Title: A Geodesic Equation for the Space of Sasakian metrics

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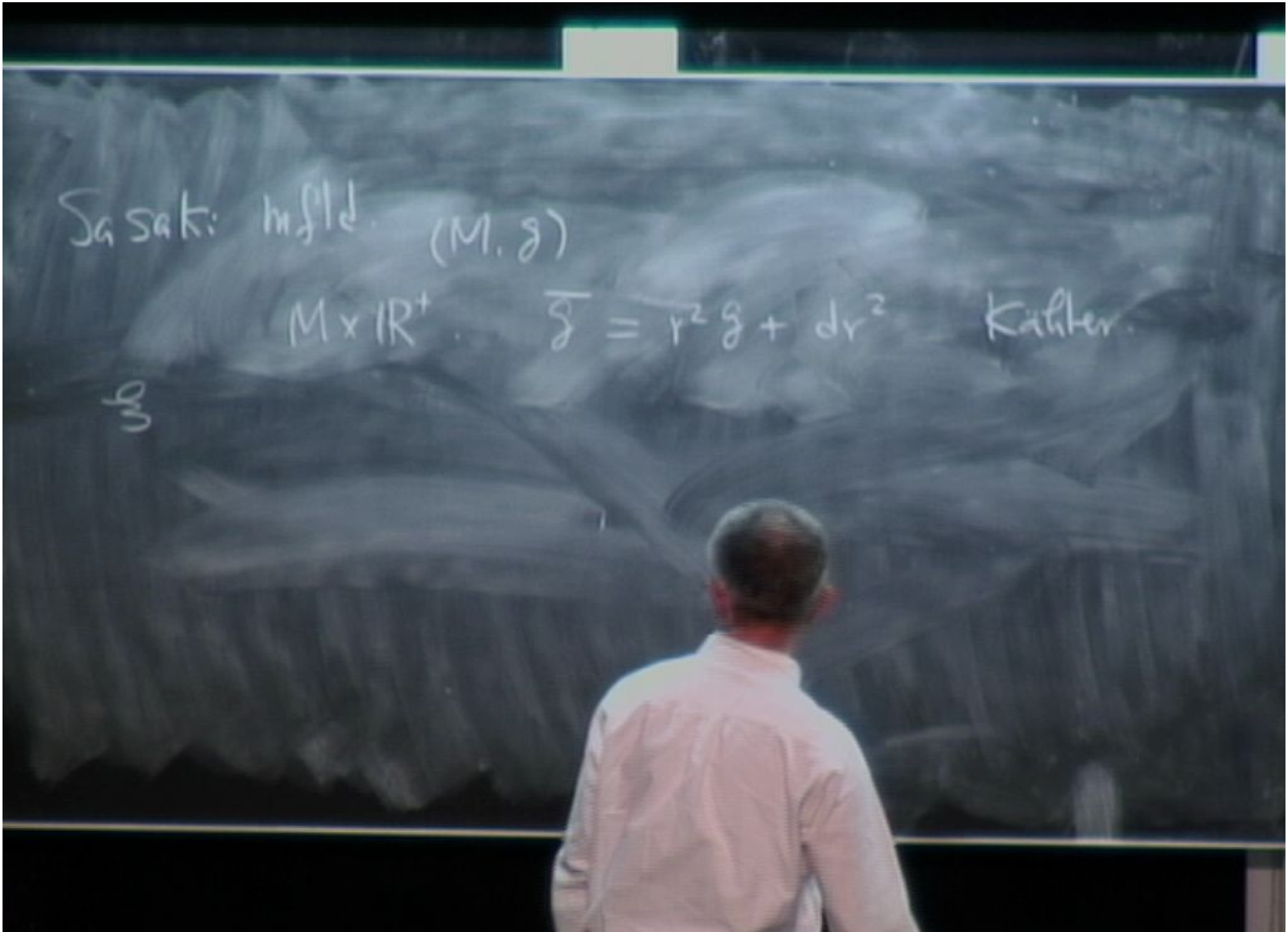
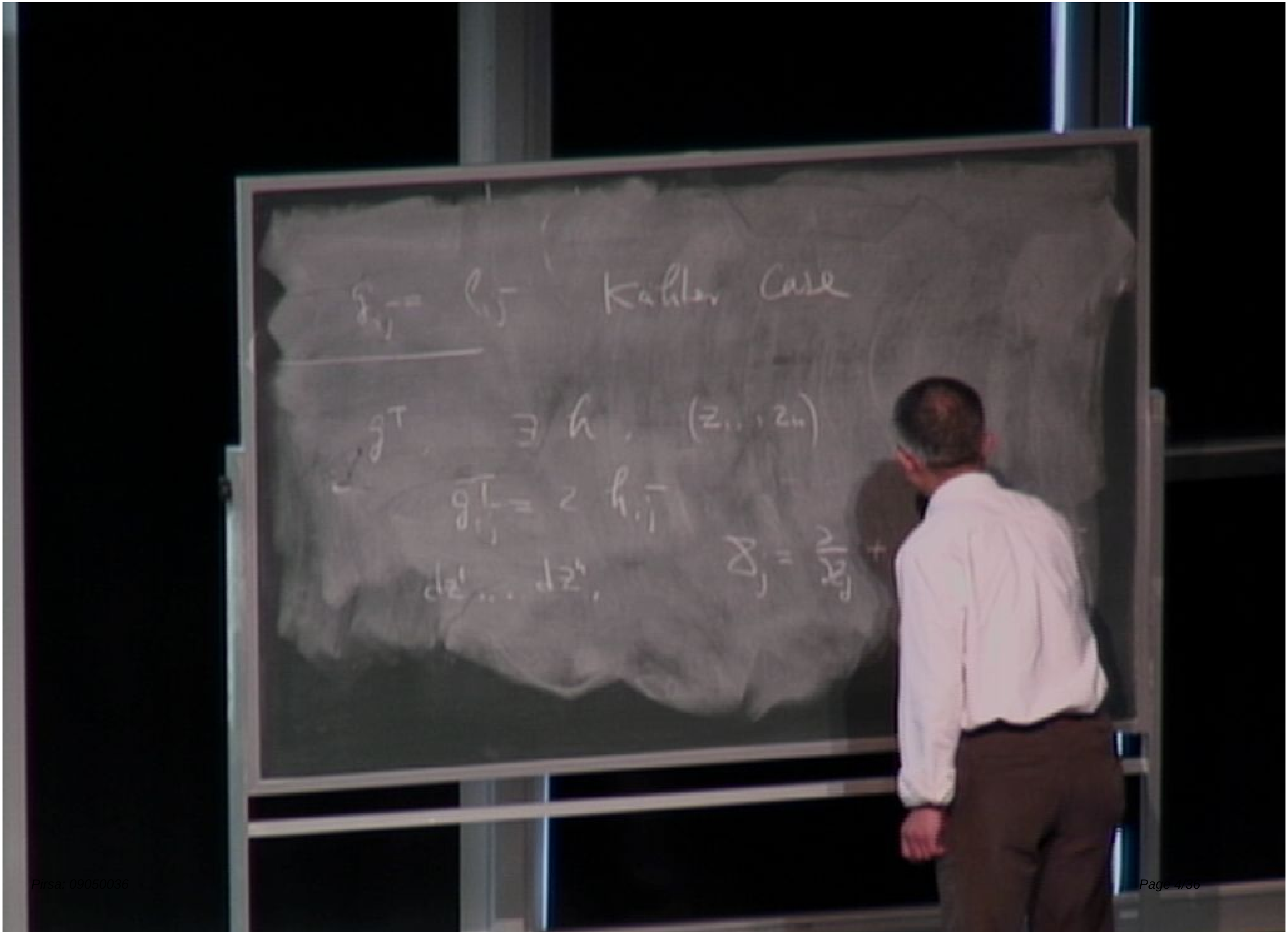
Abstract: "Sasakian geometry is often described as an odd dimensional counterparts of K\""ahler geometry. There is a natural Riemannian metric on the space of Sasakian metrics, which in turn gives a geodesic equation on this space. It can be viewed as parallel case of a well-known geodesic equation for the space of K\""ahler metrics. The equation is connected to some interesting geometric properties of Sasakian manifolds. It is a complicated complex Monge-Amp\`ere type involving
gradient terms. We discuss the problem of existence and regularity of solutions of this type of equations. This is a joint work with Xi Zhang."


Sasaki mfld. (M, g)
M × IR⁺ . g̅ = r² g + dr² Kähler.
g

Sasaki mfld. (M, g)

$M \times \mathbb{R}^+$. $\bar{g} = r^2 g + dr^2$ Kähler.

ξ. Transveres Kähler Structure on M.


Kahler case
(z1, ..., zn)
dz1 ... dzn

$g_{i\bar{j}} = h_{i\bar{j}}$ Kähler case

$\xi = \frac{\partial}{\partial x}$

$x + iy$

g^T $\exists\ h$, $(z_1, \dots, z_n)$.

$g^T_{i\bar{j}} = 2\, h_{i\bar{j}}$

$\delta_j = \frac{\partial}{\partial z_j} + \sqrt{-1}\, h_j \frac{\partial}{\partial x}$

$dz^1, \dots, dz^n$,

$\bar{\delta}_j$

$g_{i\bar{j}} = h_{i\bar{j}}$ Kähler case

$\xi = \frac{\partial}{\partial x}$

$x + iy \to w$

g^T, $\exists\ h$, $(z_1, \ldots, z_n)$,

$g^T_{i\bar{j}} = 2\, h_{i\bar{j}}$

$\delta_j = \frac{\partial}{\partial z_j} + \sqrt{-1}\, h_j \frac{\partial}{\partial x}$

$dz^1, \ldots, dz^n$,

$\bar{\delta}_j$

$(z_1, \ldots, z_n, w)$

Sasaki mfld (M^{2n+1}, g)

$M \times \mathbb{R}^+$. $\bar{g} = r^2 g + dr^2$ Kähler.

ξ. Transveres Kähler Structure on M.

$\bar{\partial}_B$. ∂_B

Sasaki: mfld (M^{2n+1}, g)

$M \times \mathbb{R}^+ \quad \bar{g} = r^2 g + dr^2 \quad$ Kähler.

$\xi \quad$ Transveres Kähler Structure on M.

$\bar{\partial}_B, \ \partial_B$

Sasaki: mfld (M^{2n}, g)

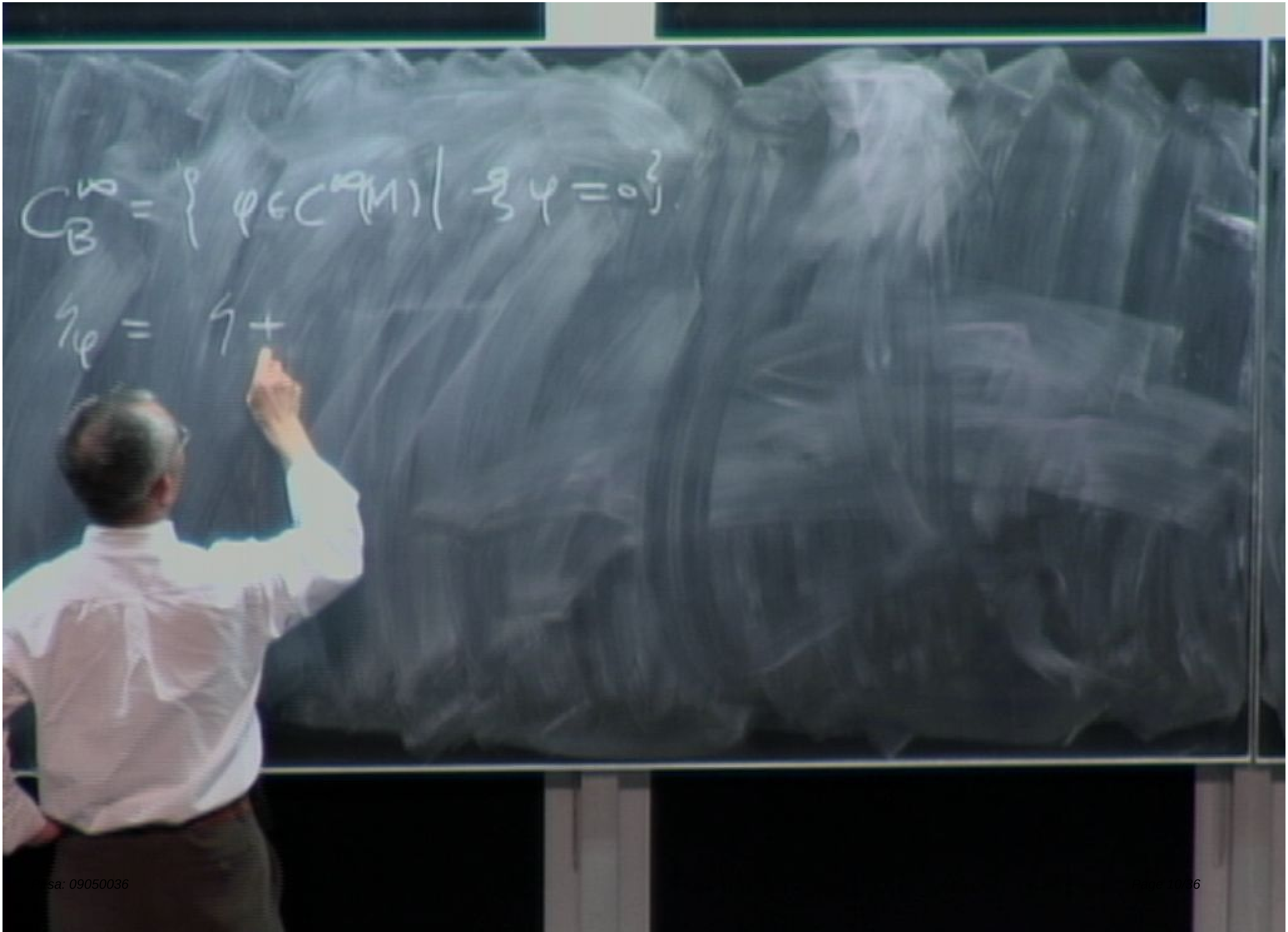
$M \times \mathbb{R}^+ \quad \bar{g} = r^2 g + dr^2 \quad$ Kähler.

ξ. Transverse Kähler Structure on M.

$\bar\partial_B, \partial_B \quad \xi, \quad \eta \quad$ 1-form

Deformation Space $\quad g^T = d\eta$

$$\mathcal{H} = \{ \varphi \in C^\infty_B(M) \mid \eta_\varphi \wedge (d\eta_\varphi)^n \neq 0 \}.$$


$C_B^\infty = \{ \varphi \in C^\infty(M) \mid \xi \varphi = 0 \}$,
$\eta_\varphi = \eta +$

$$C^{\infty}_{B} = \{ \varphi \in C^{\infty}(M) \mid \xi \varphi = 0 \}.$$

$$\eta_{\varphi} = \eta + \sqrt{-1}\, \partial_B \bar{\partial}_B \varphi.$$

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is natural measure $\varphi \in \mathcal{H}$.

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There is natural measure , $\varphi \in \mathcal{H}$.

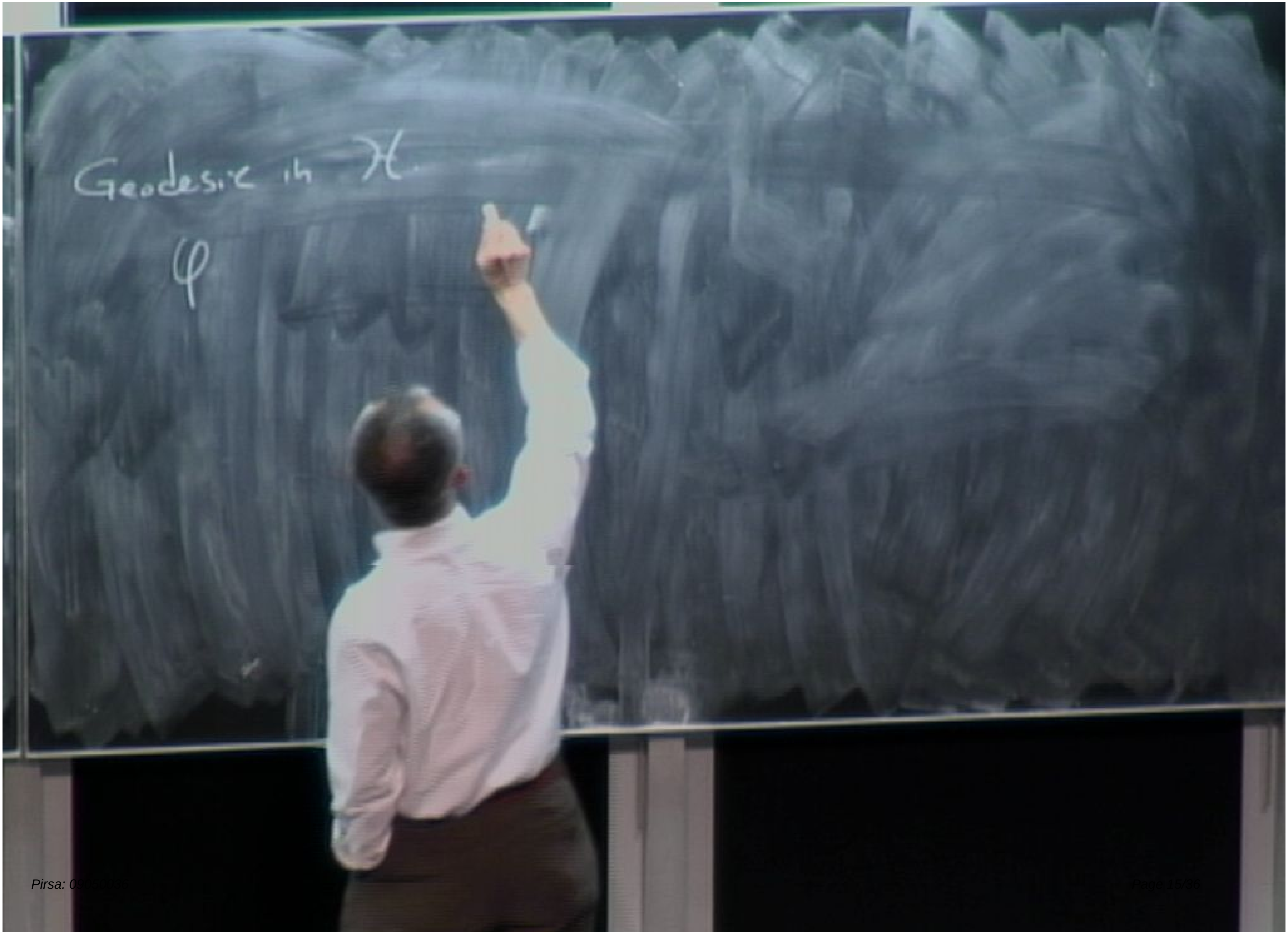
$\langle \psi_1, \psi_2 \rangle = \int$

$$C_B^\infty = \{ \varphi \in C^\infty(M) \mid \xi \varphi = 0 \}.$$

$$\eta_\varphi = \eta + \sqrt{-1}\, \partial_B \bar{\partial}_B \varphi.$$

There is natural measure, $\varphi \in \mathcal{H}$.

$$\langle \psi_1, \psi_2 \rangle_\varphi = \int_M \psi_1 \cdot \psi_2 \; \underline{\eta_\varphi \wedge (d\eta_\varphi)^n}$$


Geodesic in $\mathcal{H}$.
φ

Geodesic in $\mathcal{H}$

$$\ddot{\varphi}(t) - \frac{1}{4}\,|\nabla\dot{\varphi}|^2_{g_\varphi} = 0$$

Parallel to the Kähler case

Geodesic in $\mathcal{H}$

$$\ddot{\varphi}(x) - \frac{1}{4} |\nabla\dot{\varphi}|^2_{g_\varphi} = 0$$

Parallel to the Kähler case

Mabuchi - Semmes - Donaldson:

Geodesic in $\mathcal{H}$.

$$\ddot{\varphi}(t) - \frac{1}{4}\left|\nabla\dot{\varphi}\right|^2_{g_\varphi} = 0$$

Parallel to the Kähler case

Mabuchi - Semmes - Donaldson

Geodesic in $\mathcal{H}$.

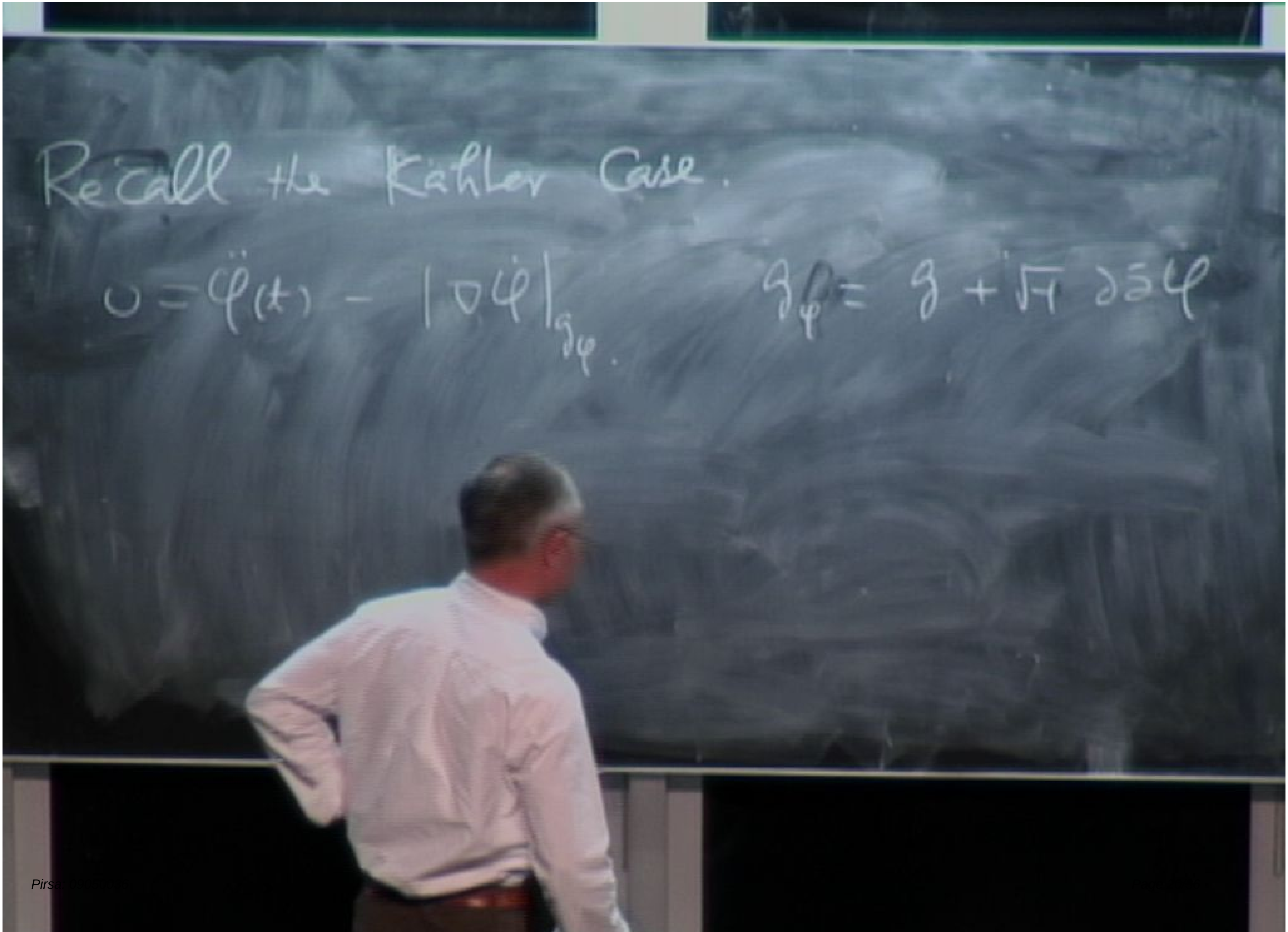
$$\ddot{\varphi}(t) - \frac{1}{4} |\nabla \dot{\varphi}|^2_{g_\varphi} = 0$$

Parallel to the Kähler case

Mabuchi - Semmes - Donaldson.

X. X. Chen $\exists!$ $C^{1,1}$ solution to geodesic equation

Cor: If $C_1 \le 0 \Rightarrow$ Uniqueness for extremal metric.


Recall the Kähler Case.
$0 = \ddot{\varphi}(t) - |\nabla\dot{\varphi}|_{g_\varphi}$.
$g_\varphi = g + \sqrt{-1}\,\partial\bar{\partial}\varphi$

Recall the Kähler Case. M

$$0 = \ddot{\varphi}(t) - |\nabla \dot{\varphi}|_{g_\varphi}. \qquad g_\varphi = g + \sqrt{-1}\,\partial\bar{\partial}\varphi$$

Recall the Kähler Case. M

$$Q = \ddot{\varphi}(t) - |\nabla \dot{\varphi}|_{g_\varphi}. \qquad g_\varphi = g + \sqrt{-1}\,\partial\bar{\partial}\varphi$$

$$Q \det(g_\varphi) = \varepsilon > 0$$

Recall the Kähler Case. M

$Q = \ddot{\varphi}(t) - |\nabla \dot{\varphi}|_{g_\varphi}$.

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Recall the Kähler Case. M

$$Q = \ddot{\varphi}(t) - |\nabla \dot{\varphi}|_{g_\varphi}. \qquad g_\varphi = g + \sqrt{-1}\,\partial\bar{\partial}\varphi$$

$$Q \det(g_\varphi) = \varepsilon > 0 \qquad w = t + is$$

$$\Updownarrow$$

Recall the Kähler Case. M

$Q = \ddot{\varphi}(t) - |\nabla \dot{\varphi}|_{g_\varphi}$. $g_\varphi = g + \sqrt{-1}\,\partial\bar{\partial}\varphi$

$\bar{g} = g + dw\wedge d\bar{w}$

$Q \det(g_\varphi) = \varepsilon > 0$ $w = t + is$

$\Updownarrow$

$i, j = 1 \ldots n.$ $\alpha, \beta = 1 \ldots n+1$

$\det(\bar{g}_{\alpha\bar{\beta}} + \varphi_{\alpha\bar{\beta}}) = \varepsilon \det(\bar{g}_{\alpha\bar{\beta}})$

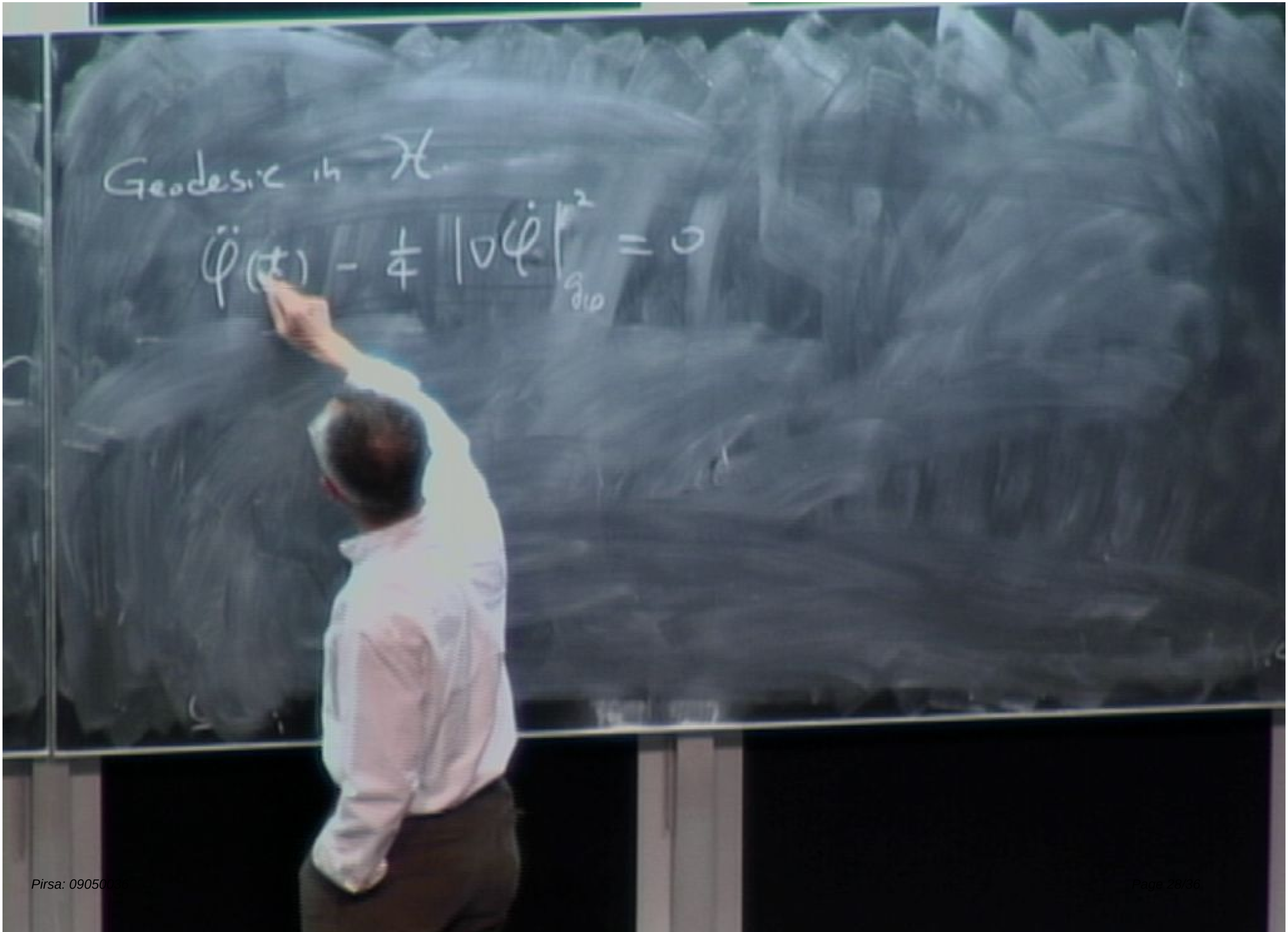
$\varphi(t)$

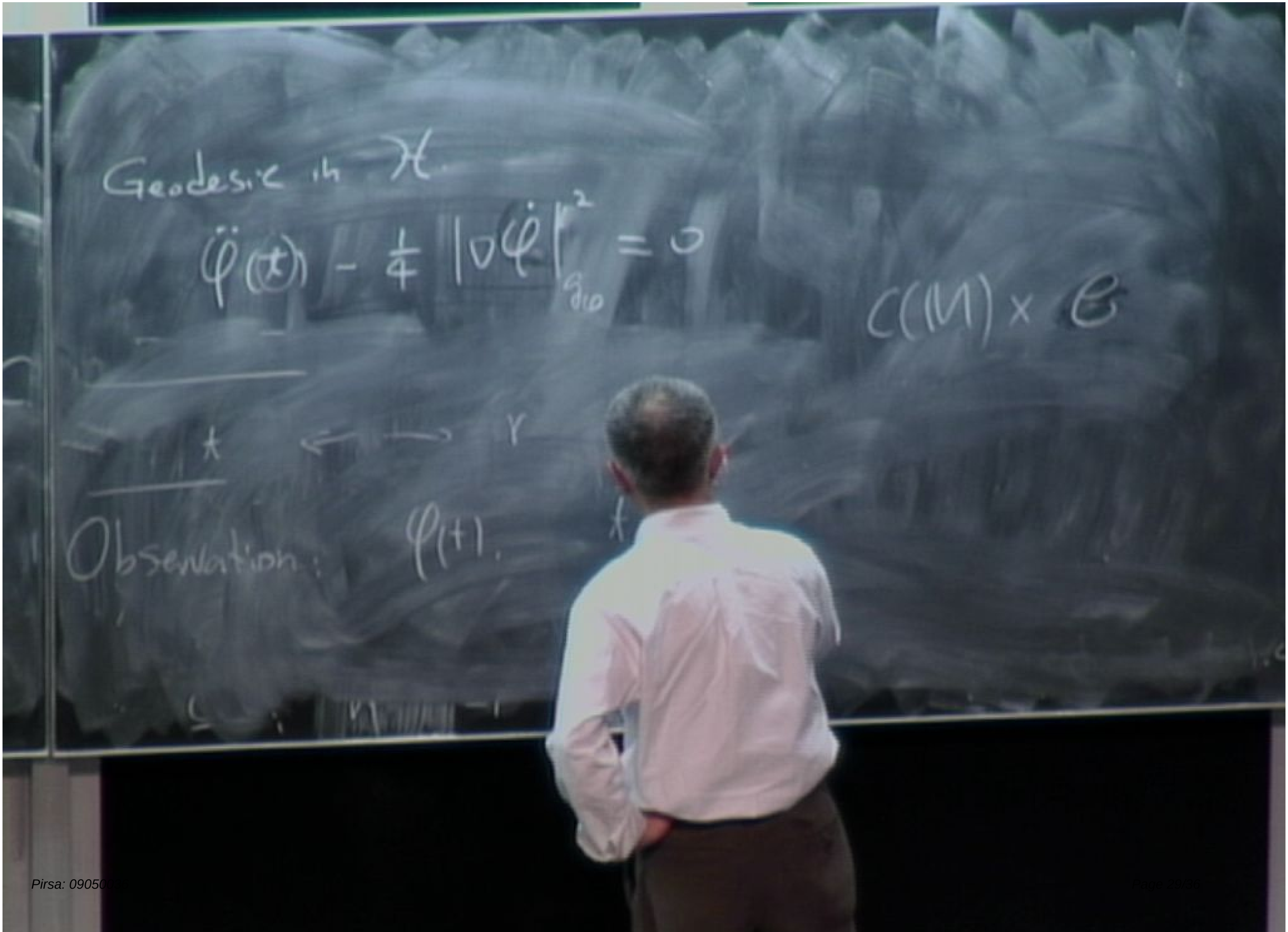
φ_1

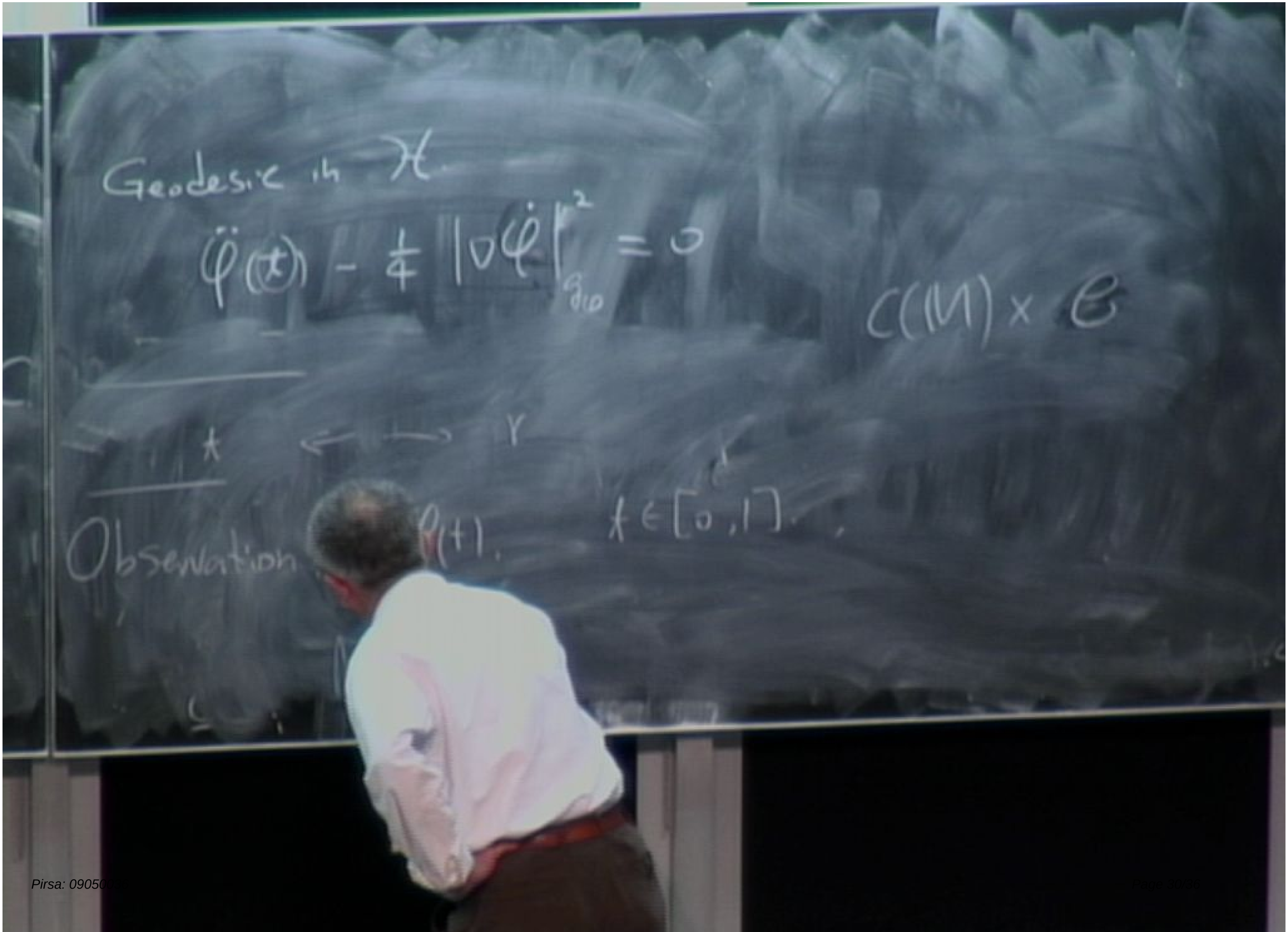
φ_0

X. X. Chen: $\exists!$ $\varphi(z, w)$. $\|\varphi\|_{C^1} + |\Delta\varphi| \leq$

C indep. of ε


Geodesic in $\mathcal{H}$.
$\ddot{\varphi}(t) - \frac{1}{4} \|\nabla\dot{\varphi}\|^2_{g_\varphi} = 0$


Geodesic in $\mathcal{H}$.
$\ddot{\varphi}(t) - \frac{1}{4} |\nabla \dot{\varphi}|^2_{g_{\varphi}} = 0$
$C(M) \times G$
$t \longleftrightarrow r$
Observation:
$\varphi(t)$.


Geodesic in $\mathcal{H}$.
$\ddot{\varphi}(t) - \frac{1}{4}\|\nabla\dot{\varphi}\|^2_{g_{\varphi}} = 0$
$C(M) \times G$
Observation
$\rho(t)$.
$t \in [0,1]$.

Geodesic in $\mathcal{H}$

$$\ddot{\varphi}(t) - \frac{1}{4} |\nabla \dot{\varphi}|^2_{g_{\varphi}} = 0$$

$$C(M) \times \mathcal{C}$$

$t \longleftrightarrow r$

Observation: $\varphi(t)$, $t \in [0, 1]$.

$$\bar{M} = M \times [1, \tfrac{3}{2}]$$

Geodesic in $\mathcal{H}$.

$$\ddot{\varphi}(t) - \frac{1}{4}\|\nabla\dot{\varphi}\|^2_{g_{\varphi}} = 0$$

$t \longleftrightarrow r$

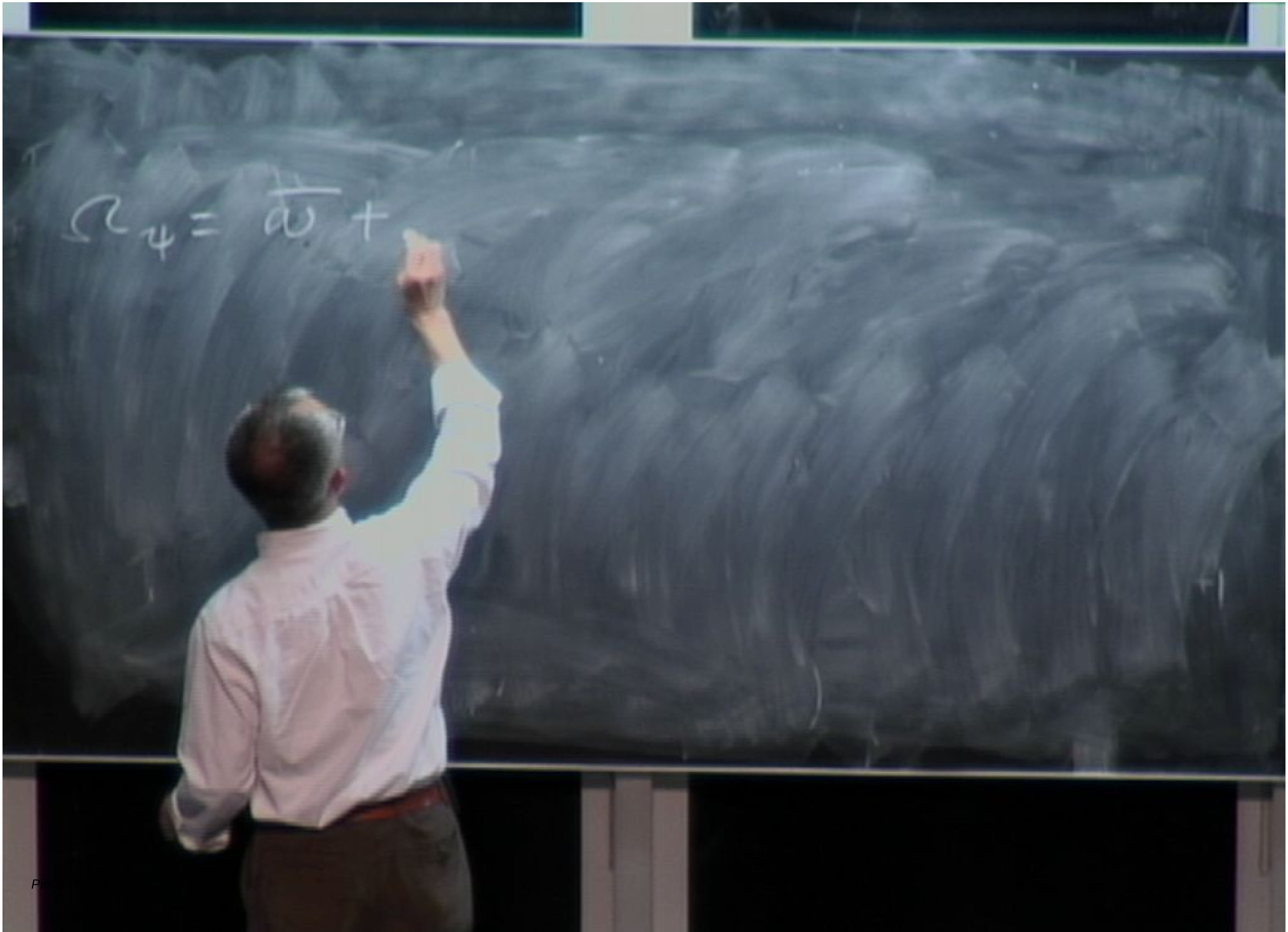
Observation: $\varphi(t)$, $t \in [0,1]$.

$$\overline{M} = M \times [1, \tfrac{3}{2}]$$

$C(M) \times \mathbb{C}$

$t = 2(r-1)$

$\psi(r,\cdot) = \varphi(2(r-1),\cdot) + 4\log r$


$\Omega_4 = \bar{\omega} +$

$$\Omega_\psi = \overline{\omega} + \frac{r^2}{2}\partial\bar{\partial}\psi - \frac{r^2}{2}\psi_r\,\partial\bar{\partial} r$$

$$\overline{\omega}$$

<u>Lemma</u>: $(\Omega_\psi)^{n+1} = 0 \iff (*)$

$$\mathcal{H} = \{ \varphi \in C^\infty_B(M) \mid \eta_\varphi \wedge (d\eta_\varphi)^n \neq 0 \}$$

$$d\eta_\varphi = d\eta + \sqrt{-1}\,\partial_B \bar{\partial}_B \varphi$$

$\mathcal{H}$ is a metric space, non-positive curved

geodesic convex.

abuchi: energy $\mathcal{M}$ if " $c_1 \leq 0$ "

$\mathcal{M} = \mathcal{M}(\varphi(t))$ is convex. $\Rightarrow$

Existence part of constant scalar curvature is gu