Title: Hamiltonian structure of isomonodromic deformations of rational connections on the Riemann sphere

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Abstract: The classical "split" rational R-matrix Poisson bracket structure on the space of rational connections over the Riemann sphere provides a natural setting for studying deformations. It can be shown that a natural set of Poisson commuting spectral invariant Hamiltonians, which are dual to the Casimir invariants of the Poisson structure, generate all deformations which, when viewed as nonautonomous Hamiltonian systems, preserve the generalized monodromy of the connections, in the sense of Birkhoff (i.e., the monodromy representation, the Stokes parameters and connection matrices). These spectral invariants may be expressed as residues of the trace invariants of the connection over the spectral curve. Applications include the deformation equations for orthogonal polynomials having "semi-classical" measures. The τ function for such isomonodromic deformations coincides with the Hankel determinant formed from the moments, and is interpretable as a generalized matrix model integral. They are also related to Seiberg-Witten invariants. (This talk is based in part on joint work with: Marco Bertola, Gabor Pusztai and Jacques Hurtubise)

Hamiltonian structure of rational isomonodromic deformations on the Riemann sphere*

J. Harnad

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Université de Montréal
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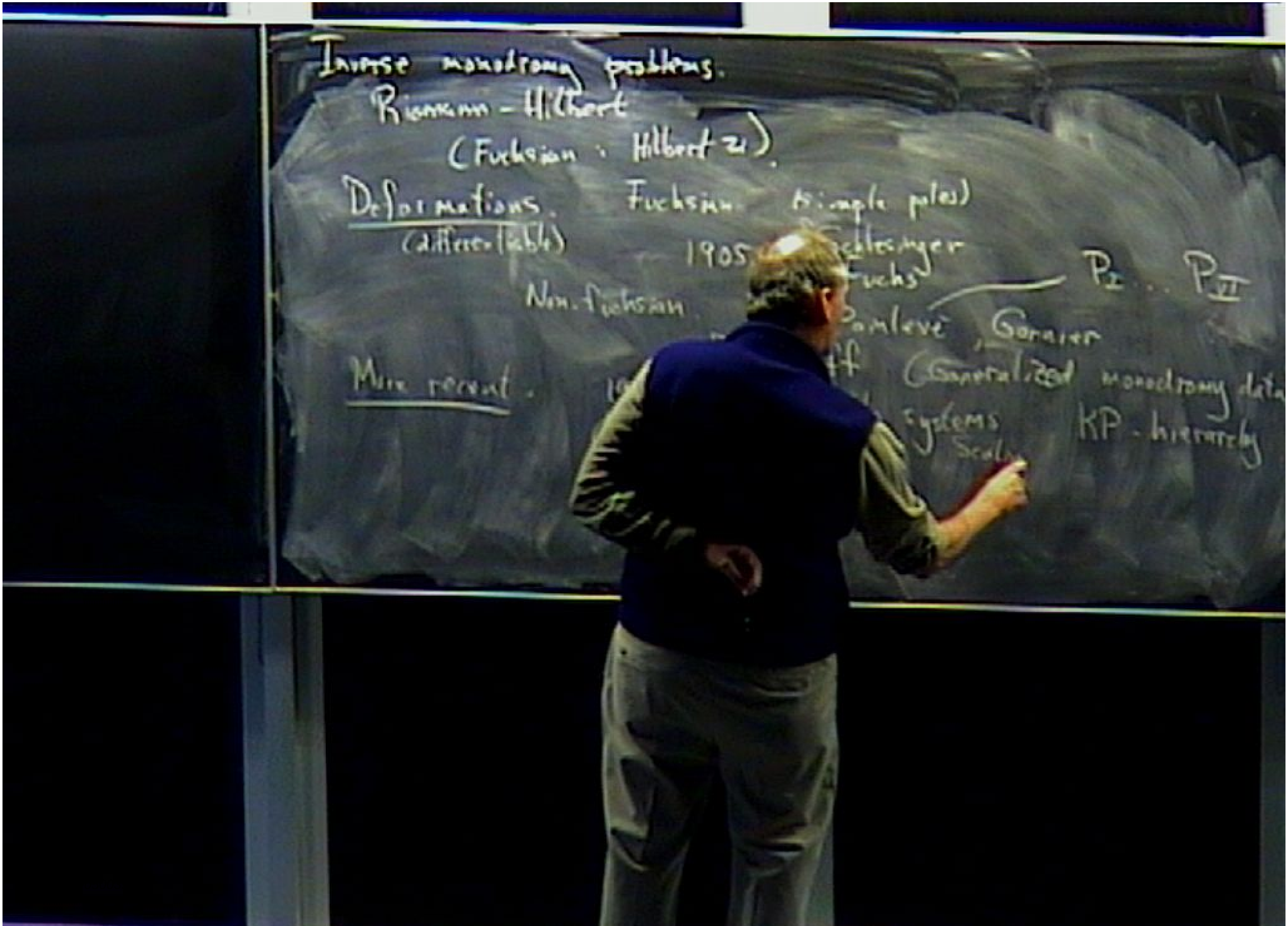
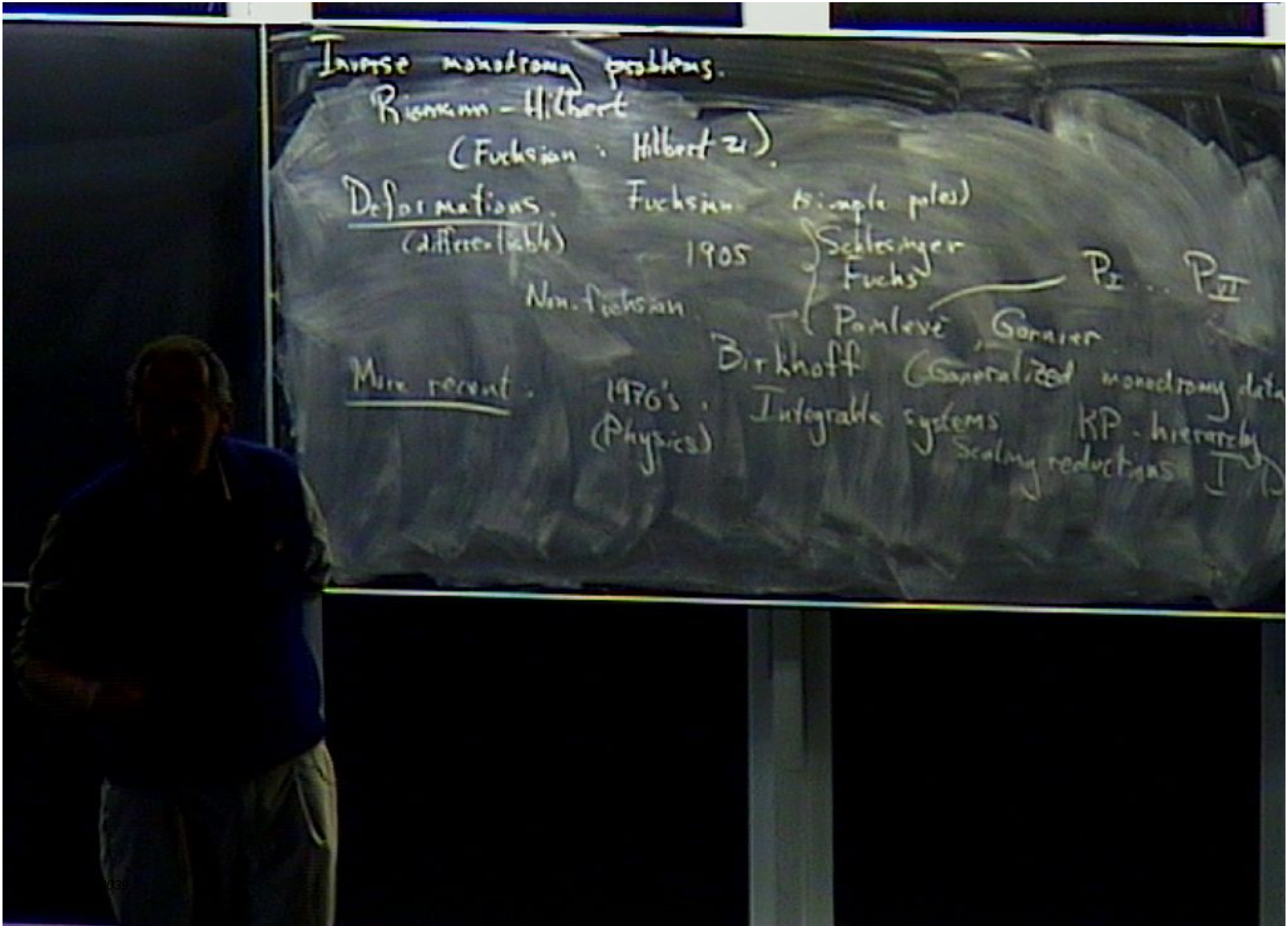
Connections in Geometry and Physics
Perimeter Institute, Waterloo, Ont. , May 8-10, 2009

(* Based on joint work with M. Bertola, J. Hurtubise and G. Pusztai)


Inverse monodromy problems.
Riemann - Hilbert
(Fuchsian : Hilbert 21).
Deformations.
(differentiable)
Fuchsian
simple poles)
1905
P_I ... P_VI
Non-fuchsian
Garnier
More recent.
monodromy data
Systems
KP - hierarchy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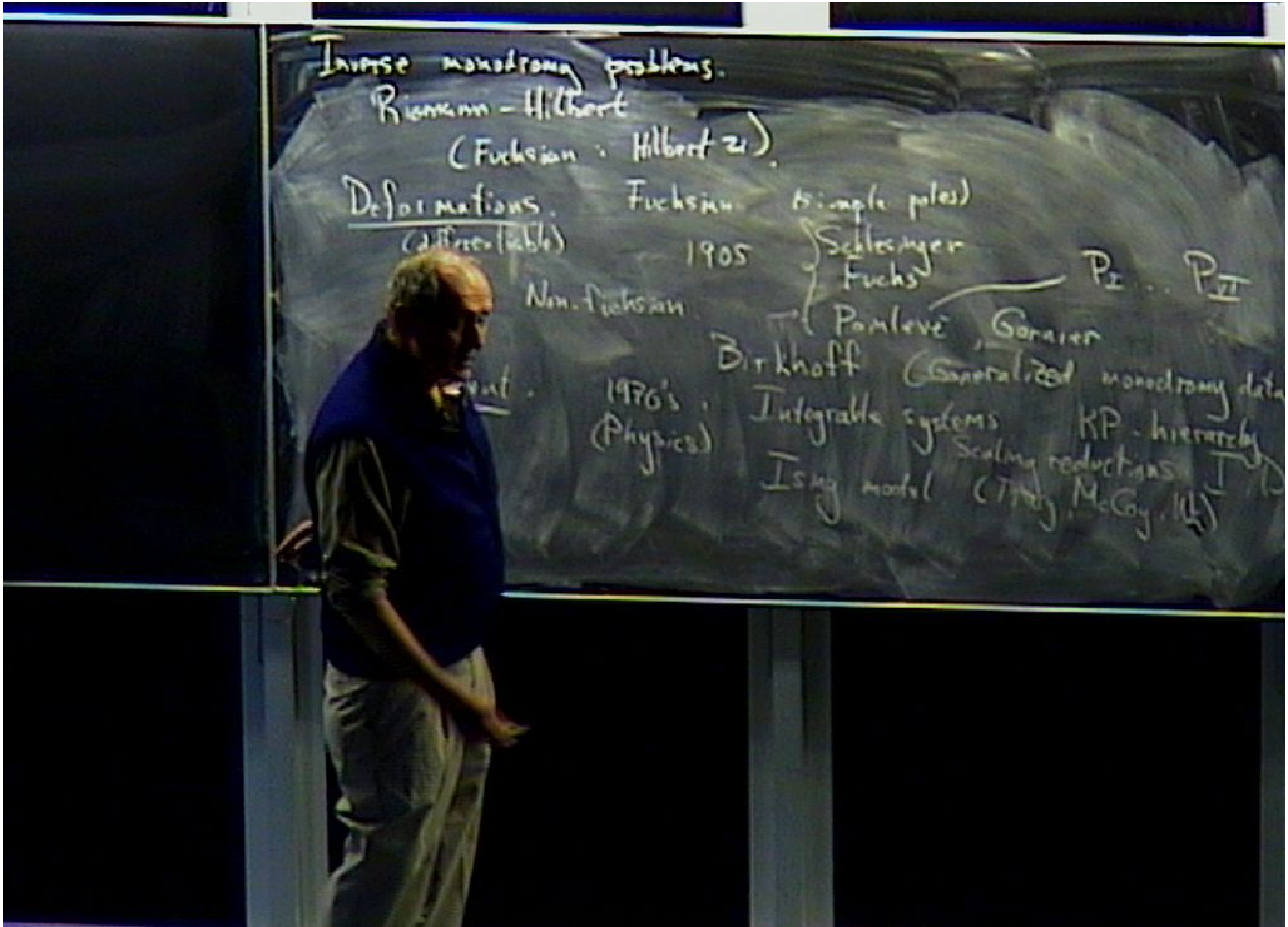
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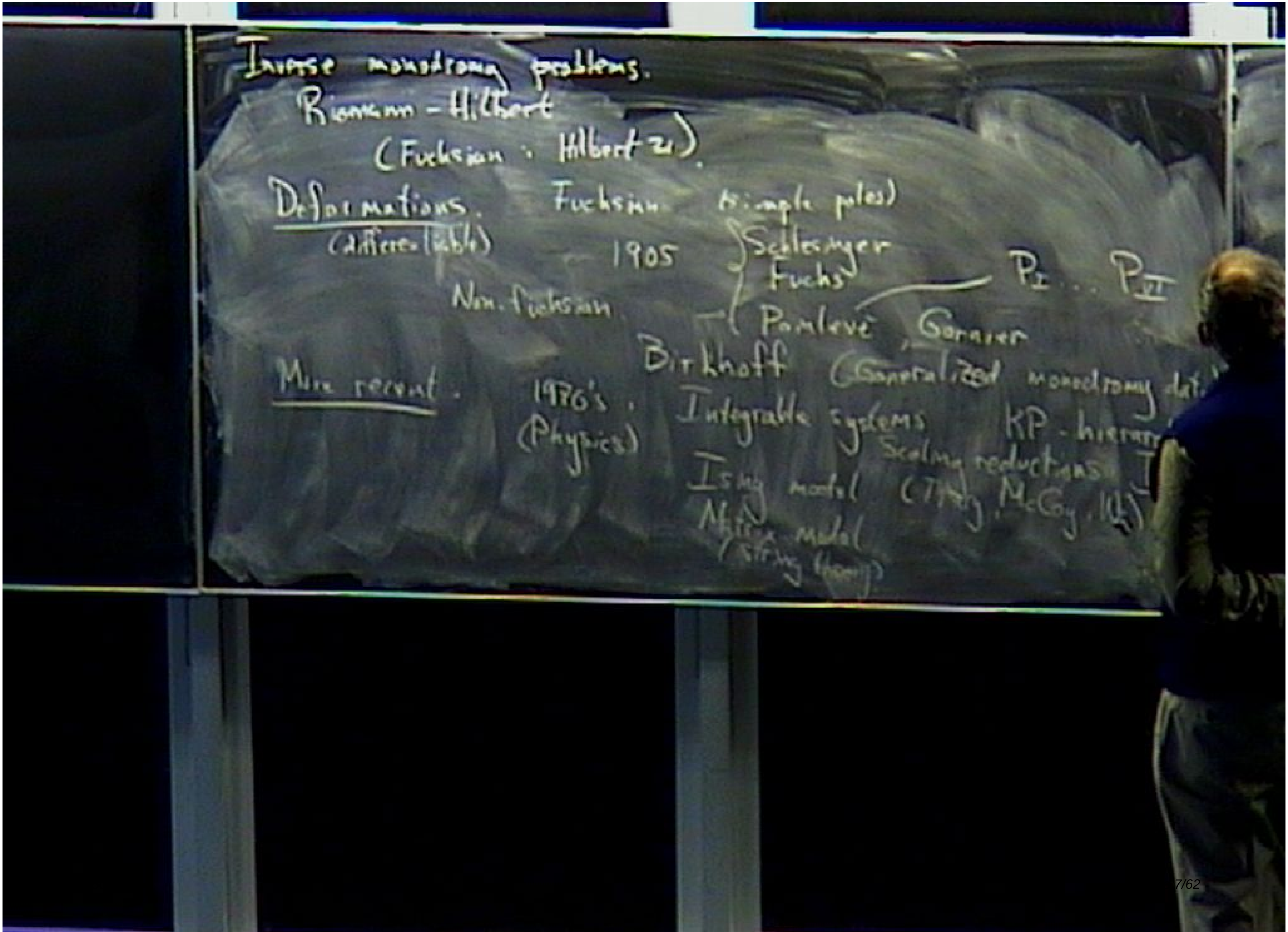
Ising model (Tracy, McCoy, Wu)

Matrix model (String theory)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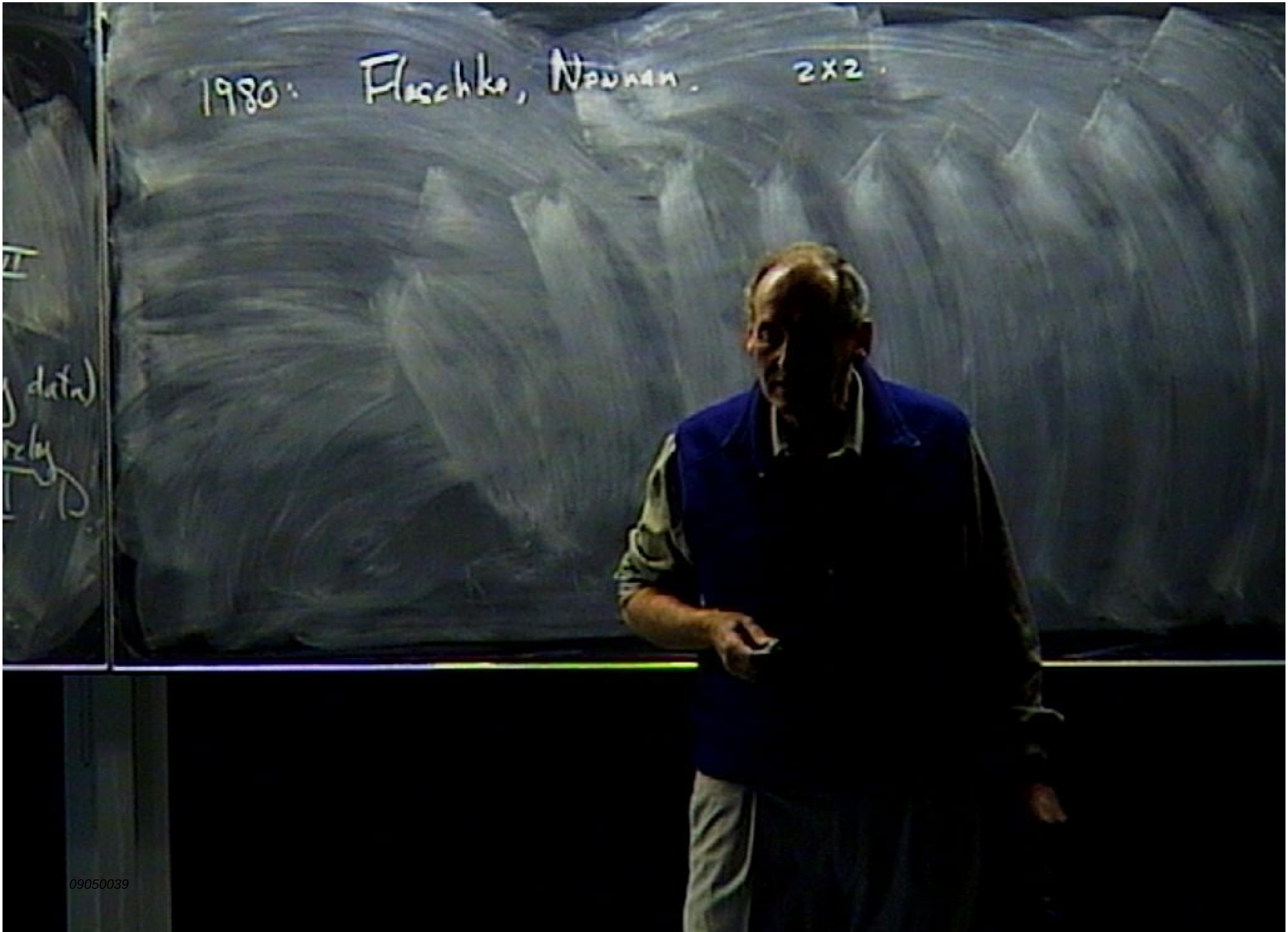

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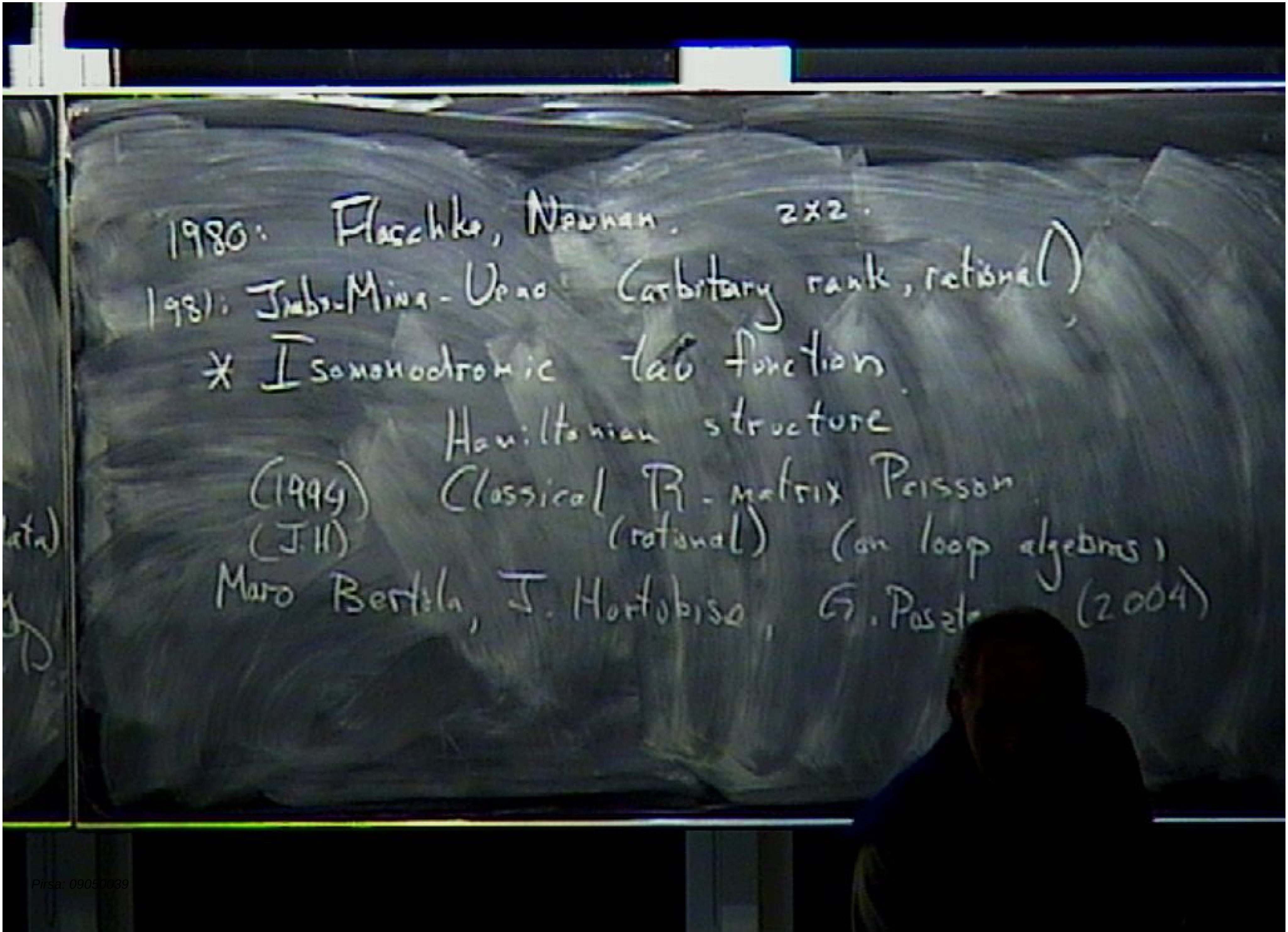
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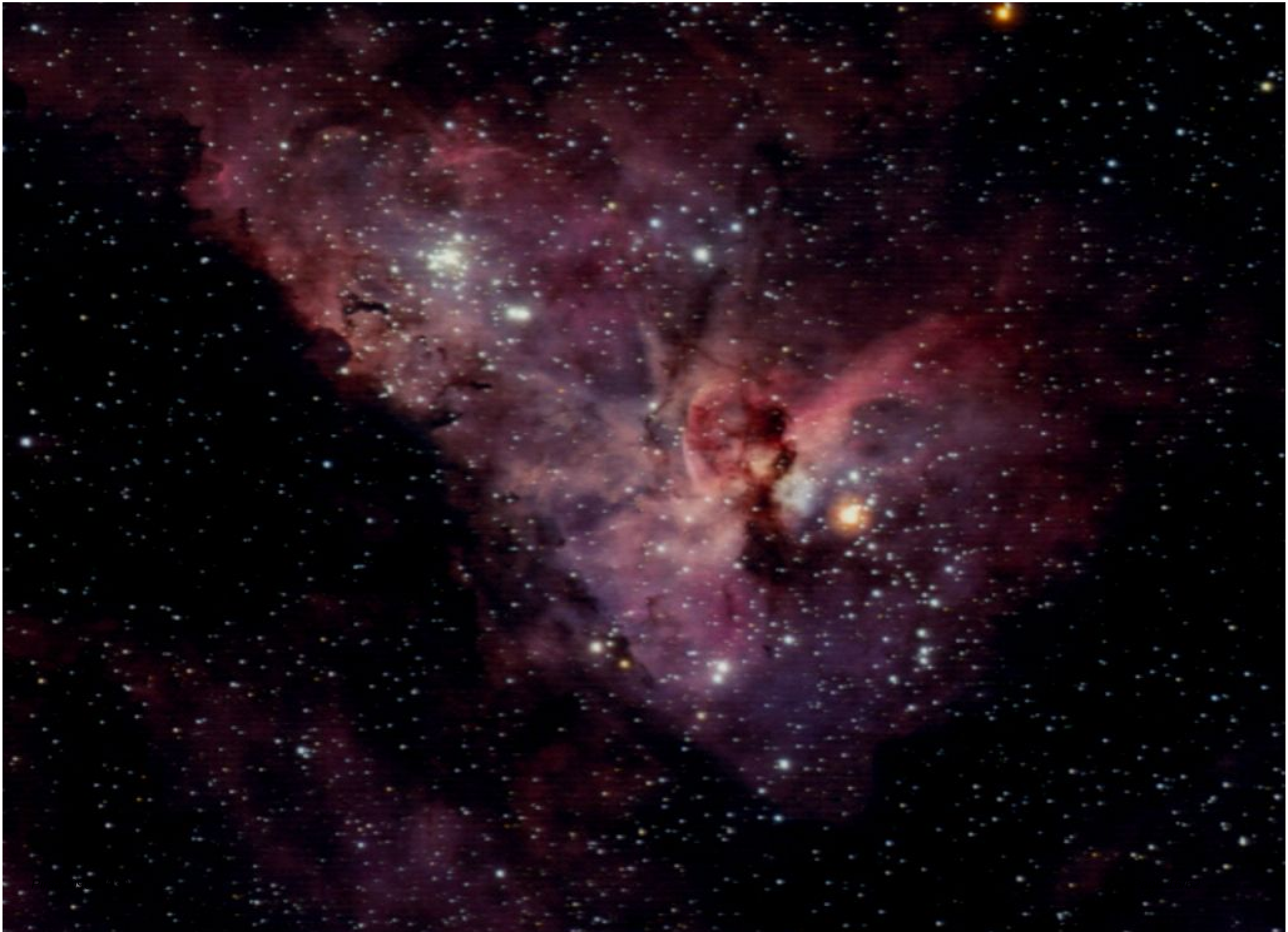

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Outline

1. Generalized isomonodromic deformations over $\mathbf{CP}^1$
2. R-matrix Hamiltonian structure
3. Spectral residue theorems and isomonodromic Hamiltonians
4. Example: Semiclassical orthogonal polynomials

Deformations preserving generalized monodromy

Question: Allowing $\{\alpha_\nu, t^a_{\nu,j}, \}_{\substack{1\le a\le m \\ 1\le j\le d_\nu,\ 0\le\nu\le n}}$ to vary, what are the most general differentiable deformations in A(x) that preserve $\{M_\nu, S_{\nu j}, C_\nu\}$?

Answer: (Schlesinger (1905) - Jimbo, Miwa, Ueno (1981))
The necessary and sufficient conditions are the compatibility of the overdetermined system:

$$\mathcal{D}_x\Psi = 0, \qquad \frac{\partial}{\partial\alpha_\nu}\Psi = \Omega_\nu\Psi, \qquad \frac{\partial}{\partial t^a_{\nu j}}\Psi = \Omega^a_{\nu j}\Psi,$$

where

$$\Omega^a_{\nu j} = \left(\frac{1}{j}(x-\alpha_\nu)^{-j}Y_\nu\frac{\partial T_\nu}{\partial t^a_{\nu j}}Y_\nu^{-1}\right)_{\mathcal{P}_{\alpha_\nu}}, \quad \Omega_\nu = -\left(Y_\nu\frac{\partial T_\nu}{\partial a_\nu}Y_\nu^{-1}\right)_{\mathcal{P}_{\alpha_\nu}}$$

where $\mathcal{P}_{\alpha_\nu}$:= Principal part at $x=\alpha_\nu$

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Monodromy preserving deformations

Overdetermined ifferential system

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$$\widetilde{d}\Psi = \widetilde{\Omega}\Psi$$

$$\widetilde{\Omega} := A(x)dx + \Omega, \quad \widetilde{d} := dx\frac{\partial}{\partial x} + d$$

$$d := \sum_{\nu=0}^{n}\sum_{j=1}^{d_\nu}\sum_{a=1}^{m} dt^a_{\nu j}\frac{\partial}{\partial t^a_{\nu j}} + \sum_{\nu=1}^{n} d\alpha_\nu \frac{\partial}{\partial \alpha_\nu}$$

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Compatibility: zero-curvature condition

$$\widetilde{d}\widetilde{\Omega} + \frac{1}{2}\left[\widetilde{\Omega}, \widetilde{\Omega}\right] = 0$$

Isomonodromic τ-function

Closed 1-form on parameter space

Define a 1-form on the space of deformation parameters $\{\alpha_\nu, t^a_{\nu j}\}$

$$\omega := \sum_{\nu=0}^{n}\left(\sum_{j=1}^{d_\nu}\sum_{a=1}^{m} h^a_{\nu j}dt^a_{\nu j} + h_\nu d\alpha_\nu\right)$$

where

$$h^a_{\nu j} := \operatorname*{res}_{x=\alpha_\nu} \operatorname{tr}\left(Y_\nu^{-1}\frac{\partial Y_\nu}{\partial x}\frac{\partial T_\nu}{\partial t^a_{\nu j}}\right)$$

$$h_\nu := \operatorname*{res}_{x=\alpha_\nu} \operatorname{tr}\left(Y_\nu^{-1}\frac{\partial Y_\nu}{\partial x}\frac{\partial T_\nu}{\partial \alpha_\nu}\right)$$

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Isomonodromic τ-function

The zero curvature equations imply this is a

Closed 1−form (Jimbo-Miwa-Ueno (1981))

$$d\omega = 0.$$

Therefore, there exists, at least locally, a function τ^{IM} determined, up to a parameter independent normalization factor, by:

Isomonodromic τ-function

$$d\ln(\tau^{IM}) = \omega.$$

Actually, τ^{IM} is globally defined, away from collision points of the poles and resonances, as a section of a line bundle, and is holomorphic where defined (Miwa, 1981).

Fuchsian case: Schlesinger equations

Fuchsian case: Schlesinger equations (all $d_\nu = 0,\ d_0 = 0$):

Schlesinger equations (1905)

$$A(x) = \sum_{j=1}^{n} \frac{A_\nu}{x-\alpha_\nu}, \quad \Omega_\nu = -\frac{A_\nu}{x-\alpha_\nu},$$

$$\frac{\partial A_\nu}{\partial \alpha_\nu} = \frac{[A_\nu, A_\mu]}{\alpha_\nu - \alpha_\mu}, \quad \nu \neq \mu,$$

$$A_\infty := -\sum_{\nu=1}^{n} A_\nu = \mathrm{cst}.$$

Logarithmic derivative

$$H_\nu := \frac{\partial \ln(\tau^{IM})}{\partial \alpha_\nu} = \frac{1}{2} \operatorname*{res}_{x=\alpha_\nu} \mathrm{tr}(A^2(x))|_{\mathbf{int}}$$

Simplest case: $m = 2$, $n = 3$, Reduction to Painlevé VI

Setting $m = 2$, $n = 3$, set $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = t$ and eigenvalues

$$\mathrm{eiv}(A_\nu) = \pm\theta_\nu, \quad s\nu = 1, 2, 3, \quad A_\infty = \begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}$$

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$$\begin{aligned} \frac{d^2u}{dt^2} &= \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right)\left(\frac{du}{dt}\right)^2 \\ &\quad - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right)\frac{du}{dt} \\ &+ \frac{u(u-1)(u-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{u^2} + \gamma\frac{t-1}{(u-1)^2} + \delta\frac{t(t-1)}{(u-t)^2}\right), \end{aligned}$$

where

$$\alpha = (\theta_\infty + \frac{1}{2})^2, \quad \beta = -\frac{1}{2}\theta_1^2, \quad \gamma = \frac{1}{2}\theta_2^2, \quad \delta = -\frac{1}{2}\theta_3^2 + \frac{1}{2}.$$

Hamiltonian structure of Schlesinger equations

Total derivative:

$$\widetilde{X}_{H_\nu} = X_{H_\nu} + X^0_{\alpha_\nu},$$

$$\text{where} \quad X^0_{\alpha_\nu} := \frac{\partial^0}{\partial \alpha_\nu} = \quad \text{"explicit derivative"}$$

Since

$$X^0_{\alpha_\nu}(A) = -\frac{A_\nu}{(x-\alpha_\nu)^2} = \frac{\partial \Omega_\nu}{\partial x},$$

Hamilton's equations ≡ Schlesinger ≡ zero curvature:

$$\frac{\partial A}{\partial \alpha_\nu} = \{A, H_\nu\} + X^0_{\alpha_\nu}(A) = [\Omega_\nu, A] + \frac{\partial \Omega_\nu}{\partial x}$$

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Commutative Hamiltonians (non-autonomous):
"time variables" = location of poles $\{\alpha_\nu\}$

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Connection form:

$$\Omega = \sum_{\nu=1}^{n} \Omega_\nu d\alpha_\nu + \sum_{a=1}^{m} \Omega_0^a dt_a,$$

$$\Omega_\nu = -\frac{A_\nu}{x-\alpha_\nu}, \quad \Omega_0^a = xE_a + \sum_{b\neq a}^{m} \frac{E_a A_\infty E_b + E_b A_\infty E_a}{t_a - t_b}$$

Hamilton's equations:

$$\frac{\partial A}{\partial \alpha_\nu} = \{A, H_\nu\} + X^0_{\alpha_\nu}(A) = [\Omega_\nu, A] + \frac{\partial \Omega_\nu}{\partial x}$$

$$\frac{\partial A}{\partial t_a} = \{A, K_a\} + X^0_{t_a}(A) = [\Omega_0^a, A] + \frac{\partial \Omega_0^a}{\partial x}$$

where the explicit derivatives'are:

$$X^0_{\alpha_\nu}(A) = -\frac{A_\nu}{(x-\alpha_\nu)^2} = \frac{\partial \Omega_\nu}{\partial x}, \quad X^0_{t_a}(A) = E_a = \frac{\partial \Omega_0^a}{\partial x}.$$

R-matrix Hamiltonian structure (JH (1994), M. Bertola, J.H., J. Hurtubuse, G. Pusztai (2004)

(Other approaches: P. Boalch **[Bo]** (2001): Goldman **[G]**(1986) structure on (generalized) monodromy data, Poisson Lie groups; N. Woodhouse **[W]**(2002): twistor space approach.)

Rational R-matrix Poisson structure

$$\{A(\xi) \overset{\otimes}{,} A(y)\} := [r(x-y), A(x) \otimes \mathbf{I} + \mathbf{I} \otimes A(y)].$$

where $A(x), A(y) \in \mathrm{End}(\mathbf{C}^{\mathbf{m}})$

$$r(x-y) := \frac{P_{12}}{x-a} \in End(\mathbf{C}^m \otimes \mathbf{C}^m), \quad P_{12}(u \otimes v) := v \otimes u$$

Poisson subspace of rational elements with fixed pole structure:

$$\{A(x)\} \subset (L\mathfrak{sl}(m, \mathbf{C})^*)_R = \mathfrak{sl}_+(m, \mathbf{C}) \ominus \mathfrak{sl}_-(m, \mathbf{C})$$

(Equivalent to $\oplus_{\nu=0}^{n} \left[\mathfrak{sl}^{d_\nu}(m, \mathbf{C})\right]^*$)

Classical split rational R-matrix

$R := \frac{1}{2}(P_+ - P_-)$

$$[X, Y] = [RX, Y] + [X, RY] = [X_+, Y_+] - [X_-, Y_-]$$
$$X_\pm := P_\pm(X), \quad Y_\pm := P_\pm(Y), \quad (P_\pm = \text{projectors})$$

Spectral curve: $\det(y\mathbf{I} - A(x)) = 0$
(m-sheeted branched cover of $\mathbf{P^1}$)

Casimir invariants: $\{t^a_{\nu j}, \alpha_r\}$ (center of Poisson algebra)

Hamiltonian vector field for spectral invariant:

$$H \in \mathcal{I}\,(Lsl(m, C)), \quad X_H(A) = [R\,(\delta(H))\,,\ A]$$

Isomonodromic Hamiltonian

Equations of motion (total derivative)

$$\frac{\partial A}{\partial t} = X_H(A) + X_t^0(A) = [\Omega_t, A] + \frac{\partial \Omega_t}{\partial x}$$

where $\Omega_t := R(\delta(H)) + f(A)$ for some $f(A)$ commuting with A:

$$[A, f(A)] = 0,$$

provided

$$X_t^0(A) \equiv \frac{\partial \Omega_t}{\partial x}.$$

Spectral residue theorem

Theorem 1. Spectral residue theorem

1) Deformation parameters (and local monodromy):

$$t^a_{\nu j} = \operatorname*{res}_{y_a(x=\alpha_\nu)} (x-\alpha_\nu)^j y dx, \quad j = 0, \dots d_\nu$$

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Hamiltonian isomonodromy theorem

Theorem 2. Hamiltonian isomonodromic deformations

$$\frac{\partial A}{\partial t^a_{\nu j}} = [\Omega^a_{\nu j}, \ A] + \frac{\partial \Omega^a_{\nu j}}{\partial x}$$
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Perturbation theory formulae

Linear operator

$$\tilde{A}(\chi) = A^{(0)} + \chi A^{(1)} + \chi^2 A^{(2)} + \cdots$$

Eigenvalues

$$\lambda(\chi) = t + \frac{1}{2\pi i}\oint_{z=t} \mathrm{d}z \sum_{p=1}^{\infty} \frac{1}{p} \mathrm{tr}\left((A(\chi) - A^{(0)})(z\mathbf{I} - A^{(0)})^{-1}\right)^p,$$

Applying this to the residues, with

$$\widetilde{A} := (x - \alpha_\nu)^{d_\nu + 1} A(x), \quad \chi := x - \alpha_\nu$$

gives:

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Rational covariant derivative

Theorem 3. Local trace formulae ([BHHP], 2004)

The Hamiltonian $h^a_{\nu J}$ can be written as

$$h^a_{\nu J} = \underset{x=\alpha_\nu}{res} \ \underset{z=t^a_{\nu d_\nu}}{res} \frac{\mathrm{tr}\left(\mathbf{1} + (x-c_\nu)^{d_\nu+1} A(x)(z\mathbf{I} - A_{\nu,d_\nu})^{-1}\right)^{d_\nu+J}}{J(d_\nu+J)(x-\alpha_\nu)^{d_\nu+J+1}}$$

and the principal part of its functional derivative as

$$\Omega^a_{\nu j} = \left[\delta h^a_{\nu j}\right]_{\mathcal{P}_\nu} = \Big[\underset{z=t^a_{\nu d_\nu}}{res} \frac{(z\mathbf{I} - A_{\nu,d_\nu})^{-1}}{j(x-\alpha_\nu)^j} \times \left(\mathbf{1} + (x-c_\nu)^{d_\nu+1}(z\mathbf{I} - A_{\nu,d_\nu})^{-1} A(x)\right)^{d_\nu+J-1} \Big]_{\mathcal{P}_\nu}.$$

Semiclassical orthogonal polynomials

(**[BEH]** (2003), **[BH]** (2004))

$$\int_{\kappa} p_J(x) p_K(x) \mu(x) \mathrm{d}x = h_J \delta_{JK},$$

where

$$p_J(x) = x^J + \cdots ,$$

$$\mu(x) = \mathrm{e}^{-\frac{1}{\hbar}V(x)}, \quad V(x) := \sum_{\nu=0}^{n} T_\nu(x) ,$$

$$T_0(x) := t_{0,0} + \sum_{j=1}^{d_0} \frac{t_{0,j}}{j} x^j$$

$$T_\nu(x) := \sum_{j=1}^{d_\nu} \frac{t_{\nu,j}}{j(x-\alpha_\nu)^j} - t_{r,0} \ln(x - \alpha_\nu)$$

$$-\hbar \partial_x \ln \mu(x) = V'(x) = \sum_{j=1}^{d_0} t_{0,j} x^{j-1} - \sum_{\nu=1}^{n} \sum_{j=0}^{d_\nu} \frac{t_{\nu,j}}{(x-\alpha_\nu)^{j+1}},$$

Fundamental system

Normalized orthogonal polynomials

$$\pi_J(x) := \frac{1}{\sqrt{h_J}} p_J(x)$$

Second kind solutions

$$\sigma_J(x) := \mathrm{e}^{\frac{1}{\hbar}V(x)} \int_{\kappa} \mathrm{e}^{-\frac{1}{\hbar}V(z)} \frac{\pi_J(z)}{x-z} \mathrm{d}z$$

Fundamental systems

$$\Psi_J(x) := \begin{pmatrix} \pi_{J-1}(x) & \sigma_{J-1}(x) \\ \pi_J(x) & \sigma_J(x) \end{pmatrix}$$

satisfy

$$\hbar\partial_x \Psi_J(x) = A_J(x)\Psi_J(x).$$

$$\hbar\partial_{\alpha_\nu} \Psi_J(x) = \Omega_{\nu;J}(x)\Psi_j(x). \qquad \hbar\partial_{t_{\nu,j}} \Psi_J(x) = \Omega_{\nu,j;J}(x)\Psi_J(x)$$

where each $A_J(x)$ has the same pole structure as $V'(x)$.

Relation to matrix models

(N.B. This is not the traceless gauge; the latter is obtained by replacing $\Psi_J(x) \to e^{-\frac{1}{2\hbar}V(x)}\Psi_J(x)$.)

Relation between Z_N and τ_N^{IM}

The partition function $Z_N(t_{\nu,j}, \alpha_j)$ of the generalized random matrix model and the isomonodromic tau function τ_N^{IM} for the associated ODE are related by

$$\ln(Z_N) = \ln(\tau_N^{IM}) + \hbar^{-2} \sum_{0 \leq \nu < \mu \leq n} \mathrm{res}_{x=\alpha_\mu} T'_\mu(x) T_\nu(x)$$

up to a term independent of the deformation parameters.

Spectral curve and matrix models

The characteristic polynomial of the matrix $\mathcal{D}_N(x)$ is given by

$$\det\big(y - \mathcal{D}_N(x)\big) = y^2 - yV'(x) + \hbar \left\langle \mathrm{tr} \frac{V'(M) - V'(x)}{M - x} \right\rangle$$

$$= y^2 - yV'(x) + N \sum_{j=1}^{d_0 - 1} t_{0,j+1} x^{j-1} - \hbar^2 \mathbf{D}(x) \ln Z_N$$

Spectral curve and Virasoro generators

where, in terms of the **Virasoro generators**

$$\mathbf{V}^{(\nu)}_{-j} := \sum_{m=1}^{d_\nu - j} m t_{\nu, m+j} \frac{\partial}{\partial t_{\nu,m}}, \quad \nu = 0, \ldots, n, \quad j = 0, \ldots, d_\nu - 1$$

the operator $\mathbf{D}(\mathbf{x})$ is:

$$\mathbf{D}(x) := \sum_{\nu=1}^{n} \frac{1}{x-\alpha_\nu} \frac{\partial}{\partial \alpha_\nu} - \sum_{j=0}^{d_0-3} x^j \mathbf{V}^{(0)}_{-j-2} - \sum_{\nu=1}^{n} \sum_{j=2}^{d_\nu+1} \frac{1}{(x-\alpha_\nu)^j} \mathbf{V}^{(\nu)}_{2-j}$$