Title: Spaces of Linear Modules on Regular Graded Clifford algebras

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Abstract: The space of regular noncommutative algebras includes regular graded Clifford algebras, which correspond to base point free linear systems of quadrics in dimension n in P^n. The schemes of linear modules for these algebras can be described in terms of this linear system. We show that the space of line modules on a 4 dimensional algebra is an Enriques surface called the Reye congruence, and we extend this result to higher dimensions.

Outline

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Noncommutative Algebras
Familiar Examples

Noncommutative Geometry — Functions do not commute

$$yx \neq xy$$

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$\mathbb{H} = \mathbb{R}\langle i, j\rangle / (i^2 = j^2 = -1, ji = -ij)$ $k = ij$

Matrices with generators and relations

$$\mathbb{C}^{n\times n} \simeq \mathbb{C}\langle x, y\rangle/(x^n = 1, y^n = 1, yx = \zeta xy)$$

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$$x = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 0 & 0 \end{pmatrix} \qquad y = \begin{pmatrix} \zeta & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \zeta^n \end{pmatrix}$$

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$\partial x - x\partial = 1$

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Properties in common with polynomial functions $\mathbb{C}[x, y]$

surface basis $\{x^i y^j\}$

irreducible domains $fg = 0 \Rightarrow f = 0$ or $g = 0$

smooth good homological algebra properties

Sklyanin algebra

Sklyanin algebra Artin, Tate, and Van den Bergh
Noncommutative $\mathbb{CP}^2$

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Problems and Methods New problems and techniques from algebraic geometry applied to noncommutative algebra.

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Classifying modules: Point modules of Sklyanin algebra are given by elliptic curve E.

Problems and Methods: New problems and techniques from algebraic geometry applied to noncommutative algebra.

Applications: Noncommutative algebraic geometry applied to commutative algebraic geometry, physics.

Artin-Schelter Regular Algebras

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2. $\dim A_n \leq cn^\delta$ for all n some c, δ polynomial growth
3. A is Gorenstein

$$\mathrm{Ext}^i_A(k, A) = \begin{cases} k & i = d \\ 0 & \text{else} \end{cases}$$

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A is commutative $\Rightarrow$ polynomial algebra
$\dim A \leq 3$ classified

Classification in Small Dimensions

Dimension 2

$$
\begin{aligned}
A &= k\langle x, y\rangle / xy - qyx \qquad q \in k^* \\
\text{or} \quad A &= \langle x, y\rangle / xy - yx + x^2
\end{aligned}
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Important Invariant - Point Modules
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Important Invariant - Point Modules
Most interesting component: Sklyanin Algebras
Dimension 4 No classification
Many know examples and families
Lu, Palmieri, Wu, Zhang
Shelton, Smith, Vancliff

Regular Clifford Algebras

V vector space dim n
$Q_1, \ldots, Q_n$ quadrics in $\mathbb{P}(V)$

$$Q_i = V(f_i(v) = 0) = V(w_i(v, v) = 0$$

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Let $A^! = k[V^*]/(f_1, \ldots, f_n)$
Assume $\dim_k A^! < \infty$ Complete intersection
$\mathbb{P}(W)$ linear system spanned by Q_i

$$W \hookrightarrow \mathrm{Sym}^2 V^*$$

Properties

Theorem (Bernstein,Gelfand,Gelfand,Buchweitz,Bondal, LeBrun,Orlov,Kuznetsov,Kapranov)

- $A = A^{!!} = \mathrm{Ext}^*_{A^!}(k, k) = k\langle V\rangle/(W + \wedge^2 V)^\perp$
- $D^b(A)$ dual to $D^b(A^!)$
- A is Artin-Schelter Regular
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$$A = k[{}^2 W^*]\langle {}^1 V\rangle$$

$$v_i v_j + v_i v_j = \sum w^i w_i(v_i, v_j)$$

$$w^i \in W^*$$

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  $$v_i v_j + v_i v_j = \sum w^i w_i(v_i, v_j)$$
  $$w^i \in W^*$$
- $\mathcal{A} = \tilde{A}$ sheaf of algebras on $\mathbb{P}(W)$
  $\mathcal{A} = \mathcal{C}l_0(W)$.
- $H_A(t) = \sum \dim A_i t^i = 1/(1-t)^n = H_{\mathbb{P}(V)}(t)$

Examples

Theorem Regular Clifford Algebras are a component in the space of Artin-Schelter Regular algebras of dimension ≥ 4
$\simeq$ open dense subset of $Gr(n, Sym^2 V^*)$.

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Examples

$$\frac{k\langle x, y\rangle}{yx + xy}$$

$$\frac{k\langle x, y, z\rangle}{\begin{pmatrix} axy + ayx + bz^2 \\ ayz + azy + bx^2 \\ azx + axz + bz^2 \end{pmatrix}}$$

$$\frac{k\langle w, x, y, z\rangle}{\text{six generic symmetric relations}}$$

Properties in dim 4

Propoerties of dim 4 regular Clifford algebras

- $Z = Z(\mathcal{A}) \xrightarrow{2} \mathbb{P}(W).$ ramified on

$$\begin{aligned} \Delta_3 &= \text{quartic symmetroid} \\ \Delta_2 &= \text{10 ordinary double points not in general position} \end{aligned}$$

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- at 10 ODPs $\mathcal{A} \simeq \mathcal{C}l_0 \begin{pmatrix} 1 & & \\ & x & y \\ & y & z \end{pmatrix}$ Atiyah Flop Algebra

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  $D^b(\mathcal{A}) \simeq D^b(\text{small resolution})$
- $\mathcal{A}$ is Azumaya outside Δ_2
- Z Fano, unirational not rational (Artin-Mumford)
- Z has no projective small resolution

Linear Modules and Subalgebras

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$$L(A, d) = \text{linear modules of dimension } d$$

$$S(A, d) = \text{linear subalgebras of dimension } d$$

$\dim A = 3$	$L(A, 1) =$ elliptic curve	point modules
$\dim A = 4$	$L(A, 1) = 20$ pts	point modules
	$L(A, 2) =$ projective surface	line modules

Reye Congruence

$$W \subseteq \mathrm{Sym}^2 V^* \qquad \dim V = \dim W = 4$$

web of quadrics in $\mathbb{P}(V)$ base point free (generic)

$$w_1, \dots, w_4 \in V^* \otimes V^*$$

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pairs of points **polar** wrt to quadrics in net
$S \cap \Delta = \emptyset \qquad S$ is K3

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$$\begin{array}{rcl} S & \to & \mathrm{Gr}(2,4) = \mathbb{G}(1,3) \\ (x,y) & \mapsto & \overline{xy} \end{array}$$

Image E is an Enriques Surface.

$L \subseteq \mathbb{P}(V)$ line $L = \overline{xy}$
Q_w quadric with bilinear form w

$$Q_w \supseteq L \iff \begin{aligned} 0 &= w(x, x) \\ &= w(x, y) \\ &= w(y, y) \end{aligned}$$

3 linear conditions on Q_w

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3 linear conditions on Q_w
Generic $L \subseteq \mathbb{P}(V)$ is in $Q_w \in \mathbb{P}(W)$ for point w

$$E' = \{L \in \mathrm{Gr}(2,V) : \{Q \in \mathbb{P}(W) : Q \supseteq L\} \text{ is a line in } W\}$$

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Theorem A graded Clifford Algebra

$$L(A, d) = \{(L, U) \in \mathrm{Gr}(n-d, V) \times \mathrm{Gr}(d, w) : L \subseteq Q \; \forall Q \in U\}$$

$$= S(A, d) \qquad A/AL \leftrightarrow k\langle L\rangle$$

Consequences

Corollary

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$$n \leq \binom{n-2d+1}{2}$$

- $B \in S(A, d)$ then $\exists$ map pullback by projection

$$L(B, k) \to L(A, k+d)$$

Toric Example

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Δ_3 coordinate tetrahedron
Δ_2 6 coordinate lines
K3 surface $S \subseteq \mathbb{P}^3 \times \mathbb{P}^3$

$$\left.\begin{array}{l} 8\ \mathbb{P}^2 s\ \Delta \\ 6\ \mathbb{P}^1 \times \mathbb{P}^1 s\ \square \end{array}\right\} \text{arranged in cuboctohedron}$$

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Questions

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- When does $A^!$ have noncommutative deformations?

$$\dim \operatorname{Ext}^2_{A^! \otimes A^!}(A^!, A^!)_0 > \dim \operatorname{cok} \ \text{Jacobian}$$

- Classify Artin-Schelter Regular Algebras
- Identify other components

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