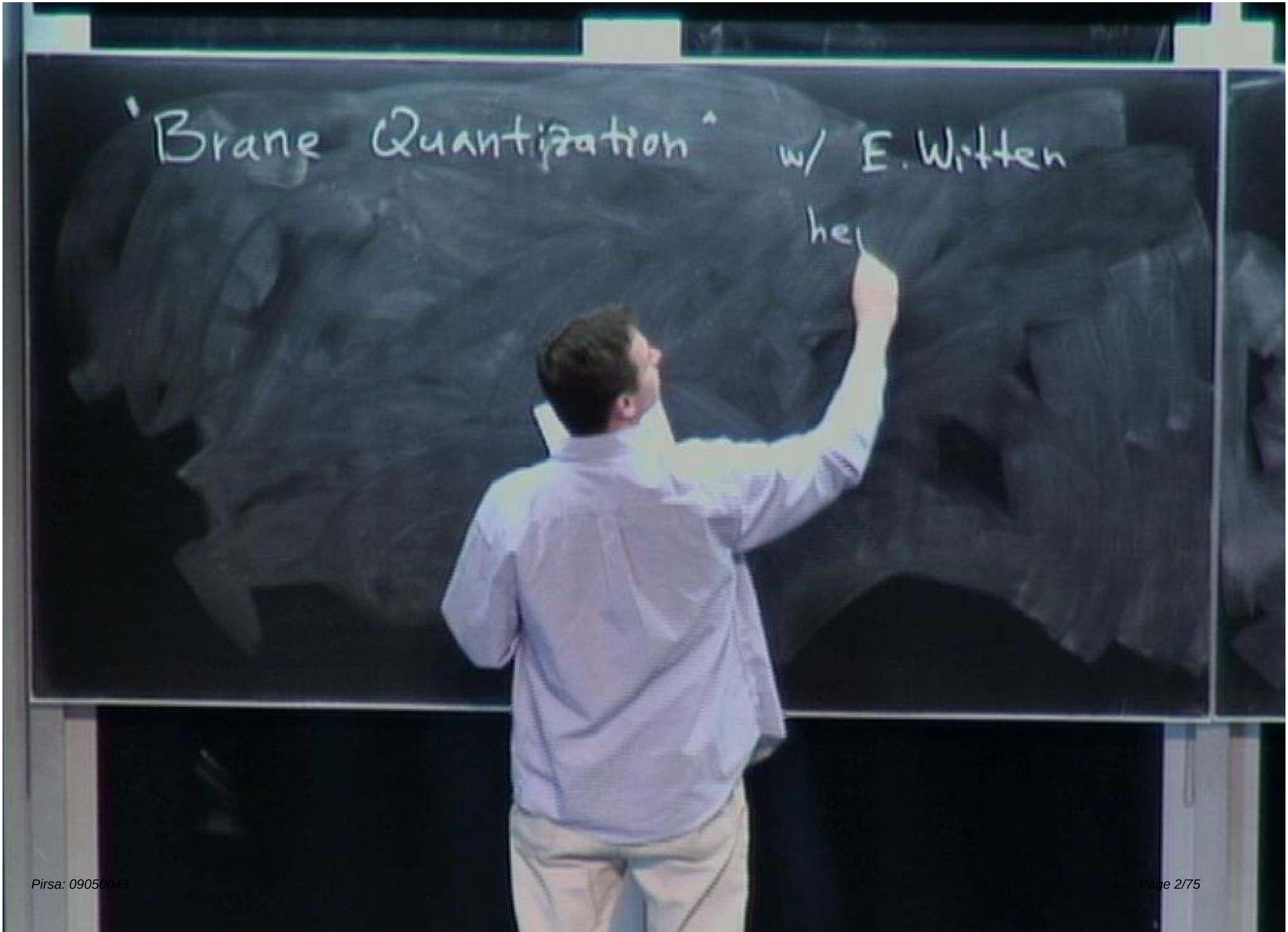
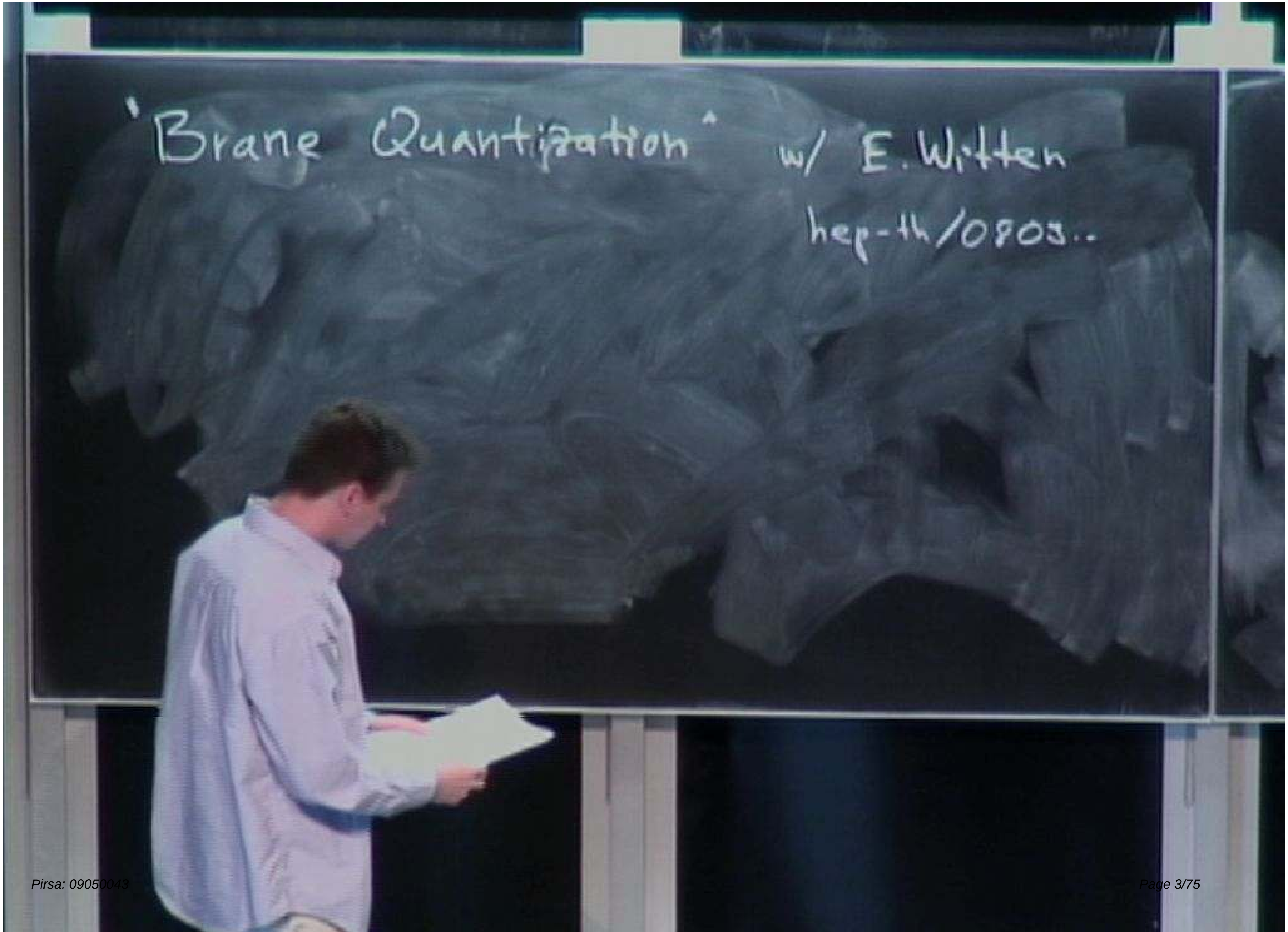
Title: Branes and Quantization

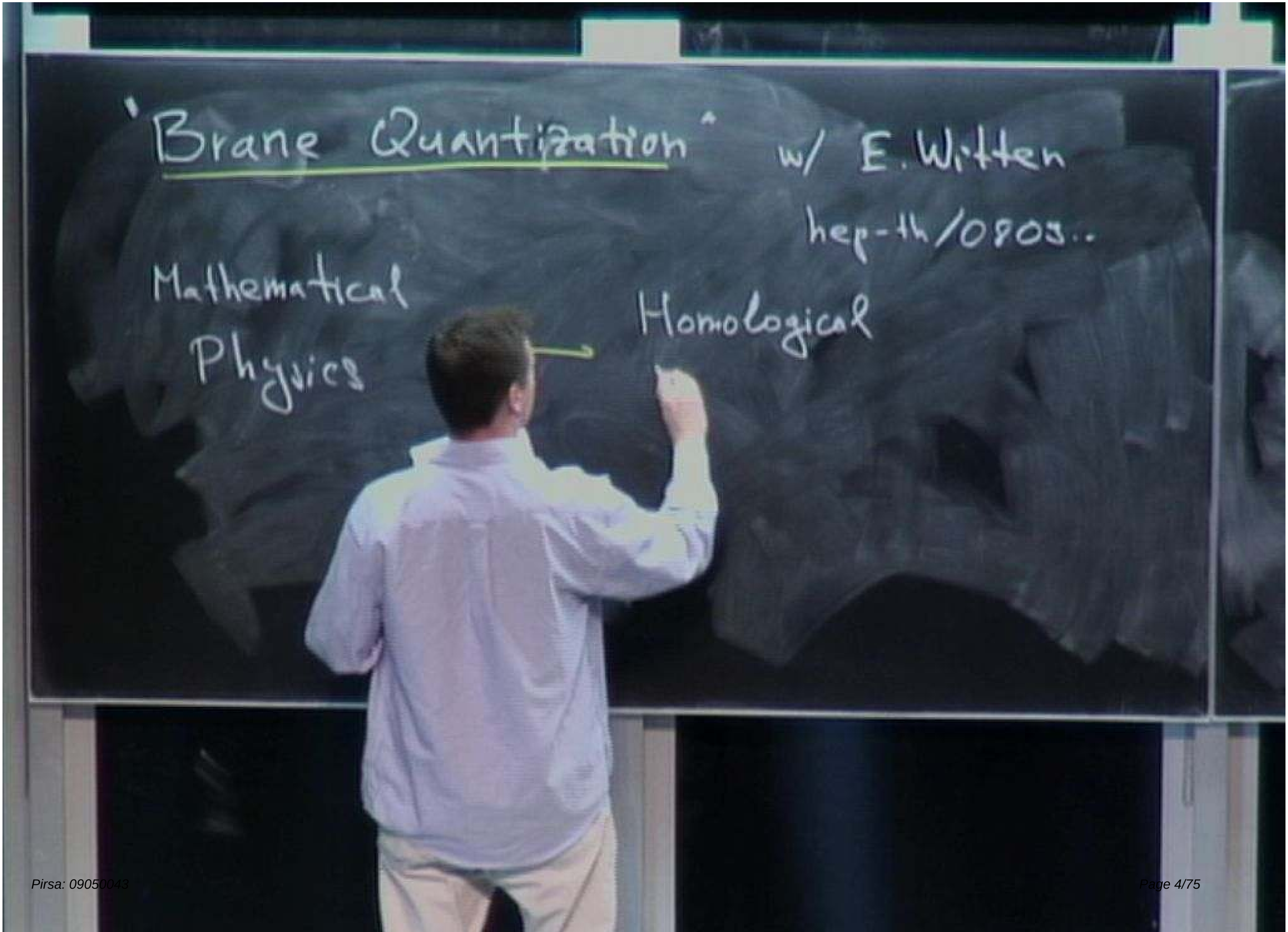
Date: May 10, 2009 01:45 PM

URL: http://pirsa.org/09050043

Abstract: TBA


'Brane Quantization'
w/ E. Witten
he


'Brane Quantization' w/ E. Witten
hep-th/0803..


"Brane Quantization" w/ E. Witten
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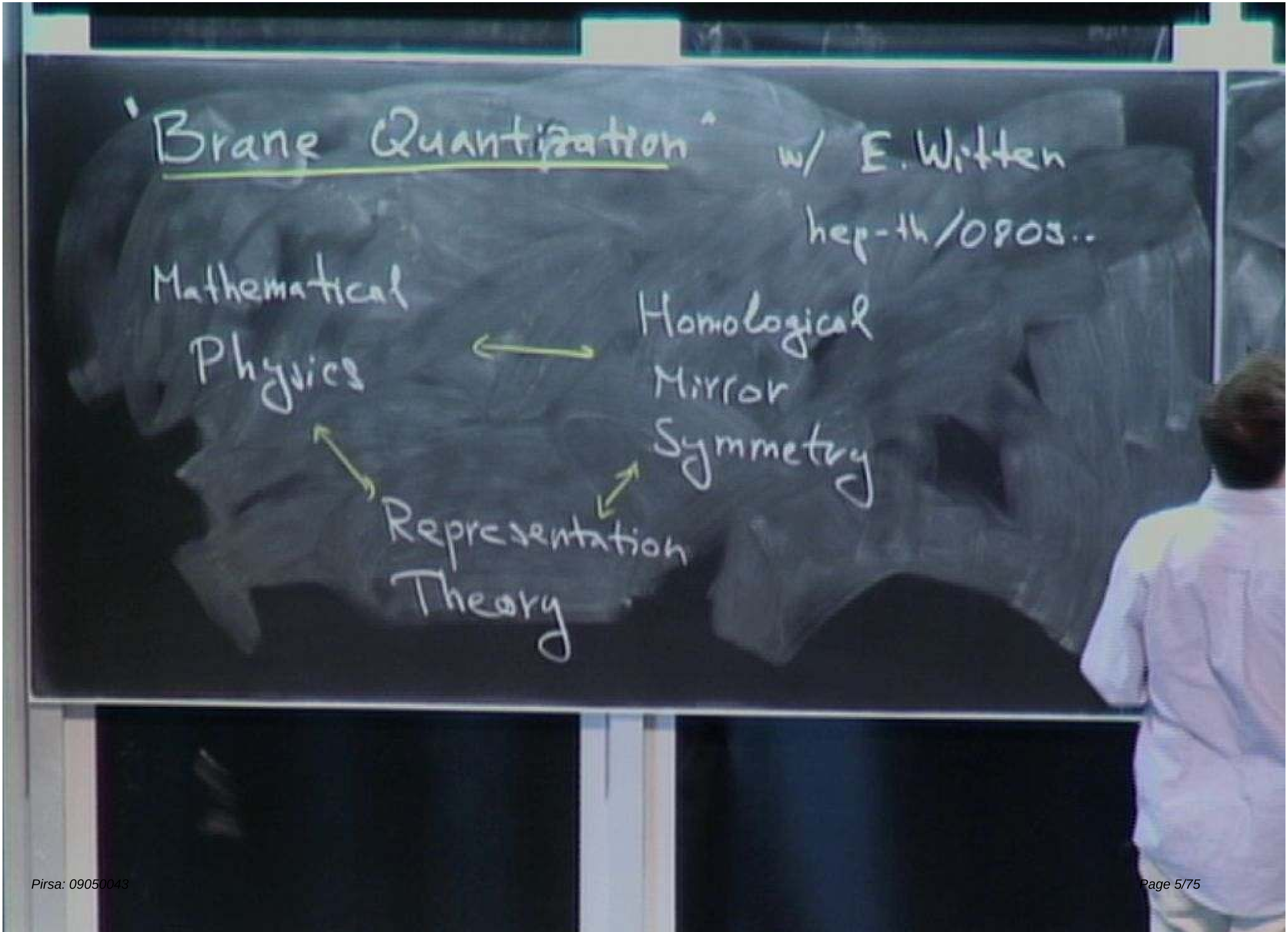
Mathematical
Physics
Homological


'Brane Quantization'
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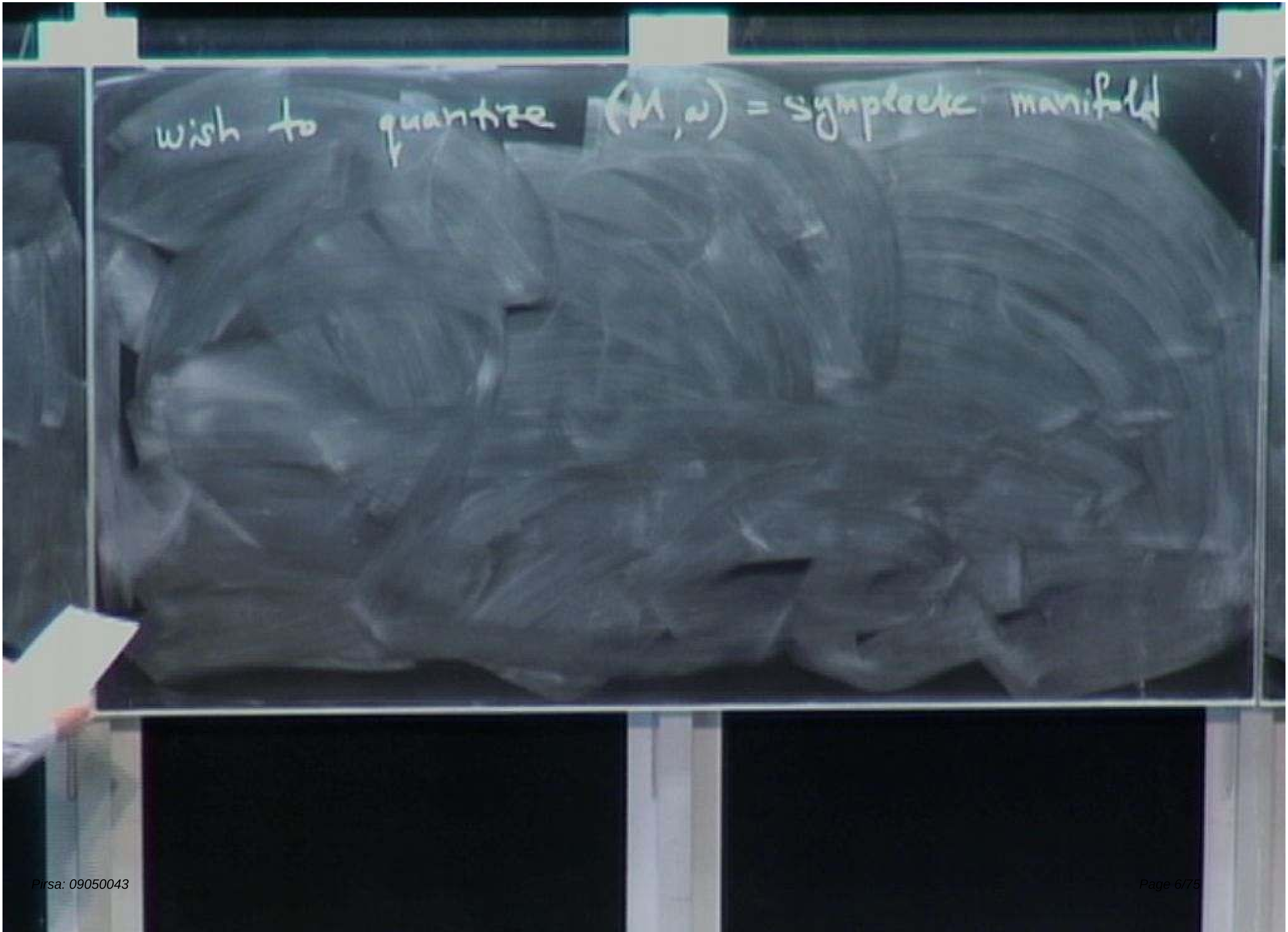
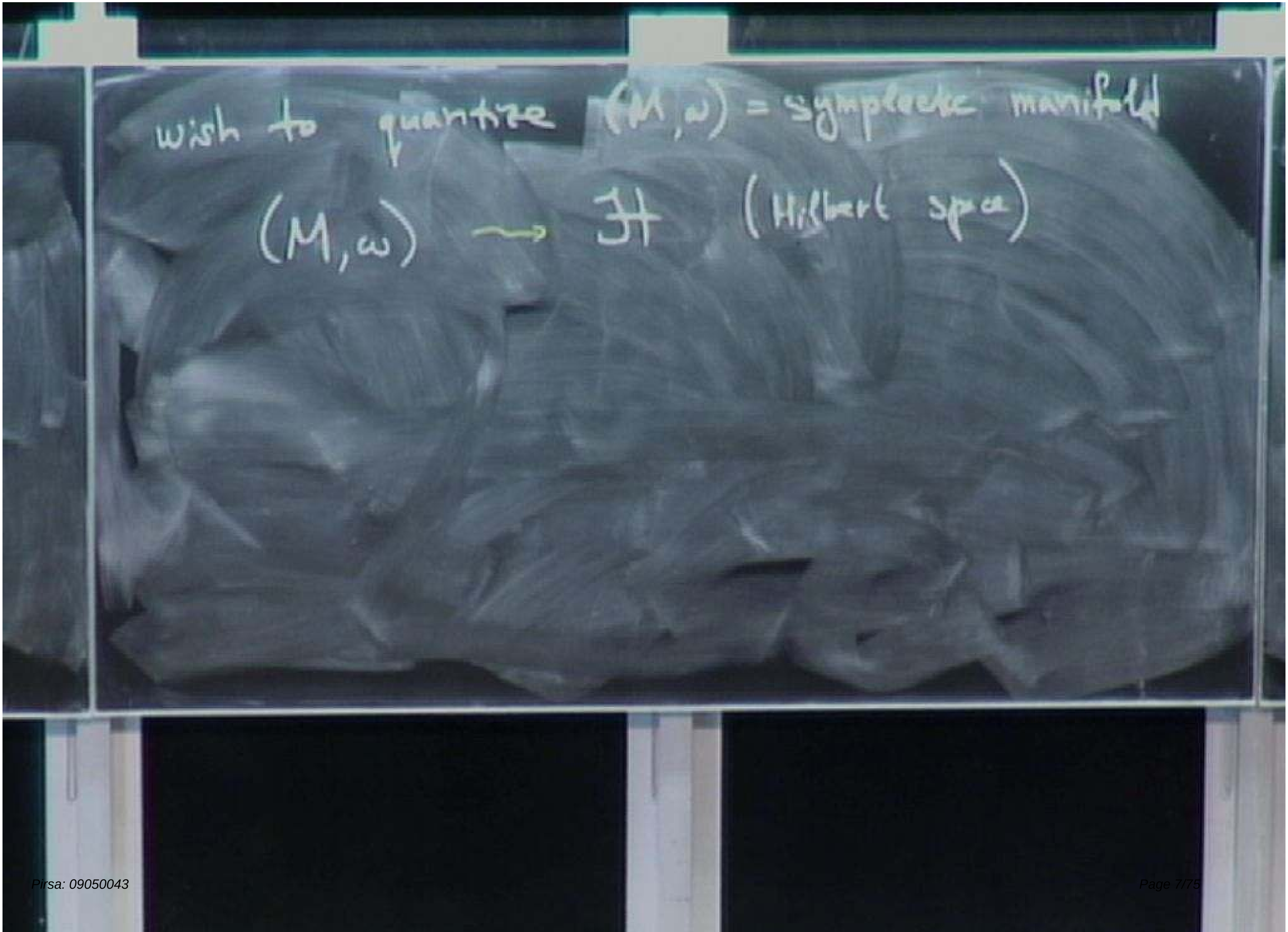
Mathematical Physics
Homological Mirror Symmetry
Representation Theory


wish to quantize (M, ω) = symplectic manifold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$(M, \omega) \rightsquigarrow \mathcal{H}$ (Hilbert space)

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$$(M, \omega) \rightsquigarrow \mathcal{H} \quad \text{(Hilbert space)}$$

$$\circlearrowright$$

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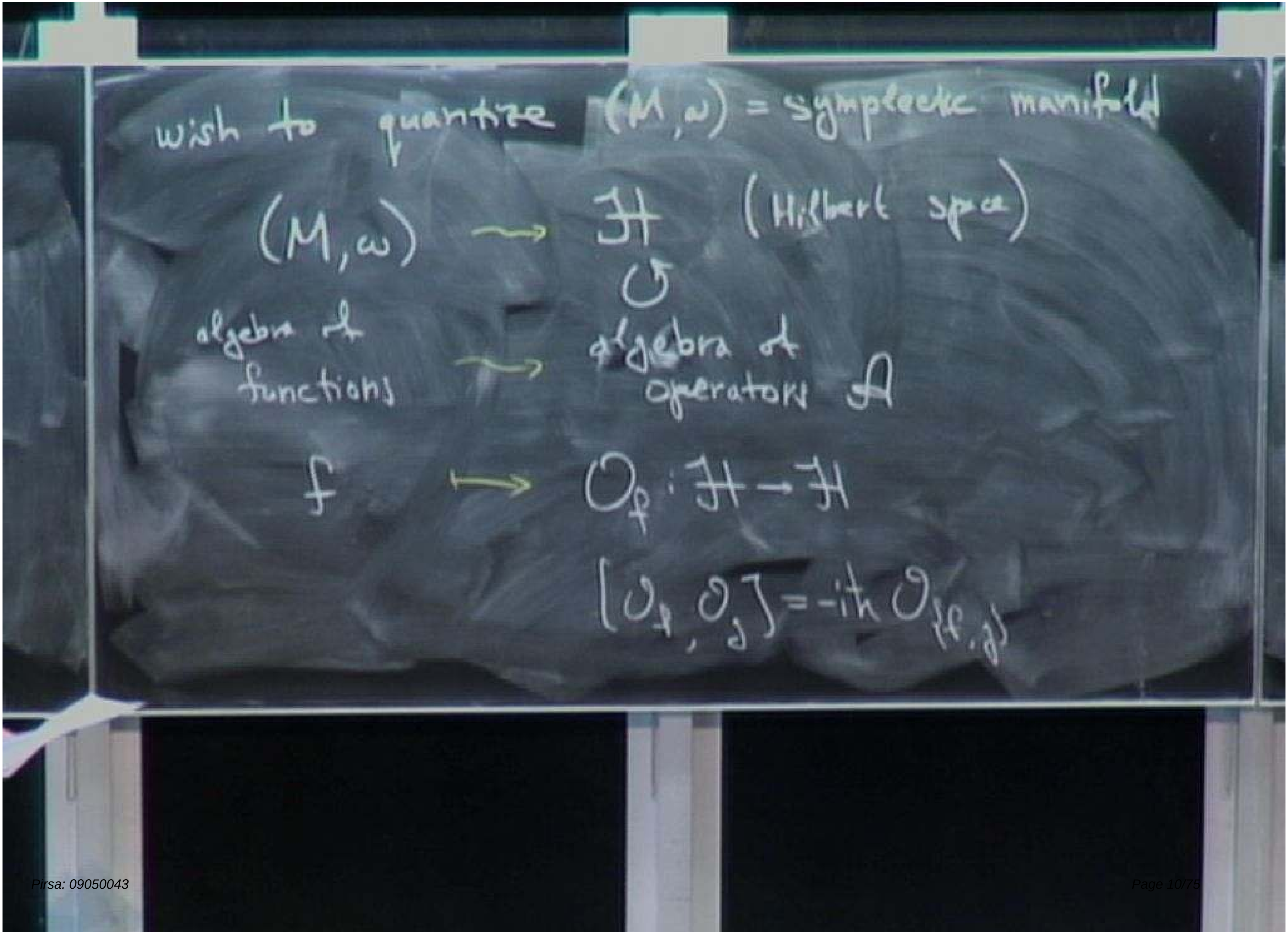
$$f \longmapsto \mathcal{O}_f : \mathcal{H} \to \mathcal{H}$$

$$[\mathcal{O}_f, \mathcal{O}_g]$$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(M, ω) ⟶ ℋ (Hilbert space)
algebra of functions ⟶ algebra of operators 𝒜
f ⟼ 𝒪_f : ℋ → ℋ
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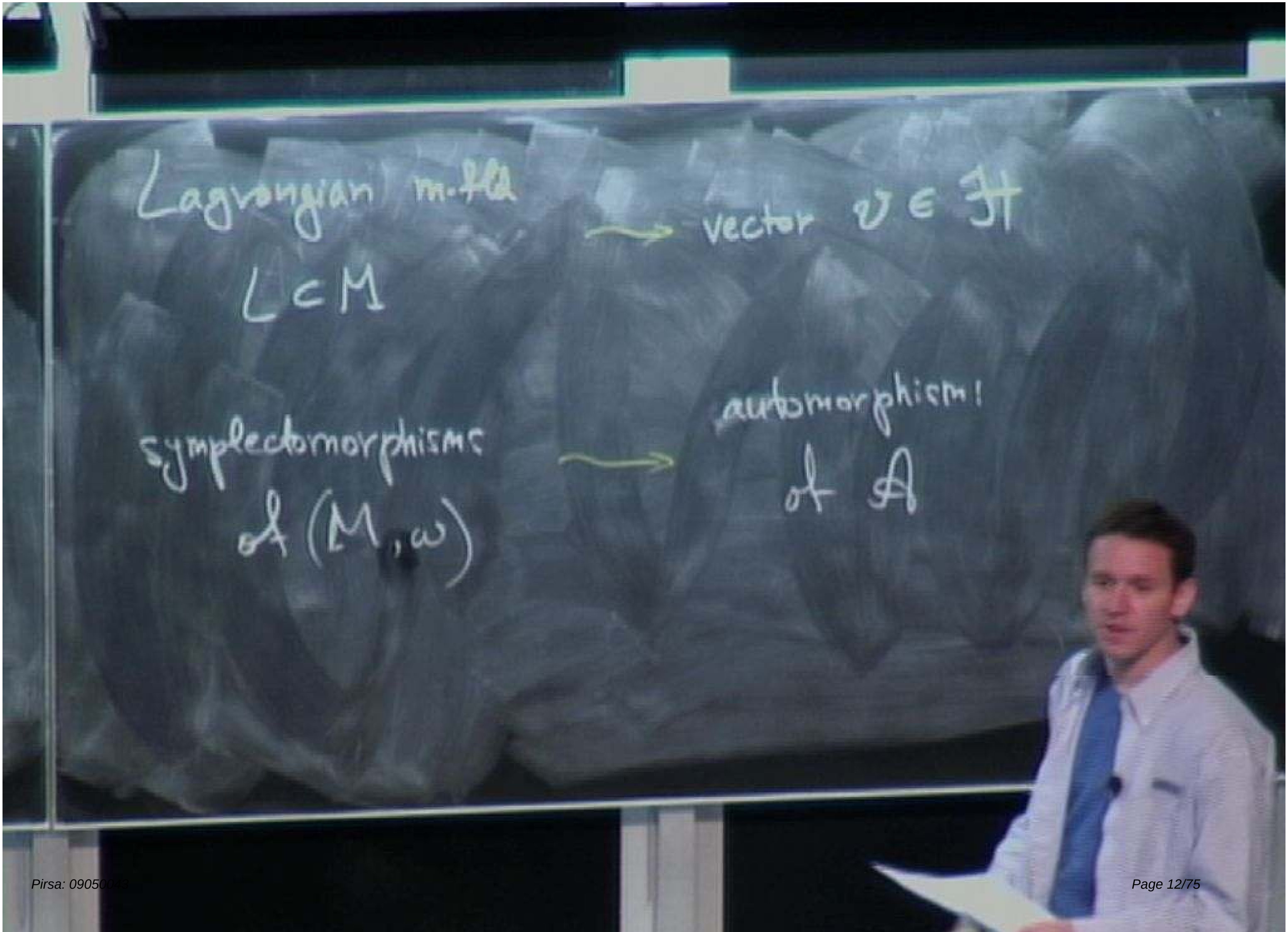
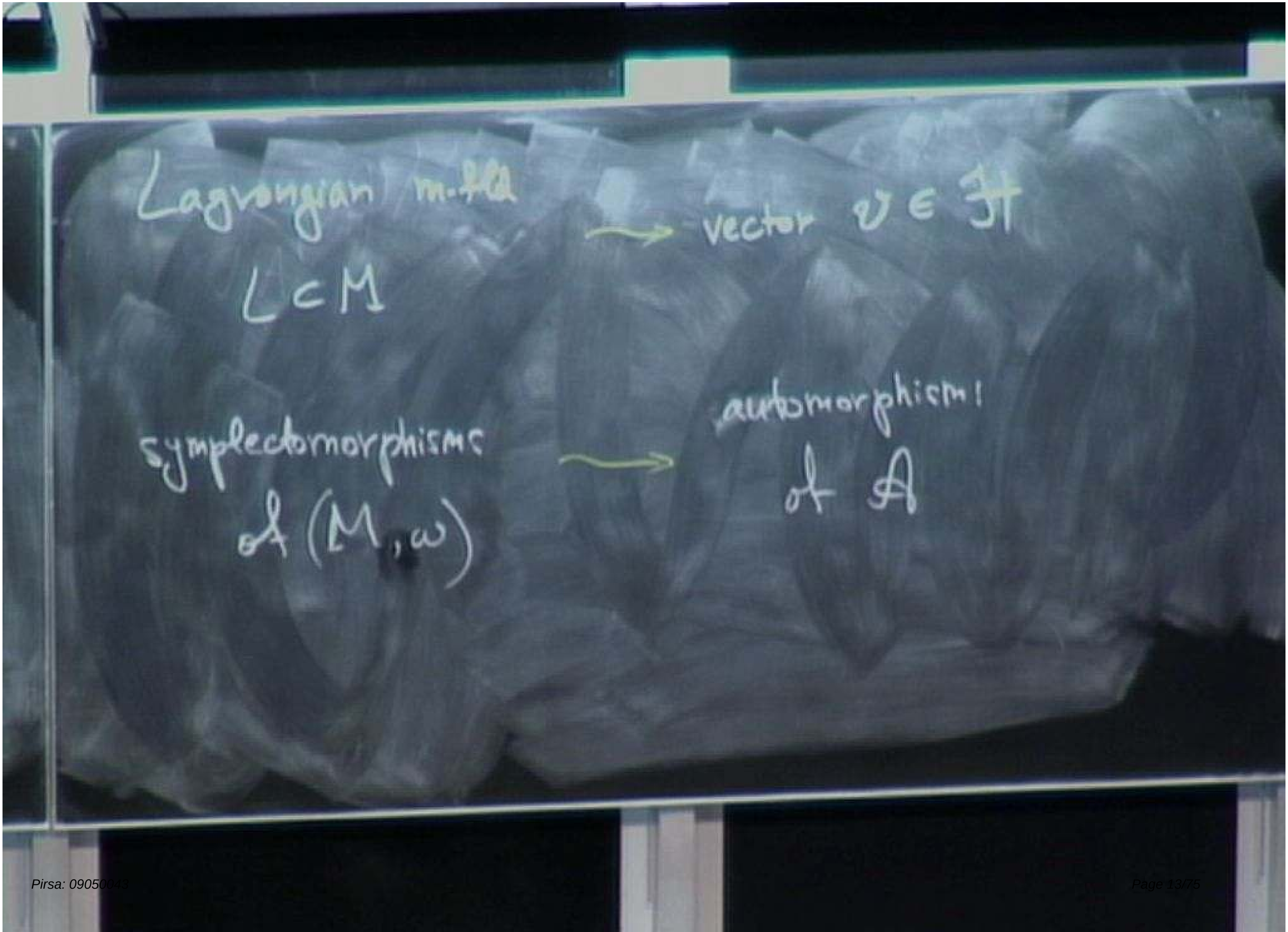
$$[\mathcal{O}_f, \mathcal{O}_g] = -i\hbar\, \mathcal{O}_{\{f,g\}} + \dots$$


Lagrangian m-fld
L ⊂ M
vector v ∈ H
symplectomorphisms
of (M, ω)
automorphism
of A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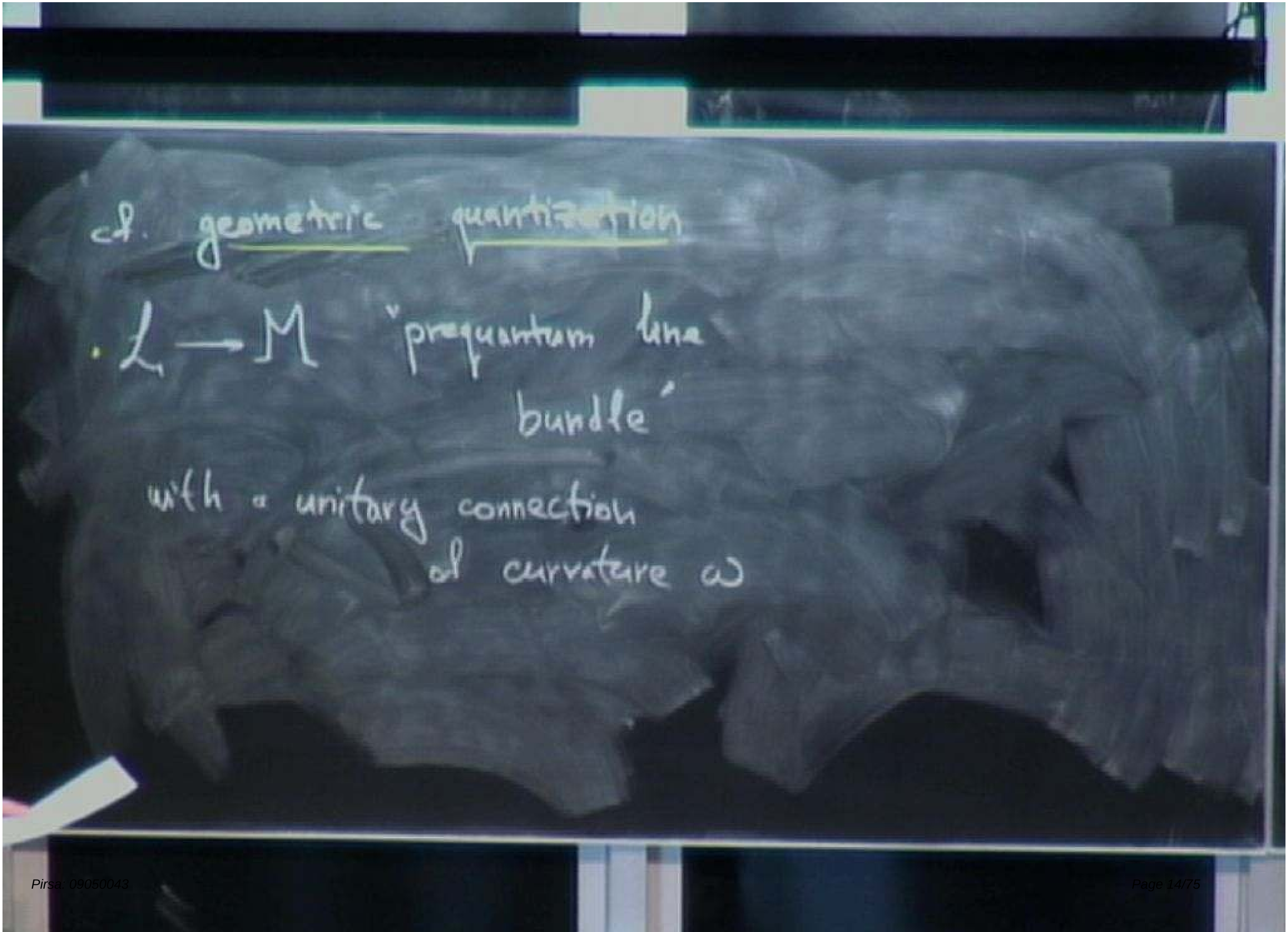
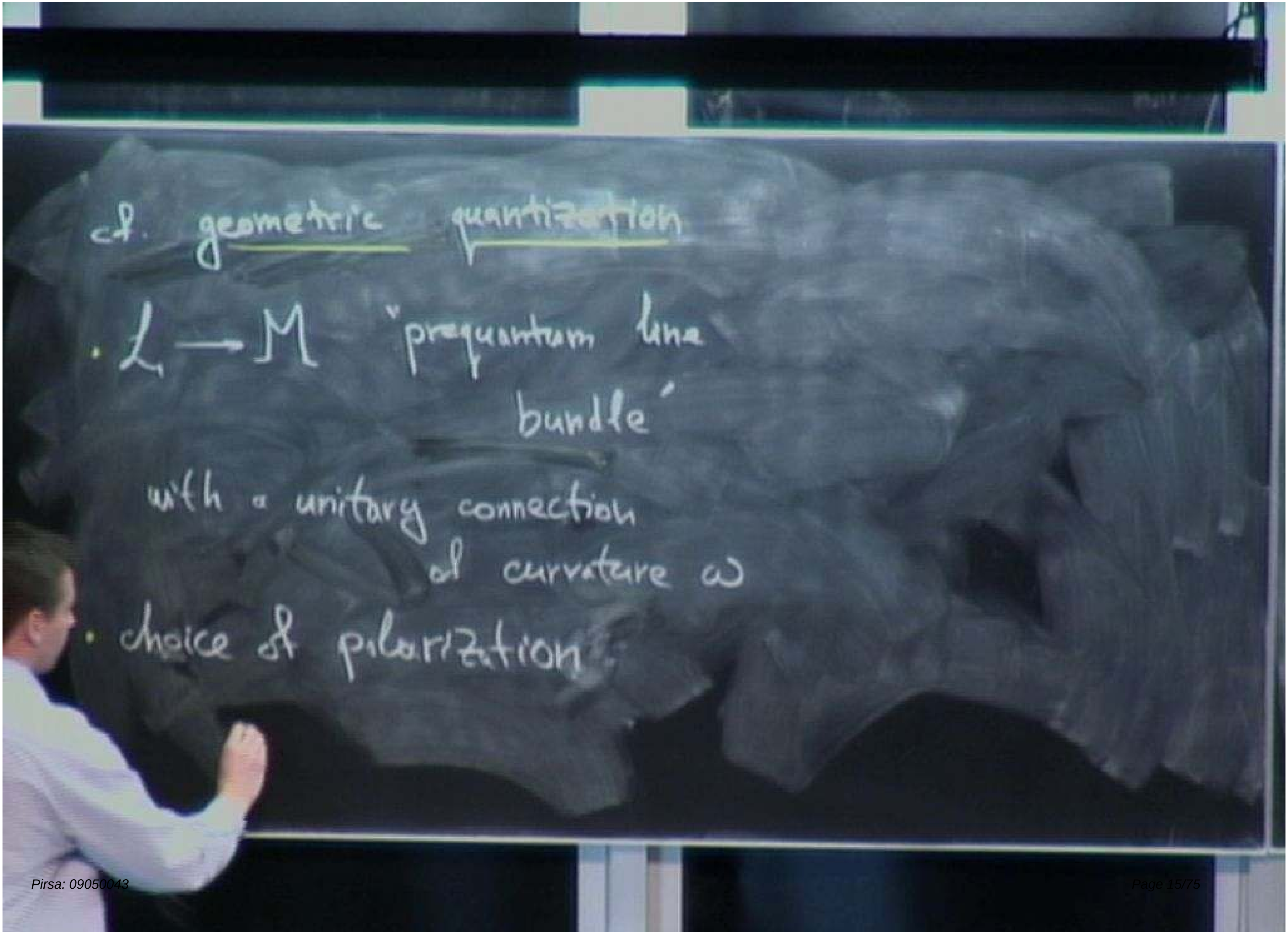

cf. geometric quantization
• L → M "prequantum line bundle"
with a unitary connection
of curvature ω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• $\mathcal{L} \longrightarrow M$ "prequantum line bundle"
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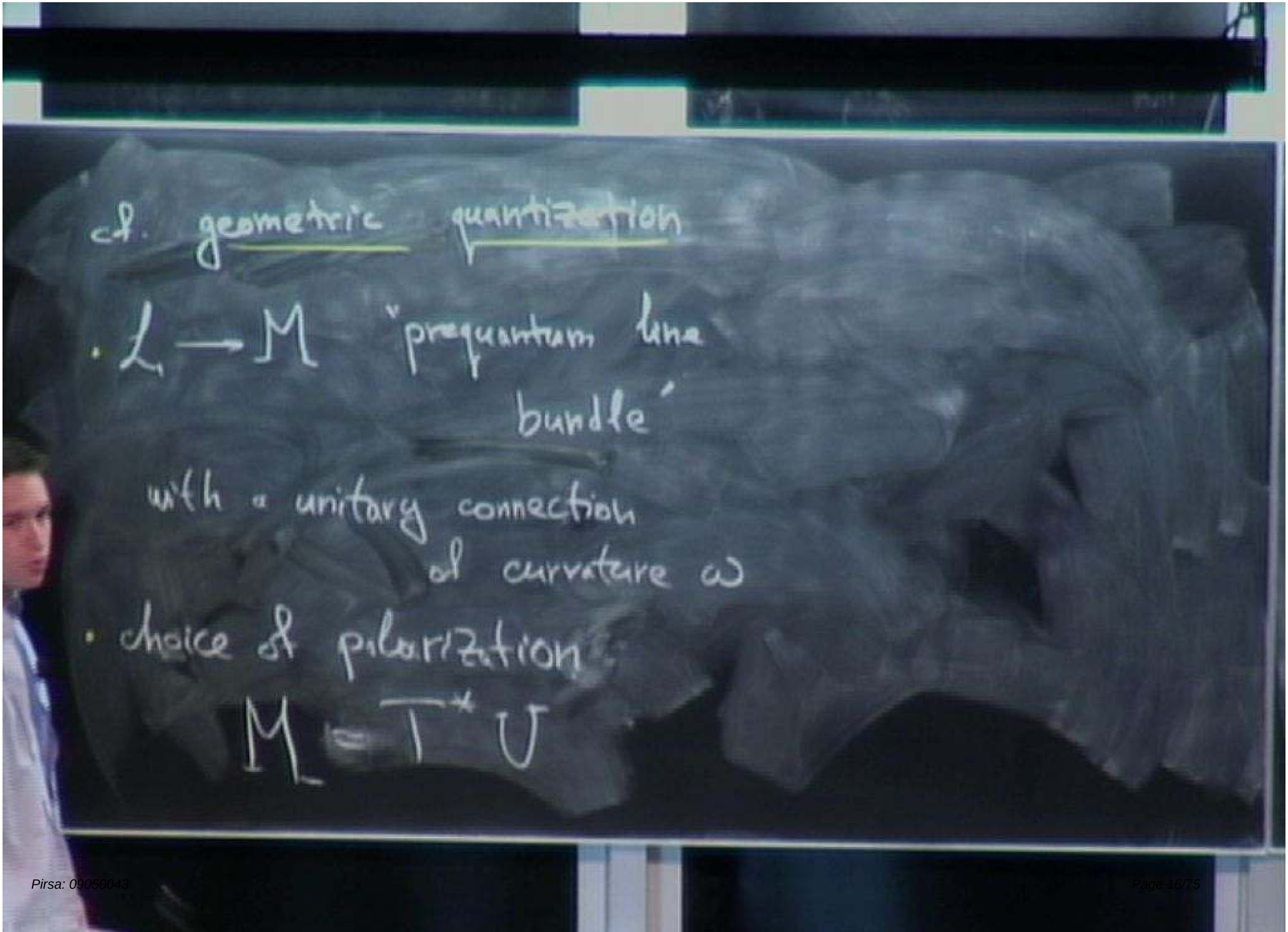
• choice of polarization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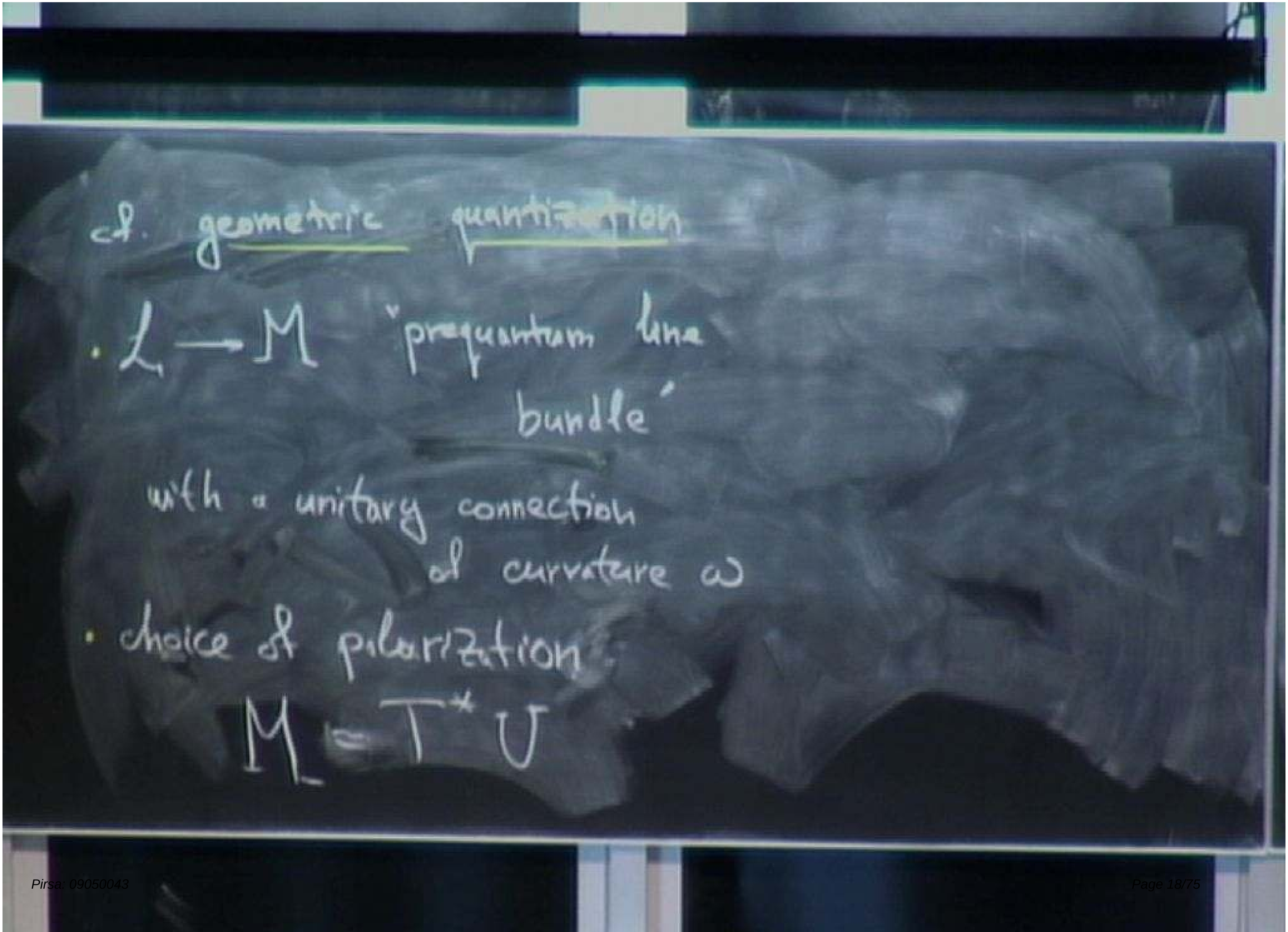
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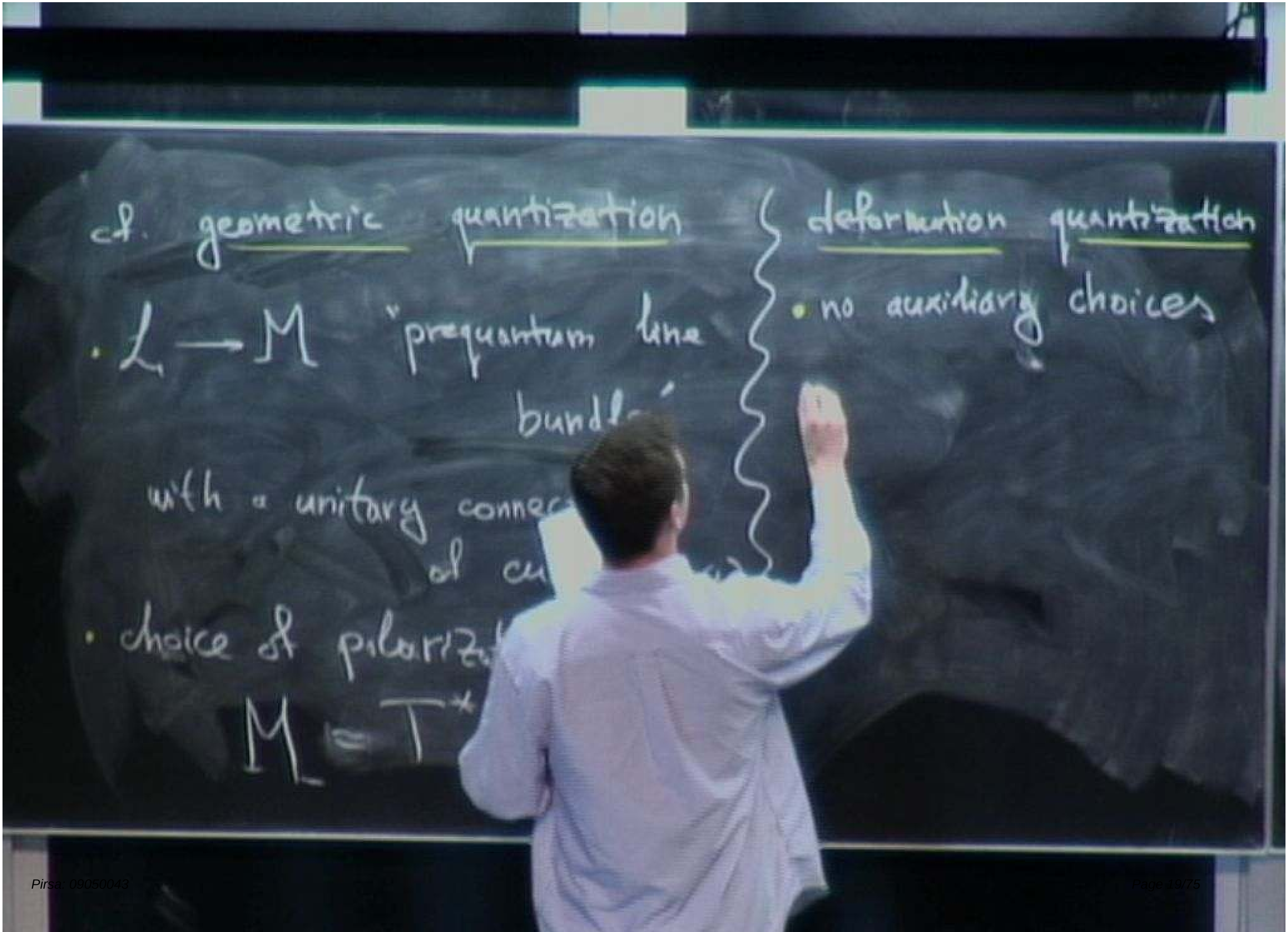
cf. geometric quantization
deformation quantization
• L → M "prequantum line bundle"
with a unitary connec
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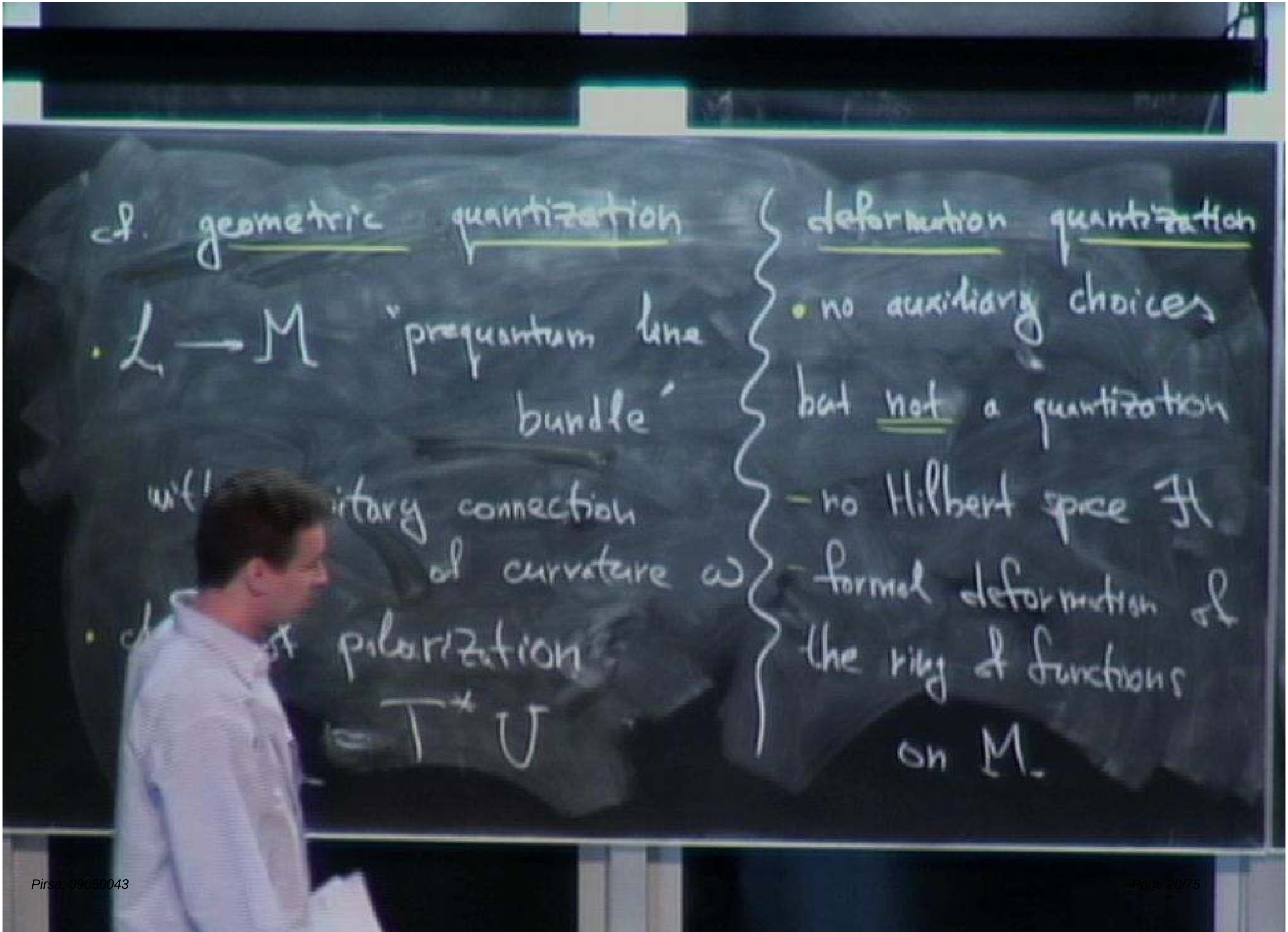
• no auxiliary choices


cf. geometric quantization
L → M "prequantum line bundle"
connection
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T*U
deformation quantization
• no auxiliary choices
but not a quantization
– no Hilbert space H
– formal deformation of
the ring of functions
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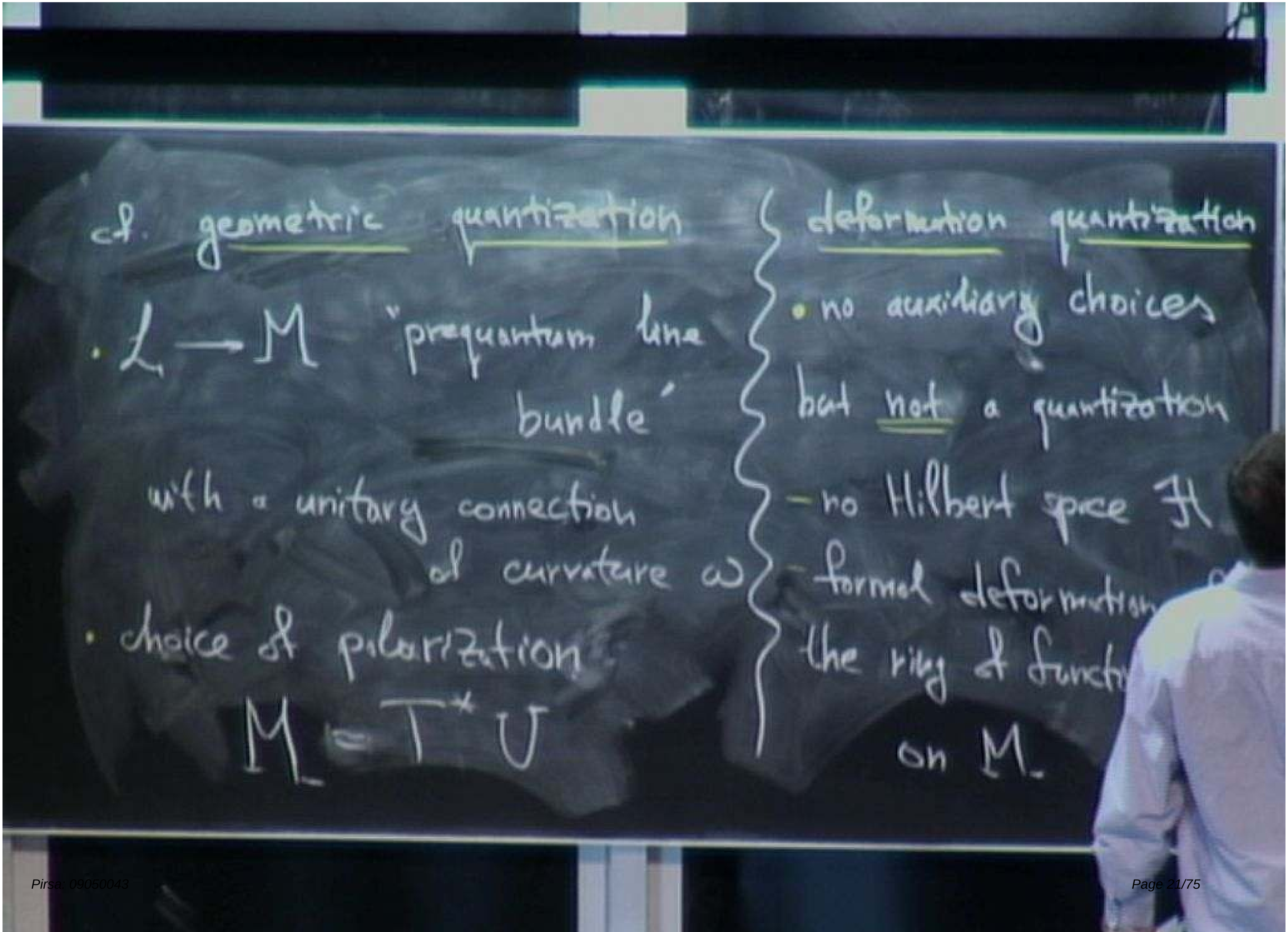
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• no auxiliary choices
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– no Hilbert space H
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Ex: $M = S^2$

$$\omega_0 = \mathrm{vol}(S^2)$$

$$\left(M, \omega = \frac{1}{\hbar}\omega_0\right) \rightsquigarrow \mathcal{H}$$

$$\hbar = 1/n, \quad n \in \mathbb{Z}$$

$$\dim \mathcal{H} = n$$

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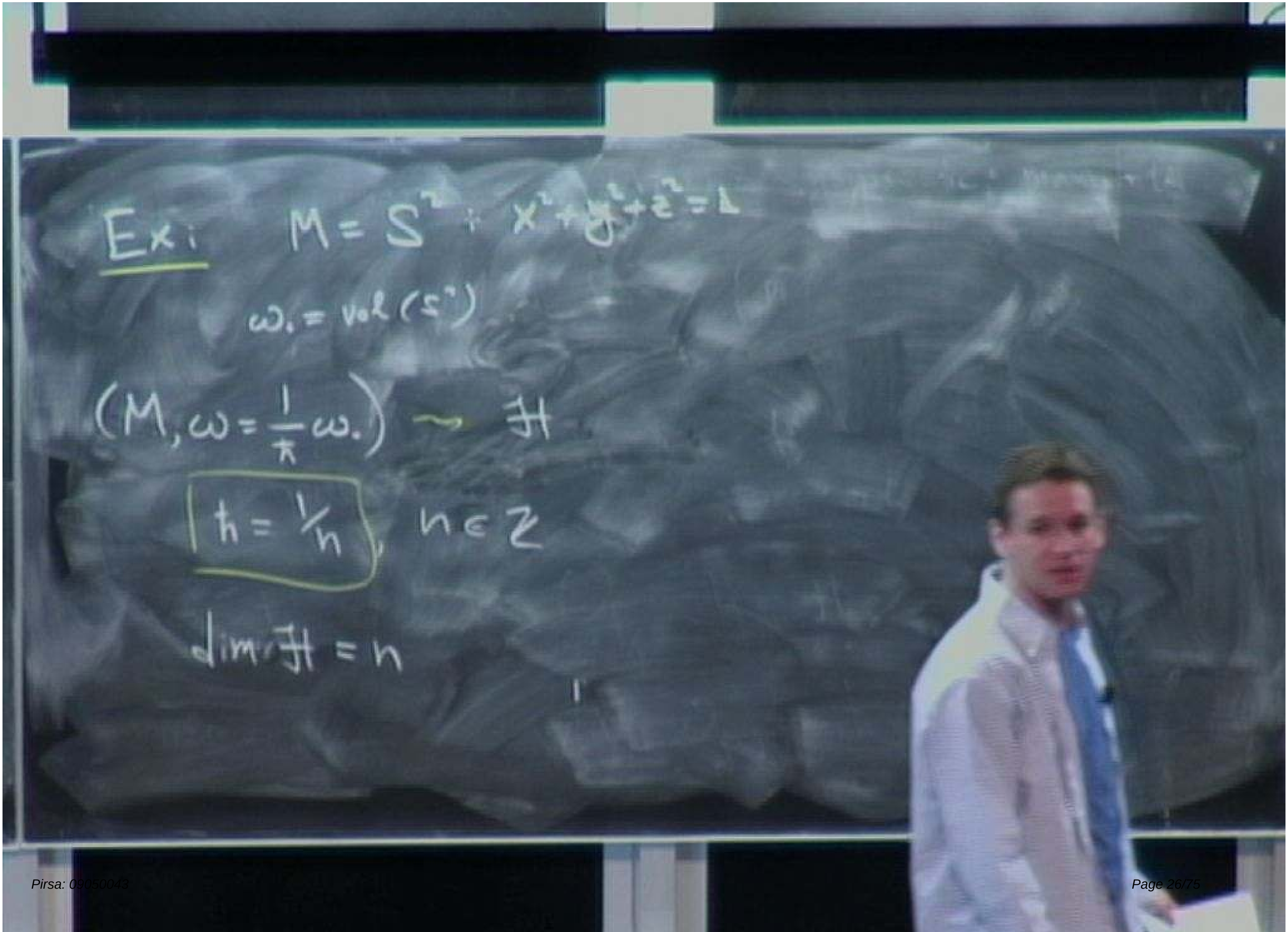
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(M, ω = 1/ħ ω₀) ⇝ H
ħ = 1/n, n ∈ Z
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Ex: $M = S^2 : x^2+y^2+z^2=1$ + brane quantization:

$\omega_0 = vol(S^2)$

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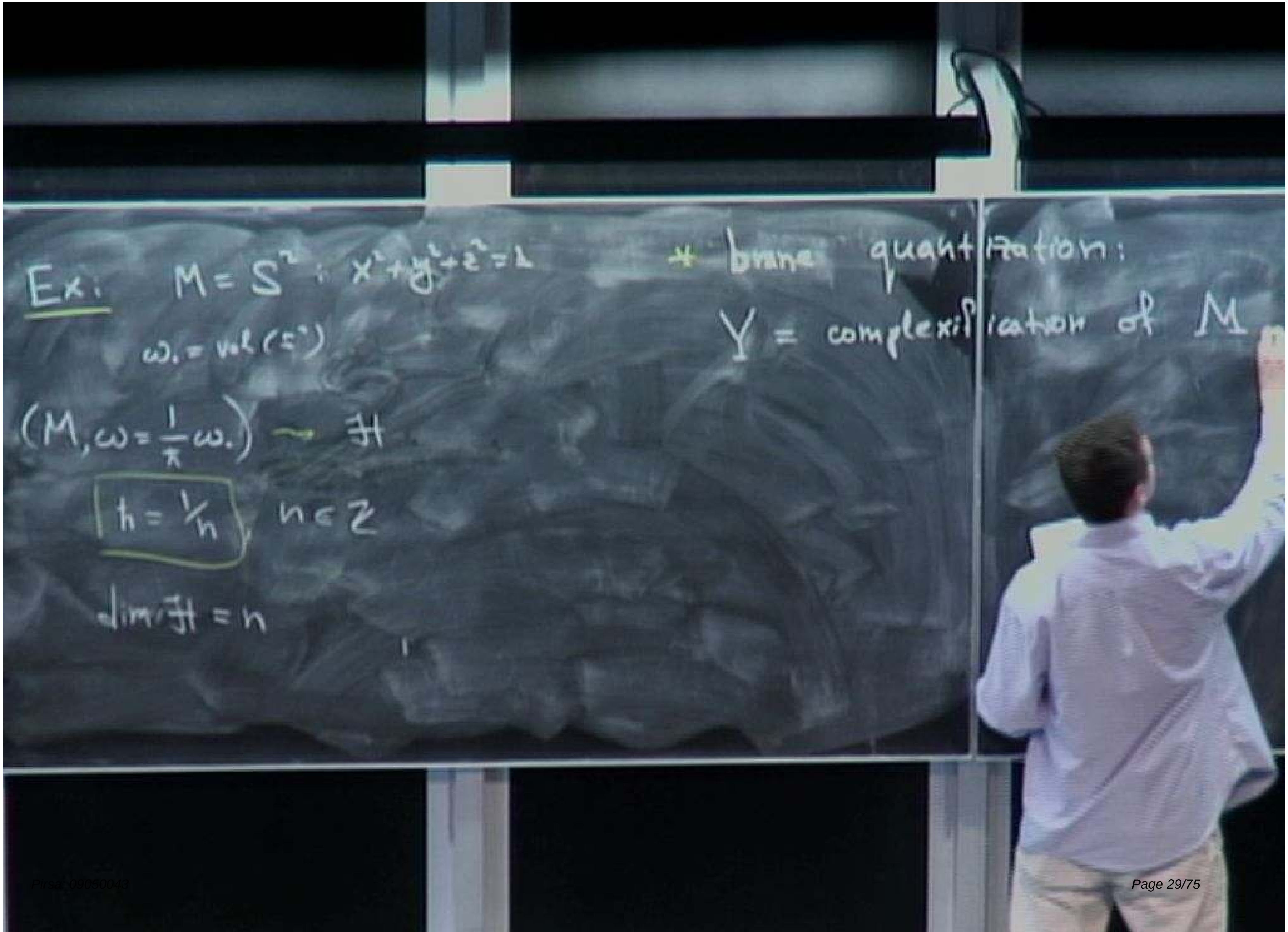
Ex: $M = S^2$: $x^2+y^2+z^2=1$
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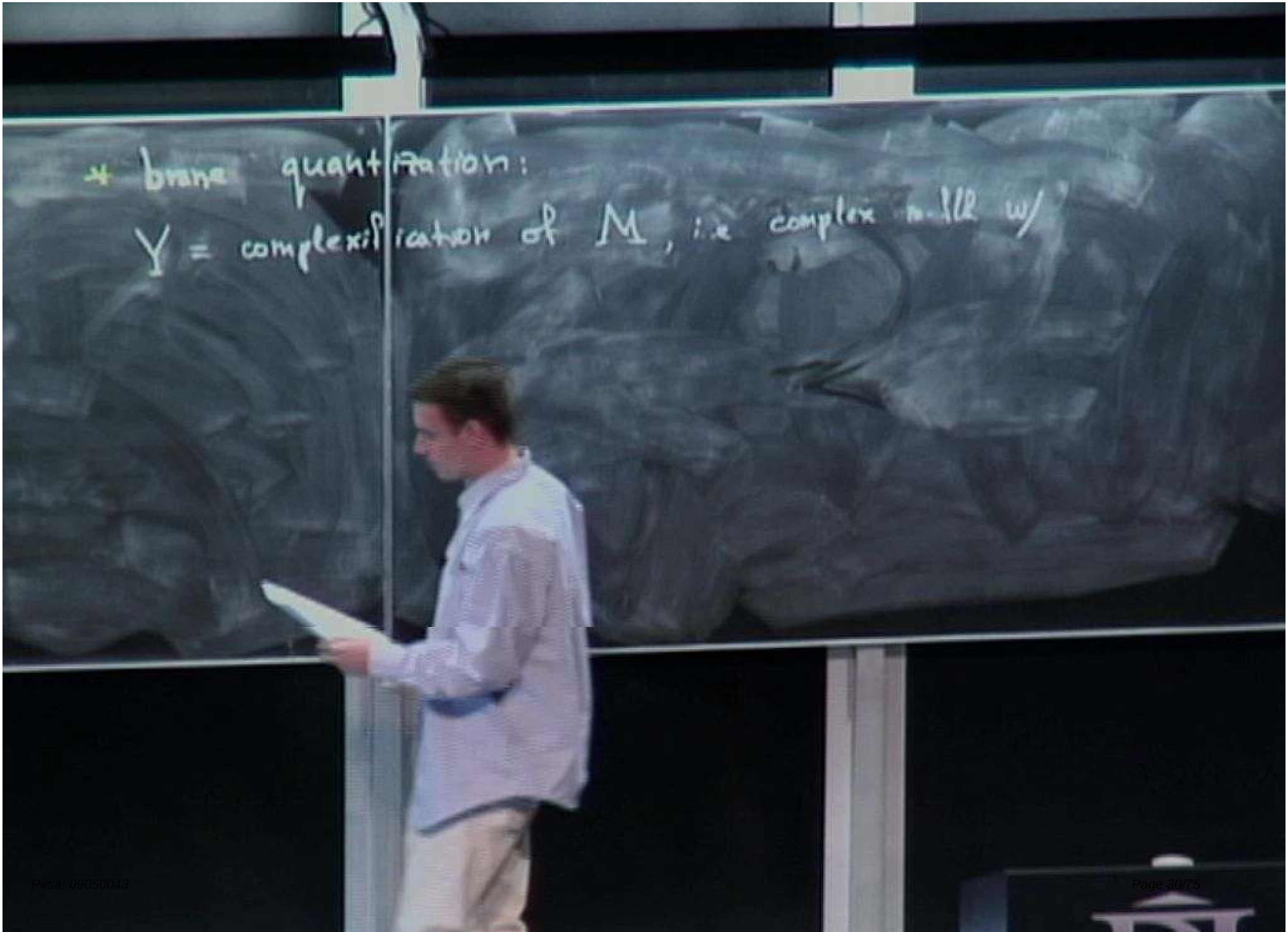
* brane quantization:
Y = complexification of M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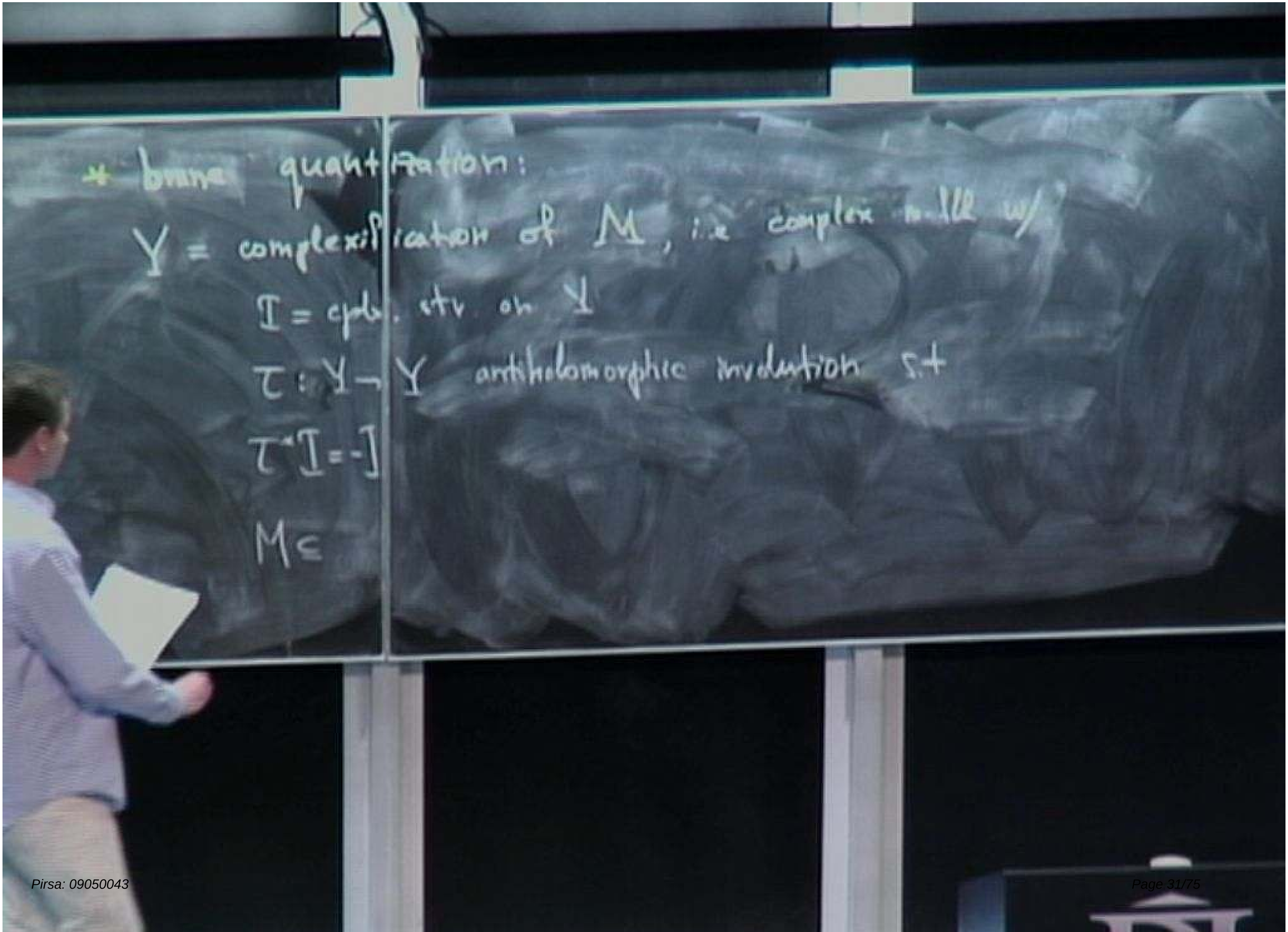
* brane quantization:
Y = complexification of M, i.e complex mfld w/
I = cplx. str. on Y
τ: Y → Y antiholomorphic involution s.t
τ*I = -I
M ⊂

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($M \subseteq$ fixed pt. set of τ)

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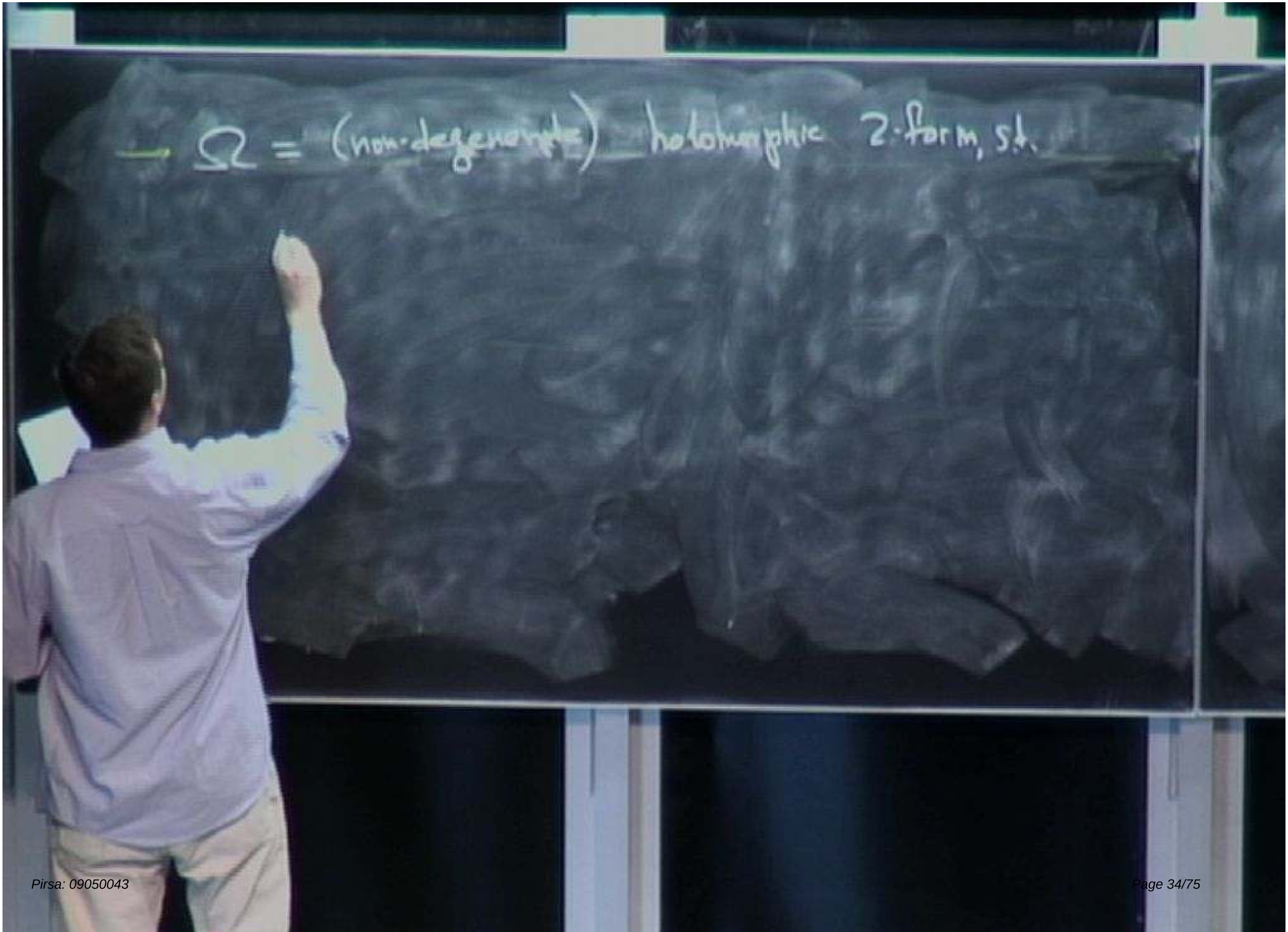
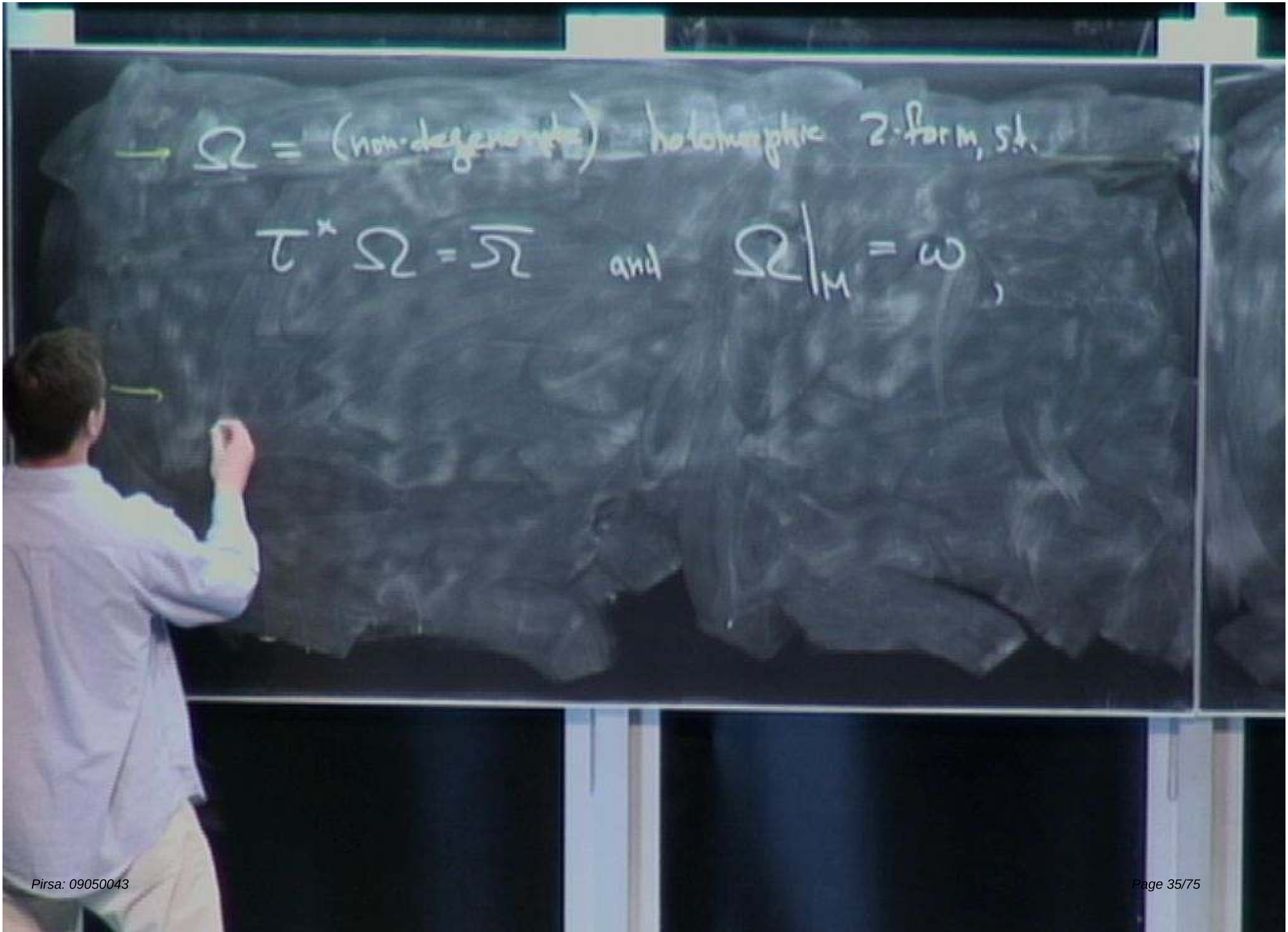
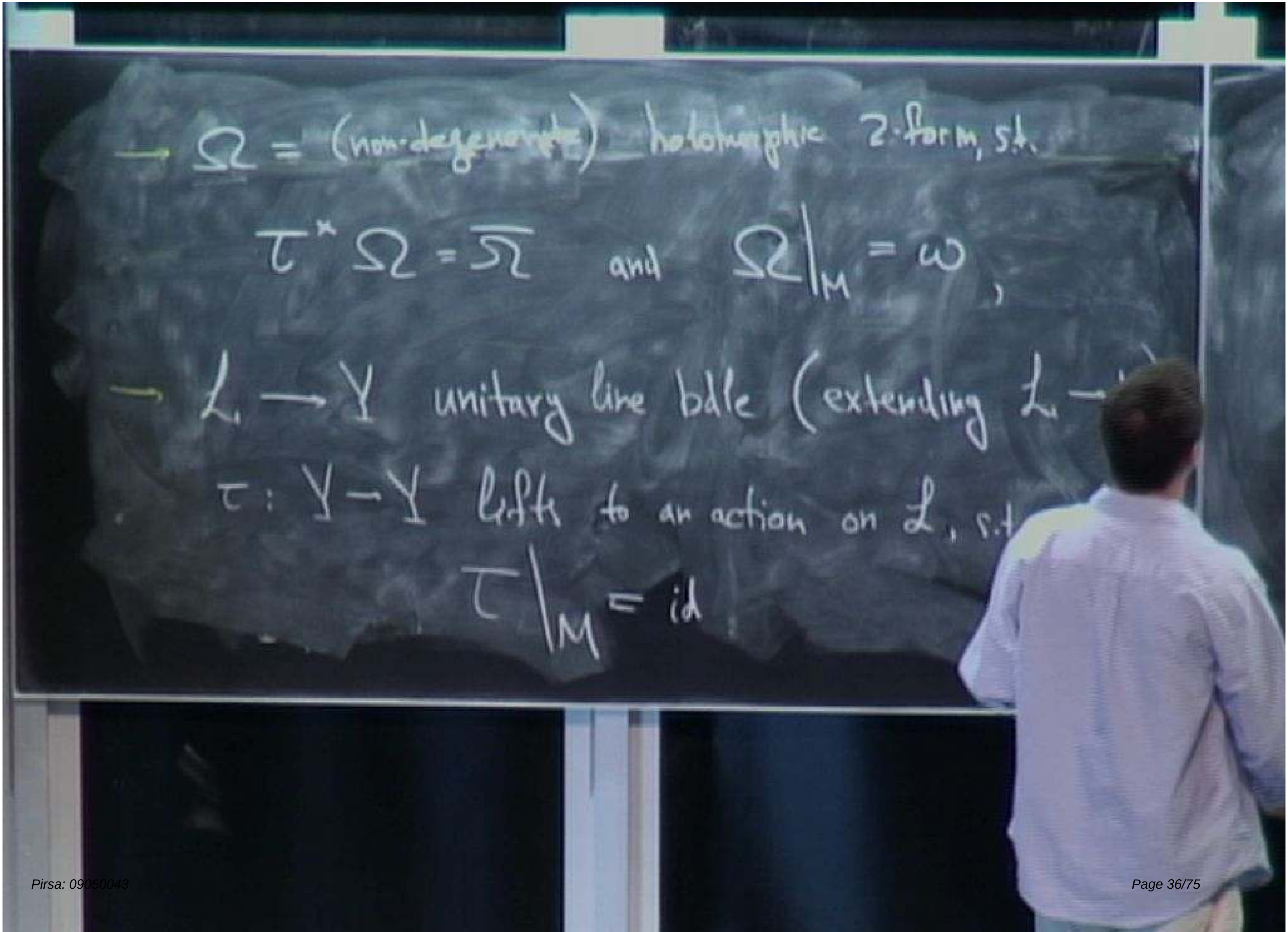

→ Ω = (non-degenerate) holomorphic 2-form, s.t.


Ω = (non-degenerate) holomorphic 2-form, s.t.
τ* Ω = Ω̄ and Ω|M = ω,


→ Ω = (non-degenerate) holomorphic 2-form, s.t.
$\tau^* \Omega = \overline{\Omega}$ and $\Omega|_M = \omega$,
→ $\mathcal{L} \longrightarrow Y$ unitary line bdle (extending $\mathcal{L} \to$
$\tau : Y \to Y$ lifts to an action on $\mathcal{L}$, s.t
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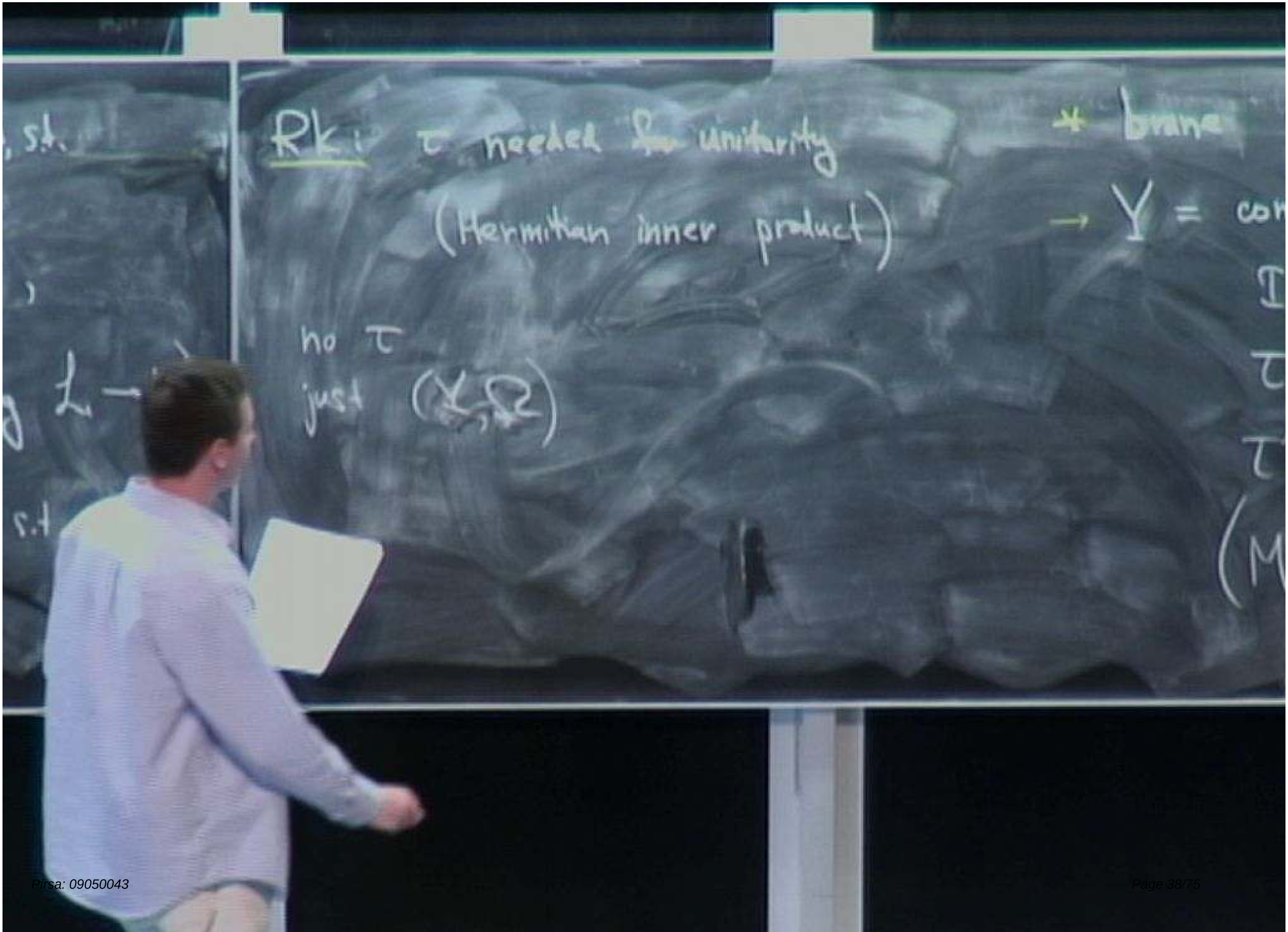
Rk: τ needed for unitarity
(Hermitian inner product)
no τ
just
* brane
→ Y =

, s.t.

)

g $\mathcal{L} \to M$)

s.t

Rk: τ needed for unitarity

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just (X, Ω) and $\mathcal{L}$

$\rightsquigarrow \mathcal{H}$ w/ no Hermitian inner product

* brane

$\to Y =$ con

I

τ

τ

(M

, s.t.

)

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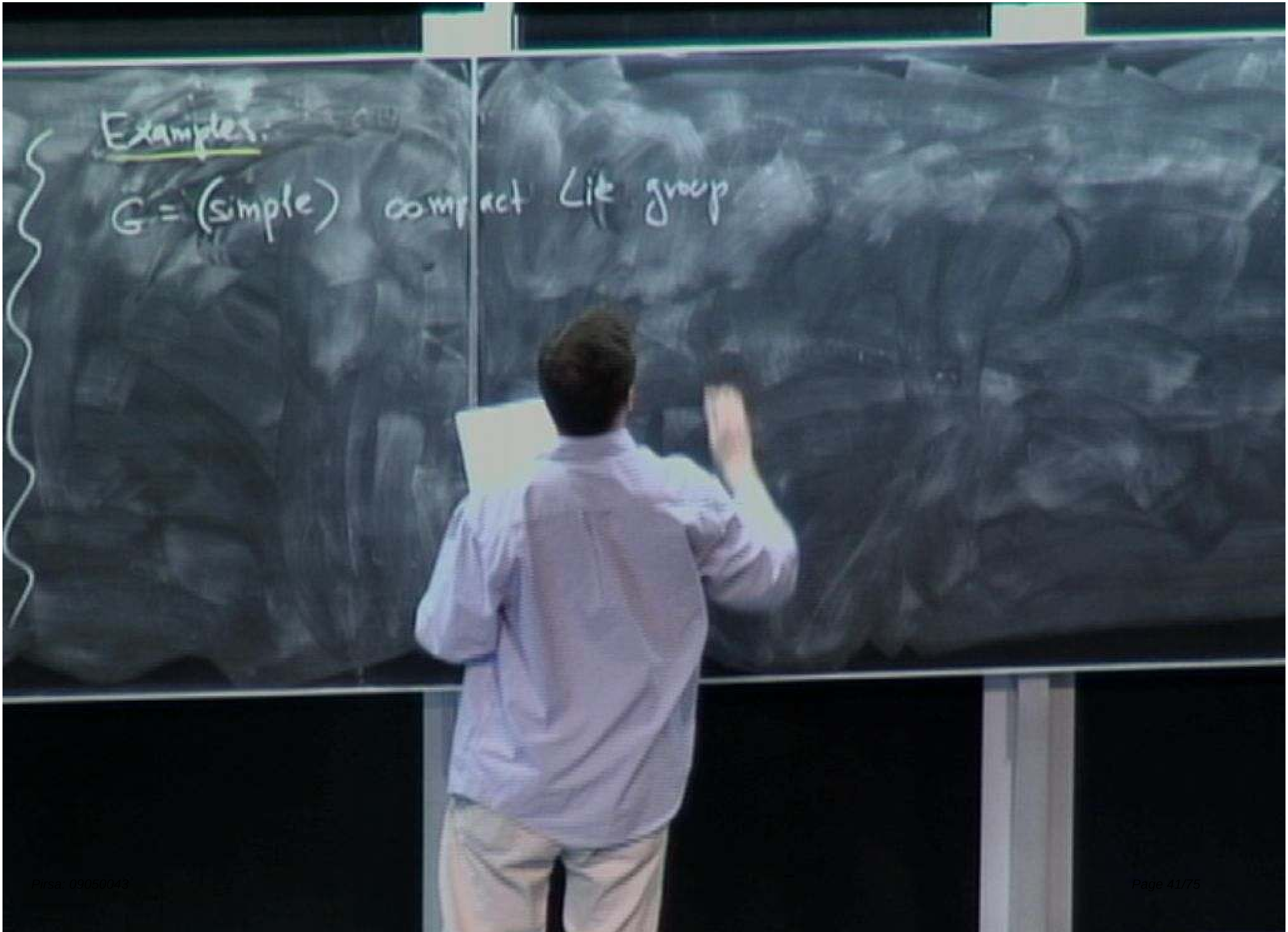
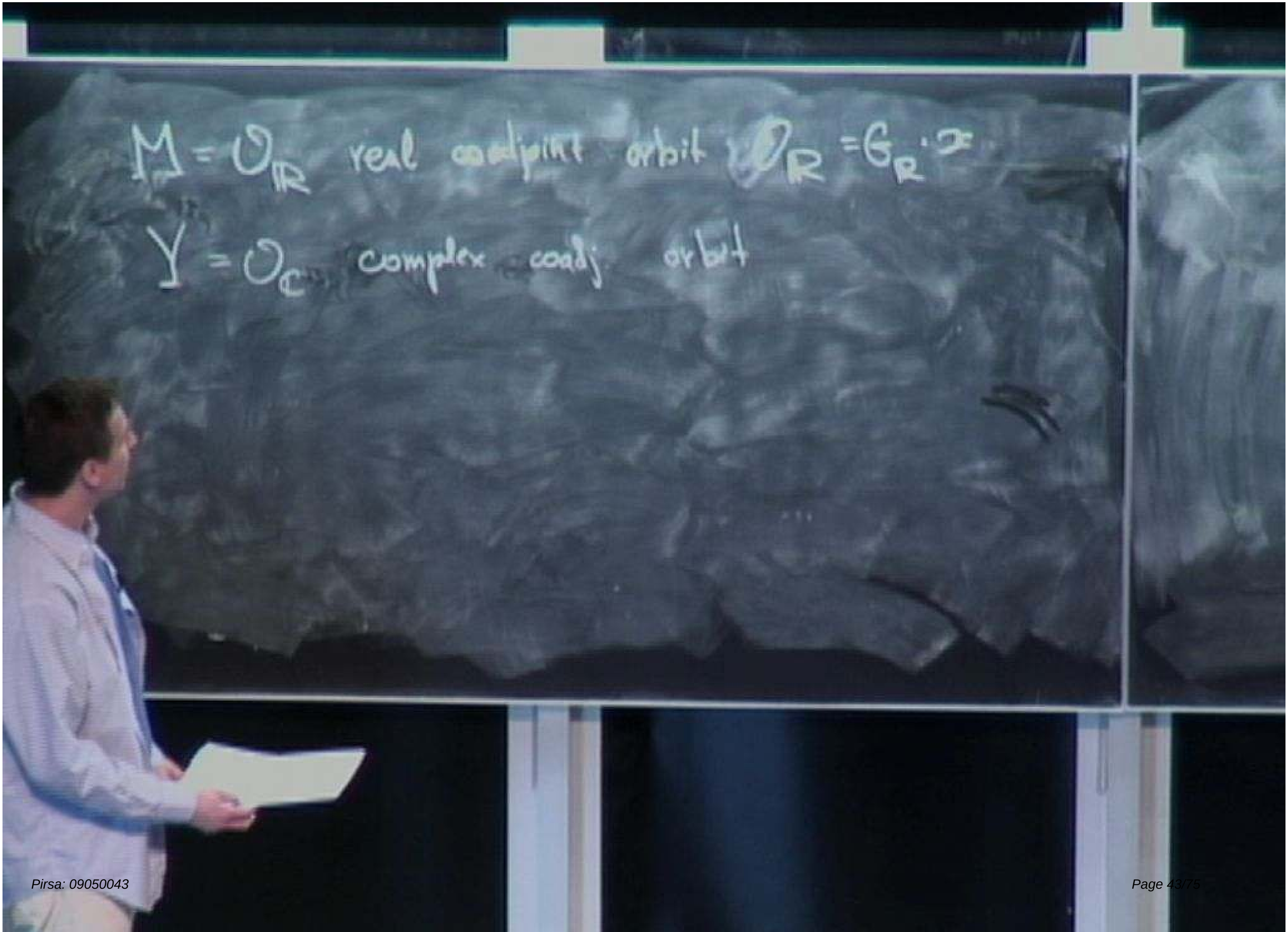

Examples.
G = (simple) compact Lie group

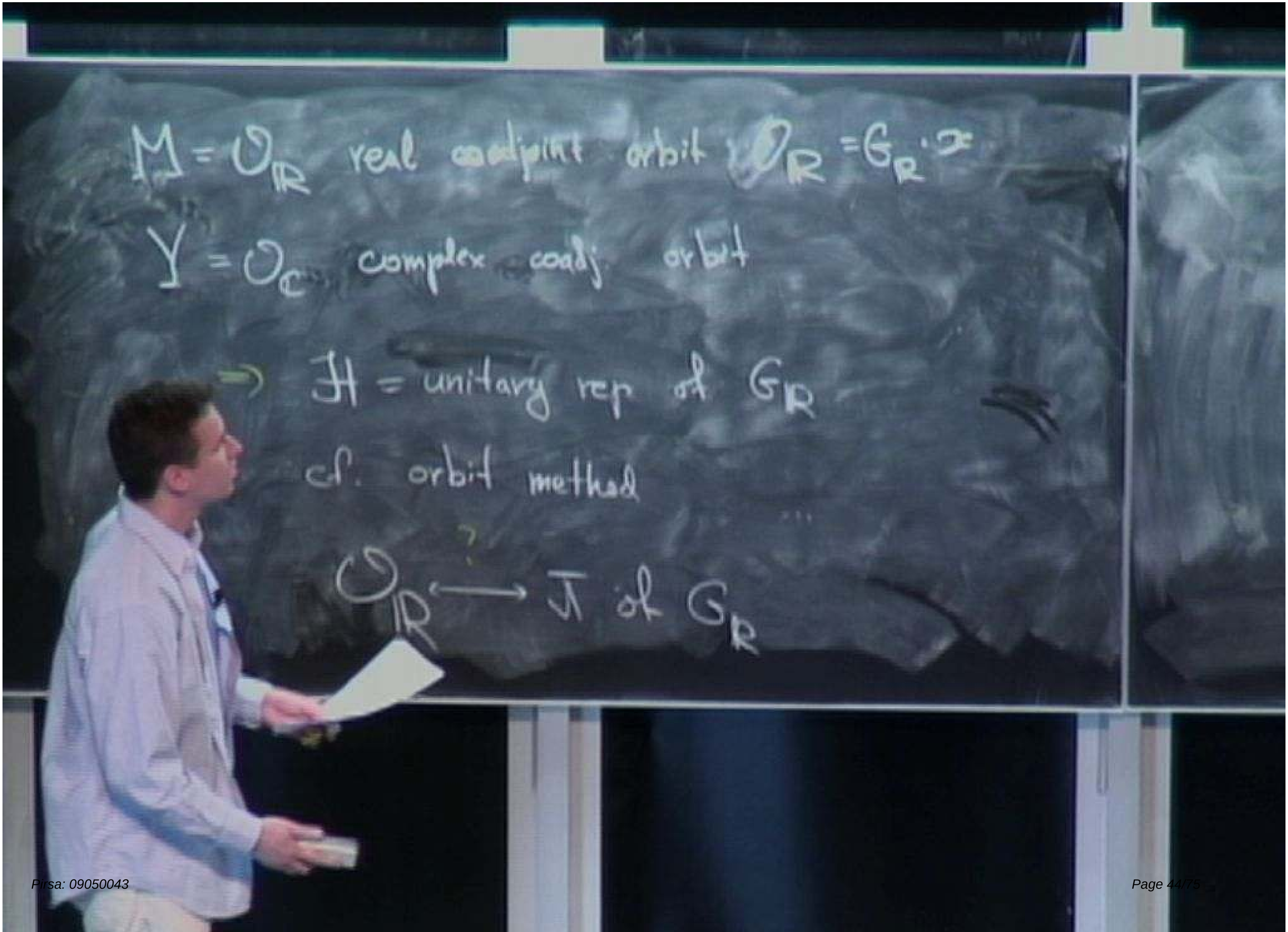
Examples:

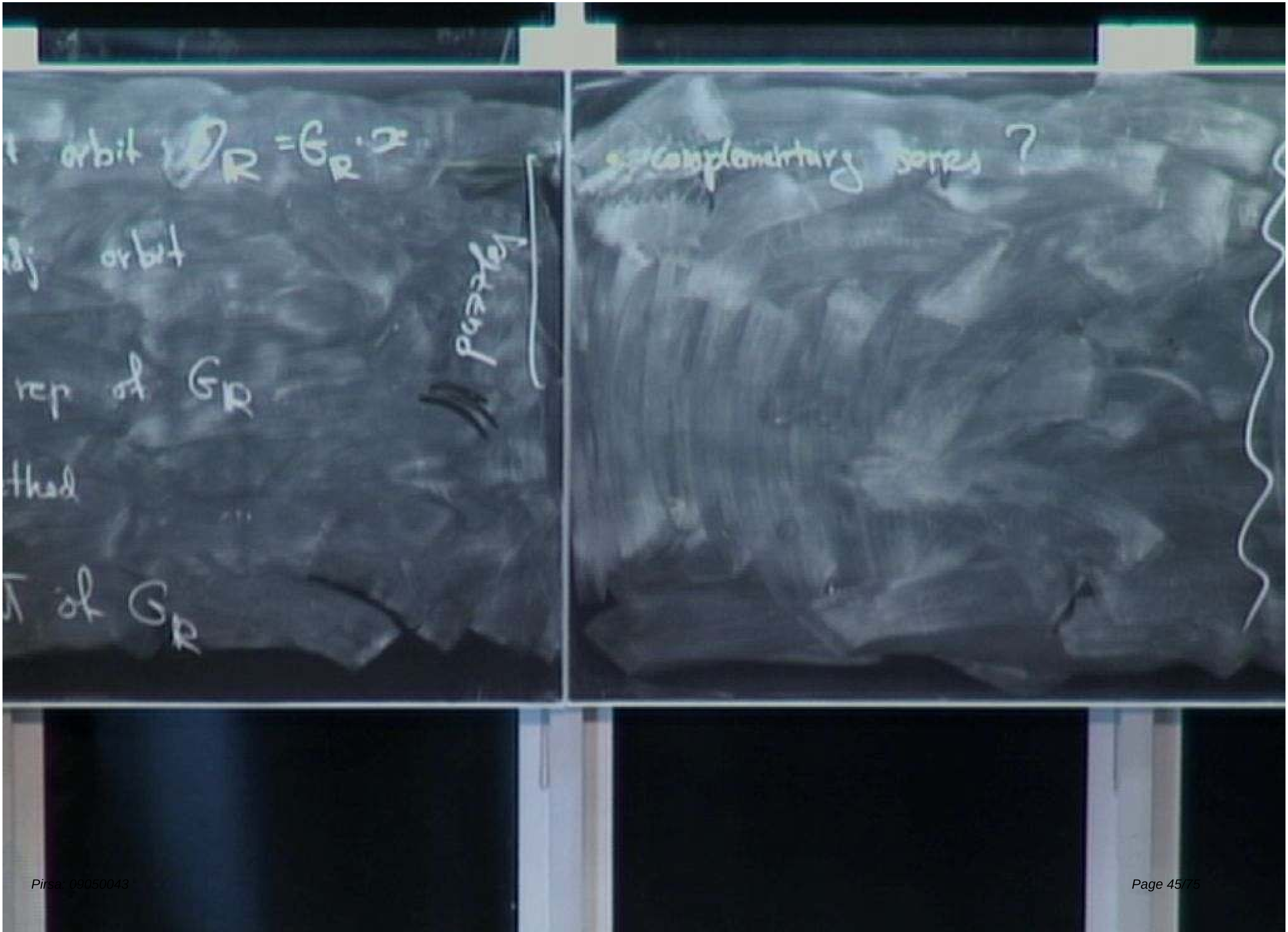
G = (simple) compact Lie group — $SU(2)$

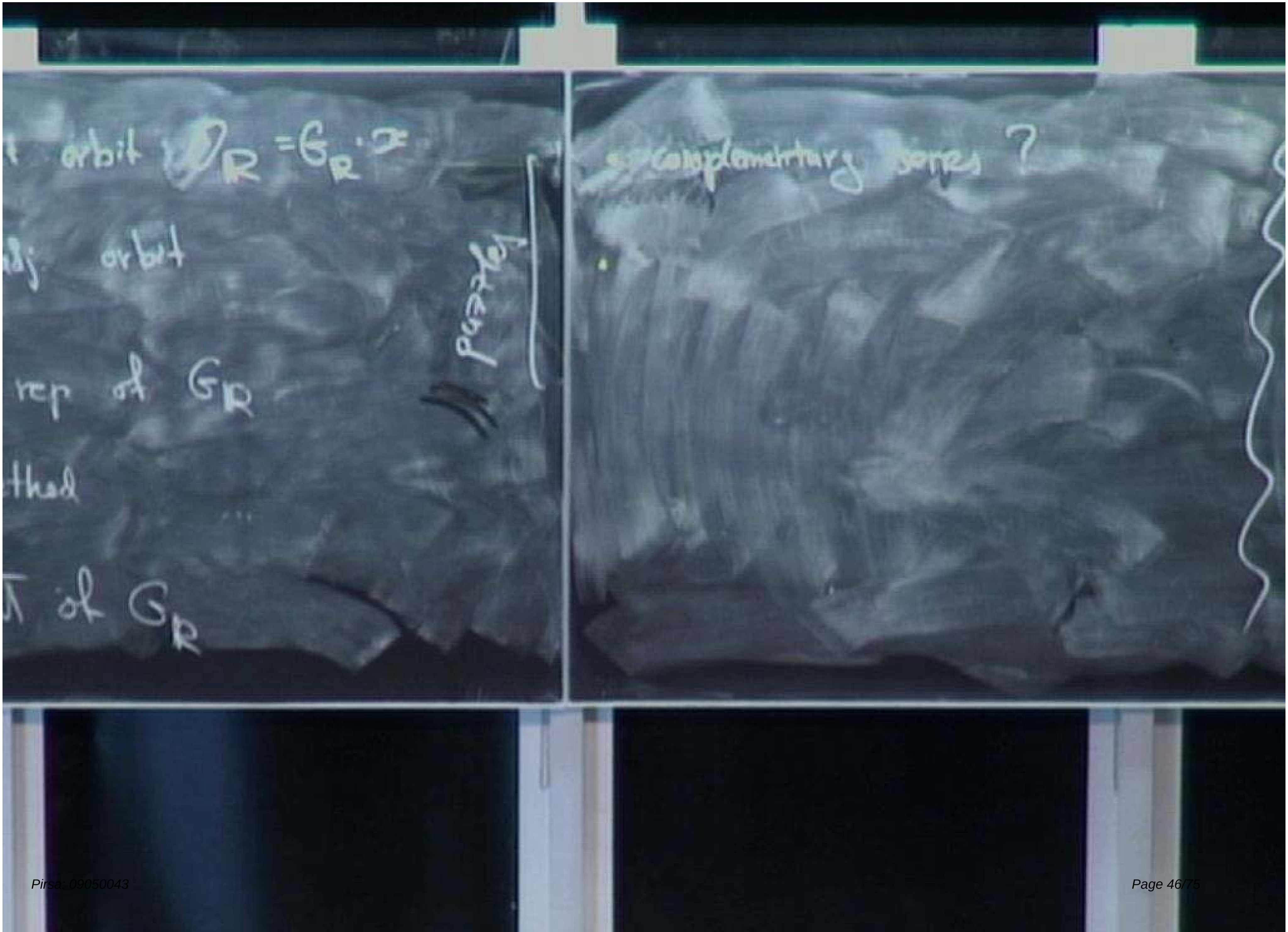
$G_{\mathbb{C}}$ = complexification of G — $SL(2,\mathbb{C})$

$G_{\mathbb{R}}$ = real form of $G_{\mathbb{C}}$ — $SU(2)$, $SL(2,\mathbb{R})$


M = O_ℝ real coadjoint orbit O_ℝ = G_ℝ · x
Y = O_ℂ complex coadj orbit


M = O_R real coadjoint orbit ; O_R = G_R · x
Y = O_C complex coadj orbit
⟹ H = unitary rep of G_R
cf. orbit method
O_R ⟷ H of G_R


orbit $O_R = G_R \cdot x$
orbit
rep of G_R
thod
of G_R
extend
complementary series ?


orbit $\mathcal{O}_{\mathbb{R}} = G_{\mathbb{R}} \cdot x$
adj. orbit
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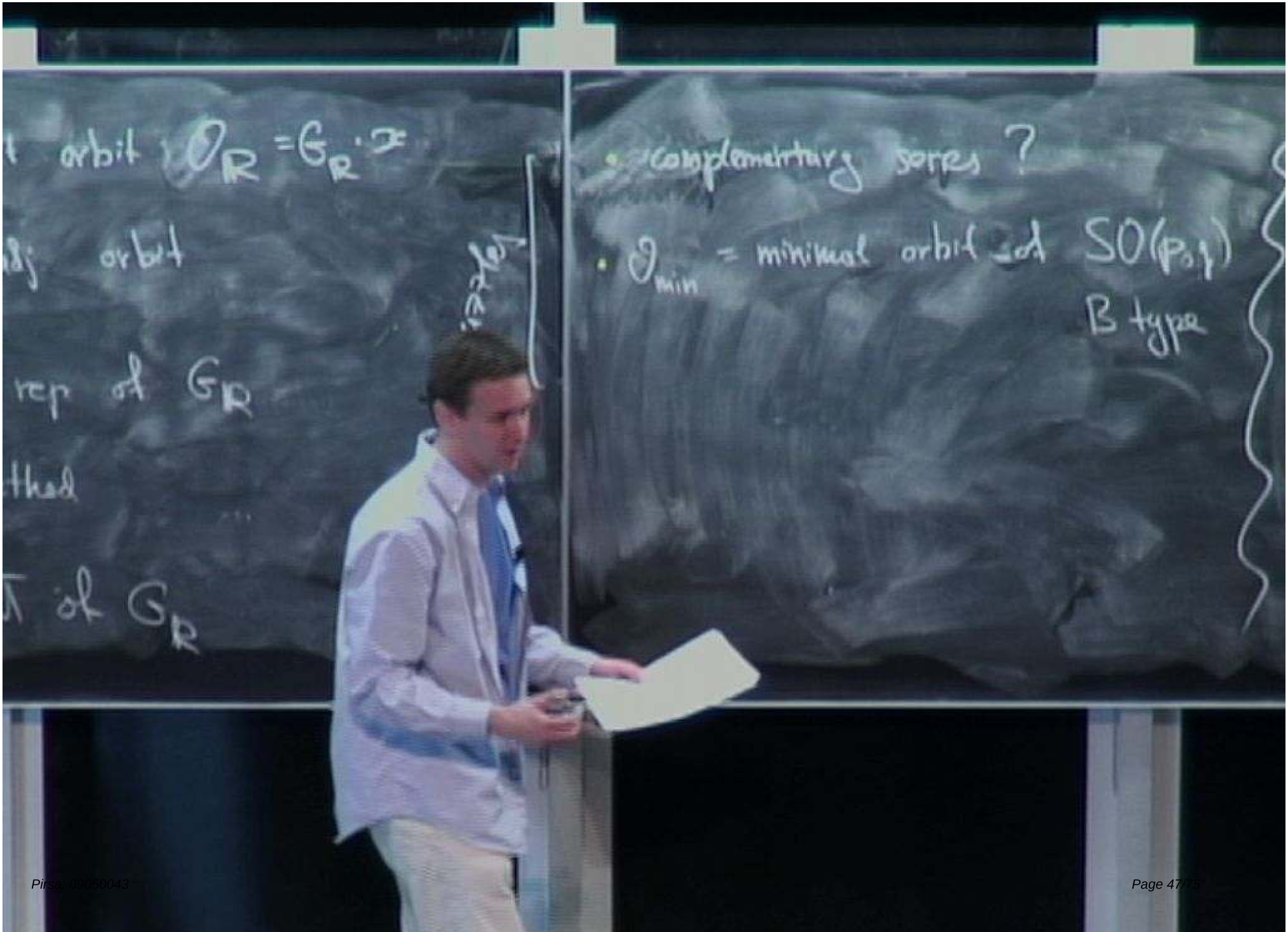
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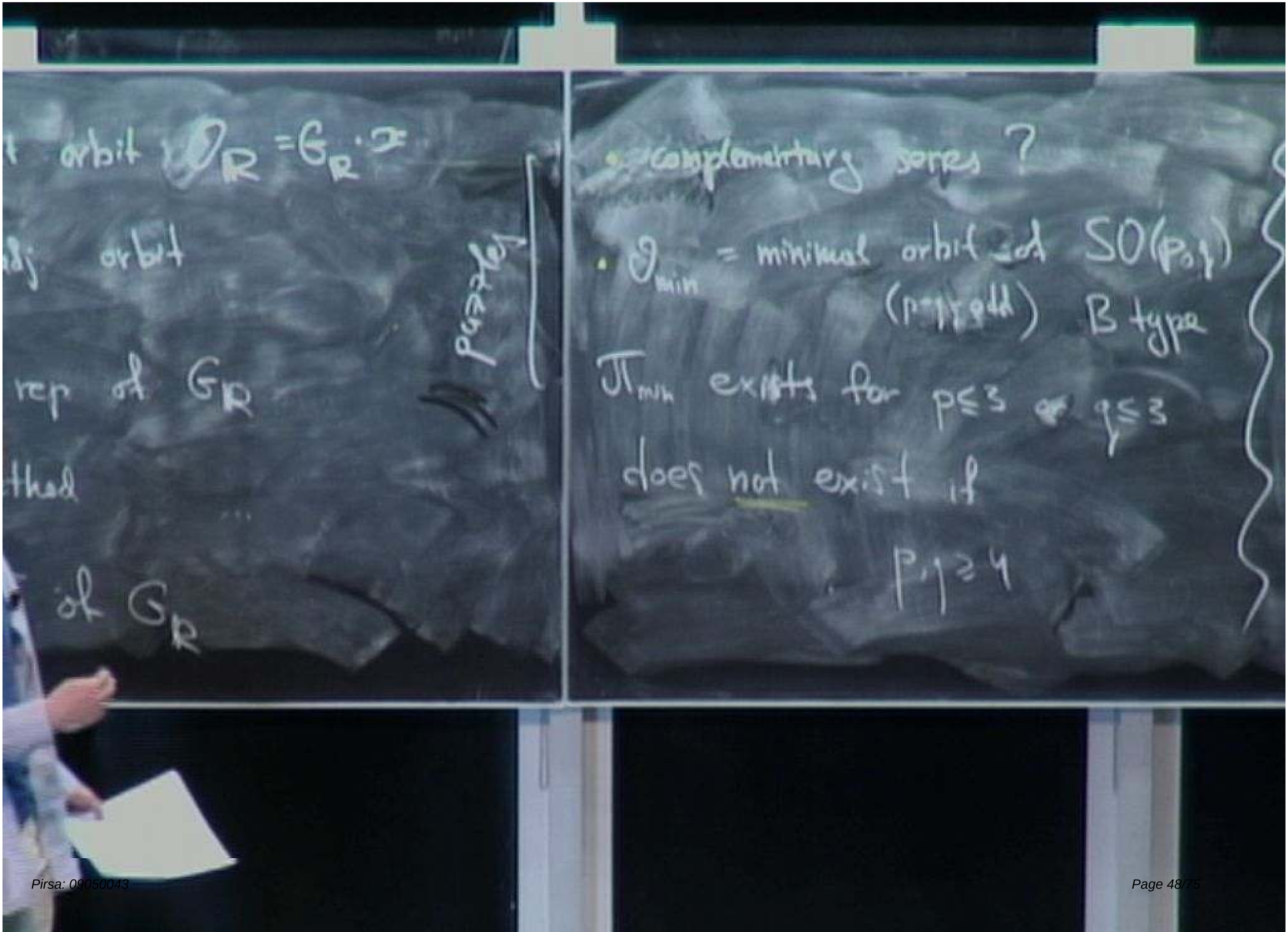
O_{min} = minimal orbit of SO(p,q)
B type


orbit
adj. orbit
rep of
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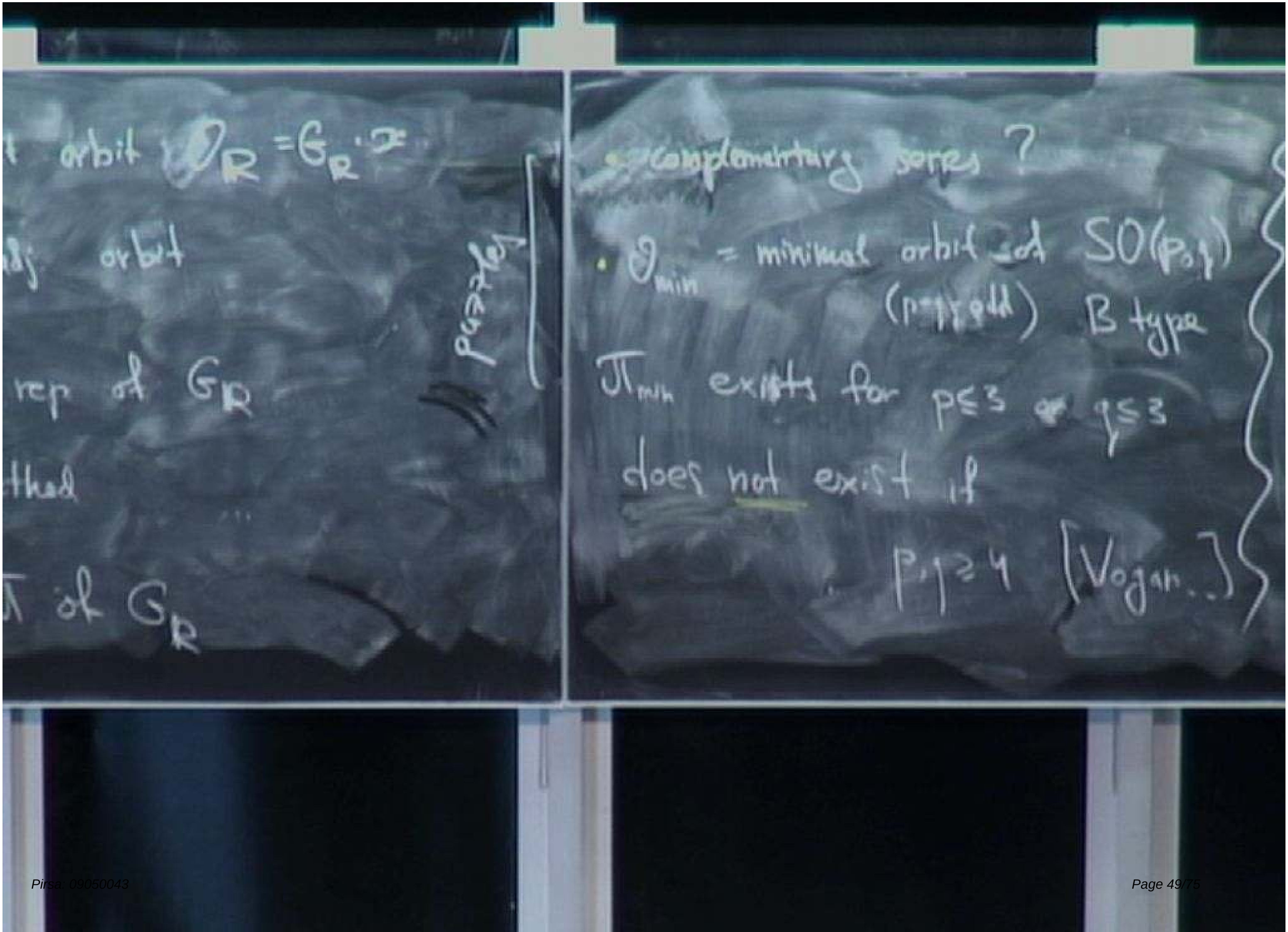
complementary series ?
minimal orbit of SO(p,q)
B type
exists for p≤3 or q≤3
does not exist if
p,q≥4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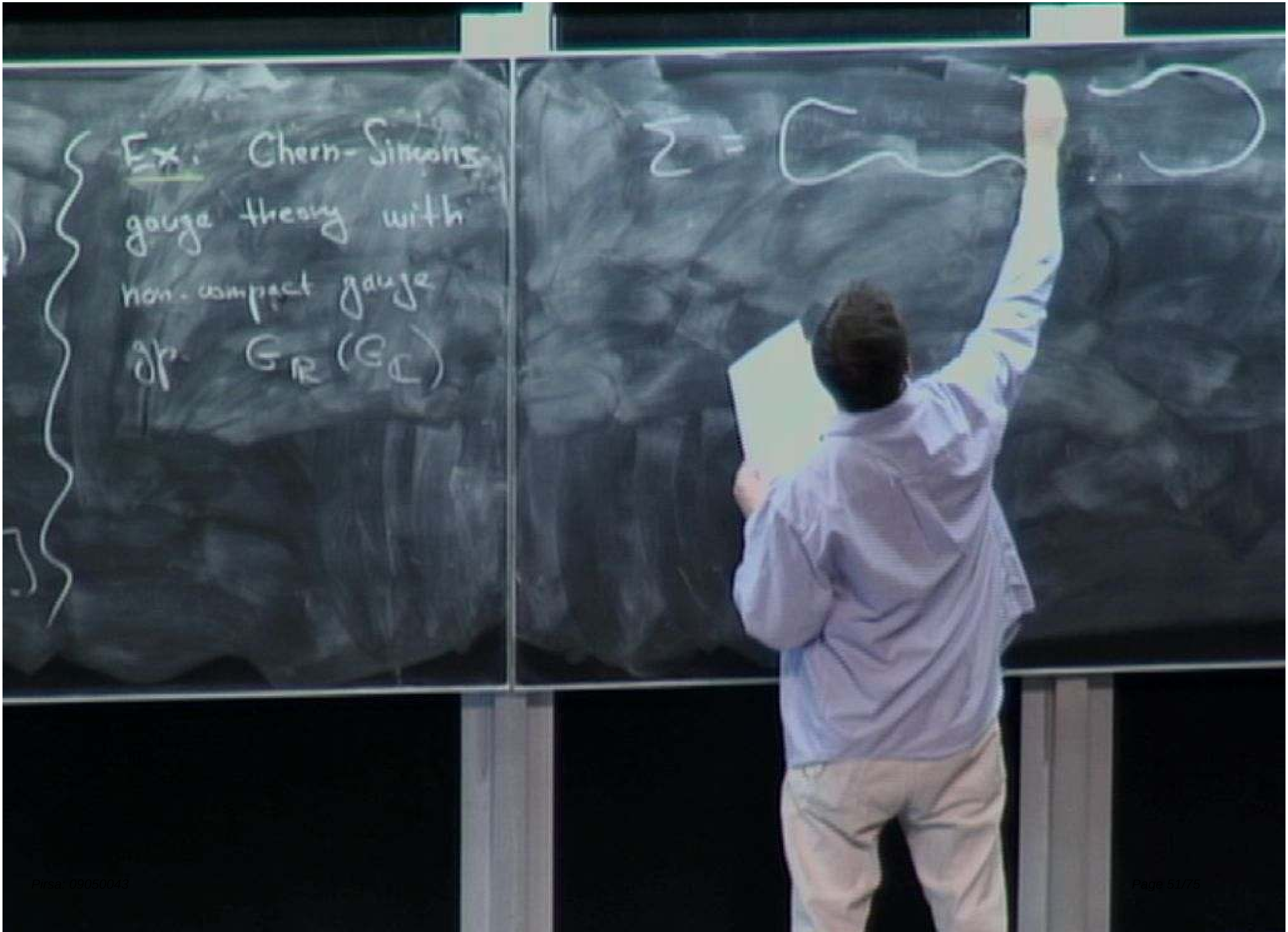
($p+q$ odd) B type
π_{min} exists for $p \leq 3$ or $q \leq 3$
does not exist if
$p,q \geq 4$ [Vogan..]

Ex: Chern-Simons gauge theory with non-compact gauge gp. $G_{\mathbb{R}}$ $(G_{\mathbb{C}})$


Ex: Chern-Simons
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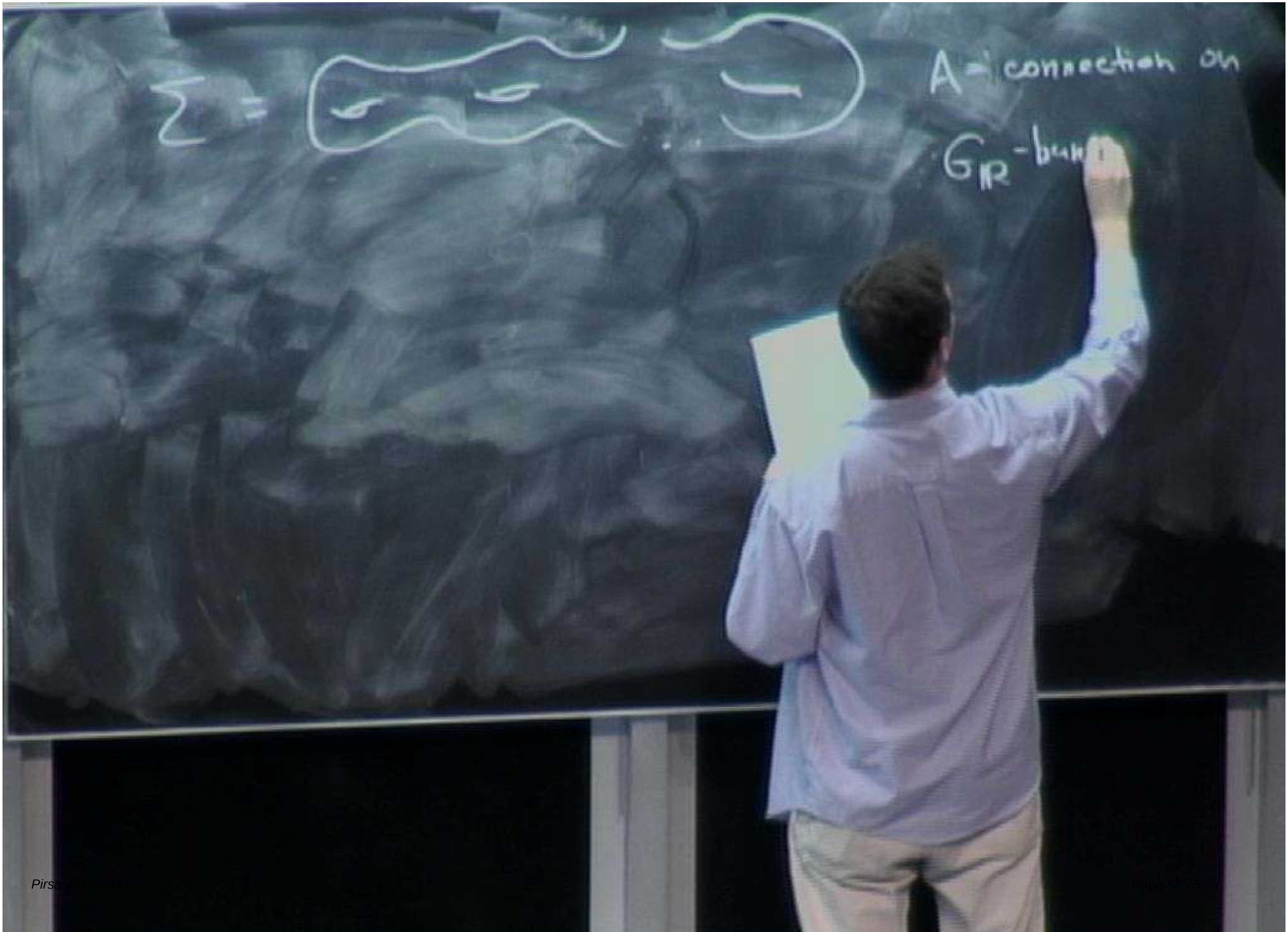
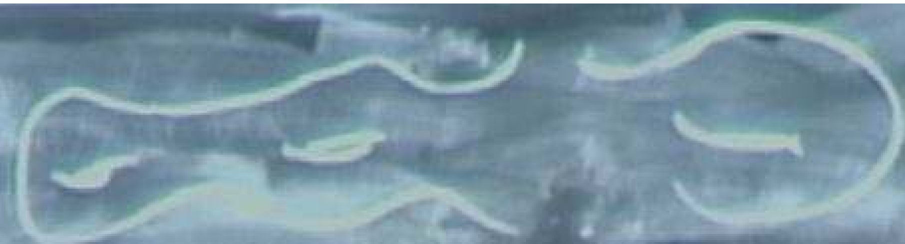
gp. G_R (G_C)
Σ =


Σ =
A = connection on
G_ℝ - bun

$\Sigma =$ 

A: connection on G_{IR}-bundle $E \to \Sigma$

$M = \mathcal{M}_{flat}(G_{IR}, \Sigma)$

$\Sigma =$ 

A = connection on $G_{\mathbb{R}}$-bundle $E \to \Sigma$

$$\mathcal{M} = \mathcal{M}_{flat}(G_{\mathbb{R}}, \Sigma)$$

$$= \mathrm{Hom}(\pi_1(\Sigma); G_{\mathbb{R}})/_{conj}$$

$$\omega = \frac{k}{4\pi}\int_{\Sigma} \mathrm{Tr}\, \delta A \wedge \delta A$$

Σ 

A = connection on G_{IR}-bundle $E \to \Sigma$

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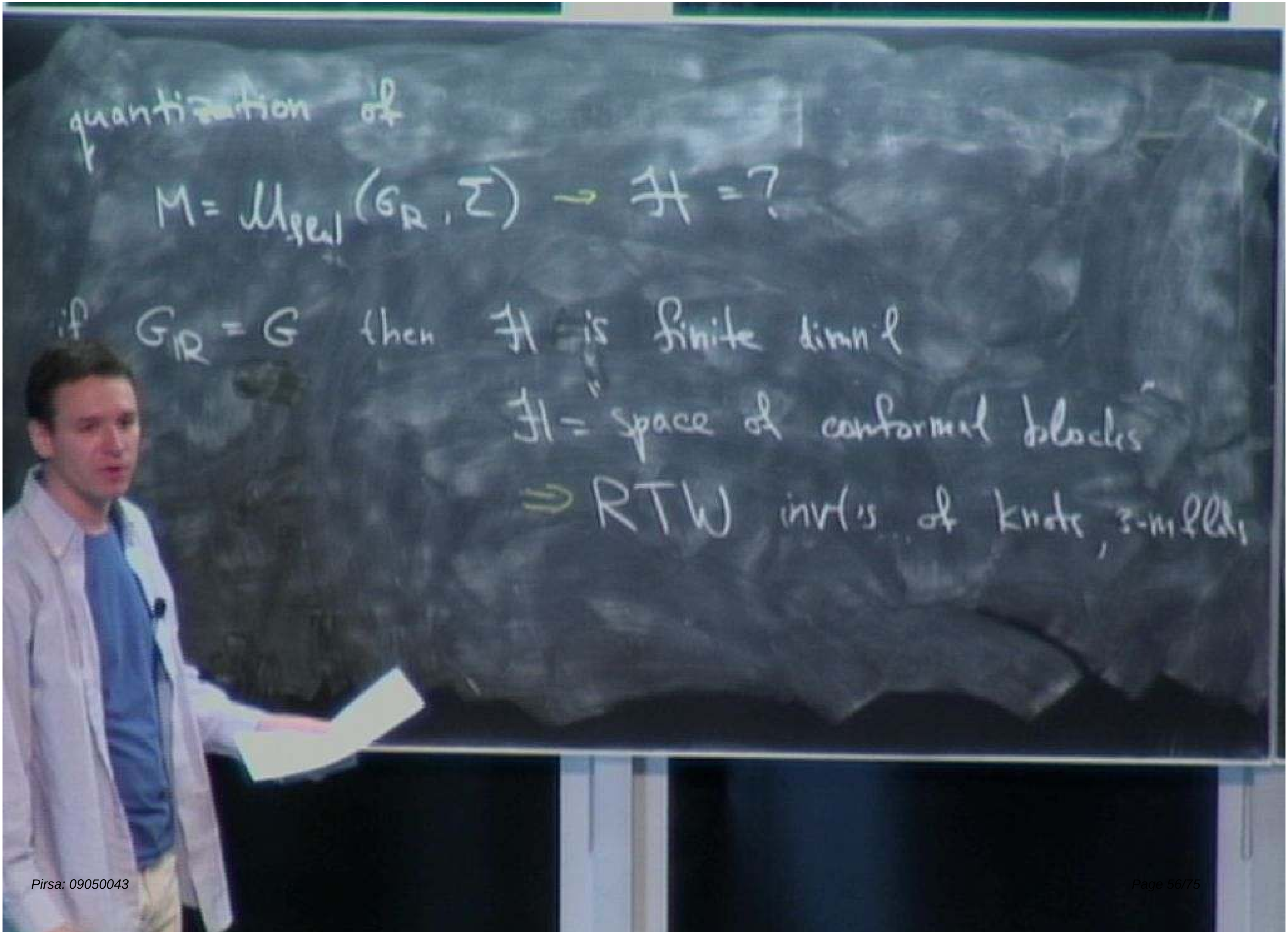
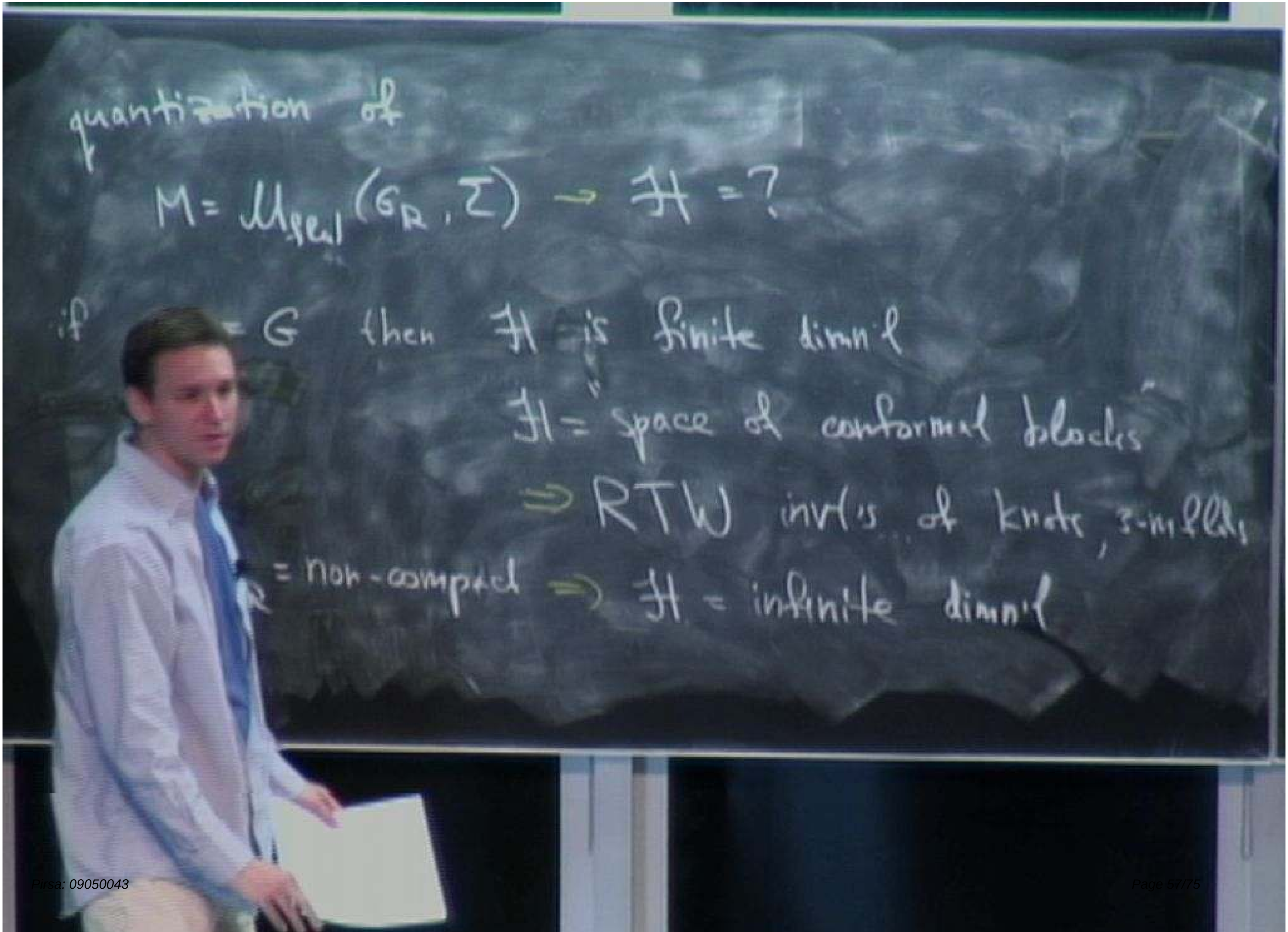

quantization of
M = M_flat (G_R, Σ) → H = ?
if G_R = G then H is finite dim'l
H = space of conformal blocks"
⇒ RTW inv'ts of knots, 3-mflds



quantization of
M = M_real (G_R , Σ) → H = ?
if = G then H is finite dimn'l
H = "space of conformal blocks"
⇒ RTW inv'ts of knots, 3-mflds
= non-compact ⇒ H = infinite dimn'l

quantization of

$$M = \mathcal{M}_{flat}(G_{\mathbb{R}}, \Sigma) \rightarrow \mathcal{H} = ?$$

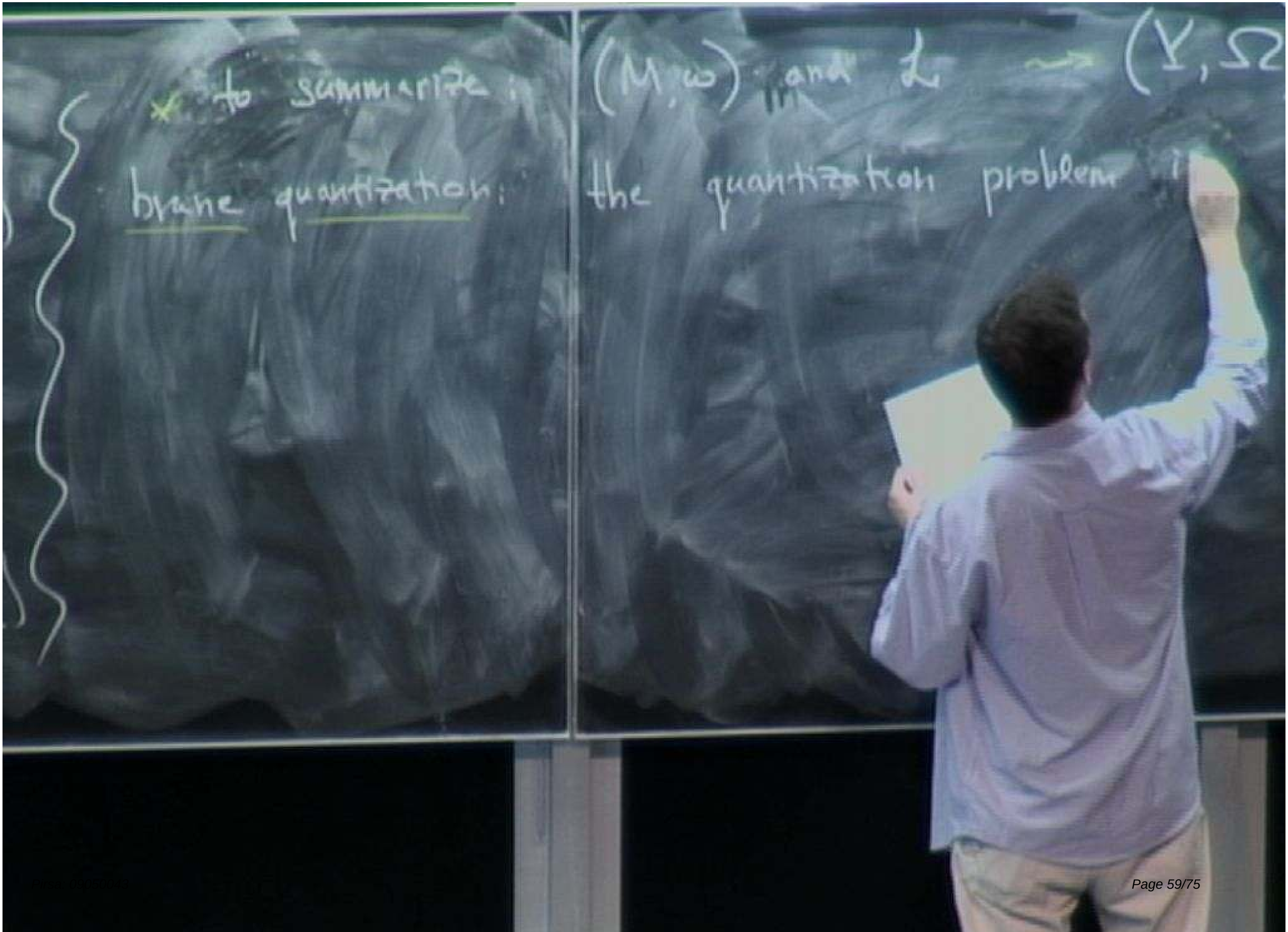
if $G_{\mathbb{R}} = G$ then $\mathcal{H}$ is finite dimn'l

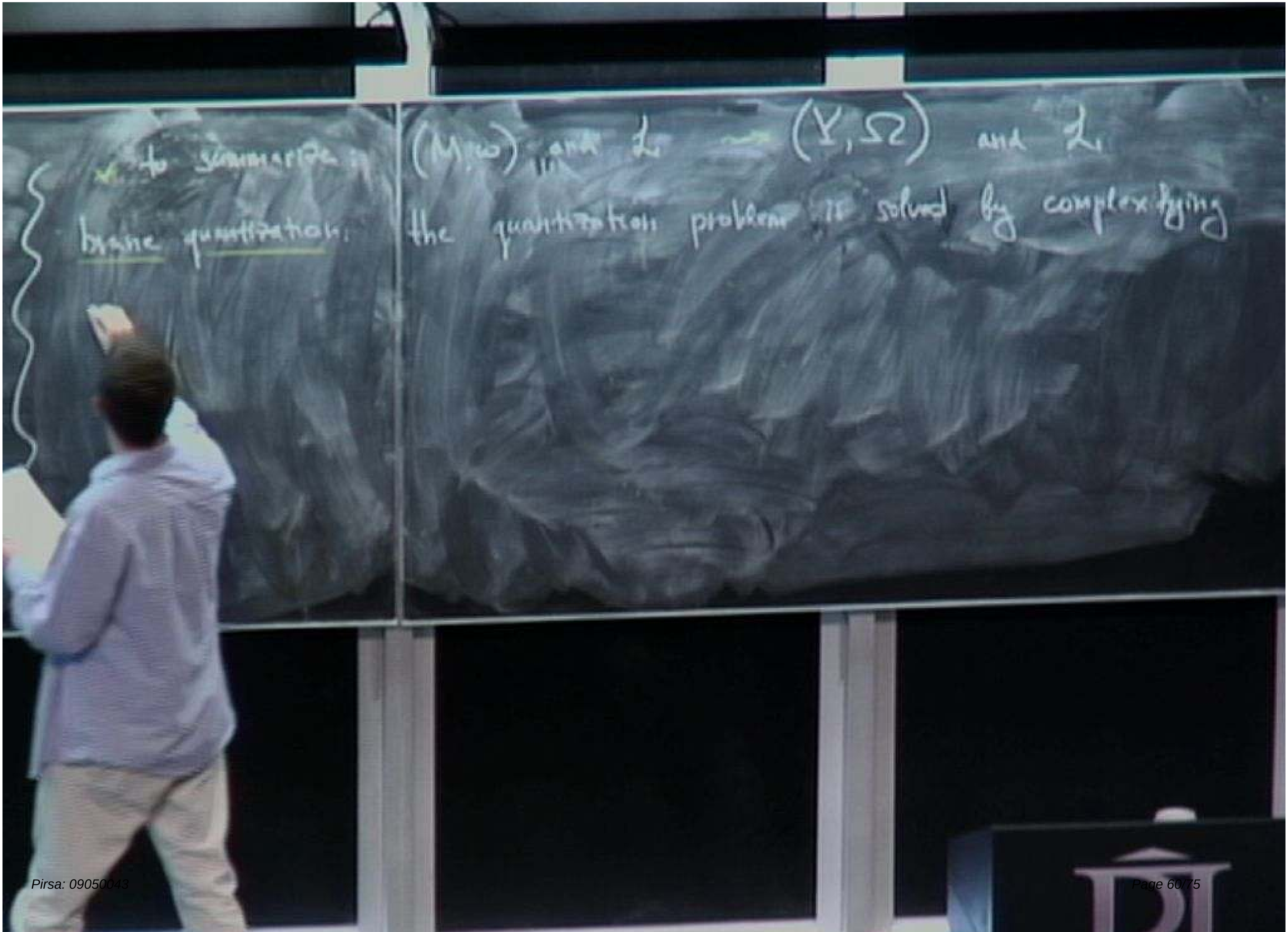
$\mathcal{H}$ = "space of conformal blocks"

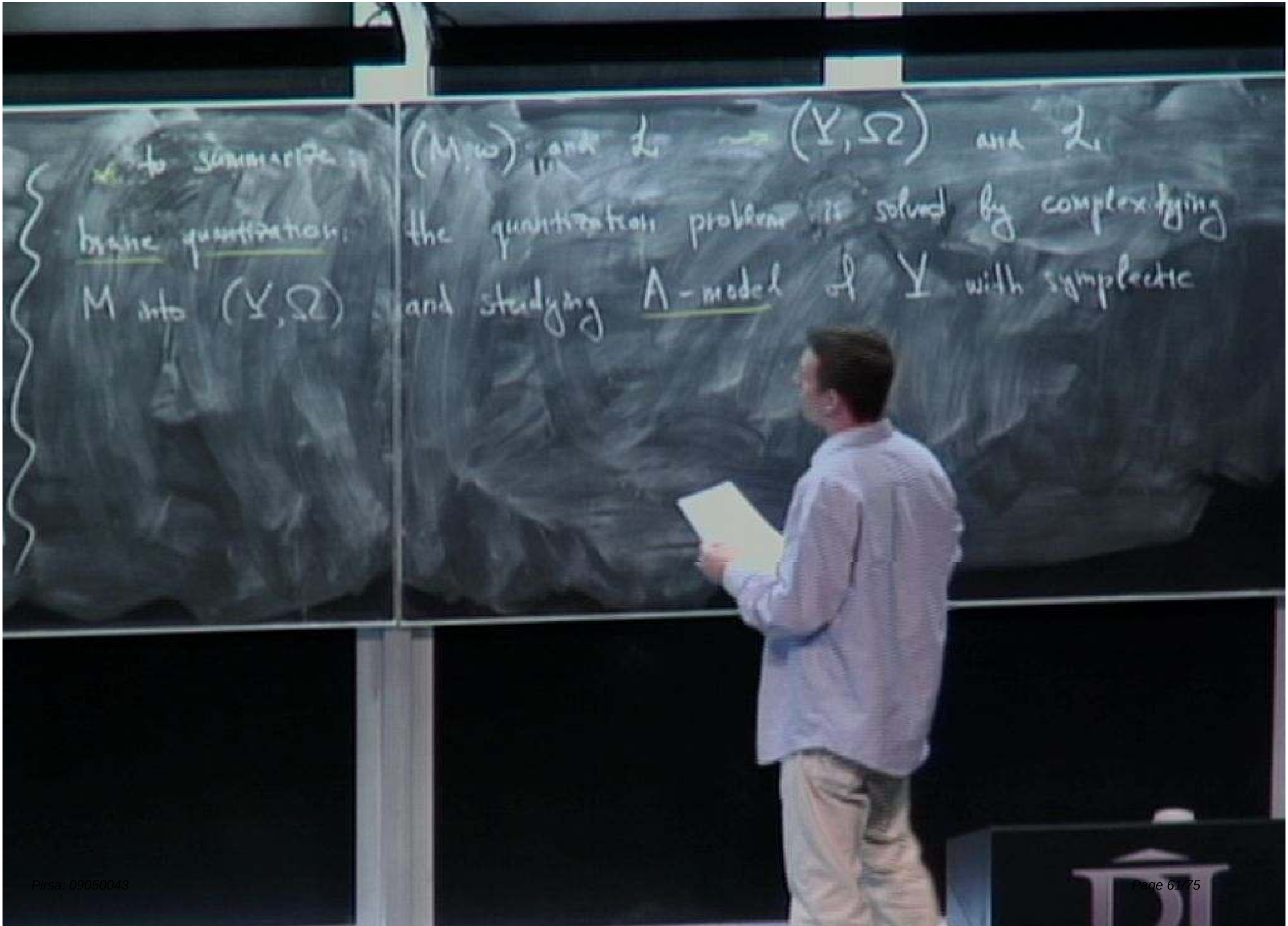
$\Rightarrow$ RTW inv'ts of knots, 3-mflds

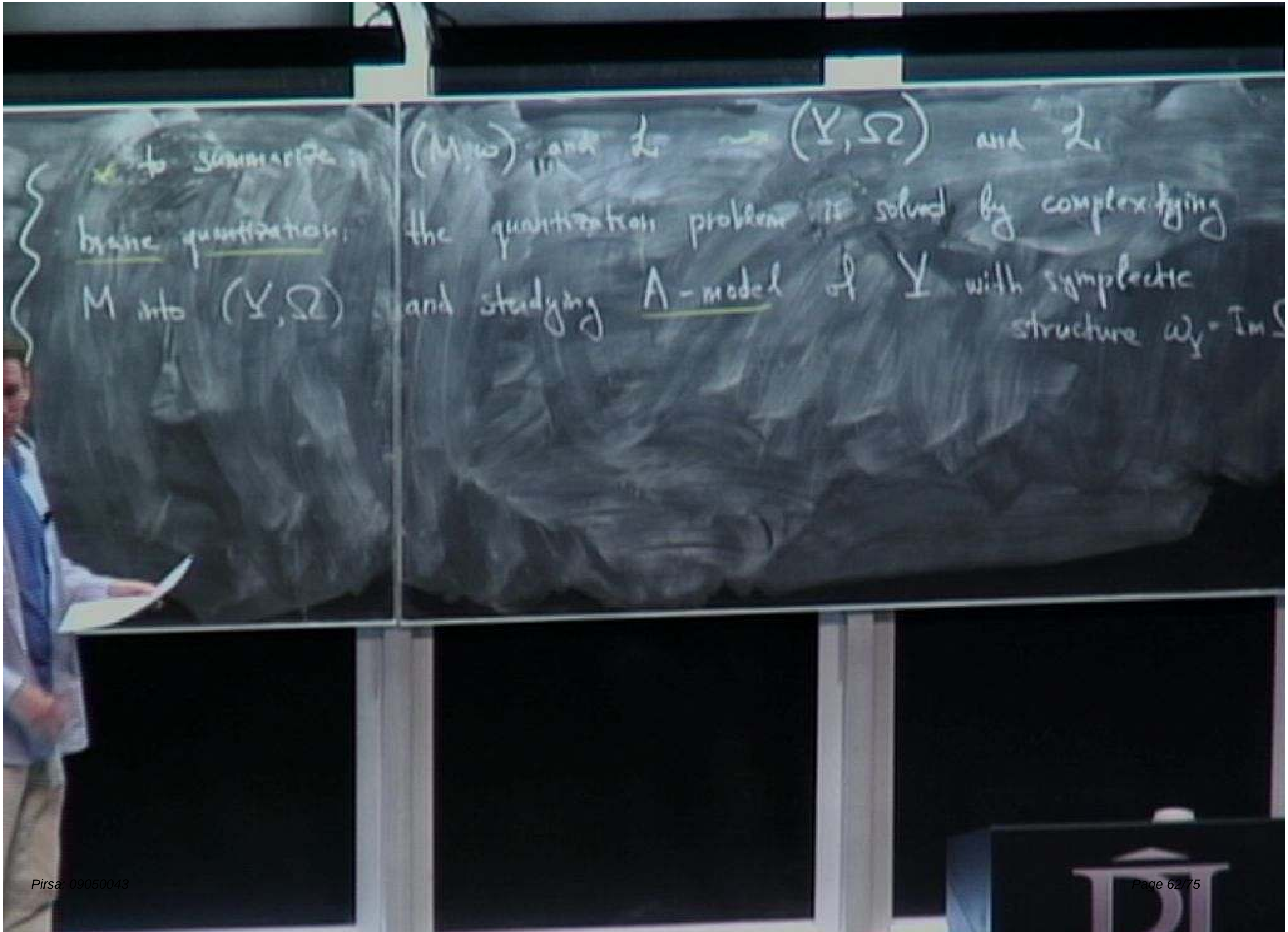
if $G_{\mathbb{R}}$ = non-compact $\Rightarrow$ $\mathcal{H}$ = infinite dimn'l

$$Y = \mathcal{M}_{flat}(G_{\mathbb{C}}, \Sigma)$$


to summarize:
(M, ω) and L ⟶ (Y, Ω
brane quantization: the quantization problem i


to summarize (M, ω) and L ⇝ (Y, Ω) and L₁
brane quantization: the quantization problem is solved by complexifying


to summarize: (M,ω) and L ⇝ (Y,Ω) and L
brane quantization: the quantization problem is solved by complexifying
M into (Y,Ω) and studying A-model of Y with symplectic


to summarize: (M,ω) and L ⟶ (Y,Ω) and L
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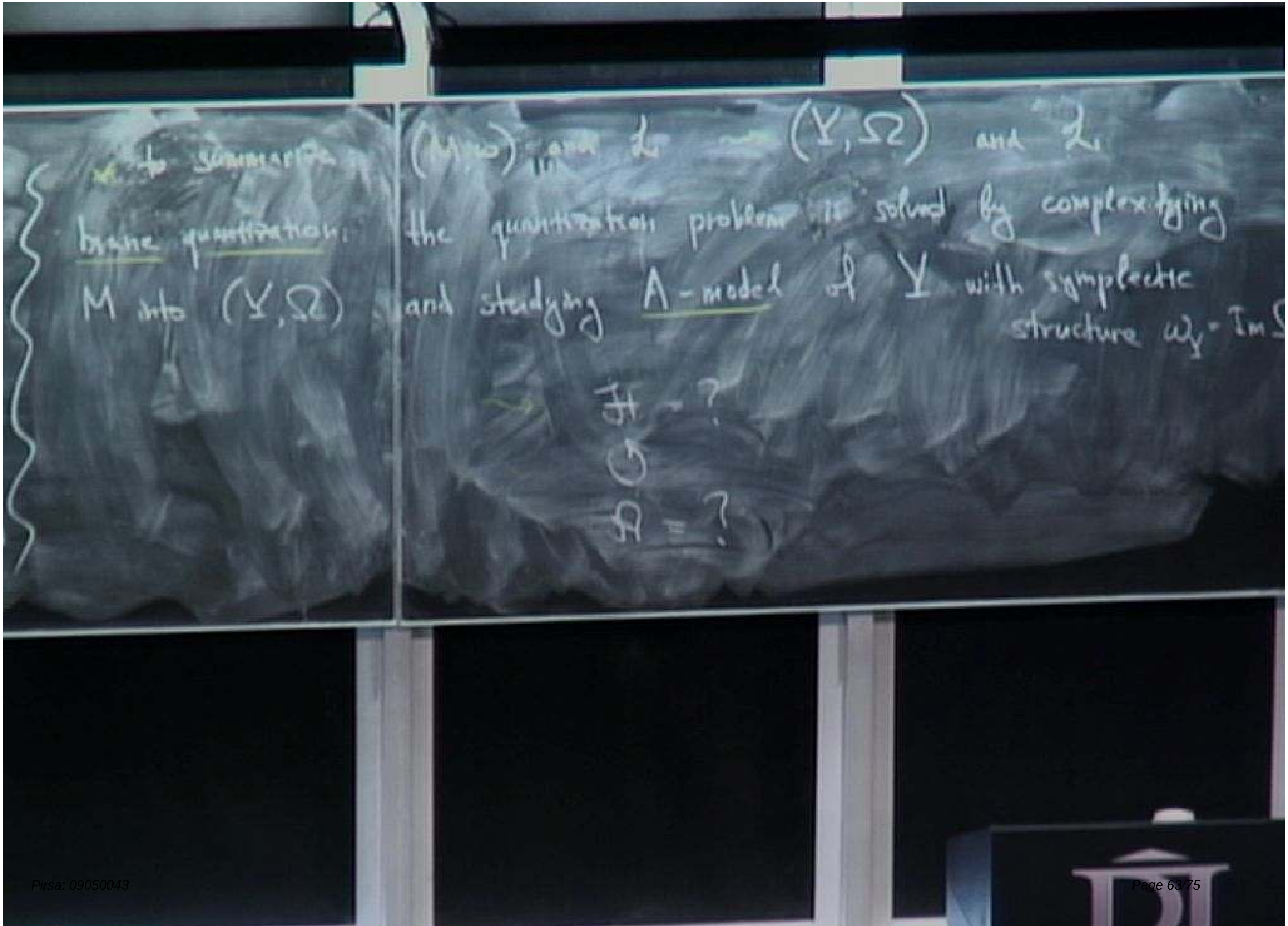
structure ω_Y = Im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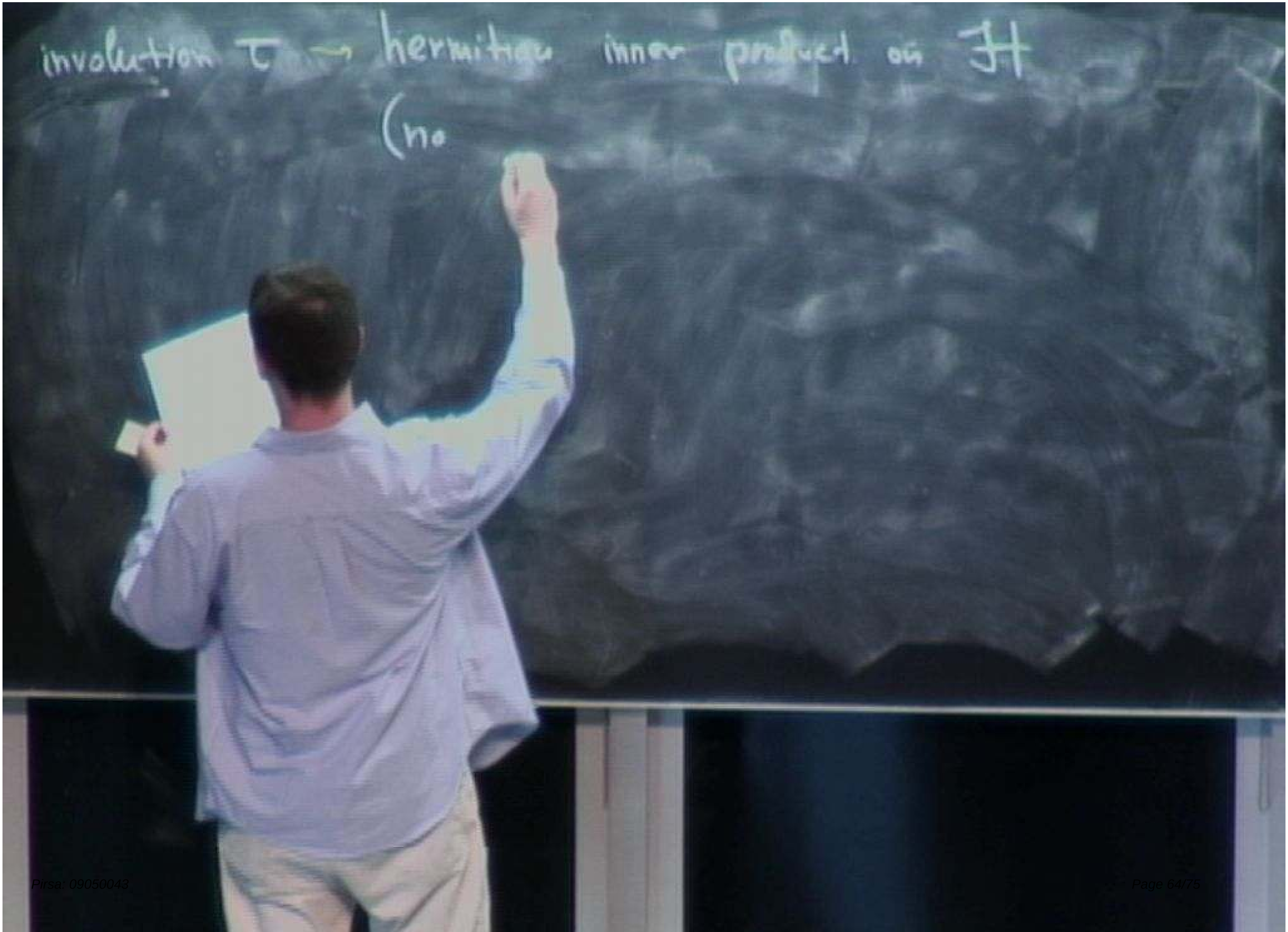
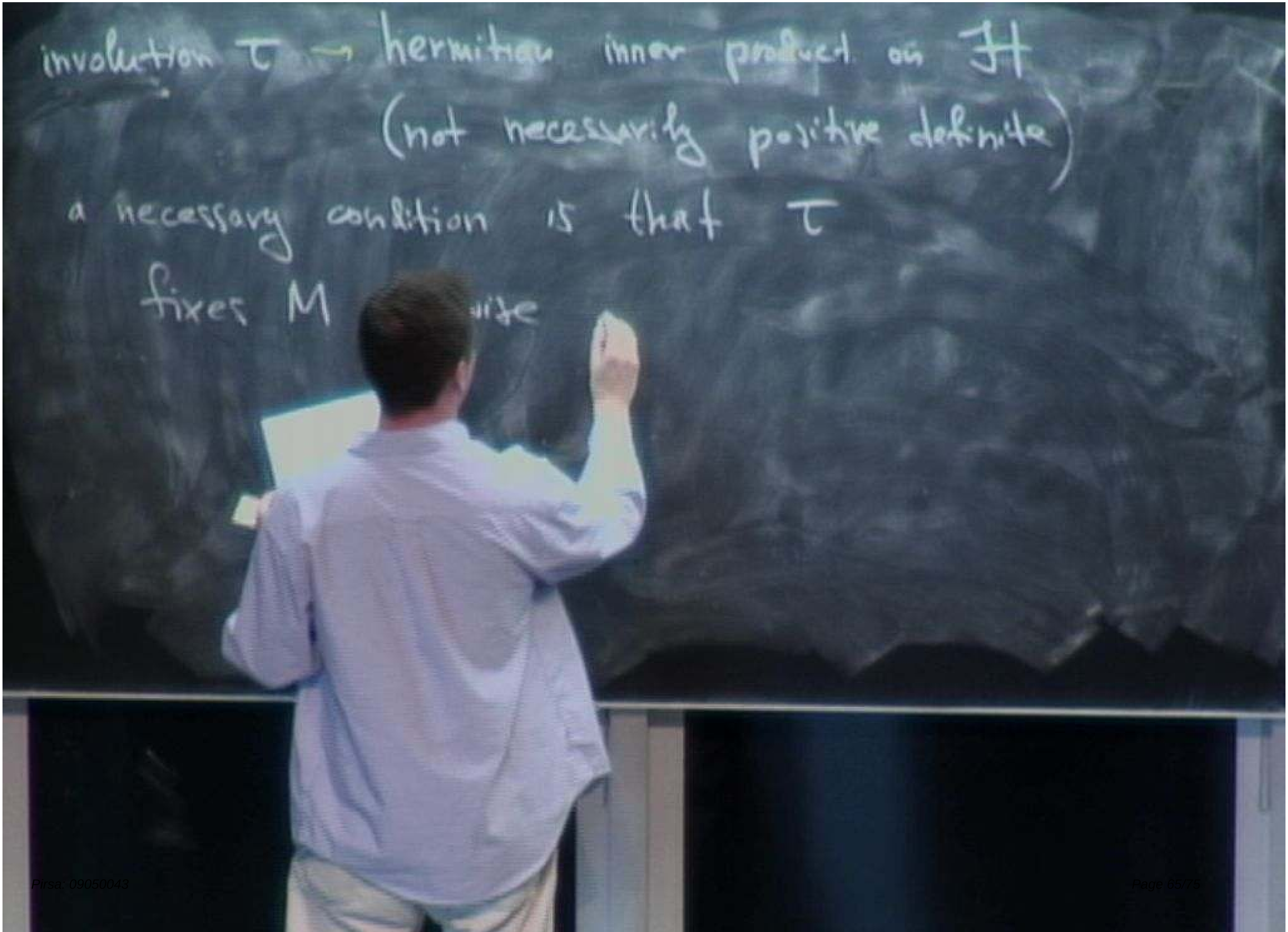

involution τ ⇝ hermitian inner product on ℋ
(no


involution τ → hermitian inner product on H
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a necessary condition is that τ
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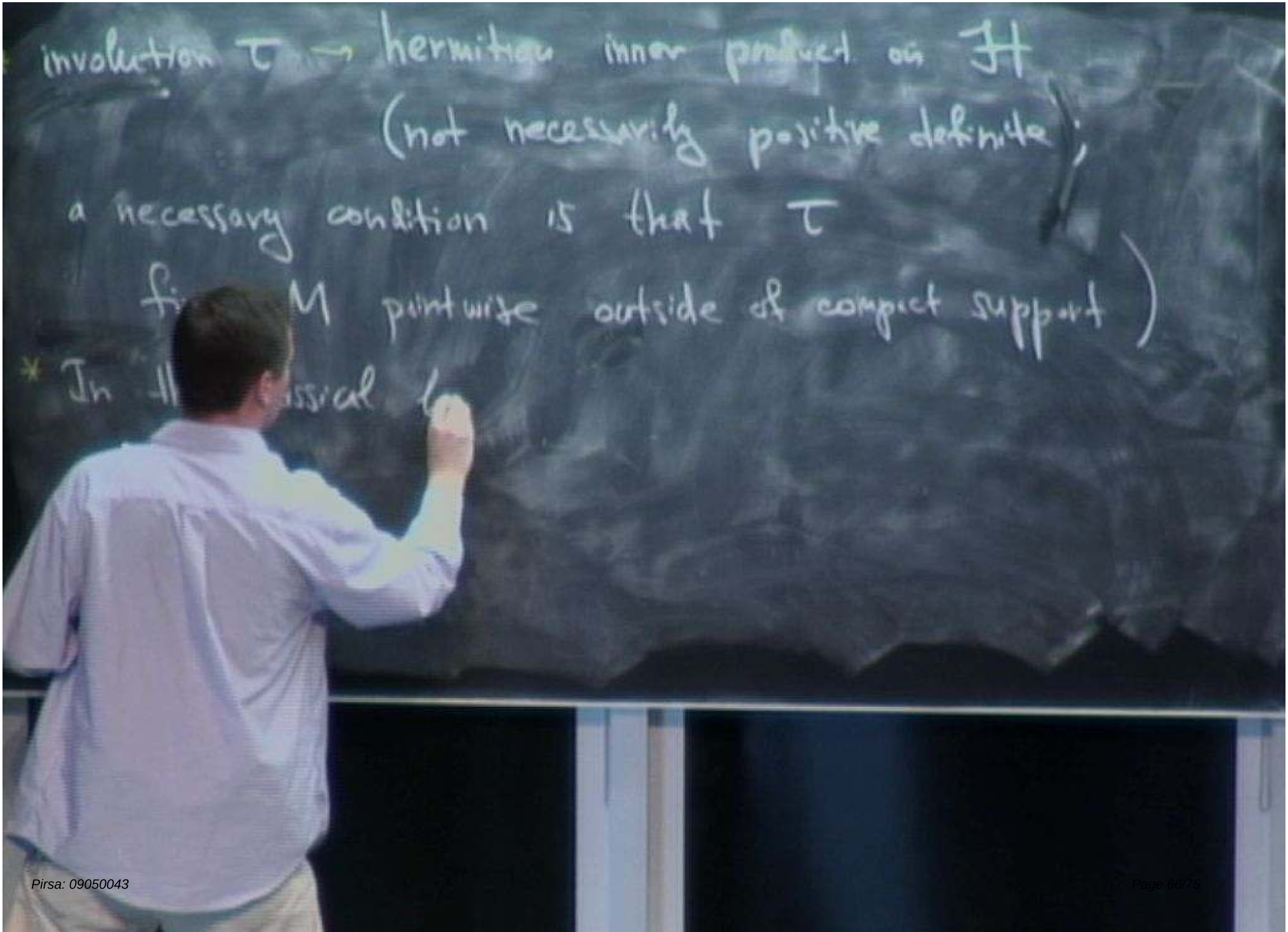
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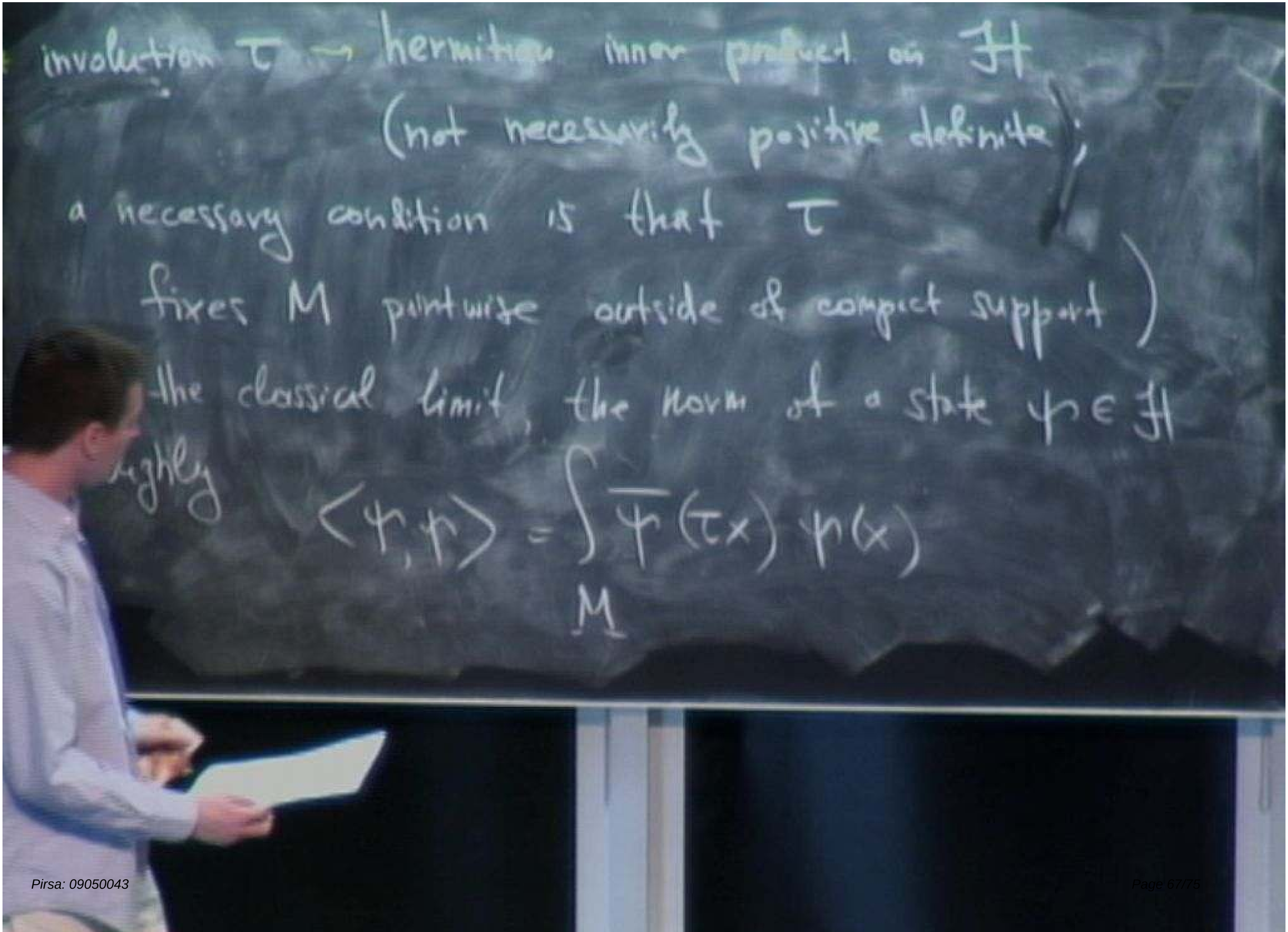
* In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the classical limit, the norm of a state ψ ∈ H
roughly
⟨ψ, ψ⟩ = ∫_M ψ̄(τx) ψ(x)

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$$\langle \psi, \psi \rangle = \int_M \overline{\psi}(\tau x)\, \psi(x)$$

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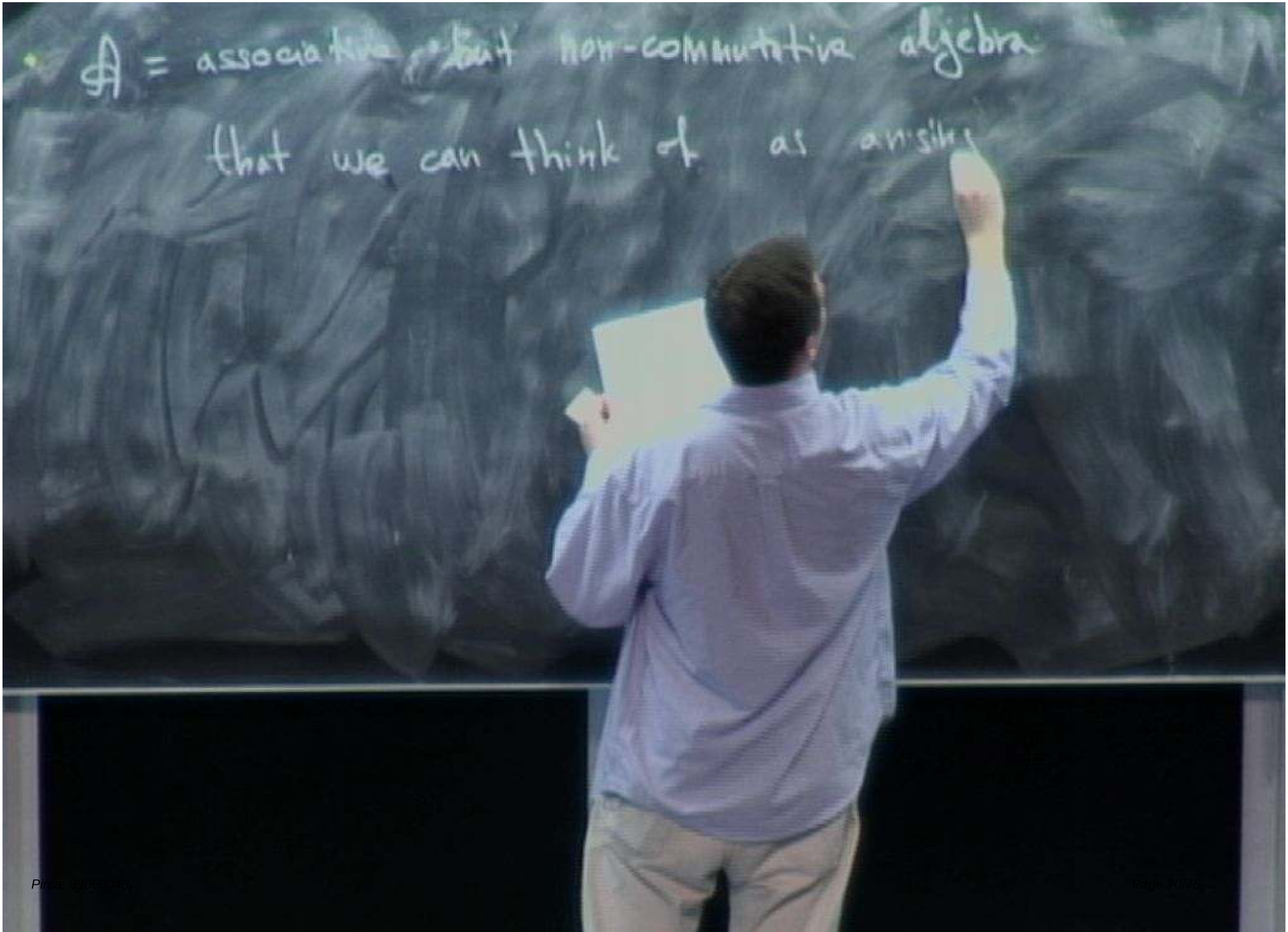
A = associative but non-commutative algebra
that we can think of as a sing

• $\mathcal{A}$ = associative but non-commutative algebra

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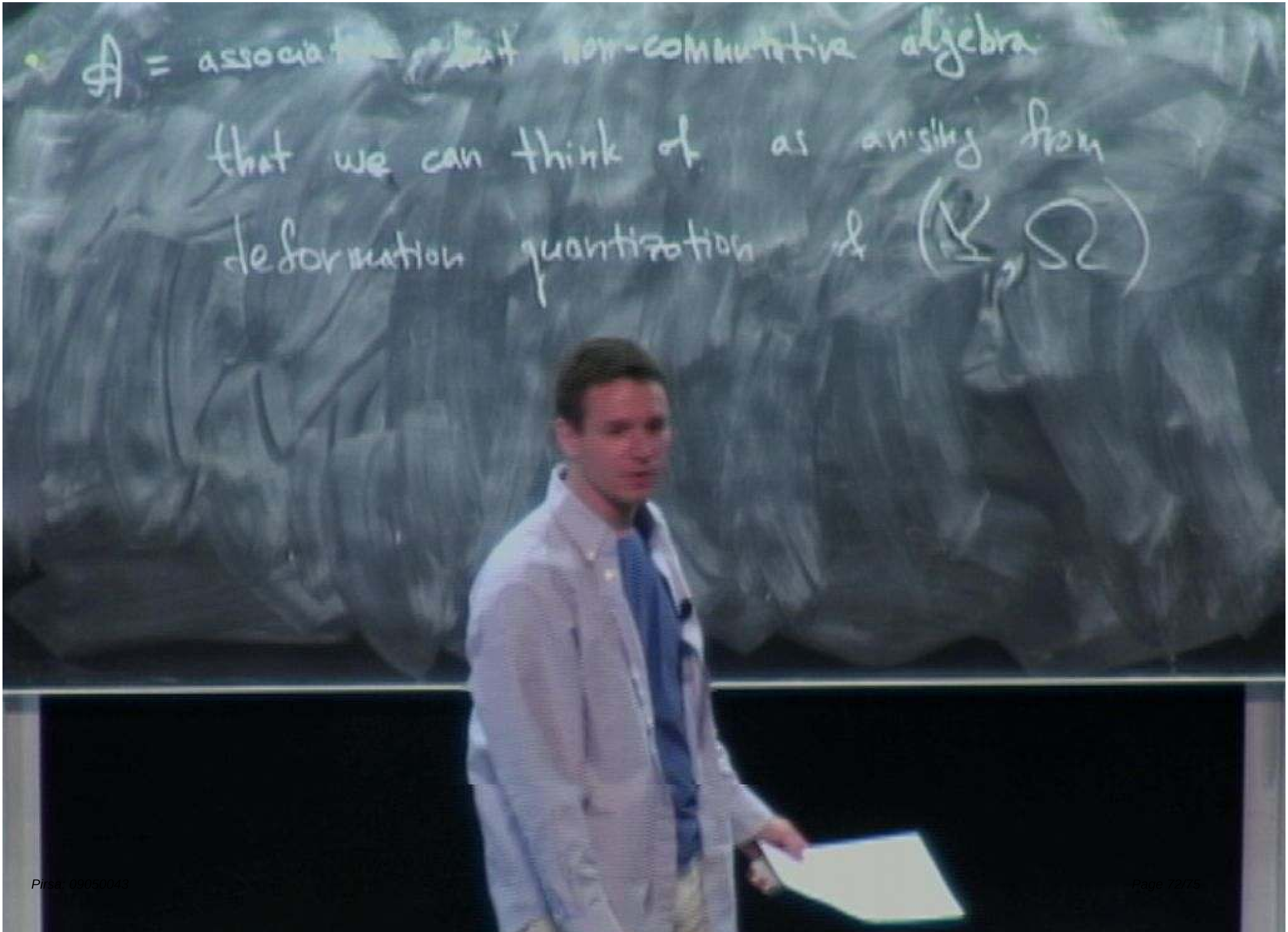
deformation quantization of (X, Ω)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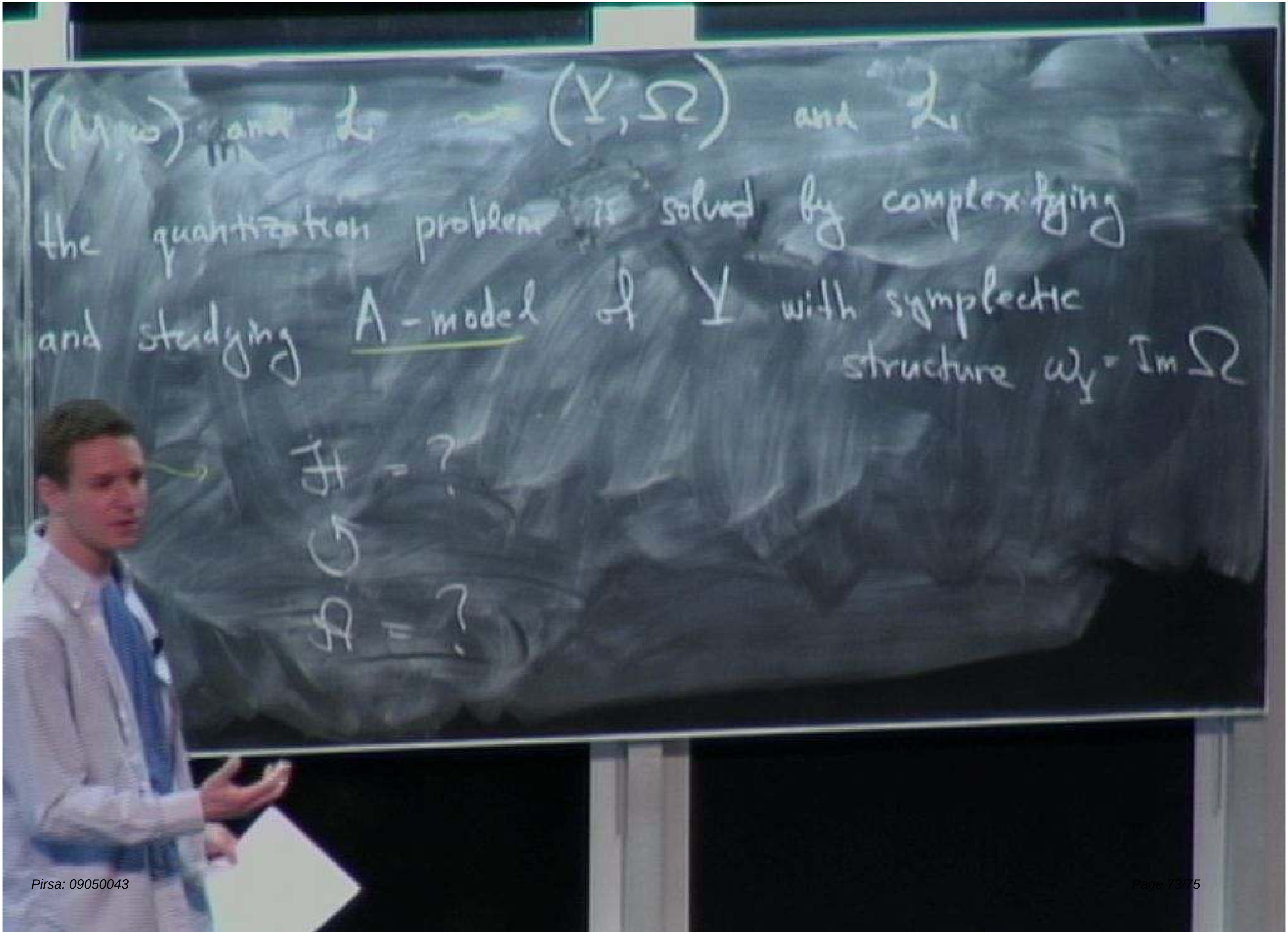
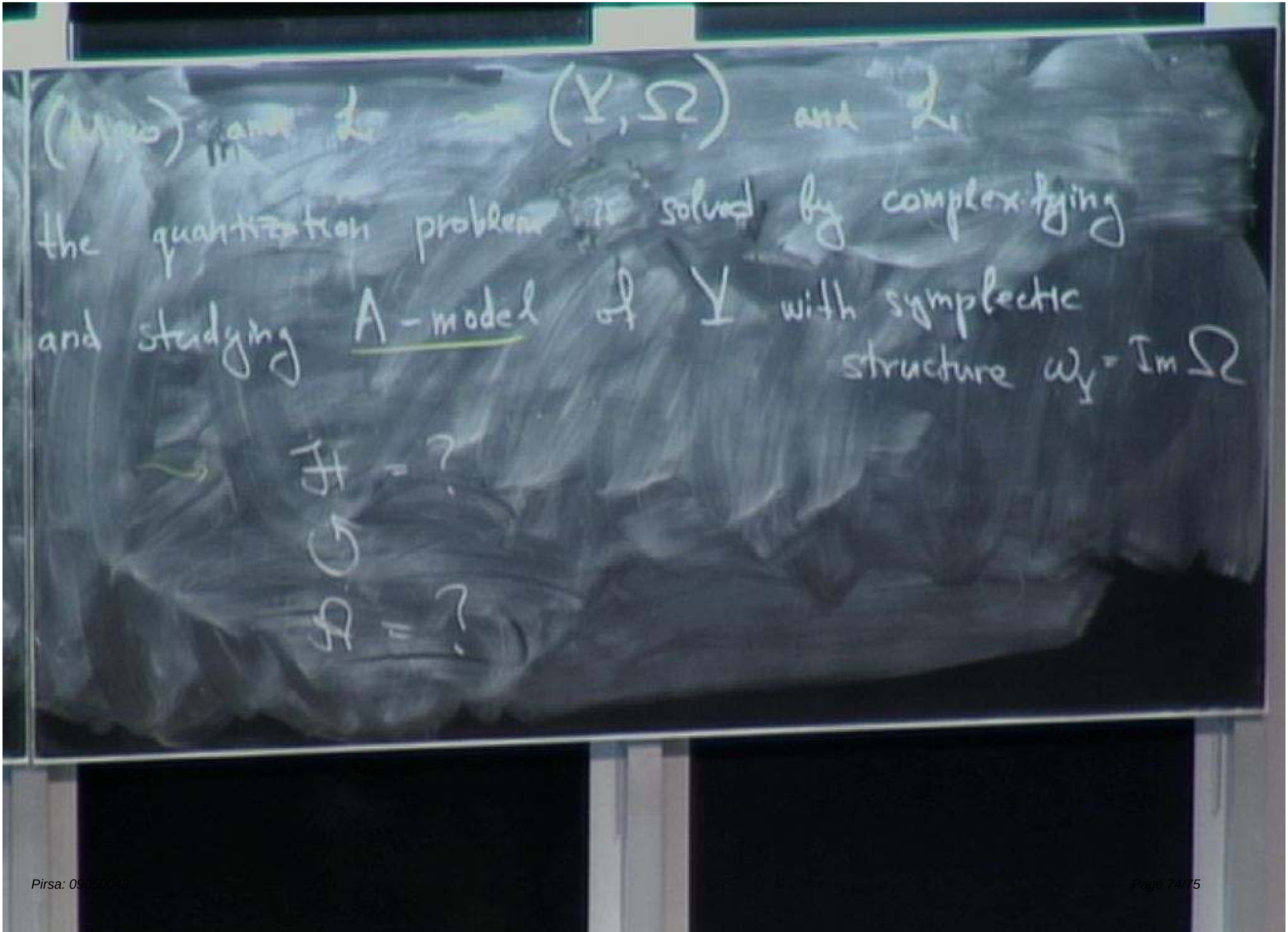


(M, ω) and $\mathcal{L}$ $\rightsquigarrow$ (Y, Ω) and $\mathcal{L}_1$
the quantization problem is solved by complexifying
and studying A-model of Y with symplectic
structure $\omega_Y = \mathrm{Im}\,\Omega$
$\mathcal{H} = ?$
$\mathcal{A} = ?$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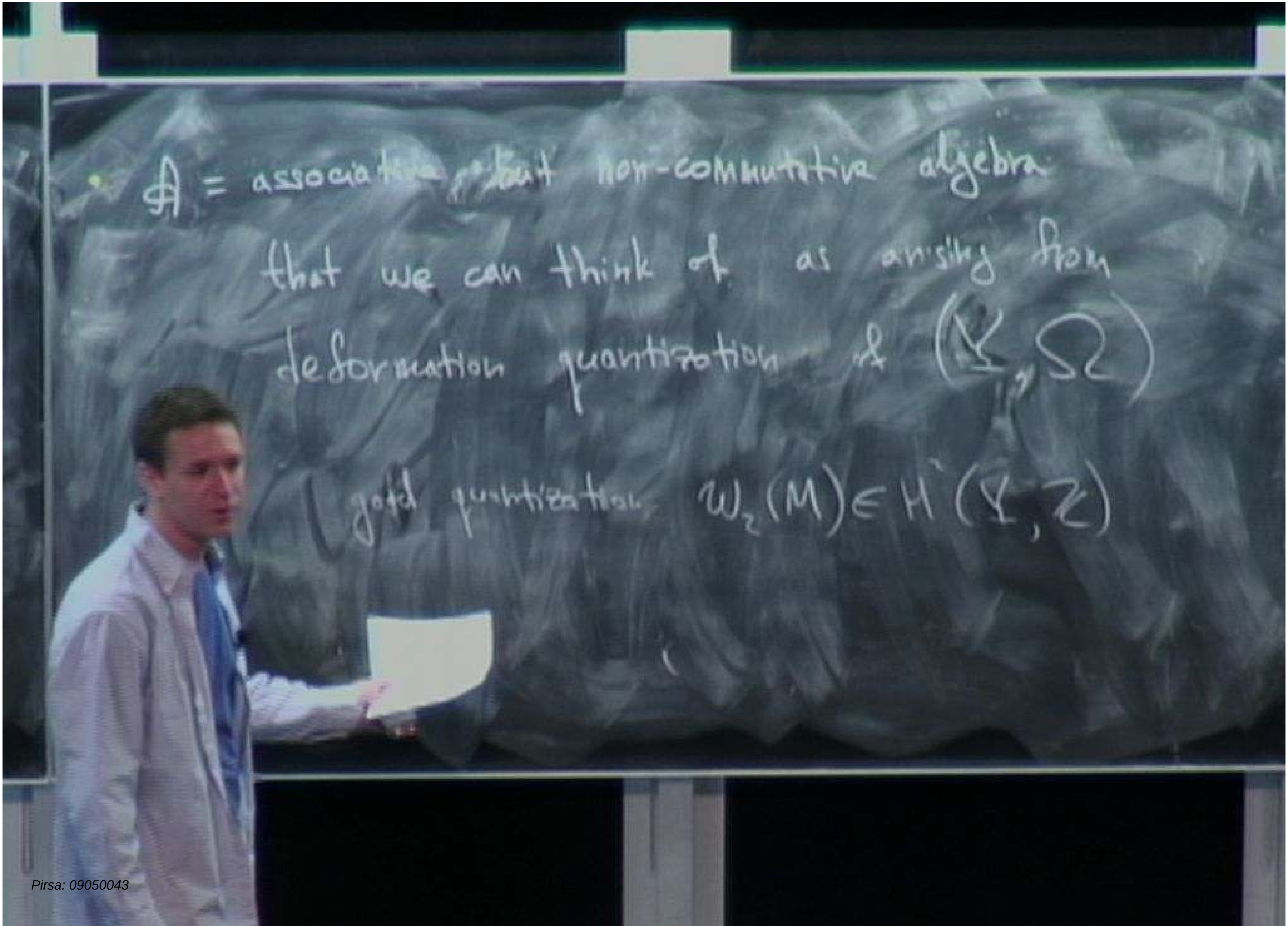
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good quantization w2(M) ∈ H2(X, Z)