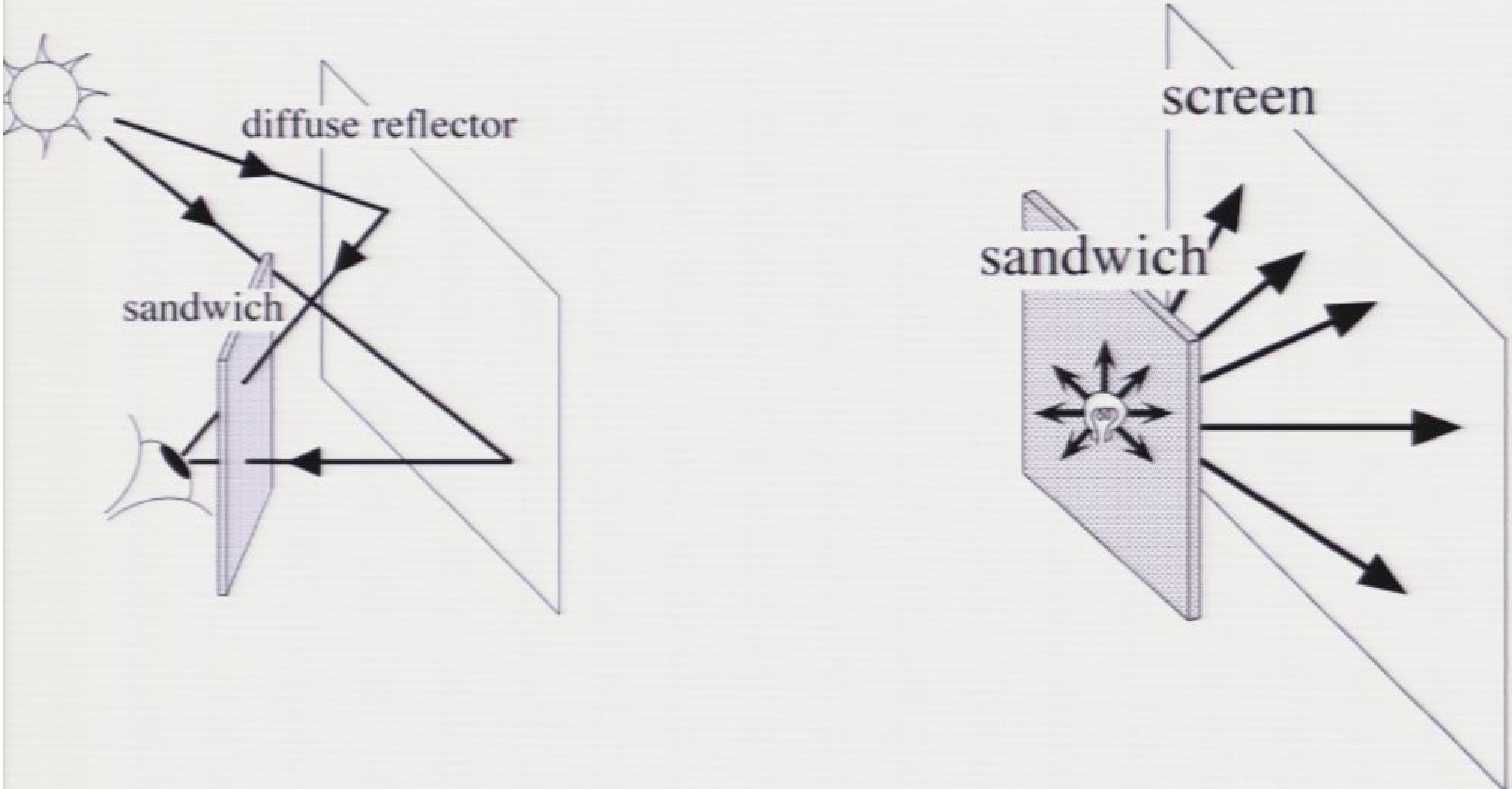
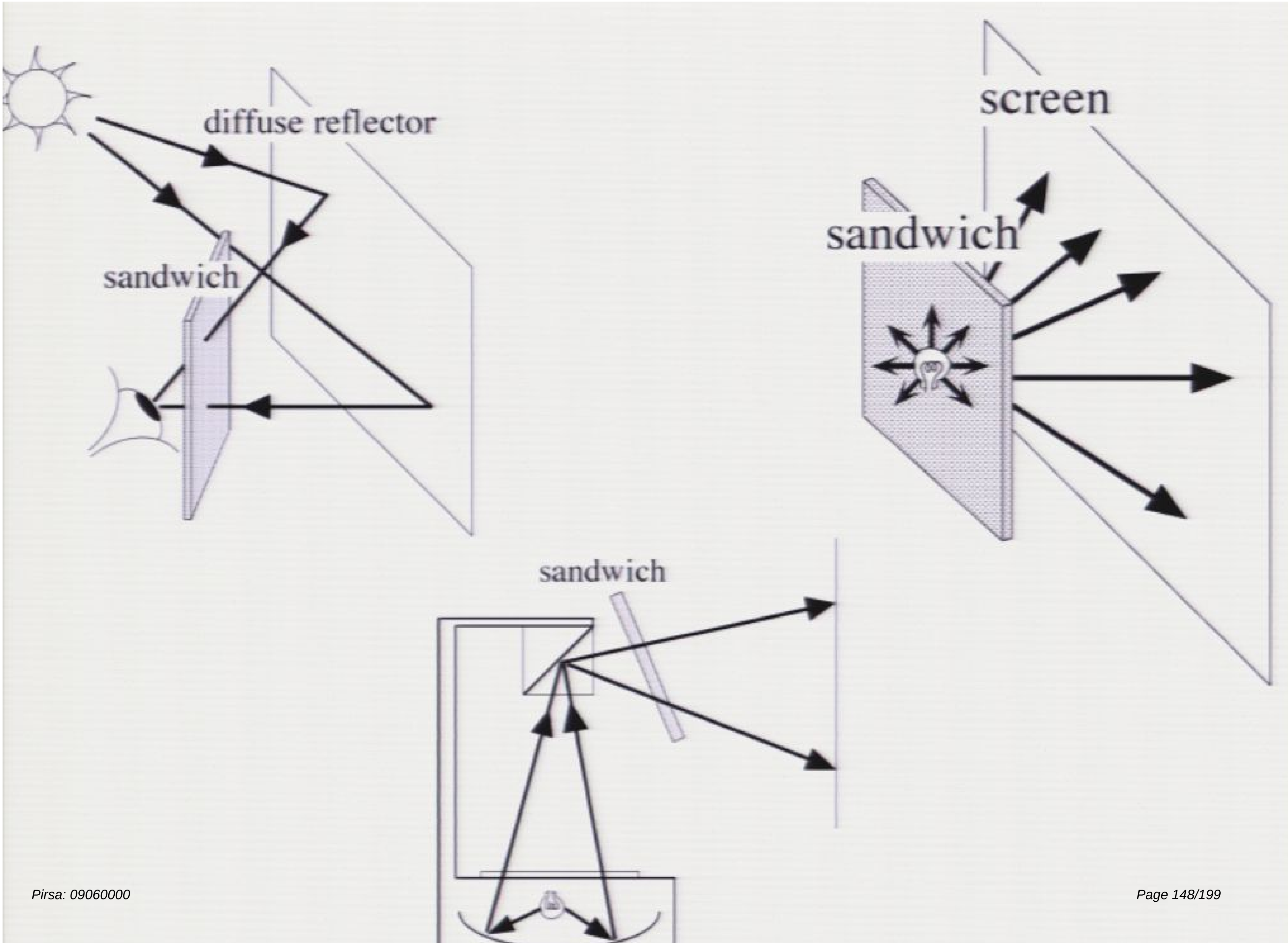



diffuse reflector
sandwich


diffuse reflector
sandwich
screen
sandwich


diffuse reflector
sandwich
screen
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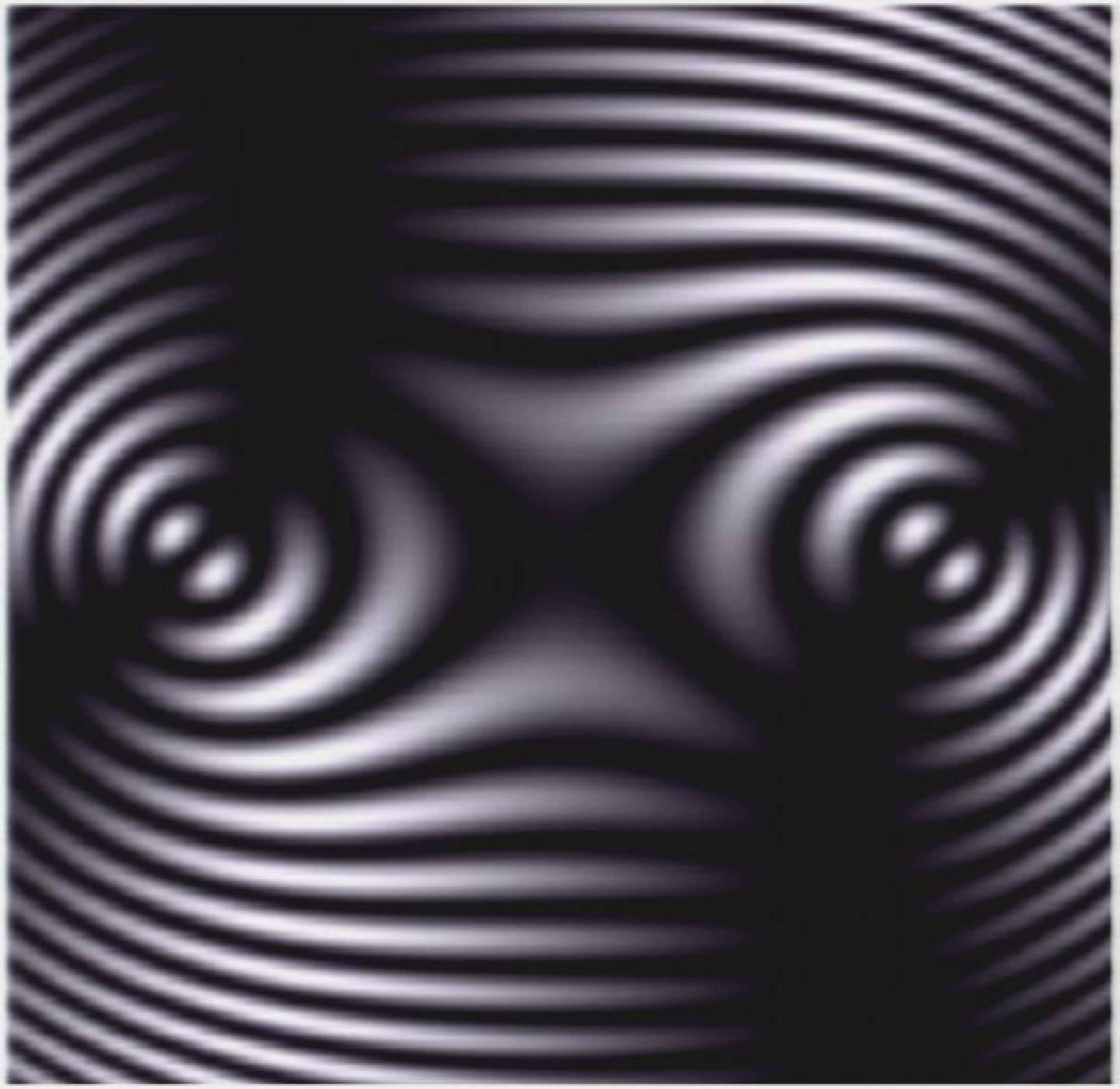
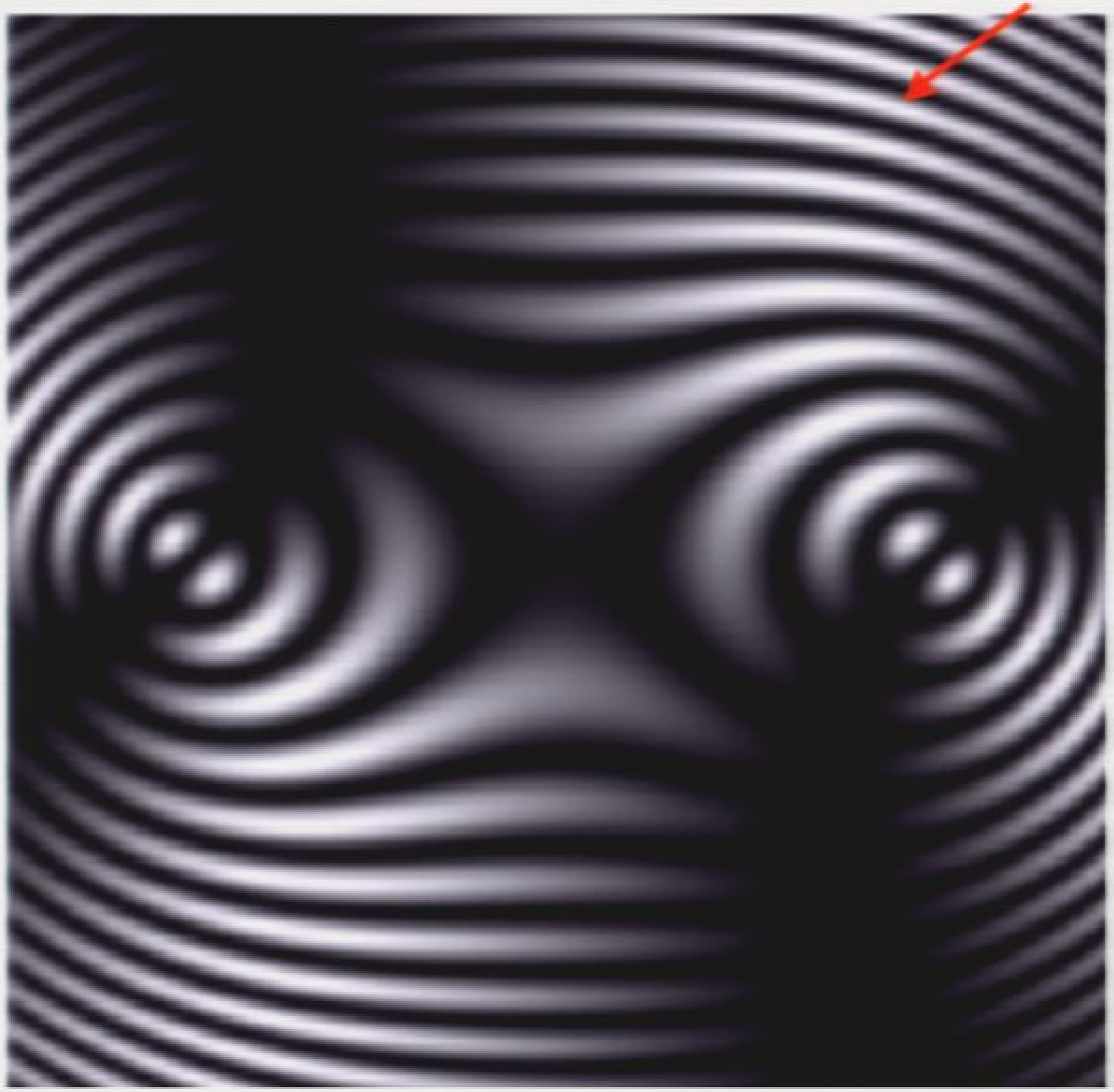
sandwich

:onoscopic figure

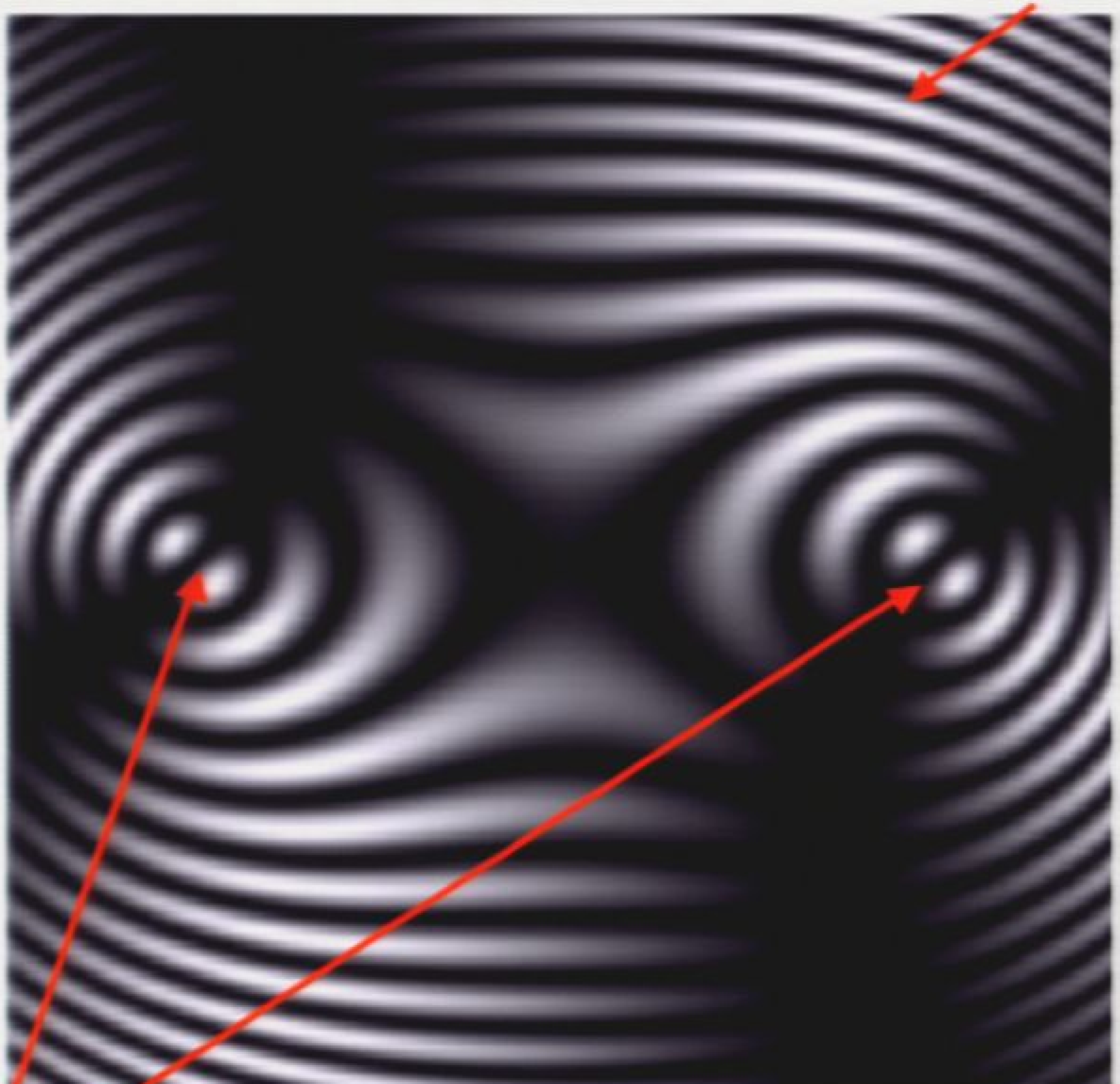


conoscopic figure



conoscopic figure

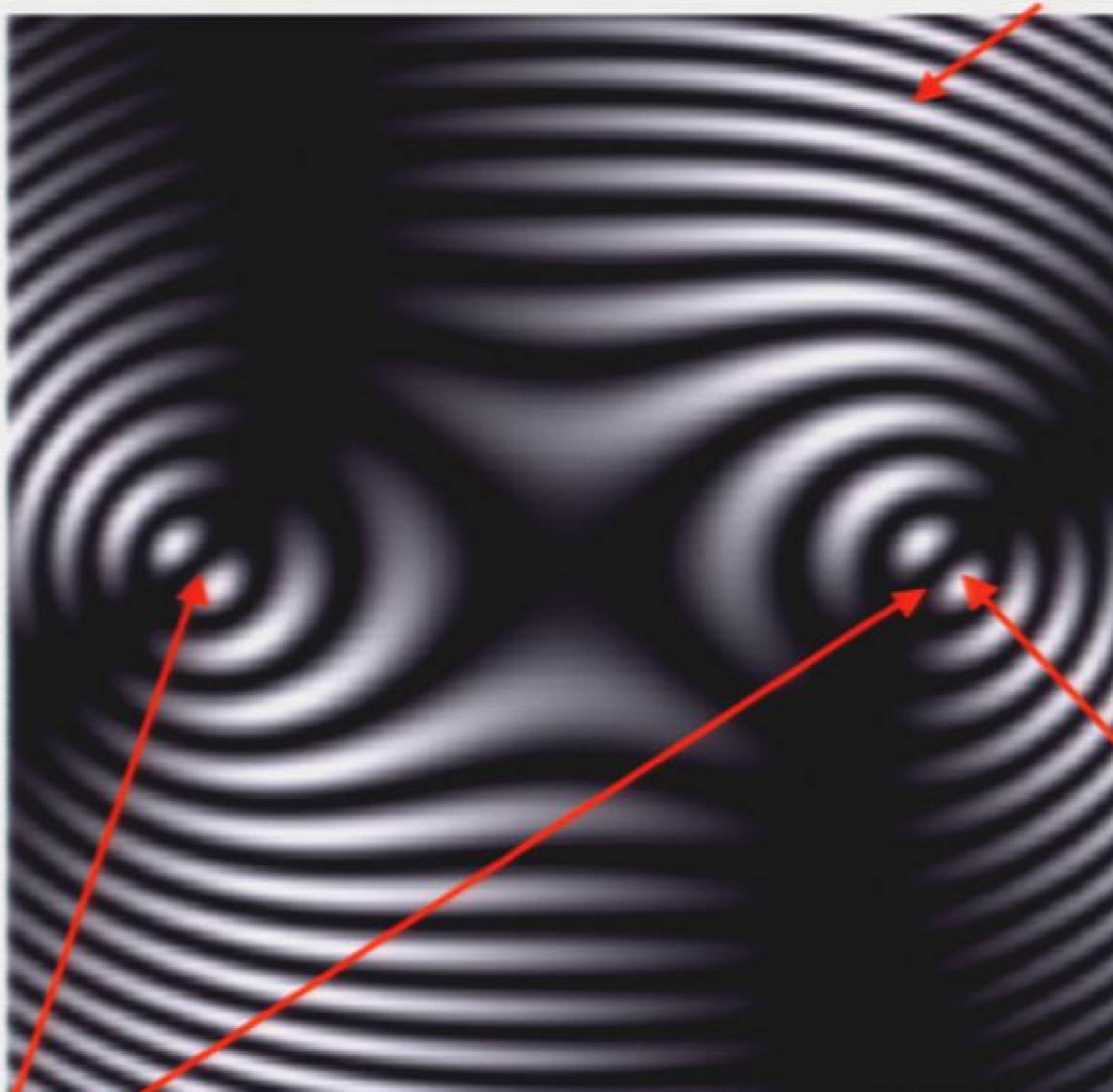
interference fringes



ullseyes at directions of diabolical points (optic axes)

conoscopic figure

interference fringes



black brush,
from polarization
geometric phase

ullseyes at directions of diabolical points (optic axes)

generalizations of Hamilton's conical refraction, radically altering the mathematical structure (with Mike Jeffrey)

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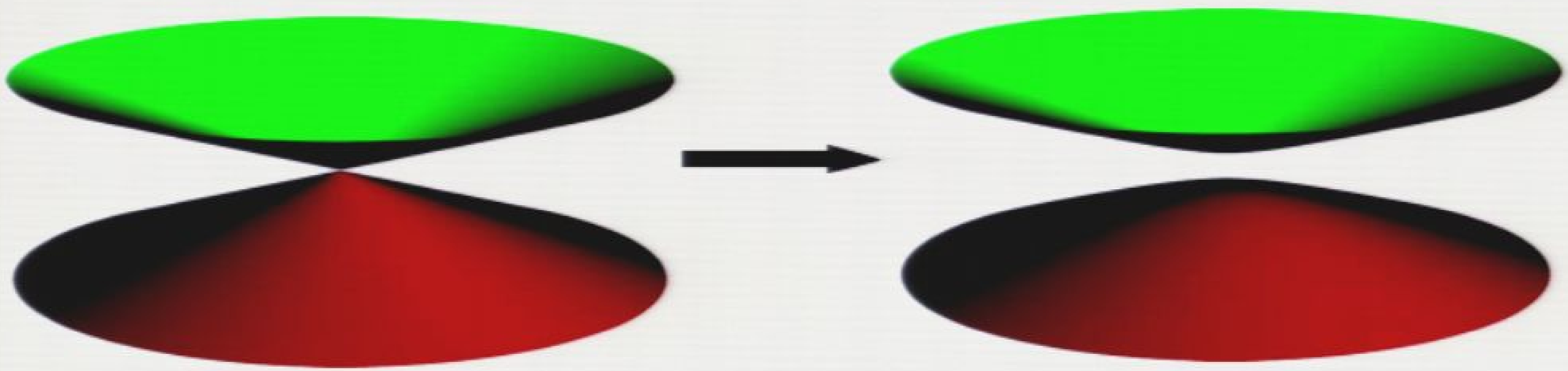

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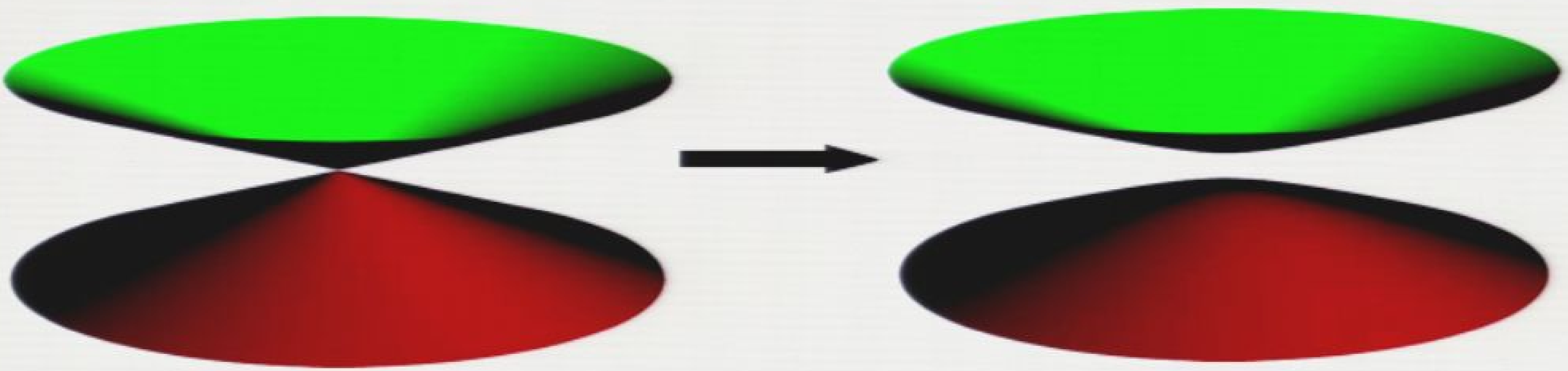

geometrical optics, with chirality parameter σ

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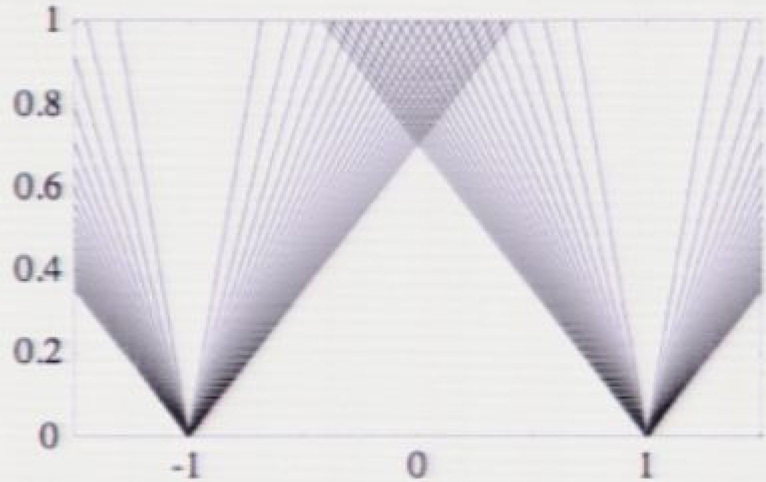
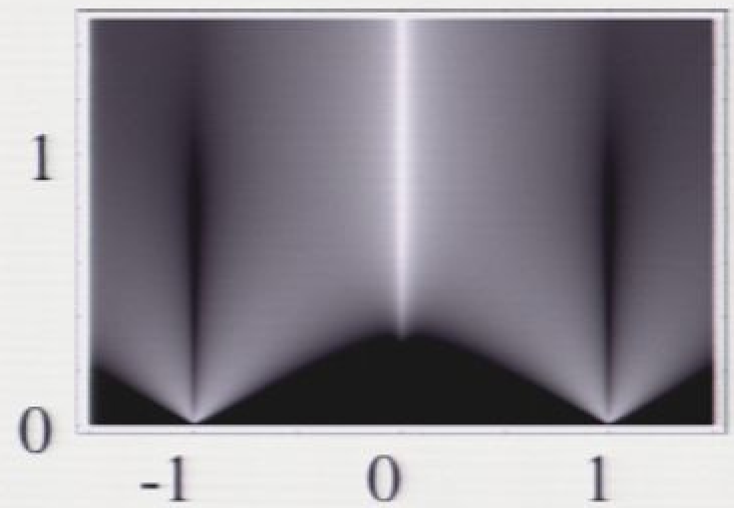
$\sigma = 0$





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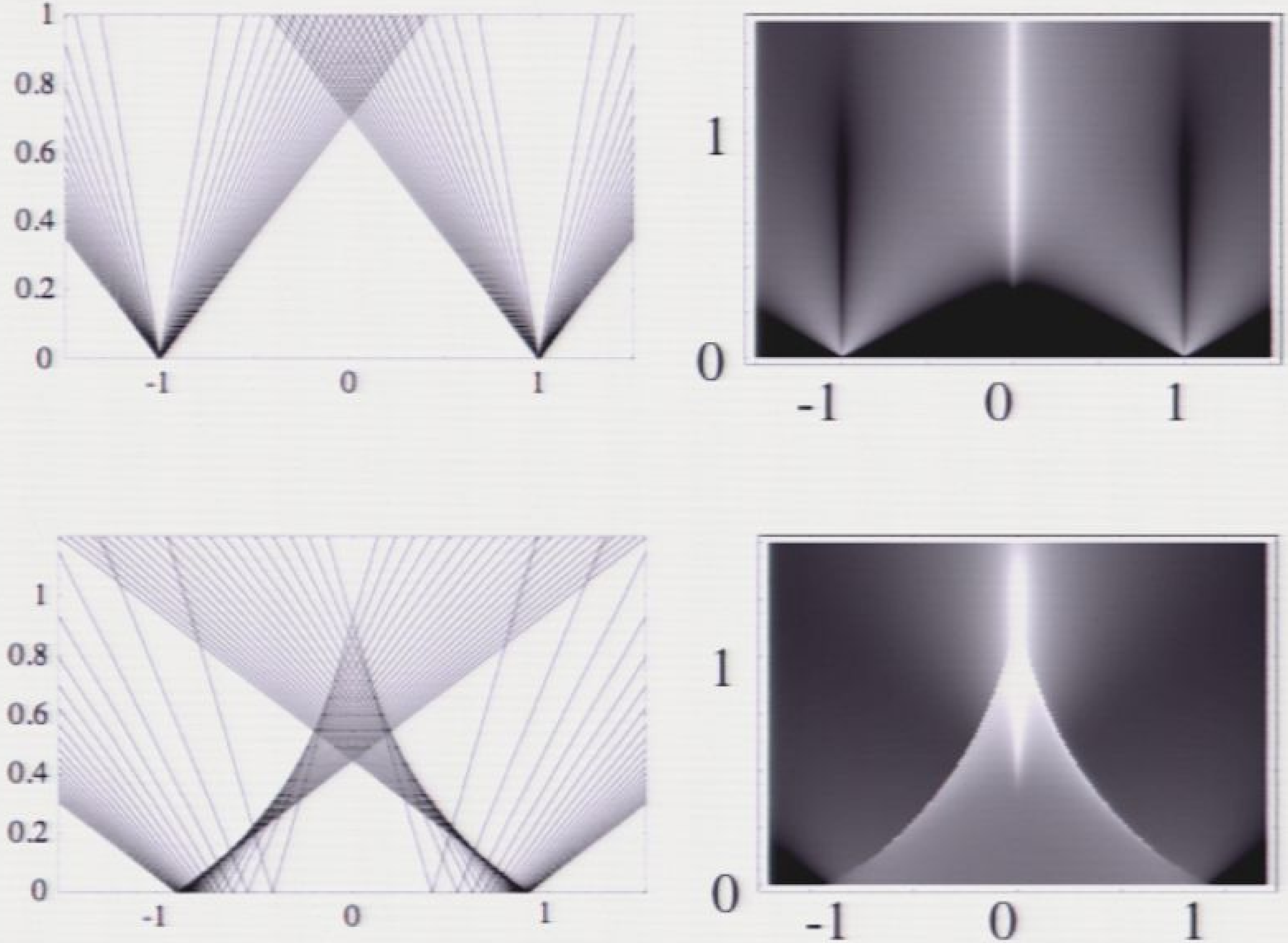
$\sigma=1$



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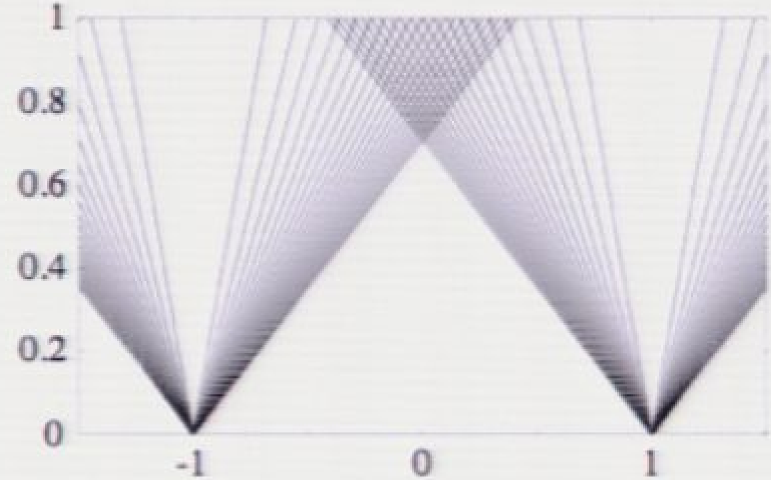

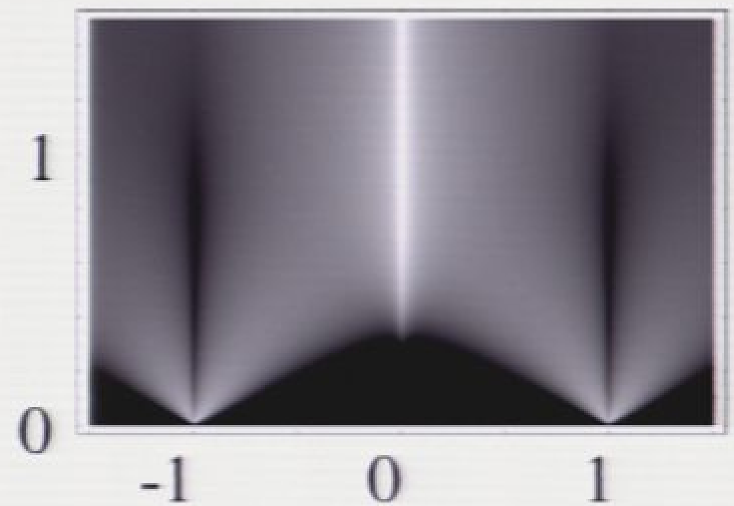

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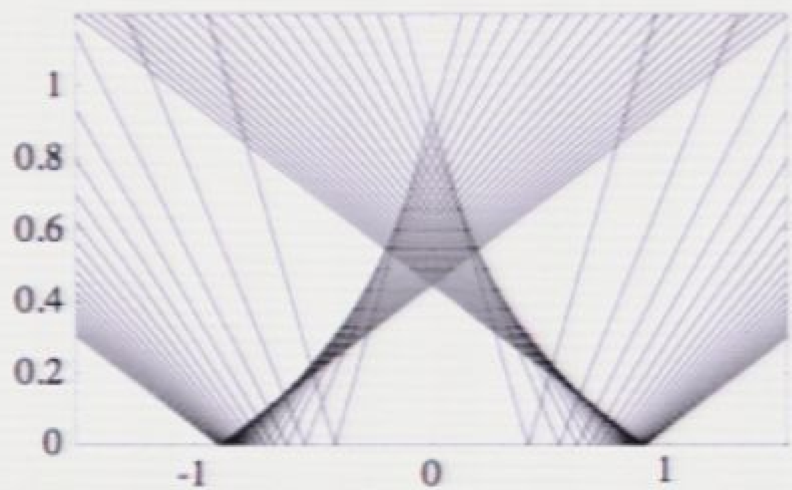

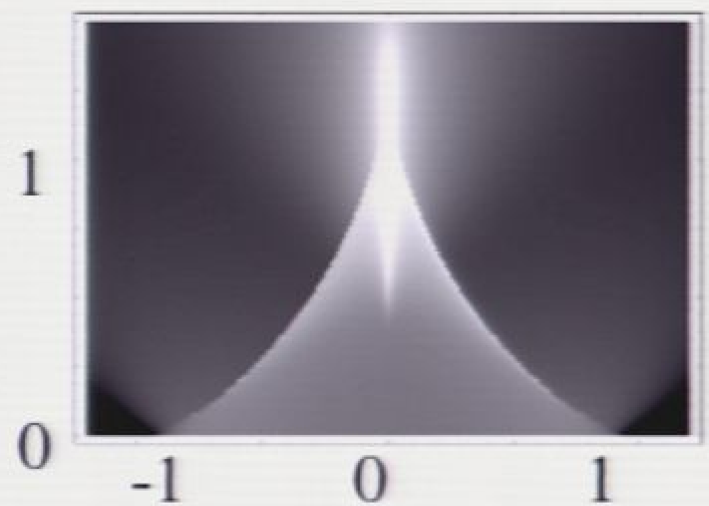

a ***caustic*** - cusped focal cone - springs out from the Hamilton-Poggendorff ring

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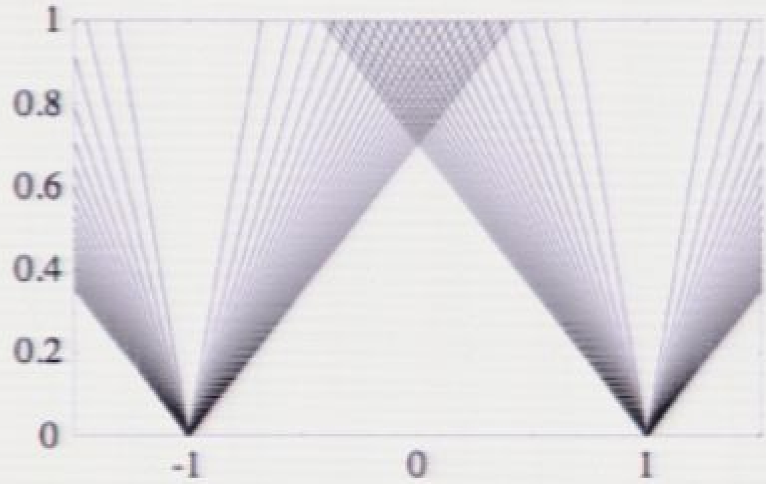

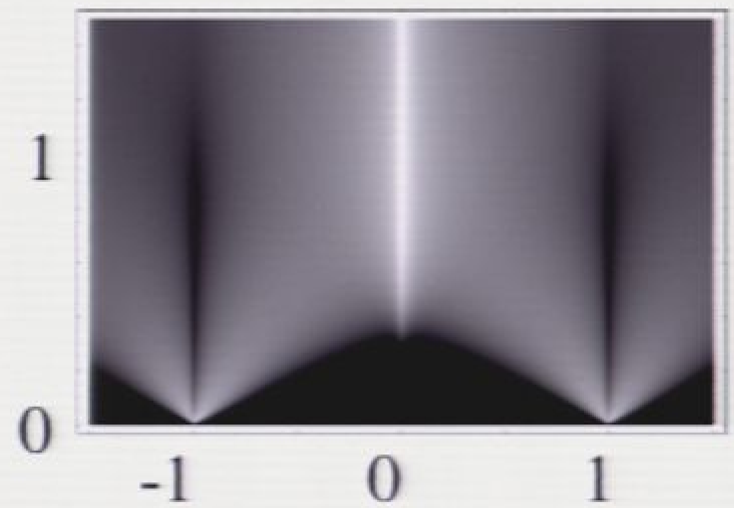

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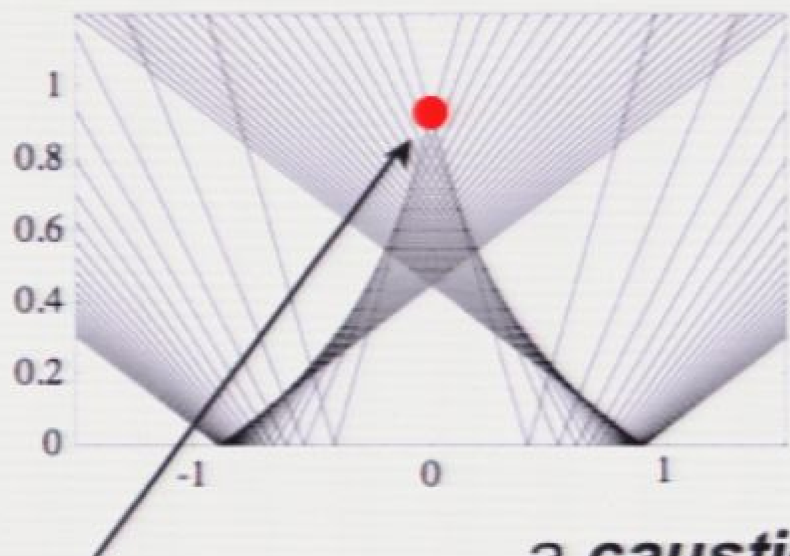
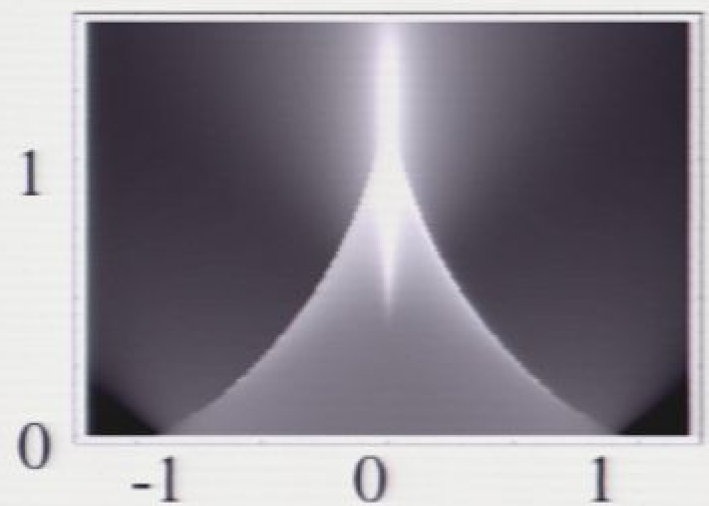
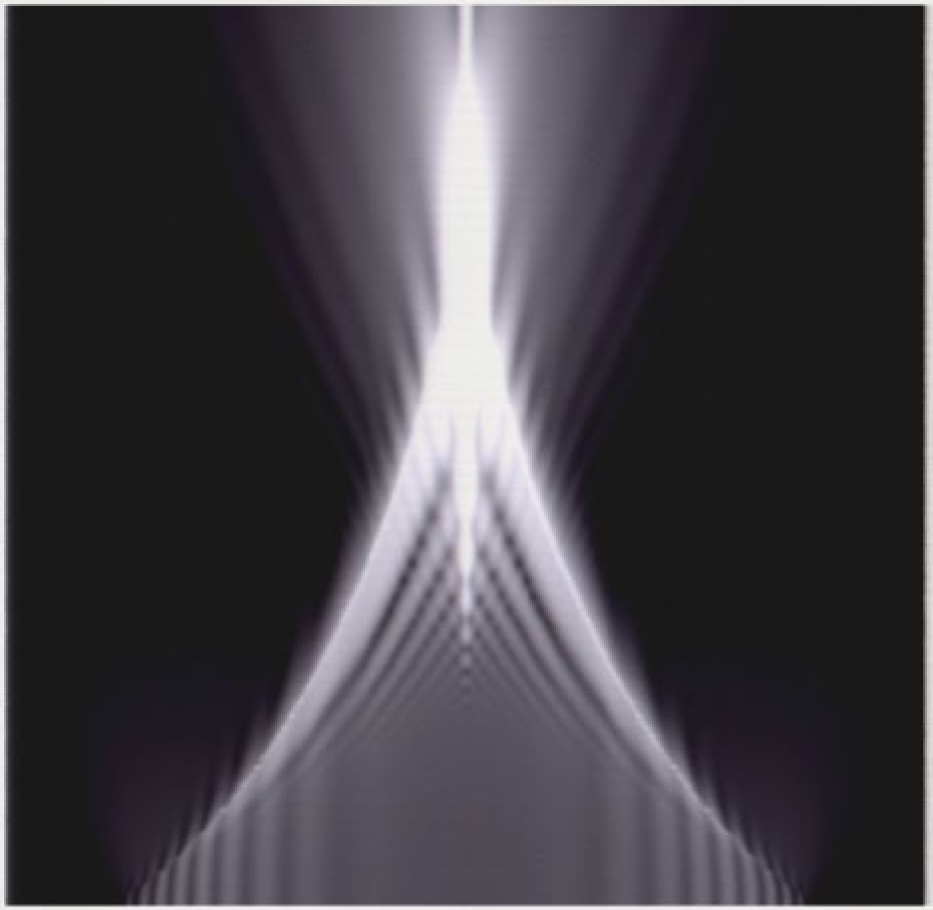


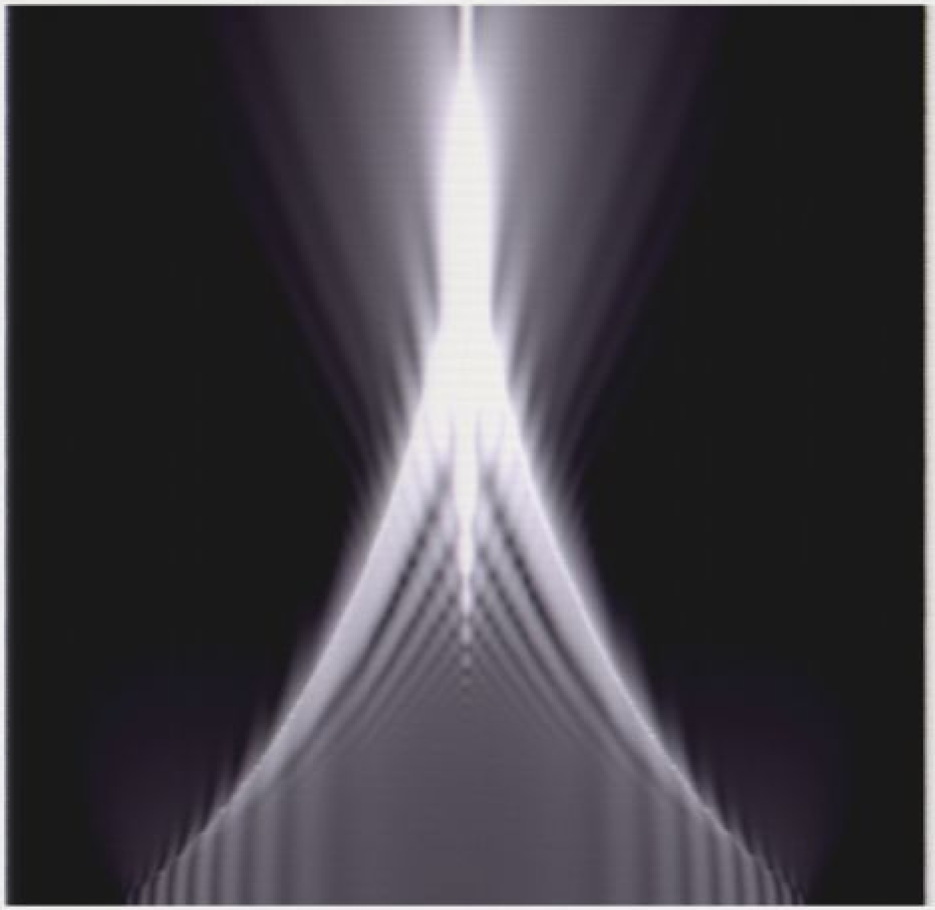


cusp height $1/\sigma$

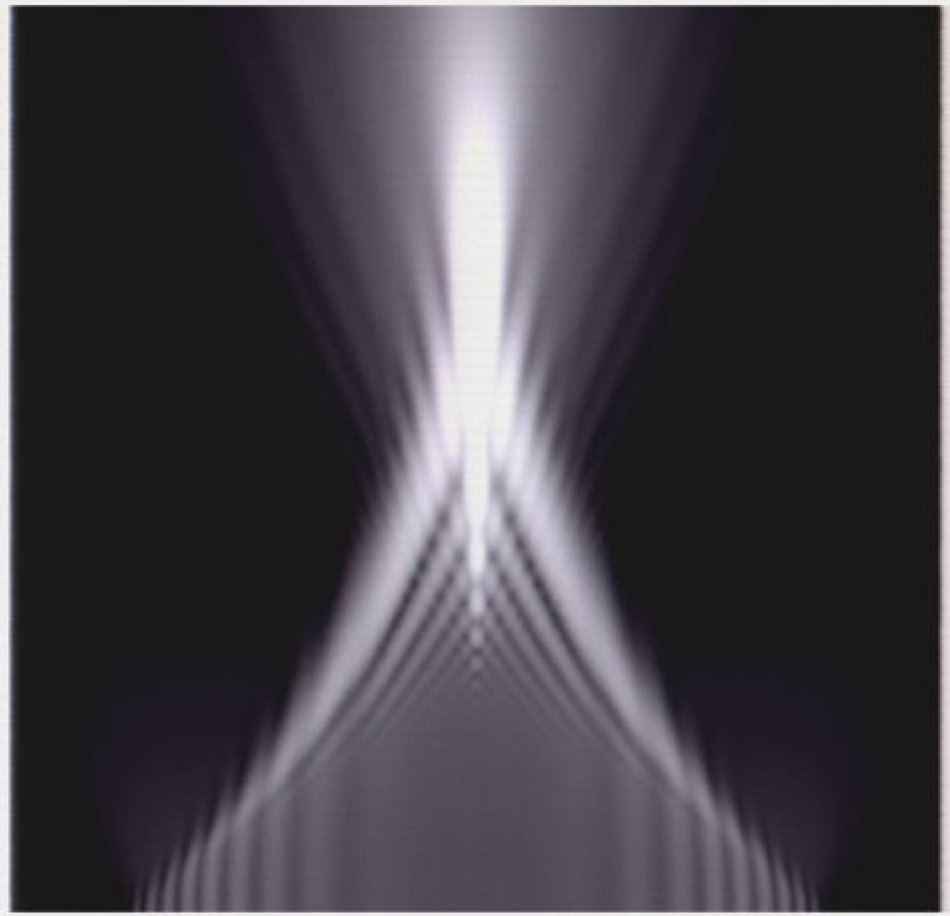
a ***caustic*** - cusped focal cone - springs out from the Hamilton-Poggendorff ring rays



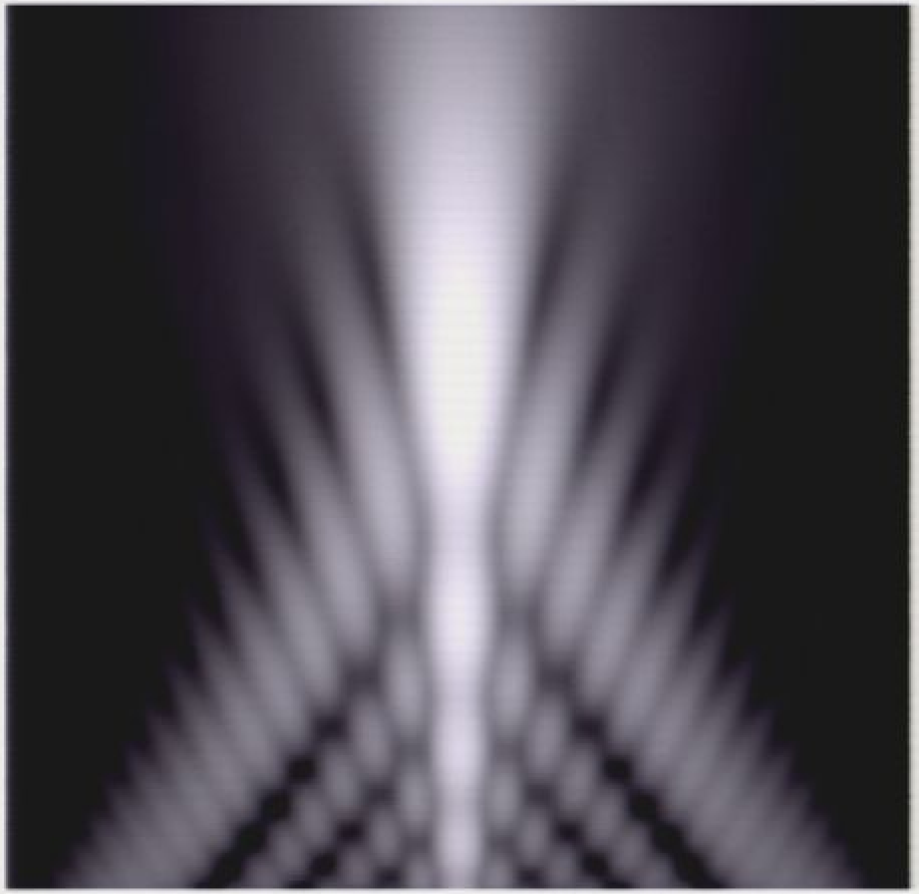
rays



wave



exact



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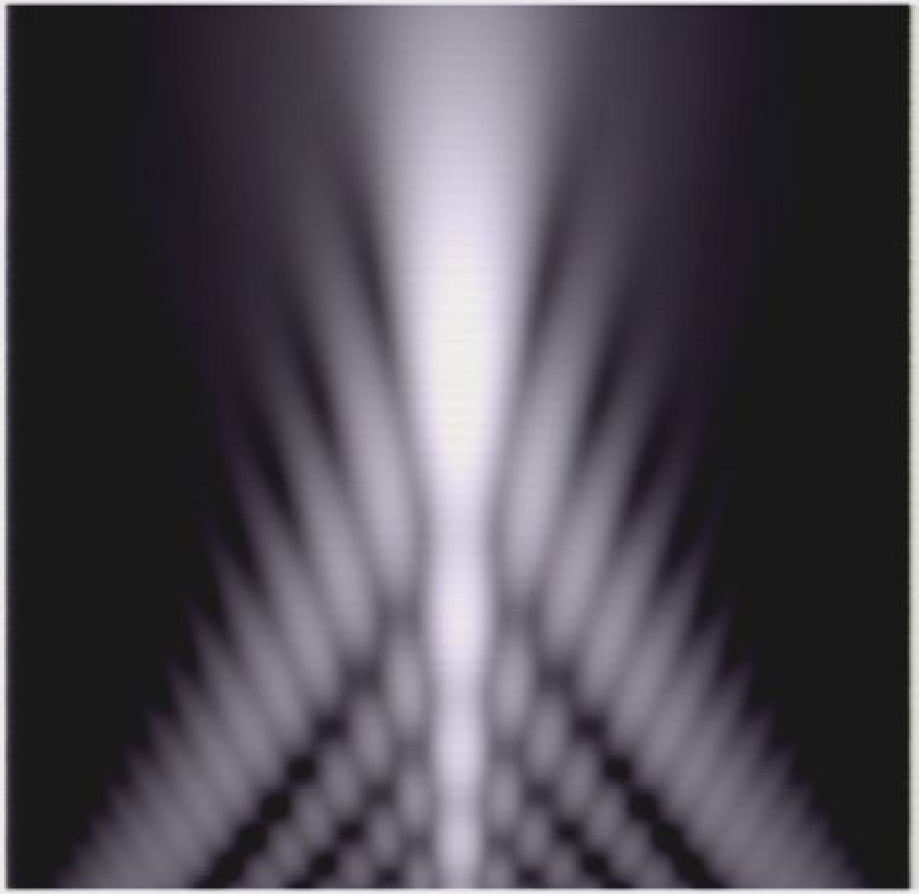
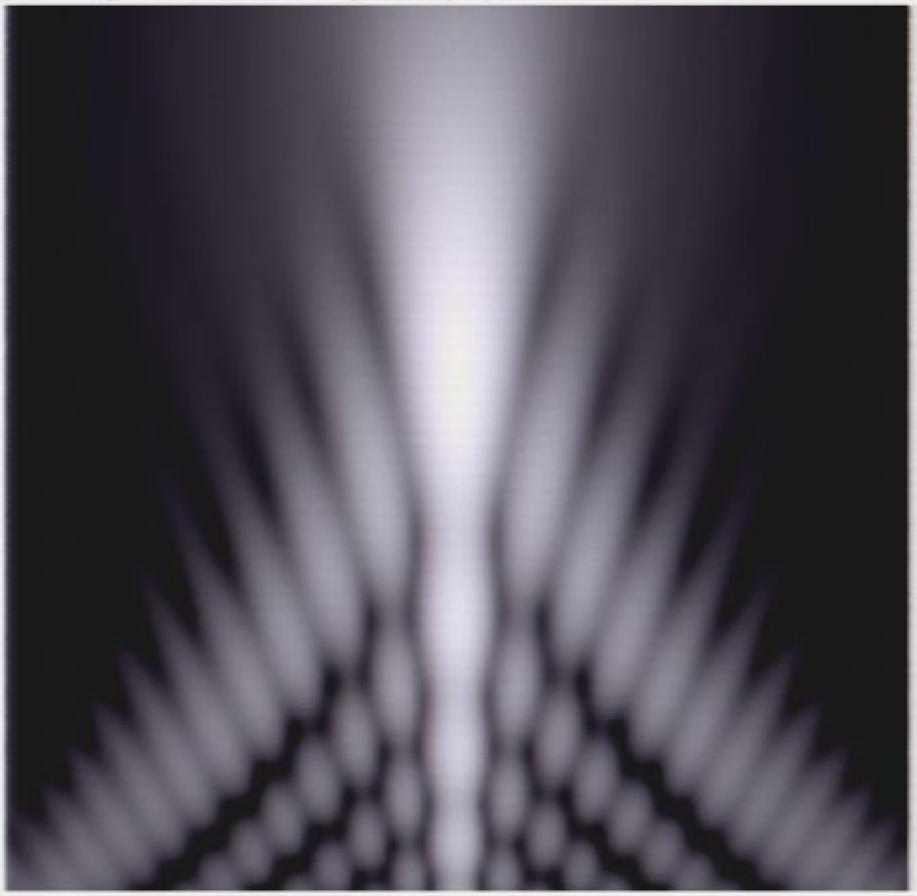
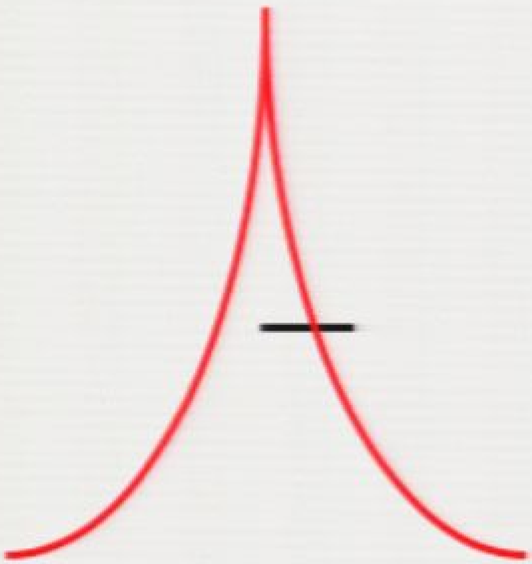


spun cusp approximation





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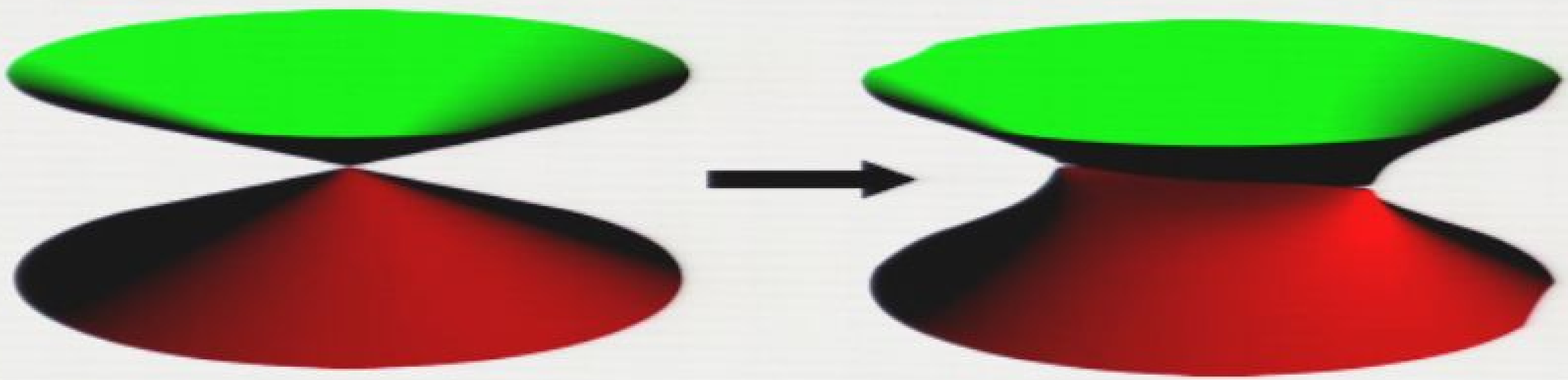

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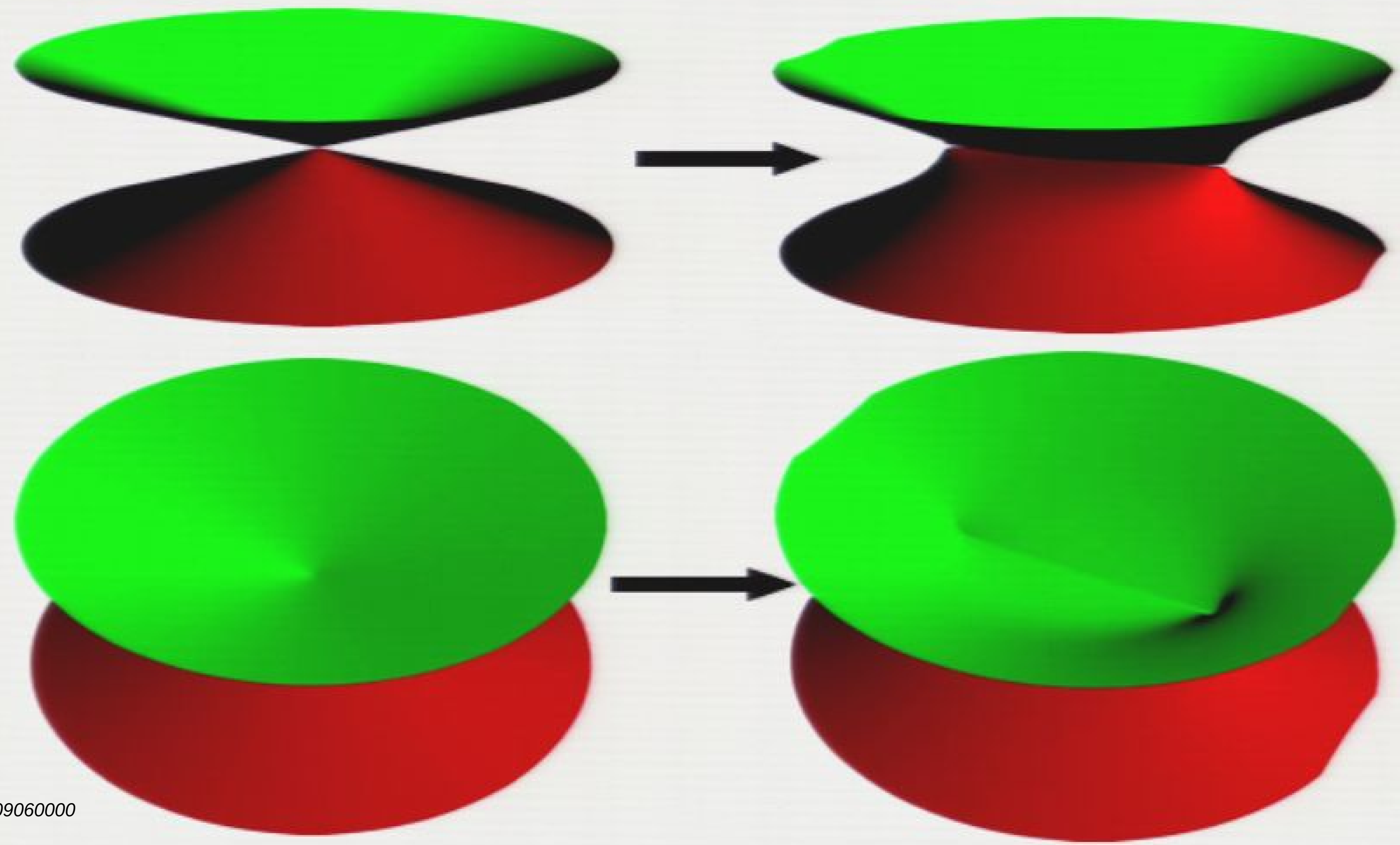

quantify absorption by a vector δ (nonhermitian traceless part of 2x2 complex symmetric matrix)

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how to choose direction on incident beam?
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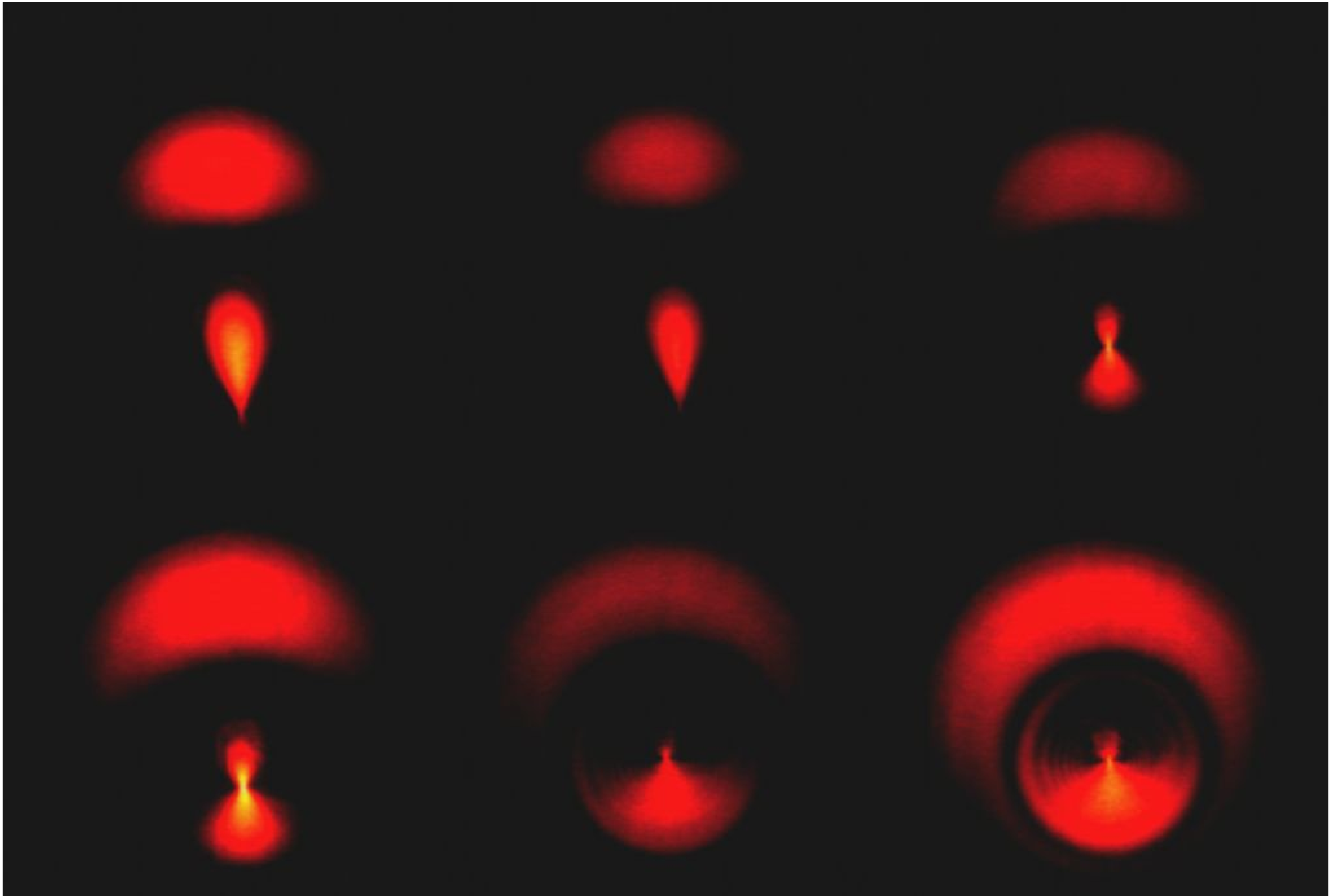
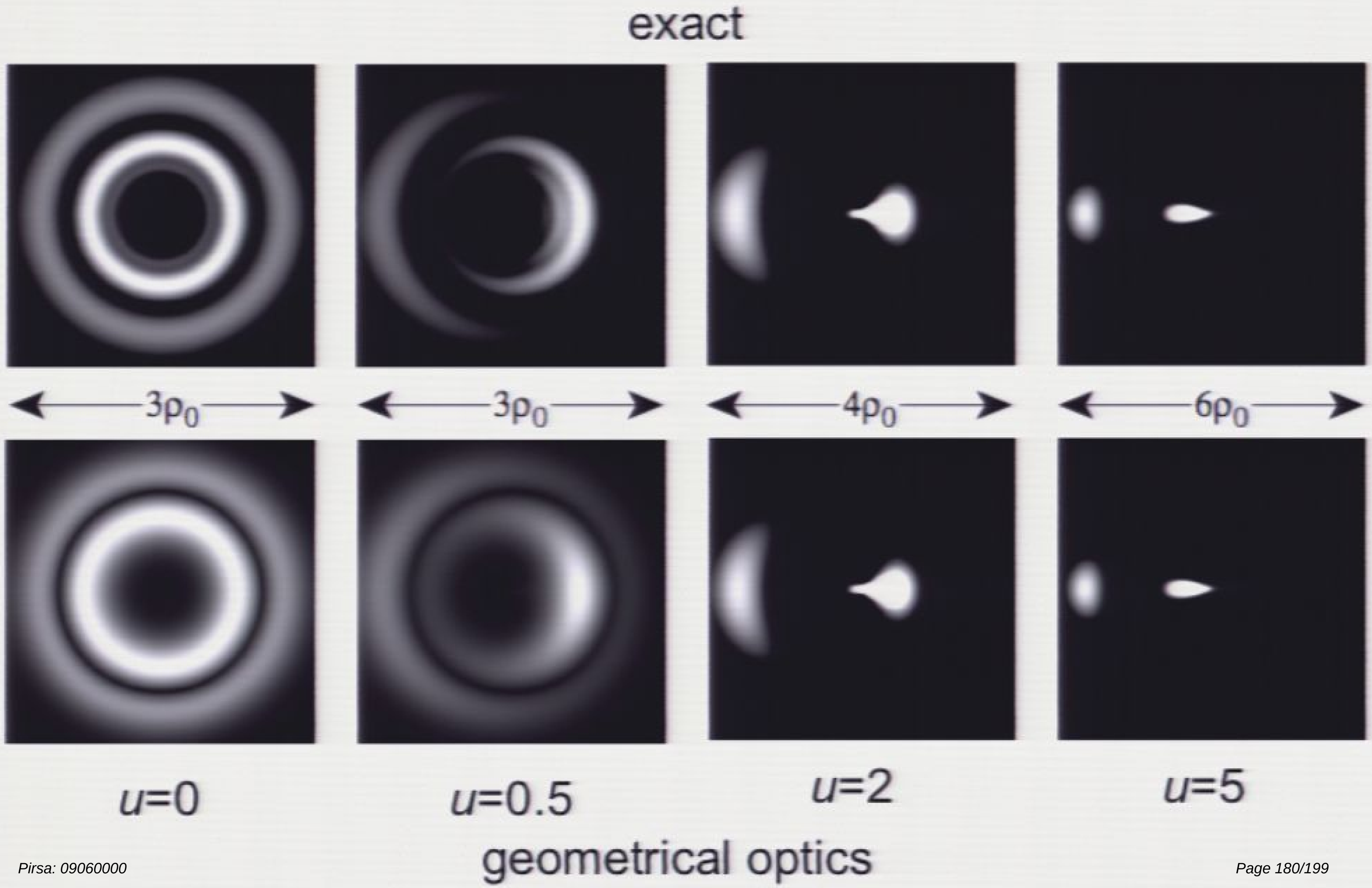
...but profound effects!



$\rho_0 = 20, \quad \zeta = 6$



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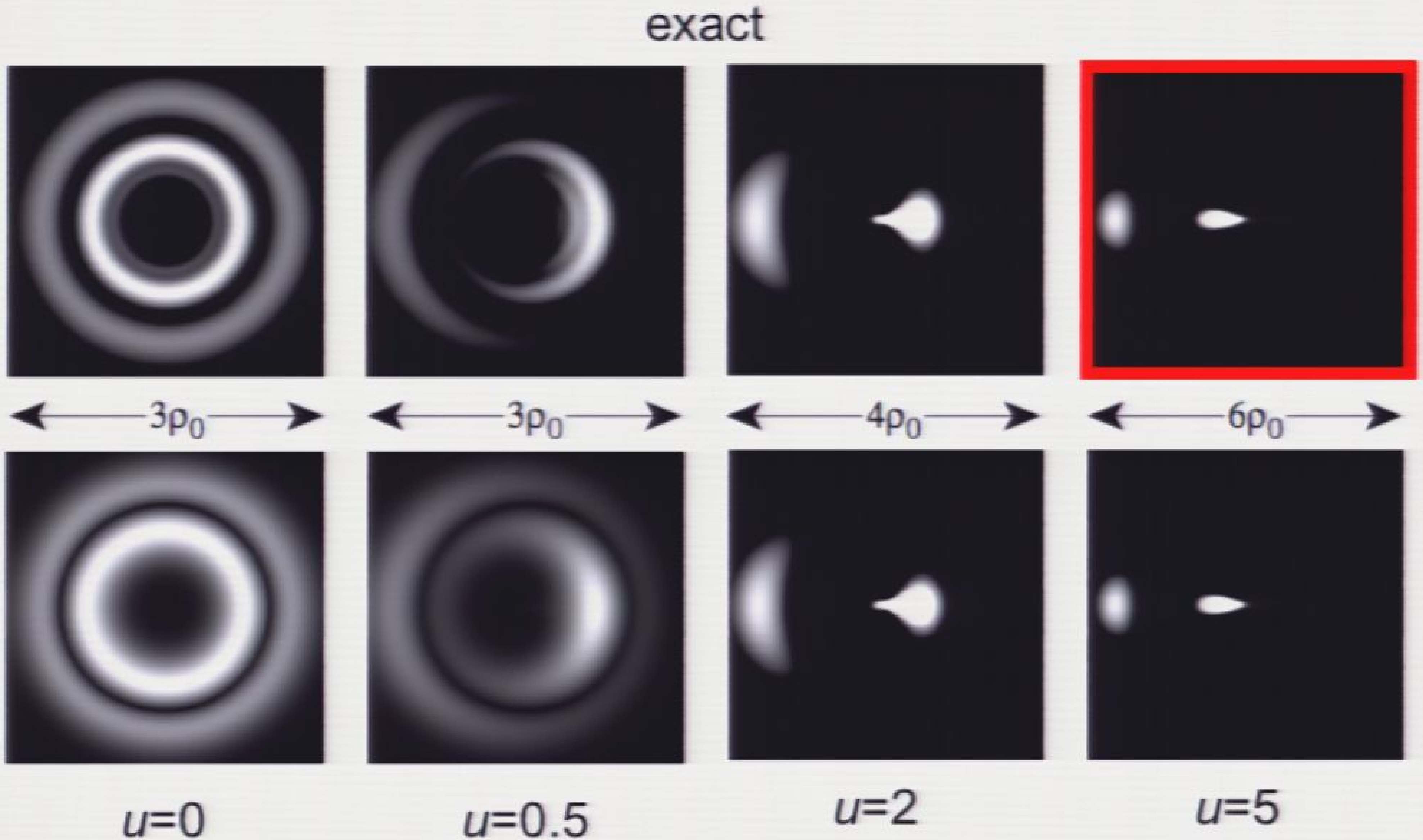

geometrical optics

surprise (3): fine structure revealed by logarithmic intensity plots

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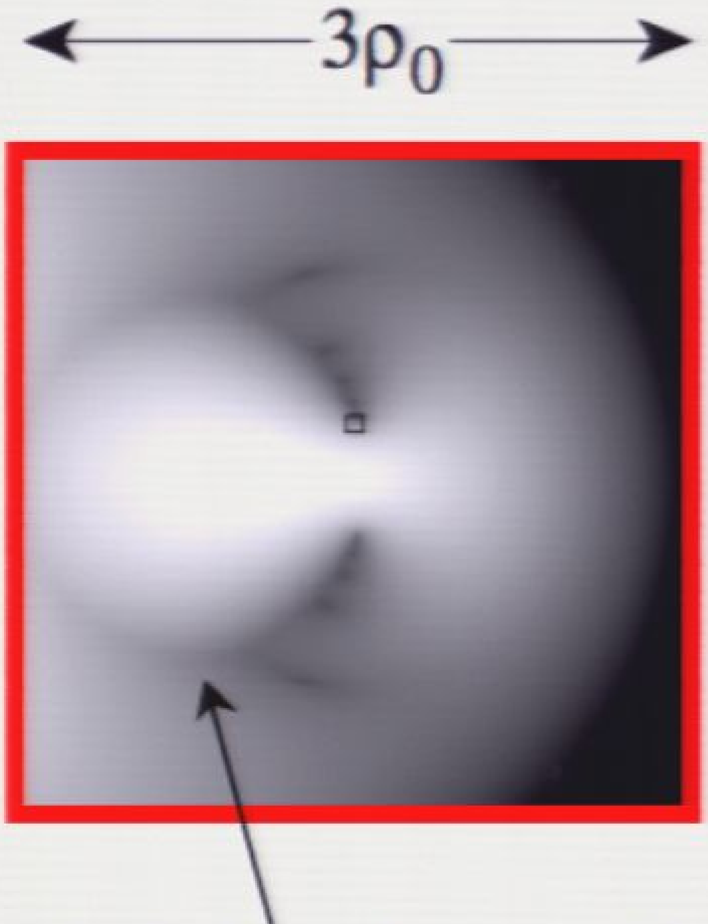

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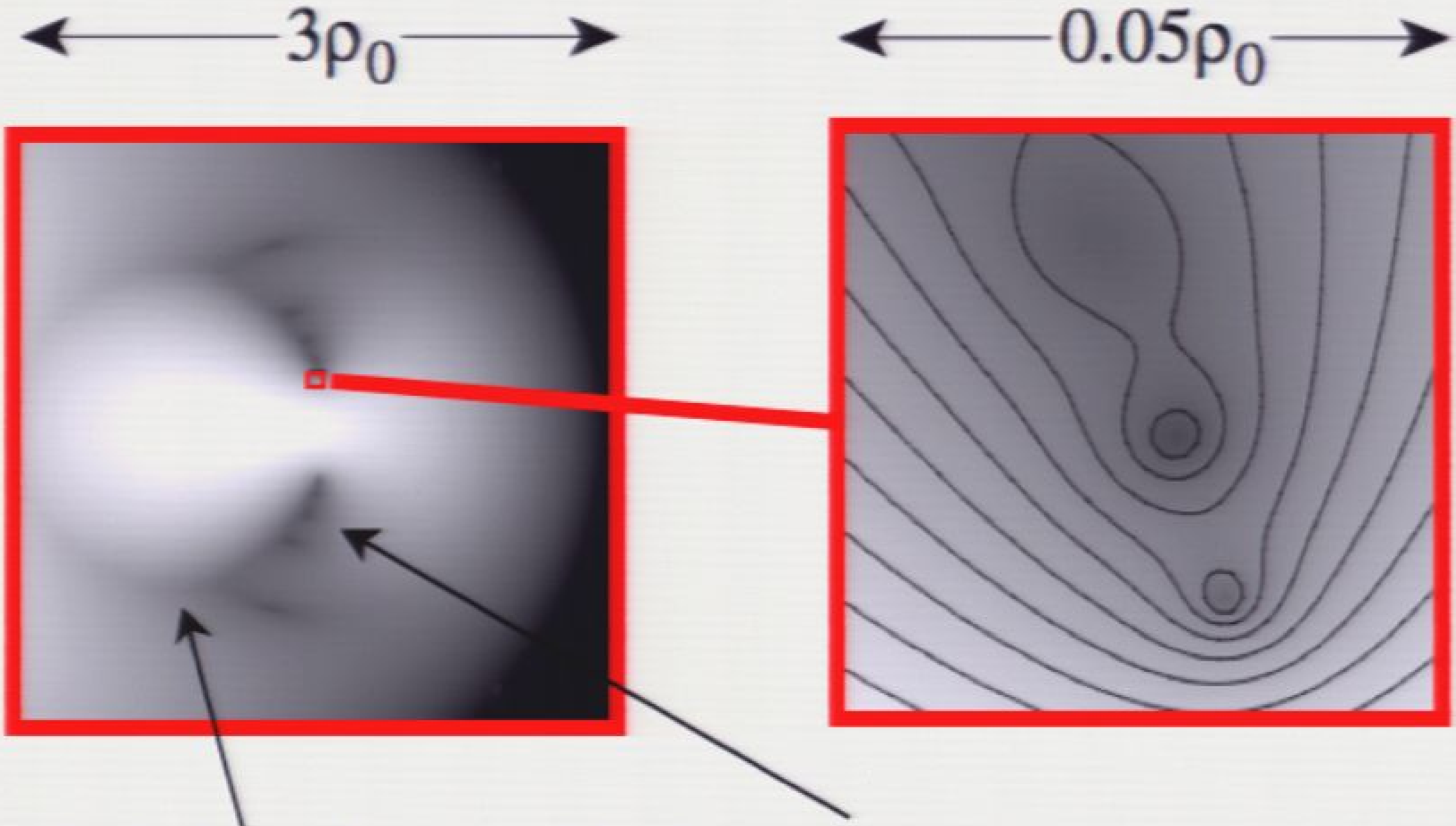

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6. leads to unusual asymptotics

7. was the first conical intersection in physics (cf. chemistry)

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12. is 175-year antidote to short-term science