Title: Ricci Solitons with Large Symmetry Group

Date: May 09, 2009 08:30 AM

URL: http://pirsa.org/09050049

Abstract: We produce new examples of Ricci solitons, including many of non-Kahler type, by looking for solutions with symmetries, thus reducing the equations to dynamical systems

Ricci solitons with large symmetry group

Perimeter Institute, May 2009

Joint work with M. Wang (McMaster)

Ricci flow

$$\frac{\partial g_\tau}{\partial \tau} = -2\text{Ric}\,(g_\tau)$$

Einstein metrics g have Ric(g)$= -\frac{\epsilon}{2}g$, so give solution evolving by homotheties

$$g_\tau = (1+\epsilon\tau)g$$

Generalise? Allow flow of diffeomorphisms

$$g_\tau = (1+\epsilon\tau)\psi_\tau^* g$$

These arise from *Ricci solitons*

$$\text{Ric}(g) + \frac{1}{2}L_X g + \frac{\epsilon}{2}g = 0$$

Equation for metric g and vector field X.

$\epsilon = 0$ steady soliton

$\epsilon = 1$ expanding

$\epsilon = -1$ shrinking

Important special case–*gradient solitons*

Now $X = \text{grad}\, u$, so equation is

$$\text{Ric}(g) + \text{Hess}(u) + \frac{\epsilon}{2} g = 0$$

How to find solutions?

Symmetries?

No compact homogeneous examples except Einstein metrics.

Weaken homogeneity assumption?

Cohomogeneity one metrics.

Take g (and u) to depend only on one variable t

eg if G acts isometrically on manifold M (preserving u), with generic hypersurface orbits.

Example

$M = \mathbb{R}^n \quad : \quad G = SO(n) \quad :$ generic orbit S^{n-1}

$$g = dt^2 + f(t) g_{\text{round}} \quad : \quad u = u(t)$$

Nonlinear ODE system.

Special orbit at origin–boundary conditions.

$n = 2$ Hamilton-Witten cigar (asymptotic to cylinder)

$n \geq 3$ Bryant soliton (asymptotic to paraboloid)

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Gradient Ricci solitons

Metric $dt^2 + g_t \ : \ X = \text{grad}\, u$.

Cohomogeneity 1 equations:

$$\begin{aligned} \text{tr}\, \dot{L} + \text{tr}\,(L^2) - \ddot{u} &= \frac{\epsilon}{2} \\ \dot{L} + (\text{tr}\, L)L - r - \dot{u}L &= \frac{\epsilon}{2}. \end{aligned}$$

L is shape operator (endomorphism wrt the metric g_t on hypersurface orbit), so

$$\dot{g}_t(Y, Z) = 2g_t(LY, Z)$$

Also r is Ricci endomorphism wrt g_t

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Kähler examples

(Cao, Koiso, Guan, Feldman-Ilmanen-Knopf, Pedersen-Tonneson-Friedman-Valent, Apostolov-Calderbank-Gauduchon-Tonneson-Friedman...)

Generalise (D-Wang):

Hypersurface circle bundle over product of Fano Kähler-Einstein manifolds

Equations exactly solvable if soliton is Kähler

Noncompact steady, expanding.

Compact shrinking.

Some complete noncompact shrinking solitons

(generalise FIK example—$\mathbb{CP}^n$ base)

Non-Kahler examples?

Multiple warped products

$$dt^2 + \sum_{i=1}^{r} g_i^2(t)\mathsf{h}_i$$

Open set is

$$(a, b) \times M_1 \times \ldots \times M_r$$

with possible collapse at one or both ends.

(M_i, h_i) are Einstein with positive Einstein constants λ_i and dimension d_i.

Generalisation of Bryant-Ivey examples.

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Consider steady case.

Suitable coordinates

$$X_i = \frac{\sqrt{d_i}\dot{g}_i}{g_i}\frac{1}{-\dot{u} + \mathrm{tr}L}$$

$$Y_i = \frac{\sqrt{\lambda_i d_i}}{g_i}\frac{1}{-\dot{u} + \mathrm{tr}L}$$

$$\frac{d}{ds} = \frac{1}{-\dot{u} + \mathrm{tr}L}\frac{d}{dt}$$

Equations become

$$X_i' = X_i\left(\sum_{j=1}^{r} X_j^2 - 1\right) + \frac{Y_i^2}{\sqrt{d_i}},$$

$$Y_i' = Y_i\left(\sum_{j=1}^{r} X_j^2 - \frac{X_i}{\sqrt{d_i}}\right).$$

Have Lyapunov function

$$\mathcal{L} = \sum_{j=1}^{r} \left(X_j^2 + Y_j^2\right) - 1$$

$$\mathcal{L}' = 2\mathcal{L}\left(\sum_{j=1}^{r} X_j^2\right)$$

Pick trajectory in unstable manifold of critical point on unit sphere

$$(X : Y) = (\frac{1}{\sqrt{d_1}}, 0..0 : \sqrt{\frac{d_1 - 1}{d_1}}, 0..0)$$

flowing into unit ball.

Analysis of ω-limit set and Lyapunov shows trajectory converges to origin.

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Must establish several limits to ensure metric is C^3. Then regularity theory gives smoothness at critical point (sphere M_1 collapses)

Obtain complete steady soliton on vector bundle over $M_2 \times \ldots \times M_r$.

Asymptotically paraboloid

$$\sim \quad dt^2 + t\mathrm{h}$$

Ricci is nonnegative.

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$$\mathcal{H} = \sum_{j=1}^{r} \sqrt{d_j} X_j$$

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Consider trajectories flowing into this region from critical point

$$(X : Y : W) = (\frac{1}{\sqrt{d_1}}, 0..0 : \sqrt{\frac{d_1 - 1}{d_1}}, 0..0 : 0)$$

on sphere in XY-space ($W = 0$).

Bounds on X_i, Y_i, W so flow exists for all s.

Analysis of smoothness near critical point is similar to steady case.

Long-time behaviour, ω-limit set, is more involved.

Can show trajectory converges to origin as $s \to \infty$.

Moreover $X_i \sim W^2$.

Recall $W' = W(\sum_j X_j^2 - \frac{\epsilon}{2}W^2)$.

Geodesic distance to $s = \infty$ is

$$\int_{t_0}^{t^{-1}(\infty)} dt = \int_{s_0}^{\infty} W\, ds = \int_{w_0}^{0} \frac{dW}{\sum_j X_j^2 - \frac{\epsilon}{2}W^2}$$

which is infinite.

So we obtain complete expanding soliton on vector bundle over $M_2 \times \ldots \times M_r$.

Note also $W' = -\frac{\epsilon}{2}W^3 + O(W^5)$

Truncating gives cone metric, in accordance with our expectation that our metrics are asymptotically conical.

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