Title: Groupoids of Connections and Higher-Algebraic QFT

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Abstract: This talk will discuss, illustrated by a toy example, how to construct "higher-algebraic" quantum field theories using groupoids. In particular, the groupoids describe configuration spaces of connections, together with their gauge symmetries, on spacetime, space, and boundaries of regions in space. The talk will describe a higher-algebraic "sum over histories", and how this construction is related to usual QFT's, and particularly the relation to the case of the Chern-Simons theory.

Groupoids of Connections and Higher-Algebraic QFT

Jeffrey C. Morton

Department of Mathematics
University of Western Ontario

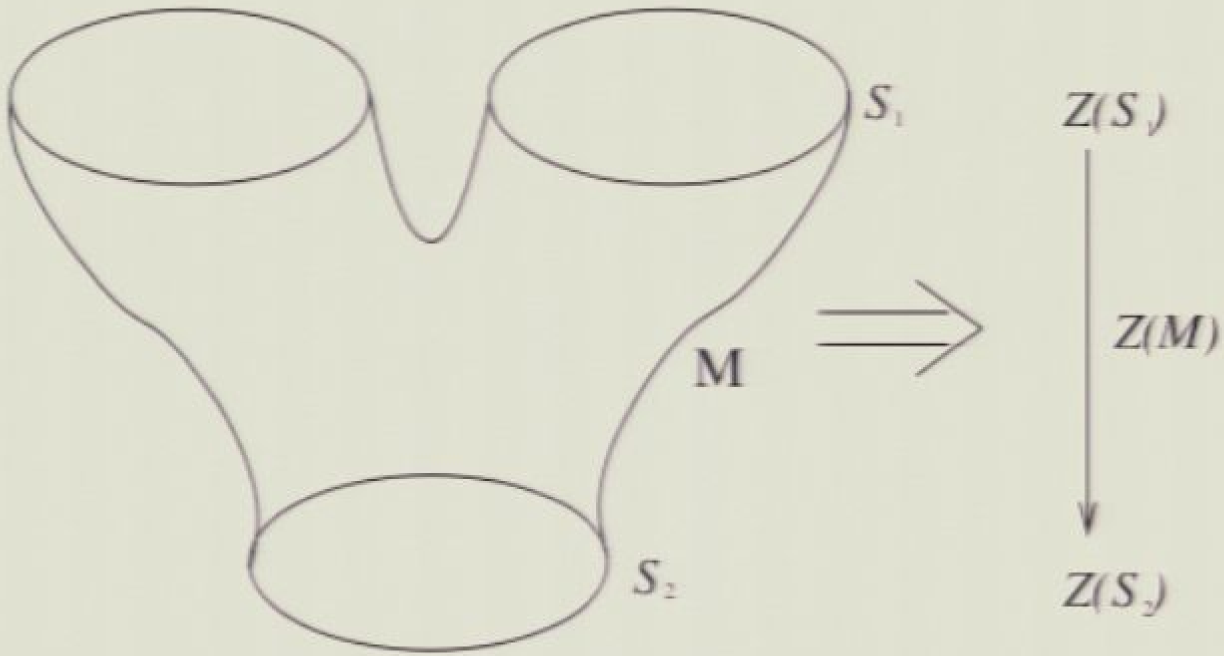
Connections in Geometry and Physics, 2009

Outline

1. TQFT and ETQFT
2. Groupoids of Connections
   - Groupoids and Moduli Spaces
   - Example: S^1
3. Constructing Z_G
   - 2-Vector Spaces for Manifolds
   - Z_G: 2-Linear Maps for Cobordisms
   - Z_G: 2-Morphisms
4. Physics - Sort Of

A Topological Quantum Field Theory can be seen as a monoidal functor:

$$Z_G : \mathbf{nCob} \to \mathbf{Vect}$$



In particular:

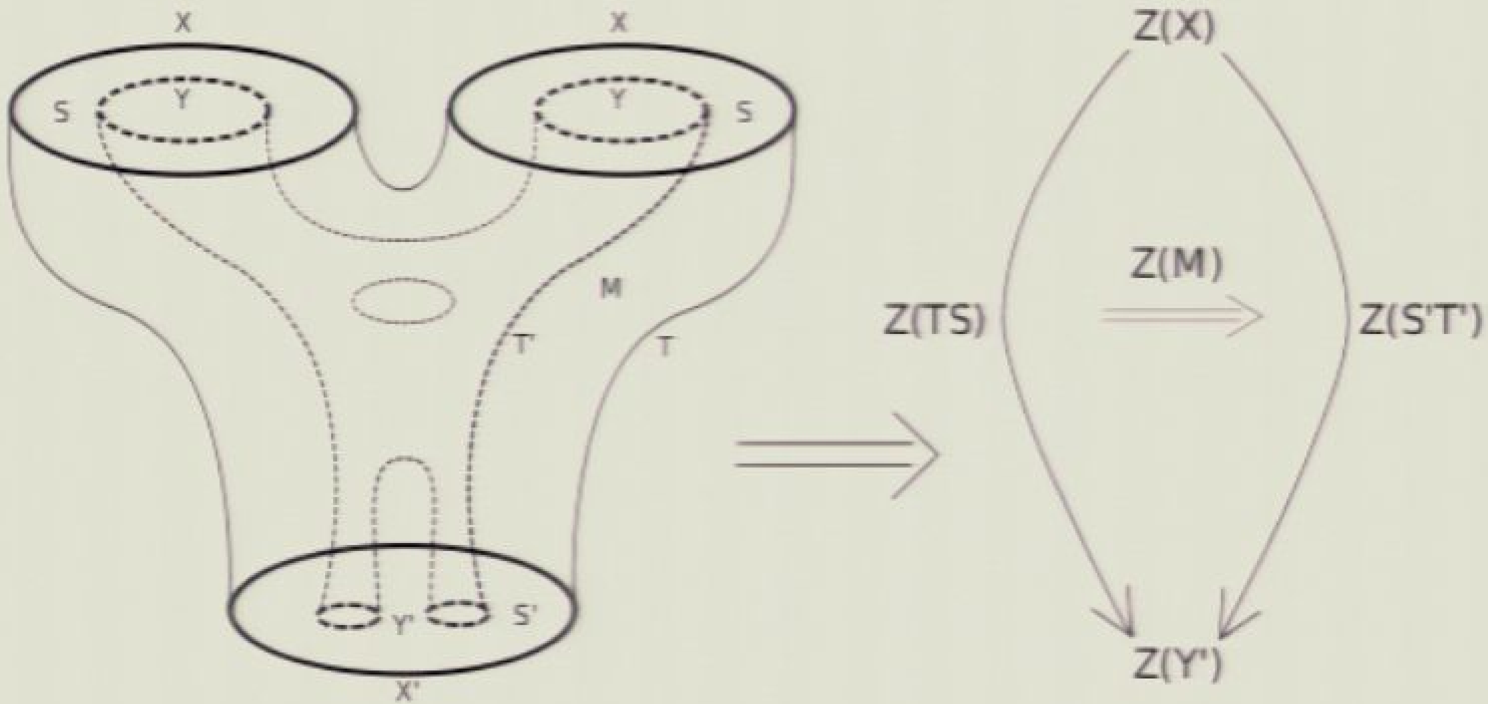
$$Z(M_2 \circ M_1) = Z(M_2) \circ Z(M_1)$$

and

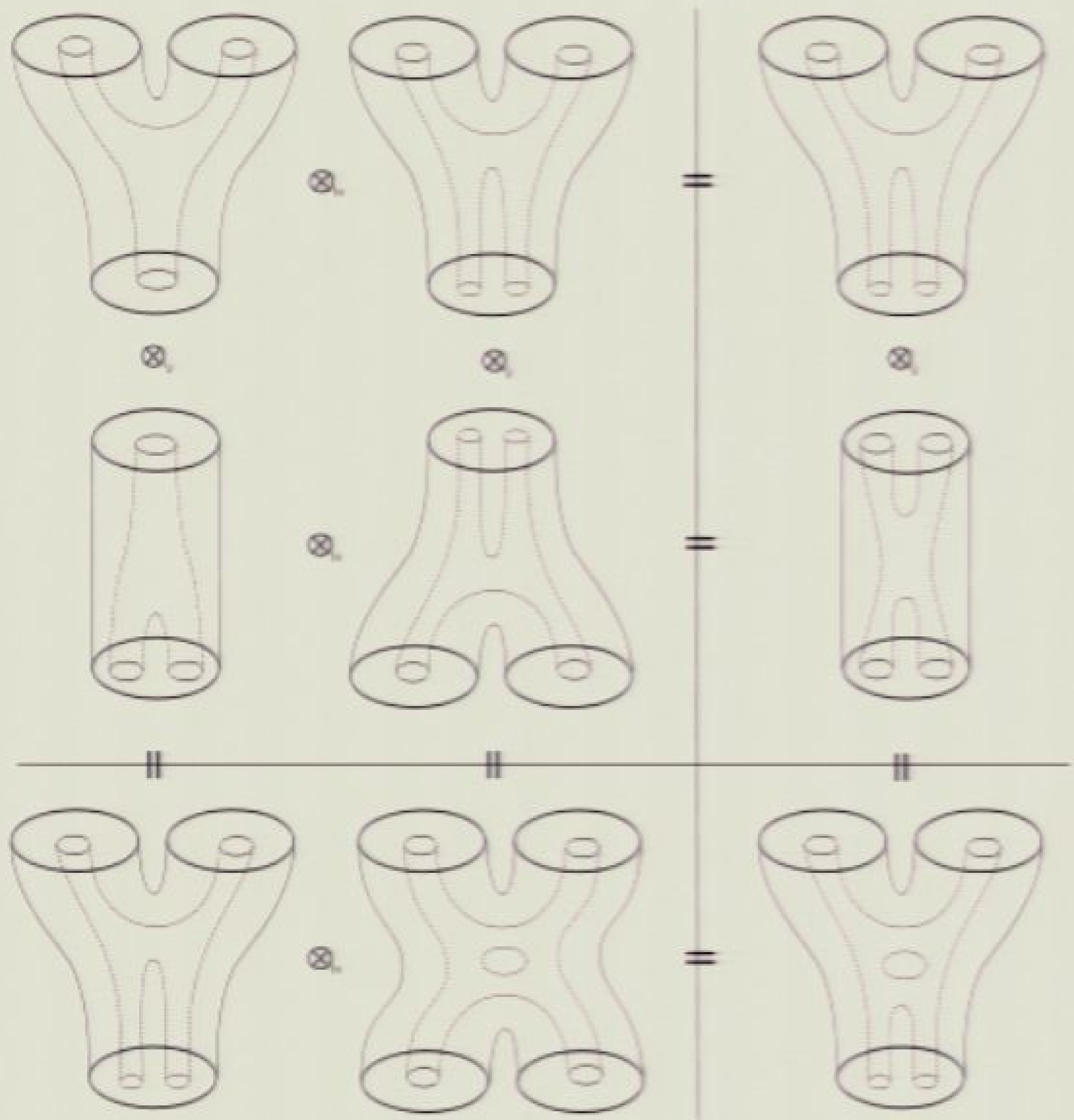
$$Z(S_1 \amalg S_2) = Z(S_1) \otimes Z(S_2) \text{ and } Z(\varnothing) = \mathbb{C}$$

We'll see that for each (finite, or compact Lie) group G, there is an *Extended TQFT*, namely a (monoidal) 2-functor:

$$Z_G : \mathbf{nCob_2} \to \mathbf{2Vect}$$



Cobordisms of cobordisms form a 2-category $\mathbf{nCob}_2$:



Definition

A **2-Vector space** is a $\mathbb{C}$-linear abelian category generated by simple elements. A 2-linear map is an exact $\mathbb{C}$-linear functor.

Finite-dimensional 2-vector spaces are all equivalent to $\mathbf{Vect}^k$. 2-linear maps then look like:

$$\begin{pmatrix} V_{1,1} & \cdots & V_{1,k} \\ \vdots & & \vdots \\ V_{l,1} & \cdots & V_{l,k} \end{pmatrix} \begin{pmatrix} W_1 \\ \vdots \\ W_k \end{pmatrix} = \begin{pmatrix} \bigoplus_{i=1}^k V_{1,i} \otimes W_i \\ \vdots \\ \bigoplus_{i=1}^k V_{l,i} \otimes W_i \end{pmatrix}$$

There are also *natural transformations* between 2-linear maps, which look like matrices with components $\alpha_{i,j} : V_{i,j} \to V'_{i,j}$.

A *groupoid* is a category in which all morphisms are invertible (a "many-object group", as a category is a "many-object monoid"). In a *Lie groupoid*, Ob and Mor $= \cup_{x,y} hom(x,y)$ are manifolds (and source, target, identity maps are surjective submersions).
If X is a set, and a group G acts on X, there is an *action groupoid* $X /\!\!/ G$ with:
• Objects: elements of X
• Morphisms: triples (x, g, y) where $gx = y$ This groupoid, up to equivalence of groupoids, represents a *quotient stack*.

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Two interesting moduli spaces:
• connections on a manifold M: $\mathcal{A}(M)$
• *flat* connections on M: $\mathcal{A}_0(M)$
Both are acted on by gauge transformations. We will mostly consider:

$$\mathcal{A}_0(M)/\!/\mathcal{G}$$

$\Pi_1(M)$ has objects $x \in M$ and morphisms homotopy classes of paths.
The groupoid of *flat* connections is equivalent to the functor category:

$$\mathcal{A}_0(B) = \mathrm{Fun}(\Pi_1(B), G)$$

(Gauge transformations are natural transformations between these functors).

For example, if $B = S^1$, $\Pi_1(S^1) \simeq \mathbb{Z}$. A G-connection g is specified by the holonomy $g(1) \in G$. A natural transformation from g to g' is given by $h \in G$, such that $g' = hgh^{-1}$. So then:

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is equivalent to the groupoid with:

- Objects: conjugacy classes $[g]$ of G
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Lemma

If **X** *is a groupoid, the functor category* $Rep(\mathbf{X}) = [\mathbf{X}, \mathbf{Vect}]$ *is a 2-vector space.*

Later on, *2-Hilbert space* structure will come from a "measure" on $\underline{\mathbf{X}}$, given using *groupoid cardinality*

$$|\mathbf{X}| = \sum_{[x]} \frac{1}{|Aut(x)|}$$

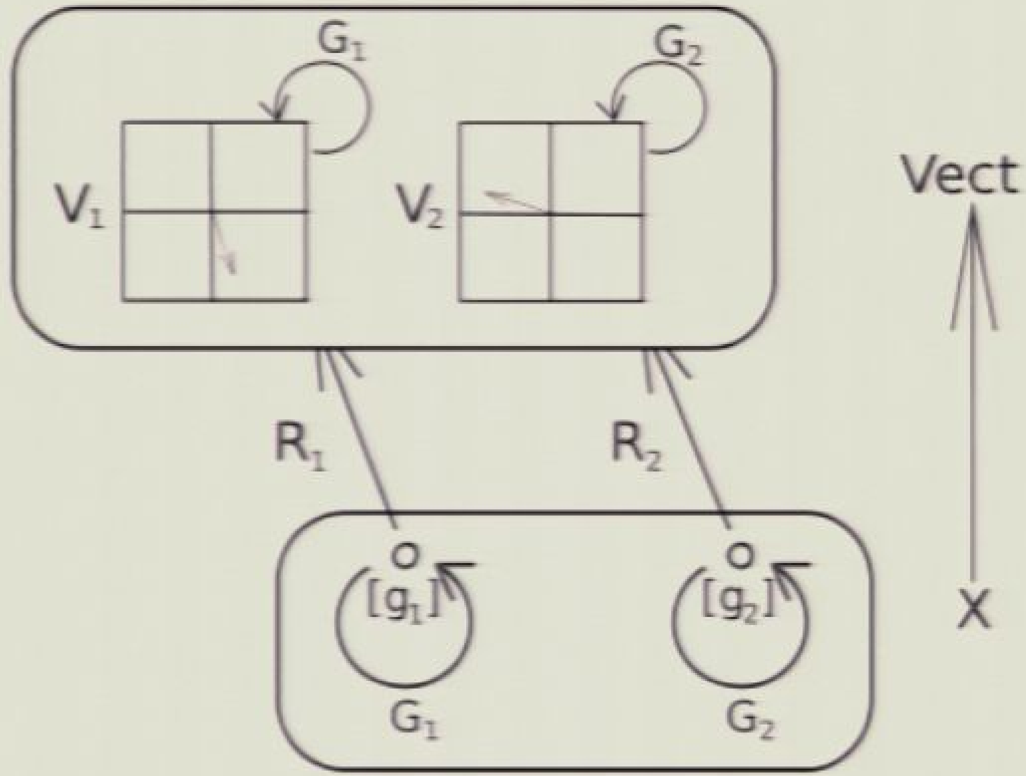
or the analog for differentiable stacks (Weinstein) from the "volume form":

$$vol(\mathbf{X}) = \int_{\underline{X}} \left(\int_{Aut([x])} d\nu \right)^{-1} d\mu$$

The methods used can also be used to apply to any theory whose *states* and *histories*, and their *symmetries* give moduli stacks of finite total volume. Here, these are connections and gauge transformations.
To build $Z_G : \mathbf{nCob}_2 \rightarrow \mathbf{2Vect}$, use a topological gauge theory with gauge group G (assume G finite, or compact Lie). Flat G-connections on manifolds can be specified by holonomies along paths.
Then the 2-vector space $Z_G(B)$ is:

$$Z_G(B) = Rep(\mathcal{A}_0(B)) = [\mathcal{A}_0(B), \mathbf{Vect}]$$

Suppose $B = S^1$. We get $Z_G(S^1) = [\mathcal{A}(S^1), \mathbf{Vect}] \simeq [G /\!/ G, \mathbf{Vect}]$. This gives a vector space for each $[g] \in G$ and an isomorphism for each conjugacy relation:

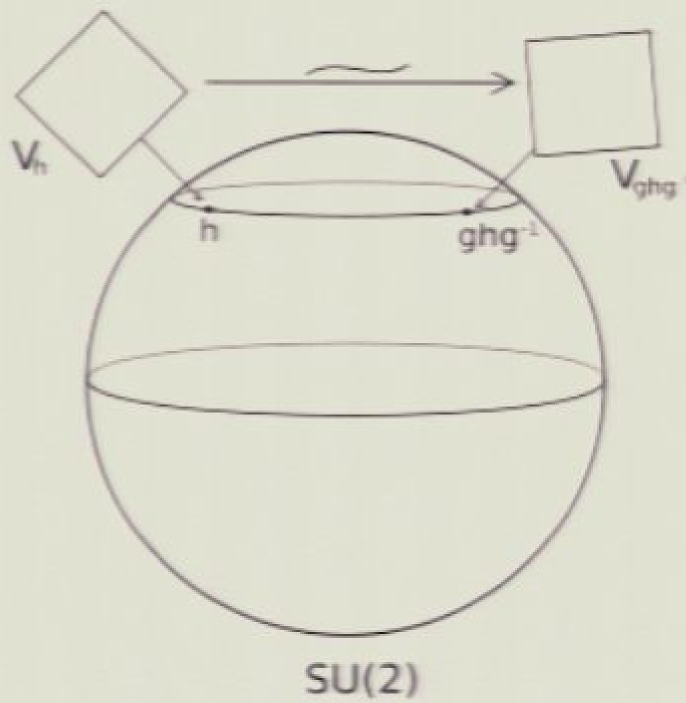


So that

$$Z_G(S^1) \simeq \prod_{[g]} \mathbf{Rep(Aut([g]))}$$

So any 2-vector in this 2-vector space is a direct sum of irreducible

A physically interesting case is $G = SU(2)$. The irreducible (basis) objects of $Z_{SU(2)}(S^1) \simeq [SU(2)/\!/SU(2), \mathbf{Vect}]$ amount to a choice of conjugacy class in $SU(2)$ (i.e. $\phi \in [0, 2\pi]$ and representation of stabilizer subgroup ($U(1)$ if $m \neq 0$, or $SU(2)$ if $m = 0$).



A general object corresponds to some coherent sheaf of vector spaces on $SU(2)/\!/SU(2)$ (i.e. equivariant).

A cobordism between manifolds can be expressed as a diagram:

$$B \xleftarrow{i} S \xrightarrow{i'} B'$$

which gives a diagram of the groupoids of connections:

$$\mathcal{A}_0(B) \xleftarrow{i^\star} \mathcal{A}_0(S) \xrightarrow{(i')^\star} \mathcal{A}_0(B')$$

since both connections and gauge transformations on S can be restricted along the inclusion maps i and i'.

So we have:

$$Z_G(B) \xrightarrow{p^*} [\mathcal{A}_0(S), \mathbf{Vect}] \xleftarrow{(p')^*} Z_G(B')$$

where p^* is the *pullback* 2-linear map, taking $F : \mathcal{A}_0(B) \to \mathbf{Vect}$ to $(F \circ p) : \mathcal{A}_0(S) \to \mathbf{Vect}$. Likewise $(p')^* : Z_G(B') \to [\mathcal{A}_0(S), \mathbf{Vect}]$.
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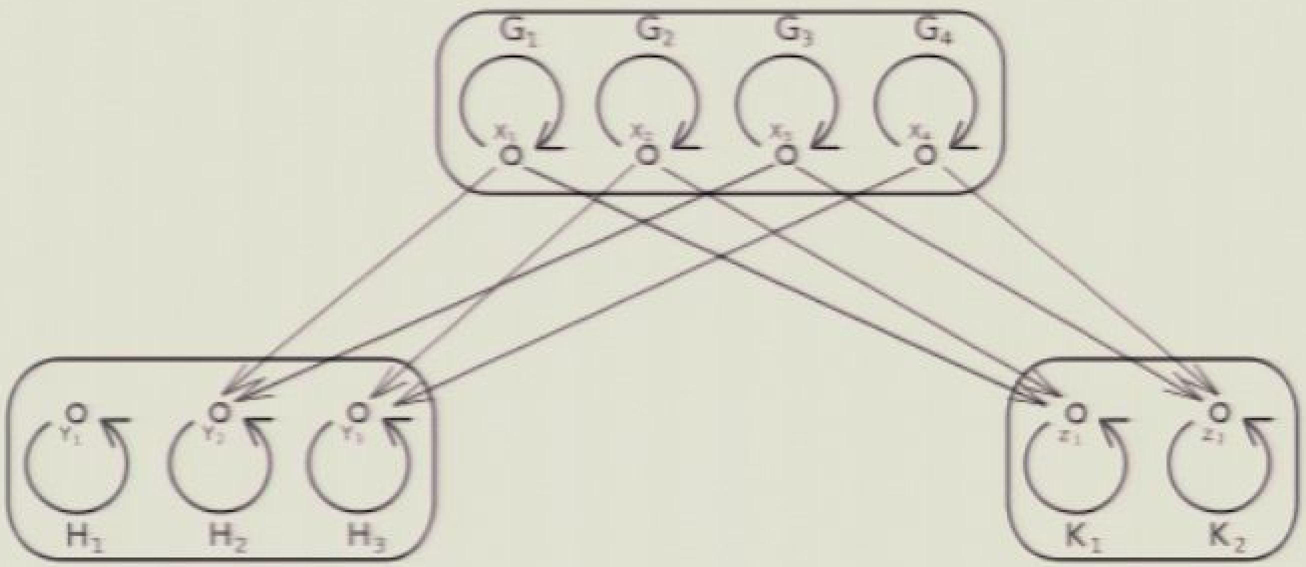

Then picking basis elements $(a, W) \in Z_G(B)$ and $(a', W') \in Z_G(B')$, we get

$$Z_G(S)_{(a,W),(a',W')} = \bigoplus_{[s]} \hom_{Rep(\mathrm{Aut}(s))}[p^*(W), (p')^*(W')]$$

for objects s with $(p, p')(s) = (a, a')$.
(By Schur's lemma, this counts the multiplicity of the irrep W' in $(p')_* \circ p^* W$.)
So the adjoint 2-linear map

$$(p')_* : [\mathcal{A}_0(S), \mathbf{Vect}] \to Z_G(B')$$

pushes forward a 2-vector $p^*F \in Rep(\mathcal{A}_0(S))$ to the *induced representation* in $Rep(\mathcal{A}_0(B'))$.

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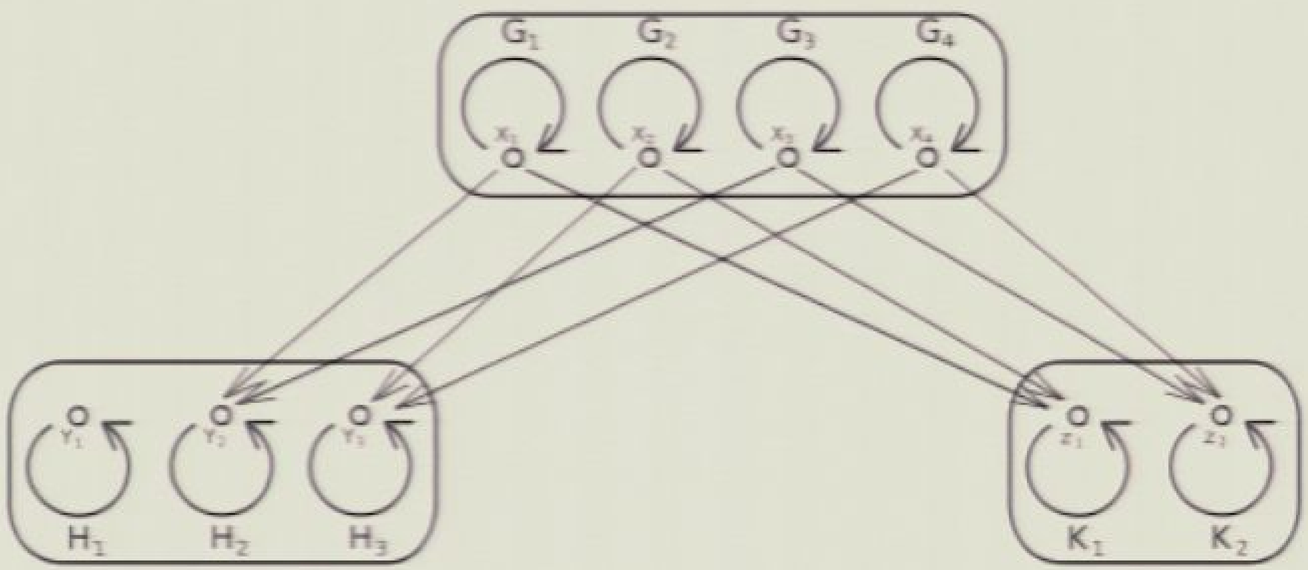

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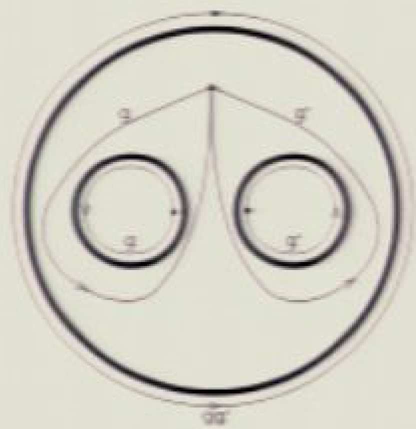

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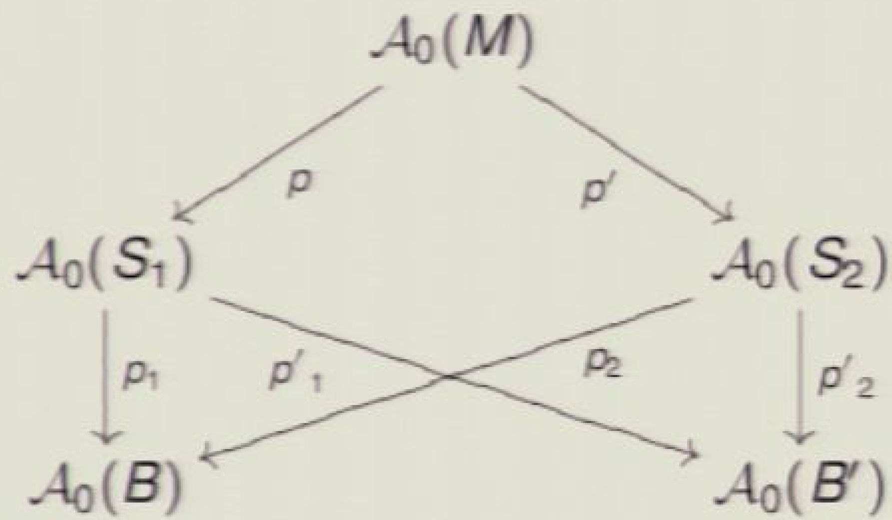
$$\begin{array}{ccc} & (G \times G) /\!\!/ G & \\ {}^{\Delta}\swarrow & & \searrow^{m} \\ (G /\!\!/ G)^2 & & G /\!\!/ G \end{array} \tag{1}$$

$Z_G(Y)$ sends a representation over $([g], [g'])$ to one with nontrivial reps over $[gg']$ for any representatives (g, g').

Given a cobordism with corners between two cobordisms with the same source and target: there is a tower of groupoids:



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has components which are given by:

$$Z_G(M)_{([a],W),([a'],W'),(s_1,s_2)}(f) = |\widehat{(s_1,s_2)}| \sum_{g \in \mathrm{Aut}(s_2)} gfg^{-1}$$

where $\widehat{(s_1,s_2)}$ is a subgroupoid of $\mathcal{A}_0(M)$, the "essential preimage" of (s_1,s_2) under (p,p'), and $|\cdot|$ is the groupoid cardinality (or stack volume).
(This comes from an analogous "pull-push" operation: cf Baez and Dolan, "Groupoidification".)

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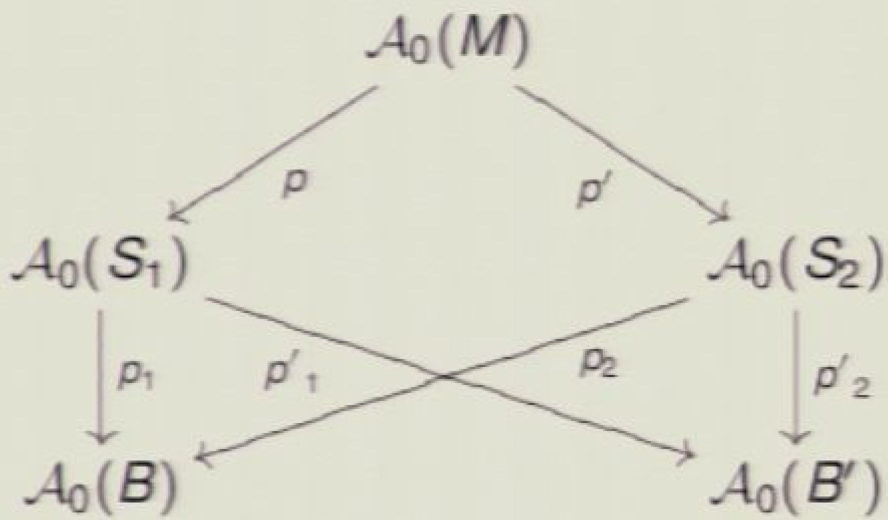

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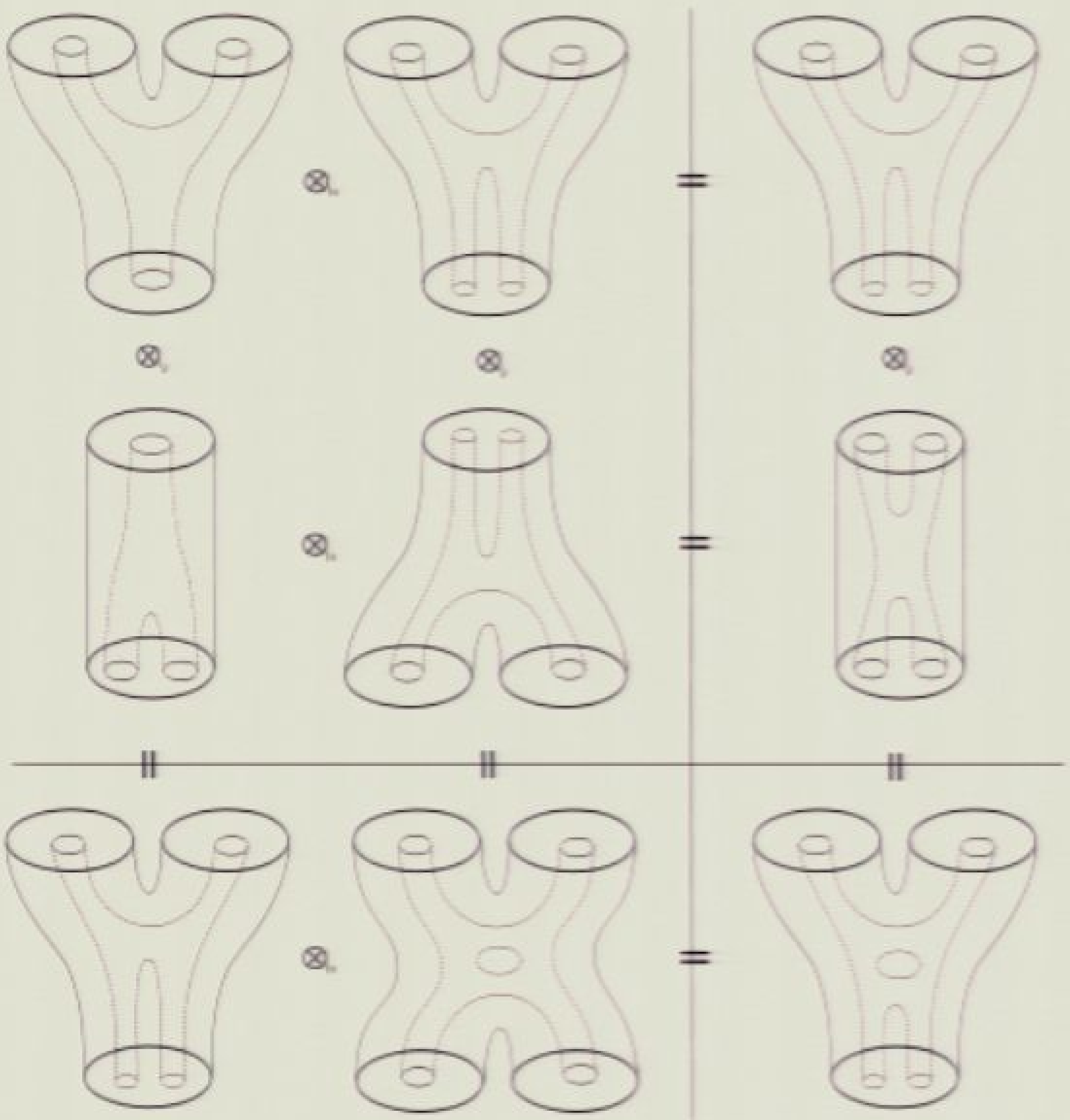

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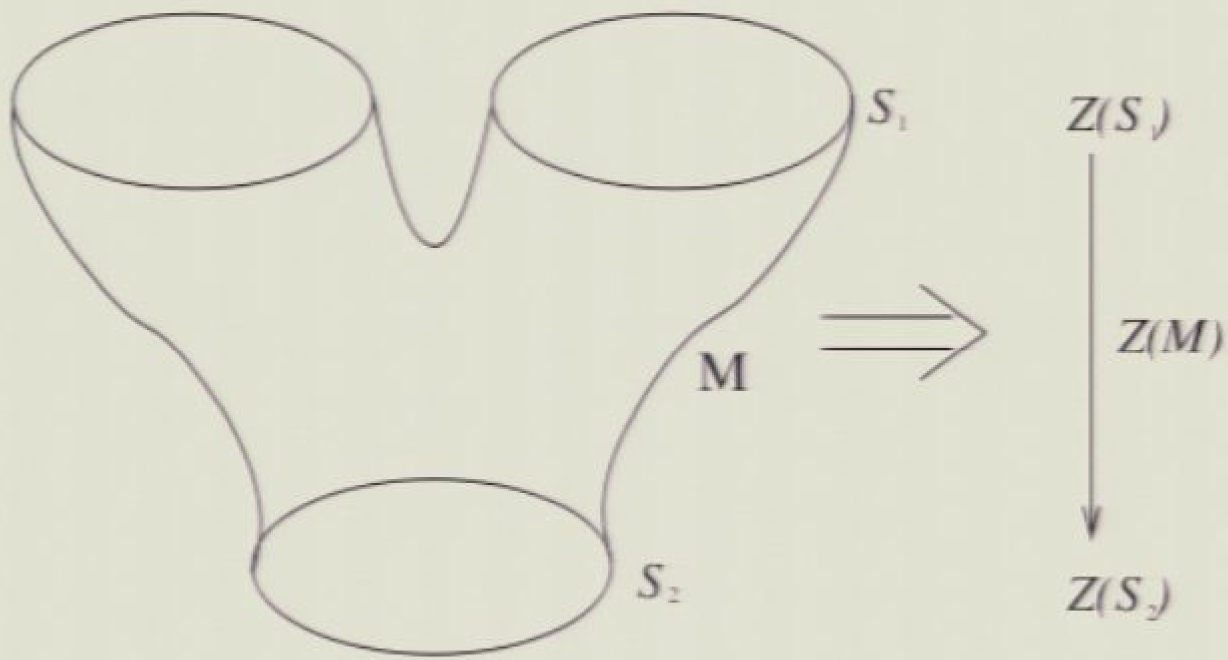

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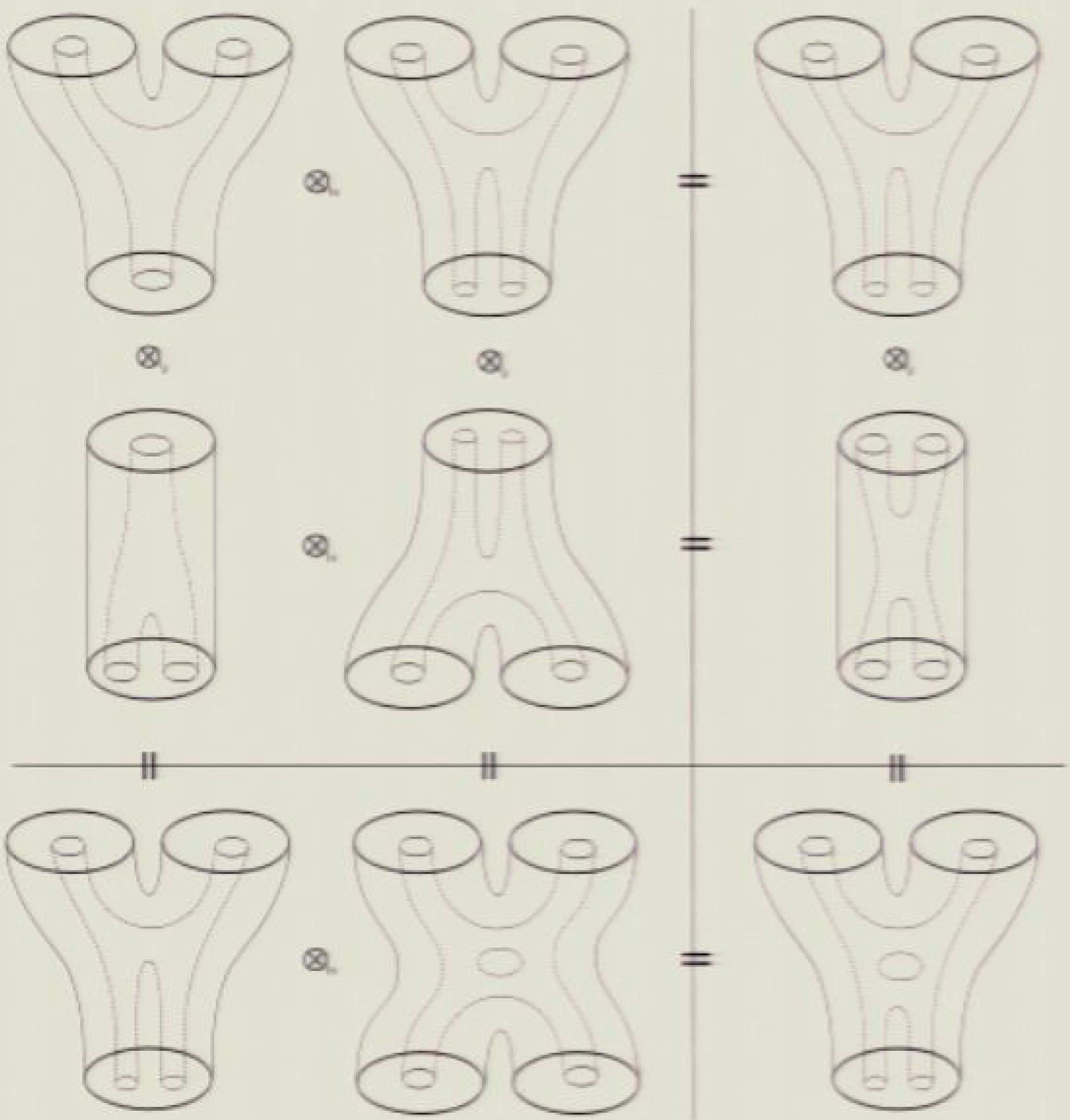

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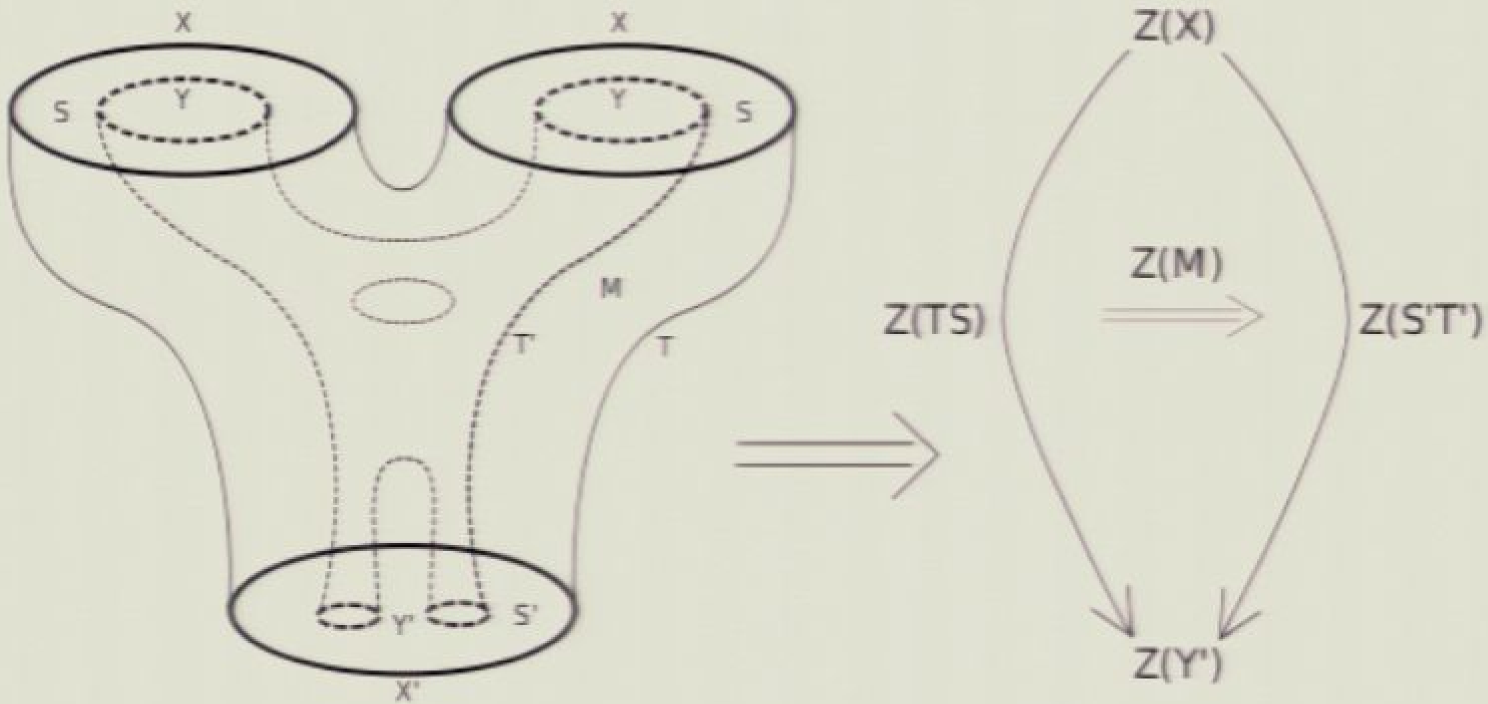

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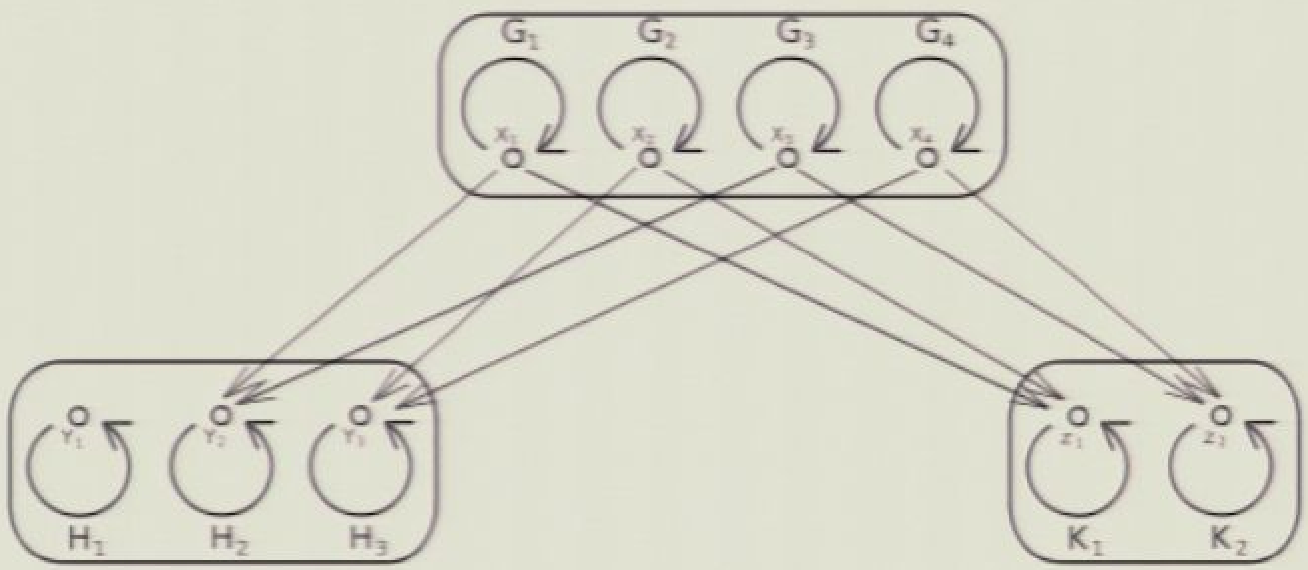

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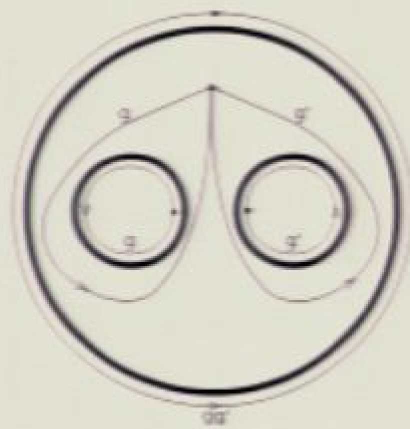

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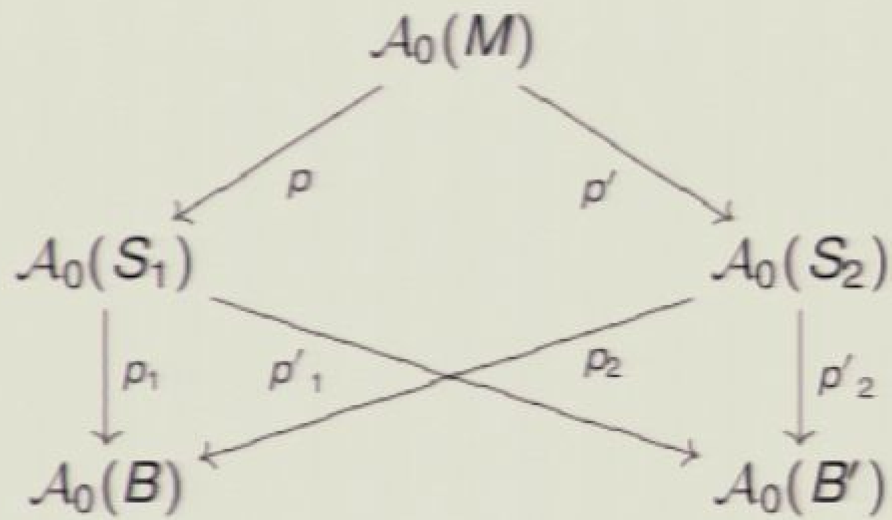

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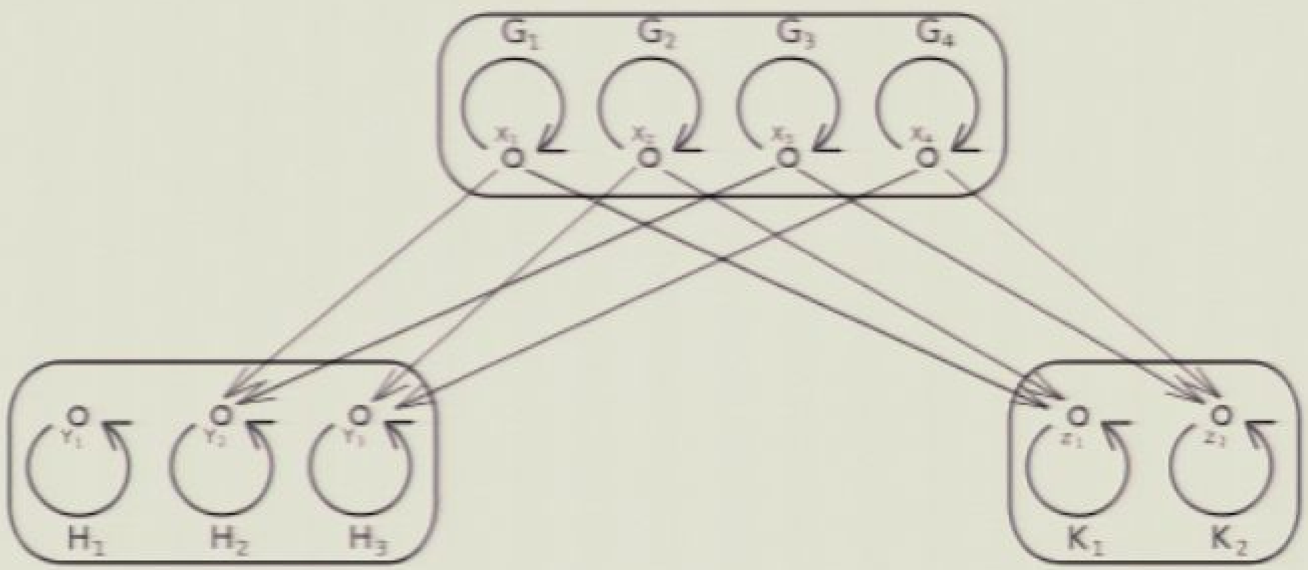

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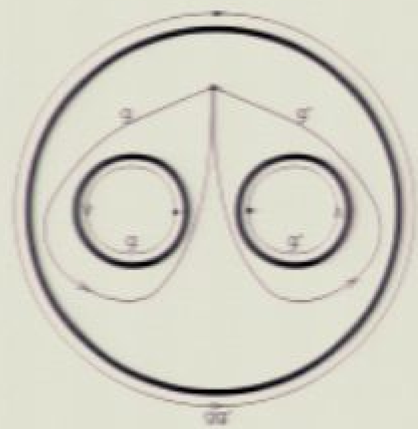

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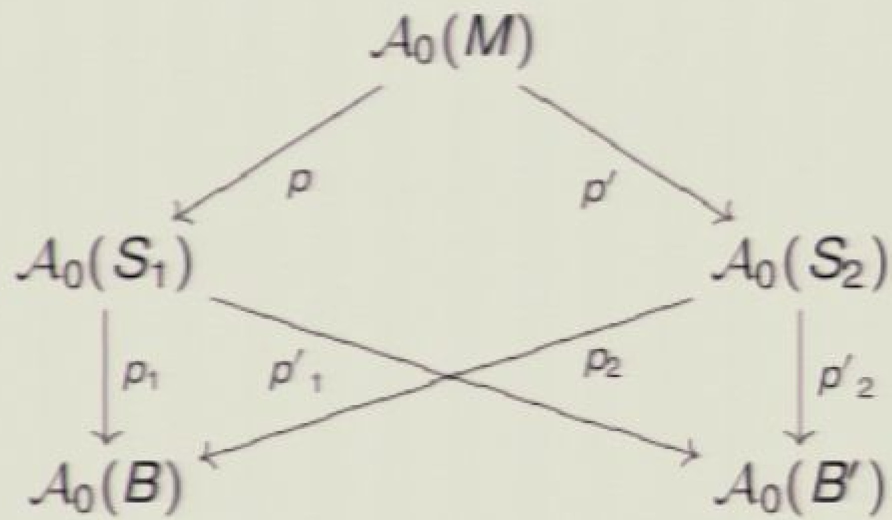

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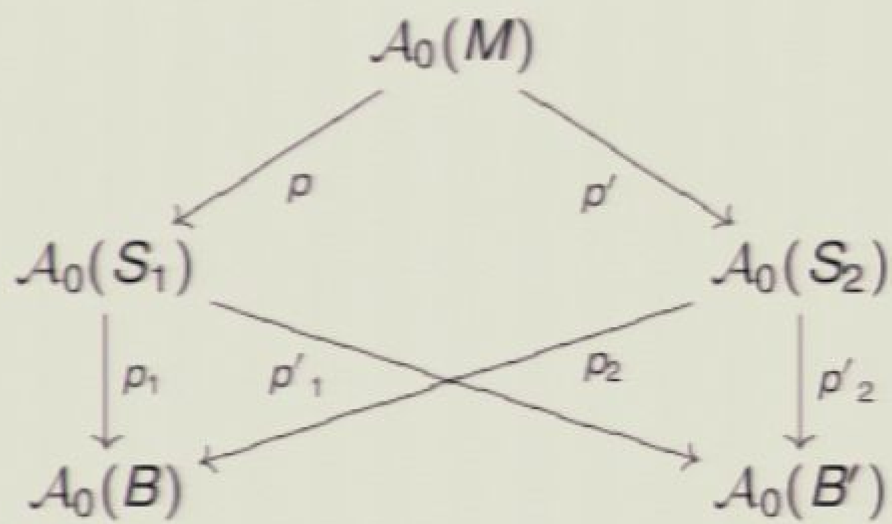

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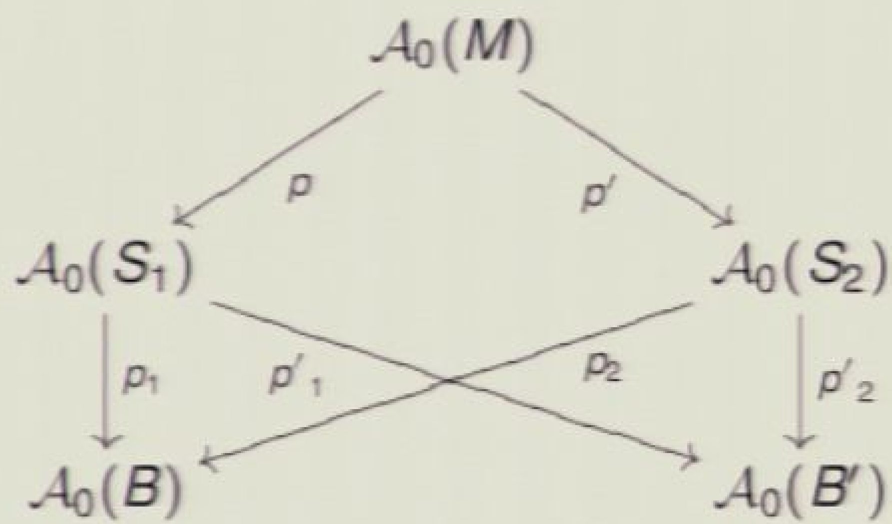

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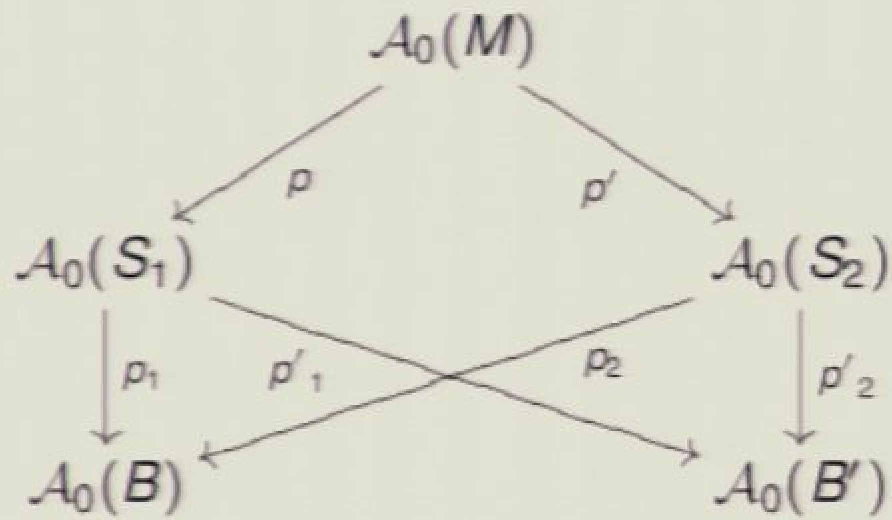

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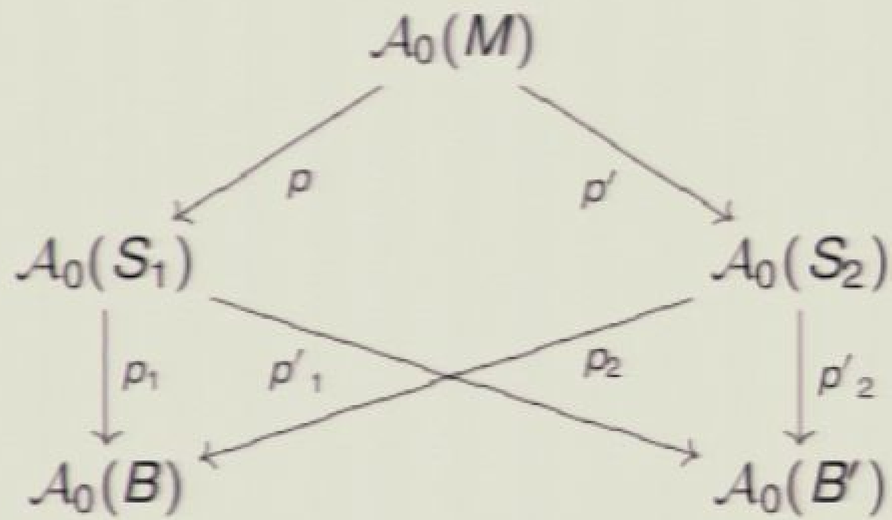

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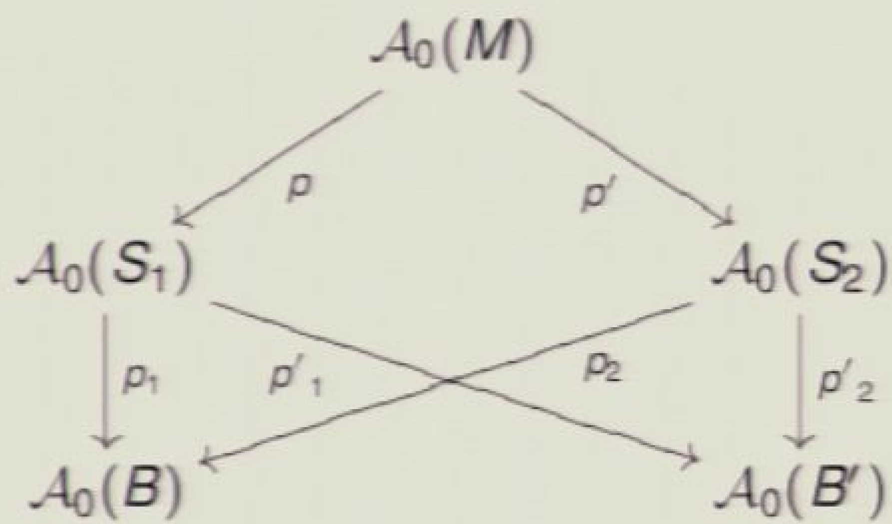

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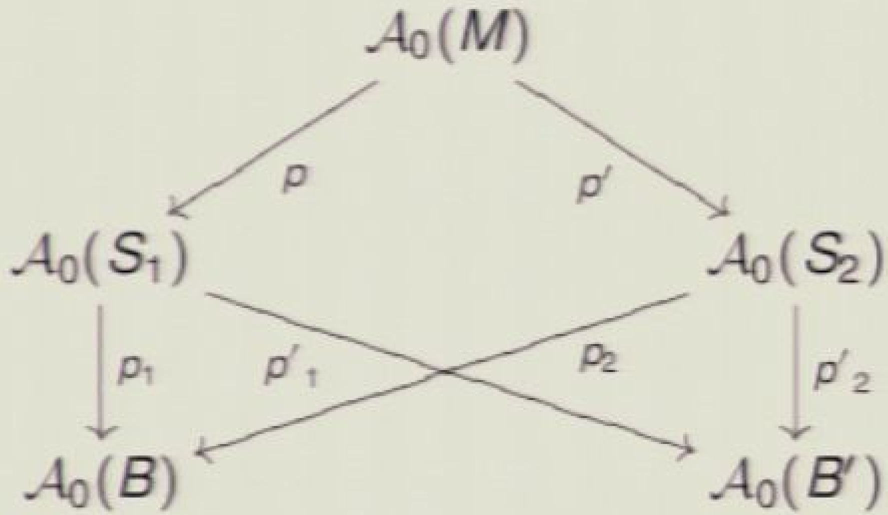

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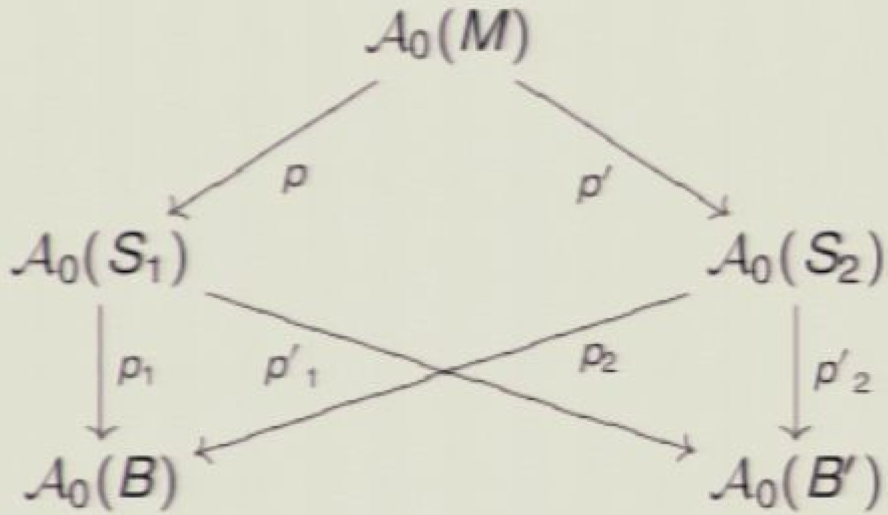

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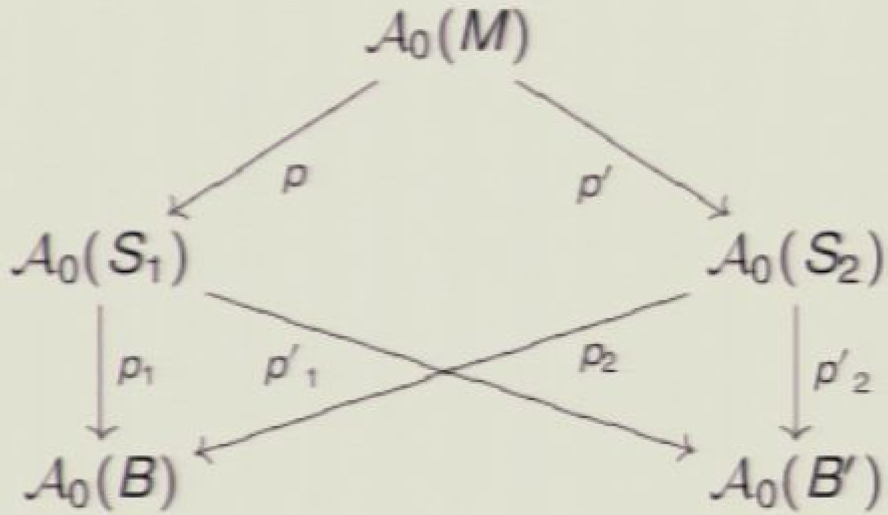

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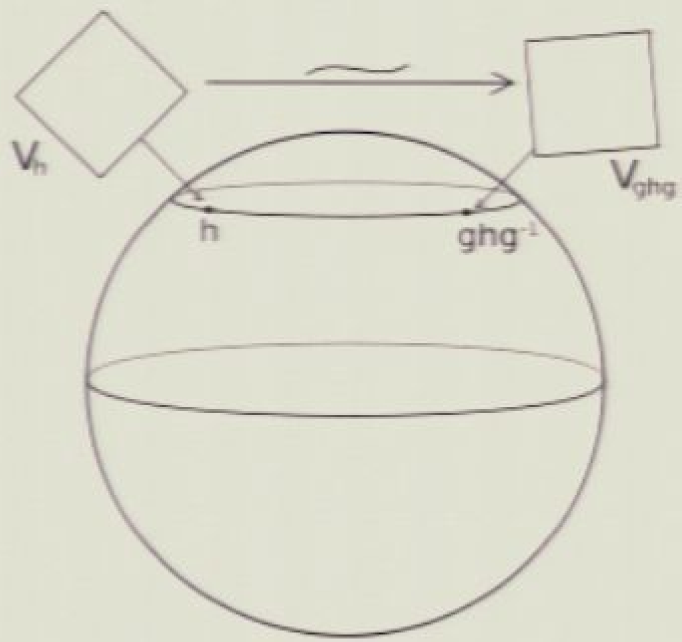
A physically interesting case is $G = SU(2)$. The irreducible (basis) objects of $Z_{SU(2)}(S^1) \simeq [SU(2)/\!/SU(2), \mathbf{Vect}]$ amount to a choice of conjugacy class in $SU(2)$ (i.e. $\phi \in [0, 2\pi]$ and representation of stabilizer subgroup ($U(1)$ if $m \neq 0$, or $SU(2)$ if $m = 0$).



A general object corresponds to some coherent sheaf of vector spaces on $SU(2)/\!/SU(2)$ (i.e. equivariant).

The methods used can also be used to apply to any theory whose *states* and *histories*, and their *symmetries* give moduli stacks of finite total volume. Here, these are connections and gauge transformations.
To build $Z_G : \mathbf{nCob_2} \rightarrow \mathbf{2Vect}$, use a topological gauge theory with gauge group G (assume G finite, or compact Lie). Flat G-connections on manifolds can be specified by holonomies along paths.
Then the 2-vector space $Z_G(B)$ is:

$$Z_G(B) = Rep(\mathcal{A}_0(B)) = [\mathcal{A}_0(B), \mathbf{Vect}]$$

Lemma

If $\mathbf{X}$ *is a groupoid, the functor category* $Rep(\mathbf{X}) = [\mathbf{X}, \mathbf{Vect}]$ *is a 2-vector space.*

Later on, *2-Hilbert space* structure will come from a "measure" on $\underline{\mathbf{X}}$, given using *groupoid cardinality*

$$|\mathbf{X}| = \sum_{[x]} \frac{1}{|Aut(x)|}$$

or the analog for differentiable stacks (Weinstein) from the "volume form":

$$vol(\mathbf{X}) = \int_{\underline{X}} \left(\int_{Aut([x])} d\nu \right)^{-1} d\mu$$

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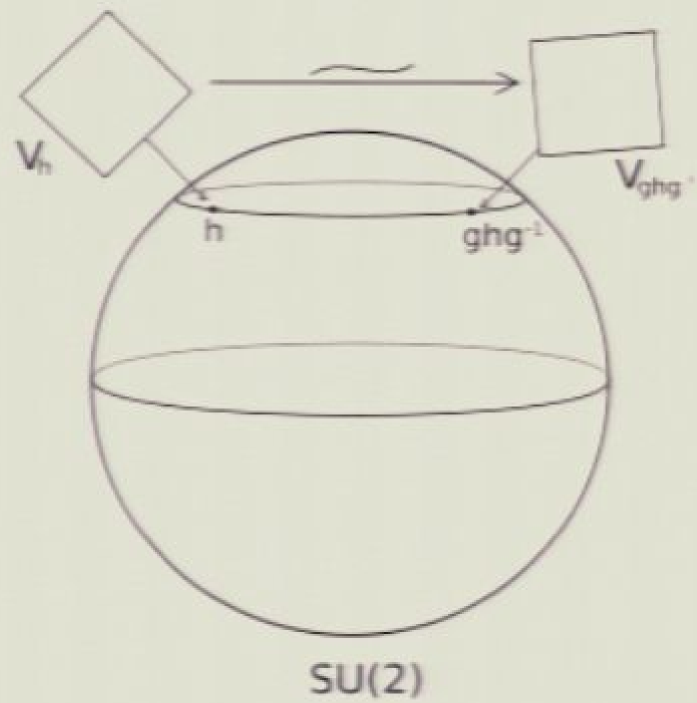

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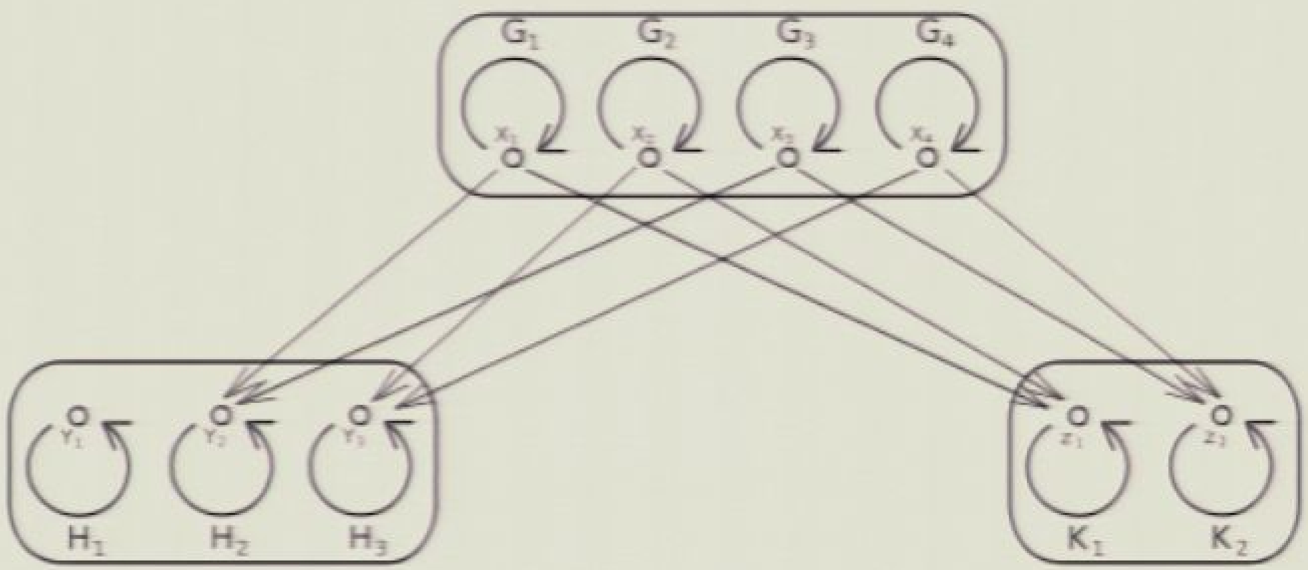
So we have:

$$Z_G(B) \xrightarrow{p^*} [\mathcal{A}_0(S), \mathbf{Vect}] \xleftarrow{(p')^*} Z_G(B')$$

where p^* is the *pullback* 2-linear map, taking $F : \mathcal{A}_0(B) \rightarrow \mathbf{Vect}$ to $(F \circ p) : \mathcal{A}_0(S) \rightarrow \mathbf{Vect}$. Likewise $(p')^* : Z_G(B') \rightarrow [\mathcal{A}_0(S), \mathbf{Vect}]$.
To push a 2-vector in $Z_G(B)$ to one in $Z_G(B')$ involves a (direct) sum over all "histories" - i.e. connections which restrict to a and a', as in this diagram:



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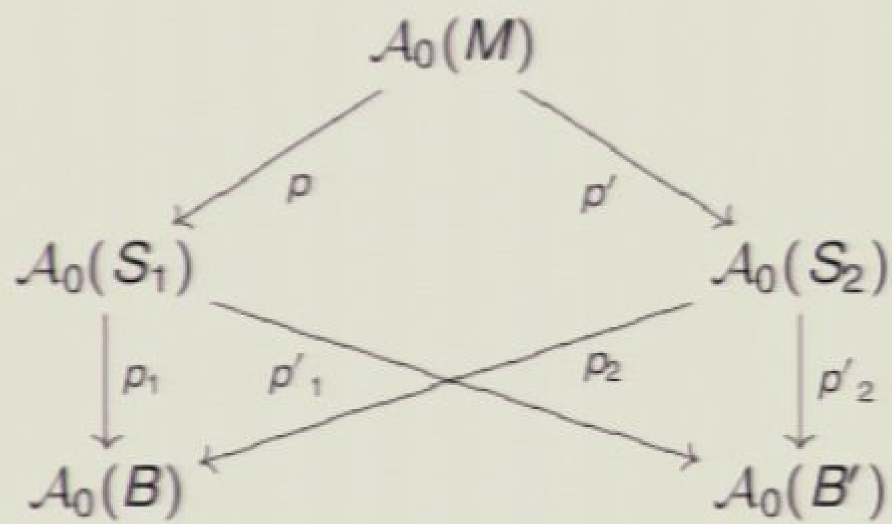

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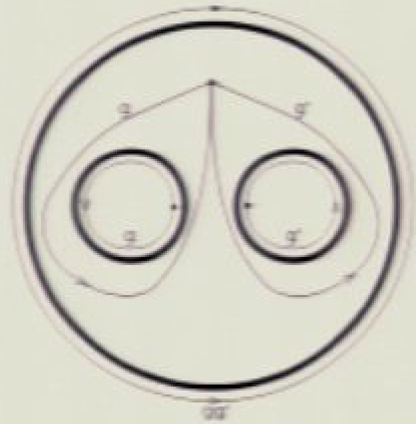
Suppose $Y : S^1 + S^1 \to S^1$ is the "pair of pants":



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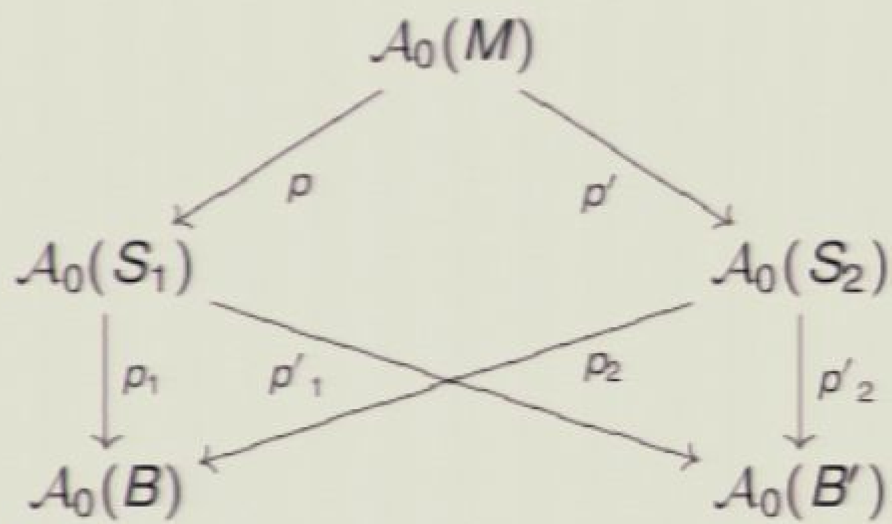

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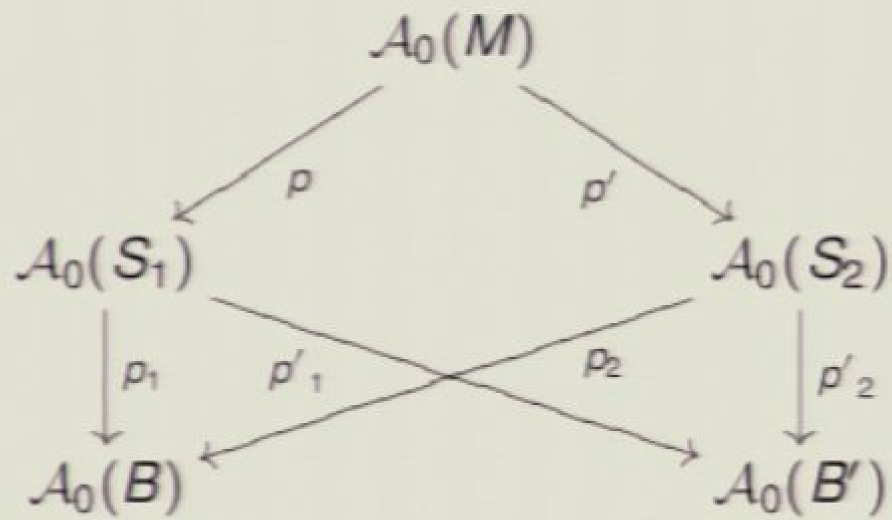

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To build $Z_G : \mathbf{nCob_2} \rightarrow \mathbf{2Vect}$, use a topological gauge theory with gauge group G (assume G finite, or compact Lie). Flat G-connections on manifolds can be specified by holonomies along paths.
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$$Z_G(B) = Rep(\mathcal{A}_0(B)) = [\mathcal{A}_0(B), \mathbf{Vect}]$$

Lemma

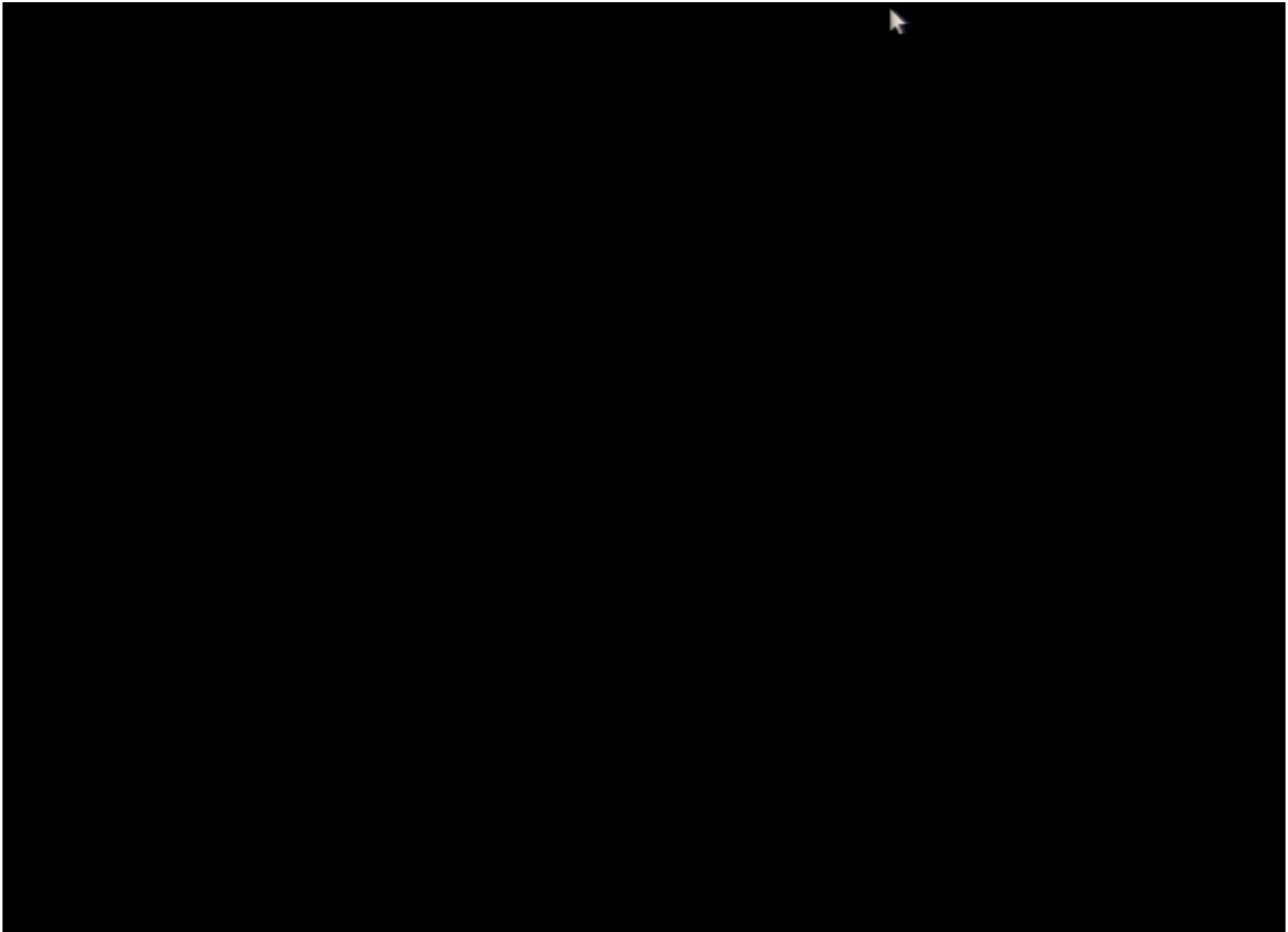
If **X** *is a groupoid, the functor category Rep*(**X**) = [**X**, **Vect**] *is a 2-vector space.*

Later on, *2-Hilbert space* structure will come from a "measure" on $\underline{\mathbf{X}}$, given using *groupoid cardinality*

$$|\mathbf{X}| = \sum_{[x]} \frac{1}{|Aut(x)|}$$

or the analog for differentiable stacks (Weinstein) from the "volume form":

$$vol(\mathbf{X}) = \int_{\underline{X}} \left(\int_{Aut([x])} d\nu \right)^{-1} d\mu$$



The natural transformation

$$Z_G(M)_{([a],W),([a'],W')} : \bigoplus_{[s_1]} \hom_{Rep(\mathrm{Aut}(s_1))}[p_1^*(W), p_2^*(W')] \rightarrow \bigoplus_{[s_2]} \hom_{Rep(\mathrm{Aut}(s_2))}[p'^*_1(W), p'^*_2(W')]$$

has components which are given by:

$$Z_G(M)_{([a],W),([a'],W'),(s_1,s_2)}(f) = |\widehat{(s_1,s_2)}| \sum_{g \in \mathrm{Aut}(s_2)} gfg^{-1}$$

where $\widehat{(s_1,s_2)}$ is a subgroupoid of $\mathcal{A}_0(M)$, the "essential preimage" of (s_1,s_2) under (p,p'), and $|\cdot|$ is the groupoid cardinality (or stack volume).
(This comes from an analogous "pull-push" operation: cf Baez and Dolan, "Groupoidification".)

Theorem

The construction we've just seen gives a 2-functor

$$Z_G : \mathbf{nCob_2} \rightarrow \mathbf{2Vect}$$

(that is, an Extended TQFT).

For physics, we really want **2-Hilbert spaces**: **Hilb**-enriched abelian $\star$-categories with all limits. Generated by simple objects (i.e. ones where $\hom(x,x) \cong \mathbb{C}$.

Typical example: a category of **fields of Hilbert spaces**, ($\mathcal{H}$ on a measure space (X,μ) consists of an X-indexed family of Hilbert spaces $\mathcal{H}_x$ (together with a good space of sections).

Morphisms are (certain) **fields of bounded operators** $\phi : \mathcal{H} \to \mathcal{K}$, with $\phi_x \in \mathcal{B}(\mathcal{H}_x, \mathcal{K}_x)$ preserving good sections.

2-linear maps: $\mathbb{C}$-linear additive $\star$-functors.

$\Phi_{\mathcal{K},\mu}$: **Meas(X)** $\to$ **Meas(Y)** is specified by:

- a field of Hilbert spaces $\mathcal{K}_{(x,y)}$ on $X \times Y$
- item a Y-family $\{\mu_y\}$ of measures on X, where:

$$\Phi_{\mathcal{K},\mu}(\mathcal{H})_y = \int_X^{\oplus} \mathcal{H}_x \otimes \mathcal{K}_{(x,y)} \mathrm{d}\mu_y(x)$$

End of presentation. Press Escape to exit.

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When $B = B' = \emptyset$, so that $\mathcal{A}_0(B) = \mathcal{A}_0(B') = 1$, the terminal groupoid, with $Rep(1) = \mathbf{Vect}$. Then the extended TQFT reduces to a TQFT. For G is a finite group, this theory reproduces the (untwisted) Dijkgraaf-Witten model. If G is compact Lie, this is *BF theory*.
For $B \neq \emptyset$, this describes a TQFT coupled to boundary conditions—"matter". Take the circle as boundary around an excised point particle!
If $G = SU(2)$ and $n = 3$, this depicts particles classified by mass ($m \in [0, 2\pi]$) and spin (unitary group representations) propagating on a background described by 3D quantum gravity (a BF theory in 3D). If $n = 4$, this is a limit of gravity as Newton's $G \to 0$.

End of presentation. Press Escape to exit.