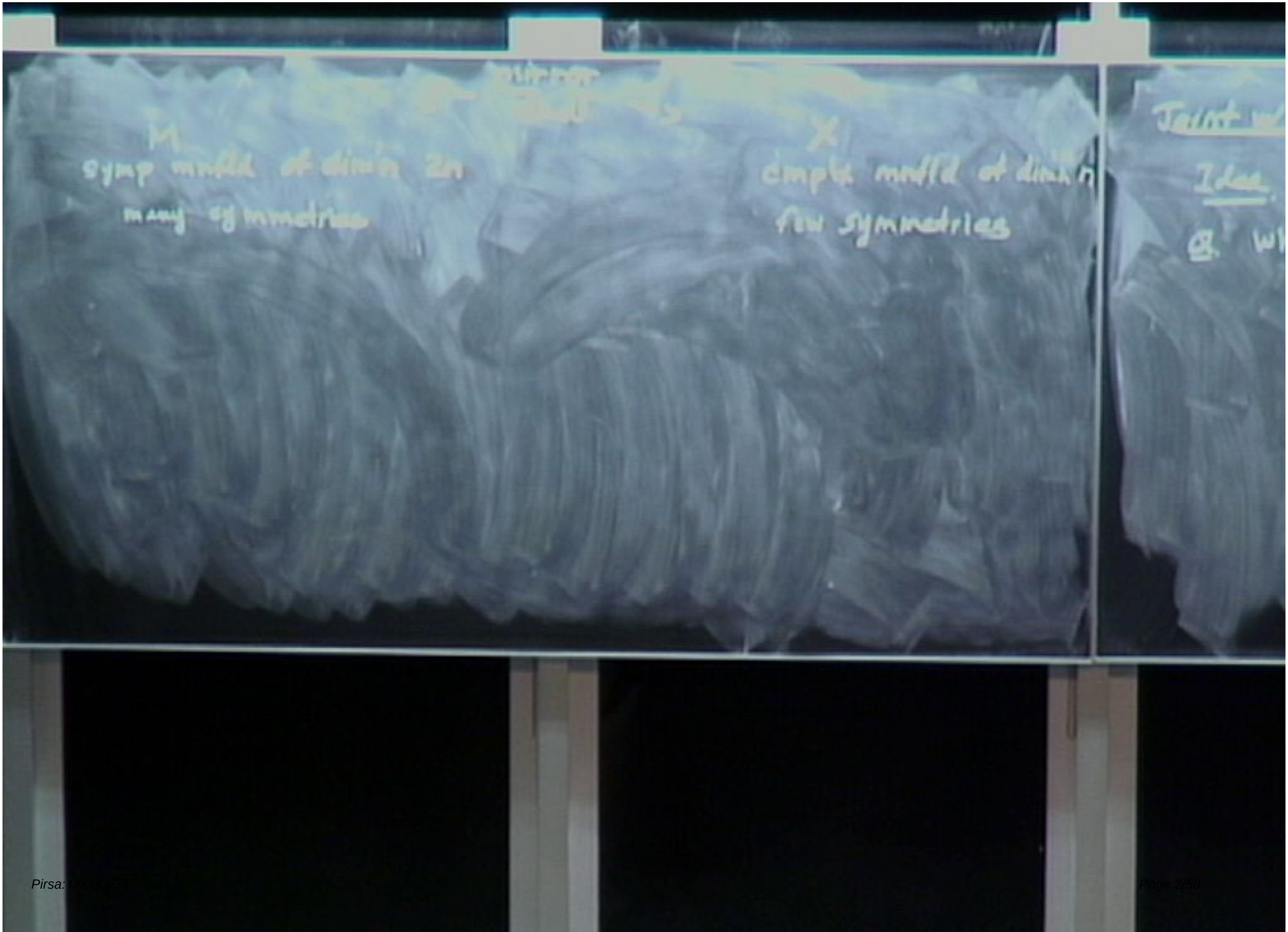
Title: Faithfulness of braid group actions on derived categories

Date: May 09, 2009 11:15 AM

URL: http://pirsa.org/09050034

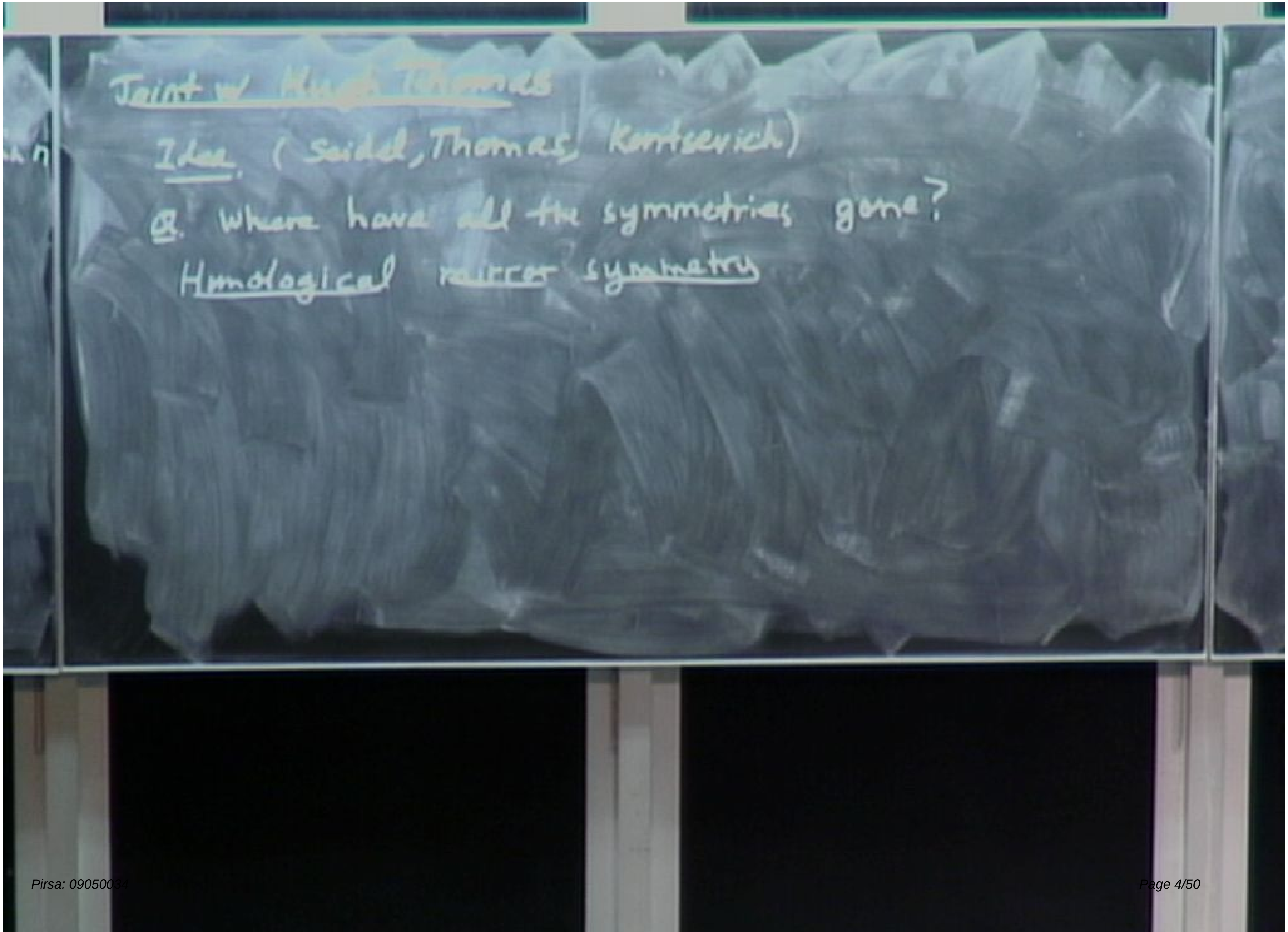
Abstract: Inspired by homological mirror symmetry, Seidel and Thomas constructed braid group actions on derived categories of coherent sheaves of various varieties and proved faithfulness of such actions for braid groups of type A. I will discuss joint work with Hugh Thomas giving some faithfulness results for derived braid group actions of types D and E


mirror
M
symp mfld of dim'n 2n
many symmetries
X
cmplx mfld of dim'n n
few symmetries

Joint w/ Hugh Thomas

Idea (Seidel, Thomas, Kontsevich)

Q. Where have all the symmetries gone?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Homological mirror symmetry

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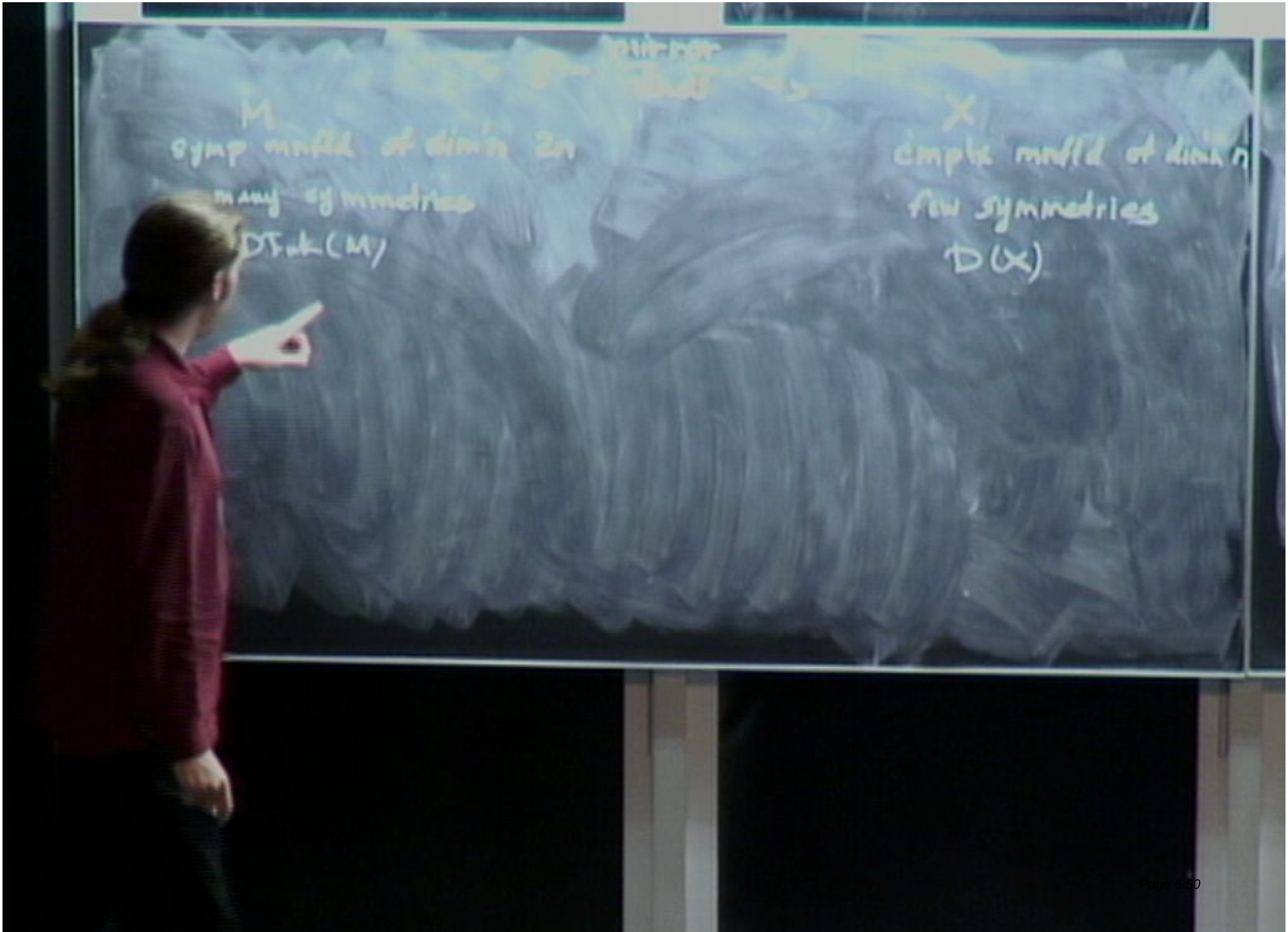
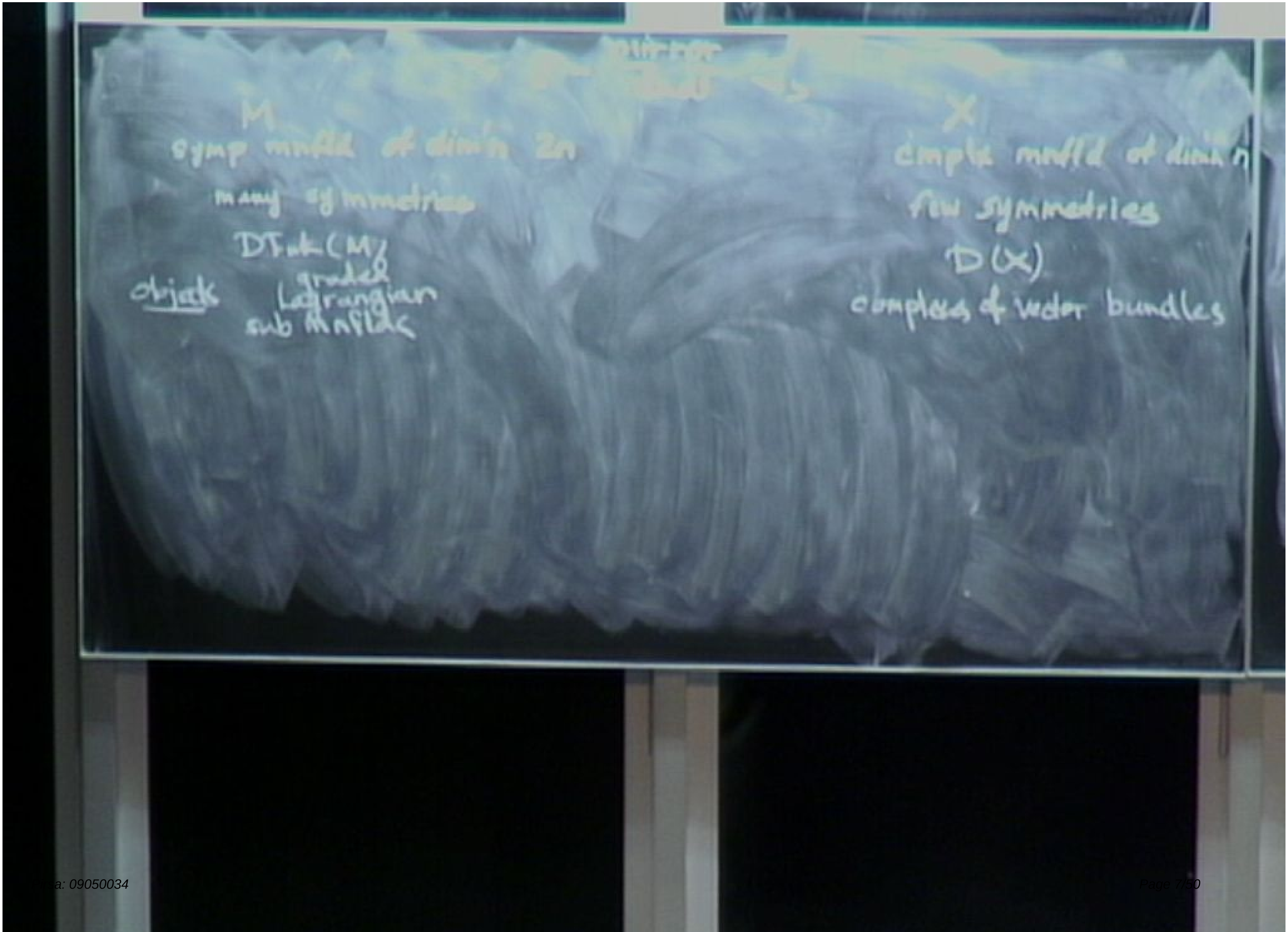
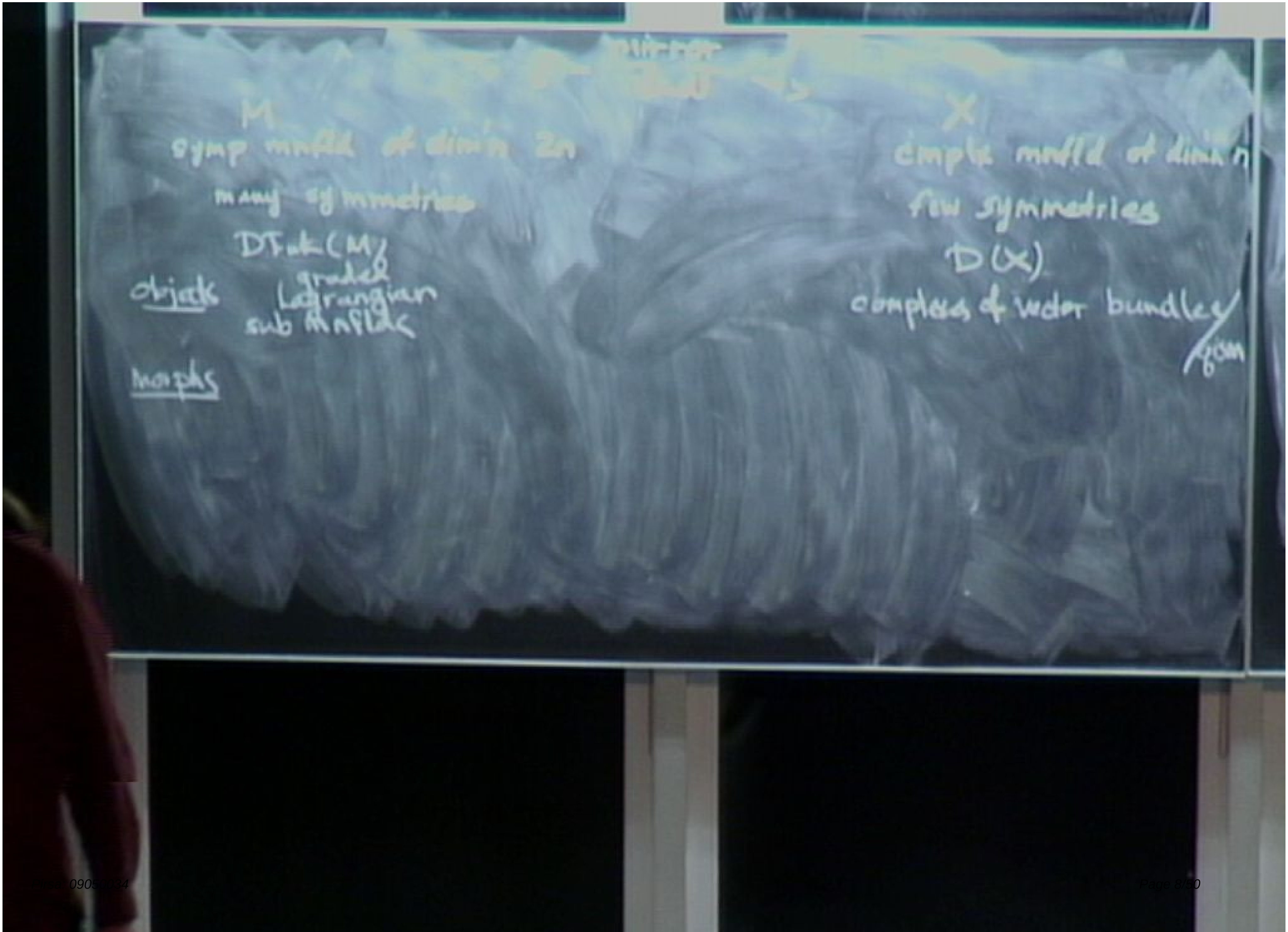
Homological mirror symmetry

If M, X mirror dual, then there is an equiv between derived Fukaya category of M and derived category of coh. sheaves on X



mirror
M
symp mnfld of dim'n 2n
many symmetries
DFuk(M)
X
cmplx mnfld of dim'n n
few symmetries
D(X)


mirror
M
symp mnfld of dim'n 2n
many symmetries
DFuk(M)
objects
graded
Lagrangian
sub mnflds
X
cmplx mnfld of dim'n n
few symmetries
D(X)
complexes of vector bundles


mirror
M
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morphs
X
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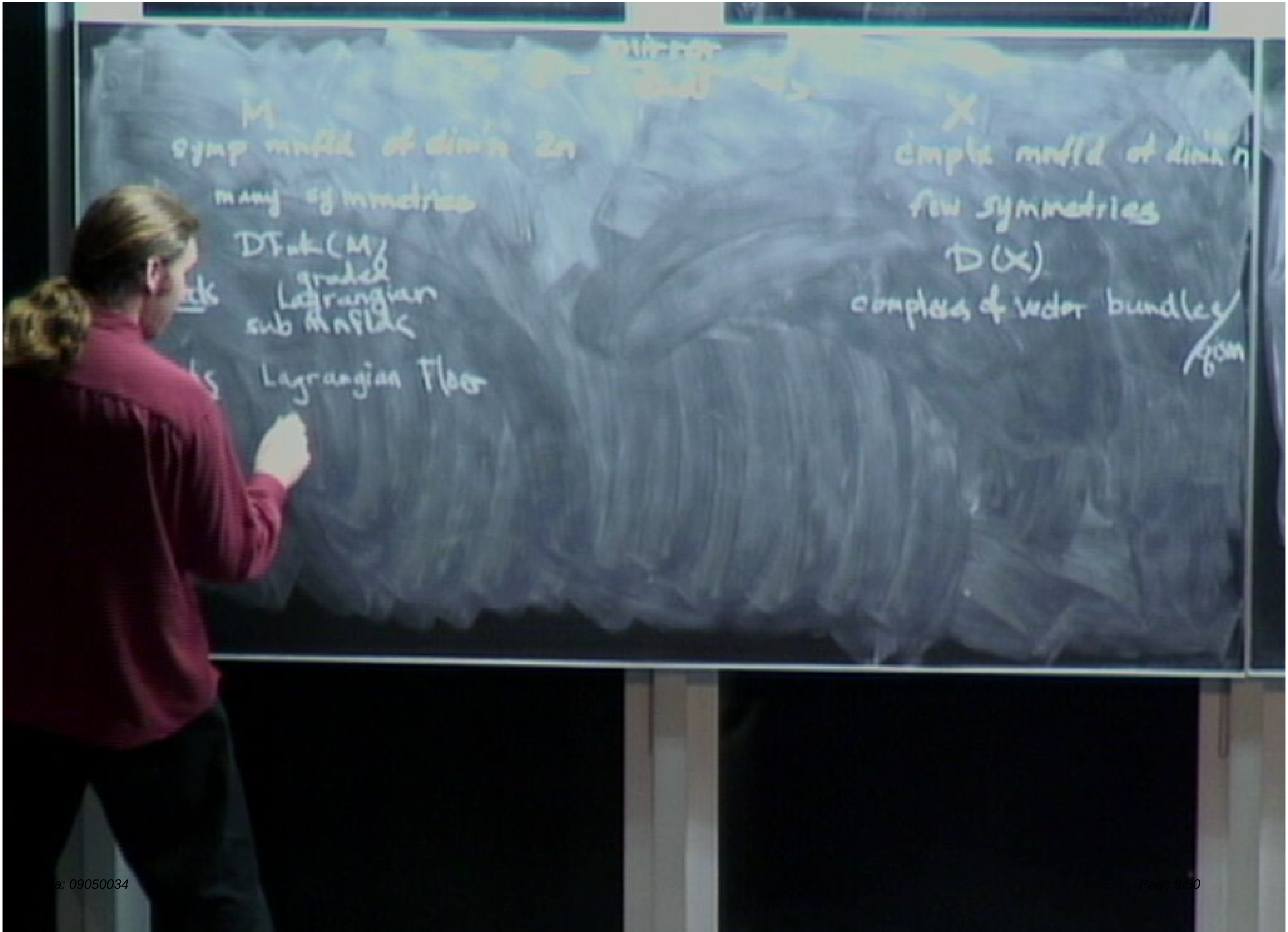
D(X)
complexes of vector bundles / ~


mirror
M
symp mnfld of dim'n 2n
many symmetries
DFuk(M)
graded Lagrangian sub mnflds
Lagrangian Floer
X
cmplx mnfld of dim'n n
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Hom


mirror
M
symp mnfld of dim'n 2n
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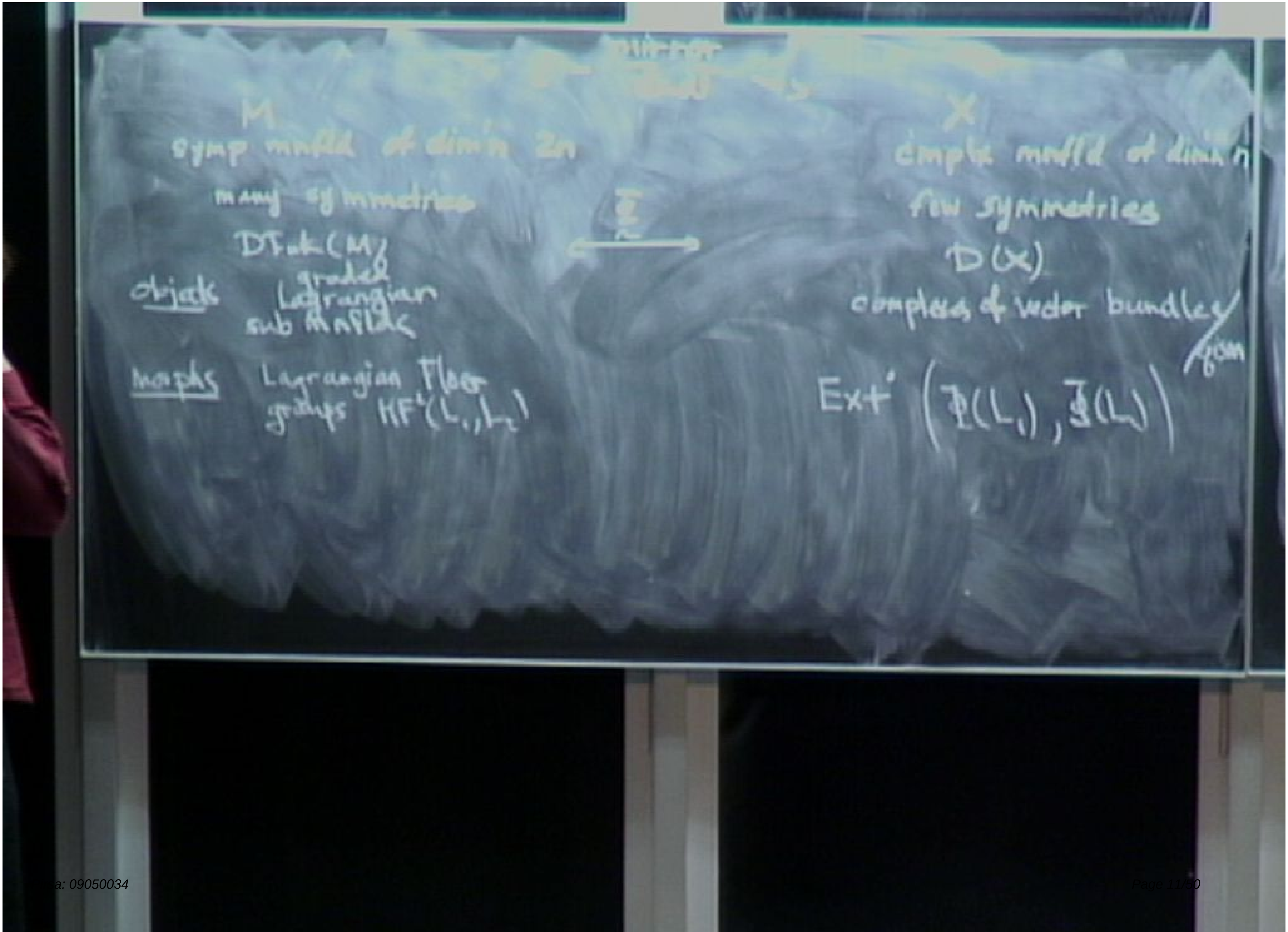
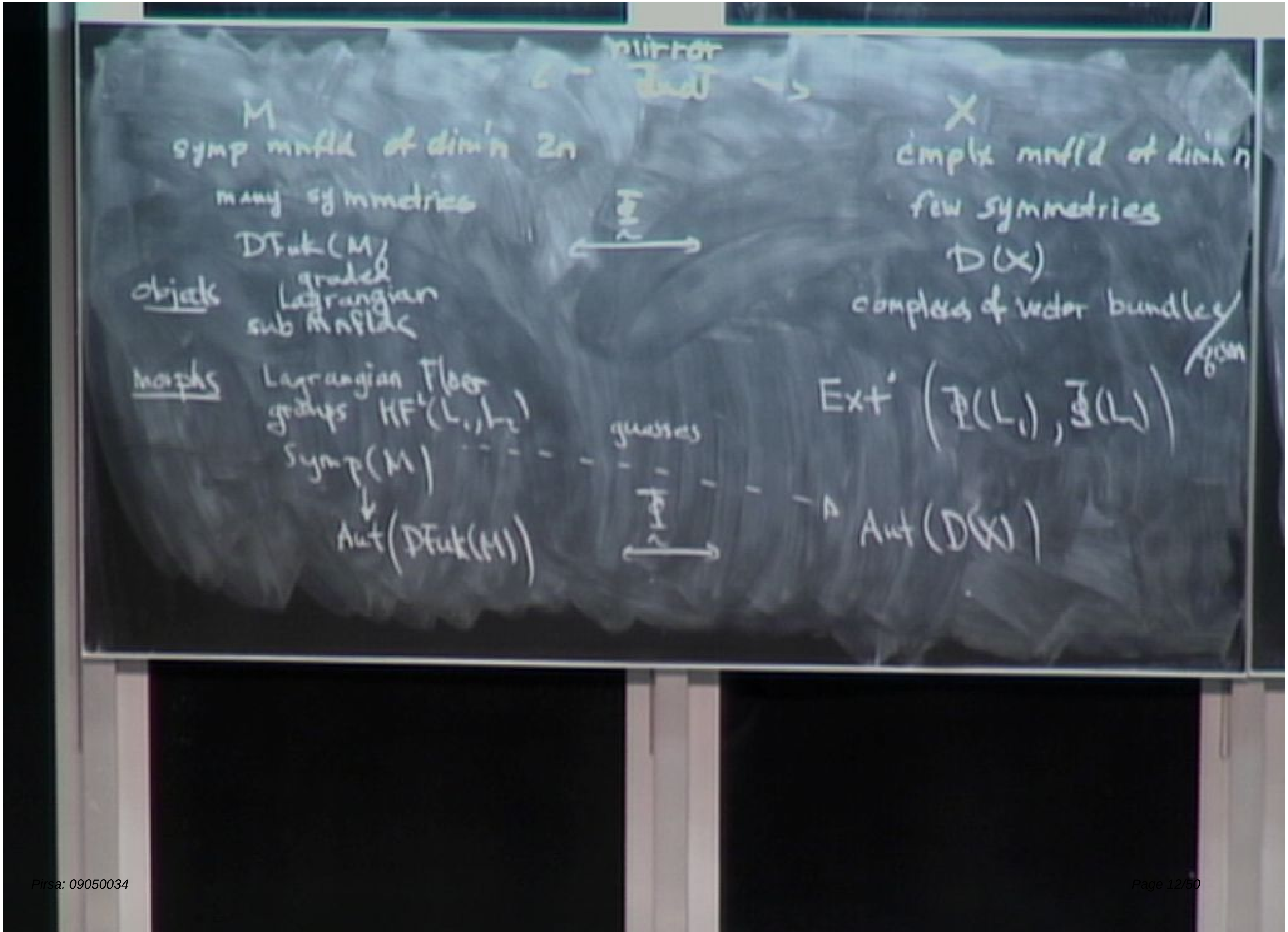
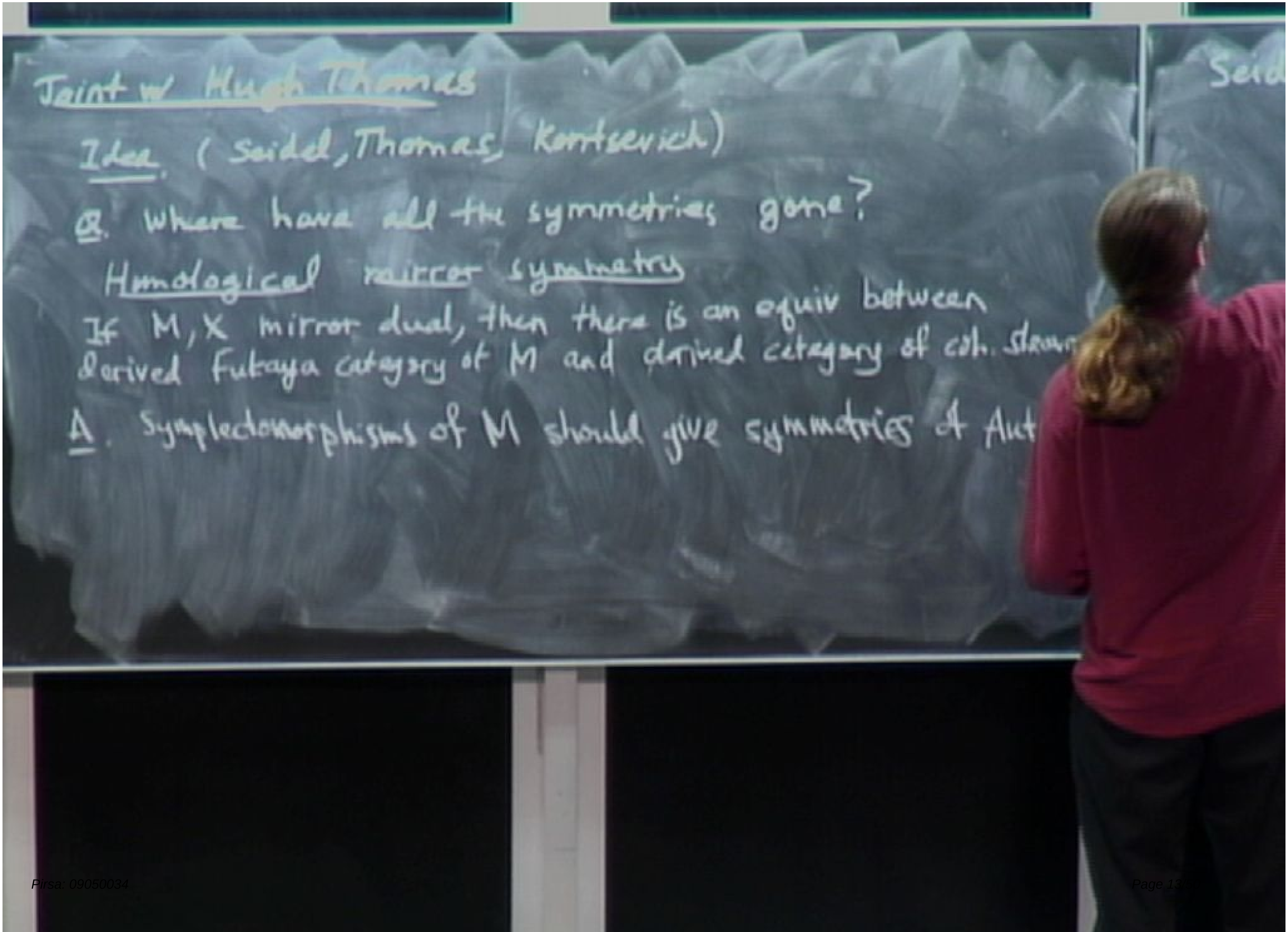
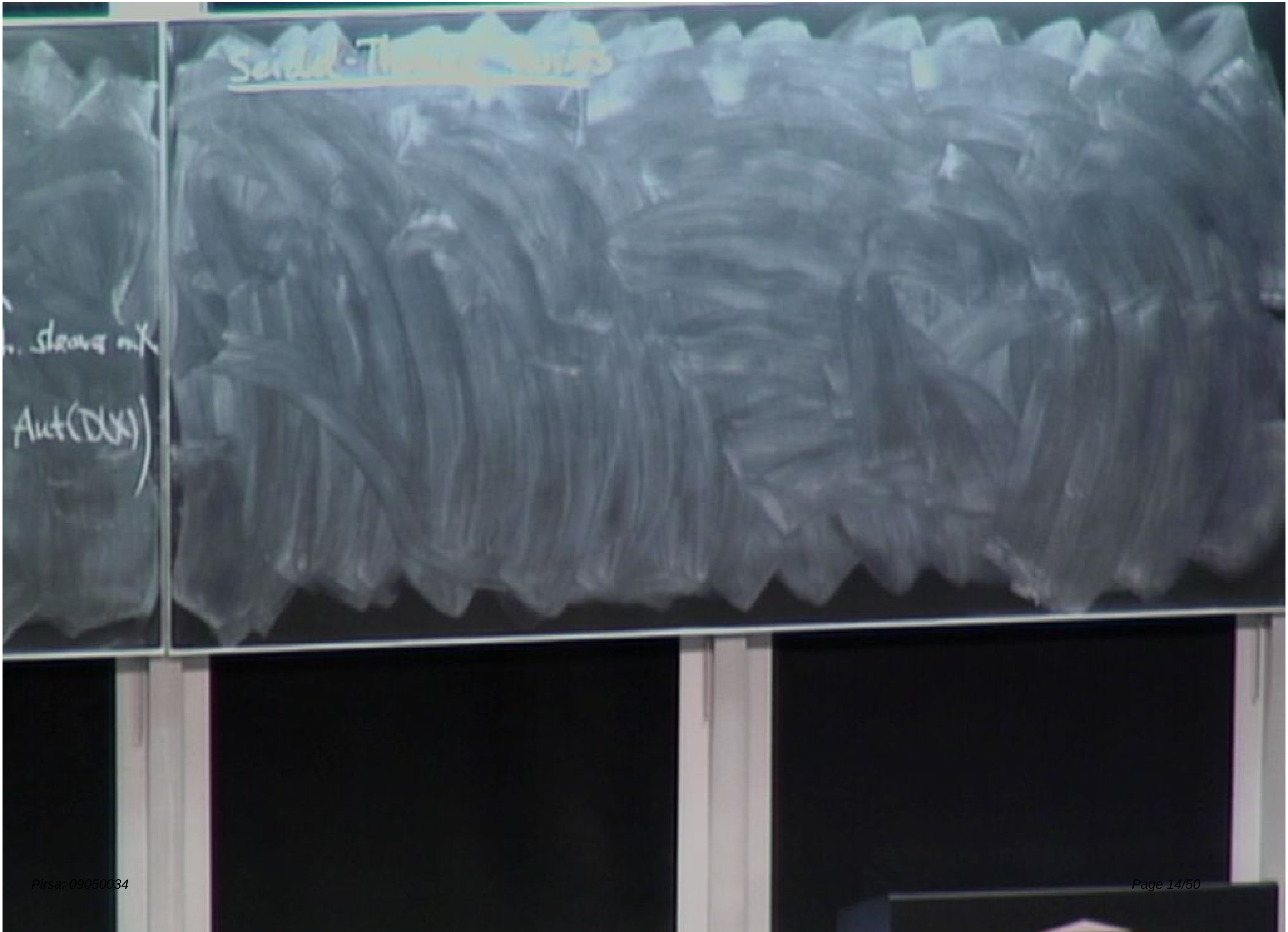
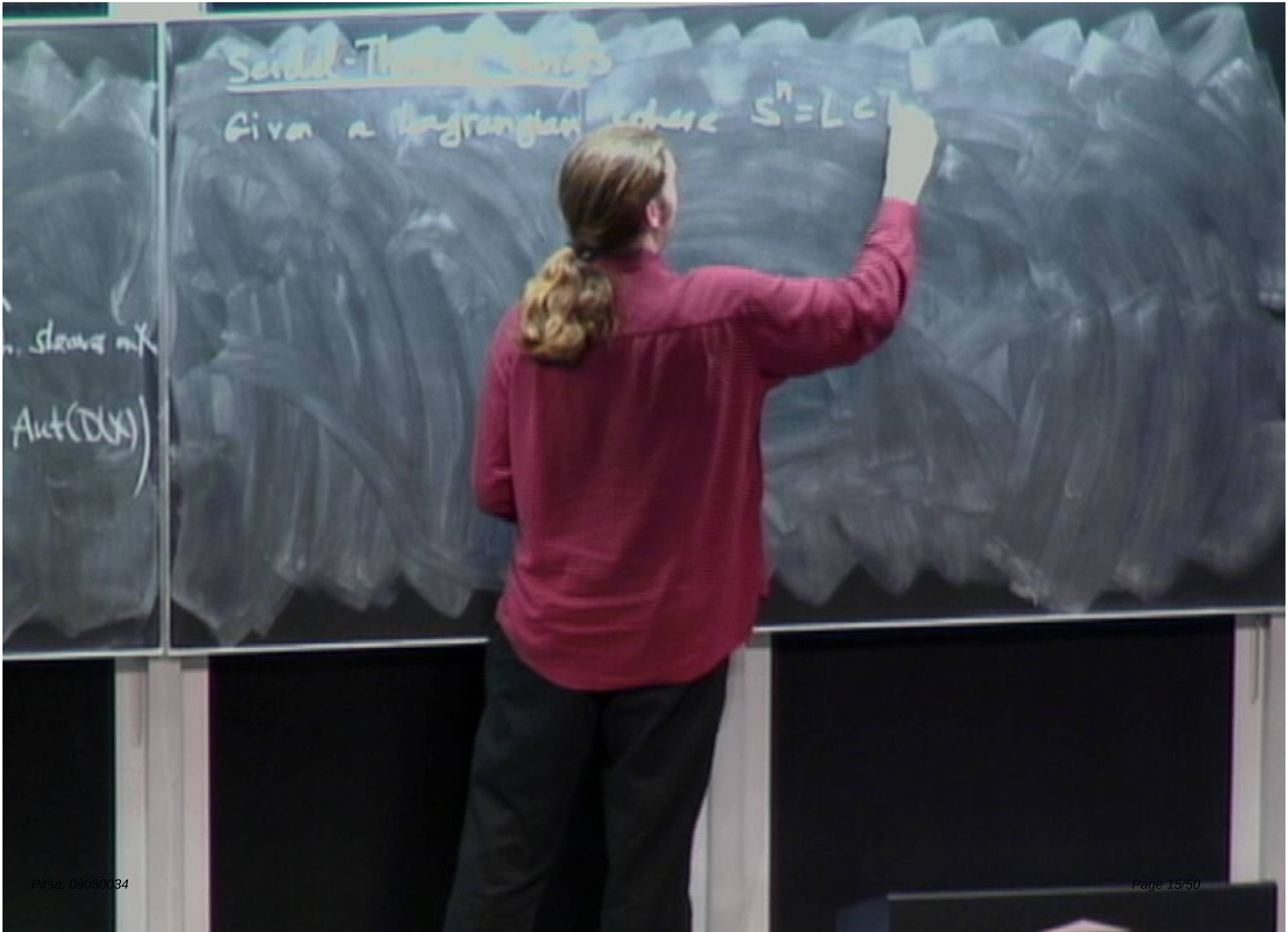
qism


M
symp mnfld of dim'n 2n
many symmetries
DFuk(M)
objects graded Lagrangian sub mnflds
morphs Lagrangian Floer groups HF*(L1, L2)
X
cmplx mnfld of dim'n n
few symmetries
D(X)
complexes of vector bundles
Ext*(Φ(L1), Φ(L2))

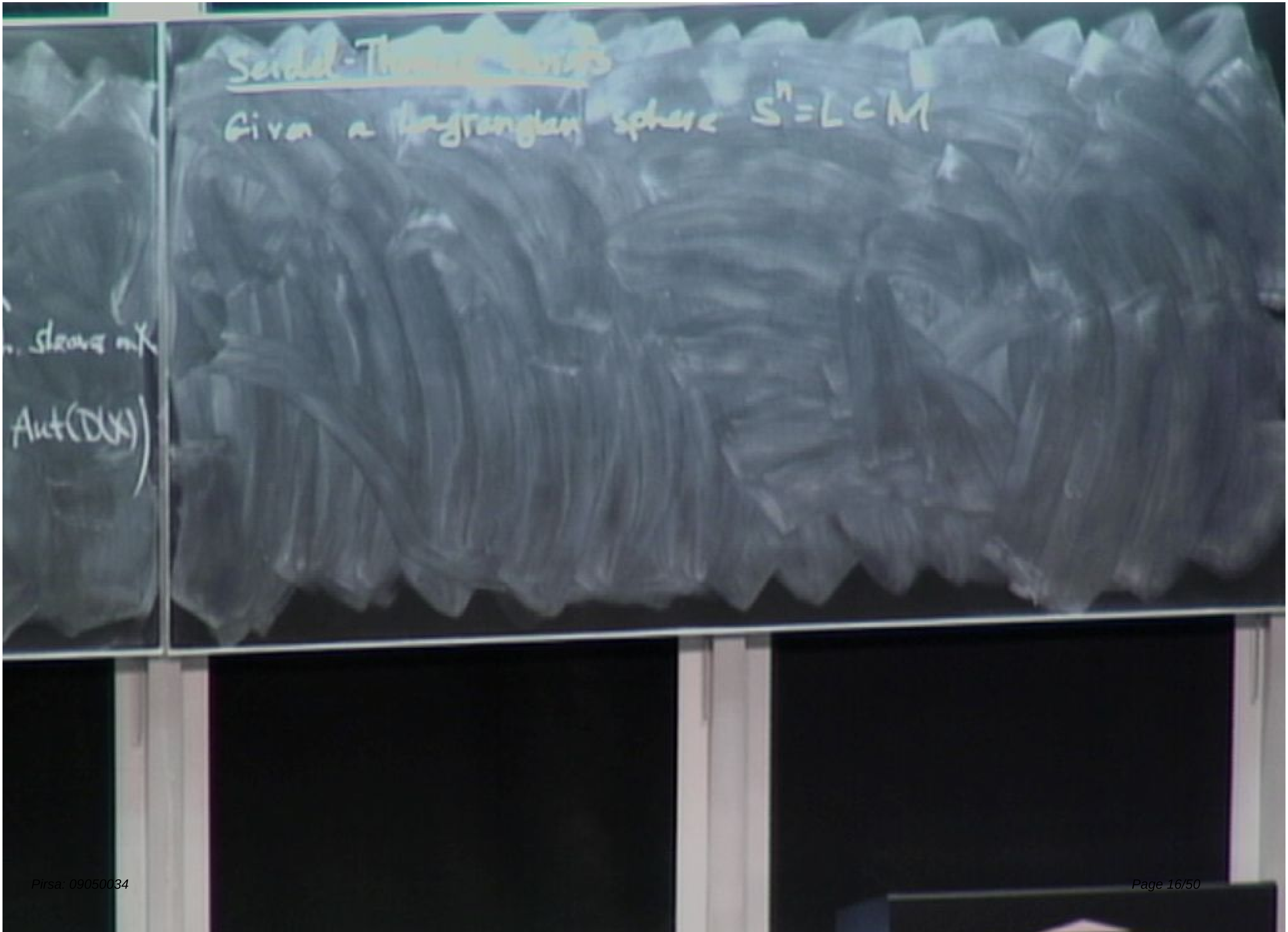

mirror
duality
M
symp mnfld of dim'n 2n
many symmetries
DFuk(M)
objects
graded
Lagrangian
sub mnfldc
morphs
Lagrangian Floer
groups HF*(L1,L2)
Symp(M)
Aut(DFuk(M))
X
cmplx mnfld of dim'n n
few symmetries
D(X)
complexes of vector bundles
qism
Ext*(Φ(L1), Φ(L2))
guesses
Aut(D(X))


Joint w/ Hugh Thomas
Idea. (Seidel, Thomas, Kontsevich)
Q. Where have all the symmetries gone?
Homological mirror symmetry
If M, X mirror dual, then there is an equiv between
derived Fukaya category of M and derived category of coh. sheaves
A. Symplectomorphisms of M should give symmetries of Aut

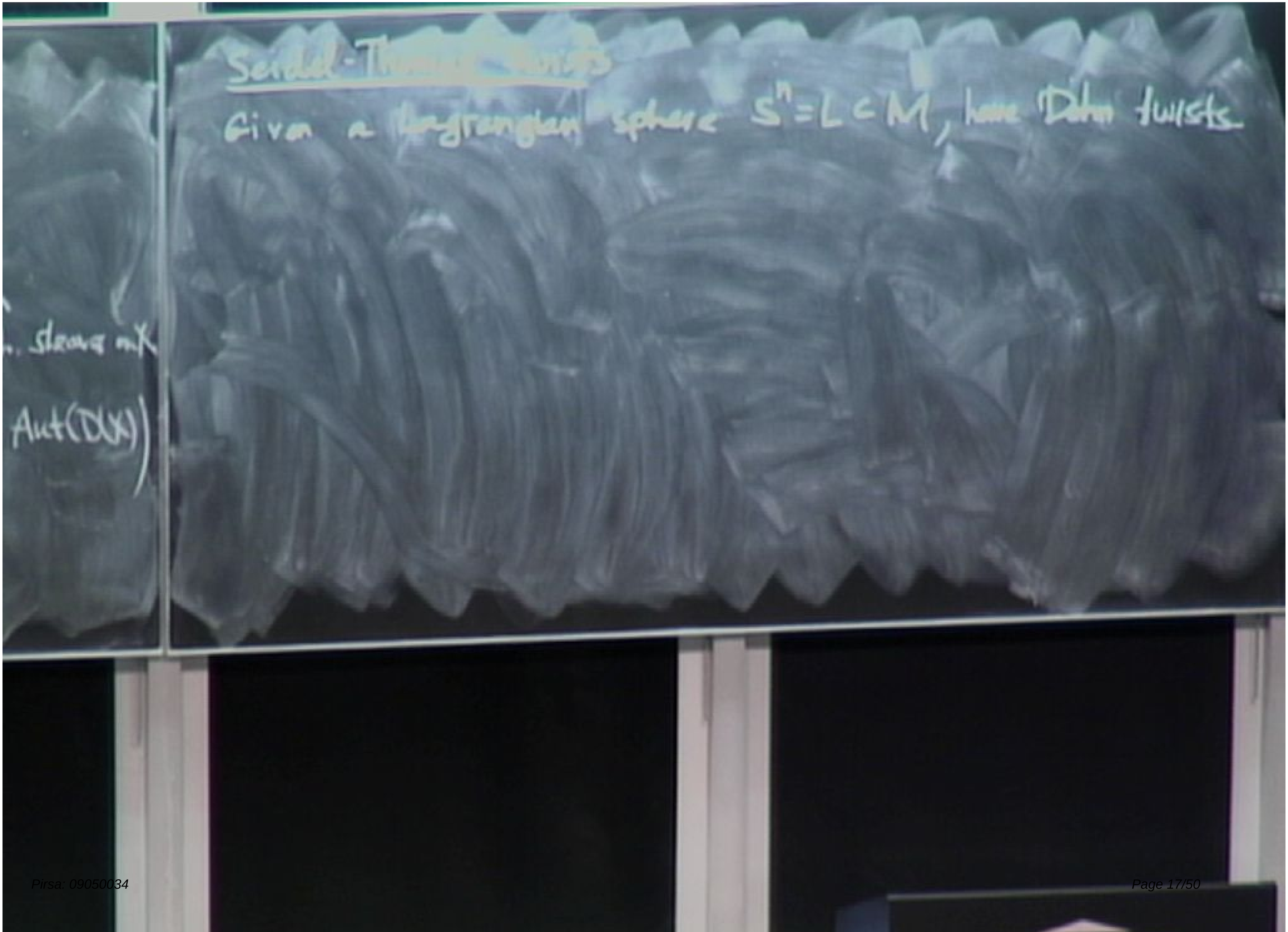

Aut(D(X))

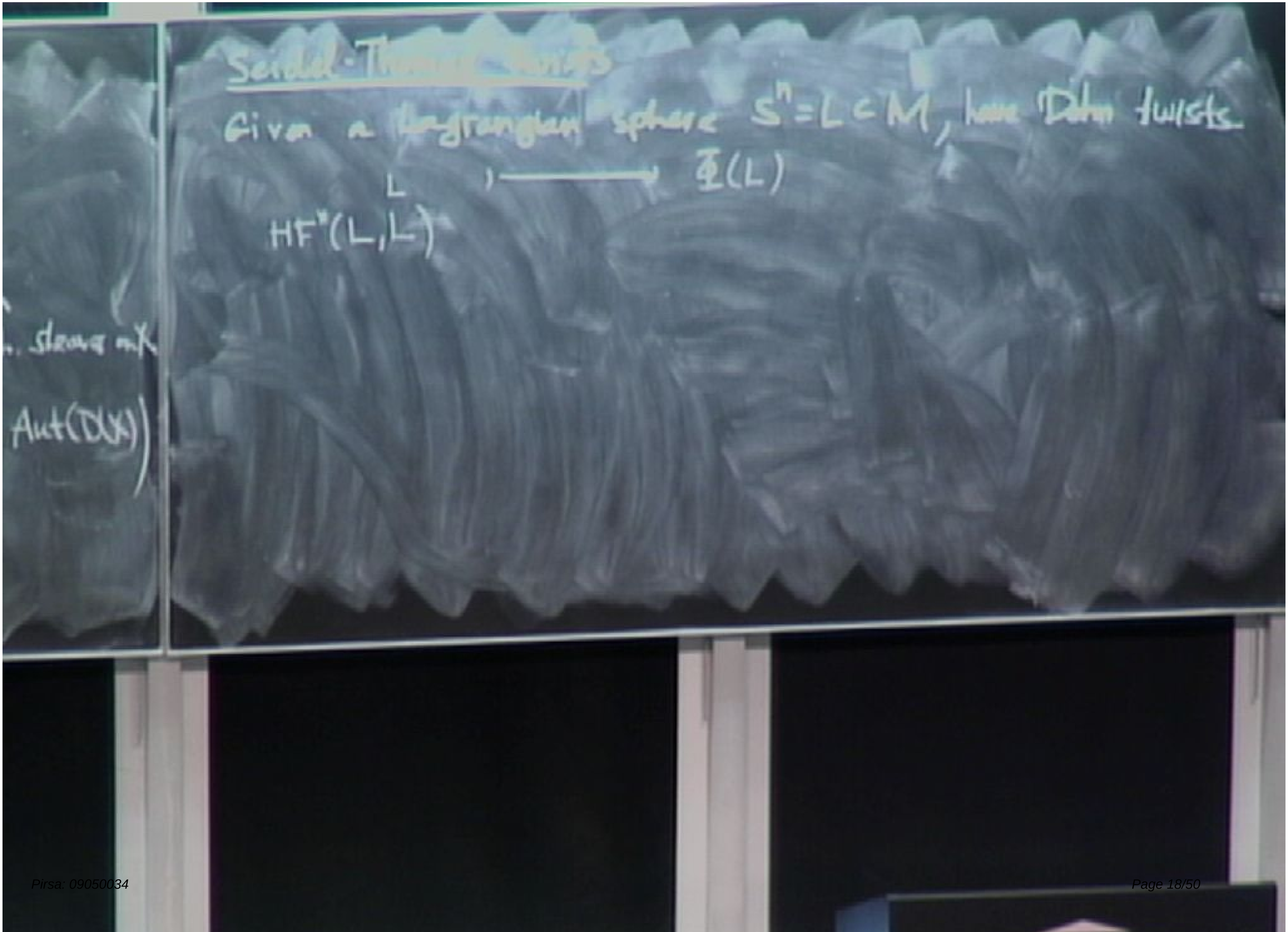


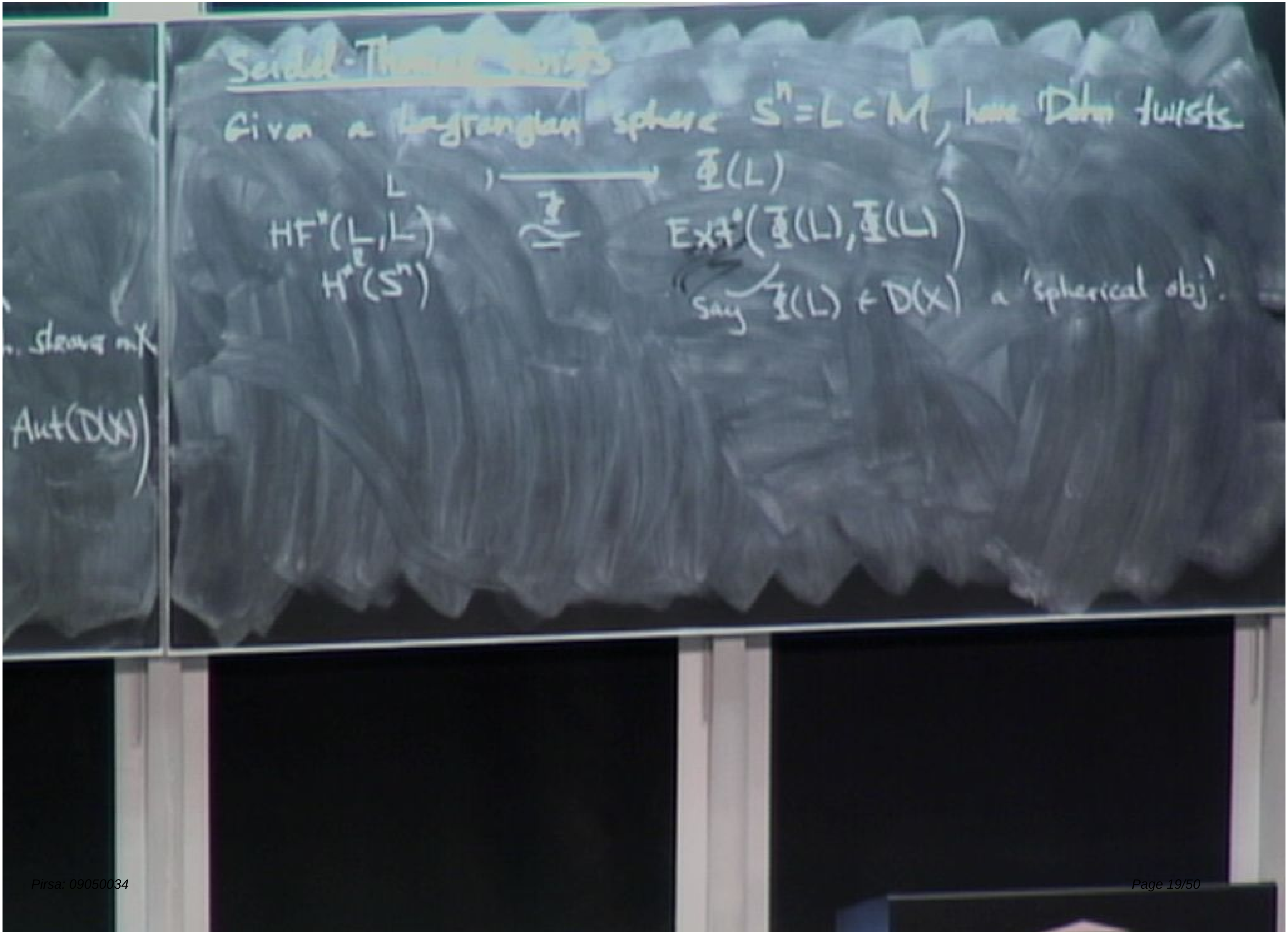
Aut(D(X))

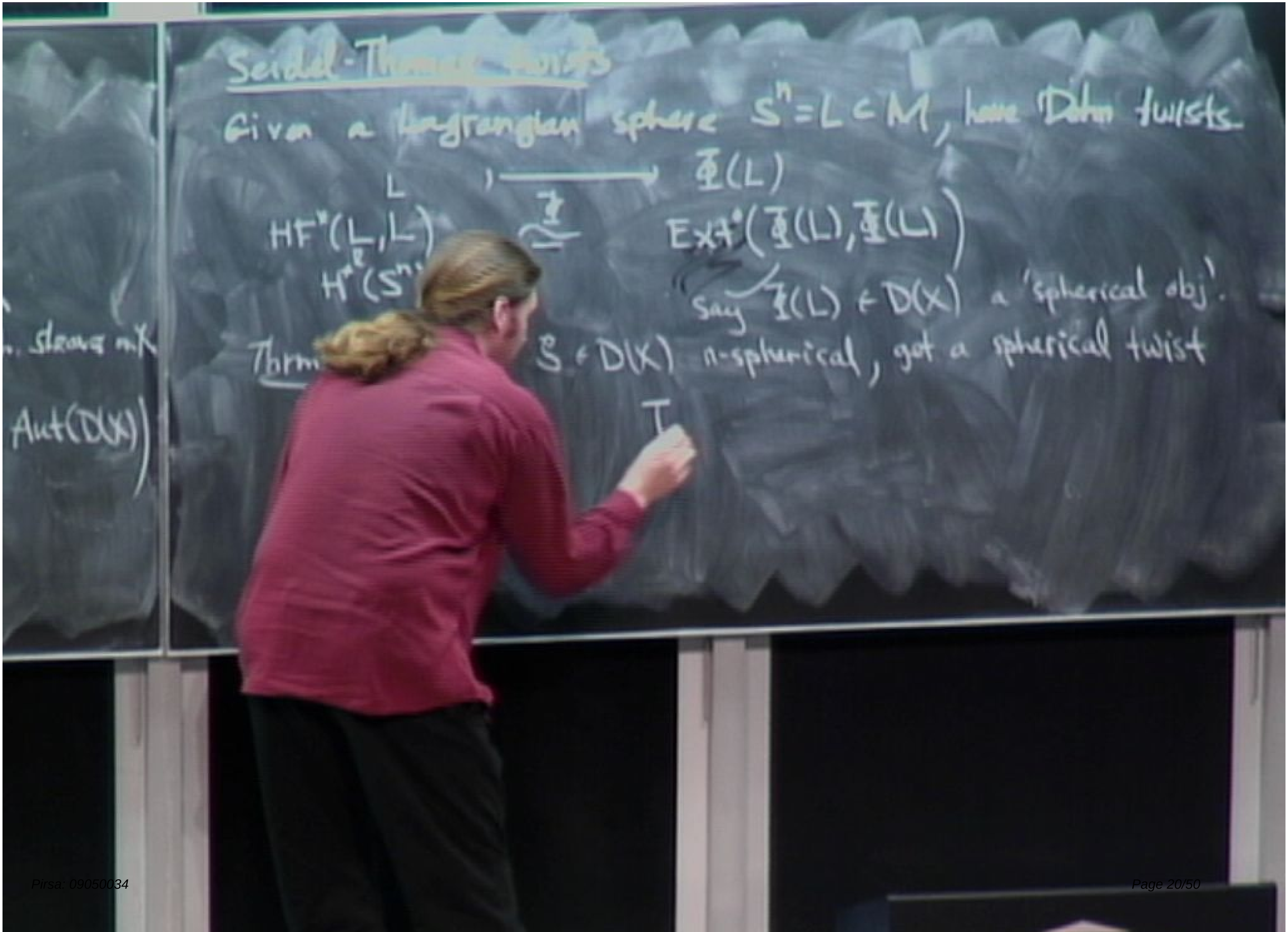


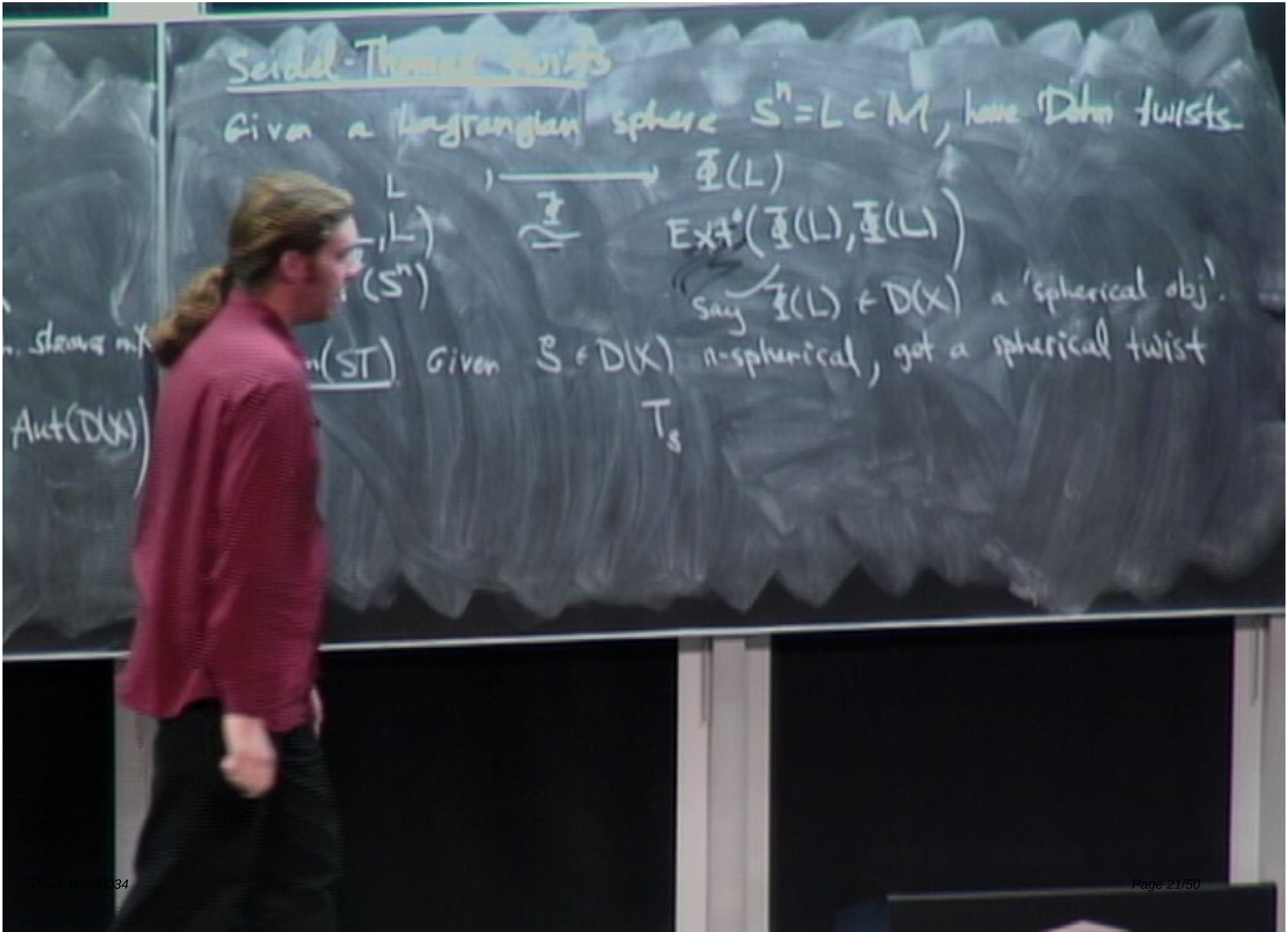
Given a Lagrangian sphere S^n = L ⊂ M
sheaves on X
Aut(D(X))


Seidel-Thomas twists
Given a Lagrangian sphere S^n = L ⊂ M, have Dehn twists
Aut(D(X))


Given a Lagrangian sphere $S^n = L \subset M$, have Dehn twists
$L \longmapsto \Phi(L)$
$HF^*(L,L)$
sheaves on X
Aut(D(X))


Seidel-Thomas twists
Given a Lagrangian sphere S^n = L ⊂ M, have 'Dehn twists'
L ↦ Φ(L)
HF*(L,L) ≅ Ext*(Φ(L),Φ(L))
H*(S^n)
Say Φ(L) ∈ D(X) a 'spherical obj'.
sheaves on X
Aut(D(X))


Seidel-Thomas twists
Given a Lagrangian sphere $S^n = L \subset M$, have Dehn twists
$L \longmapsto \Phi(L)$
$HF^*(L,L) \cong Ext^*(\Phi(L),\Phi(L))$
Say $\Phi(L) \in D(X)$ a 'spherical obj'.
$S \in D(X)$ n-spherical, got a spherical twist
Aut(D(X))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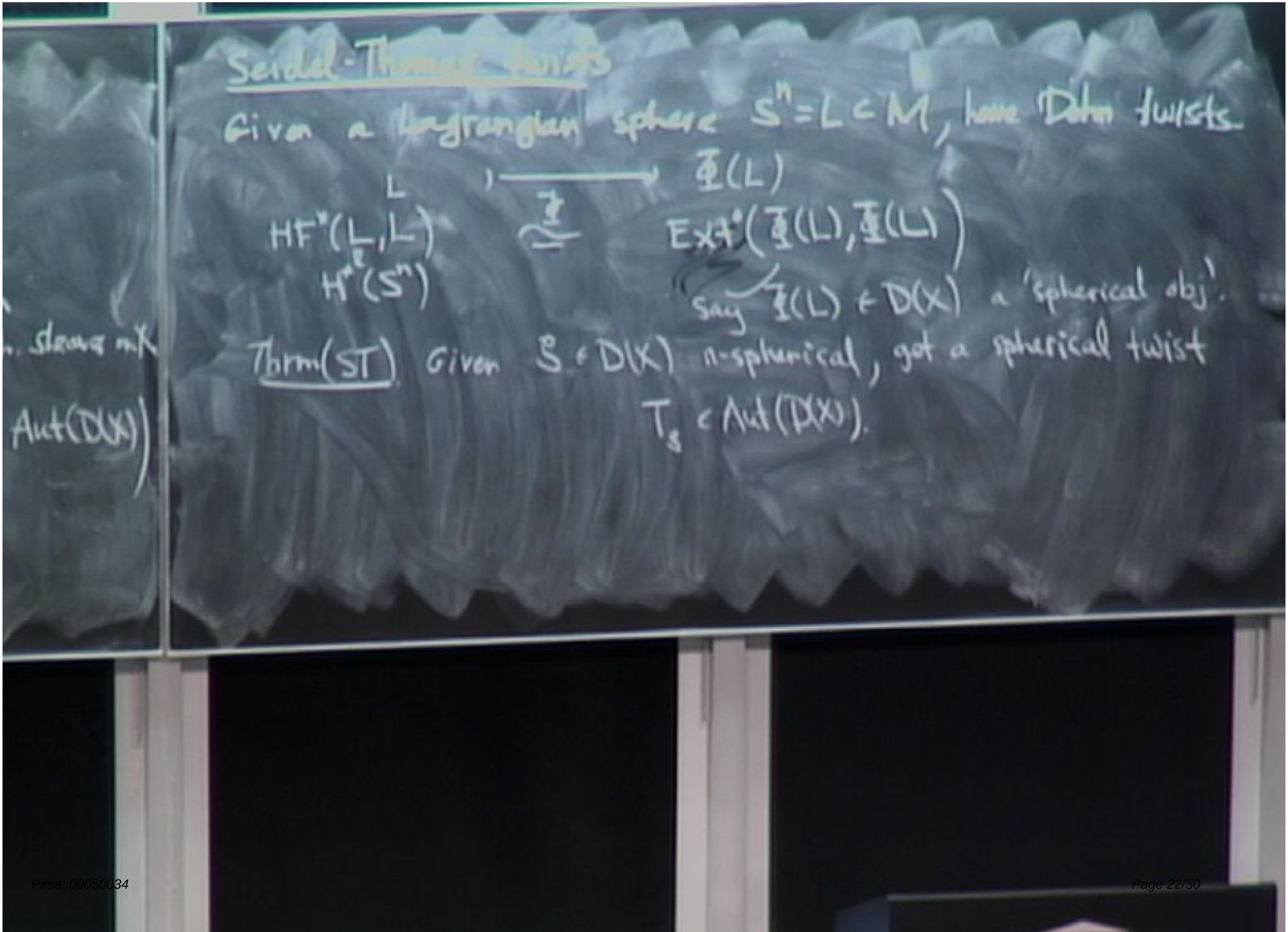
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T_S ∈ Aut(D(X)).
sheaves on X
Aut(D(X))

... sheaves on X

Aut(D(X)))

Seidel-Thomas twists

Given a Lagrangian sphere $S^n = L \subset M$, have 'Dehn twists'

$$L \mapsto \Phi(L)$$

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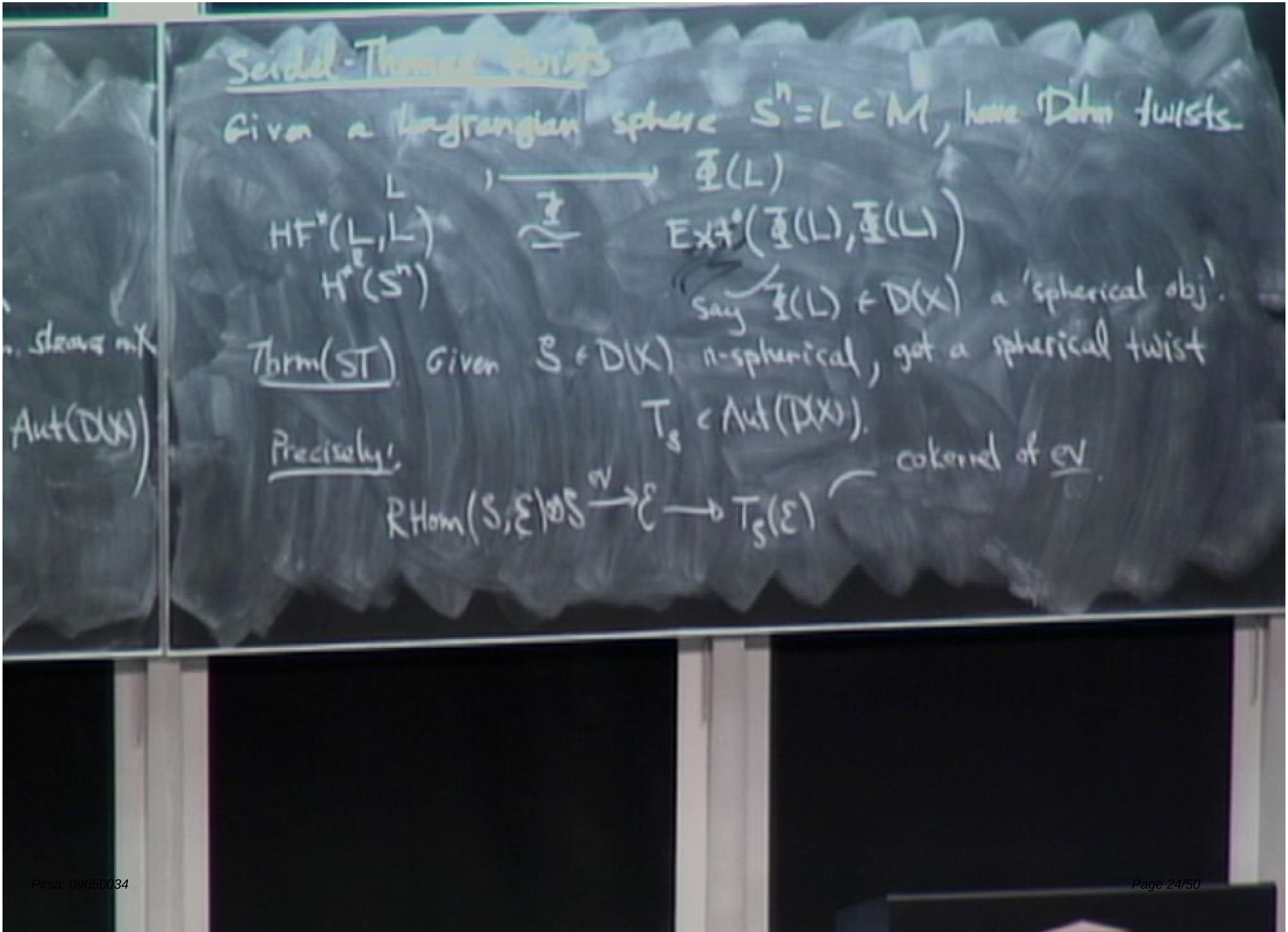
$$\cong H^*(S^n)$$

Say $\Phi(L) \in D(X)$ a 'spherical obj'.

Thrm (ST) Given $S \in D(X)$ n-spherical, get a spherical twist

$$T_S \in Aut(D(X)).$$

Precisely: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T_S ∈ Aut(D(X)).
Precisely:
RHom(S,E)⊗S →ev E → T_S(E)
cokernel of ev
sheaves on X
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... sheaves on X

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Given a Lagrangian sphere $S^n = L \subset M$, have 'Dehn' twists.

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Say $\Phi(L) \in D(X)$ a 'spherical obj'.

Thrm(ST) Given $S \in D(X)$ n-spherical, get a spherical twist $T_S \in Aut(D(X))$.

Precisely:

$$RHom(S,\mathcal{E})\otimes S \xrightarrow{ev} \mathcal{E} \longrightarrow T_S(\mathcal{E})$$

(homotopy) cokernel of ev

Seidel-Thomas twists

Given a Lagrangian sphere $S^n = L \subset M$, have Dehn twists

$$L \longmapsto \Phi(L)$$

$$HF^*(L,L) \overset{\Phi}{\cong} Ext^*(\Phi(L), \Phi(L))$$

$$H^*(S^n)$$

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... sheaves on X

Aut(D(X))

Joint w/ Hugh Thomas

Example. $G \subset SL_2(\mathbb{C})$. $G \curvearrowright \mathbb{C}^2$,

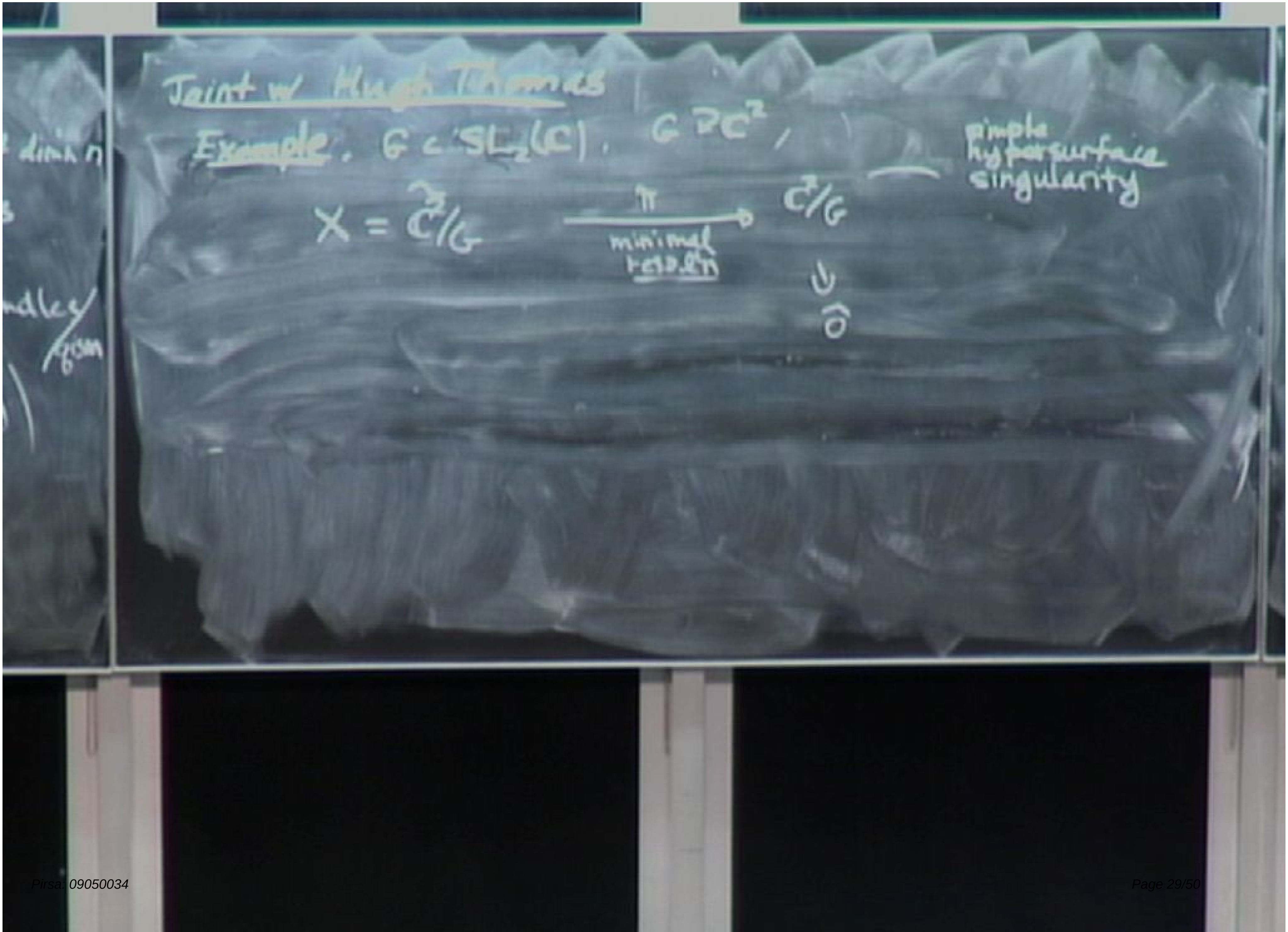
$$\xrightarrow[\text{minimal resol'n}]{\pi} \mathbb{C}^2/G \quad \text{— simple hypersurface singularity}$$

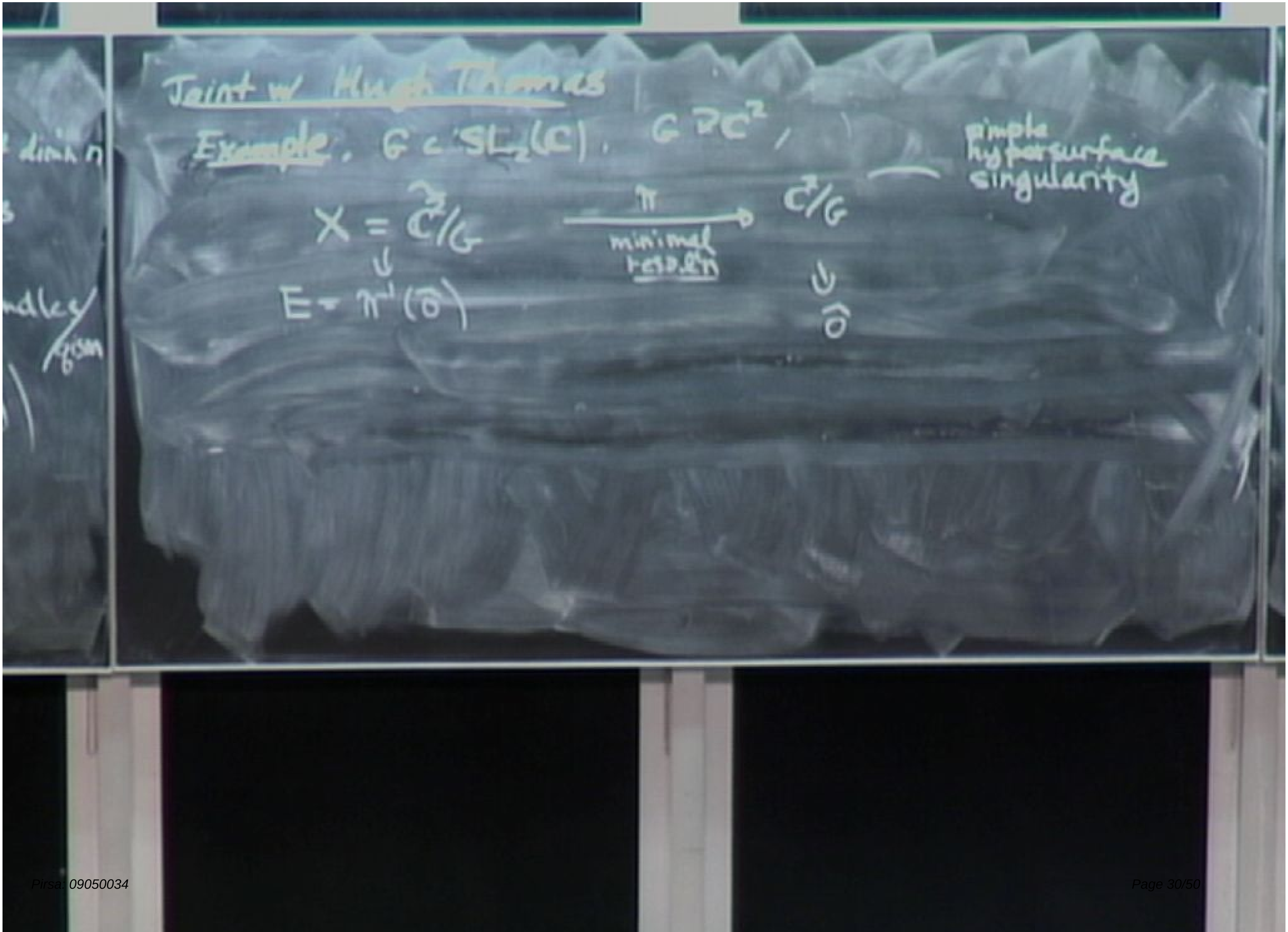
Joint w/ Hugh Thomas

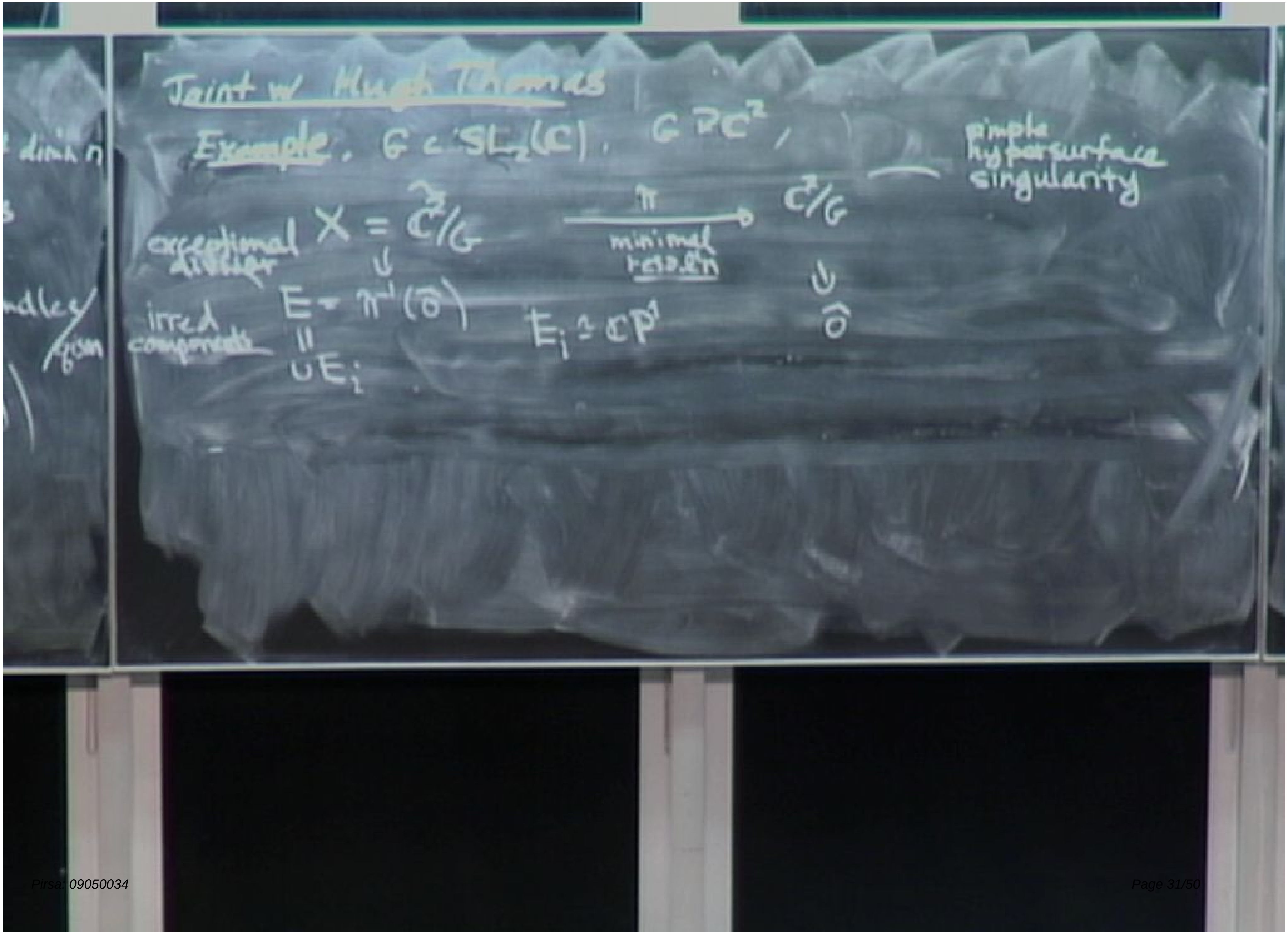
Example. $G \subset SL_2(\mathbb{C})$. $G \curvearrowright \mathbb{C}^2$,

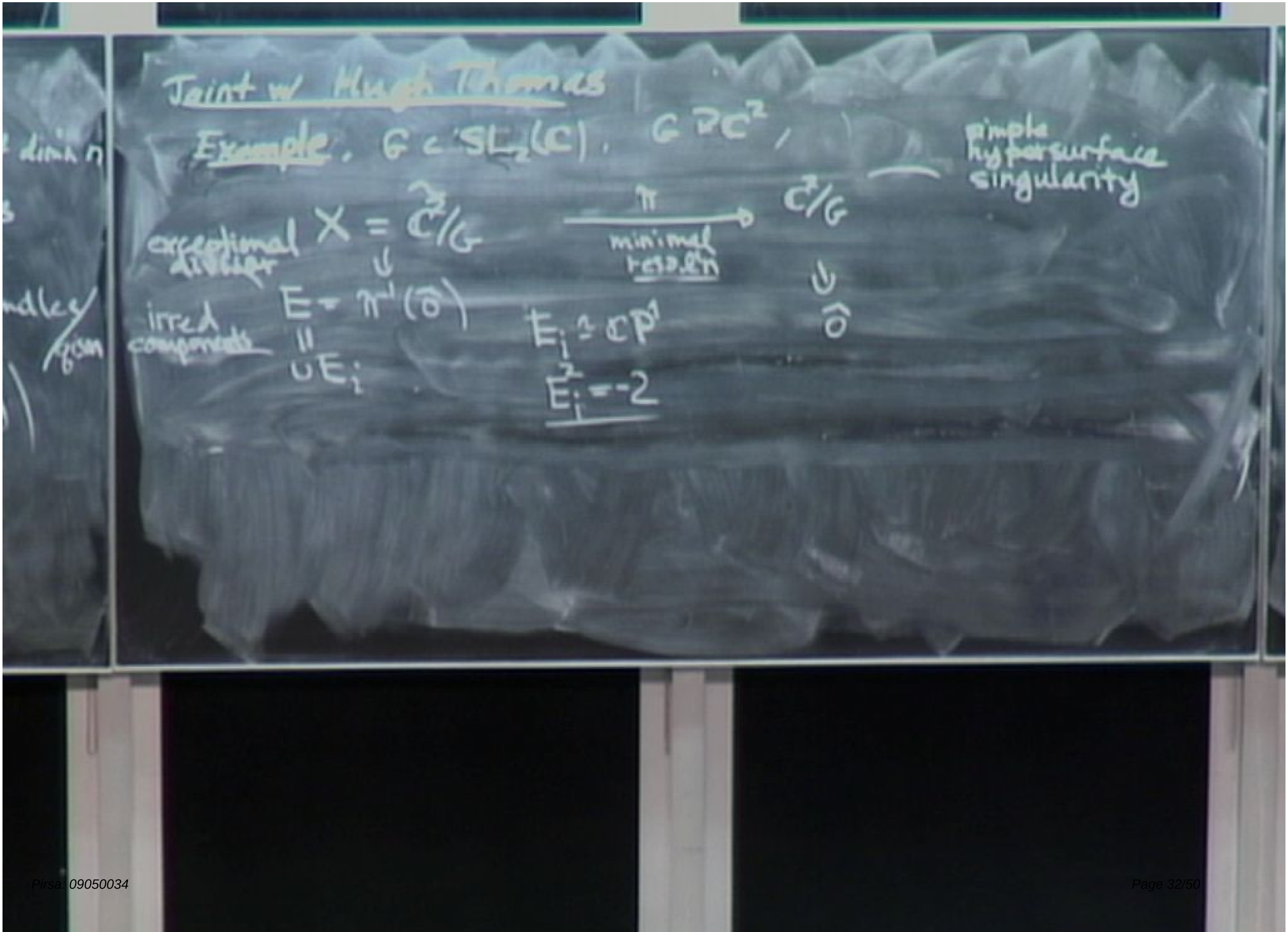
$$X = \widetilde{\mathbb{C}^2/G} \xrightarrow[\text{minimal resol'n}]{\pi} \mathbb{C}^2/G$$

$\mathbb{C}^2/G$ — simple hypersurface singularity

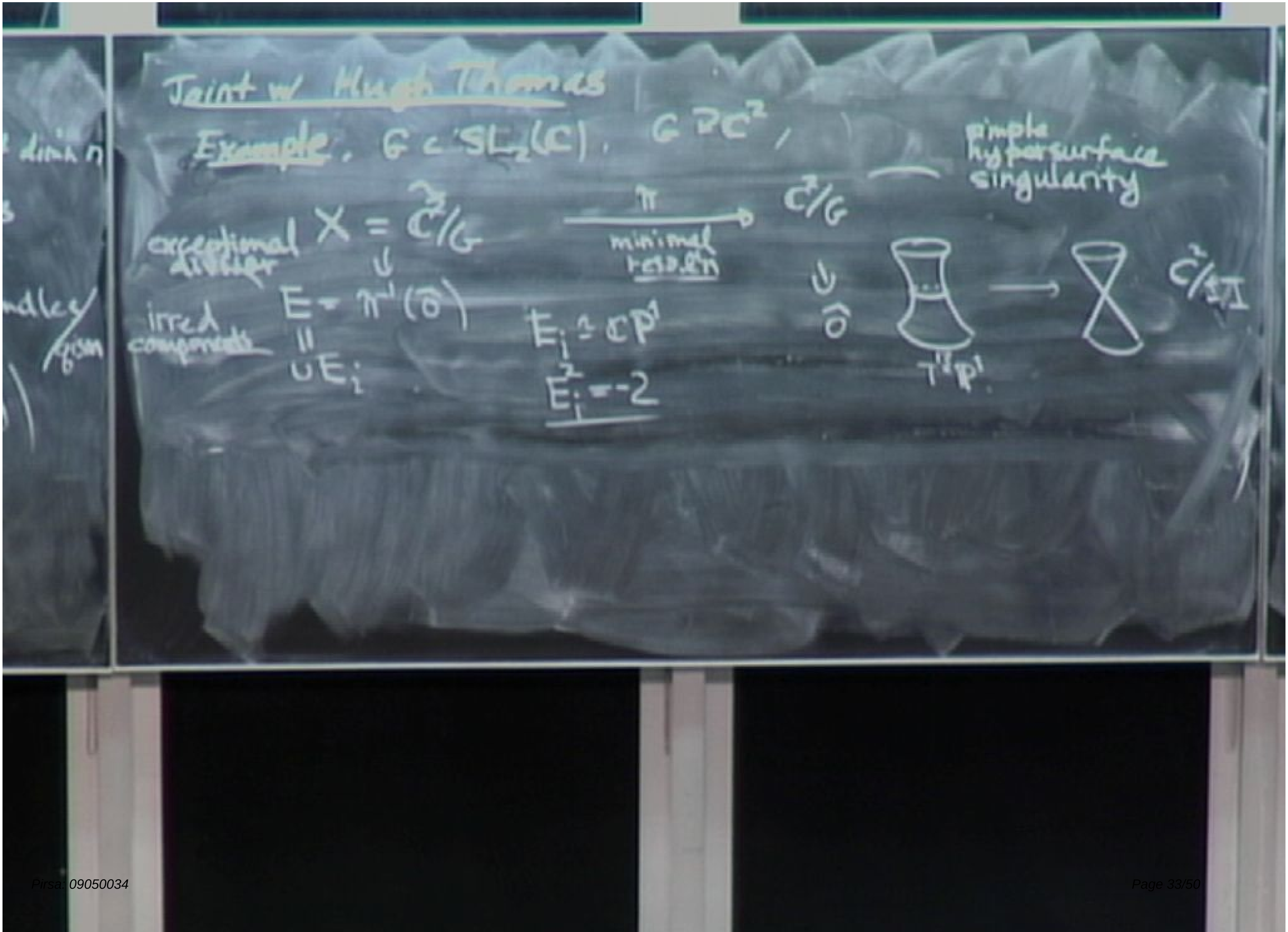

Joint w/ Hugh Thomas
Example. G ⊂ SL2(C), G ↻ C2,
simple hypersurface singularity
X = C2/G
π
minimal resol'n
C2/G
0

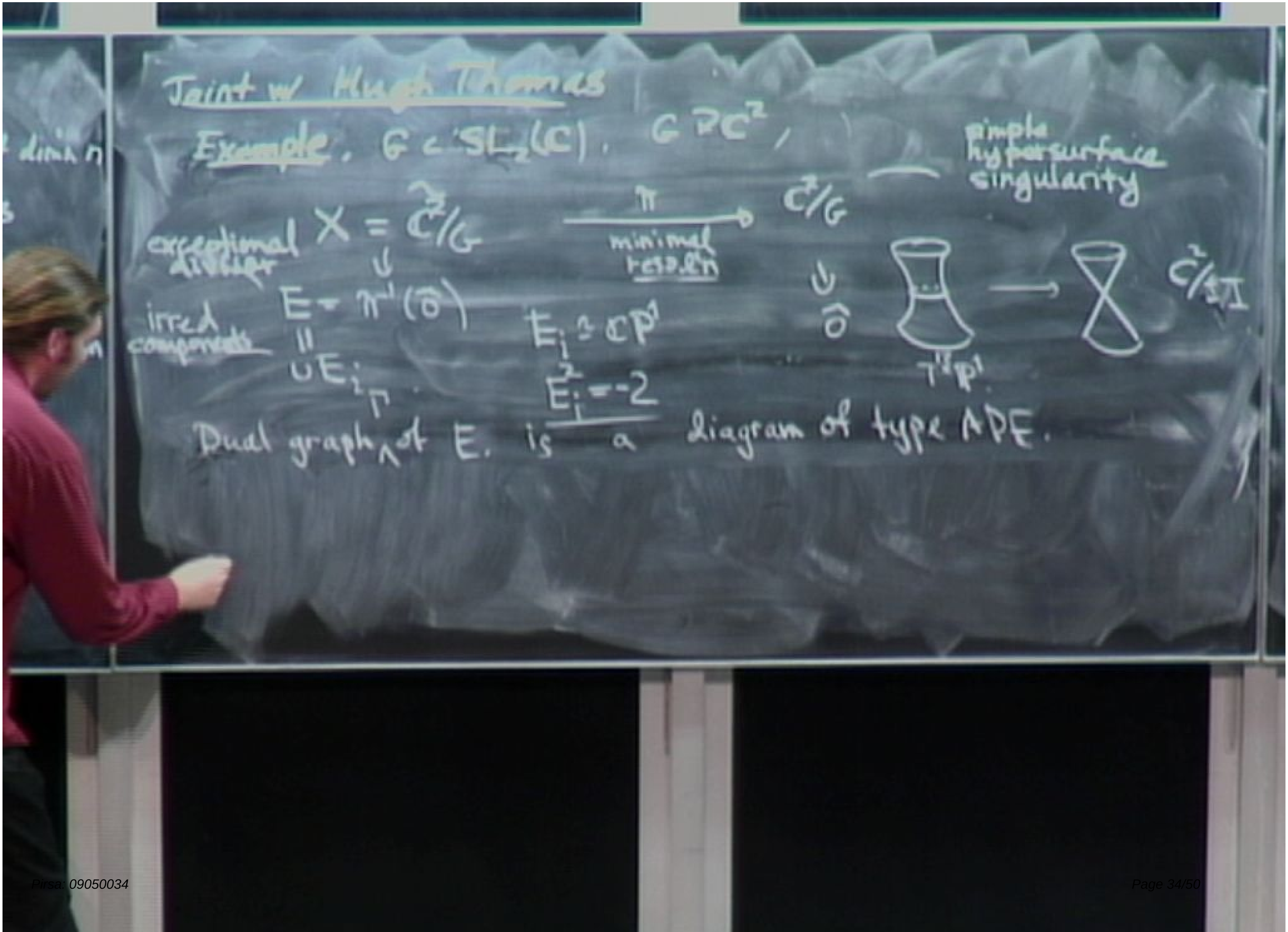

Joint w/ Hugh Thomas
Example. G ⊂ SL2(C). G ↻ C2,
simple hypersurface singularity
X = C2/G ~
π
minimal resol'n
C2/G
E = π-1(0)
0


Joint w/ Hugh Thomas
Example. G ⊂ SL₂(ℂ), G ↻ ℂ²,
simple
hypersurface
singularity
exceptional divisor
X = C̃²/G
π
minimal resol'n
ℂ²/G
E = π⁻¹(0̂)
irred components
= ∪ E_i
E_i ≅ ℂP¹
0̂



Joint w/ Hugh Thomas
Example. G ⊂ SL2(C). G ↻ C2
simple hypersurface singularity
X = C̃2/G
π
minimal resol'n
C2/G
exceptional divisor
irred components
E = π-1(0̂)
Ei ≅ CP1
Ei2 = -2
∪Ei


Joint w/ Hugh Thomas
Example. G ⊂ SL_2(ℂ), G ↻ ℂ²,
simple hypersurface singularity
X = ℂ̃²/G
π
minimal resol'n
ℂ²/G
exceptional divisor
E = π⁻¹(0̂)
irred components
∪E_i
E_i ≅ ℂP¹
E_i² = −2
0̂
T*P¹
ℂ²/±I


Joint w/ Hugh Thomas
Example. G < SL_2(C). G ↻ C^2,
simple hypersurface singularity
X = C̃^2/G —π→ C^2/G
minimal resol'n
exceptional divisor
E = π^{-1}(0̂)
irred components
∪E_i
E_i ≅ CP^1
E_i^2 = -2
T^*P^1
C^2/±I
Dual graph of E. is a diagram of type ADE.

Joint w/ Hugh Thomas

Example. $G \subset SL_2(\mathbb{C})$, $G \circlearrowright \mathbb{C}^2$, $\mathbb{C}^2/G$ simple hypersurface singularity

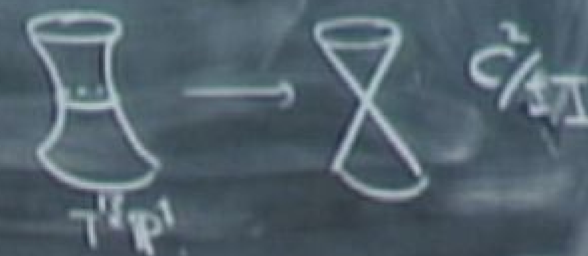
exceptional divisor $X = \widetilde{\mathbb{C}^2/G} \xrightarrow[\text{minimal resol'n}]{\pi} \mathbb{C}^2/G$

$\Downarrow \qquad\qquad \Downarrow$

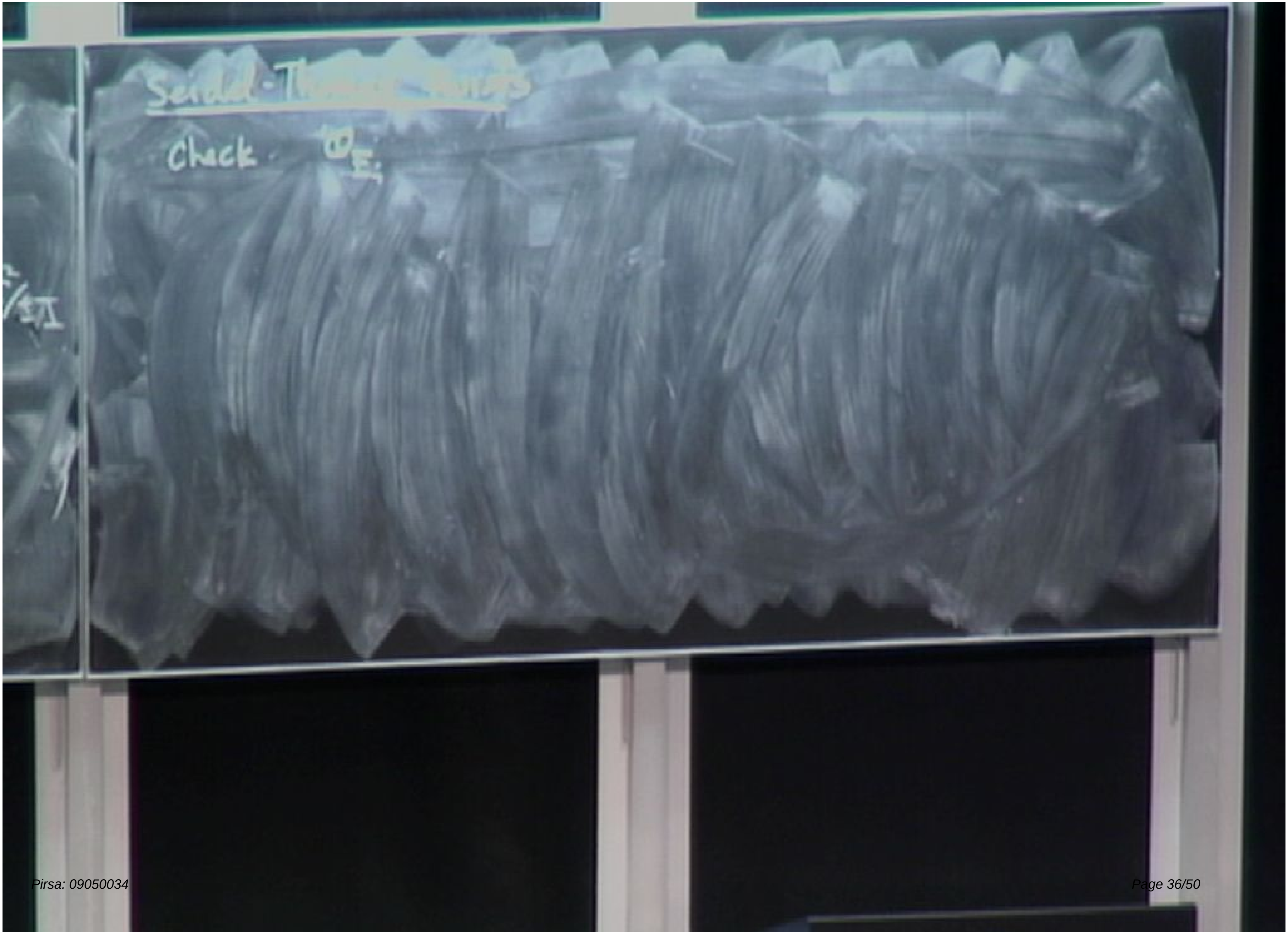
$E = \pi^{-1}(\hat{0}) \qquad \hat{0}$

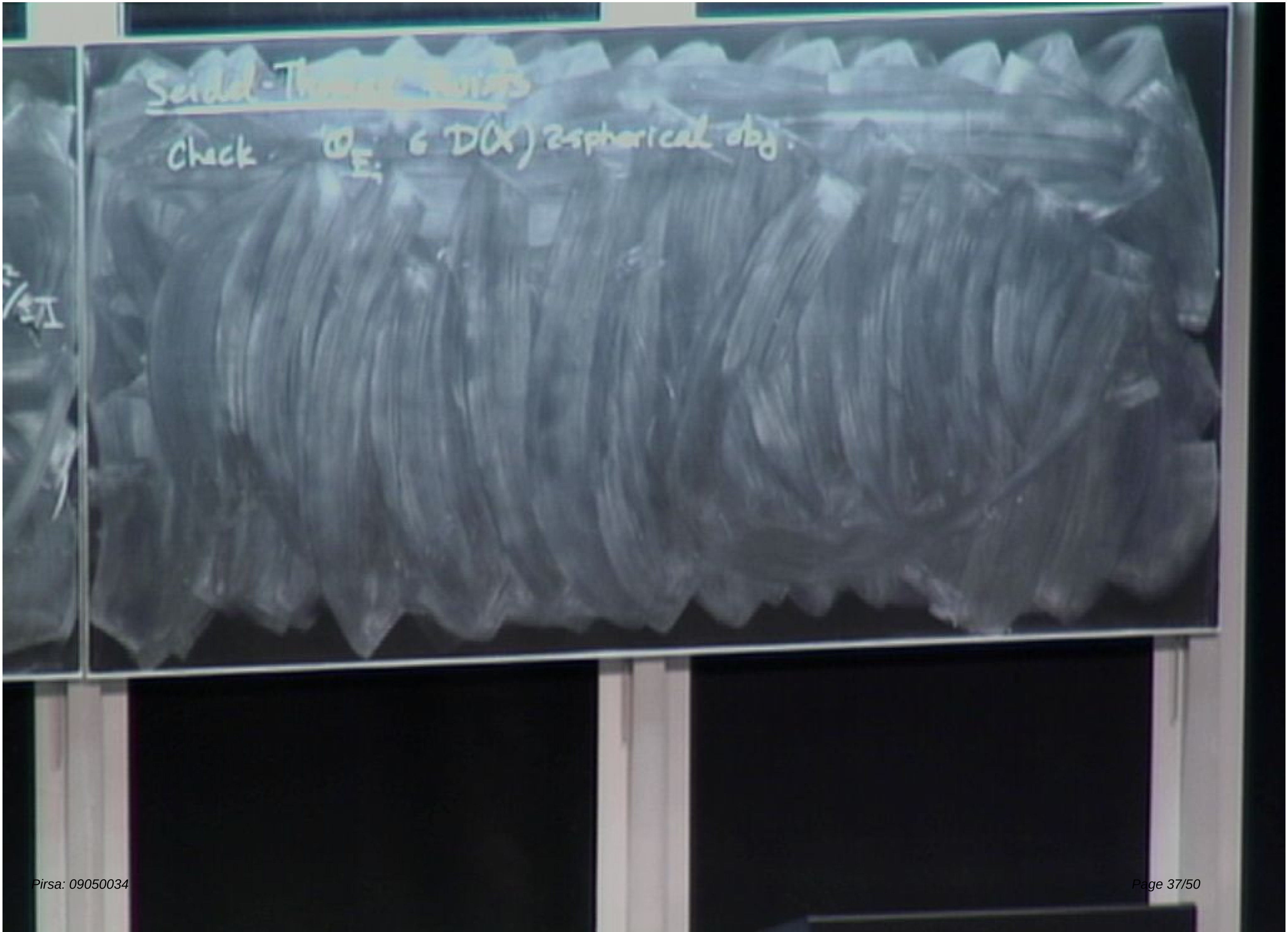
irred components $\| \quad E_i \cong \mathbb{CP}^1$

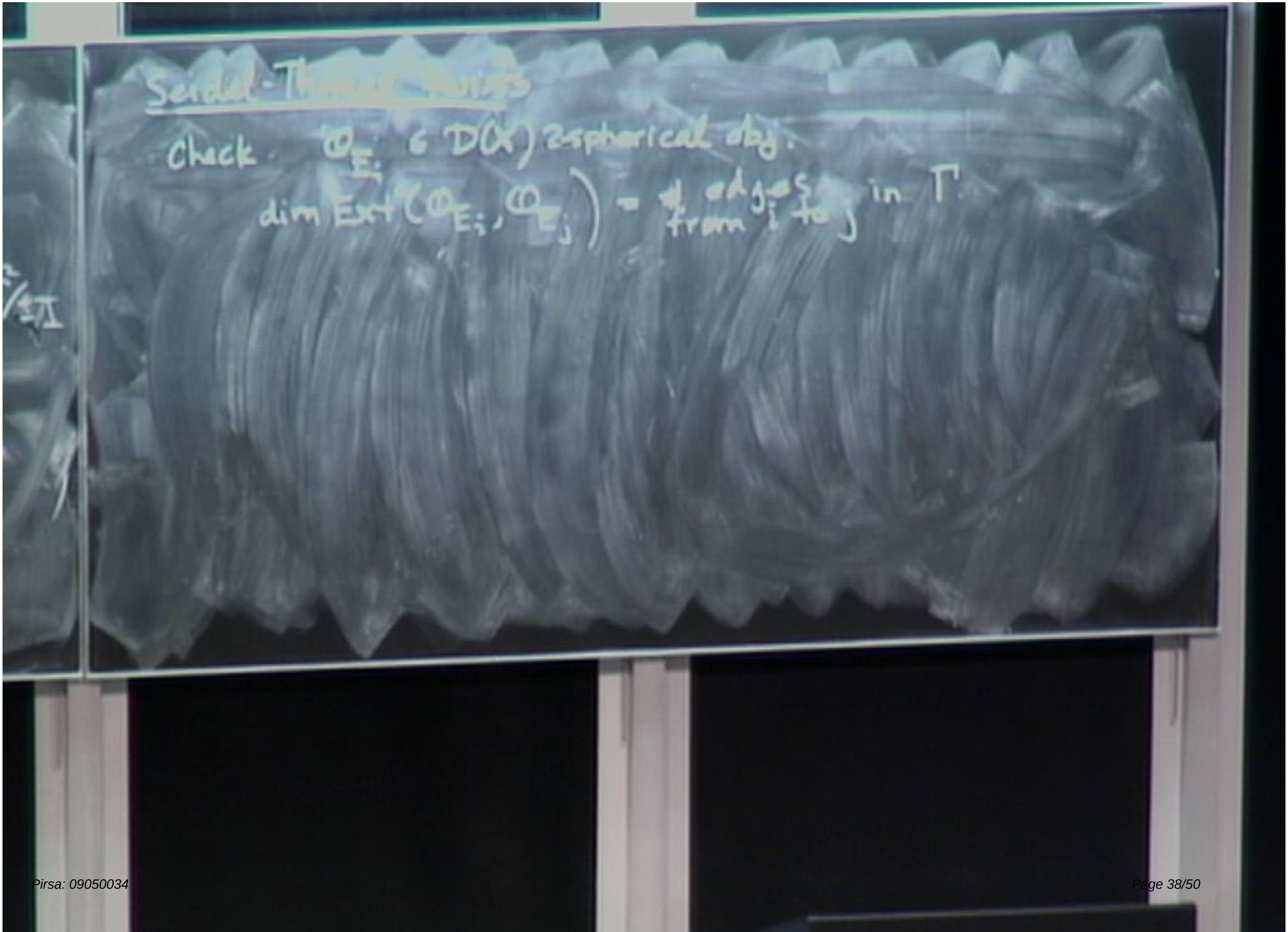
$\cup E_i, \; i \in \Gamma \qquad E_i^2 = -2$

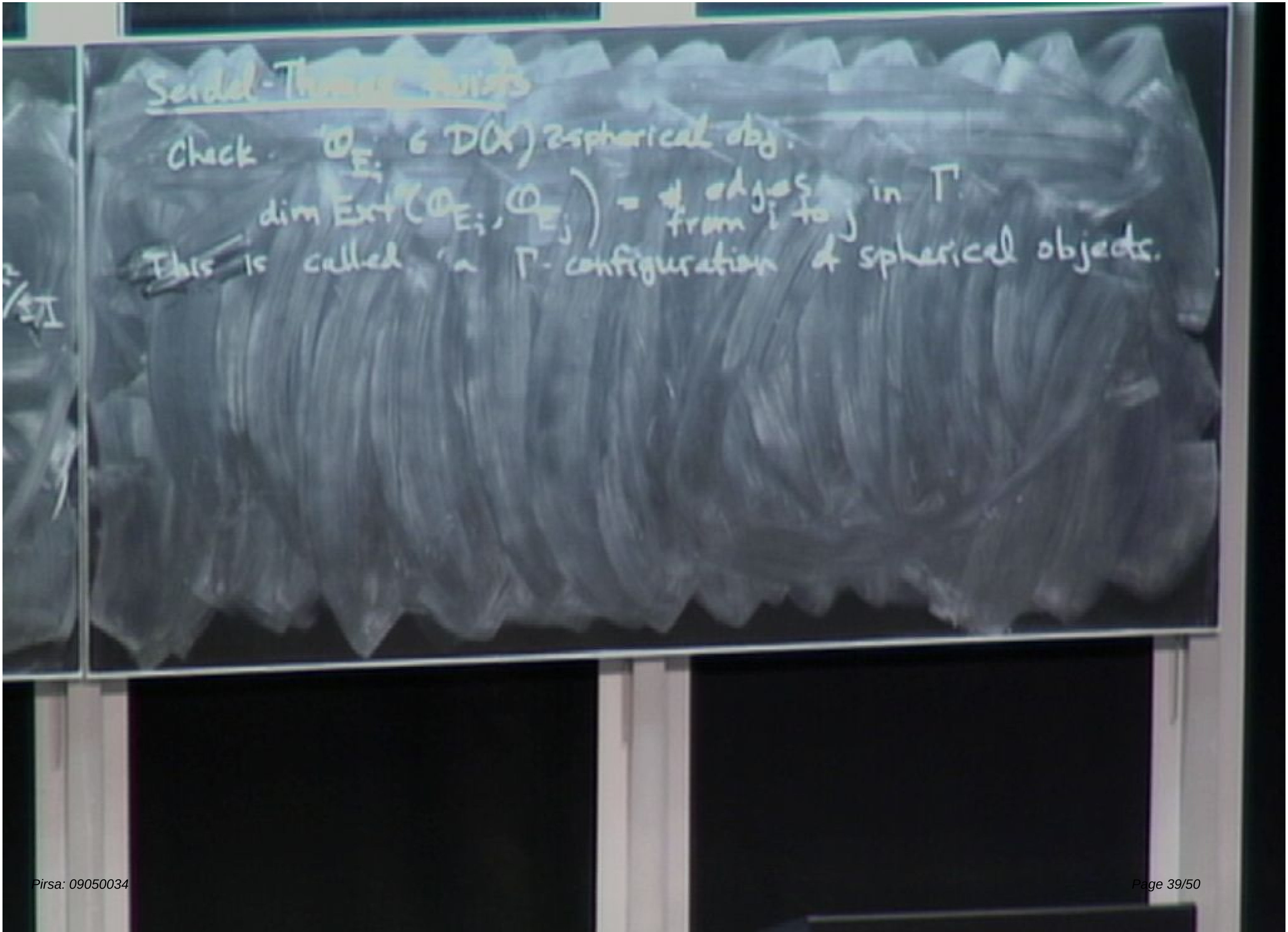


Dual graph of E is a diagram of type ADE.


Check


Seidel
Check. $\mathcal{O}_{E_i} \in D(X)$ 2-spherical obj.


Seidel-Thomas Twists
Check: $\mathcal{O}_{E_i} \in D(X)$ 2-spherical obj.
$\dim \mathrm{Ext}^*(\mathcal{O}_{E_i}, \mathcal{O}_{E_j})$ = # edges from i to j in Γ


Seidel-Thomas Twists
Check: O_{E_i} ∈ D(X) 2-spherical obj.
dim Ext^1(O_{E_i}, O_{E_j}) = # edges from i to j in Γ.
This is called a Γ-configuration of spherical objects.

Seidel-Thomas Twists

Check. $\mathcal{O}_{E_i} \in D(X)$ 2-spherical obj.

$\dim \mathrm{Ext}^1(\mathcal{O}_{E_i}, \mathcal{O}_{E_j})$ = # edges from i to j in Γ.

This is called a Γ-configuration of spherical objects.

Thrm (S-T) If $\{S_i\}$ Γ-config of spher objects,

then the twists

Seidel-Thomas twists

Check: $\mathcal{O}_{E_i} \in D(X)$ 2-spherical obj.

$\dim \mathrm{Ext}^1(\mathcal{O}_{E_i}, \mathcal{O}_{E_j}) = \#$ edges from i to j in Γ.

This is called a Γ-configuration of spherical objects.

<u>Thrm</u> (S-T) If $\{S_i\}$ Γ-config of spher objects, then the twists T_i satisfy braid relations (

Seidel-Thomas twists

Check: $\mathcal{O}_{E_i} \in D(X)$ 2-spherical obj.

$\dim \operatorname{Ext}^1(\mathcal{O}_{E_i}, \mathcal{O}_{E_j}) = \#$ edges from i to j in Γ.

This is called a Γ-configuration of spherical objects.

Thrm (S-T) If $\{S_i\}$ Γ-config of spher objects, then the twists T_i satisfy braid relations ($T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$)

Seidel-Thomas twists

Check. $\mathcal{O}_{E_i} \in D(X)$ 2-spherical obj.

$\dim \mathrm{Ext}^1(\mathcal{O}_{E_i}, \mathcal{O}_{E_j})$ = # edges from i to j in Γ.

This is called a Γ-configuration of spherical objects.

Thrm (S-T) If $\{S_i\}$ Γ-config of spher objects, then the twists T_i satisfy braid relations $(T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1})$

So $B_\Gamma \to D(X)$.

Seidel-Thomas twists

Check. $\mathcal{O}_{E_i} \in D(X)$ 2-spherical obj.

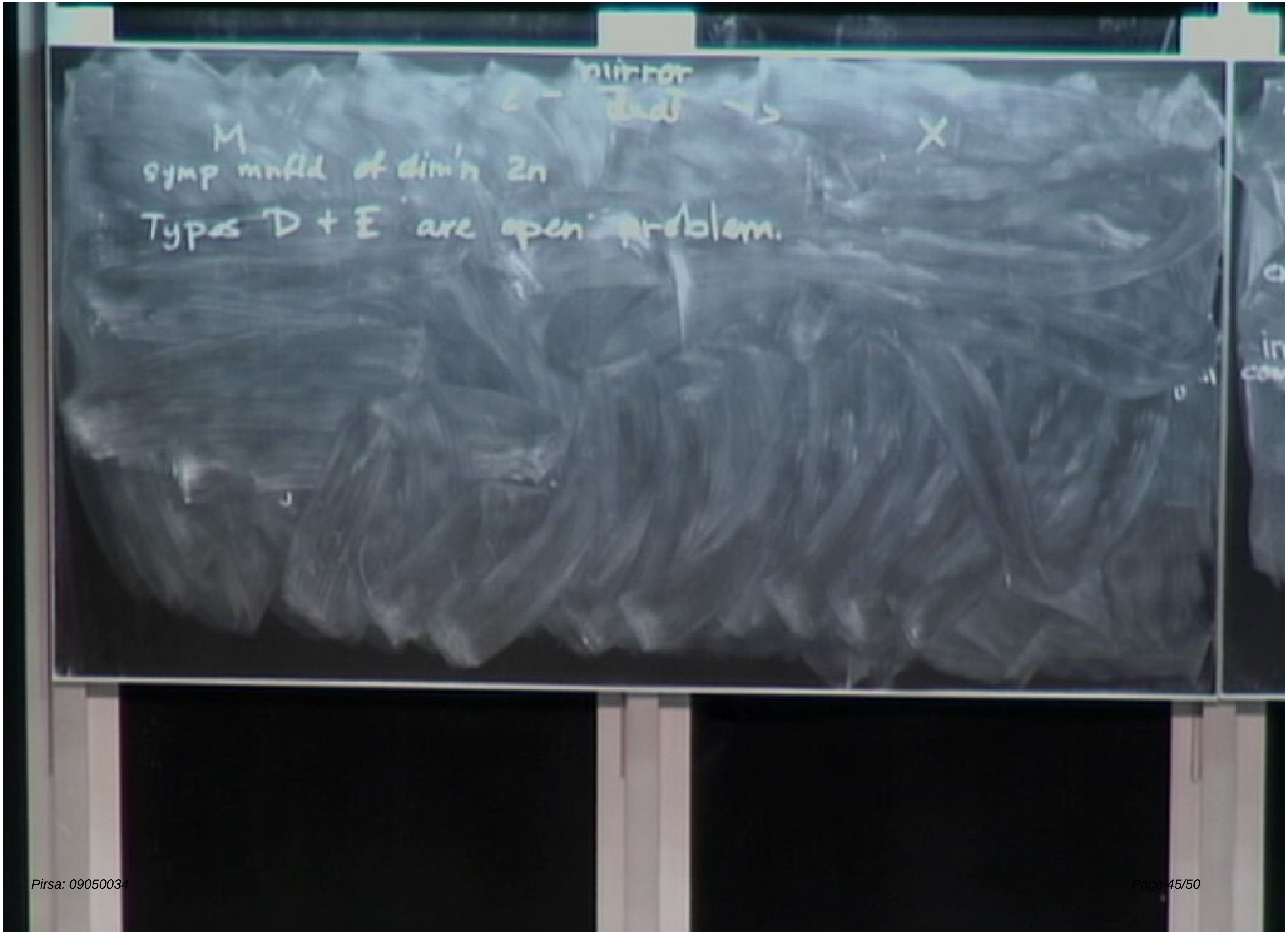
$\dim \mathrm{Ext}^1(\mathcal{O}_{E_i}, \mathcal{O}_{E_j}) = \#$ edges from i to j in Γ.

This is called a Γ-configuration of spherical objects.

Thrm (S-T) If $\{S_i\}$ Γ-config of spher objects, then the twists T_i satisfy braid relations $\left(T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \right)$

So $B_\Gamma \curvearrowright D(X)$. If $\dim X \geq 2$ and $\Gamma = A_m$, then this action is faithful.

(B_Γ: braid grp)


mirror
M
X
symp mnfld of dim'n 2n
Types D + E are open problem.

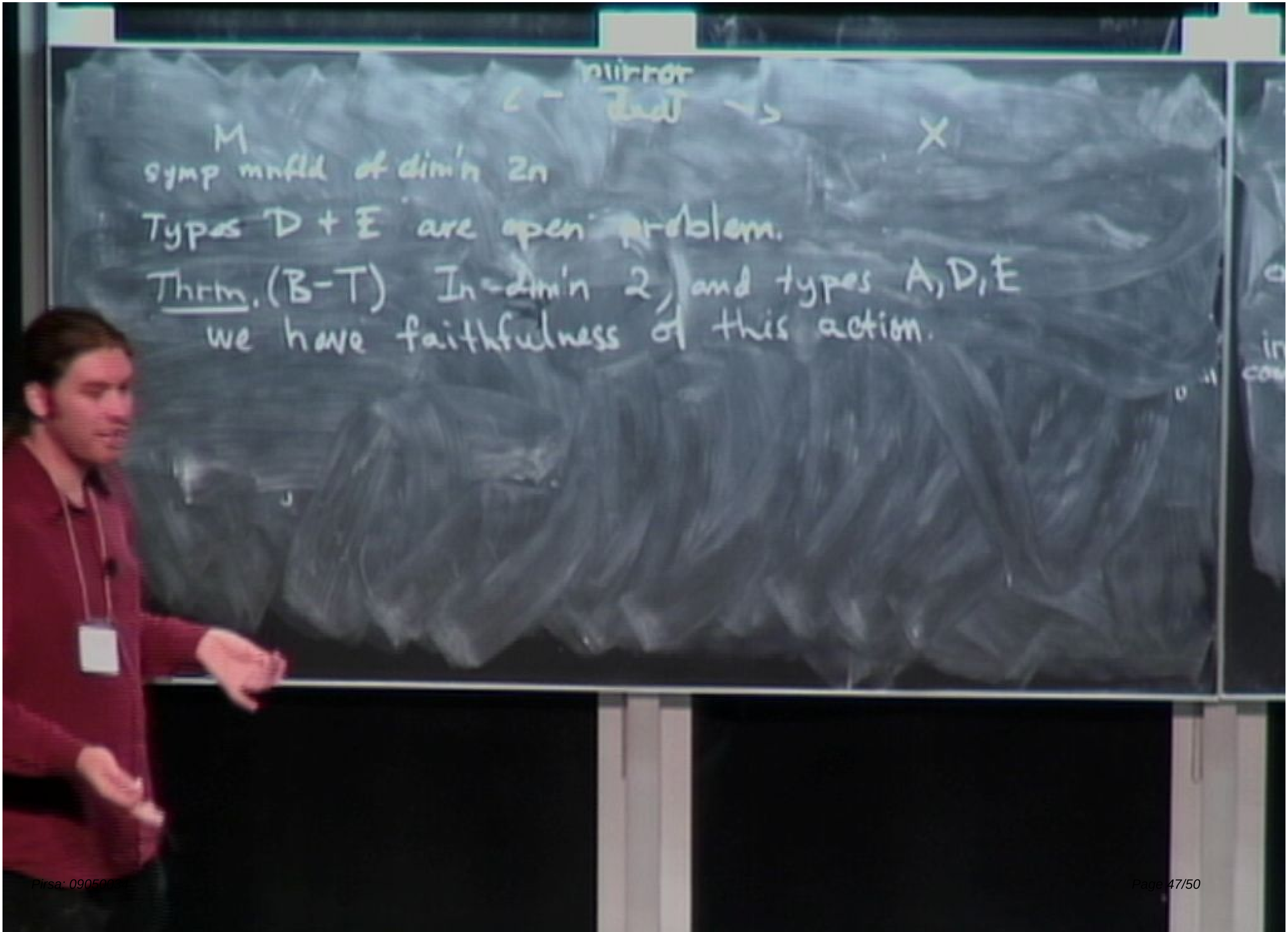
mirror
⟵ dual ⟶

M　　　　　　　　　　　　　　　X

symp mnfld of dim'n 2n

Types D + E are open problem.

Thrm. (B-T) In dim'n 2, and types A, D, E
we have faithfulness of this action.


mirror
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Types D + E are open problem.
Thrm. (B-T) In dim'n 2, and types A, D, E
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mirror
$\longleftarrow$ dual $\longrightarrow$

M $\qquad$ X

symp mnfld of dim'n $2n$

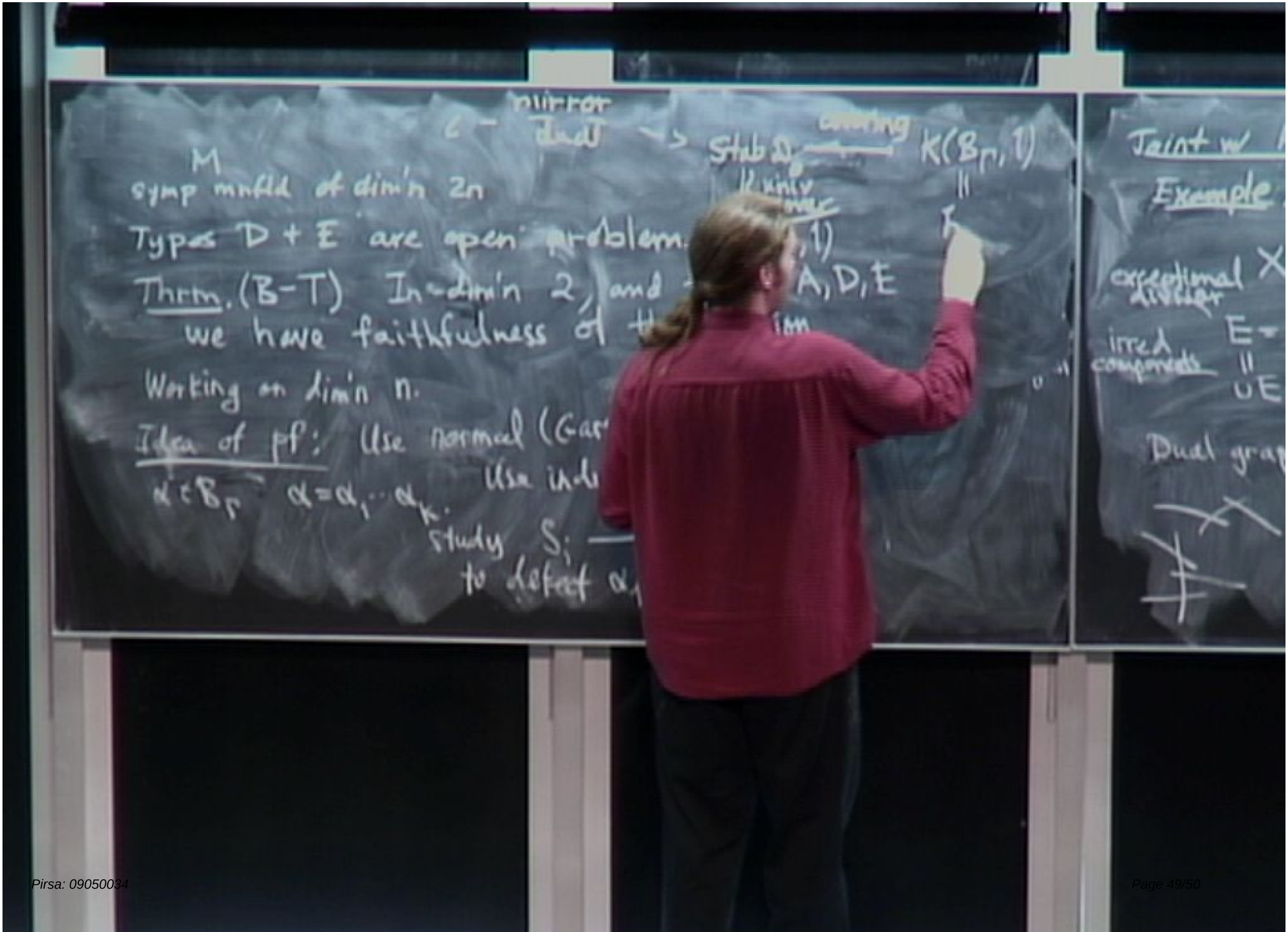
Types D + E are open problem.

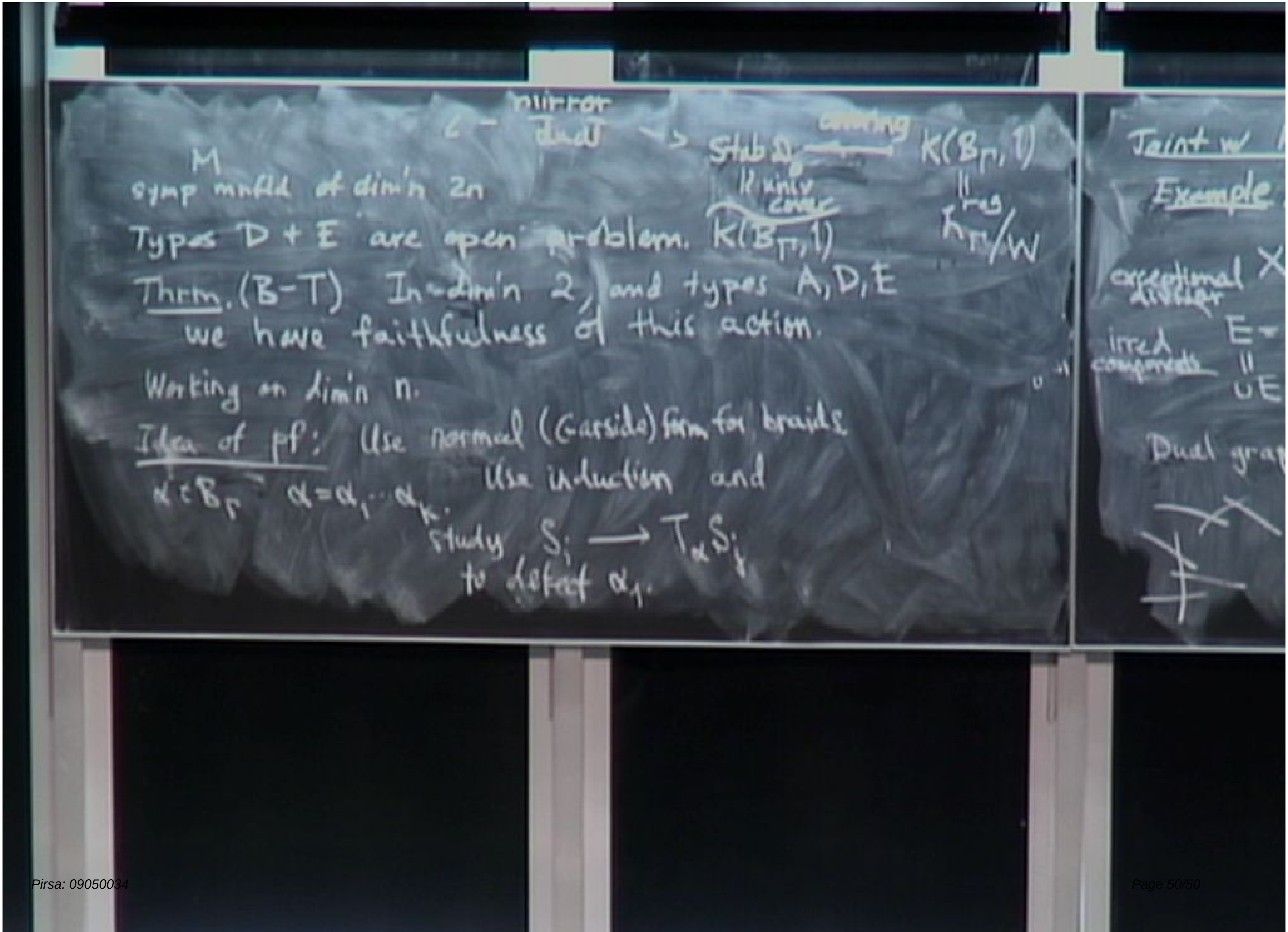
Thrm. (B-T) In dim'n 2, and types A, D, E we have faithfulness of this action.

Working on dim'n n.

Idea of pf: Use normal (Garside) form for braids.

$\alpha \in B_r$ $\quad \alpha = \alpha_1 \cdots \alpha_k$. Use induction and study $S_i \longrightarrow T_\alpha S_j$ to detect α_1.


mirror
dual
Stab
covering
K(B
M
symp mnfld of dim'n 2n
Types D + E are open problem.
Thrm. (B-T) In dim'n 2 and
A, D, E
we have faithfulness
Working on dim'n n.
Idea of pf: Use normal
Use ind
study
to detect
Example
exceptional divisor
irred components
Dual grap


mirror
Dual
Stab
covering
$K(B_\Gamma, 1)$
M
symp mnfld of dim'n 2n
univ cover
reg
Types D + E are open problem. $K(B_\Gamma, 1)$
$\mathfrak{h}_\Gamma / W$
Thrm. (B-T) In dim'n 2, and types A, D, E
we have faithfulness of this action.
Working on dim'n n.
Idea of pf: Use normal (Garside) form for braids
$\alpha \in B_\Gamma$ $\alpha = \alpha_1 \cdots \alpha_k$. Use induction and
study $S_i \longrightarrow T_\alpha S_j$
to detect α_1.
Joint w/
Example
exceptional divisor
irred components
Dual grap