Title: Zero cosmological Constant from Normalized General Relativity

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Abstract: Normalizing the Einstein-Hilbert action by the volume functional makes the theory invariant under constant shifts in the Lagrangian. The field equations then resemble unimodular gravity whose otherwise arbitrary cosmological constant is now determined as a Machian universal average. We first prove that an empty space-time is necessarily flat, and then demonstrate, by invoking the scalar field paradigm, that the cosmological constant is actually zero. Normalized general relativistic cosmology is then discussed at the mini-superspace level, where we confirm the generic zero cosmological constant result, and furthermore prove that if matter is attractive then the Universe cannot be closed. The only non-vanishing cosmological constant solutions are associated with static Einstein-like closed universes.

Λ=0 from Normalized General Relativity

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Since dinner is almost ready, it will (hopefully) be a short presentation...

Warning: At this stage, I can only offer a naive idea, not yet a full theory.

Rationale

- Unlike dark energy, the cosmological constant is a real constant. Its value must be the same everywhere, and it is furthermore time independent. To fix its exact universal value, one may need to take into account the entire space-time manifold.

- To calculate the tiny non-zero value of the cosmological constant, and thus overcome the worst fine-tunning problem ever, it seems mandatory (as well as logical) to first decode the zero cosmological constant limit, which cannot reflect fine details of some smart model.

- There is a symmetry principle underlying any field theoretical quantity which vanishes identically. Are we somehow missing a fundamental symmetry principle?

After a century of general relativity, decades of superstring/M theory, and with lots of extra dimensions floating around, the (bifurcated) cosmological constant puzzle is still very much alive...

The resolution may wait for the so-called quantum gravity,
or else may in fact have a more naive origin...

On field theoretical grounds, an underlying symmetry principle $\mathcal{L} \to \mathcal{L} + const$ seems mandatory to account for the limit $\Lambda = 0$.

As otherwise, how can one honestly expect to bridge a discrepancy of 123 orders of magnitude (or just 53 orders of magnitude when broken SUSY is called to the rescue)?

- Uni-modular gravity (in some sense, Einstein): $\sqrt{-g} = 1$
- Alternative measure (Guendelman): $\sqrt{-g} \to \frac{1}{4!}\epsilon^{\mu\nu\lambda\sigma}\partial_\mu\phi_1\partial_\nu\phi_2\partial_\lambda\phi_3\partial_\sigma\phi_4$
- Signature reversal symmetry ('tHooft): $g_{\mu\nu} \to -g_{\mu\nu}$

Unimodular gravity highlights

In place of the full Einstein equations one obtains only their traceless part

$$\mathcal{R}^{\mu\nu} - \frac{1}{4} g^{\mu\nu} \mathcal{R} = -\frac{1}{2} \left(T^{\mu\nu} - \frac{1}{4} g^{\mu\nu} T \right)$$

These equations, first introduced by Einstein when discussing the possible role played by gravitational fields in the structure of elementary particles, give the false impression that TrR has been left out.

Assuming energy-momentum conservation, the Bianchi identities imply

$$-\mathcal{R}_{,\mu} + \frac{1}{2} T_{,\mu} = 0 \implies -\mathcal{R} + \frac{1}{2} T = 4\tilde{\Lambda}$$

The newborn constant of integration $\tilde{\Lambda}$ replaces the original put by hand Λ , which has disappeared from the equations, as the physical cosmological constant.

Normalized General Relativity

Consider the so-called normalized Einstein-Hilbert action

$$I_{NGR} = \frac{I_{GR}}{\epsilon\, I_V} = \frac{\int (-\mathcal{R} + \mathcal{L}_m)\sqrt{-g}\; d^4x}{\epsilon \int \sqrt{-g}\; d^4x}$$

- The initial Λ absorbed, as designed, by redefining $I \to I + \frac{1}{2}\epsilon\Lambda$.
- ϵ, having units of (length)$^{-4}$, introduced on dimensional grounds.
- Adding a total derivative to the Lagrangian is not necessarily trivial.

The above action is by definition non-local, at least in the Gel'fand-Fomin sense. Counter intuitively, however, we will show that the associated equations of motion are de-facto local, such that the sole effect of the apparent non-locality is expressed in this case by globally fixing the value of some newly emerging $\widetilde{\Lambda}$.

Non-local functionals of this kind have been considered in the literature. Two examples of which are in order:

Degravitation (Arkani-Hamed, Dimopoulos, Dvali, Gabadadze '02):
A carefully designed so-called 'filter function' has been postulated, such that in the far infrared

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = \frac{1}{4} \eta^2 g_{\mu\nu} \langle \mathcal{R} \rangle - \frac{1}{2} T_{\mu\nu}$$

$$\langle \mathcal{R} \rangle = \frac{\int \mathcal{R} \sqrt{-g}\, d^4x}{\int \sqrt{-g}\, d^4x}$$ denoting the space-time average of the Ricci scalar.

Yamabe problem ('60): $$\frac{\int \mathcal{R} \sqrt{-g}\, d^n x}{\left(\int \sqrt{-g}\, d^n x\right)^{\frac{n-2}{n}}}$$

This functional was invoked to prove that any compact Riemannian manifold of dimension n>3 can be conformally mapped into a constant scalar curvature manifold.
The power $\frac{n-2}{n}$ so chosen to guarantee the invariance under global rescaling of the metric $g_{\mu\nu} \to k g_{\mu\nu}$.

• All classical matter field equations and geodesic trajectories remain intact.

• The gravitational field equations, on the other hand, are deceptively local and highly resemble uni-modular gravity,

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R} = \widetilde{\Lambda} g_{\mu\nu} - \frac{1}{2} T_{\mu\nu}$$

but unlike in uni-modular gravity, $\widetilde{\Lambda} \equiv \frac{1}{2}\epsilon I$ does not stay arbitrary.

The solution $g_{\mu\nu}(x;\widetilde{\Lambda})$ of the field equations can be re-used to calculate the value $I(\widetilde{\Lambda})$ of the action along the classical path.
This way, $\widetilde{\Lambda}$ must be a solution of the functional equation

$$I(\widetilde{\Lambda}) = \frac{2\widetilde{\Lambda}}{\epsilon}$$

We find it remarkable that the entire non-locality of the theory goes into fixing one single constant parameter, namely $\widetilde{\Lambda}$.

Revival of Mach Principle

Together, the gravitational field equations are non-local (and acausal) since they contain a space-time average of the Einstein-Hilbert action. However, once $\tilde{\Lambda}$ gets fixed, they become practically local.

A situation where the equations of motion keep a local character, but have a 'minimal touch' of non-locality, can be considered a realization of the Mach principle.

The latter refers to the vague hypothesis (several definitions exist), which has fascinated Einstein, namely

Mass there governs inertia here,

or as interpreted by Hawking and Ellis,

Local physical laws are determined by
the large-scale structure of the Universe.

With this in mind, we recall the sensitivity of the action to the domain of integration. Following the Mach philosophy, the integration should be carried out over the entire space-time manifold.

The case of an Empty Space-time

$$T_{\mu\nu} = 0 \quad \Longrightarrow \quad \mathcal{R} = -4\widetilde{\Lambda}$$

Associated with some (still arbitrary) $\widetilde{\Lambda}$, is the value $I(\widetilde{\Lambda}) = \frac{4\widetilde{\Lambda}}{\epsilon}$ of the action.

Confronting the latter with the master functional equation establishes one of our main results

$$\frac{4\widetilde{\Lambda}}{\epsilon} = \frac{2\widetilde{\Lambda}}{\epsilon} \quad \Longrightarrow \quad \widetilde{\Lambda} = 0$$

Within the framework of NGR, and consistent with Einstein philosophy,

An empty space-time is Ricci tensor flat

(if maximal symmetry is imposed, than it is totally flat)

Bear in mind, however, that $T_{\mu\nu} \neq 0$ does not necessarily imply $\widetilde{\Lambda} \neq 0$.

To demonstrate the non-trivial role of adding a total derivative to the Lagrangian, consider a (4-dim) Gauss-Bonnet term

$$I = \frac{\int (-\mathcal{R} - \xi B)\sqrt{-g}\; d^4x}{\epsilon \int \sqrt{-g}\; d^4x}$$

$$B = \mathcal{R}^2 - 4\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} + \mathcal{R}^{\mu\nu\lambda\sigma}\mathcal{R}_{\mu\nu\lambda\sigma}$$

The equations of motion remain intact, but the functional equation for the cosmological constant is now quadratic:

$$I = 2I + \xi\left(\frac{2}{3}4 - 2\right) I^2$$

$$I_1 = 0 \quad , \quad I_2 = -\frac{3}{2\xi}$$

- *I=0* is still a solution.
- The non vanishing solution explodes at the $\xi \to 0$ limit.

Topological origin for the cosmological constant?
Non-zero cosmological constant from *f(R)* theories?

Probing the effect of Matter

To study the effect of matter, it seems pedagogical to first derive the NGR extension of Schwarzschild geometry surrounding a point-like particle of mass M. However, following the Geroch-Traschen theorem, GR (and hence NGR) is not capable of consistently dealing with co-dimension n>2 gravitating sources.

With this in mind, to probe the effect of matter we

- Calculate the space-time average $\langle \mathcal{R} \rangle$ of the Ricci scalar directly from the metric tensor in vacuum, and
- Neglect the gravitational self-force effects, and approximate the source contribution by $-M \int dt$.

$$ds^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + r^2 d\Omega^2$$

$$F(r) = 1 - \frac{M}{8\pi r} - \frac{\widetilde{\Lambda} r^2}{3}$$

it is the Laplacian piece $r^{-2}\partial_r\left(r^2\partial_r F(r)\right)$, residing in the Ricci scalar, which gives rise (like in the weak field limit) to the source $-M\delta(r)/8\pi r^2$, such that

$$\int -\mathcal{R}\sqrt{-g}\, d^4x = 4\tilde{\Lambda}V_4 + \frac{1}{2}M\int dt$$

Altogether, collecting the various pieces which constitute $I(\tilde{\Lambda})$, one can verify that the master functional now reads

$$\frac{4\tilde{\Lambda}}{\epsilon} - \frac{M\int dt}{2\epsilon V_4} = \frac{2\tilde{\Lambda}}{\epsilon}$$

$$V_4 = V_3\int dt = \frac{4}{3}\pi R^3\int dt \quad \Longrightarrow \quad \tilde{\Lambda} = \frac{M}{4V_3}$$

The combination of a finite M and an infinite spatial volume, which is the case here, clearly dictates $\tilde{\Lambda} \to 0$, thereby singling out the Schwarzschild solution.

By coincidence(?), exactly the same formula, albeit for a finite spatial volume, characterizes the static so-called Einstein Universe.

Minimally Coupled Scalar Field

$$\mathcal{L}_\phi = -\frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) \; ,$$
$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} + g_{\mu\nu}\mathcal{L}_\phi \; .$$

Tracing and then space-time averaging the gravitational field equations gives rise to

$$-\langle \mathcal{R} \rangle = 4\widetilde{\Lambda} + \frac{1}{2}\langle g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\rangle + 2\langle V \rangle$$

Calculating the value of the action along the classical solution, one finds

$$I(\widetilde{\Lambda}) = \frac{4\widetilde{\Lambda}}{\epsilon} + \frac{\langle V \rangle}{\epsilon} \qquad \Longrightarrow \qquad \widetilde{\Lambda} + \frac{1}{2}\langle V \rangle = 0$$

The theory becomes equivalent to a general relativistic minimally coupled scalar field theory governed by the effective scalar potential

$$V_{eff}(\phi) = V(\phi) - \langle V \rangle$$

which is manifestly invariant under $V \to V + const$.

Of particular interest is the case where the potential is bounded from below, that is $V(\phi) \geq V_{min}$. The configuration with the lowest energy density is then associated with

$$V(\phi(t)) = V_{min} = \langle V \rangle$$

and is thus characterized, irrespective of the value of V_{min} , by $\Lambda_{phys} = 0$.

The latter result holds even in cases where V_{min} is only asymptotically approached at $t \to +\infty$, e.g. an expanding FRW universe where the positive Hubble constant induces a friction term in the scalar field equation

$$\langle V \rangle = \frac{\int V(\phi(t)) a(t)^3 dt}{\int a(t)^3 dt} \to V_{min}$$

In particular, this quintessence category does not exclude an inflationary episode which only adds a finite contribution to the numerator.

Recalling that the cosmological constant (in contrast with dark energy) is characterized by being the space-time average of itself, and appreciating the fact that in a scalar field theory, the cosmological constant solely resides in the scalar potential, we are driven to the conclusion that

$$\Lambda_{phys} = \frac{1}{2} \langle V_{eff} \rangle \equiv 0$$

One may further argue that when adding the variety of standard model fields into the matter Lagrangian, the above formula establishes a simple yet a powerful mechanism for canceling out their vacuum's zero-point energy contribution ρ_{vac} to the cosmological constant. In fact, there is no need to calculate the zero-point energy, or even correctly identify its physical cutoff. The only vital piece of information is that it is a conserved constant energy density. This way, with $V_{eff}(\phi)$ being inert to $V(\phi) \to V(\phi) + \rho_{vac}$, the above result $\Lambda_{phys} = 0$ prevails.

A cosmological observer equipped with GR, but being unaware of NGR, would presumably interpret the slowly varying dark energy

$$\Lambda_{eff}(t) = \frac{1}{2} V_{eff}(\phi(t)) \geq 0$$

as todays 'cosmological constant', and consistent with recent observations, would find its tiny value to be positive definite. Unfortunately, its exact value is beyond the scope of the present approach.

Normalized General Relativistic Cosmology

The fact that the standard FRW energy/momentum tensor

$$T^{\mu}_{\nu} = Diag\,(-\rho, P, P, P)$$

is not derivable from an underlying matter Lagrangian is not an issue in GR, at least at the practical level.

The problem is effectively bypassed by invoking the mini-superspace formalism, with $\rho(a)$ serving as the potential term

$$\begin{aligned} I^{GR}_{mini} &= 2\int\left(3\frac{\ddot{a}a+\dot{a}^2+k}{a^2}-\Lambda-\rho(a)\right)a^3dt \\ &= 2\int\left[\left(3\frac{-\dot{a}^2+k}{a^2}-\Lambda-\rho(a)\right)a^3+\frac{d}{dt}\left(3a^2\dot{a}\right)\right]dt\;. \end{aligned}$$

The total derivative term can be eliminated from the action, leaving us with the more familiar second derivative free form.
The latter is used e.g when constructing the Hawking-Hartle wave function of the no-boundary universe.

Invoking the mini-superspace formalism to study normalized general relativistic cosmology, our starting point is the action

$$I_{mini}^{NGR} = \frac{2\int \left(3\frac{\ddot{a}a + \dot{a}^2 + k}{a^2} - \rho(a)\right) a^3 dt}{\epsilon \int a^3 dt}$$

- The initially subscribed Λ can now be absorbed, without any lose of generality, by redefining $I \to I - 2\Lambda$.

This, however, does not necessarily mean solving the cosmological constant puzzle since, like in uni-modular gravity, a newly born $\tilde{\Lambda} = \frac{1}{2}\epsilon I$ is about to make its appearance.

- The total derivative in the numerator cannot be removed since it is divided now by the volume functional.

The field equation minimizes the variation

$$\delta \int \left(6\frac{\ddot{a}a + \dot{a}^2 + k}{a^2} - 2\rho(a)\right) a^3 dt - 2\tilde{\Lambda}\delta \int a^3 dt = 0$$

The 'non-local' functional condition acquires a simple form

$$\Lambda = \frac{\int (\rho + 3P) a^3 dt}{2 \int a^3 \, dt} = \frac{1}{2} \langle \rho + 3P \rangle$$

which is in fact kinematical in origin, namely

$$\left\langle \frac{\ddot{a}}{a} \right\rangle = 0$$

We have to check the self-consistency of the following cases:

- **Conventional matter**

$$\rho + 3P > 0 \, , \quad e.g. \quad \rho = \frac{\alpha^2}{a^4} \, , P = \frac{1}{3} \rho \quad \Longrightarrow \quad \Lambda \geq 0$$

- **Stringy-like network (self-consistency guaranteed)**

$$\rho + 3P = 0 \, , \quad e.g. \quad \rho = \frac{\beta^2}{a^2} \, , P = -\frac{1}{3} \rho \quad \Longrightarrow \quad \Lambda = 0$$

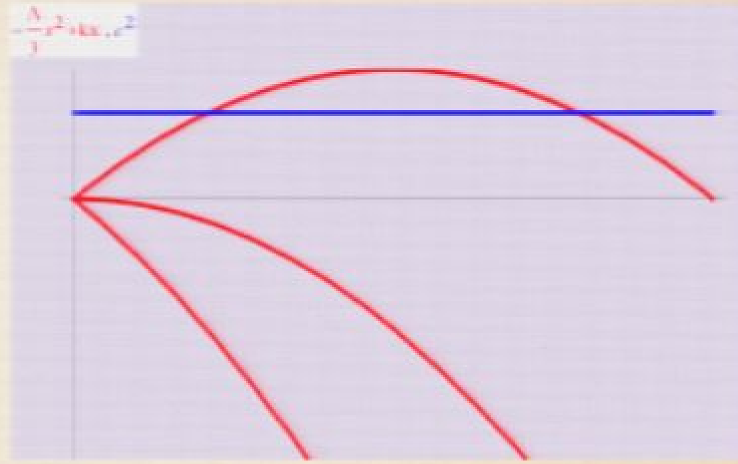
- **Ghosty matter,**

$$\rho + 3P < 0 \, , \quad e.g. \quad \rho = \frac{\gamma^2}{a} \, , P = -\frac{2}{3} \rho \quad \Longrightarrow \quad \Lambda \leq 0$$

Consider conventional matter,

is $\Lambda > 0$ **possible?**

$$\dot{x}^2 + \left(4kx - \frac{4\Lambda}{3}x^2\right) = 4c^2 \quad , \; x = a^2$$



$$for \;\; k \leq 0 \, , \quad \langle \rho + 3P \rangle \to 0 \;\; as \;\; T \to \infty$$

$$for \;\; k > 0 \;\; and \;\; k^2 < \frac{4\Lambda}{3}c^2, \quad \langle \rho + 3P \rangle \to 0 \;\; as \;\; T \to \infty$$

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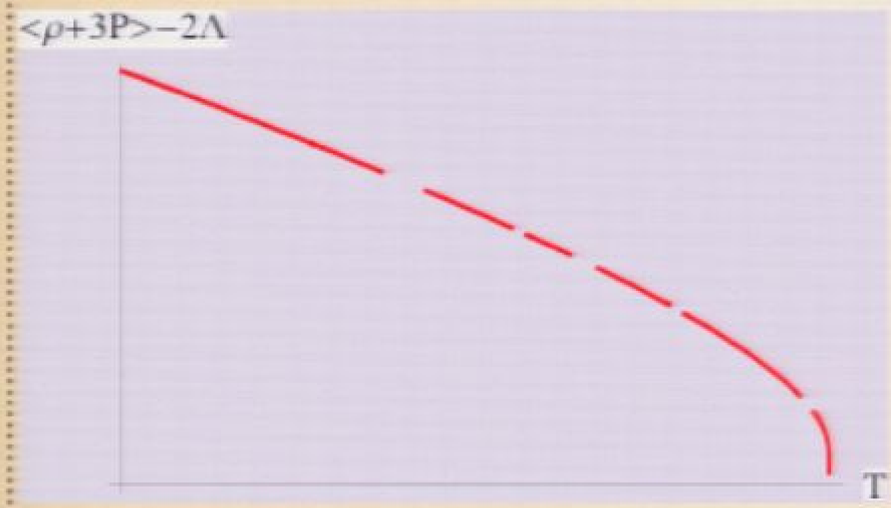

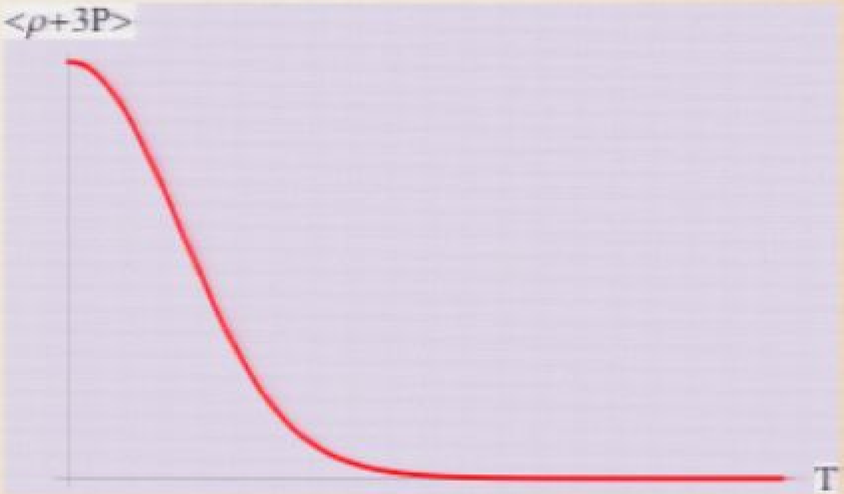

$$\langle \rho + 3P \rangle \to 0 \;\; as \;\; T \to \infty$$

The only $\Lambda > 0$ solution is the 'instanton'

$$\Lambda = \frac{3k^2}{4c^2} \;\; (k > 0)$$

associated with an expanding Universe, asymptotically approaching a static Einstein like Universe.
(This non-generic solution is unacceptable, since it explodes at the c=0 limit).

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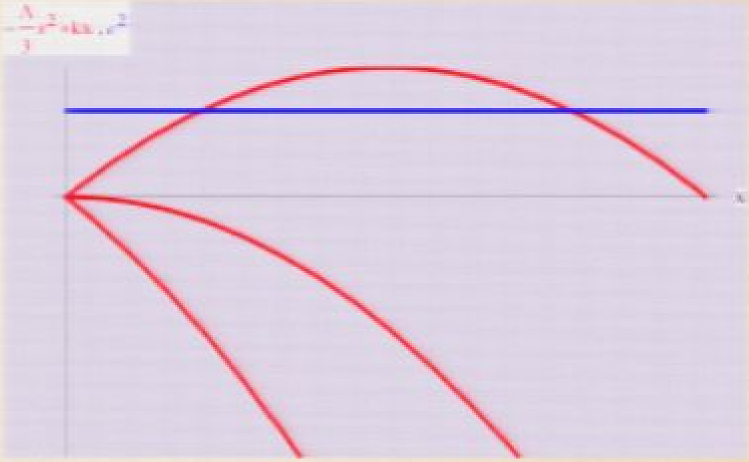

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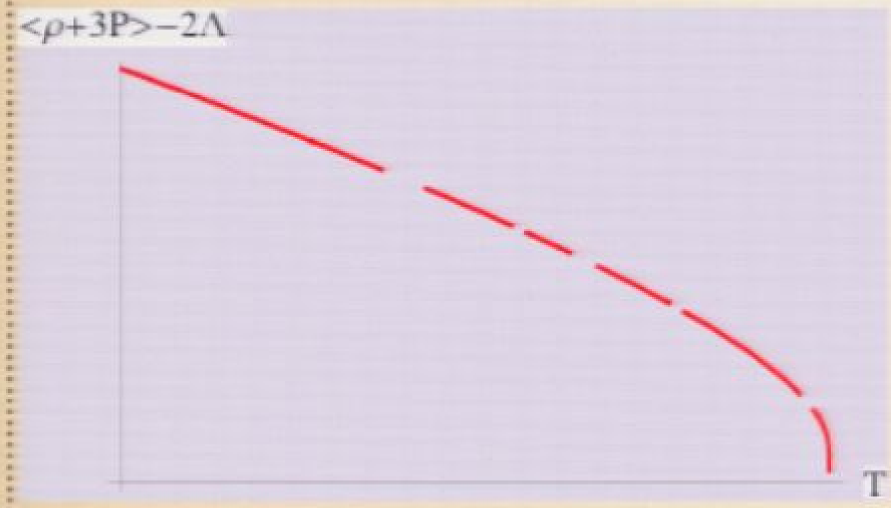

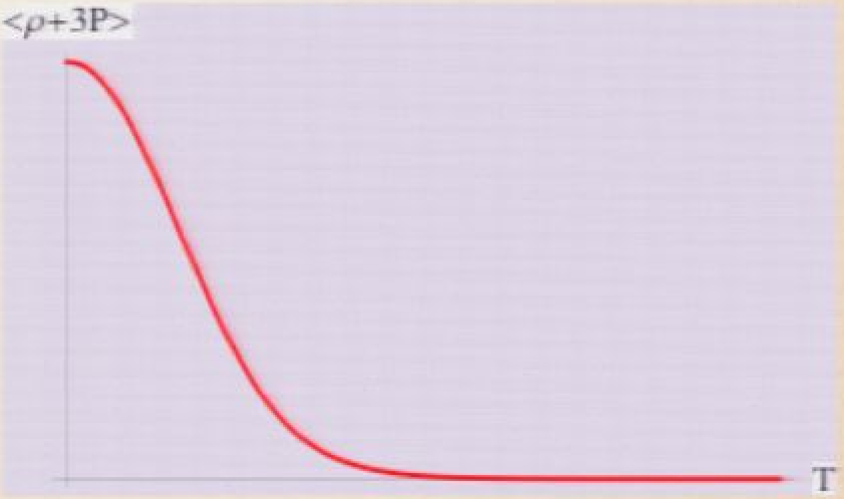

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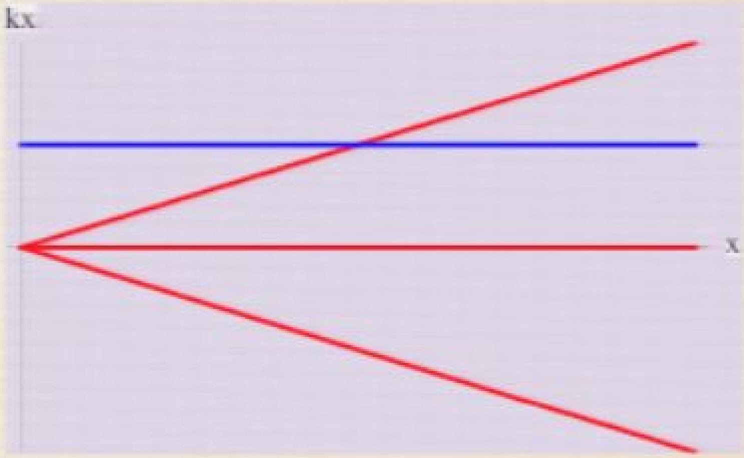

for k>0

$$3\frac{\int_0^{\frac{c}{k}} \frac{c^2}{\sqrt{2ct-kt^2}}\, dt}{\int_0^{\frac{c}{k}} \left(\sqrt{2ct-kt^2}\right)^3 dt} = \frac{8k^2}{c^2} \neq 0$$

for k=0

$$3\frac{\int_0^{T} \frac{c^2}{\sqrt{2ct}}\, dt}{\int_0^{T} \left(\sqrt{2ct}\right)^3 dt} = \frac{152}{8T^2} \to 0$$

for k<0

$$3\frac{\int_0^{T} \frac{c^2}{\sqrt{2ct-kt^2}}\, dt}{\int_0^{T} \left(\sqrt{2ct-kt^2}\right)^3 dt} \simeq \frac{12c^2}{k^2T^4}\mathrm{Log}\left[-4k\sqrt{T}\right] \to 0$$

The generic solution is $\Lambda = 0$. The bonus being the restriction $k \leq 0$, i.e. the Universe cannot be spatially closed.

As far as ghosty matter is concerned, the situation is quite similar

$$\langle \rho + 3P \rangle < 0 \quad \Longrightarrow \quad \Lambda = \frac{\int (\rho + 3P) a^3 dt}{2 \int a^3\, dt} \leqslant 0$$

The $\Lambda = 0$ **solution is singled out generically again, but now without any restrictions on k.**

The only non-generic solution is the anti-Einstein Universe characterized by

$$\Lambda = -\frac{3c^2}{4k} \quad (k > 0)$$

The case $\langle \rho + 3P \rangle = 0$ **is trivial**

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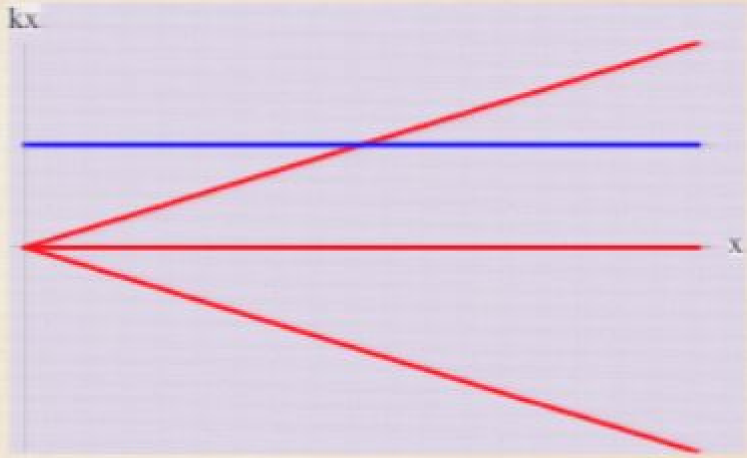

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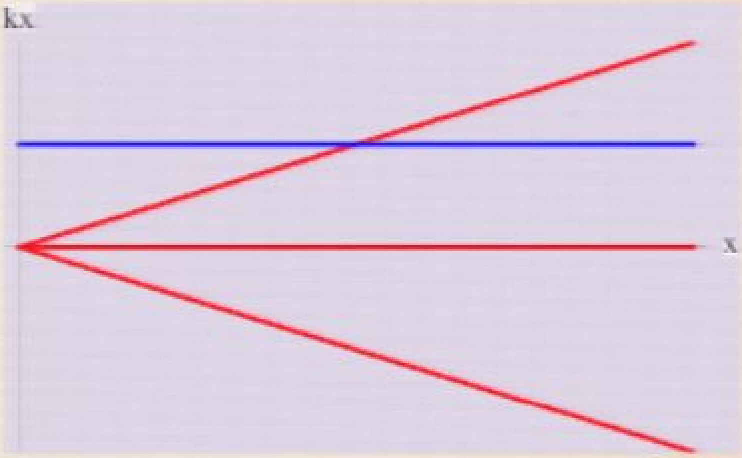

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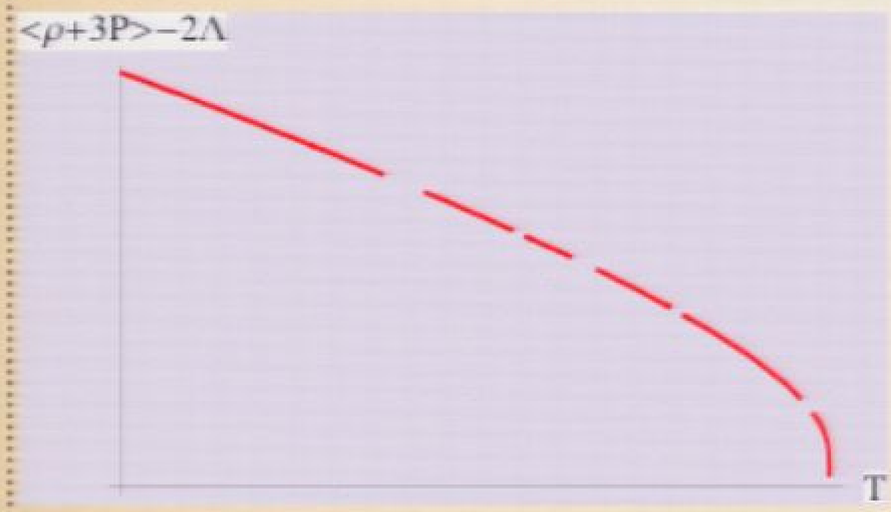

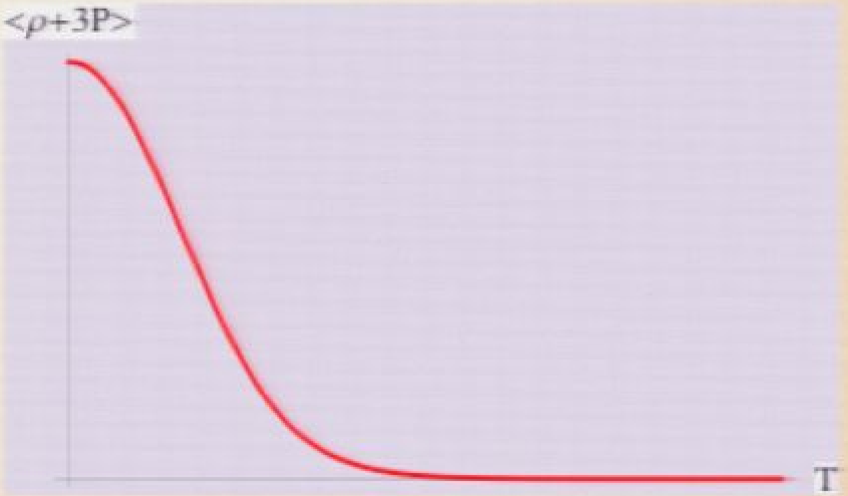

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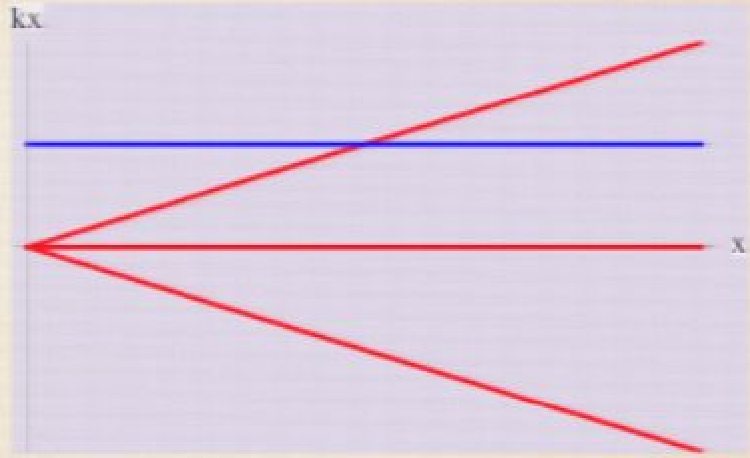

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$$3\frac{\int_0^{T} \frac{c^2}{\sqrt{2ct}}\,dt}{\int_0^{T} \left(\sqrt{2ct}\right)^3 dt} = \frac{152}{8T^2} \to 0$$

for k<0

$$3\frac{\int_0^{T} \frac{c^2}{\sqrt{2ct-kt^2}}\,dt}{\int_0^{T} \left(\sqrt{2ct-kt^2}\right)^3 dt} \simeq \frac{12c^2}{k^2T^4}\mathrm{Log}\left[-4k\sqrt{T}\right] \to 0$$

The generic solution is $\Lambda = 0$. The bonus being the restriction $k \leq 0$, i.e. the Universe cannot be spatially closed.

As far as ghosty matter is concerned, the situation is quite similar

$$\langle \rho + 3P \rangle < 0 \quad \Longrightarrow \quad \Lambda = \frac{\int (\rho + 3P) a^3 dt}{2 \int a^3 \, dt} \leqslant 0$$

The $\Lambda = 0$ **solution is singled out generically again, but now without any restrictions on k.**

The only non-generic solution is the anti-Einstein Universe characterized by

$$\Lambda = -\frac{3c^2}{4k} \quad (k > 0)$$

The case $\langle \rho + 3P \rangle = 0$ **is trivial**

$$\Lambda = \frac{\int (\rho + 3P) a^3 dt}{2 \int a^3 \, dt} = 0$$

To summarize our main points, they are:

- The cosmological constant is a concrete non-local realization of Mach principle.
- An empty space-time is necessarily Ricci tensor flat.
- Owing to the non-negativity of mass, the cosmological constant is non-negative definite.
- Invoking the scalar field paradigm, while the physical cosmological constant strictly vanishes, it is the dark energy which resembles a positive decaying 'cosmological constant'.
- At the mini-superspace level, results confirmed with a bonus: if matter is attractive then the Universe cannot be closed.
- The only non-vanishing cosmological constant solutions are associated with static Einstein-like closed universes.

To summarize our main drawbacks, they are:

- None...