Title: Combinatorics inspired by Donaldson-Thomas theory

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Abstract: I will describe some combinatorial problems which arise when computing various types of partition functions for the Donaldson-Thomas theory of a space with a torus action. The problems are all variants of the following: give a generating function which enumerates the number of ways to pile n cubical boxes in the corner of a room. Often the resulting generating functions are nice product formulae, as predicted by the recent wall-crossing formulae of Kontsevich-Soibelman. There are now a variety of techniques, both geometric and combinatorial, to compute these formula. My work uses the entirely combinatorial techniques, namely vertex operators and the planar dimer model; these techniques can be applied essentially "bare-handed" and rely very little upon the underlying algebraic geometry.

Combinatorics inspired by Donaldson-Thomas theory

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Outline

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2. Brane tilings
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   - Examples
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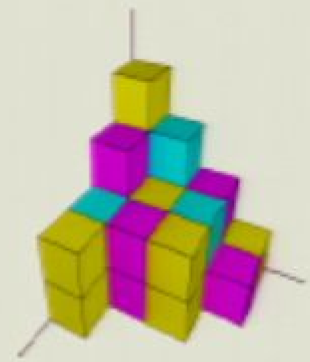
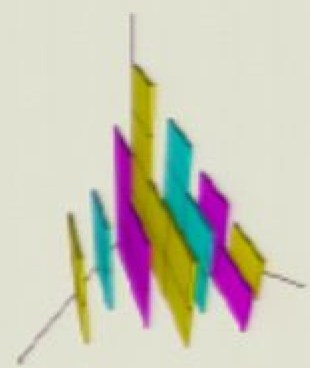
This is a *3D Young diagram*.



MacMahon (1916) gave the following generating function for 3D Young diagrams:

$$Z = \sum_{\substack{\pi \text{ 3D Young} \\ \text{diagram}}} q^{\#\{\text{cubes in } \pi\}} = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n} \right)^n .$$

One way to prove this (due to Okounkov) is to regard each π as a sequence of interlacing 2D Young diagrams.



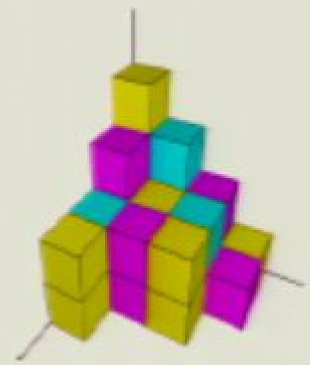
The transfer matrix (from one slice to the next) turns out to be a *vertex operator*.

To prove MacMahon's result, one computes a vacuum expectation value of a product of these opertors

We can get more from this method. For example:

If yellow (pink, blue) boxes contribute q_0 (resp. q_1, q_2) to Z, then

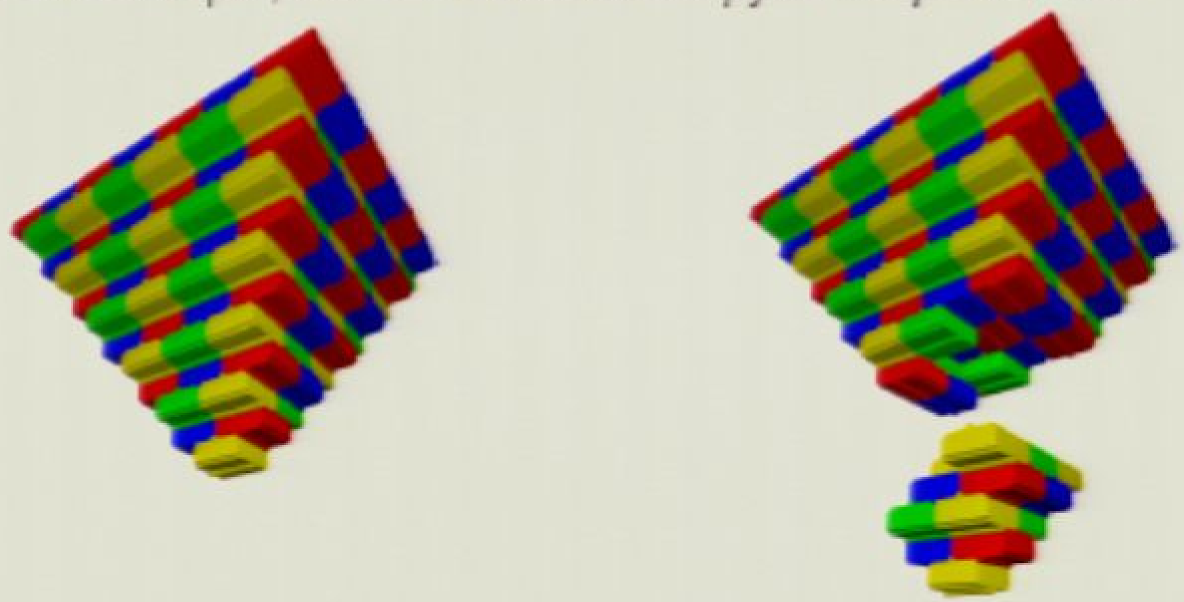
$$Z = M(1, q)^3 \widetilde{M}(q_1, q)\widetilde{M}(q_1 q_2, q)\widetilde{M}(q_2, q)$$



where $q = q_0 q_1 q_2$ and

$$M(a, x) = \prod_{n=1}^{\infty} \left(\frac{1}{1 - a x^n} \right)^n, \quad \widetilde{M}(a, x) = M(a, x) M(a^{-1}, x).$$

We can also deal with certain other lattices.
For example, we can enumerate *pyramid partitions*:



$$Z = \frac{M(1, q)^4 \widetilde{M}(q_a q_b, q) \widetilde{M}(q_b q_c, q)}{\widetilde{M}(-q_a, q) \widetilde{M}(-q_b, q) \widetilde{M}(-q_c, q) \widetilde{M}(-q_a q_b q_c, q)}.$$

These problems arise when computing *Donaldson-Thomas (DT) invariants*: signed counts of subschemes of a Calabi-Yau threefold.

For example, if the space is $\mathbb{C}^3$, the *DT partition function* turns out to be $M(1, -q)$.

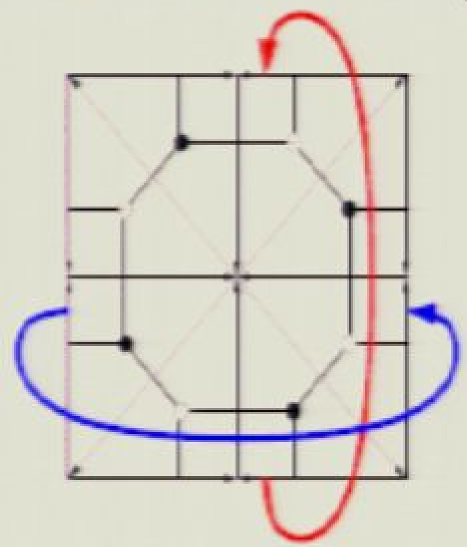
The other two generating functions are related to the DT partition functions of $\mathbb{C}^3/\mathbb{Z}_n$ and the conifold modulo $\mathbb{Z}_2$, respectively (Bryan-Y., Y.)

Where else can we apply vertex operators in this way?

We should replace the 3D Young diagrams with *framed cyclic A-modules*, where A is an algebra associated to a *brane tiling*.

A *brane tiling* consists of:

- F, a bipartite graph on a torus
- Q, the quiver dual to F

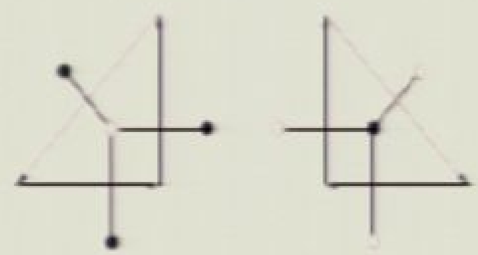


Arrows in Q go clockwise around black vertices of F.

$\mathcal{P}Q$ is the algebra of paths in Q.

Associate to each arrow a a vector $\mathrm{wt}(a) \in \mathbb{R}^3$ whose projection to the xy plane is a, and whose z-coordinate is positive.

Choose the z coordinates of the arrows so that loops around graph vertices increase z by 1.



Extend the quiver to cover the whole plane, and pick a base point v.

Paths starting at v determine points in $\mathbb{R}^3$ by summing the vectors associated to the arrows in the path. This generates a cone in a real 3-dimensional lattice.

The algebra A is $\mathcal{P}Q/I$, where I is an ideal associated to a *superpotential* of A, which encodes the torus embedding.

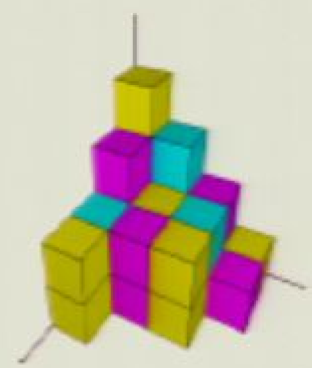
In nice cases (A *consistent*), the monomials in A are identified with this lattice. We will study ideals in this lattice (corresponding to certain representations of A).

Example: this brane tiling has only one vertex v, and three arrows.

One can compute that $A \simeq \mathbb{C}[x, y, z]$. The arrows map to $\hat{i}, \hat{j}, \hat{k} \in \mathbb{R}^3$, and $\mathrm{wt} : A \xrightarrow{\sim} \mathbb{N}^3$.

This brane tiling is consistent. An ideal of v-paths is a monomial ideals of $\mathbb{C}[x, y, z]$.

These are the same as 3D Young diagrams.

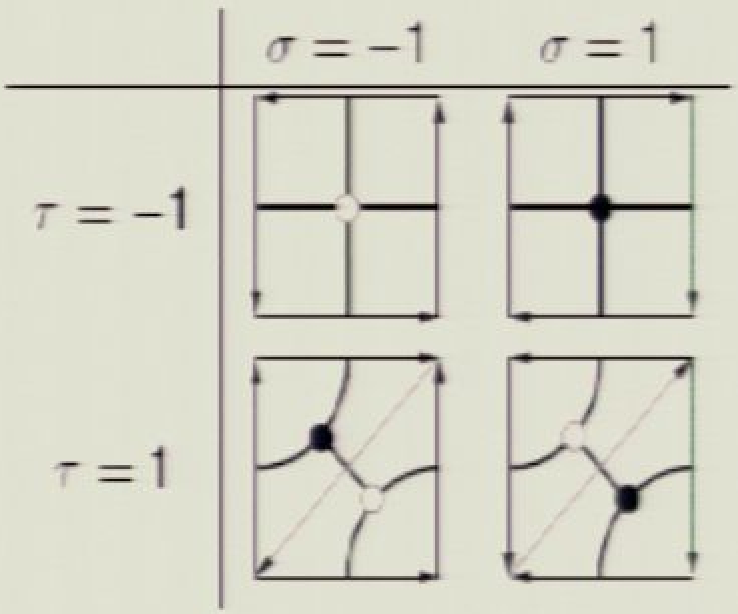


One can define Donaldson-Thomas invariants assoicated to A (Szendrői, Mozgovoy-Reineke), as a weighted Euler characteristic of a Hilbert scheme associated to A.

The *DT partition function* is the generating function for these invariants. For "nice" brane tilings (consistent and nondegenerate), we can compute the DT partition function as a signed sum over lattice ideals. .

The sign of each ideal can be computed purely formally, using the *Ringel form* of the quiver. If this form is diagonal, then a change of variables in Z produces the partition function.

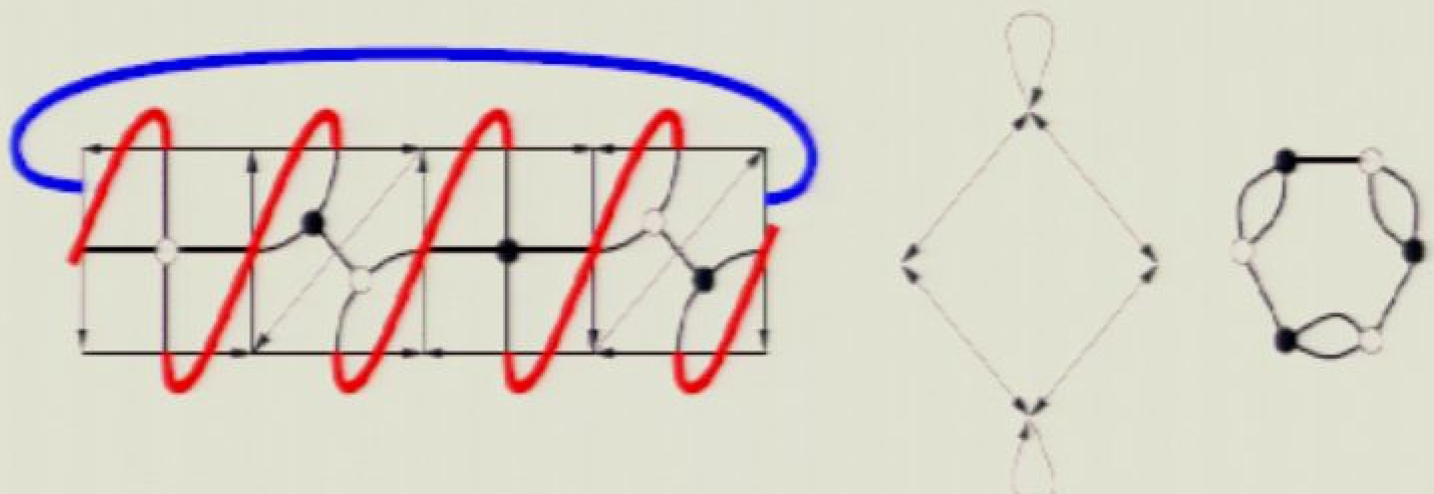
We now describe all of brane tilings whose partition functions are computable with vertex operators.



These tiles will be used to cover the fundamental domain of the torus and produce a brane tiling.

They are indexed by two signs, σ and τ.

The tiles should be assembled in a loop, so that the arrows match up. The loop is then wound around a torus. Each tile shares two arrows with the next.



Theorem: Combinatorial generating function

Let us establish the following notation:

- $T_0, \ldots, T_{k-1}$ is a loop of tiles, with $T_i = (\sigma_i, \tau_i)$
- $q_0, \ldots, q_{n-1}$ are indeterminates associated with each tile
- $q_{[a,b]} = q_a q_{a+1} q_{a+2} \cdots q_b$
- $q = q_0 q_1 \cdots q_{n-1}$

Then the generating function for A-partitions based at q_0 is

$$Z_A(q_1, \ldots, q_{n-1}, q) = M(1, q)^k \prod_{0<i<j<n} \widetilde{M}(\sigma_i \sigma_j q_{[i,j]}, q)^{\sigma_i \sigma_j}.$$

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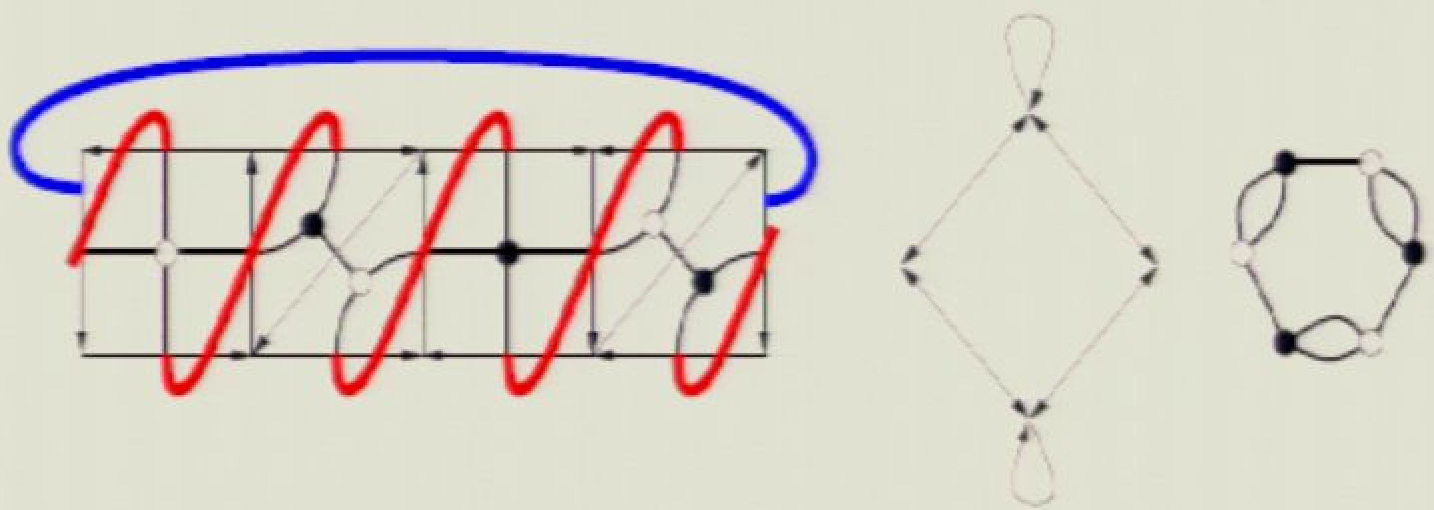

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Corollary: DT partition function

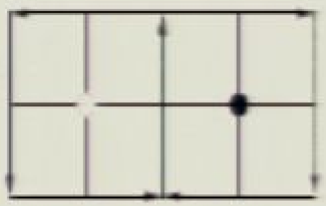
Same notation as the Theorem:

- $T_0, \ldots, T_{k-1}$ is a loop of tiles, with $T_i = (\sigma_i, \tau_i)$
- $q_0, \ldots, q_{n-1}$ are indeterminates associated with each tile
- $q = q_0 q_1 \cdots q_{n-1}$

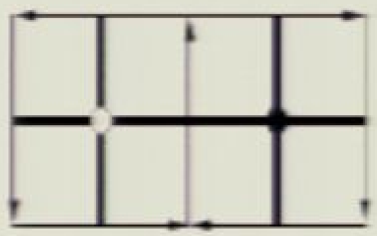
Then the Donaldson-Thomas partition function for A is

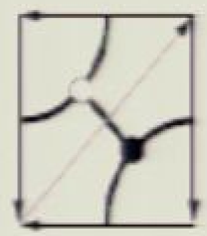
$$Z_A^{DT}(q_1, \ldots, q_{n-1}; q) = Z_A(\tau_1 q_1, \tau_2 q_2, \ldots, \tau_{n-1} q_{n-1}; -q).$$

This covers two of the old examples:

- Conifold: 
- $\mathbb{Z}_2$ orbifold of the conifold: two copies of the above
- $\mathbb{C}^3/\mathbb{Z}_n$: n copies of 

We also get some new examples, including the family $L^{a,b,a}$ (physics terminology):





Use a copies of this… and $b-a$ copies of this.

The most notable new example is the "suspended pinch point", $L^{1,2,1}$. Its DT partition function is

$$\frac{M(1,-q)^3\widetilde{M}(q_1,-q)}{\widetilde{M}(q_2,-q)\widetilde{M}(q_1q_2,-q)}$$

Mozgovoy and Reineke conjectured that for all consistent brane tilings,

$$\mathrm{PLog}(Z_A(x, \ldots, x))$$

is a rational function (here, PLog is the plethystic logarithm). Our results support their conjecture (not surprising).

Kontsevich-Soibelman: the product structure of $Z_A^{(DT)}$ is due to "wall-crossing" in a certain moduli space. Dimofte and Zukov use their approach to compute some "refined" DT invariants for the conifold.

My approach will likely also compute refined invariants.

It is unlikely that we can handle other quivers this way.

One could likely use n-component vertex operators to describe certain orbifolds of these quivers, but it is unlikely that the formulae would be as explicit, or that the combinatorial and DT functions would be related.

I conjecture that whenever the Ringel form is diagonal, there will be a nice product structure to both partition functions.