Title: The Kahler-Ricci flow on Hirzebruch surfaces

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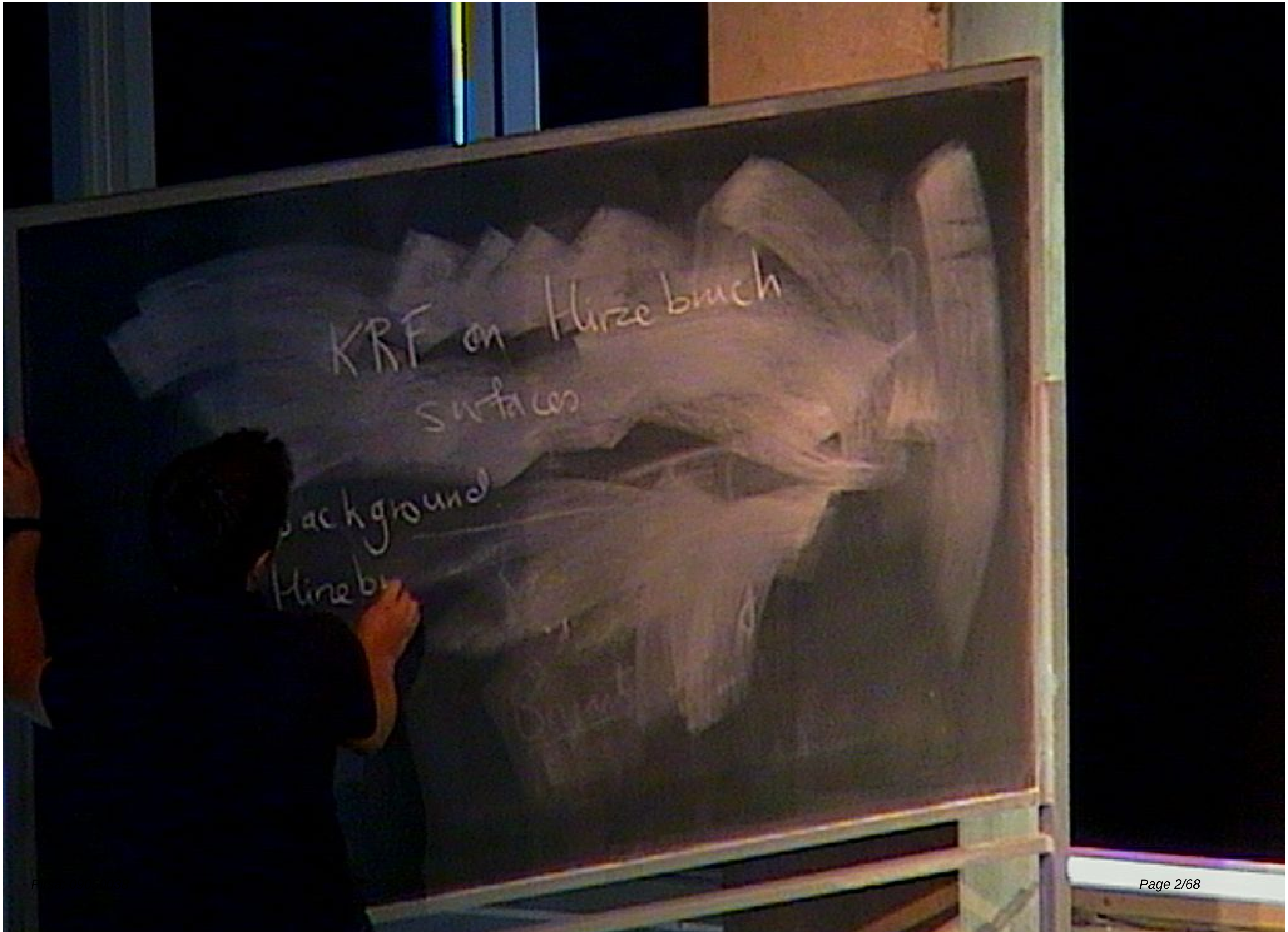
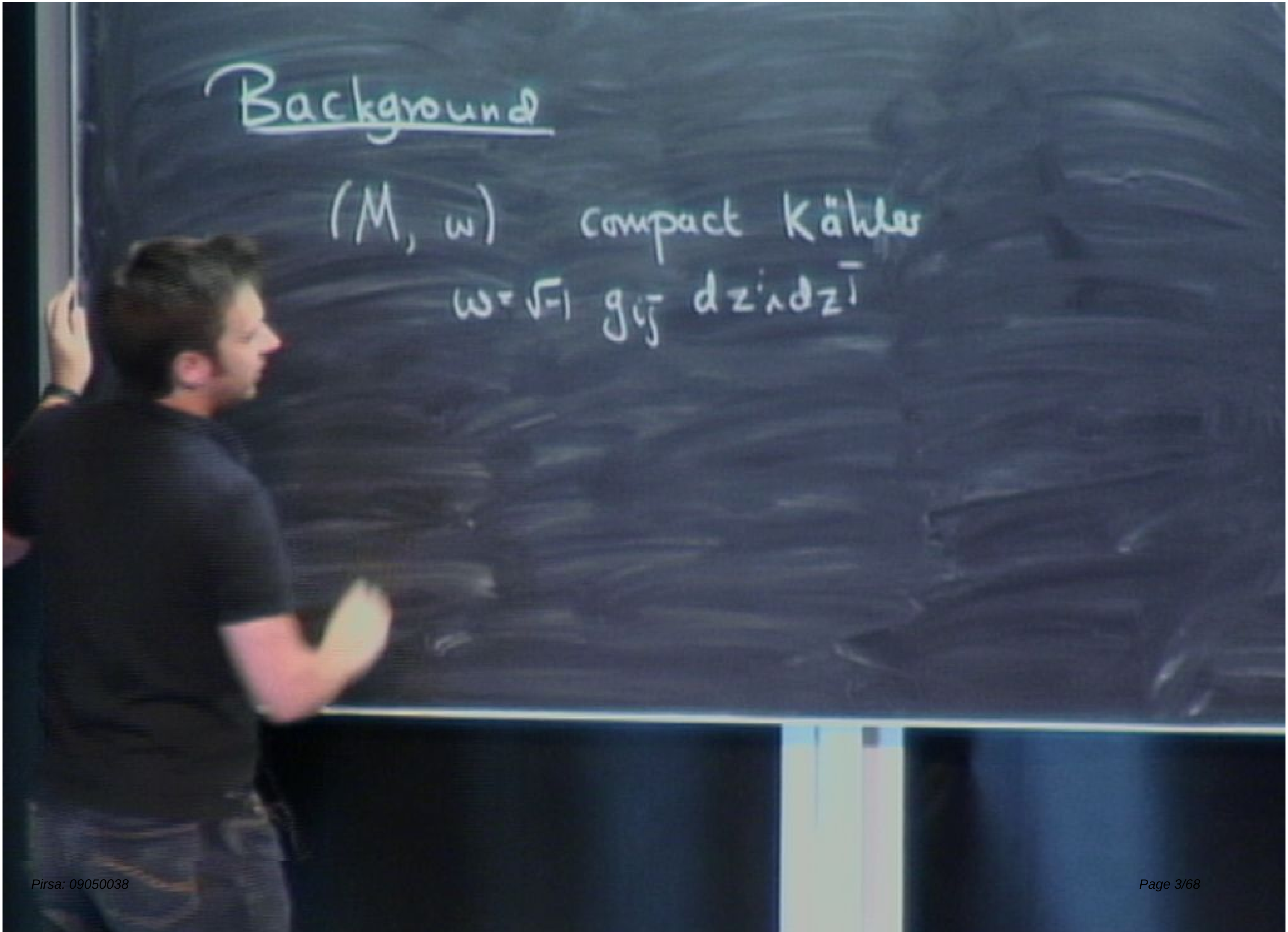
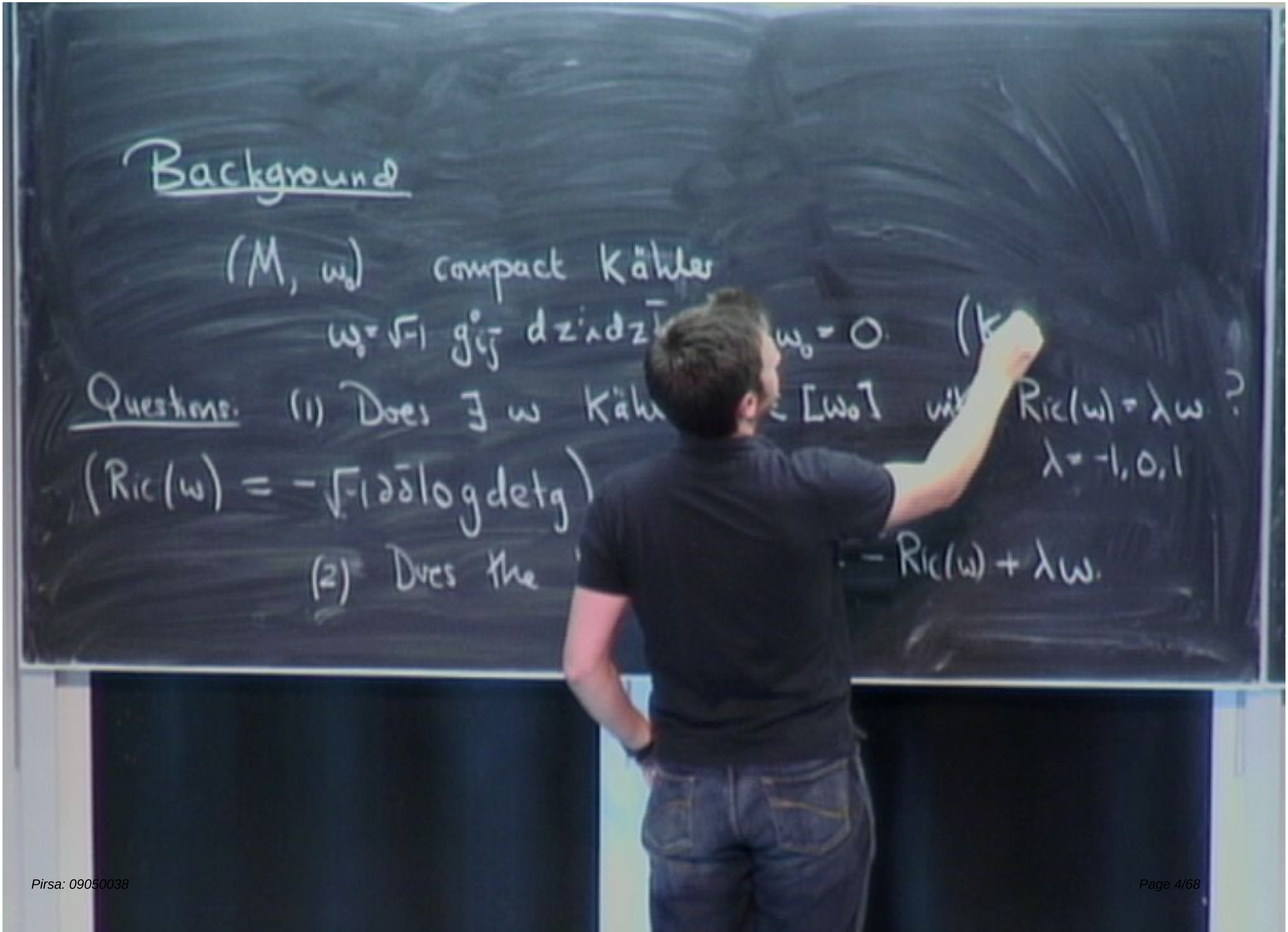
Abstract: I will discuss the metric behavior of the Kahler-Ricci flow on Hirzebruch surfaces assuming that the initial metric is invariant under a maximal compact subgroup of the automorphism group. I will describe how, in the sense of Gromov-Hausdorff, the flow either shrinks to a point, collapses to P^1 or contracts an exceptional divisor. This confirms a conjecture of Feldman-Ilmanen-Knopf. This is a joint work with Jian Song.


KRF on Hirzebruch
surfaces


Background
(M, ω) compact Kähler
ω = √-1 g_{ij̄} dz^i ∧ dz^j̄


Background
(M, ω₀) compact Kähler
ω₀ = √-1 g_ij̄ dz^i ∧ dz̄^j
ω₀ = 0. (K
Questions. (1) Does ∃ ω Käh
[ω₀]
Ric(ω) = λω ?
λ = -1, 0, 1
(Ric(ω) = -√-1 ∂∂̄ log det g)
(2) Does the
- Ric(ω) + λω.

## Background

$(M, \omega_0)$ compact Kähler

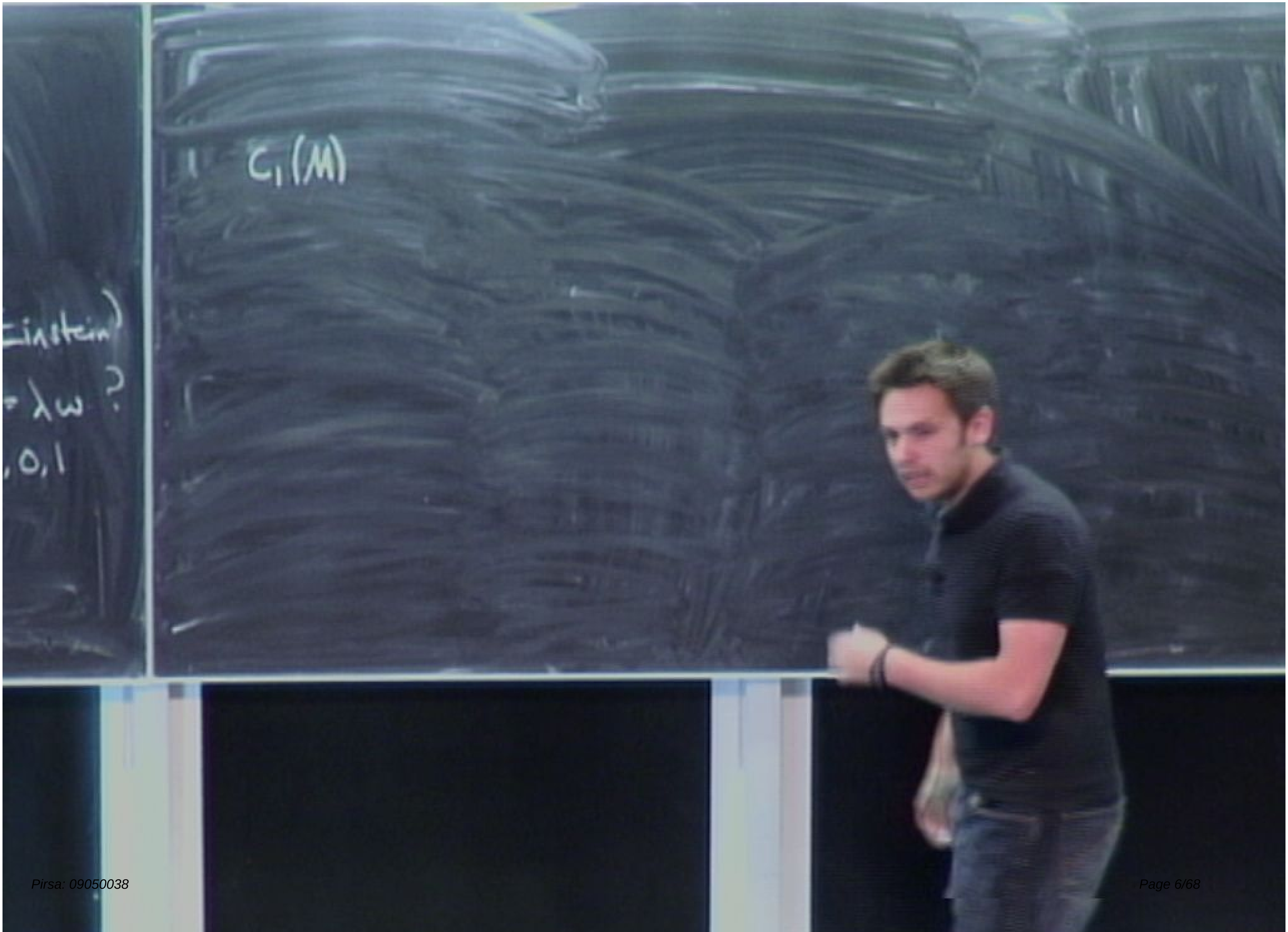
$\omega_0 = \sqrt{-1}\, g^0_{i\bar{j}}\, dz^i \wedge dz^{\bar{j}}$, $d\omega_0 = 0$. (Kähler - E

Questions. (1) Does $\exists\ \omega$ Kähler, $\omega \in [\omega_0]$ with $\mathrm{Ric}(\omega) =$

$\lambda = -1, 0,$

$\left(\mathrm{Ric}(\omega) = -\sqrt{-1}\,\partial\bar{\partial} \log \det g\right)$

(2) Does the KRF $\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega) + \lambda\omega$ converge to a KE metric?

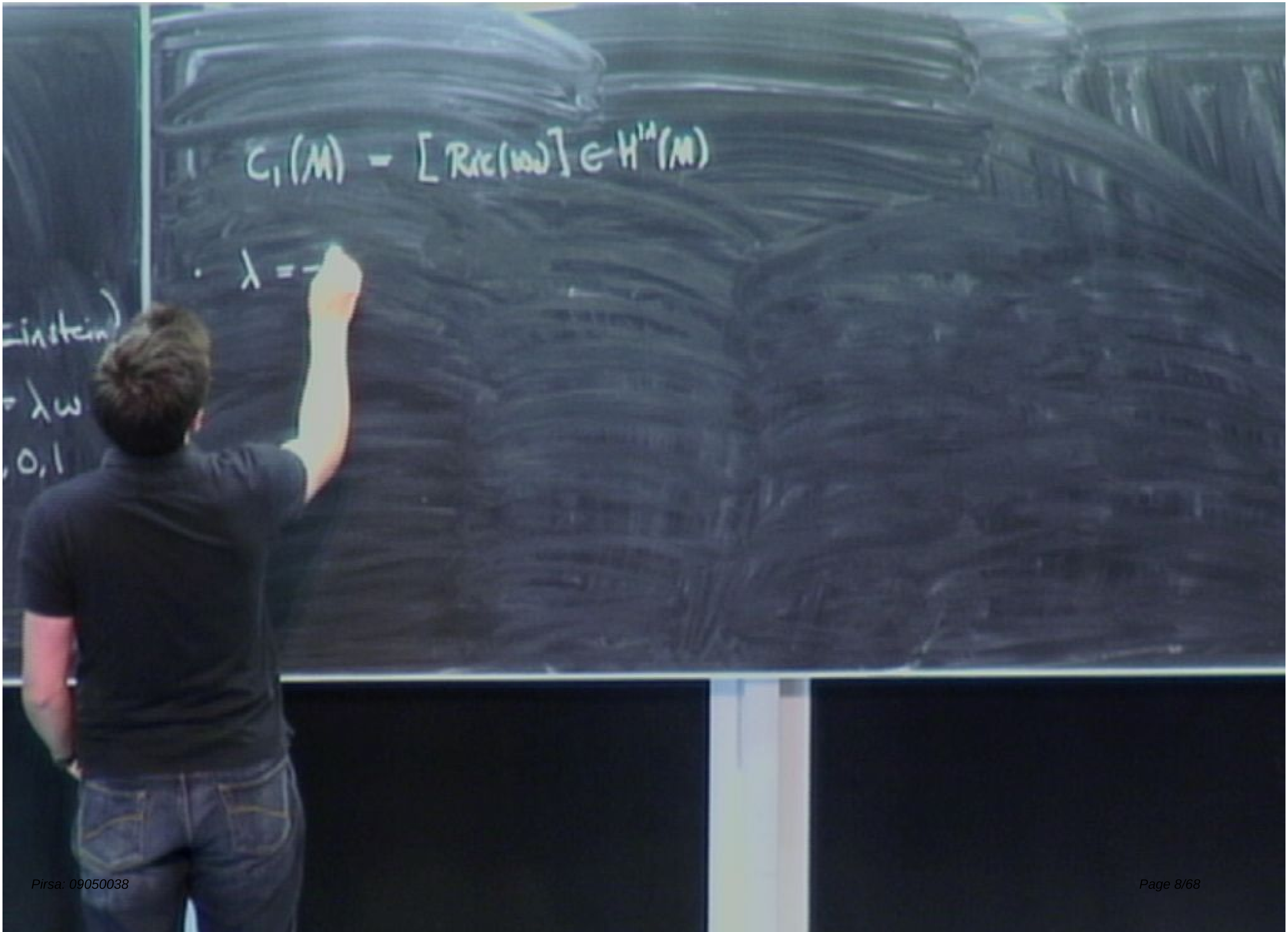

$c_1(M)$
instein)
$\lambda\omega$ ?
0, 1

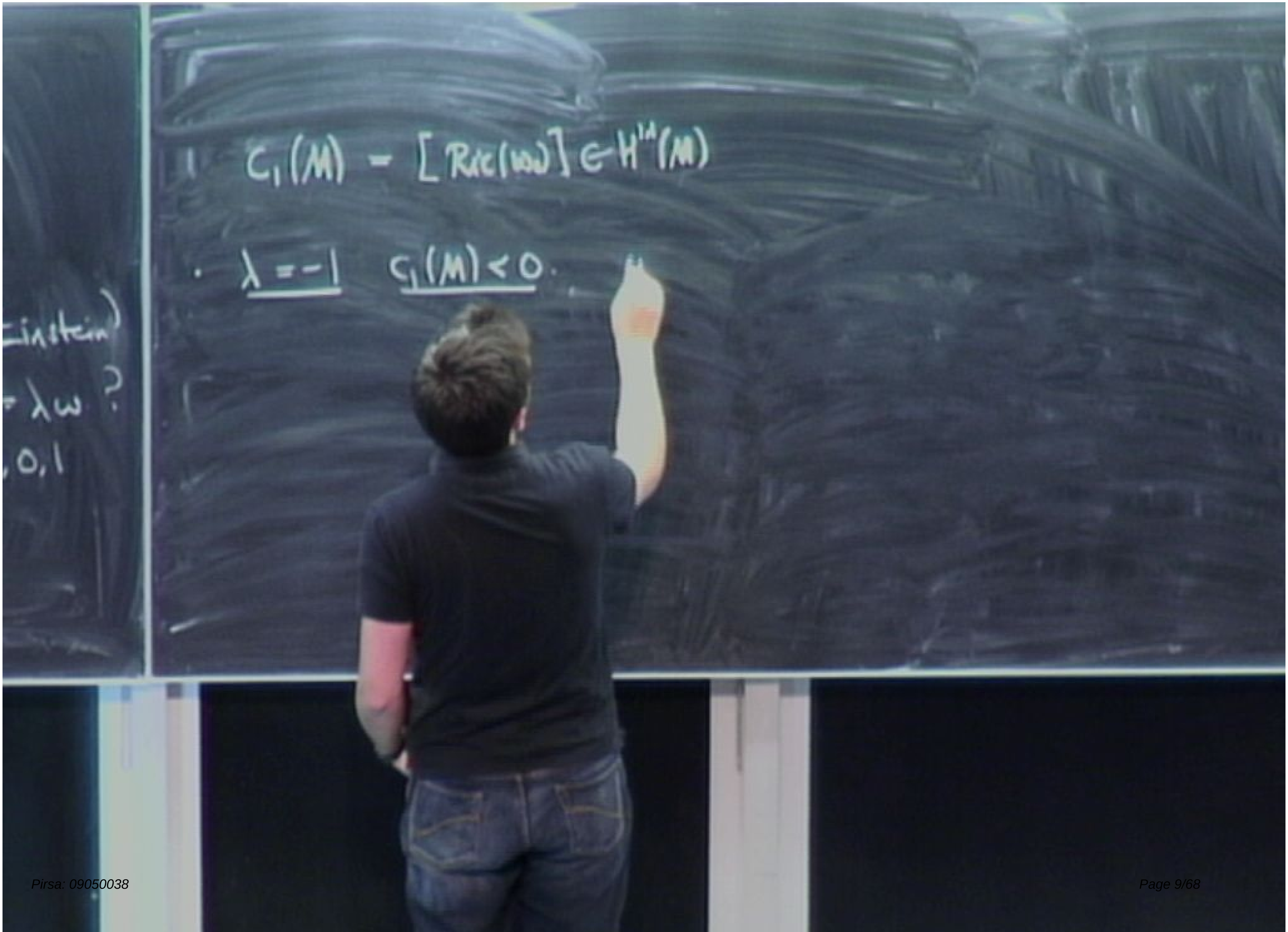
Einstein)

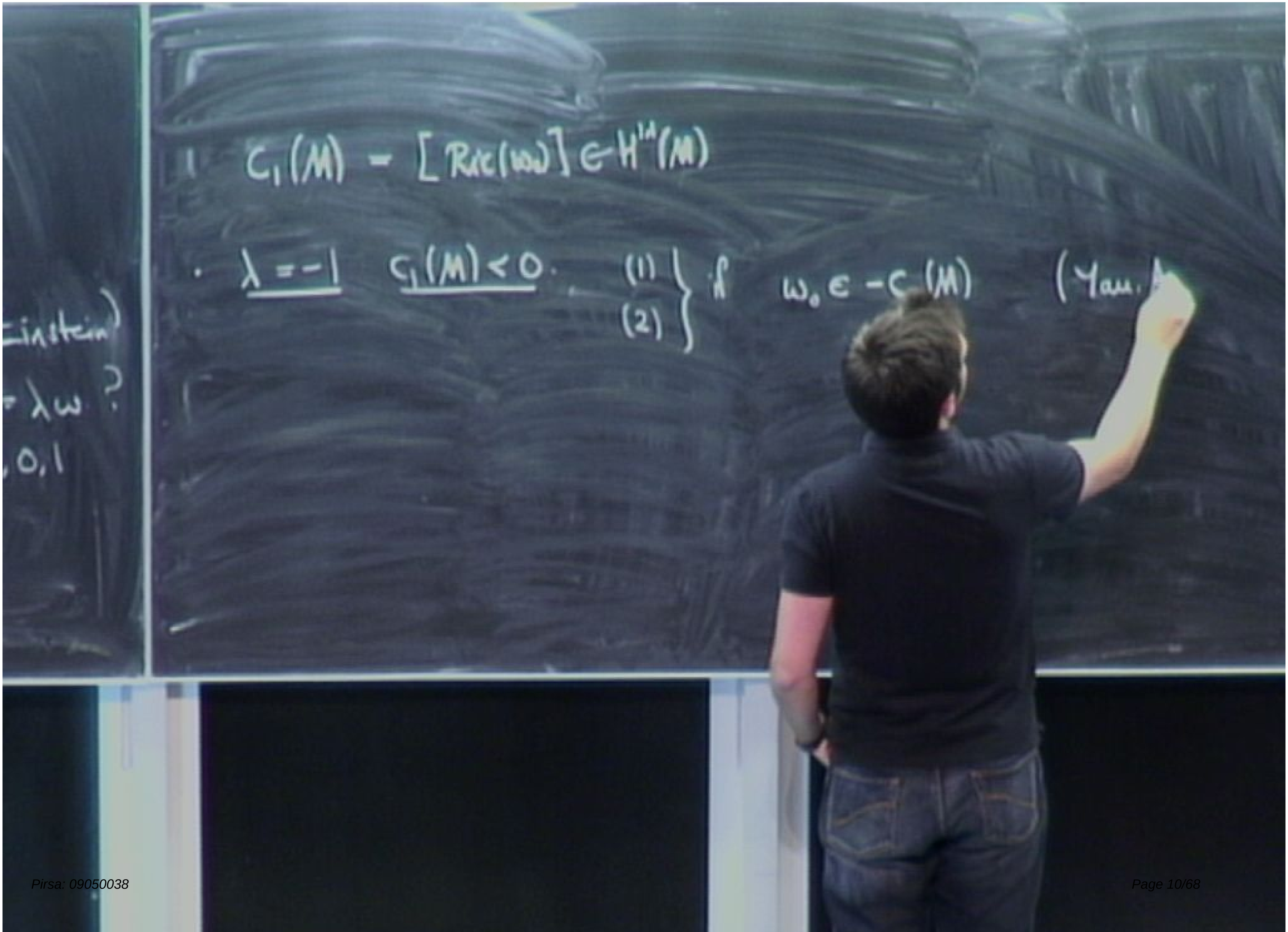
$= \lambda\omega$ ?

, 0, 1

$c_1(M) = [\mathrm{Ric}(\omega)] \in H^{1,1}(M)$


Einstein)
= λω
, 0, 1
c1(M) = [Ric(ω)] ∈ H^{1,1}(M)
λ = -


$c_1(M) = [\mathrm{Ric}(\omega)] \in H^{1,1}(M)$
$\lambda = -1 \quad c_1(M) < 0.$
$= \lambda\omega$ ?
, 0, 1

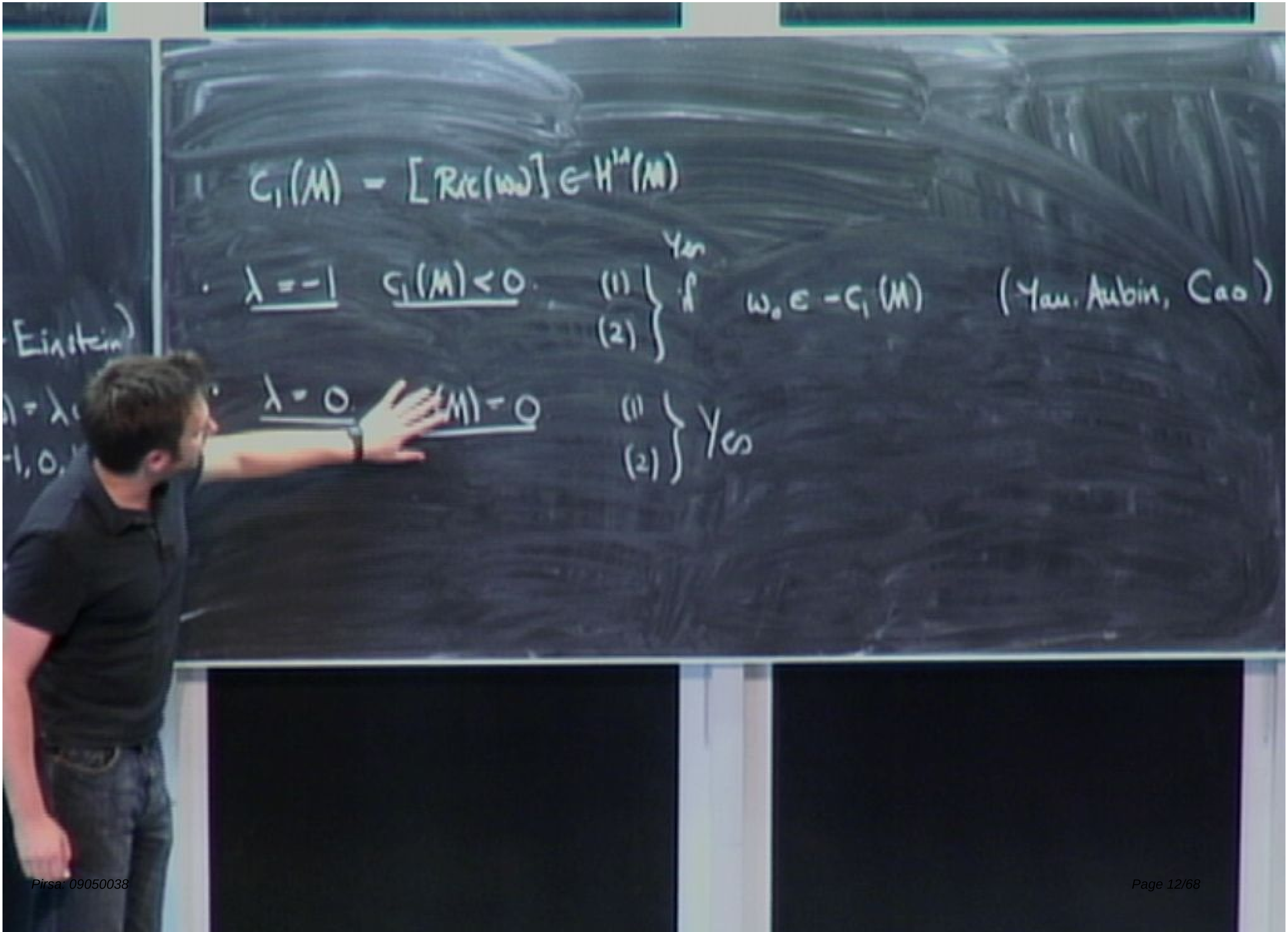

Einstein)
= λω ?
, 0, 1
c1(M) = [Ric(ω)] ∈ H^1,1(M)
· λ = -1   c1(M) < 0.   (1) (2) } if   ω0 ∈ -c1(M)   (Yau.

- Einstein)
$) = \lambda\omega$ ?
$-1, 0, 1$

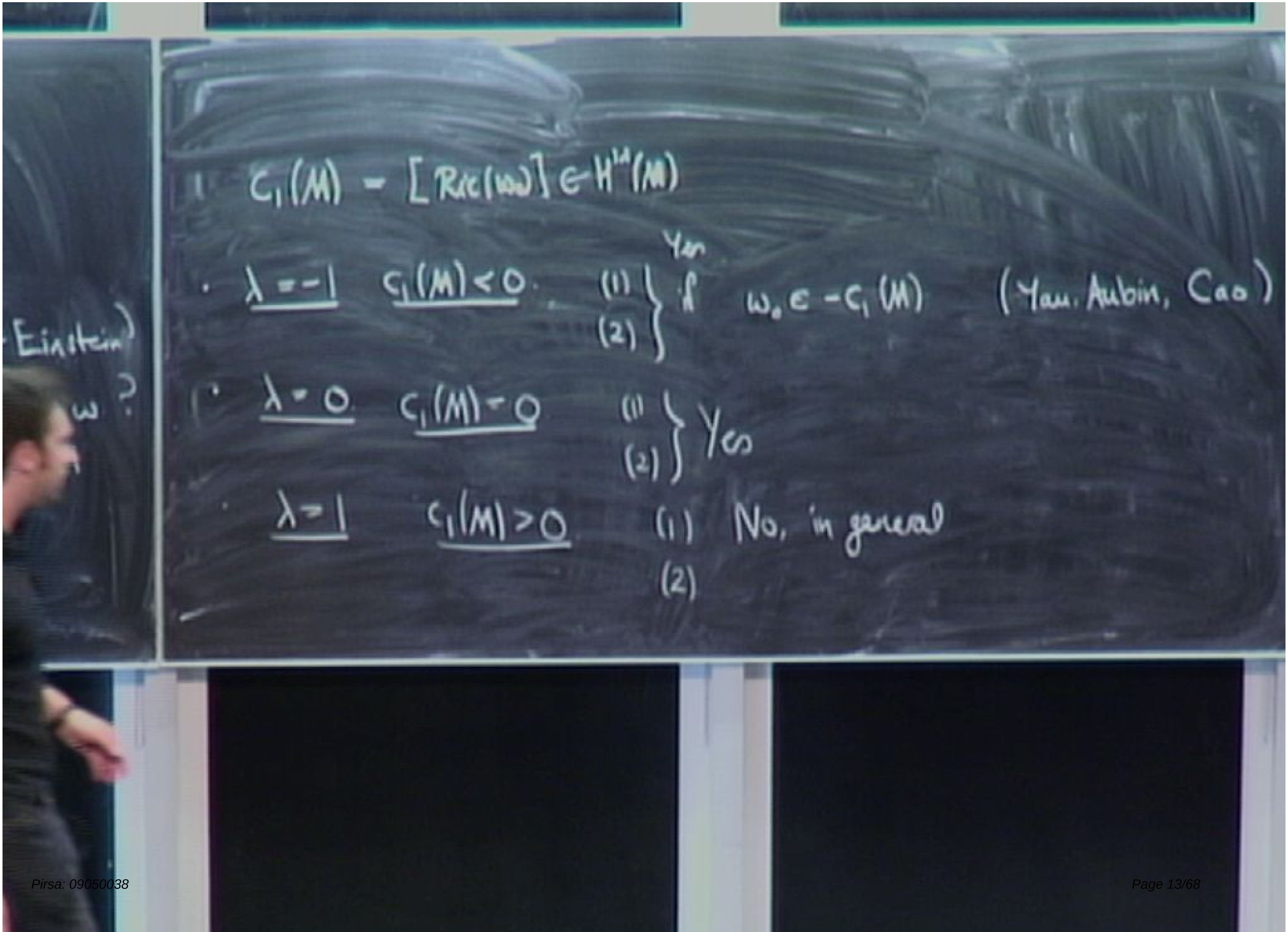
$c_1(M) = [\mathrm{Ric}(\omega)] \in H^{1,1}(M)$

- $\underline{\lambda = -1}$ $\quad \underline{c_1(M) < 0}$. $\quad \left.\begin{matrix}(1)\\(2)\end{matrix}\right\}$ if $\omega_0 \in -c_1(M)$ $\quad$ (Yau, Aubin, Cao)

- $\underline{\lambda = 0}$.



c1(M) = [Ric(ω)] ∈ H^{1,1}(M)
λ = -1   c1(M) < 0
(1) (2) Yes if ω0 ∈ -c1(M)
(Yau, Aubin, Cao)
Einstein)
λ = 0   (M) = 0
(1) (2) Yes


Einstein)
ω ?
$c_1(M) = [Ric(\omega)] \in H^{1,1}(M)$
Yes
• $\underline{\lambda = -1}$ $\underline{c_1(M) < 0}$ (1) (2) } if $\omega_0 \in -c_1(M)$ (Yau, Aubin, Cao)
• $\underline{\lambda = 0}$ $\underline{c_1(M) = 0}$ (1) (2) } Yes
• $\underline{\lambda = 1}$ $\underline{c_1(M) > 0}$ (1) No, in general
(2)

Einstein)
$) = \lambda\omega$ ?
$-1, 0, 1$

$c_1(M) = [\mathrm{Ric}(\omega)] \in H^{1,1}(M)$

- $\underline{\lambda = -1}$ $\underline{c_1(M) < 0}$ (1) (2) } Yes if $\omega_0 \in -c_1(M)$ (Yau, Aubin, Cao)
- $\underline{\lambda = 0}$ $\underline{c_1(M) = 0}$ (1) (2) } Yes (Yau, Cao)
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  (2) If $\exists$ KE then KRF converges to a KE.

Einstein)
$\ldots = \lambda\omega$ ?
$-1, 0, 1$

$c_1(M) = [\mathrm{Ric}(\omega)] \in H^{1,1}(M)$

- $\underline{\lambda = -1}$ $\underline{c_1(M) < 0}$ (1) (2) } $\overset{\text{Yes}}{\text{if}}$ $\omega_0 \in -c_1(M)$ (Yau, Aubin, Cao)

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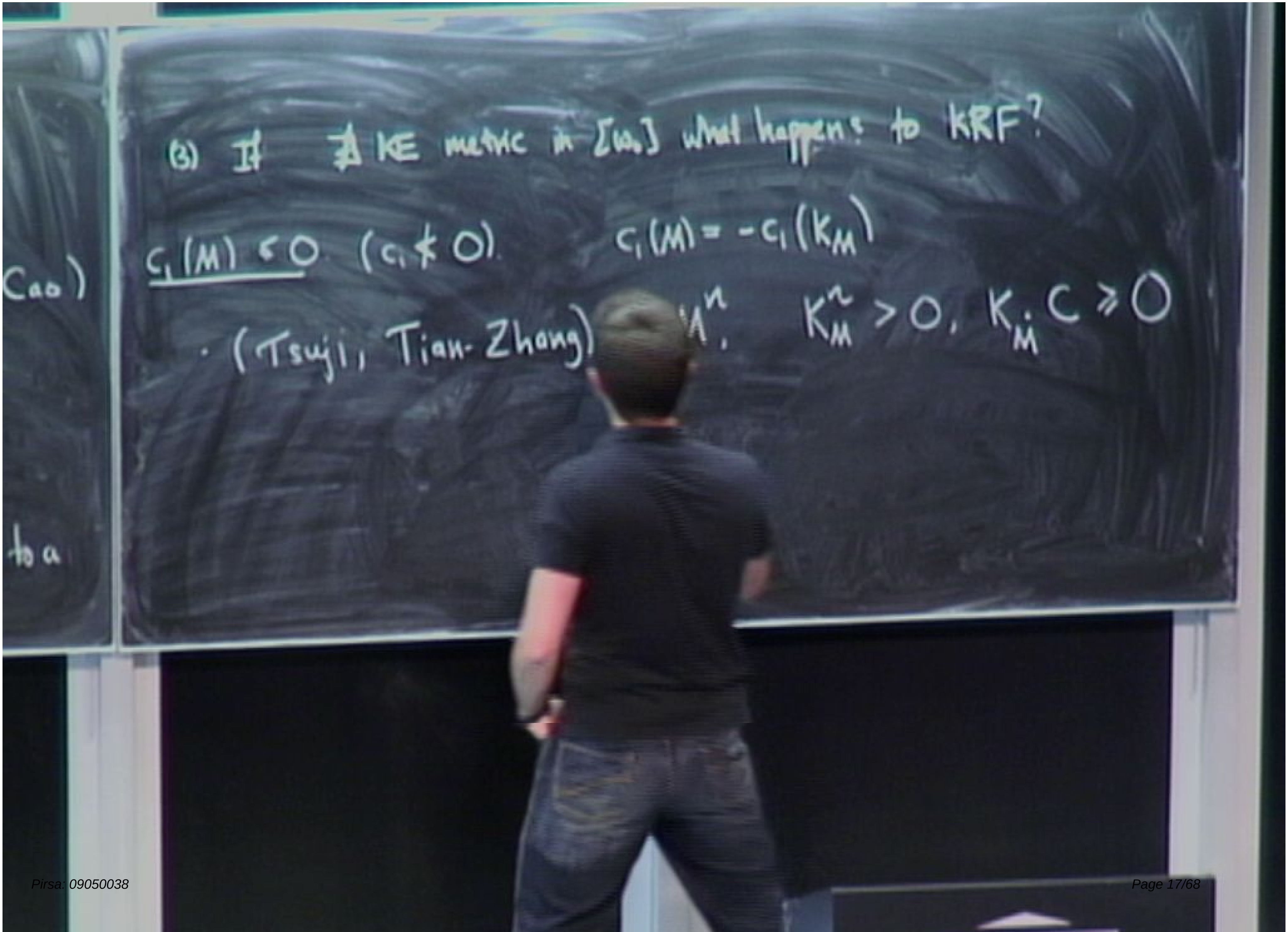
- $\underline{\lambda = 1}$ $\underline{c_1(M) > 0}$ (1) No, in general
(2) If $\exists$ KE then KRF converges to a KE. (Perelman, Tian-Zhu)

(3) If $\nexists$ KE metric in $[\omega_0]$ what happens to KRF?


(3) If $\nexists$ KE metric in $[\omega_0]$ what happens to KRF?
$c_1(M) \leq 0$ $(c_1 \not< 0)$.
$c_1(M) = -c_1(K_M)$
· (Tsuji, Tian-Zhang)
$K_M^n > 0$, $K_M \cdot C \geq 0$
Cao)
to a

$C_\infty$)

to a

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· (Tsuji, Tian-Zhang) $\quad M^n$. $\quad K_M^n > 0$, $K_M \cdot C \geq 0$
for all curves $C$ (nef)

KRF $\frac{\partial}{\partial t}\omega = -Ric(\omega) - \omega$ exists for all time & converges to a $\underline{\text{singular}}$ $\underline{\text{KE}}$ metric (smooth outside a subvariety)

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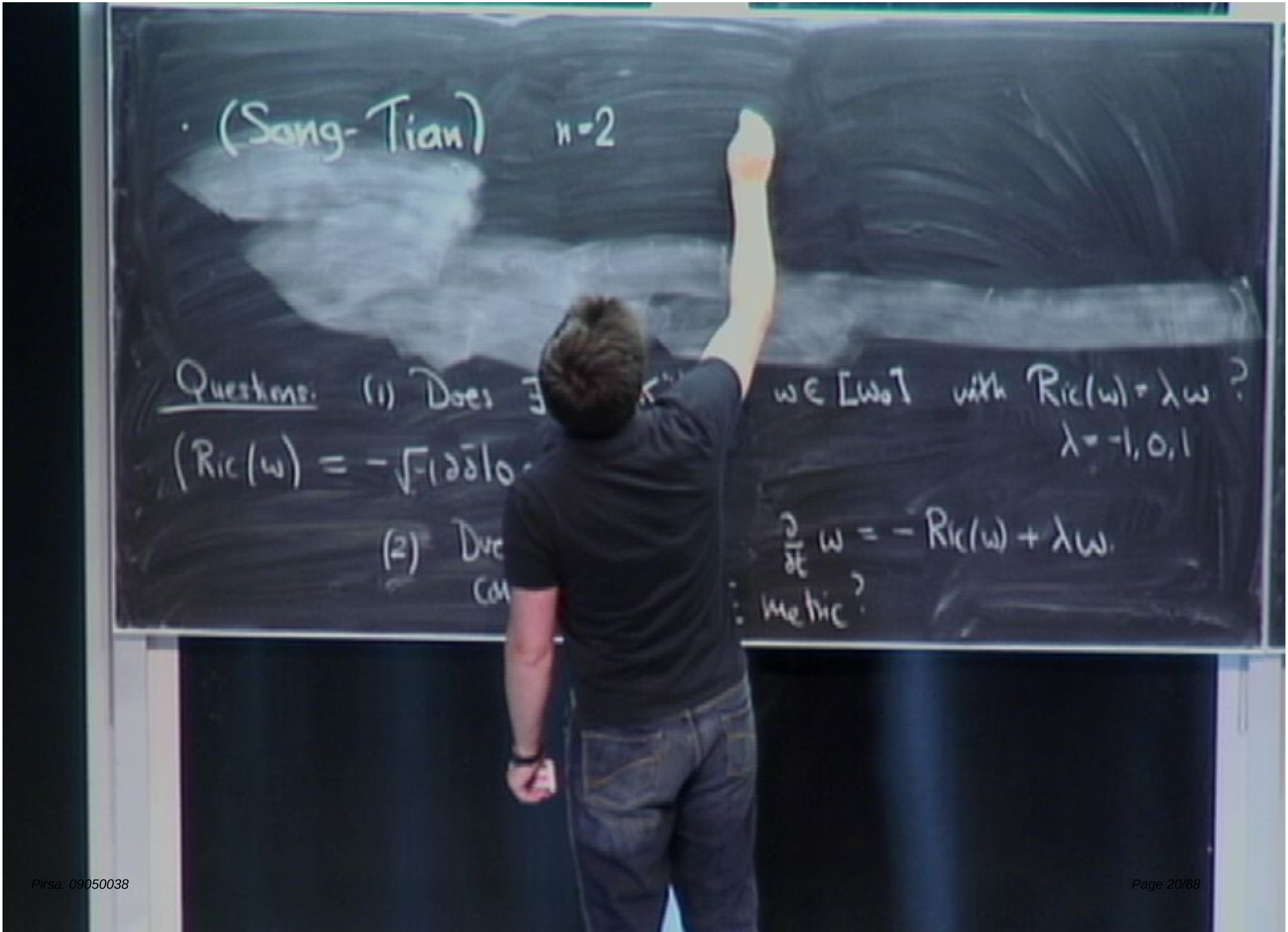
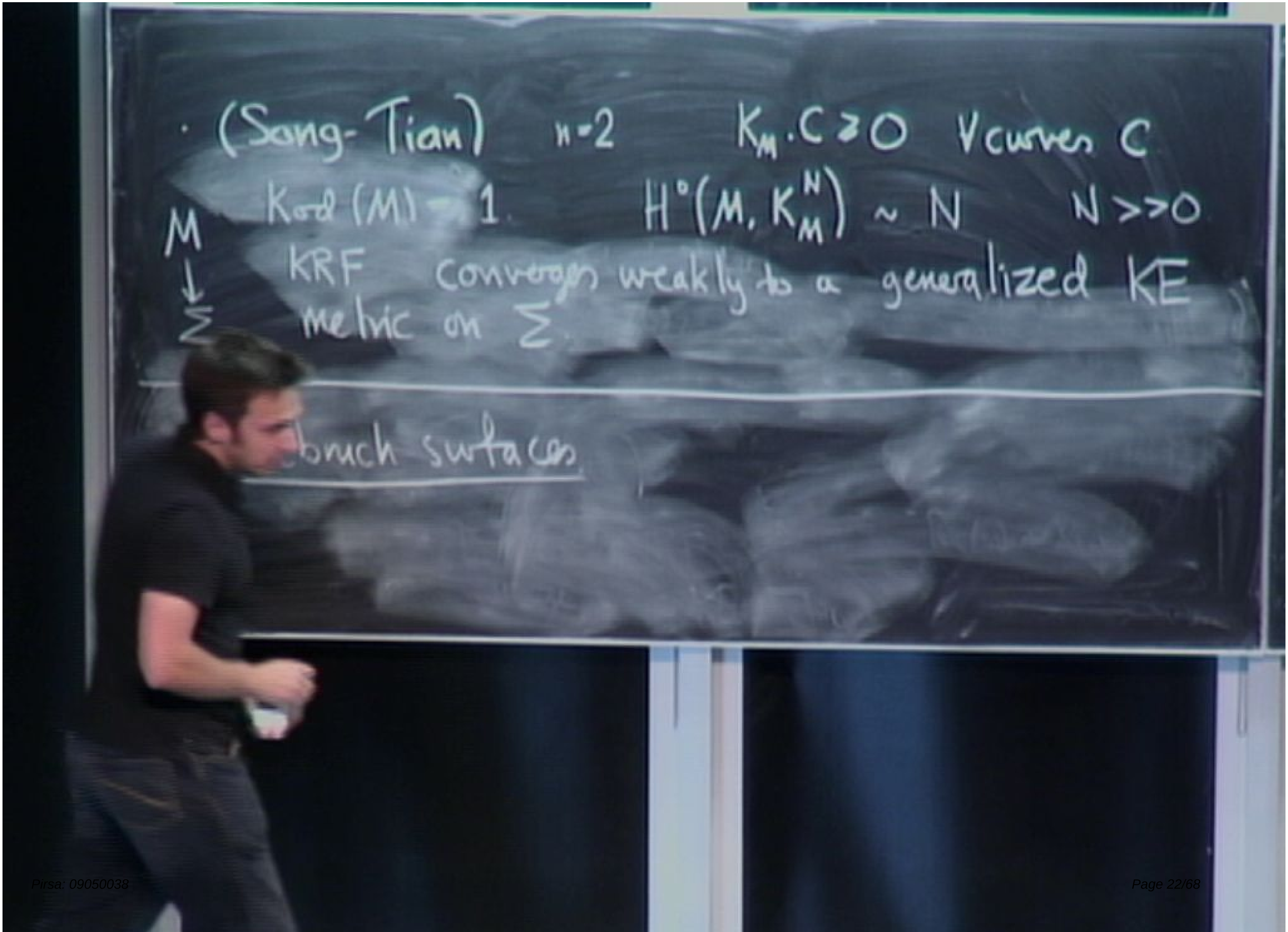

· (Song-Tian) $n=2$
Questions: (1) Does $\exists$
$\omega \in [\omega_0]$ with $Ric(\omega) = \lambda\omega$ ?
$\lambda = -1, 0, 1$
$(Ric(\omega) = -\sqrt{-1}\partial\bar{\partial}$
(2)
$\frac{\partial}{\partial t}\omega = -Ric(\omega) + \lambda\omega.$
metric?

· (Song-Tian) $n=2$ $\quad K_M \cdot C \geq 0 \quad \forall$ curves $C$

$\mathrm{Kod}(M) = 1$. $\quad H^0(M, K_M^N) \sim N \quad N >> 0$

$M$
$\downarrow$
$\Sigma$

KRF converges weakly to a generalized KE metric on $\Sigma$.


· (Song-Tian) n=2 K_M · C ≥ 0 ∀ curves C
Kod(M) = 1 H^0(M, K_M^N) ~ N N >> 0
M
↓
Σ
KRF converges weakly to a generalized KE
metric on Σ.
bruch surfaces

· (Song-Tian) $n=2$ $K_M \cdot C \geq 0$ $\forall$ curves $C$

$\mathrm{Kod}(M) = 1$ $H^0(M, K_M^N) \sim N$ $N >> 0$

$M \downarrow \Sigma$

KRF converges weakly to a generalized KE metric on $\Sigma$

Hirzebruch surfaces

· (Song-Tian) $n=2$ $K_M . C \geq 0$ $\forall$ curves $C$

Kod $(M) = 1$ $H^0(M, K_M^N) \sim N$ $N >> 0$

$M$
$\downarrow$
$\Sigma$

KRF converges weakly to a generalized KE metric on $\Sigma$

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Hirzebruch surfaces $k = 0, 1, 2, 3, \ldots$

$$M_k = \mathbb{P}(H^k \oplus \mathbb{C}_{\mathbb{P}^1})$$

$H$ = hyperplane bundle over $\mathbb{P}^1$.

$\mathbb{C}_{\mathbb{P}^1}$ = trivial line bundle over $\mathbb{P}^1$.

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IGNORE

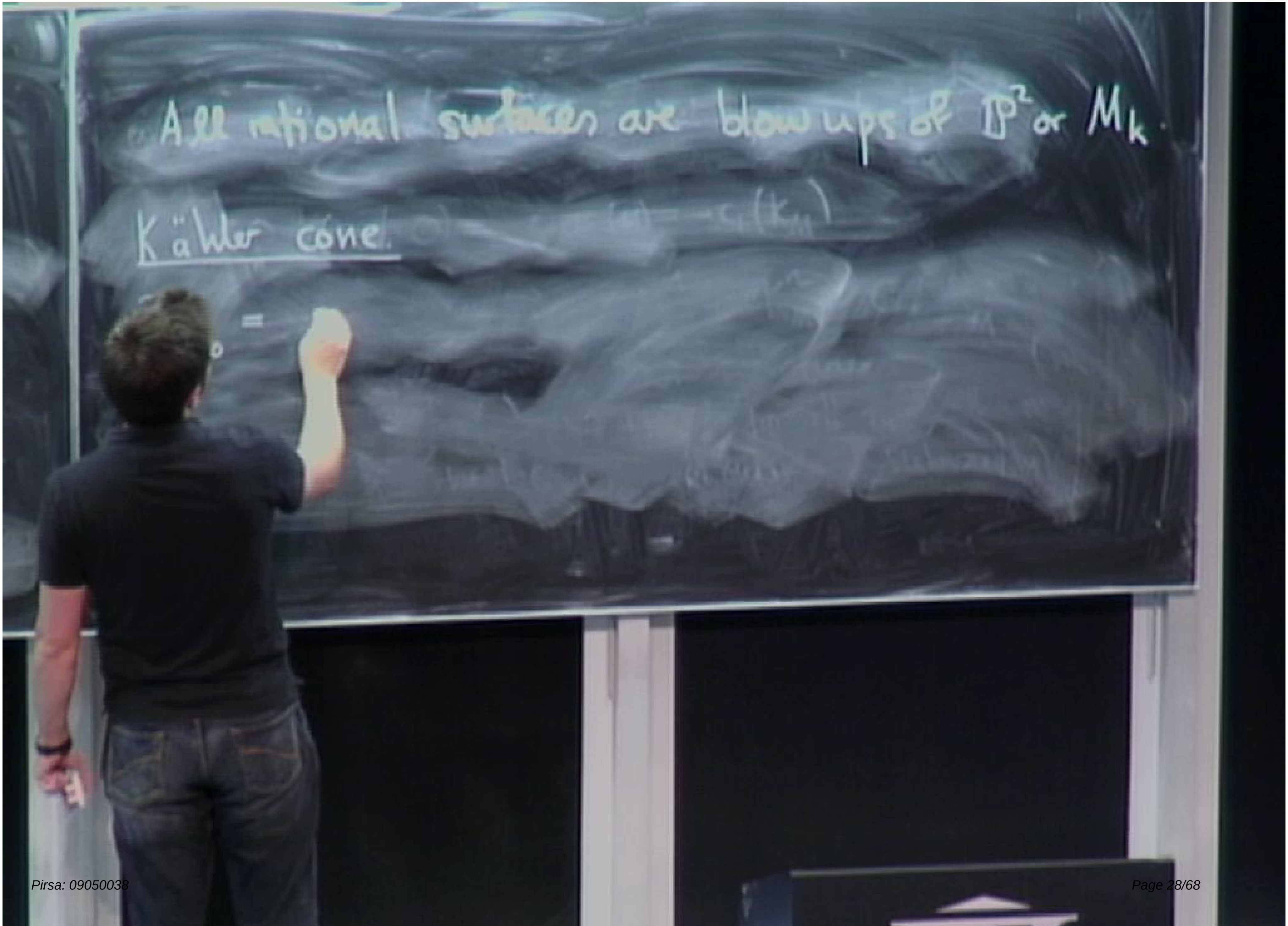
$k = 0$ $\quad \mathbb{P}^1 \times \mathbb{P}^1 \leftarrow$ IGNORE

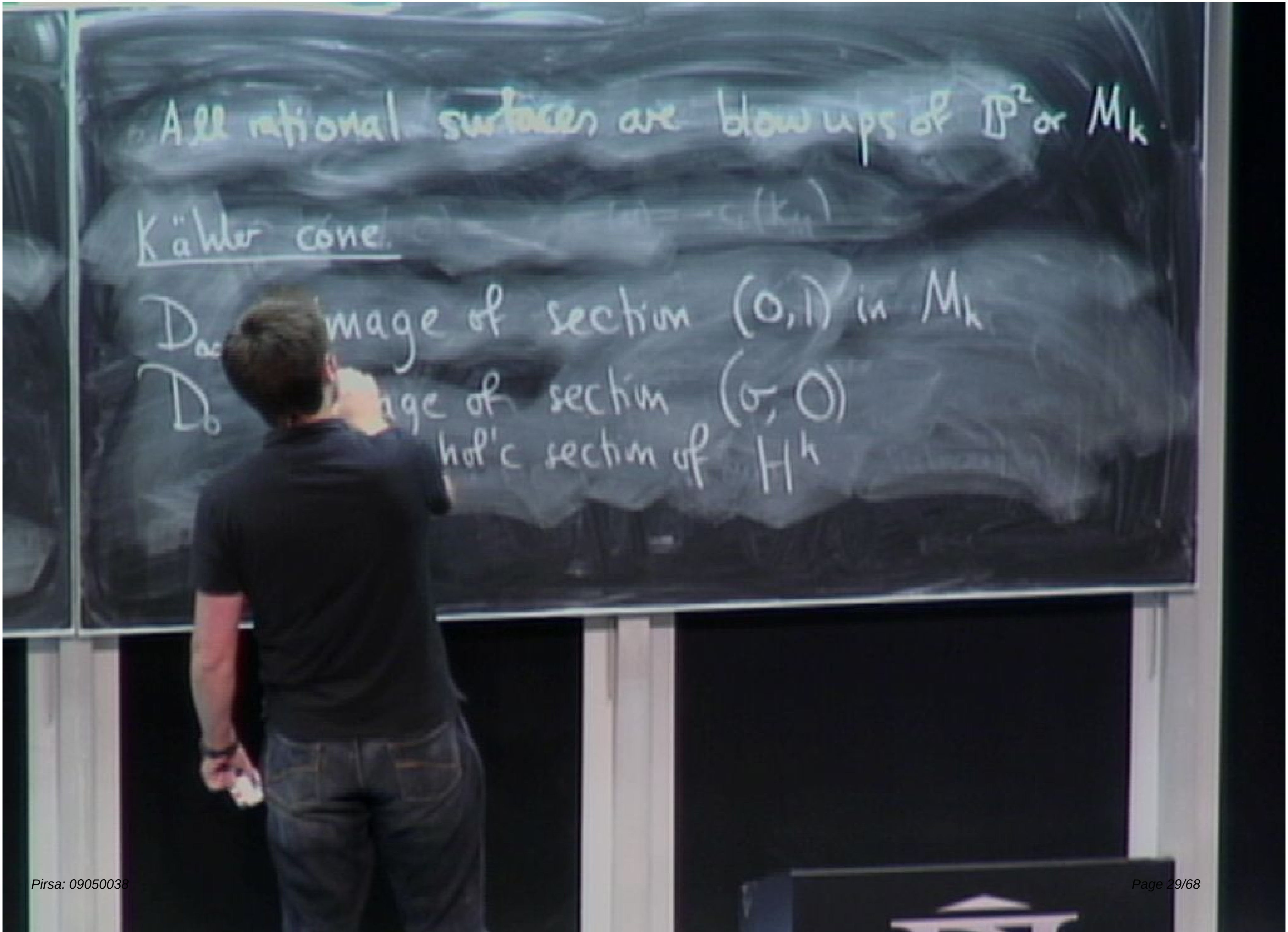
$k = 1$ $\quad \mathbb{P}^2$ blown up at 1 pt. $\Big\}$ $c_1(M) > 0$

$k = 2$ $\quad c_1(M) \geq 0$

$k \geq 3$ $\quad c_1(M)$ indefinite

• All rational surfaces are blow ups of $\mathbb{P}^2$ or $M_k$.


All rational surfaces are blow ups of $\mathbb{P}^2$ or $M_k$.
Kähler cone


All rational surfaces are blow ups of $\mathbb{P}^2$ or $M_k$.
Kähler cone
mage of section $(0,1)$ in $M_k$
$D_0$
age of section $(\sigma, 0)$
hol'c section of $H^k$

All rational surfaces are blow ups of $\mathbb{P}^2$ or $M_k$.

Kähler cone

$D_\infty$ = image of section $(0,1)$ in $M_k$

$D_0$ = image of section $(\sigma, 0)$

$\sigma$ hol'c section of $H^k$

All rational surfaces are blow ups of $\mathbb{P}^2$ or $M_k$.

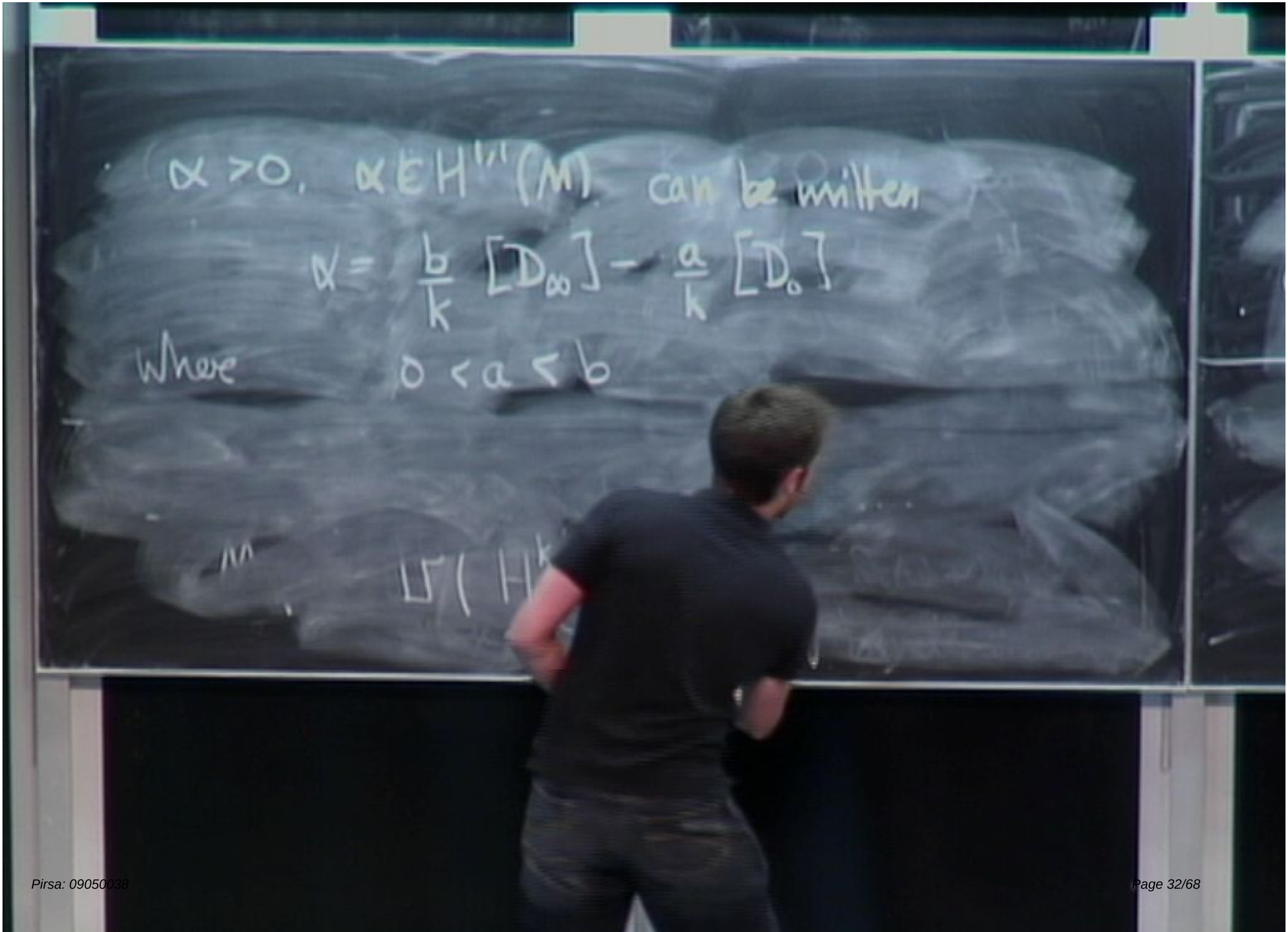
<u>Kähler cone</u>

$D_\infty$ = image of section $(0,1)$ in $M_k$

$D_0$ = image of section $(\sigma, 0)$

$\sigma$ hol'c section of $H^k$

Any Kähler class


$\alpha > 0$, $\alpha \in H^{1,1}(M)$ can be written
$\alpha = \frac{b}{k}[D_\infty] - \frac{a}{k}[D_0]$
where $0 < a < b$

$\alpha > 0$, $\alpha \in H^{1,1}(M)$ can be written

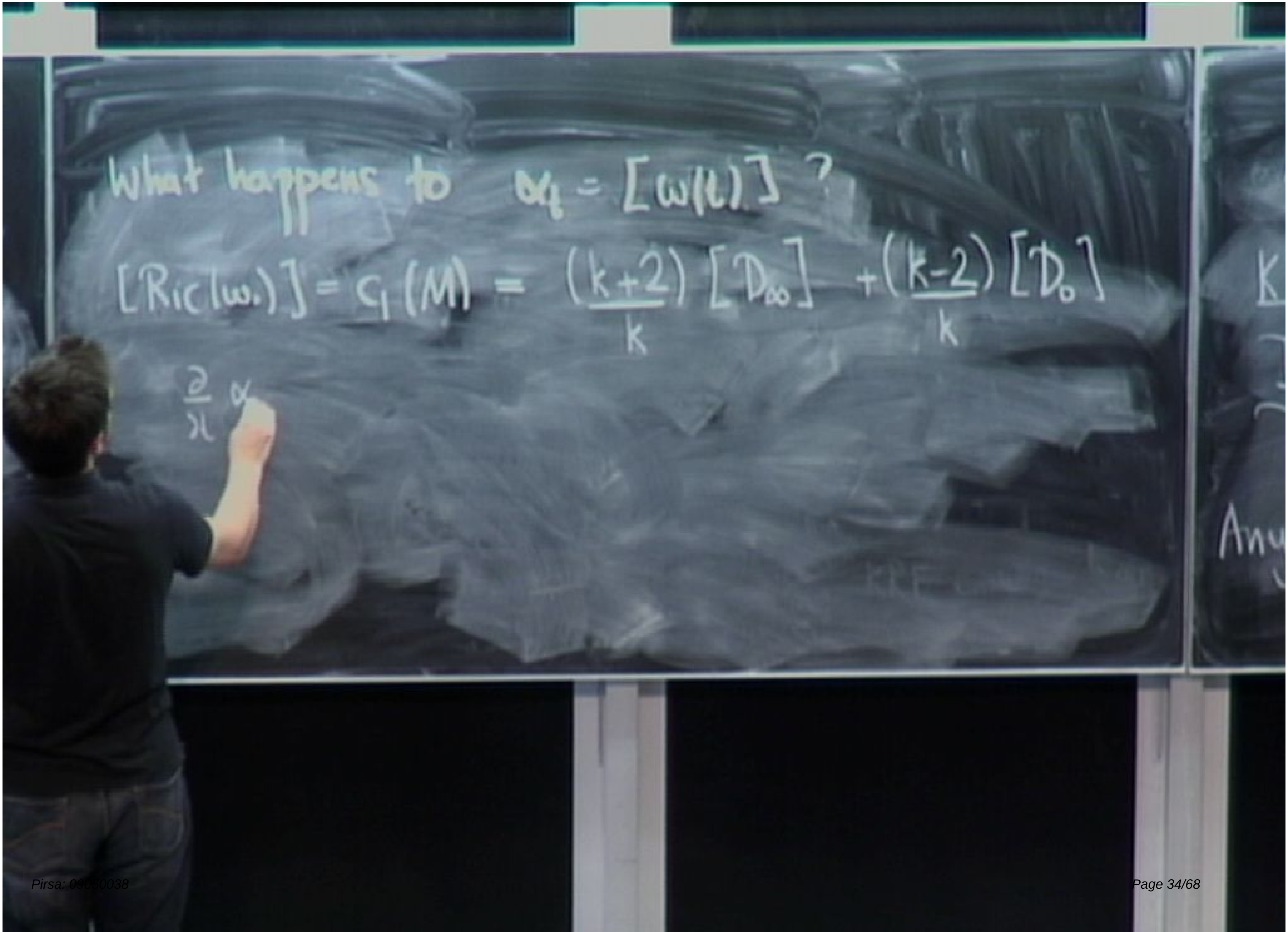
$$\alpha = \frac{b}{k}[D_\infty] - \frac{a}{k}[D_0]$$

where $0 < a < b$

Consider

$$\begin{cases} \frac{\partial}{\partial t}\omega = -Ric(\omega) \\ \omega|_{t=0} = \omega_0 \in \alpha_0 = \frac{b_0}{k}[D_\infty] - \frac{a_0}{k}[D_0] \end{cases}$$

<u>Assume</u>: $\omega_0$ is invariant under $G_k \cong U(2)/\mathbb{Z}_k$

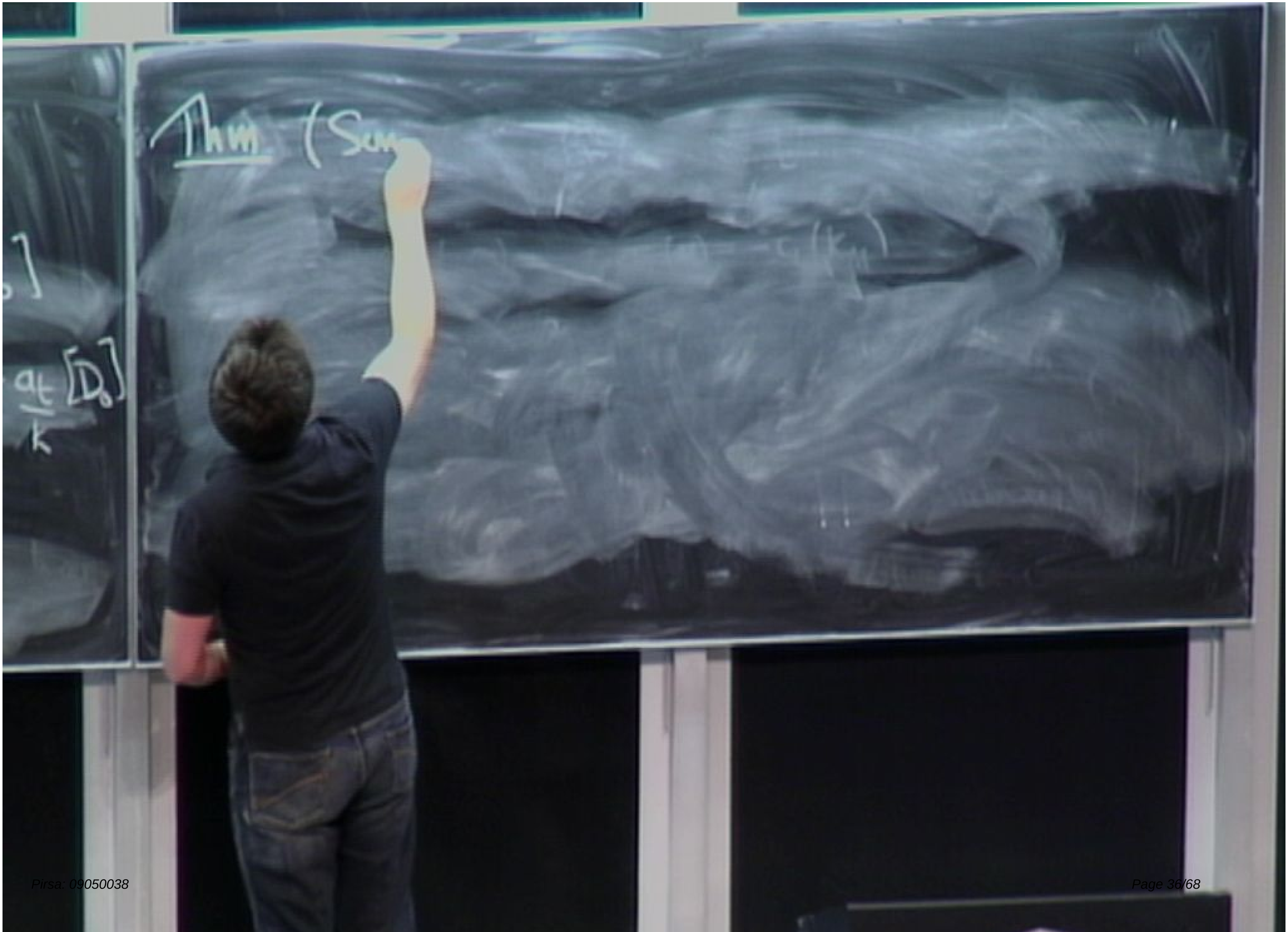

What happens to $\alpha_t = [\omega(t)]$ ?
$[Ric(\omega_0)] = c_1(M) = \left(\frac{k+2}{k}\right)[D_\infty] + \left(\frac{k-2}{k}\right)[D_0]$
$\frac{\partial}{\partial t}\alpha$

What happens to $\alpha_t = [\omega(t)]$ ?

$$[\mathrm{Ric}(\omega_0)] = c_1(M) = \left(\frac{k+2}{k}\right)[D_\infty] + \left(\frac{k-2}{k}\right)[D_0]$$

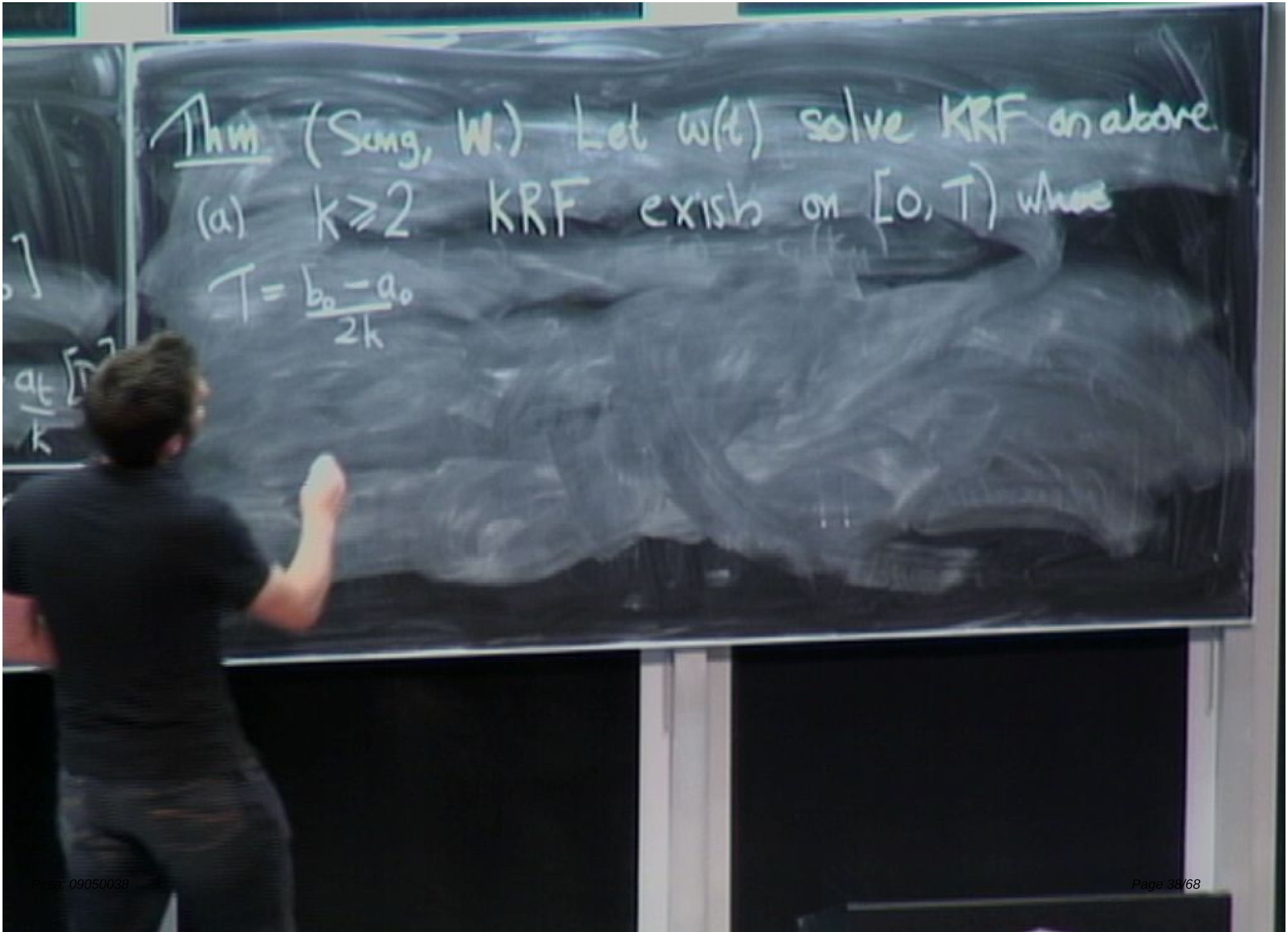
$$\frac{\partial}{\partial t}\alpha_t = -c_1(M), \quad \text{so if } \alpha_t = \frac{b_t}{k}[D_\infty] - \frac{a_t}{k}[D_0]$$

$$\boxed{\begin{aligned} b_t &= b_0 - (k+2)t \\ a_t &= a_0 + (k-2)t \end{aligned}}$$


Thm

Thm (Song, W.) Let $\omega(t)$ solve KRF as above.

(a)


Thm (Song, W.) Let $\omega(t)$ solve KRF as above.
(a) $k \geq 2$ KRF exists on $[0, T)$ where
$T = \frac{b_0 - a_0}{2k}$

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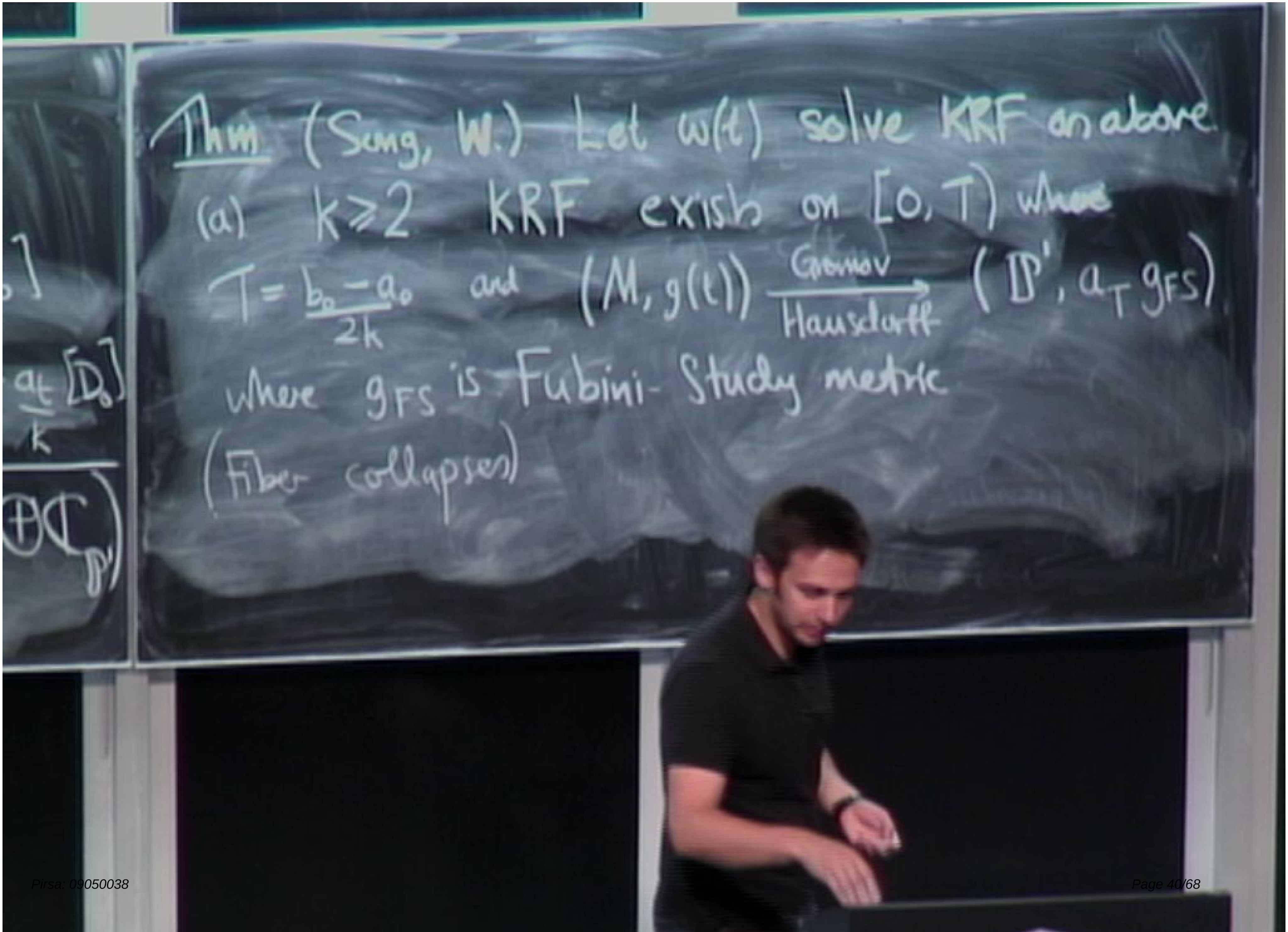
$$T = \frac{b_0 - a_0}{2k} \quad \text{and} \quad (M, g(t)) \xrightarrow[\text{Hausdorff}]{\text{Gromov}} (\mathbb{P}^1, a_T\, g_{FS})$$

where $g_{FS}$ is Fubini-Study metric.


Thm (Song, W.) Let $\omega(t)$ solve KRF as above.
(a) $k \geq 2$ KRF exists on $[0, T)$ where
$T = \frac{b_0 - a_0}{2k}$ and $(M, g(t)) \xrightarrow[\text{Hausdorff}]{\text{Gromov}} (\mathbb{P}^1, a_T g_{FS})$
where $g_{FS}$ is Fubini-Study metric
(Fiber collapses)

**Thm** (Song, W.) Let $\omega(t)$ solve KRF as above.

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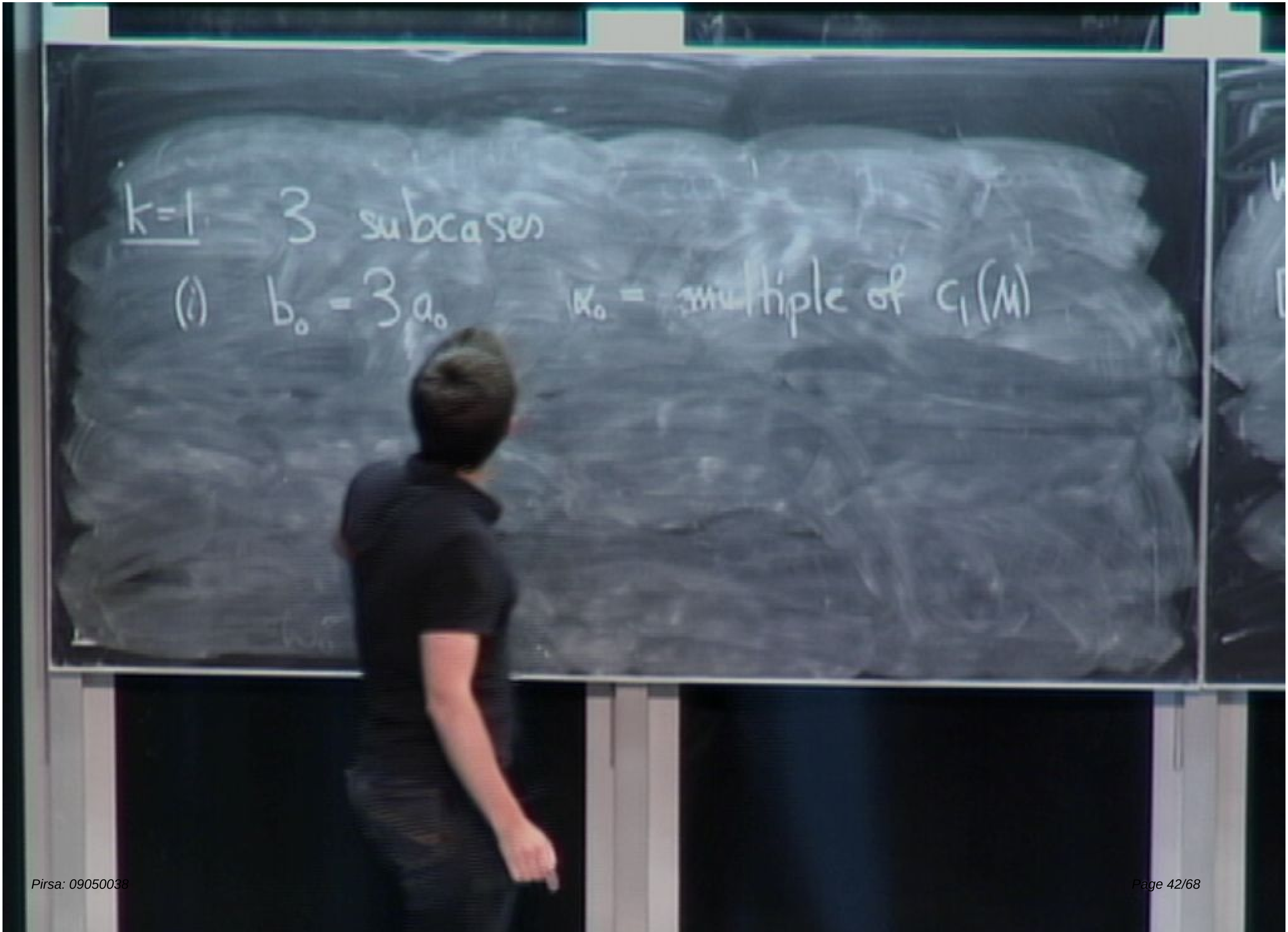

k=1 3 subcases
(i) b₀ = 3a₀ α₀ = multiple of c₁(M)

$\mathbb{D}^2$ blown up at one pt.

$\underline{k=1}$ 3 subcases

(i) $b_0 = 3a_0$ $\alpha_0 =$ multiple of $c_1(M)$

$\mathbb{P}^2$ blown up at one pt.

$\underline{k=1}$ 3 subcases

(i) $b_0 = 3a_0$ $\quad \alpha_0 =$ multiple of $c_1(M)$

$(M, g(t)) \xrightarrow{GH} \{pt\}$ as $t \to T = a_0$

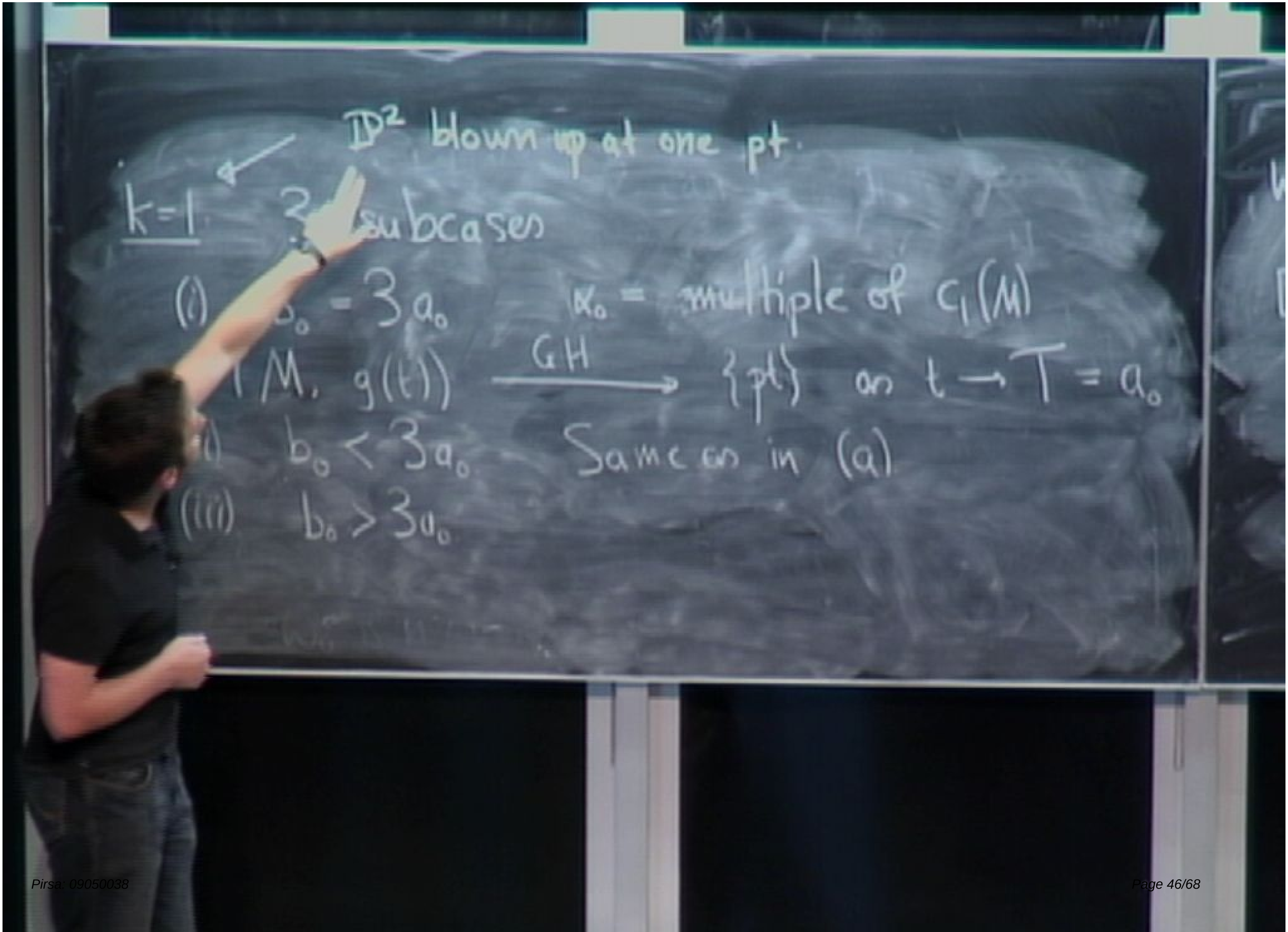
$\mathbb{P}^2$ blown up at one pt.

**$k=1$** 3 subcases

(i) $b_0 = 3a_0$ $\qquad \alpha_0 =$ multiple of $c_1(M)$

$$(M, g(t)) \xrightarrow{GH} \{pt\} \text{ as } t \to T = a_0$$

(ii) $b_0 < 3a_0$ $\qquad$ Same as in (a)


$\mathbb{P}^2$ blown up at one pt.
k=1  3 subcases
(i) $b_0 = 3a_0$  $\alpha_0$ = multiple of $c_1(M)$
$(M, g(t)) \xrightarrow{GH} \{pt\}$ as $t \to T = a_0$
(ii) $b_0 < 3a_0$  Same as in (a)
(iii) $b_0 > 3a_0$

$\mathbb{P}^2$ blown up at one pt.

$\underline{k=1}$ 3 subcases

(i) $b_0 = 3a_0$ $\quad \alpha_0 =$ multiple of $c_1(M)$

$(M, g(t)) \xrightarrow{GH} \{pt\}$ as $t \to T = a_0$

(ii) $b_0 < 3a_0$ Same as in (a)

(iii) $b_0 > 3a_0$

$\mathbb{P}^2$ blown up at one pt.

$\underline{k=1}$ 3 subcases

(i) $b_0 = 3a_0$ $\quad \alpha_0 =$ multiple of $c_1(M)$

$(M, g(t)) \xrightarrow{GH} \{pt\}$ as $t \to T = a_0$

(ii) $b_0 < 3a_0$ Same as in (a)

(iii) $b_0 > 3a_0$ KRF exists on $[0,T)$, $T = a_0$

$g(t) \to g_T$ smoothly on $M - D_0$

Write $(\bar{M}, d_T)$ as completion of

then as $t \to T$

$$(M, g(t)) \xrightarrow{GH} (\bar{M}, d_T)$$

finite diameter

$\cong D^2$

$a_0$

then as $t \to T$

$$(M, g(t)) \xrightarrow{GH} (\bar{M}, d_T)$$

finite diameter

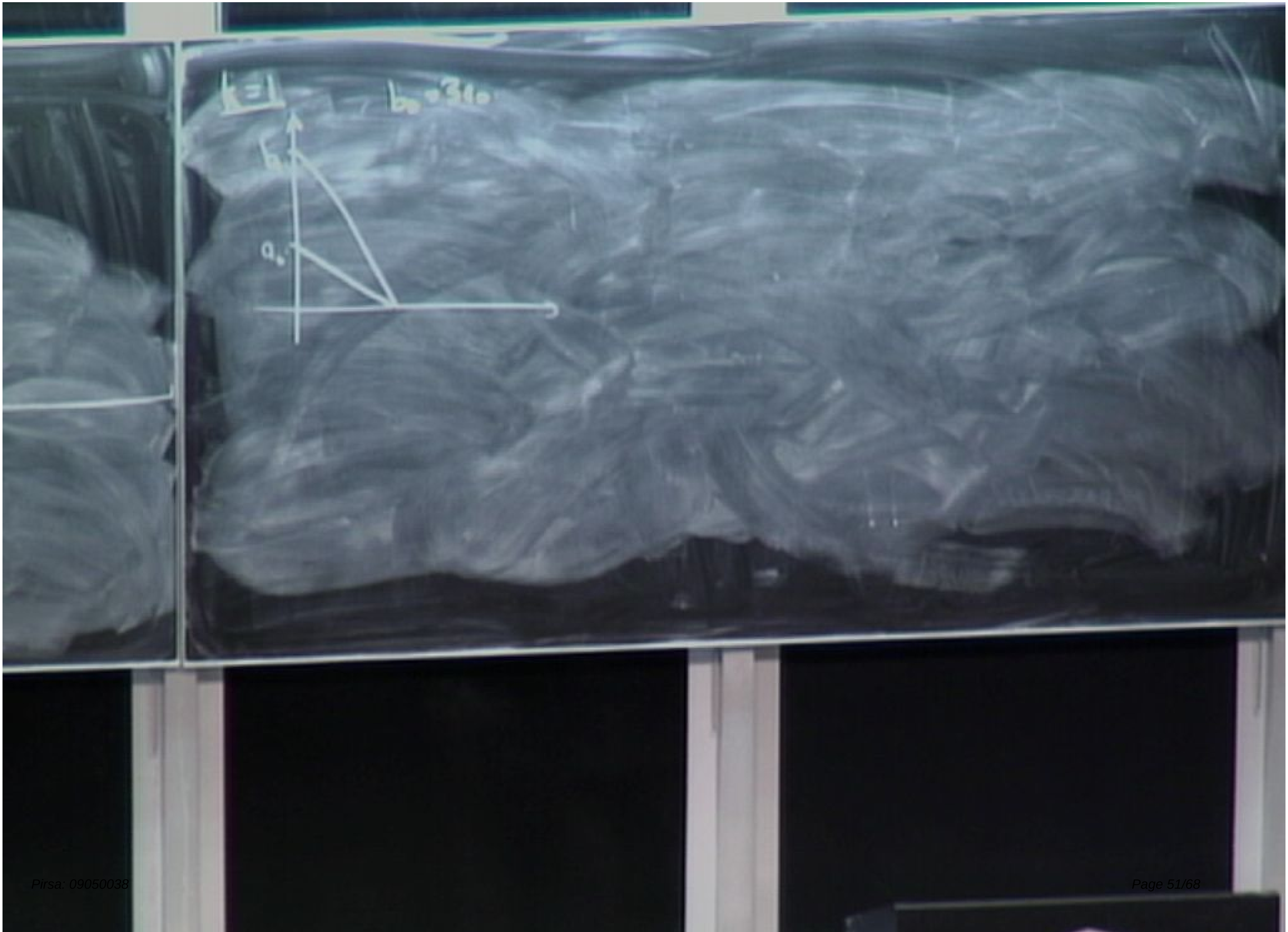
$\cong D^2$

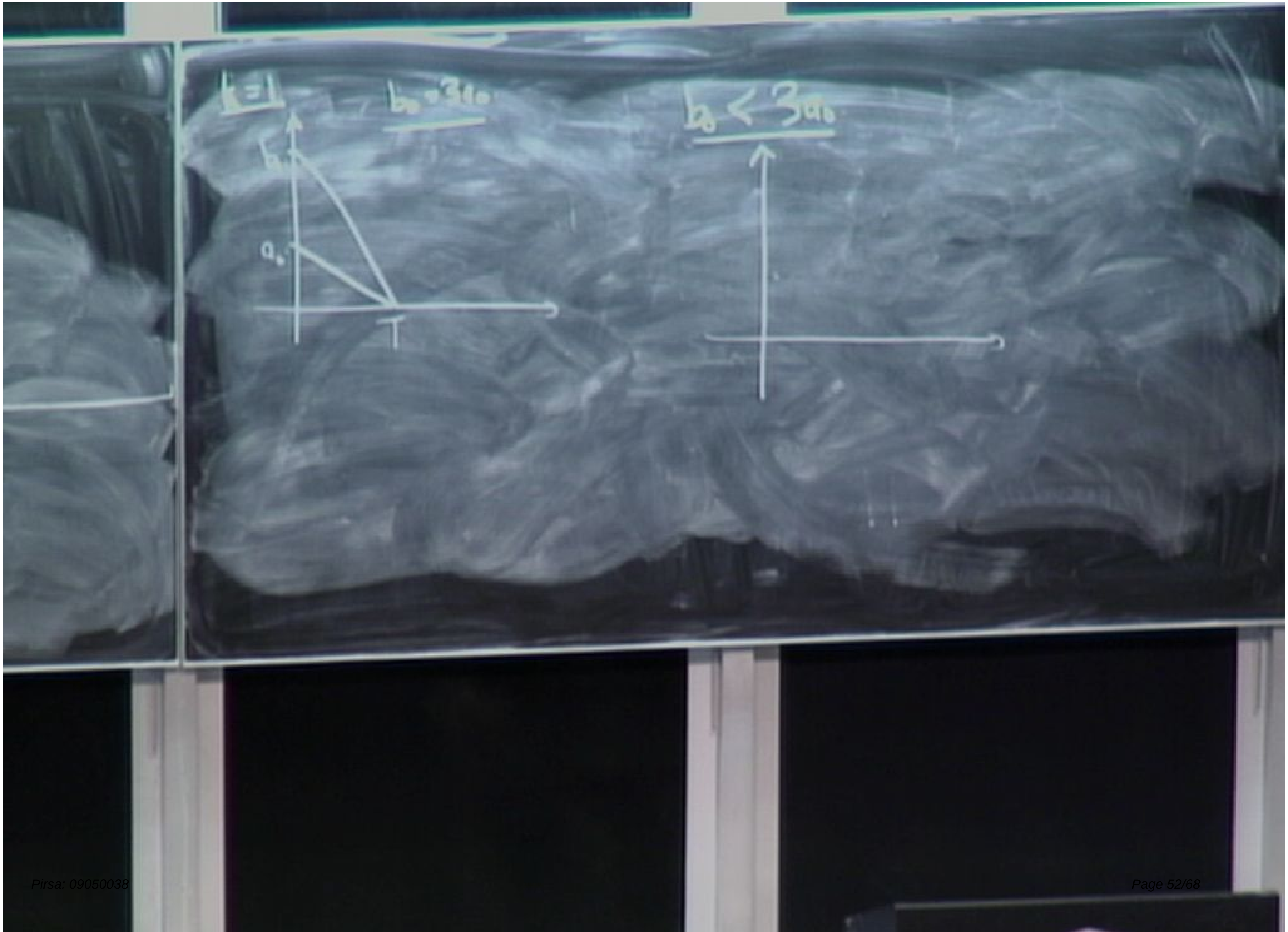
Recall $\alpha_t = \frac{b_t}{k}[D_\infty] - \frac{a_t}{k}[D_0]$

$b_t = b_0 - (k+2)t$

$a_t = a_0 + (k-2)t$

$a_0$


k=1
b0 = 3a0
b0
a0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b0 = 3a0
b0
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T
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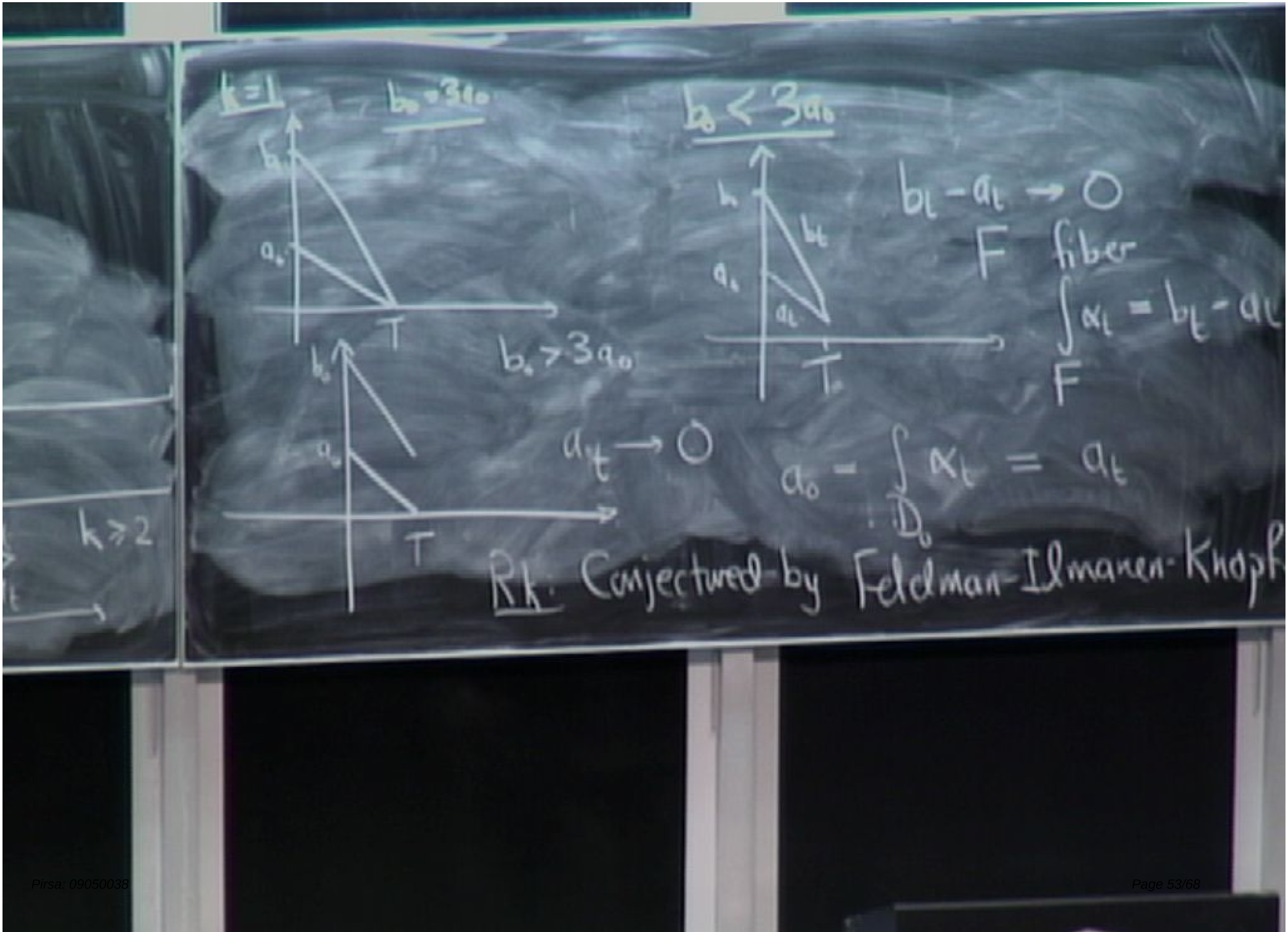

k=1
$b_0 = 3a_0$
$b_0 < 3a_0$
$b_t - a_t \to 0$
F fiber
$\int_F \alpha_t = b_t - a_t$
$b_0 > 3a_0$
$a_t \to 0$
$a_0 = \int_{D_0} \alpha_t = a_t$
$k \geq 2$
Rk: Conjectured by Feldman-Ilmanen-Knopf

## Higher dim'l analog

$n \geq 2$,

$$M_{n,k} = \mathbb{P}(H^k \oplus \mathbb{C}_{\mathbb{P}^{n-1}})$$

$H$, $\mathbb{C}_{\mathbb{P}^{n-1}}$ line bundles over $\mathbb{P}^{n-1}$

**Thm** (Song, W.) KRF, $\omega_0$ inv. $G_k = U(n)/\mathbb{Z}_k$

(a) $k \geq n$ $\quad (M, g(t)) \xrightarrow{GH} (\mathbb{P}^{n-1}, a_T\, g_{FS})$

as $t \to T = \frac{b_0 - a_0}{2k}$

(b) $1 \leq k \leq n-1$,

(i) $b_0(n-k) = a_0(n+k)$ $\quad (M, g(t)) \xrightarrow{GH} \{pt\}$

Thm (Song, W.) KRF, $\omega_0$ inv. $G_k = U(n)/\mathbb{Z}_k$

(a) $k \geq n$ $\quad (M, g(t)) \xrightarrow{GH} (\mathbb{P}^{n-1}, a_T g_{FS})$

as $t \to T = \frac{b_0 - a_0}{2k}$

(b) $1 \leq k \leq n-1$.

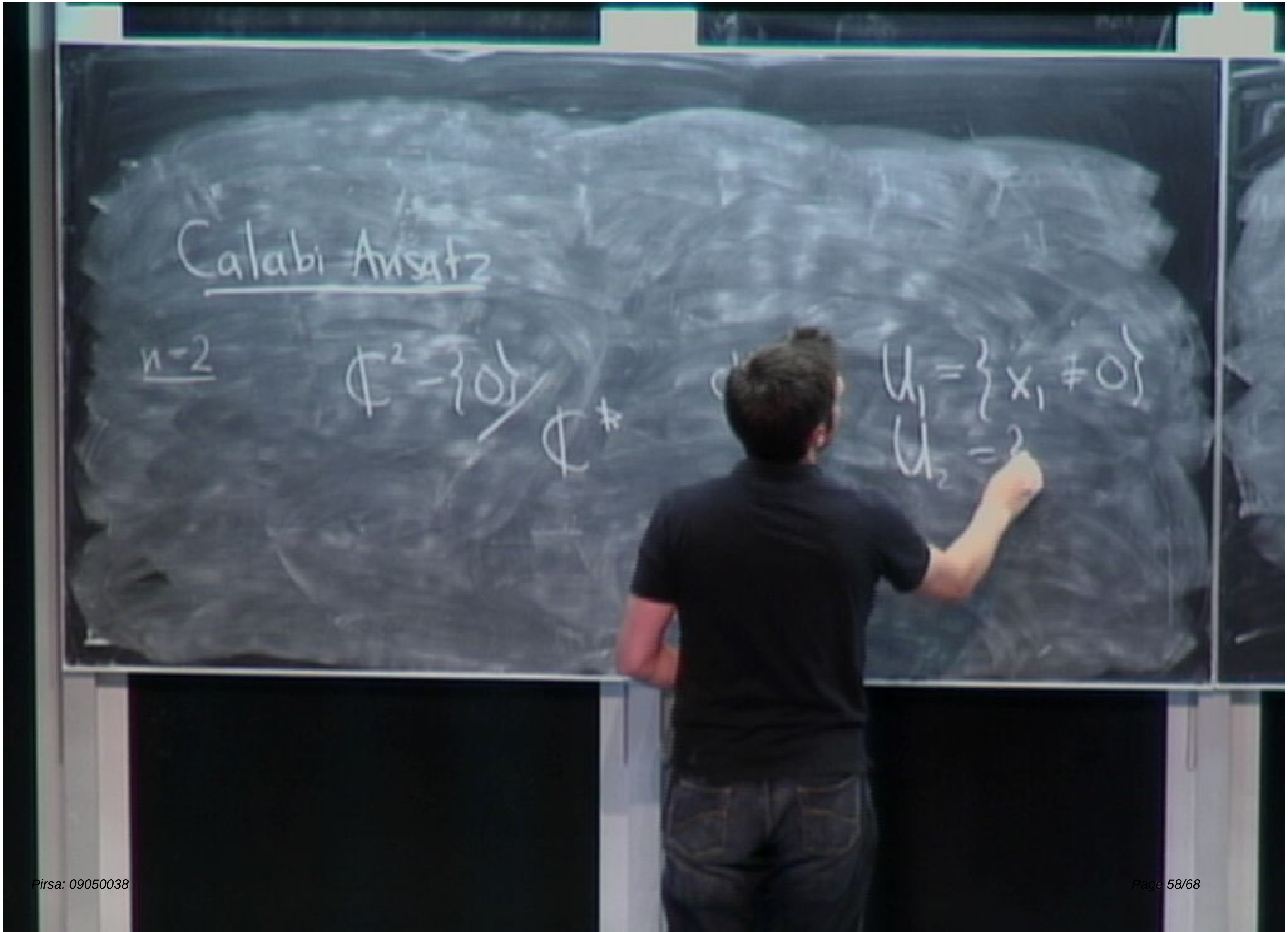
(i) $b_0(n-k) = a_0(n+k)$ $\quad (M, g(t)) \xrightarrow{GH} \{pt\}$

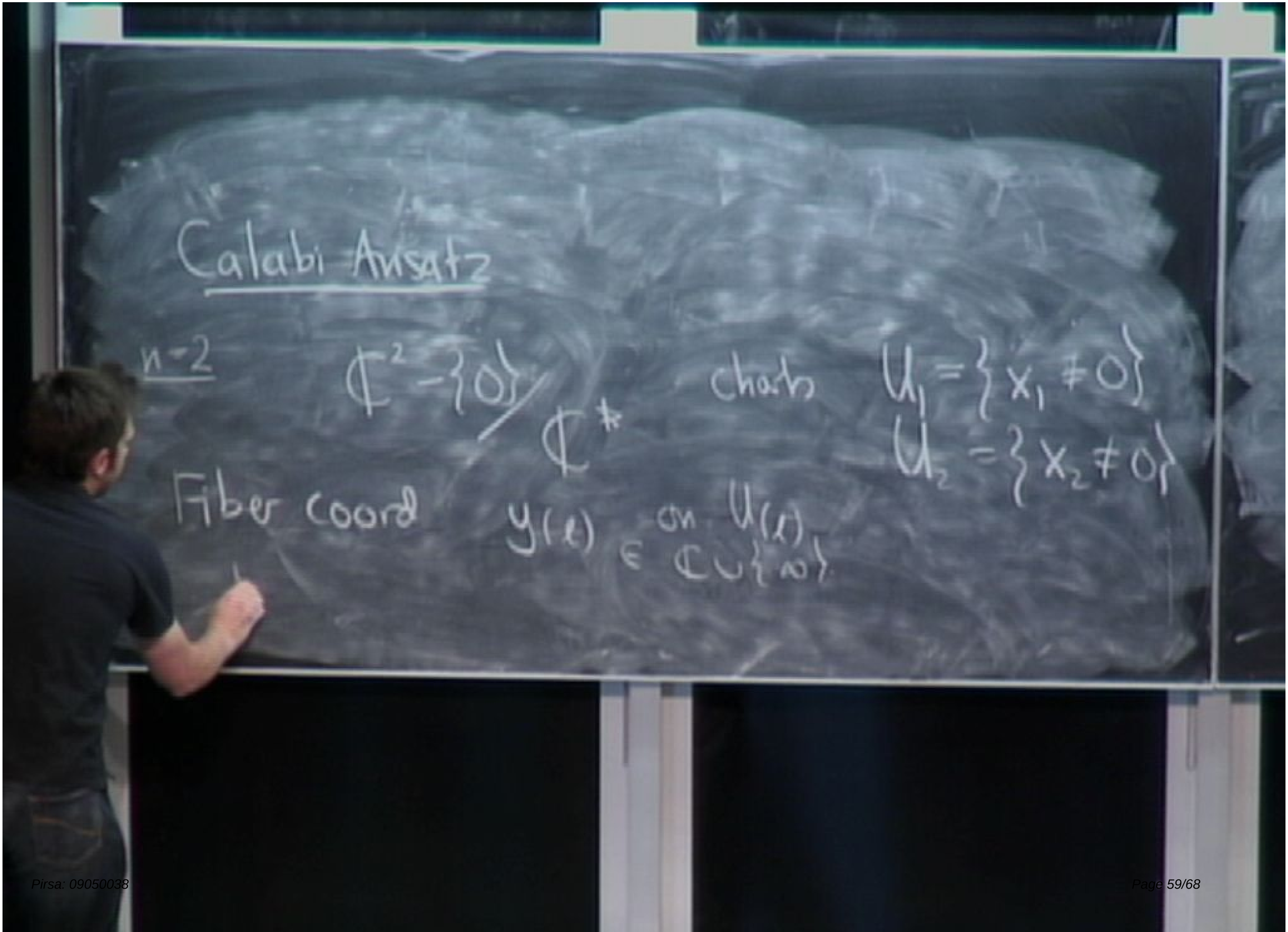
(ii) $b_0(n-k) < a_0(n+k)$, Same as (a)

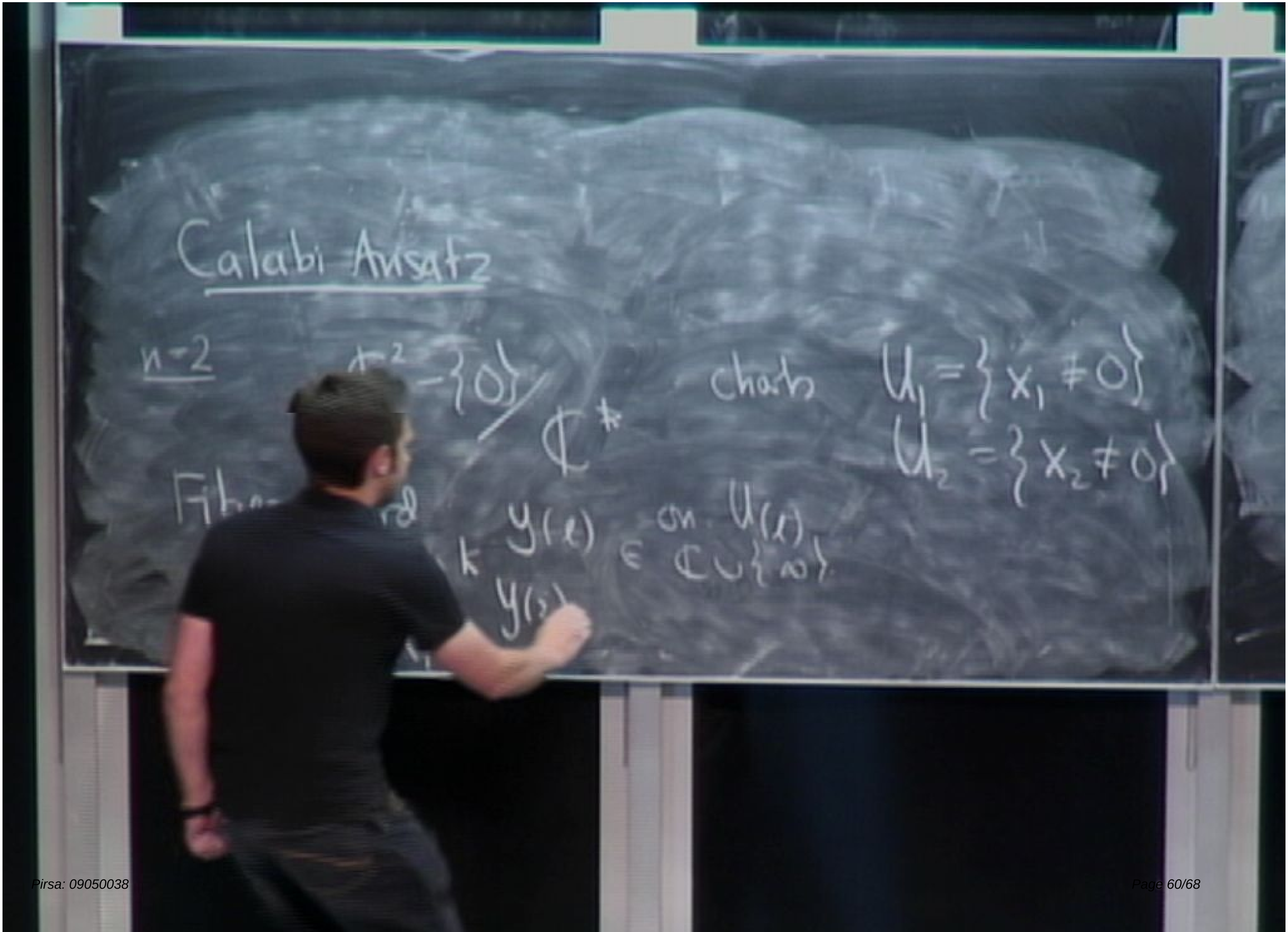
(iii) $b_0(n-k) > a_0(n+k)$

$$(M, g(t)) \xrightarrow{GH} (\bar{M}, d_T)$$

orbifold $\mathbb{B}^n / \mathbb{Z}_k$


Calabi Ansatz
n=2
ℂ² − {0} / ℂ*
U₁ = {x₁ ≠ 0}
U₂ =


Calabi Ansatz
n=2
$\mathbb{C}^2 - \{0\} / \mathbb{C}^*$
charts $U_1 = \{x_1 \neq 0\}$
$U_2 = \{x_2 \neq 0\}$
Fiber coord $y_{(\ell)}$ on $U_{(\ell)}$
$\in \mathbb{C} \cup \{\infty\}$


Calabi Ansatz
n=2
-{0}
ℂ*
charts
U₁ = {x₁ ≠ 0}
U₂ = {x₂ ≠ 0}
y(ℓ) on U(ℓ)
y(ℓ) ∈ ℂ∪{∞}
y(2)

## Calabi Ansatz

$n=2$ $\quad \mathbb{C}^2 - \{0\} / \mathbb{C}^*$ $\quad$ charts $U_1 = \{x_1 \neq 0\}$

$U_2 = \{x_2 \neq 0\}$

Fiber coord $\quad y_{(\ell)}$ on $U_{(\ell)}$ $\quad \in \mathbb{C} \cup \{\infty\}$

$$y_{(\ell)} = \left(\frac{x_\ell}{x_1}\right)^k y_{(2)} \quad \text{on } U_1 \cap U_\ell$$

Calabi Ansatz

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$$\mathbb{C}^2 - \{0\} \xrightarrow{k\text{-to-one}} M - (D_0 \cup D_\infty)$$

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$$(x_1, x_2) \longmapsto \text{in } U_1, \quad z_{(1)} = \frac{x_2}{x_1}, \; y_{(1)} = x_1^{\,k}$$

$U(2)$ acts on $\mathbb{C}^2$ corresponds to $G_k = U(2)/\mathbb{Z}_k$ on $M_k$

$0\}$

$\neq 0\}$

$0\} = D_0$

$0\} = D_\infty$

$$\mathbb{C}^2 - \{0\} \xrightarrow{k\text{-to-one}} M - (D_0 \cup D_\infty)$$

$$(x_1, x_2) \longmapsto \text{in } U_1, \quad z_{(1)} = \frac{x_2}{x_1}, \quad y_{(1)} = x_1^k$$

$U(2)$ acts on $\mathbb{C}^2$ corresponds to $G_k = U(2)/\mathbb{Z}_k$ on $M_k$

Kähler metric inv under $G_k$ is

$$\omega = \sqrt{-1}\partial\bar{\partial} u, \quad u = u(\rho), \quad \rho = \log(|x_1|^2 + |x_2|^2)$$

$u' > 0, \; u'' > 0$

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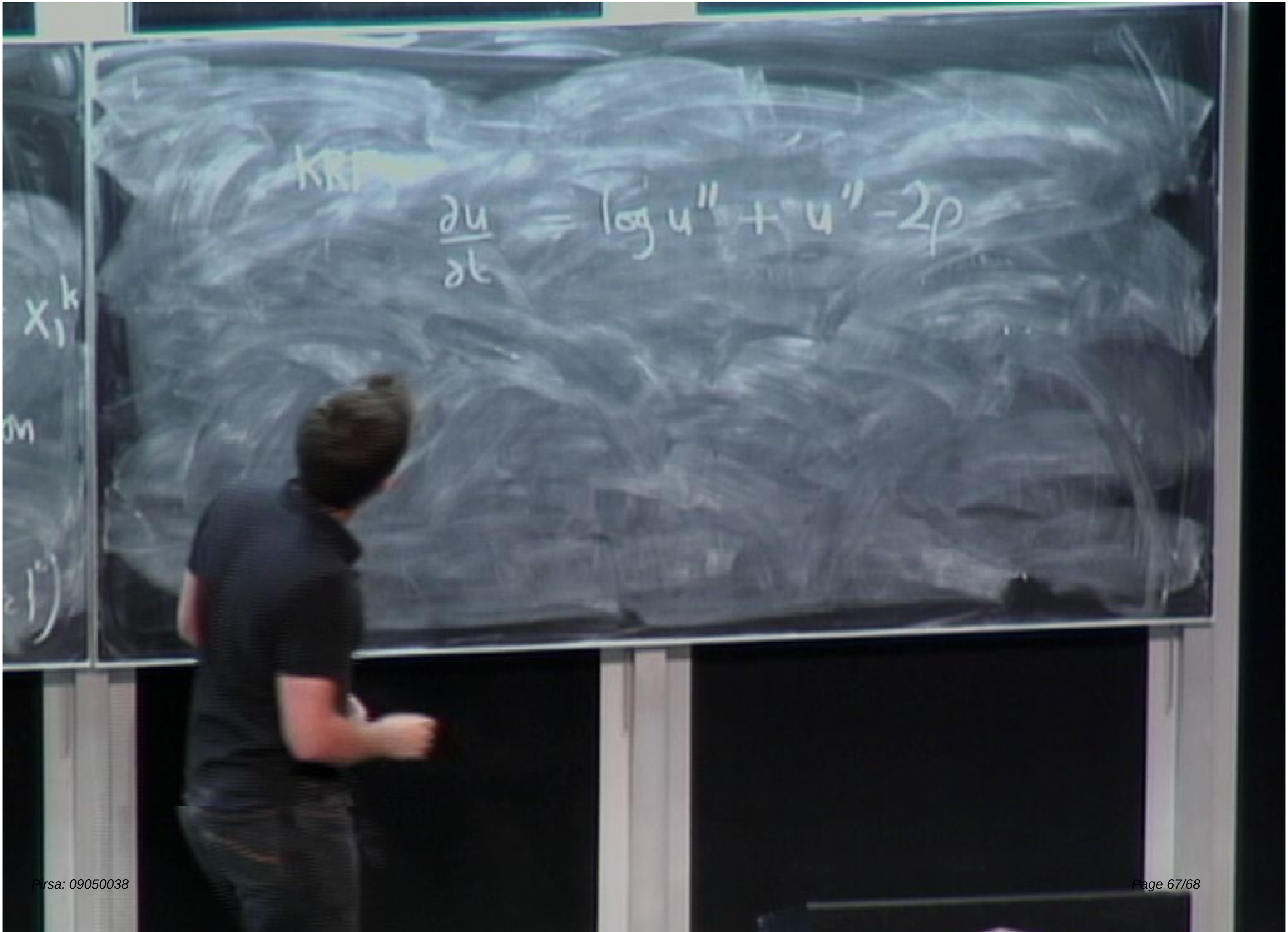
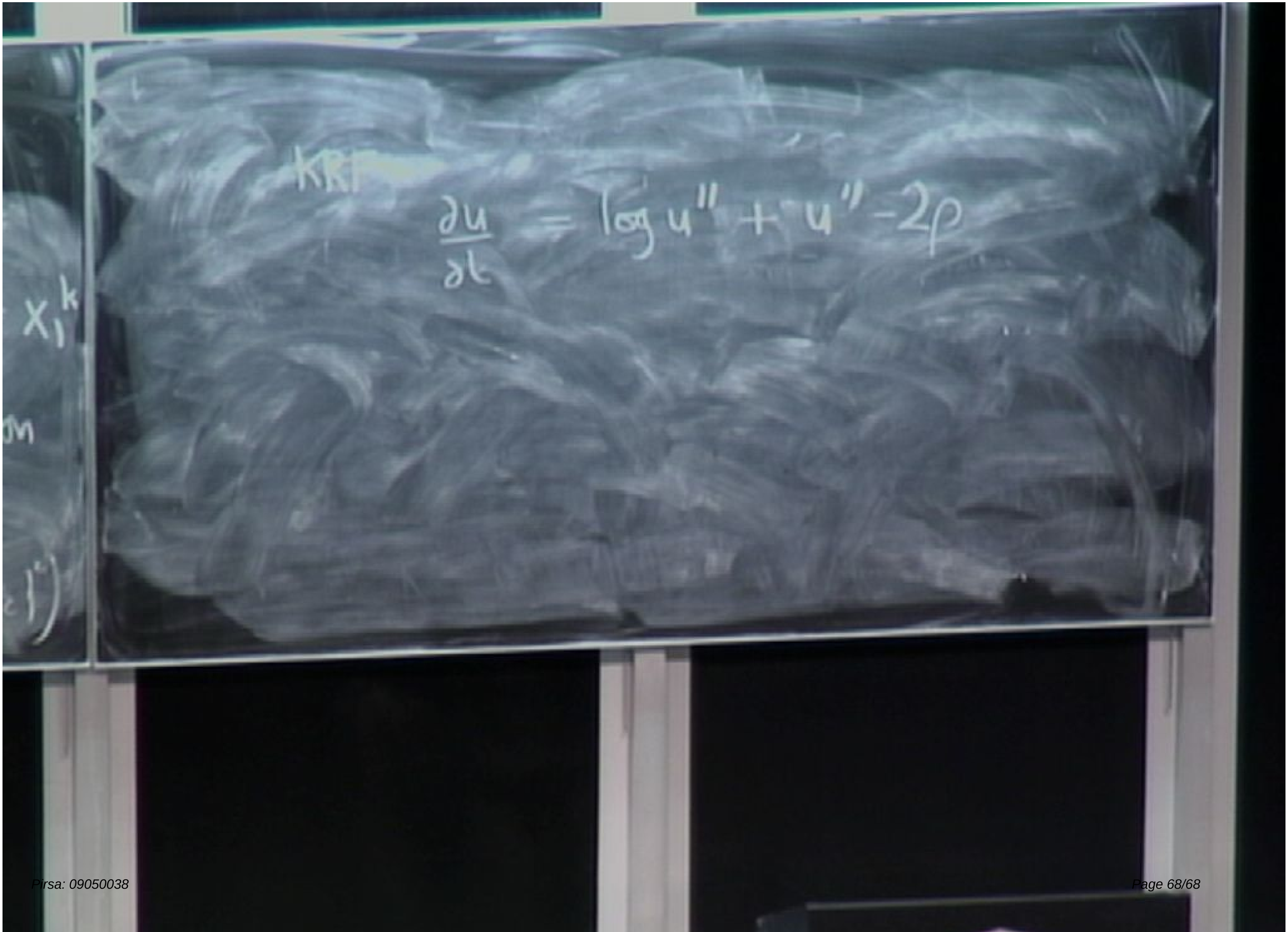
$u' > 0$, $u'' > 0$ $\quad u$ has behavior at $0$ & $\infty$


KR
$\frac{\partial u}{\partial t} = \log u'' + u'' - 2\rho$


KRF
$\frac{\partial u}{\partial t} = \log u'' + u'' - 2\rho$
$X_1^k$