Title: Perturbation Theory Out of Equilibruim

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Abstract:

Non-Equilibrium Dynamics in Economics and Finance

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Motivation and Goals

I. Perturbation theory for non-equilibrium economics (with ***Simone Severini, IQC***)

1. *What are the symmetries of the economic system?*
2. *What is the space of (gauge in-equivalent) equilibria of the economy?*
3. *How small fluctuation are "dissipated"? (perturbation theory)*
4. *What is the role of the trading network topology?*
5. *What is the relaxation time?*
6. *Will use example of pure exchange economy.*

II. Non-equilibirum finance (work in progress with S. Farinelli, UBS Zurich)

1. *Gauge symmetries in finance.*
2. *Curvature and arbitrage.*
3. *Arbitrage in the fixed income world.*
4. *Arbitrage in the spot market.*
5. *From agents to financial arbitrage (micro-macro connection)*

References

1. ***S. Severini and S. E. Vazquez, "Perturbation theory in a pure exchange non-equilibrium economy", arXiv:0904.1402.***
2. ***S. E. Vazquez, "Scale invariance, bounded rationality and non-Equilibrium economics", arXiv: 0902.3840.***
3. L. Smolin, "Time and symmetry in models of economic markets", arXiv: 0902.4272
4. P. Malandey, ``The Index Number Problem: A Differential Geometric Approach", PhD Thesis, Harvard University Economics Department, (1996).
5. Simone Farinelli, "Geometric Arbitrage Theory and Calibration of a Generator of Consistent Economic Scenarios", http://ssrn.com/abstract=1115860, (2008).
6. Simone Farinelli, "Geometric Arbitrage Theory and Market Dynamics", http://ssrn.com/abstract=1113292, (2009).
7. Simone Farinelli, "Stress Test and Consistent Aggregation of Market, Credit and Transfer Risk by Geometric Arbitrage Theory", http://ssrn.com/abstract=1103882, (2009).
8. Main inspiration: Mike Brown, Zoe-Vonna Palmrose, Stu Kauffman and Jim Herriot PARTECON model.

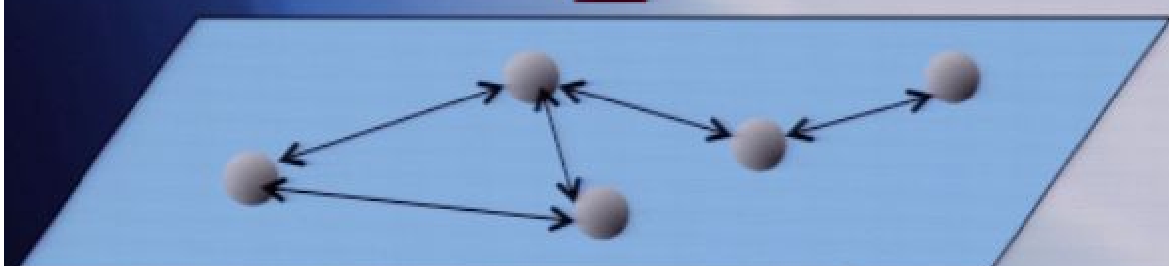
Basic Picture…

I. Micro/Macro connection:

$$dX_t = \mu_t X_t dt + \sigma_t X_t dW$$

$$\partial_t V + r\left(X\partial_X V - V\right) + \frac{1}{2}\sigma^2\partial_X^2 V = 0$$

Micro/Macro
- *Hard to understand the transition*



Finance
- *Effective macro theory*
- *Well defined principles of equilibrium (no-arbitrage)*
- *How to describe out of equilibrium? (any basic principles or constraints?)*

Interacting Agents
- *Well defined concept of equilibrium.*
- *Poor understanding of out of equilibrium dynamics. (usually approached using computer simulations)*
- *Any basic principles out of equilibrium?*
- *Inspired by kinetic theory*

Pure Exchange Economy

I. Basic set up:

1. There are N agents and P products in the economy. Greek letters will go over agents: $\alpha,\beta = 1,2,\ldots,N$, and latin letters over products: $i,j = 1,2,\ldots,P$. Will use notation $\alpha \in \mathcal{A}\,; i,j \in \mathcal{P}$

$$n^i_\alpha = \text{inventory of product } i \text{ of agent } \alpha$$

2. ***Symmetry Principle:*** Each product can be measured in any units we want, and the dynamics of the economy should be invariant under a change of unit measure.

$$n^i_\alpha \mapsto \phi^i n^i_\alpha, \text{ where } \phi^i \in \mathbb{R}^+$$

3. This induces an equivalence class of all n^i_α and $\tilde{n}^i_\alpha$ such that $n^i_\alpha = \phi^i \tilde{n}^i_\alpha$ even at different times.

Pure Exchange Economy

I. Basic set up:

4. We assume minimal rationality so that agents have an index of satisfaction $\Omega_\alpha(\vec{n}_\alpha, t)$ which they seek to maximize. However, we allow ***heterogeneous*** and ***changing*** preferences.

5. Basic properties: $\partial_i := \partial/\partial n^i_\alpha$

$$\partial_i \Omega_\alpha > 0\,, \quad \partial_i^2 \Omega_\alpha < 0 \quad \lim_{n^i_\alpha \to \infty} \partial_i \Omega_\alpha = \infty$$

6. Agents might have an opinion of the relative value of their goods. We denote such exchange rate by $(M_\alpha)^i{}_j$ and obeys (***this will be part of the index of satisfaction, and might have nothing to do with actual prices***)

$$(M_\alpha)^i{}_j = \frac{1}{(M_\alpha)^j{}_i}\,, \quad \forall\, i,j \in \mathcal{P}_\alpha$$

$$(M_\alpha)^i{}_j = (M_\alpha)^i{}_k (M_\alpha)^k{}_j\,, \quad \forall\, i,j,k \in \mathcal{P}_\alpha$$

Pure Exchange Economy

I. Basic set up:

7. Under the gauge transformation we have the following properties:

$$\partial_i \Omega_\alpha \rightarrow (\phi^i)^{-1} \partial_i \Omega_\alpha$$

$$(M_\alpha)^i{}_j \rightarrow \phi^i (\phi^j)^{-1} (M_\alpha)^i{}_j$$

8. We work in continuous time, and we assume that agents trade probabilistically.

$\sigma^{ij}_{\alpha\beta}(t)$ = Probability ***per unit time*** that agent α encounters β and make an exchange involving products *i* and *j*. (***like cross section in particle physics***)

Equilibrium Landscape

I. A pure exchange equilibrium is defined by a set of exchange rates $\{\bar{M}^i_{\ j}\}$ and inventories $\{\bar{n}^i_\alpha\}$ such that every agent is maximally satisfied with its inventory:

$$\frac{\partial}{\partial \Delta n^i}\Omega_\alpha(n^i_\alpha + \Delta n^i, n^j_\alpha - M^j_i \Delta n^i, \ldots)\Big|_{\Delta n^i = 0} = 0$$

II. We get the following equations:

$$\left(\partial_i \Omega_\alpha - \bar{M}^j_i \partial_j \Omega_\alpha\right)\big|_{\bar{n}_\alpha} = 0\,,\ \forall\ \alpha \in \mathcal{A}\,; i, j \in \mathcal{P}$$

III. ***How much can we learn just from scale invariance?***

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I. One can show ***only using symmetry arguments*** that the space of equivalence classes of equilibria at any point in time is the manifold: (Vazquez and Severini (2009))

$$\boxed{\mathcal{V} = (\mathbb{R}_+)^{N-1}}$$

II. For many agents $N \to \infty$ we have that ***the landscape of gauge in-equivalent equilibria, at any point in time, is isomorphic to the space of wealth distributions.***

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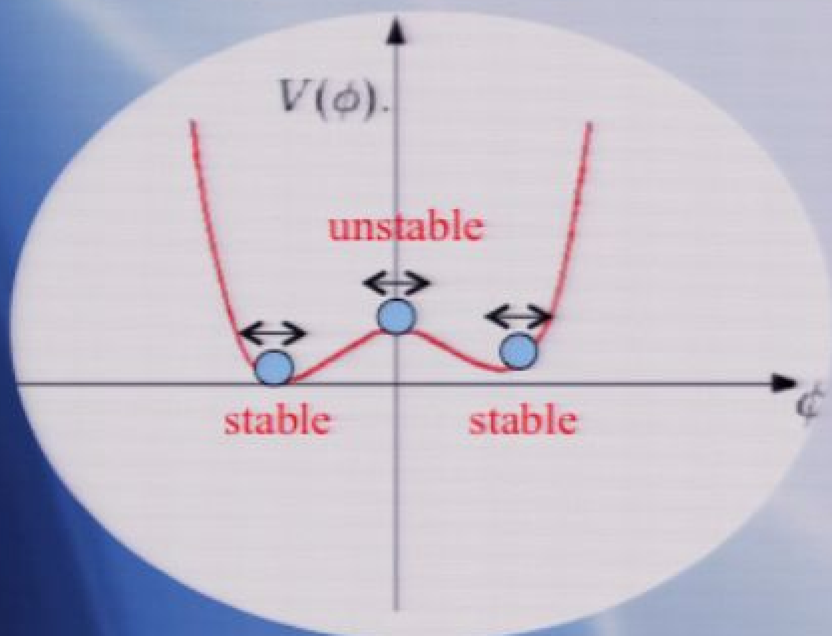
Perturbation Theory

I. One would like to have a “complete” theory of economics out of equilibrium. Given the complexity of the economic system, this is very unlikely. Instead, one can develop some kind of “kinetic” theory of small fluctuations around equilibrium. In physics, we call this ***perturbation theory***. Some theories in physics are indeed best understood using this technique (e.g. String Theory).



Perturbation theory allow us to understand the (in) stability of the equilibrium

Non-Equilibrium Dynamics

I. In general we can decompose the product number as

$$n^i_\alpha(t) = \bar{n}^i_\alpha(t) + \delta n^i_\alpha(t)$$

Arbitrary one-parameter family of equilibria

Small deviation from equilibrium

II. Gauge invariant fluctuation (x^i_α):

$$\delta n^i_\alpha \equiv \bar{n}^i_\alpha x^i_\alpha$$

III. Perturbation theory is an expansion in powers of the small fluctuation. We take the linearized approximation.

Non-Equilibrium Dynamics

I. One can show, given our assumptions about the probabilistic nature of the economy, that the Master Equations which govern the evolution of the ***linearized*** fluctuations are

$$\boxed{\left(\frac{d}{dt}+\frac{1}{\bar{n}^i_\alpha}\frac{d\bar{n}^i_\alpha}{dt}\right)\langle x^i_\alpha(t)\rangle=\frac{1}{\bar{n}^i_\alpha}\sum_{j,\beta}\sigma^{ij}_{\alpha\beta}\langle\Delta n^{ij}_{\alpha\beta}(t)\rangle}$$

where

$$\Delta n^{ij}_{\alpha\beta} \approx L\left(\partial_i\delta\Omega\partial_j\Omega-\partial_i\Omega\partial_j\delta\Omega\right) , \quad \Delta n^{ji}_{\alpha\beta}=-\frac{\partial_i\Omega}{\partial_j\Omega}\Delta n^{ij}_{\alpha\beta} ,$$

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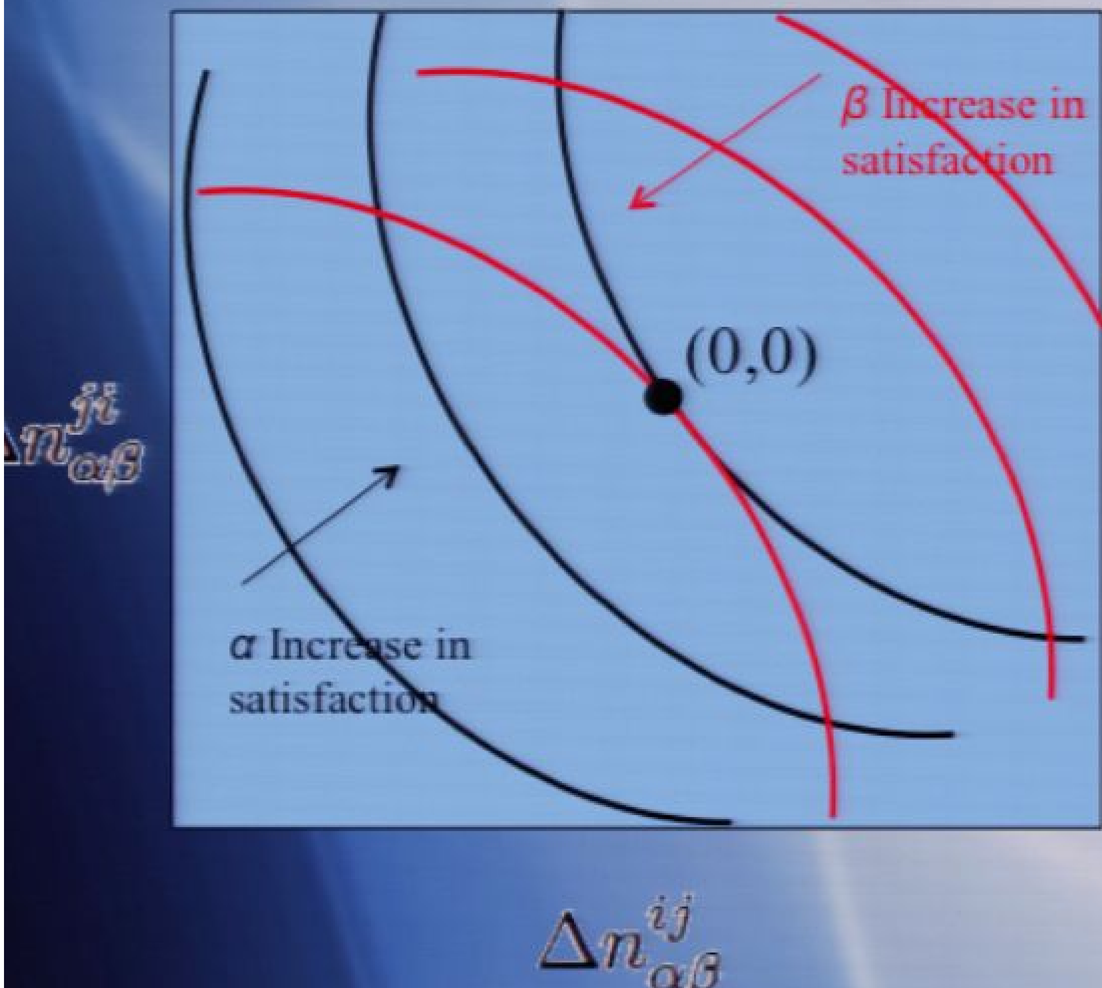
$$\Omega=\Omega_\alpha+\Omega_\beta , \quad \delta\Omega=\Omega_\alpha-\Omega_\beta .$$

In Pictures…

I. How we derive the right hand side of the master equation



In equilibrium there is no exchange that would benefit both agents.

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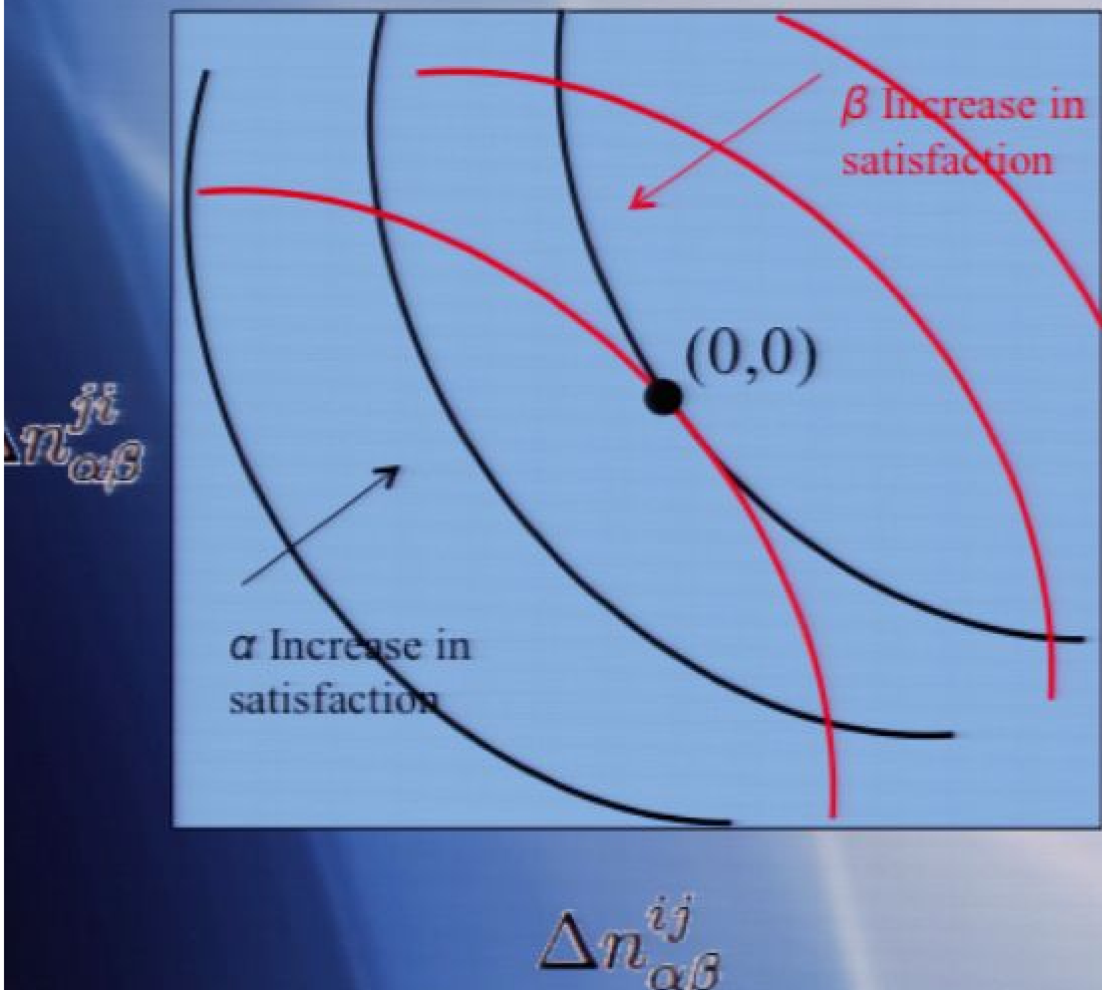

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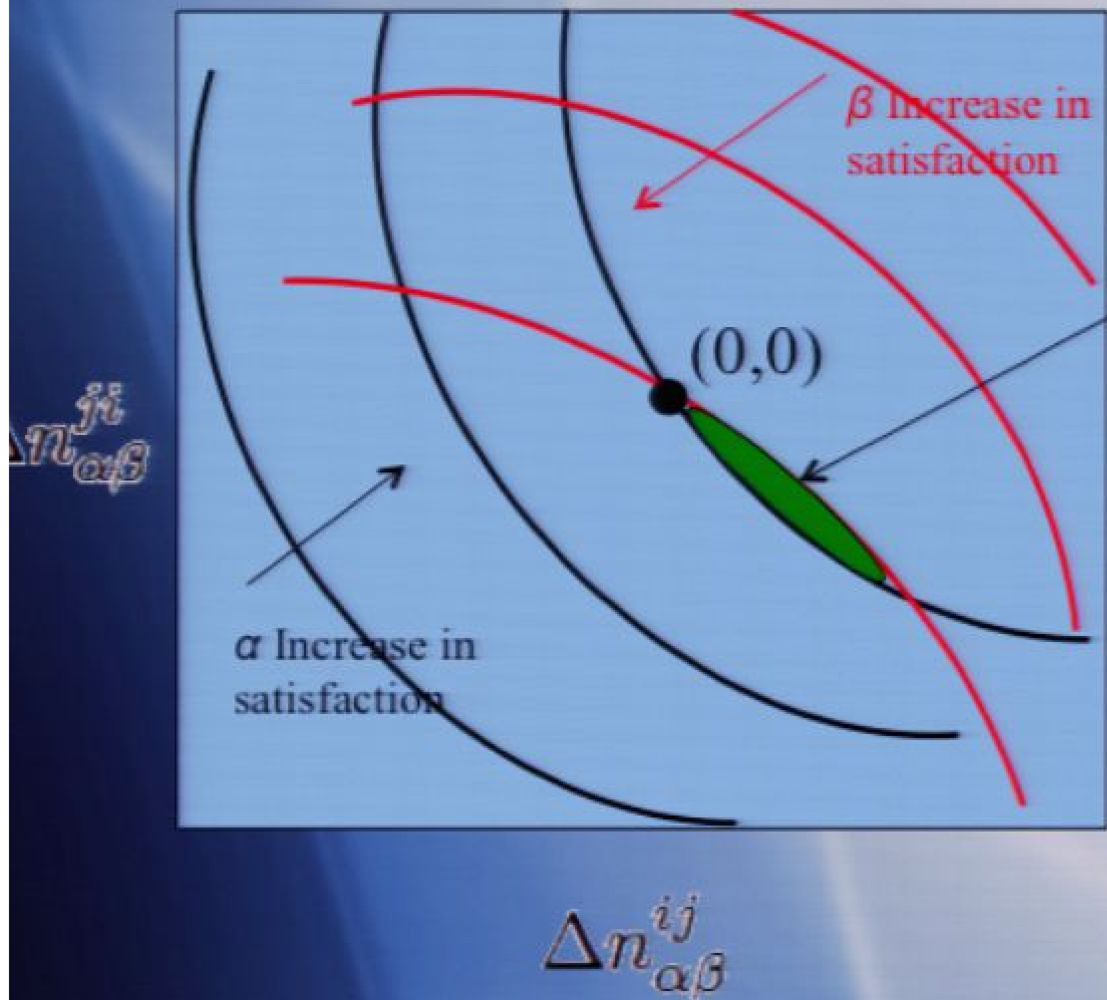

Out of equilibrium, now there is a region in which a trade will benefit both agents

In perturbation theory we assume that this region is small.

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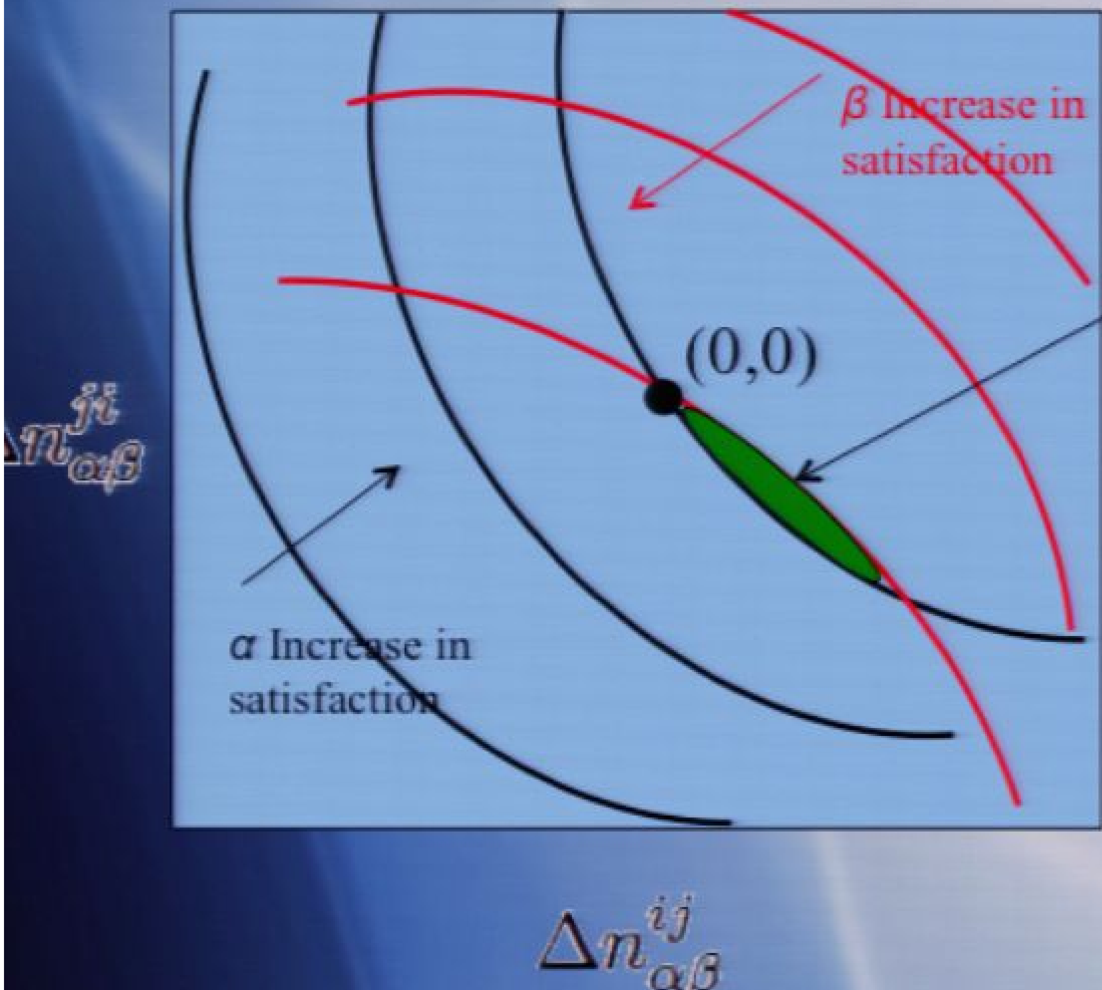

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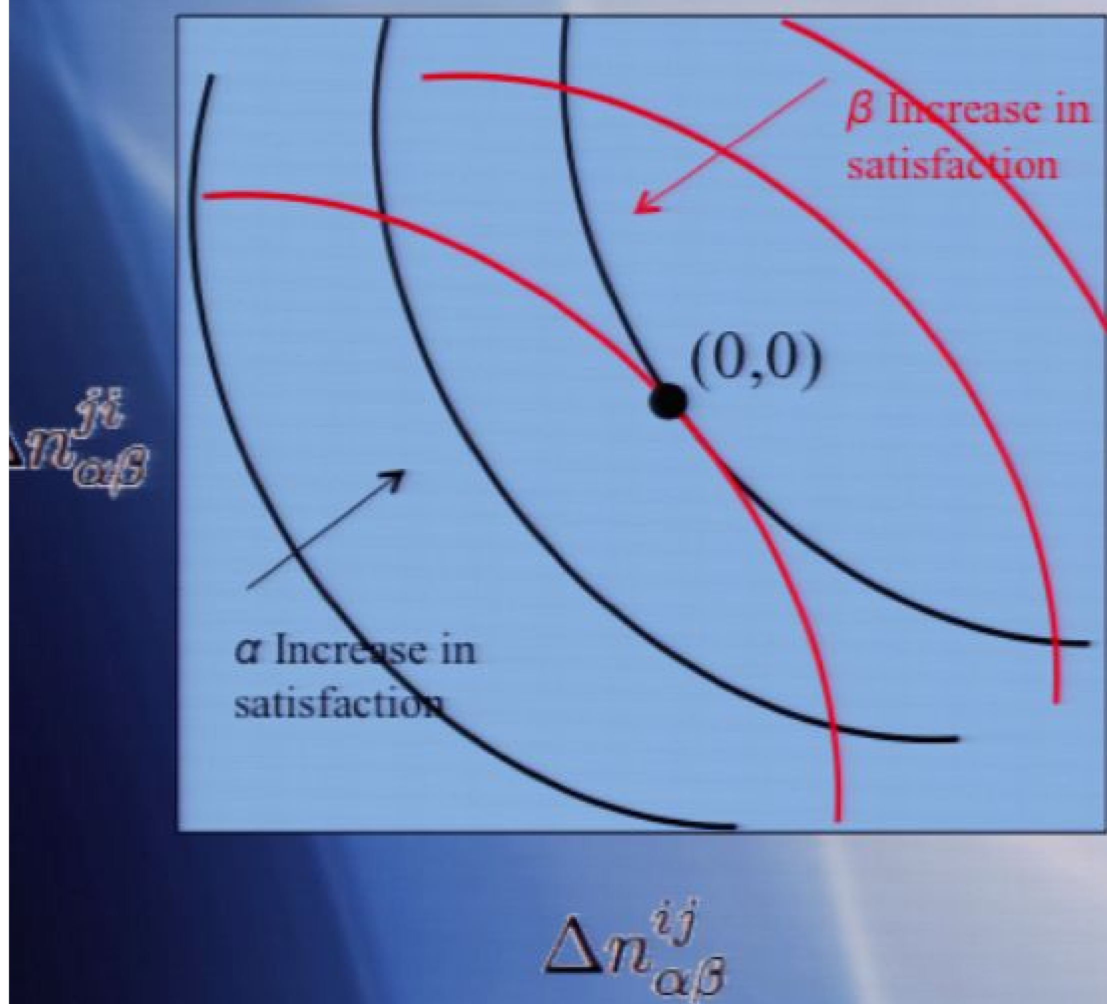

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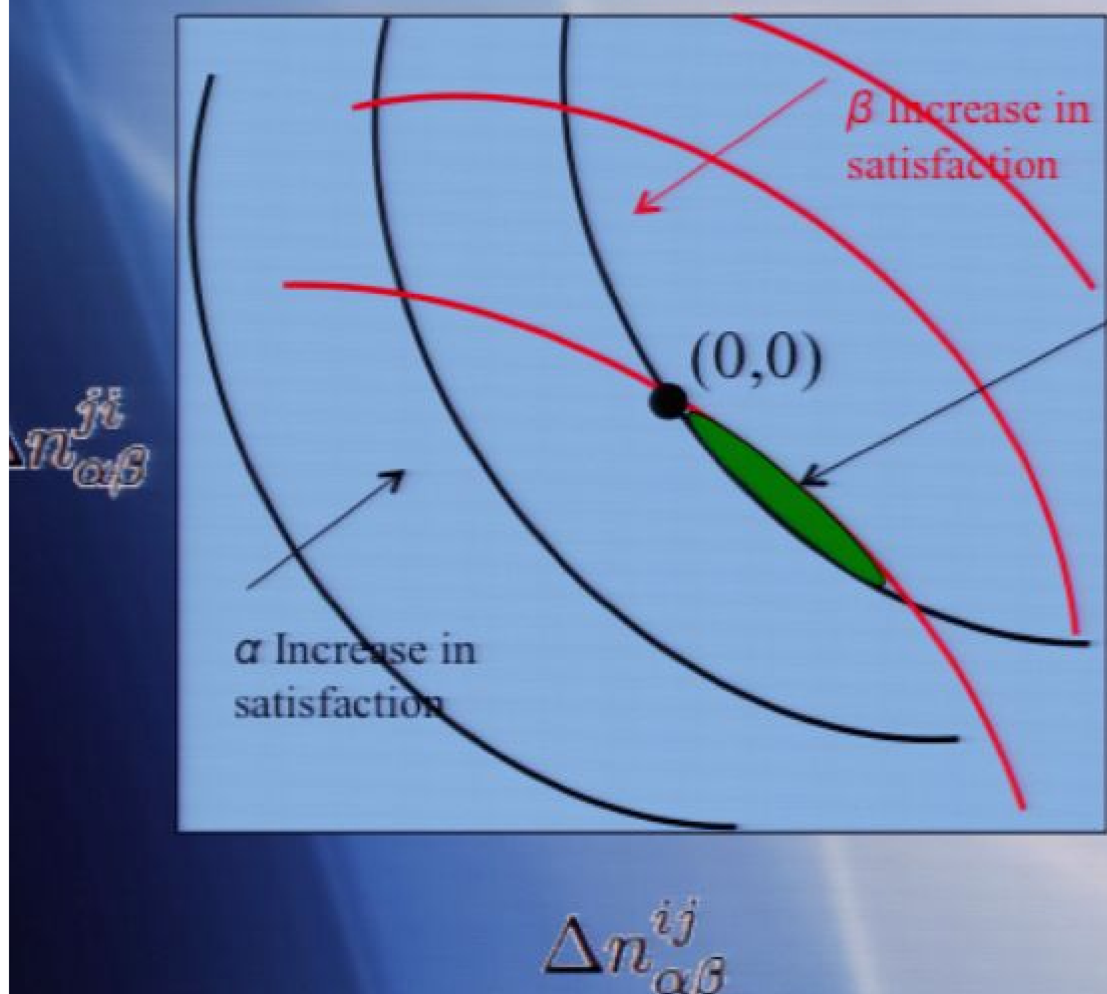

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A Universality Theorem

I. Consider now a more special case where, in any of the possible equilibria the following conditions are met:

1) The index of satisfaction is time independent $\partial_t \Omega_\alpha|_{\bar{n}} = 0$
2) Agents have homogeneous preferences (in equilibrium).

II. One finds from the evolution equations, that there are new conserved quantities which allow us to prove the following result:

Given an initial perturbation, the final wealth distribution is independent of the dynamics of the non-equilibrium theory. In particular, it is completely determined in terms of the initial conditions, and it is independent of the probability, and the network of interaction between agents.

I. In particular, a homogeneous economy is unstable under small fluctuations, in the sense that the final wealth distribution will not be homogeneous.

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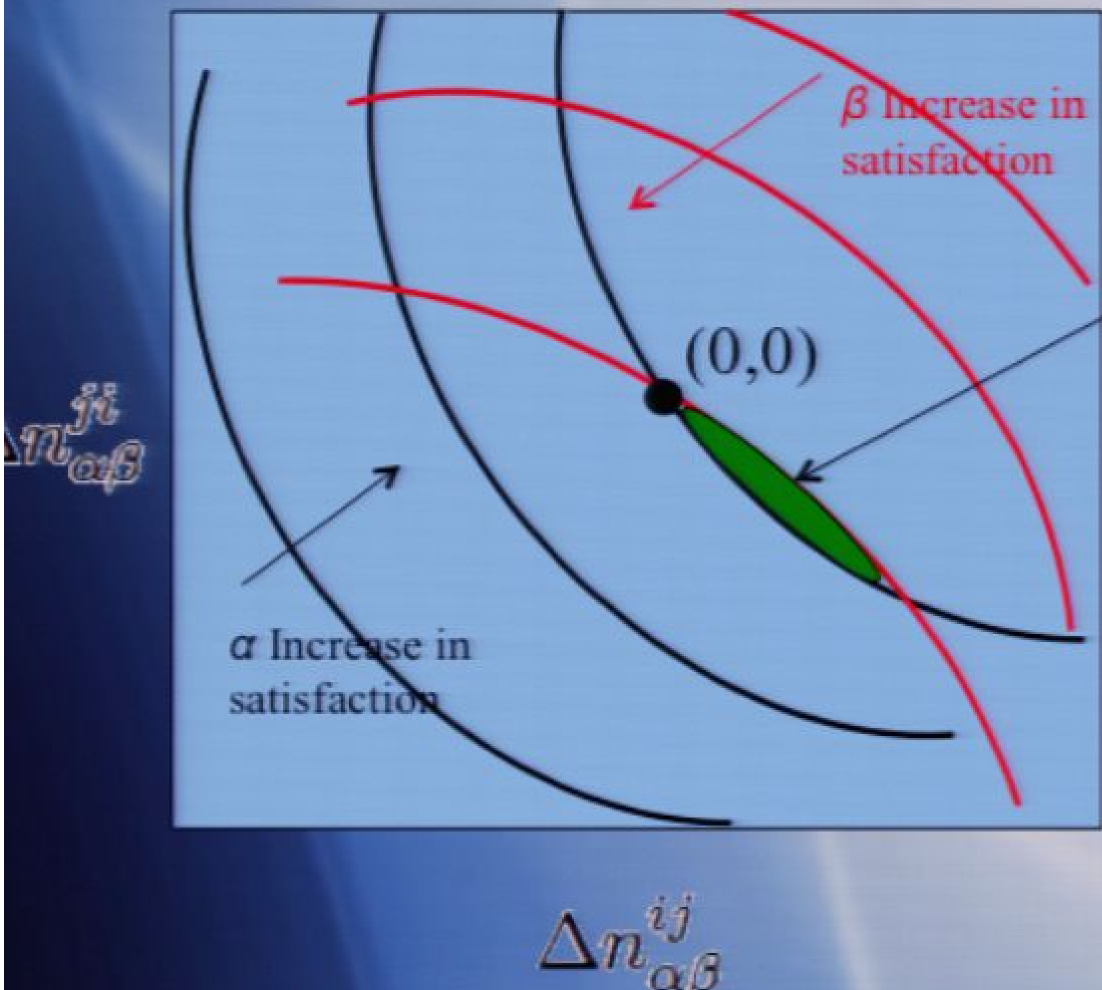

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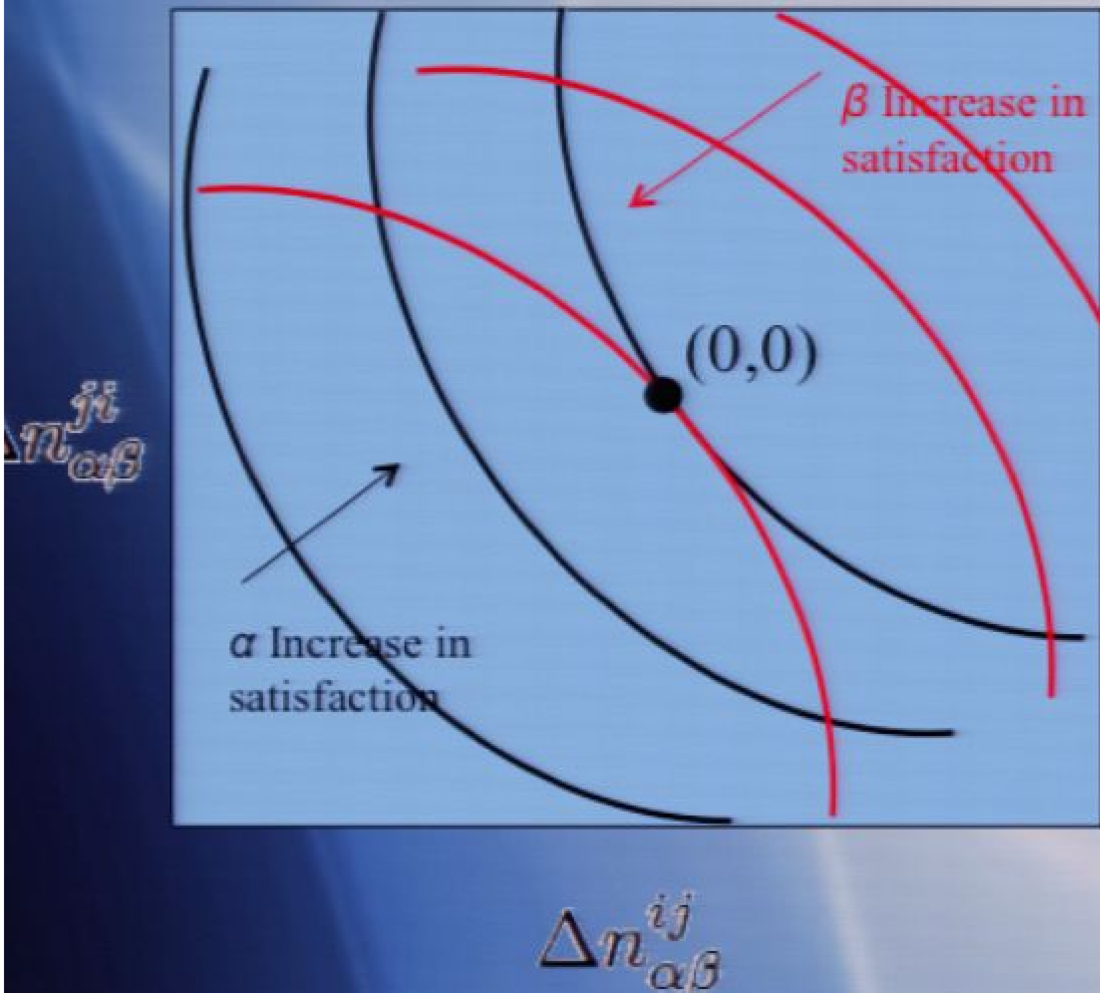

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The Role of the Trading Network

I. We want to study what effect does the trading network topology has in the non-equilibrium dynamics.

II. We also want to study if there are instabilities in any of the possible equilibria.

III. Consider following simple example with a ***regular graphs of degree d and P products. Agents choose their trading partners at random from their neighbors in the Graph. They also choose a pair of products at random.***

d = 4

$$\Omega_\alpha = \sum_i \log n_\alpha^i$$

d = 2

$$\sigma_{\alpha\beta}^{ij} = \frac{A_{\alpha\beta}}{d(P-1)}$$

Graph adjacency matrix. $A_{\alpha\beta} = 1$ if (α,β) are connected and zero otherwise.

The Role of the Trading Network

I. One can show that the evolution equation for the perturbations becomes

$$\frac{d}{dt}\langle x^i_\alpha\rangle = \frac{1}{4d(p-1)}\sum_{\beta,j} A_{\alpha\beta}(\langle x^j_\alpha\rangle - \langle x^i_\alpha\rangle - \langle x^j_\beta\rangle + \langle x^i_\beta\rangle)$$

II. From studying this equation one can show the following results:

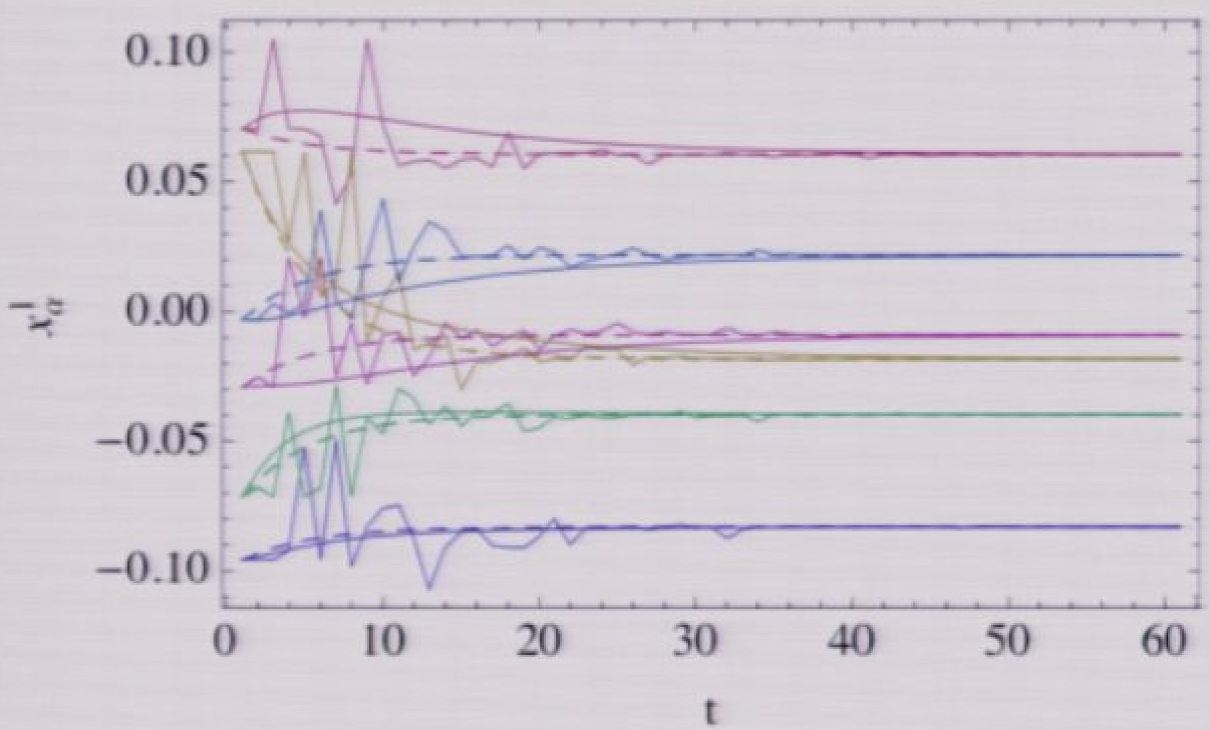
1. The system is stable in that perturbations relax exponentially in time as,

$$x \sim e^{-\gamma\Delta\lambda t} \qquad \gamma\Delta\lambda \geq 0$$

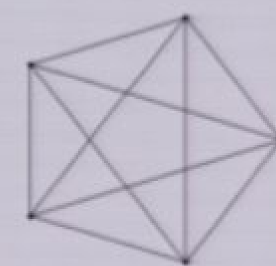
2. Here γ are constants, and $\Delta\lambda$ is the ***eigenvalue gap*** of the adjacency matrix A.
3. ***This implies that networks with more nodes (higher digree d) will relax faster (they have larger Δλ)***

The Role of the Trading Network

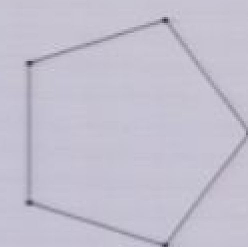
I. Comparison with numerical simulation (agents trading at random)



Dashed lines = complete graph



Solid lines = cycle



II. This formalism can be applied to more complicated examples of agents with changing preferences. See S. Vazquez and S. Severini (2009).

Heterogeneous Changing Preferences

I. A slightly more complicated model allow us to model heterogeneous agents with changing preferences. In this model agents have a prior arbitrary opinion on the relative value of products but learn the true value by bartering. ***This is one of the basic ideas of the PARTECON model.***

II. Consider convex index of satisfaction:

$$\Omega_\alpha = \log \sum_j [(M_\alpha)^i_j n^j_\alpha]^\nu \qquad 0 < \nu < 1$$

III. In equilibrium, one will end with

$$\bar{\Omega}_\alpha = \log \sum_j [\bar{M}^i_j \bar{n}^j_\alpha]^\nu$$

IV. Now we need to provide equations to describe the dynamics of the agents' preferences.

Heterogeneous Changing Preferences

I. We assume that when two agents α and β meet and trade, they update their internal preferences as for all $i, j \in \mathcal{P}$ (***they learn from each other***).

$$(M_\alpha)^i_j = (M_\beta)^i_j = -\frac{\Delta n^{ij}_{\alpha\beta}}{\Delta n^{ji}_{\alpha\beta}} \approx \frac{\partial_j(\Omega_\alpha + \Omega_\beta)}{\partial_i(\Omega_\alpha + \Omega_\beta)}$$

II. Can define gauge invariant variable (can use product # 1 for reference without loss of generality):

$$y^i_\alpha := \frac{(M_\alpha)^1_i(t) - \bar{M}^1_i}{\bar{M}^1_i}$$

Heterogeneous Changing Preferences

I. The dynamics perturbations obey linear equations

$$\frac{d}{dt}\langle x^i_\alpha\rangle = \frac{1}{4(p-1)}\sum_{\alpha,\beta} T_{\alpha\beta}[\langle x^i_\alpha\rangle - \langle x^j_\beta\rangle - \langle x^i_\alpha\rangle + \langle x^i_\beta\rangle + \frac{\nu}{1-\nu}(\langle y^i_\alpha\rangle - \langle y^j_\beta\rangle - \langle y^i_\alpha\rangle + \langle y^i_\beta\rangle)].$$

Perturbation of the product number around homogeneous equilibrium

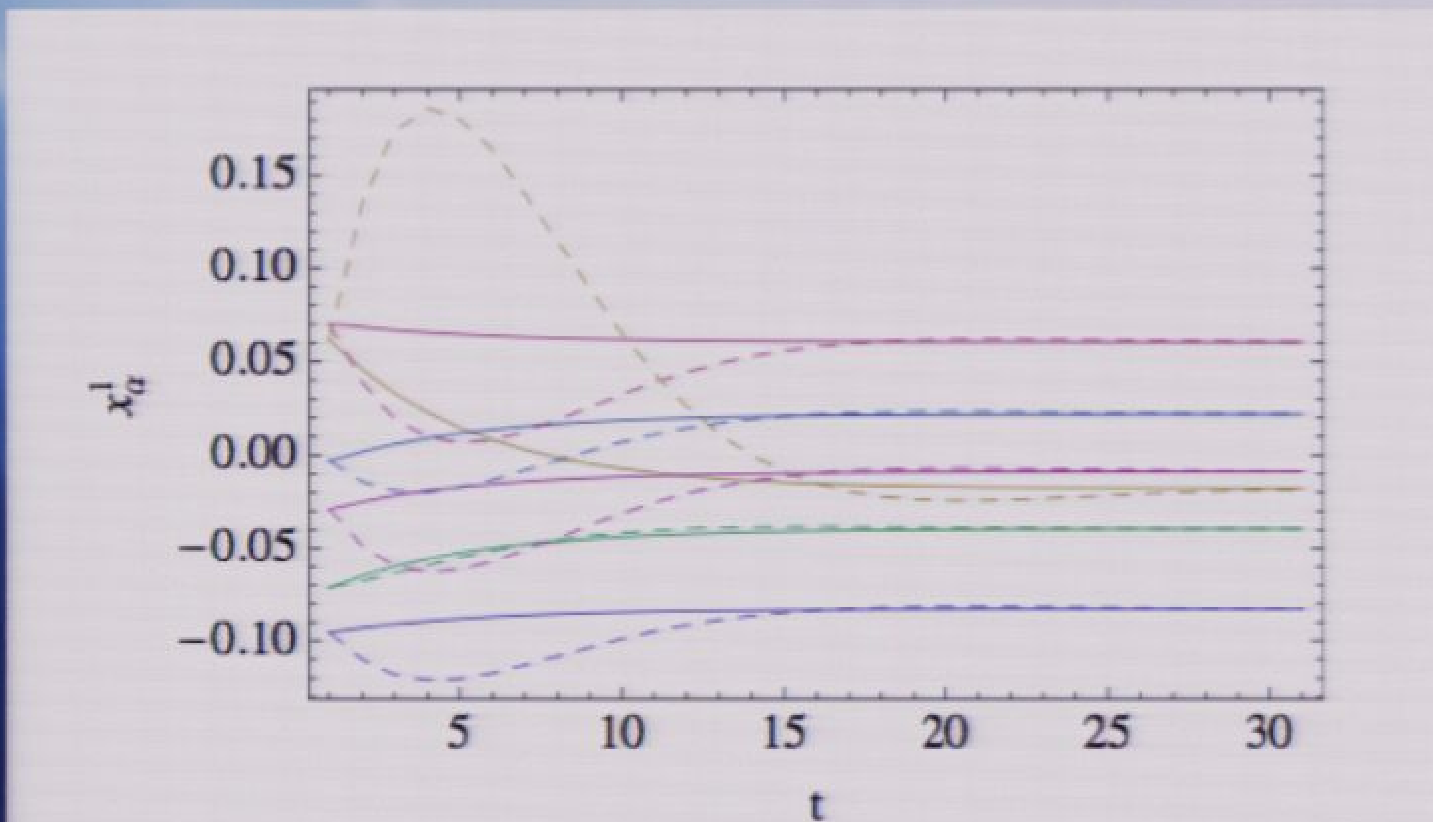
$$\frac{d}{dt}\langle y^i_\alpha\rangle = -\langle y^i_\alpha\rangle + \frac{1}{2}\sum_\beta T_{\alpha\beta}[(1-\nu)(\langle x^1_\alpha\rangle + \langle x^1_\beta\rangle - \langle x^i_\alpha\rangle - \langle x^i_\beta\rangle) + \nu\left(\langle y^i_\alpha\rangle + \langle y^i_\beta\rangle\right)],$$

Perturbation of the price opinion around homogeneous equilibrium

$$\sum_j \sigma^{ij}_{\alpha\beta} = T_{\alpha\beta}.$$

Heterogeneous Changing Preferences

I. Perturbation dynamics. Dashed lines are for heterogeneous changing preferences with $\nu = 0.9$. The solid lines are for the homogeneous unchanging preferences. ***As $\nu \rightarrow 1$ (wealth maximization), the relaxation time goes to infinity and the economy never reaches equilibrium.***



Heterogeneous Changing Preferences

I. A slightly more complicated model allow us to model heterogeneous agents with changing preferences. In this model agents have a prior arbitrary opinion on the relative value of products but learn the true value by bartering. ***This is one of the basic ideas of the PARTECON model.***

II. Consider convex index of satisfaction:

$$\Omega_\alpha = \log \sum_j [(M_\alpha)^i_j n^j_\alpha]^\nu \qquad 0 < \nu < 1$$

III. In equilibrium, one will end with

$$\bar{\Omega}_\alpha = \log \sum_j [\bar{M}^i_j \bar{n}^j_\alpha]^\nu$$

IV. ***Now we need to provide equations to describe the dynamics of the agents' preferences.***

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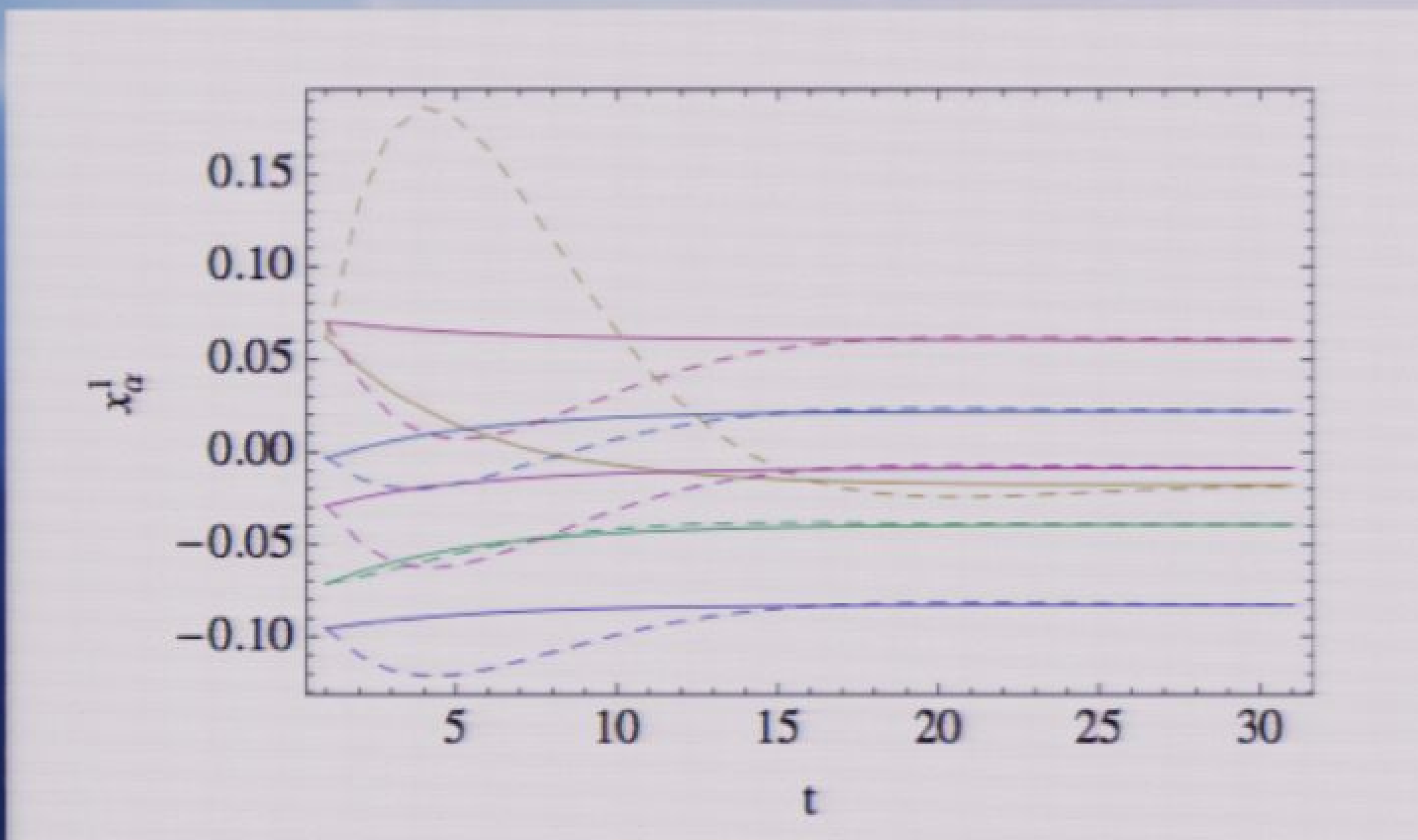

Other Models

I. One can also model production processes, out of equilibrium using similar techniques. One defines production as a metabolic process that takes some products as input and produces an output, e.g.

$$i + j \rightarrow k$$

II. One can also define a production equilibrium, and show that near such equilibrium, the output of the production process is given by

$$\begin{aligned}
\Delta n^k &\approx K_\alpha^{kk}\left(-\partial_i\Omega_\alpha P^i_{\ k} - \partial_j\Omega_\alpha P^j_{\ k} + \partial_k\Omega_\alpha\right) \\
\Delta n^i &= -P^i_{\ k}\Delta n^k\,, \\
\Delta n^j &= -P^j_{\ k}\Delta n^k\,,
\end{aligned}$$

where

$$\begin{aligned}
K_\alpha^{kk} = -\Big[&\partial_i^2\Omega_\alpha (P^i_{\ k})^2 + \partial_j^2\Omega_\alpha (P^j_{\ k})^2 + \partial_k^2\Omega_\alpha + 2\partial_i\partial_j\Omega_\alpha P^i_{\ k}P^j_{\ k} \\
&-2\partial_i\partial_k\Omega_\alpha P^i_{\ k} - 2\partial_j\partial_k\Omega_\alpha P^j_{\ k}\Big]^{-1}.
\end{aligned}$$

Conversion rates (technology)

$$\frac{dn^i}{dn^k} = -P^i_{\ k}\,,\quad \frac{dn^j}{dn^k} = -P^j_{\ k}\,,$$

$$P^i_{\ k}, P^j_{\ k} > 0$$

Other Models

I. One can introduce a Market Maker which is a profit maximizer. He/ she adjust prices in respond to excess demand.

II. One can show that, when agents are in a near-equilibrium state, the dynamics of the prices is given by:

$$D_t M^j_i \approx \frac{1}{2} \frac{\sum_\alpha J^{ii}_\alpha \left(\partial_i \Omega_\alpha - M^j_i \partial_j \Omega_\alpha\right)}{\sum_\alpha J^{ii}_\alpha \partial_j \Omega_\alpha}$$

where

$$J^{ii}_\alpha = (\partial_j \Omega_\alpha)^2 \left[2\partial_i \Omega_\alpha \partial_j \Omega_\alpha \partial_i \partial_j \Omega_\alpha - \partial_i^2 \Omega_\alpha (\partial_j \Omega_\alpha)^2 - \partial_j^2 \Omega_\alpha (\partial_i \Omega_\alpha)^2\right]^{-1}$$

Conclusions

I. We have presented a semi-analytic approach to agent-based economic modeling, assuming near-equilibrium dynamics.

II. This allow us to study the stability of economic equilibria.

III. The role of gauge symmetries is crucial to restrict the class of possible models and utility functions.

IV. The models still have enough flexibility to simulate changing preferences, speculations etc.

V. Prices only exist after a trade: they describe the relative flow of product between agents.

VI. Can also include external couplings (e.g. an external flow of products).

VII. The models can be interpreted as a kind of kinetic or fluid theory for agent based models.

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