Title: Symplectic Vortices and a Quantum Kirwan map

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Abstract: "A Hamiltonian action of a Lie group on a symplectic manifold (M,ω) gives rise to a gauge theoretic deformation of the Cauchy-Riemann equations, called the symplectic vortex equations. Counting solutions of these equations over the complex plane leads to a quantum version of the Kirwan map. In joint work with Christopher Woodward, we interpret this map as a weak morphism of cohomological field theories."

A Quantum Kirwan Map and Symplectic Vortices

Fabian Ziltener,
partly joint with C. Woodward

Overview:

1. A quantum Kirwan map

2. Symplectic Vortices

3. Morphisms of Cohomological Field Theories

1. A quantum Kirwan map

Consider

(M, ω): symplectic manifold,
G: compact connected Lie group with Lie algebra $\mathfrak{g}$.

We fix a Hamiltonian action of G on M, with moment map $\mu : M \to \mathfrak{g}^*$. This means:

$$\langle d\mu(x)v, \xi \rangle = \omega(X_\xi(x), v), \quad \mu(gx) = \mathrm{Ad}^*_{g^{-1}}\mu(x),$$

$$\forall x \in M,\ v \in T_xM,\ \xi \in \mathfrak{g},\ g \in G.$$

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Assume:

(H) μ is proper and G acts freely on $\mu^{-1}(0)$.

$\Longrightarrow$
symplectic quotient $(\bar{M},\bar{\omega})$ well-defined and closed.

Example:

$$M := \mathbb{C}^{k\times n}, \quad \omega := \omega_0,$$
$$G := \mathrm{U}(k), \quad \mu(\Theta) := \frac{i}{2}(1 - \Theta\Theta^*)$$

$\Longrightarrow \bar{M} = \mathrm{G}(k,n)$. (For $k = 1$: $\bar{\omega}$ = Fubini-Studi form.)

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I'm interested in the **quantum cohomology**

$$\mathrm{QH}^*(\bar{M}, \bar{\omega}) := H^*(\bar{M}) \otimes \Lambda.$$

The quantum cup product $\bar{*}$ on $\mathrm{QH}^*(\bar{M}, \bar{\omega})$ counts (pseudo-)holomorphic maps $S^2 \to \bar{M}$.

Example:

$$\mathrm{QH}^*(\mathbb{C}\mathrm{P}^n, \omega_{\mathrm{FS}}) = \mathbb{Z}[p, q]/(p^{n+1} - q).$$

Equivariant quantum cohomology
(introduced by A. Givental and B. Kim):

$$\mathrm{QH}^*_G(M, \omega) := H^*_G(M) \otimes \Lambda^\mu_\omega.$$

$*_G$ counts holomorphic maps from S^2 to the fibers of $(M \times \mathrm{EG})/G$.

Question:

How are $\mathrm{QH}^*_G(M, \omega)$ and $\mathrm{QH}^*(\bar{M}, \bar{\omega})$ related?

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Kirwan map:

$$\kappa_G : H^*_G(M) \to H^*(\bar{M}),$$

induced by

$$\bar{M} \to (\mu^{-1}(0) \times \mathrm{EG})/G \to (M \times \mathrm{EG})/G.$$

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Assume:

(M, ω) is weakly monotone and equivariantly convex at ∞,
$(\bar{M}, \bar{\omega})$ is semi-positive.

Conjecture. *There exists a Λ^{μ}_{ω}-algebra homomorphism*

$$\varphi : \mathrm{QH}^*_G(M, \omega) \to \mathrm{QH}^*(\bar{M}, \bar{\omega})$$

of the form

$$\varphi = \kappa_G \otimes \mathrm{id} + \sum_{0 \neq B \in H^G_2(M)} \varphi_B \otimes e^B.$$

Remark: Assume that the action is monotone, (M, ω) is symplectically aspherical, and $H^*_G(M)$ is generated as a ring by classes of degree < twice minimal Maslov number. Then A. R. Gaio and D. A. Salamon proved the conjecture.

K. Cieliebak and D. A. Salamon used this to calculate $QH^*(\bar{M}, \bar{\omega})$ for monotone torus actions on $\mathbb{R}^{2n}$.

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2. Symplectic Vortices

Idea of proof of conjecture:
We define $\varphi = \mathrm{Q}\kappa_G$ by counting symplectic vortices on $\mathbb{R}^2$.

Fix $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$: invariant inner product on $\mathfrak{g}$,

J: G-invariant and ω-compatible almost complex structure on M,

(Σ,ω_Σ,j): real surface with compatible area form and complex structure,

$G \hookrightarrow P \to \Sigma$: principal bundle.

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$(\Sigma, \omega_\Sigma, j)$: real surface with compatible area form and complex structure,

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A **symplectic vortex** is a solution $(u, A) \in C^\infty_G(P, M) \times \mathcal{A}(P)$ of

$$\begin{cases} \bar{\partial}_{J,A}(u) & = 0 \\ F_A + (\mu \circ u)\omega_\Sigma & = 0. \end{cases} \qquad (1)$$

Equations introduced by K. Cieliebak, A.R. Gaio and D. A. Salamon, and independently by I. Mundet i Riera.

No F_A in 2. equation:
Cauchy-Riemann equations for map $\Sigma \to \bar{M}$.

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Consider $\Sigma := \mathbb{R}^2, \omega_{\mathbb{R}^2} := \omega_0, j := i$.

Fact: Then every finite energy vortex (u, A) such that $\overline{u(P)}$ is compact, naturally carries a class $[u]_G \in H_2^G(M)$.

Let $B \in H_2^G(M)$.

$\mathcal{M}_B := \Big\{\text{vortex } (u, A) : [u]_G = B\Big\}/\mathcal{G}(P)$.

Fact: There are natural evaluation maps

$\mathrm{ev}_z : \mathcal{M}_B \to (M \times \mathrm{EG})/G$, for $z \in \mathbb{R}^2$,

$\overline{\mathrm{ev}}_\infty : \mathcal{M}_B \to \bar{M}$ (at $\infty \in \mathbb{R}^2 \cup \{\infty\}$).

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Heuristically, for $\alpha \in H_G^*(M)$ and $\bar{\beta} \in H^*(\bar{M})$, we define

$$\mathrm{Q}\kappa_G^B(\alpha, \bar{\beta}) := \int_{\mathcal{M}_B} \mathrm{ev}_0^* \alpha \smile \overline{\mathrm{ev}}_\infty^* \bar{\beta}.$$

We fix dual bases $(\bar{e}_i)_{i=1,\dots,N}$ and $(\bar{e}^i)_{i=1,\dots,N}$ of $H^*(\bar{M})$.

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$$\mathrm{Q}\kappa_G(\alpha) := \sum_{i,B} \mathrm{Q}\kappa_G^B(\alpha, \bar{e}_i)\bar{e}^i e^B.$$

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Reason why $Q\kappa_G$ should intertwine $*_G$ and $\bar{*}$:

Let $B \in H_2^G(M)$, $\alpha_1, \alpha_2 \in H_G^*(M)$, $\bar{\beta} \in H^*(\bar{M})$.

Claim: see blackboard.

Idea of proof of claim:

Set $z_\nu^+ := \nu$, $z_\nu^- := 1/\nu$

Consider a sequence W_ν of gauge classes of vortices on $\mathbb{R}^2$ such that $[W_\nu]_G = B$ and

$\mathrm{ev}_{z_\nu^\pm}(W_\nu) \in X_1$, $\mathrm{ev}_{-z_\nu^\pm}(W_\nu) \in X_2$, $\overline{\mathrm{ev}}_\infty(W_\nu) \in \bar{X}$.

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Goal: Define $Q\kappa$ rigorously.

Compactification:

Fix $p > 2$. We define

$$\widetilde{\mathcal{B}}^p_{\mathrm{loc}} := W^{1,p}_{\mathrm{loc},G}(P, M) \times \mathcal{A}^{1,p}_{\mathrm{loc}}(P),$$

and for $w := (u, A) \in \widetilde{\mathcal{B}}^p_{\mathrm{loc}}$:

$$e_w := \frac{1}{2}\left(|d_A u|^2 + |F_A|^2 + |\mu \circ u|^2\right),$$

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Consider $\Sigma := \mathbb{R}^2$.

Theorem 1 (Z.). *Assume (H) and that (M, ω) is equivariantly convex at ∞, and*

$$\int u^*\omega = 0, \quad \forall u \in C^\infty(S^2, M).$$

Let $k \in \mathbb{N} \cup \{0\}$, and for $\nu \in \mathbb{N}$ let $w_\nu = (u_\nu, A_\nu)$ be a vortex such that $\overline{u_\nu(P)}$ is compact, and $z_1^\nu, \ldots, z_k^\nu \in \mathbb{R}^2$.

Assume

$$\sup_{\nu \in \mathbb{N}} E(w^\nu) < \infty,$$

$$\limsup_{\nu \to \infty} |z_i^\nu - z_j^\nu| > 0, \quad \forall i \neq j.$$

Then there exists a subsequence of $\big(w^\nu, z_0^\nu := \infty, z_1^\nu, \ldots, z_k^\nu\big)$ that converges to some stable map of vortices on $\mathbb{R}^2$ and holomorphic maps $S^2 \to \bar{M}$.

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Proof:

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Fredholm theory:

View LHS of (2) as $\tilde{\mathcal{S}}(w)$, where $\tilde{\mathcal{S}} : \tilde{\mathcal{B}} \to \tilde{\mathcal{E}}$ is a $\mathcal{G}$-equivariant section.

This descends to $\mathcal{S} : \mathcal{B} := \tilde{\mathcal{B}}/\mathcal{G} \to \mathcal{E} := \tilde{\mathcal{E}}/\mathcal{G}$.

Vertical differential of $\mathcal{S}$ at $W = \mathcal{G}^* w \in \mathcal{S}^{-1}(0)$:

$$\mathcal{D}_w : \ker L_w^* \cong T_W \mathcal{B} \to \tilde{\mathcal{E}}_w \cong \mathcal{E}_W,$$

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Explanations:

$L_w:\mathrm{Lie}\,\mathcal{G}\to T_w\bar{\mathcal{B}}$, $L_w\xi=(L_u\xi,-d_A\xi)$.

We denote

$TM^u:=(u^*TM)/G\to\Sigma$, $\quad\mathfrak{g}_P:=(P\times\mathfrak{g})/G\to\Sigma$.

$$\begin{aligned}E_u &:= TM^u\oplus\textstyle\bigwedge^1(\mathfrak{g}_P)\to\Sigma,\\ E'_u &:= \textstyle\bigwedge^{0,1}(TM^u)\oplus\bigwedge^2(\mathfrak{g}_P)\to\Sigma.\end{aligned}$$

Then $T_w\bar{\mathcal{B}}=\Gamma(E_u)$, $\bar{\mathcal{E}}_w=\Gamma(E'_u)$.

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Consider $\Sigma := \mathbb{R}^2$.

Fix $p > 2$ and $\lambda \in \mathbb{R}$.

For $f : \mathbb{R}^n \to \mathbb{R}$ we denote

$$\|f\|_{p,\lambda} := \left\| f(1 + |\cdot|^2)^{\frac{\lambda}{2}} \right\|_p .$$

We define:

$$\tilde{\mathcal{B}}^p_\lambda := \left\{ (u, A) \in \tilde{\mathcal{B}}^p_{\mathrm{loc}} \,\middle|\, \overline{u(P)} \text{ compact, } \left\| \sqrt{e_{(u,A)}} \right\|_{p,\lambda} < \infty \right\}$$

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Theorem 2 (Z.). *Assume (H) and* $\dim M > 2 \dim G$.

Let $p > 2$, $\lambda > 1 - 2/p$ *and* $w := (u, A) \in \tilde{\mathcal{B}}^p_\lambda$ *be a smooth pair.*

Then:

(i) $\mathcal{X}^{p,\lambda}_w$ *and* $\mathcal{Y}^{p,\lambda}_w$ *are complete.*

(ii) If $1 - 2/p < \lambda < 2 - 2/p$ *then* $\mathcal{D}_w : \mathcal{X}^{p,\lambda}_w \to \mathcal{Y}^{p,\lambda}_w$ *is Fredholm, and*

$$\mathrm{ind}\, \mathcal{D}^{p,\lambda}_w = \dim M - 2 \dim G + 2\,\mathrm{Maslov}(w).$$

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Proof:

1. Fredholm result for augmented vertical differential and
2. surjectivity of L^*_w.

The proof of 1. is based on:

- Existence of a good trivialization of E_u. (Respects splitting
$$T_{u(p)}M = (\operatorname{im} L^{\mathbb{C}})^\perp \oplus \operatorname{im} L^{\mathbb{C}}_{u(p)}$$
for $p \in \pi^{-1}(z) \subseteq P$, $z \in \mathbb{R}^2 \setminus B_R$ and R large.)
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Main difficulty:

Kondrachov compactness fails on $\mathbb{R}^2$. In fact:

The terms $\alpha \mapsto (L_u\alpha)^{0,1}$ and $v \mapsto \omega_0\, d\mu(u)v$ occuring in $\mathcal{D}_w$ are not compact.

Explanations:

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$W = \mathcal{G}^* w \in \mathcal{S}^{-1}(0)$:

$$\mathcal{D}_w : \ker L_w^* \cong T_W \mathcal{B} \to \tilde{\mathcal{E}}_w \cong \mathcal{E}_W.$$

$$\mathcal{D}_w(v, \alpha) := \begin{pmatrix} \left(\nabla^A v + L_u \alpha\right)^{0,1} + \frac{1}{2}(\nabla_v J)\, d_A u\, j \\ d_A \alpha + \omega_\Sigma\, d\mu(u) v \end{pmatrix}.$$

Explanations:

$L_w : \mathrm{Lie}\,\mathcal{G} \to T_w \tilde{\mathcal{B}}$, $L_w \xi = (L_u \xi, -d_A \xi)$.

We denote

$TM^u := (u^* TM)/G \to \Sigma$, $\quad \mathfrak{g}_P := (P \times \mathfrak{g})/G \to \Sigma$.

$$\begin{aligned} E_u &:= TM^u \oplus \textstyle\bigwedge^1(\mathfrak{g}_P) \to \Sigma, \\ E'_u &:= \textstyle\bigwedge^{0,1}(TM^u) \oplus \bigwedge^2(\mathfrak{g}_P) \to \Sigma. \end{aligned}$$

Then $T_w \tilde{\mathcal{B}} = \Gamma(E_u)$, $\tilde{\mathcal{E}}_w = \Gamma(E'_u)$.

$\nabla :=$ Levi-Civita connection of $\omega(\cdot, J\cdot)$.

(∇, A) induces a connection ∇^A on E_u.

Proof:

1. Fredholm result for augmented vertical differential and
2. surjectivity of L_w^*.

The proof of 1. is based on:

- Existence of a good trivialization of E_u. (Respects splitting

$$T_{u(p)}M = (\operatorname{im} L^{\mathbb{C}})^{\perp} \oplus \operatorname{im} L^{\mathbb{C}}_{u(p)}$$

  for $p \in \pi^{-1}(z) \subseteq P$, $z \in \mathbb{R}^2 \setminus B_R$ and R large.)

- Work by R. B. Lockhart and R. C. McOwen.

Main difficulty:

Kondrachov compactness fails on $\mathbb{R}^2$. In fact:

The terms $\alpha \mapsto (L_u\alpha)^{0,1}$ and $v \mapsto \omega_0\, d\mu(u)v$ occuring in $\mathcal{D}_w$ are not compact.

Consider $\Sigma := \mathbb{R}^2$.

Fix $p > 2$ and $\lambda \in \mathbb{R}$.

For $f : \mathbb{R}^n \to \mathbb{R}$ we denote

$$\|f\|_{p,\lambda} := \left\| f(1 + |\cdot|^2)^{\frac{\lambda}{2}} \right\|_p.$$

We define:

$$\tilde{\mathcal{B}}^p_\lambda := \left\{ (u, A) \in \tilde{\mathcal{B}}^p_{\mathrm{loc}} \,\middle|\, \overline{u(P)} \text{ compact, } \left\| \sqrt{e_{(u,A)}} \right\|_{p,\lambda} < \infty \right\}$$

Let $w := (u, A) \in \tilde{\mathcal{B}}^p_\lambda$.

For $\zeta := (v, \alpha) \in W^{1,p}_{\mathrm{loc}}(E_u)$ we define

$$\|\zeta\|_{w,p,\lambda} := \|\zeta\|_\infty + \left\| |\nabla^A \zeta| + |d\mu(u)v| + |\alpha| \right\|_{p,\lambda}.$$

We define

$$\mathcal{X}^{p,\lambda}_w := \left\{ \zeta \in W^{1,p}_{\mathrm{loc}}(E_u) \,\middle|\, L^*_w \zeta = 0, \ \|\zeta\|_{w,p,\lambda} < \infty \right\},$$

$$\mathcal{Y}^{p,\lambda}_w := \left\{ \zeta' \in L^p_{\mathrm{loc}}(E'_u) \,\middle|\, \|\zeta'\|_{p,\lambda} < \infty \right\}.$$

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3. Morphisms of Cohomological Field Theories (joint with C. Woodward)

We denote by $\overline{\mathcal{M}}_n$ the moduli space of stable n-marked curves of genus 0.

An (even) **genus 0 CohFT** consists of

V: an $\mathbb{R}$-vector space,

$\langle \cdot ; \cdot \rangle_n : V^n \times H^*(\overline{\mathcal{M}}_n) \to \mathbb{R}, \quad n \geq 3,$

$(\cdot,\cdot) : V \times V \to \mathbb{R}$: symmetric non-degenerate.

Condition:

For $\alpha \in V^n$, $\beta \in H^*(\overline{\mathcal{M}}_n)$ and $I \subseteq \{1,\dots,n\}$ satisfying $2 \leq |I| \leq n - |I|$,

$$\left\langle \alpha; \beta \smile \mathrm{PD}(\mathrm{im}(\iota_{n,I})) \right\rangle_n =$$

$$\sum_{i,j} \langle \alpha_I, e_i; \beta'_j \rangle_{|I|+1} \left\langle \alpha_{\{1,\dots,n\}\setminus I}, e^i; \beta''_j \right\rangle_{n-|I|+1}.$$

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Morphisms:

Consider

$$\{\infty\}\times\left\{(z_1,\ldots,z_n)\in\mathbb{C}^n \,\middle|\, z_i\neq z_j,\ \forall i\neq j\right\}/\text{translation}.$$

This has a natural compactification $\overline{\mathcal{M}}_{n,1}(\mathbb{C})$.

Let V and W be CohFT.

A **morphism of CohFT's** from V to W is a collection

$$\varphi_n : V^n \times H^*(\overline{\mathcal{M}}_{n,1}(\mathbb{C})) \to W, \quad n = 0,1,\ldots$$

satisfying conditions that correspond to collapsing some of the marked points or moving them away from each other.
(See blackboard.)

This notion was introduced by C. Woodward.

Heuristically, for $n \geq 0$ we define a map

$$\mathrm{Q}\kappa_G^n : \mathrm{QH}_G(M,\omega)^n \times H^*(\overline{\mathcal{M}}_n) \to \mathrm{QH}^*(\bar{M},\bar{\omega}),$$

similarly as above.

Conjecture. *$(\mathrm{Q}\kappa_G^n)_{n\geq 0}$ is a morphism of CohFT's from $\mathrm{GW}_G(M,\omega)$ to $\mathrm{GW}(\bar{M},\bar{\omega})$.*