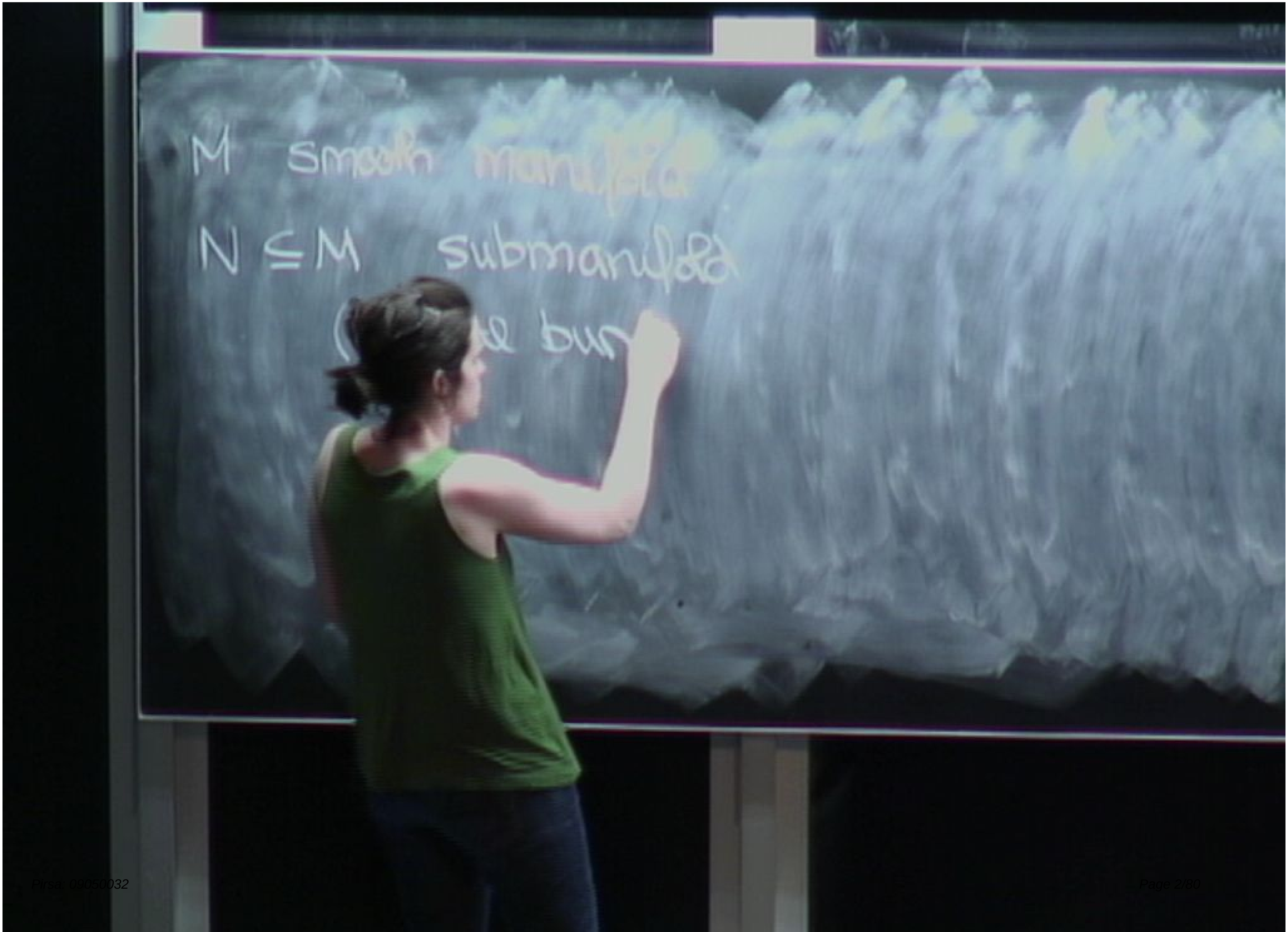
Title: Relative string topology

Date: May 08, 2009 03:15 PM

URL: http://pirsa.org/09050032

Abstract: I'll discuss how to get an interesting invariant of submanifolds by using the ideas of string topology.


M smooth manifold
N ⊆ M submanifold

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$N \subseteq M$ submanifold

(normal bundle is oriented)

$\Omega_* M = \{ \gamma : S^1 \to M \quad \gamma(1) = * \}$

$LM =$

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$$P_N M = \{\gamma : I \to M \quad \gamma(0), \gamma(1) \in N\}$$

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$N \subseteq P_N M$ as constant paths

$C_*(P_N M; N)$

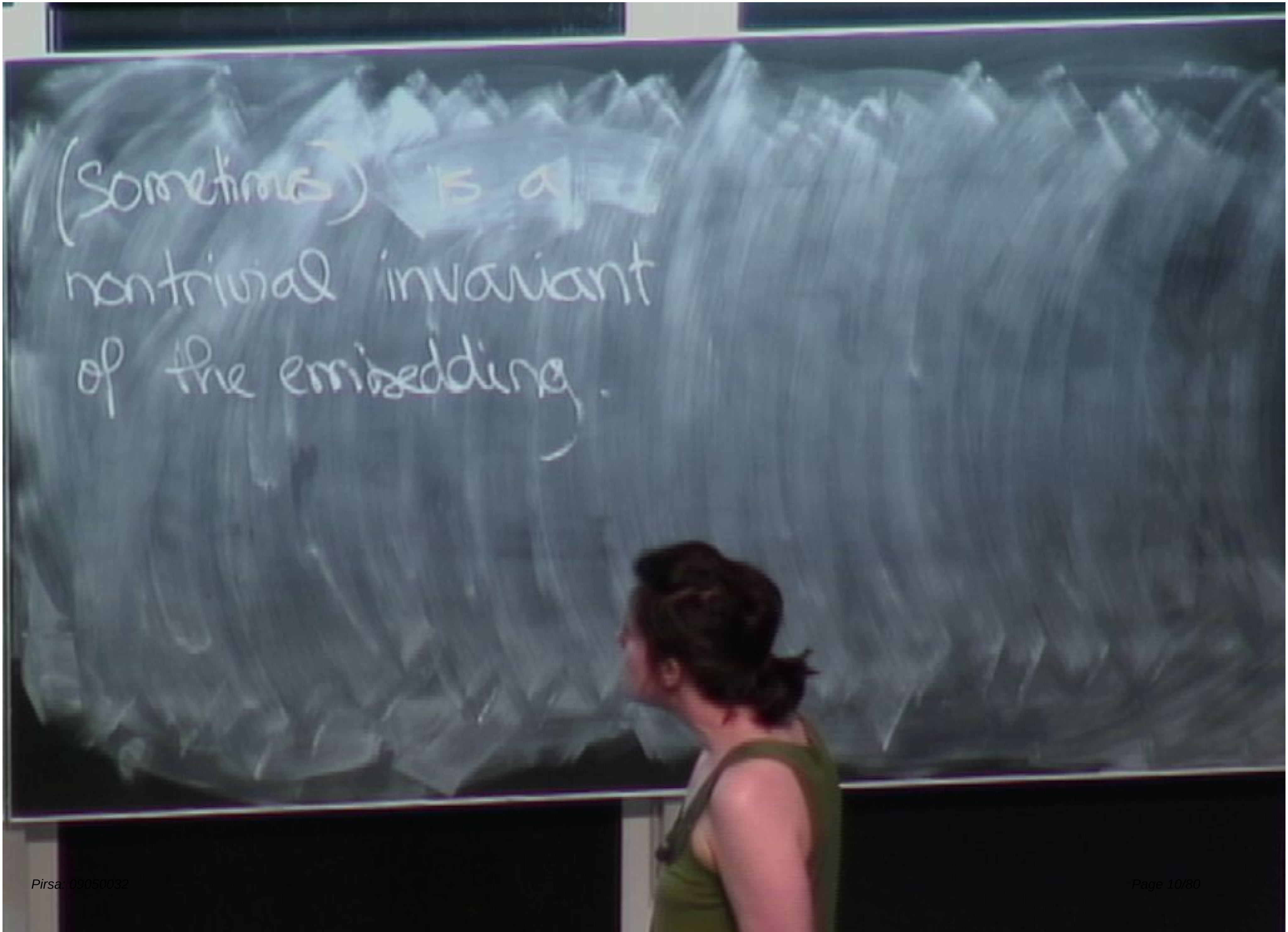
$N \subseteq P_N M$ as constant paths

$C_*(P_N M; N)$

Thm (1) There is an A_∞-coproduct on $C_*(P_N M; N)$

(2) Using this, we can build $\mathcal{H}(N \subseteq M)$ which

(sometimes) is a
nontrivial invariant
of the embedding.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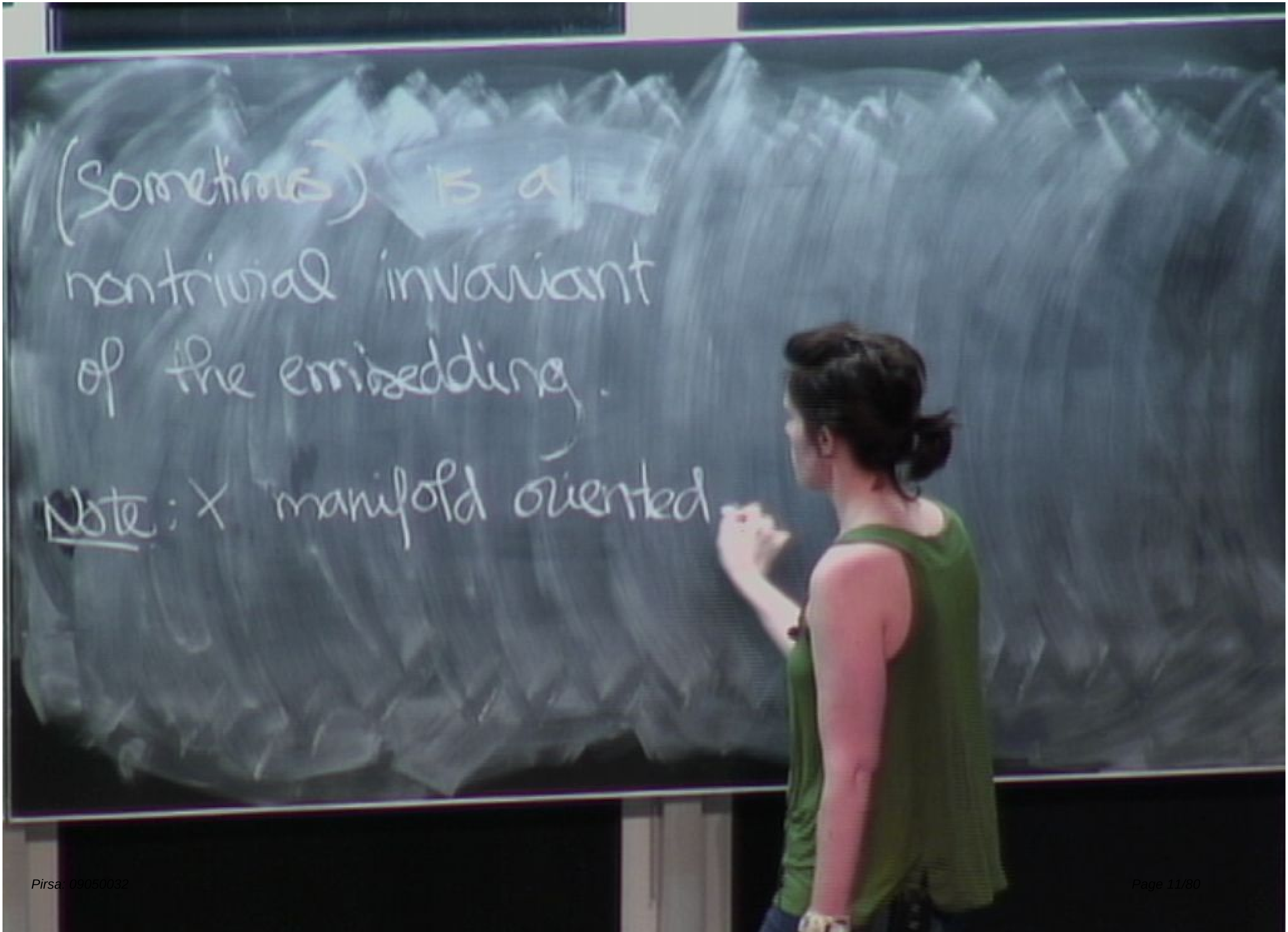
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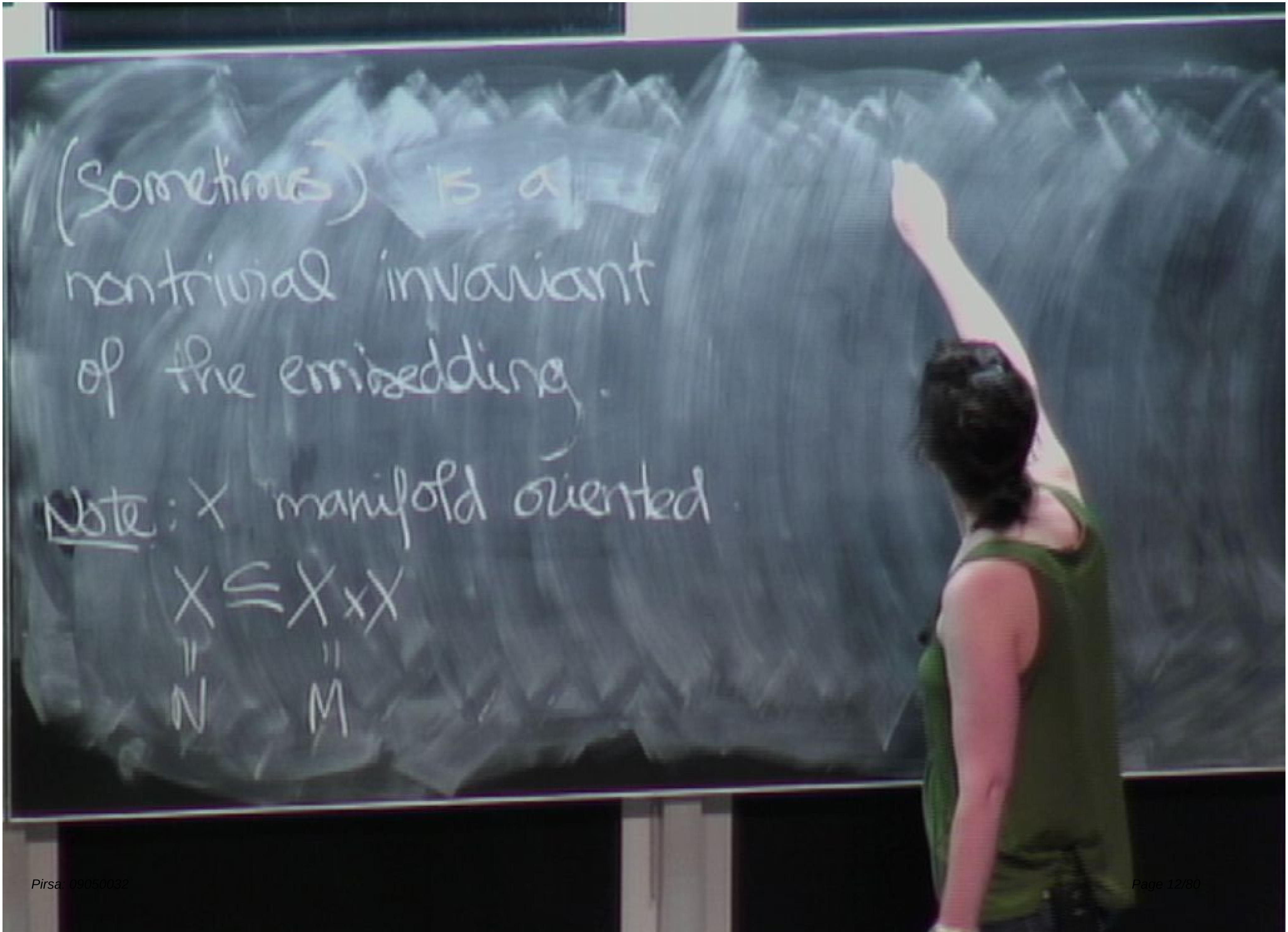
Note: X manifold oriented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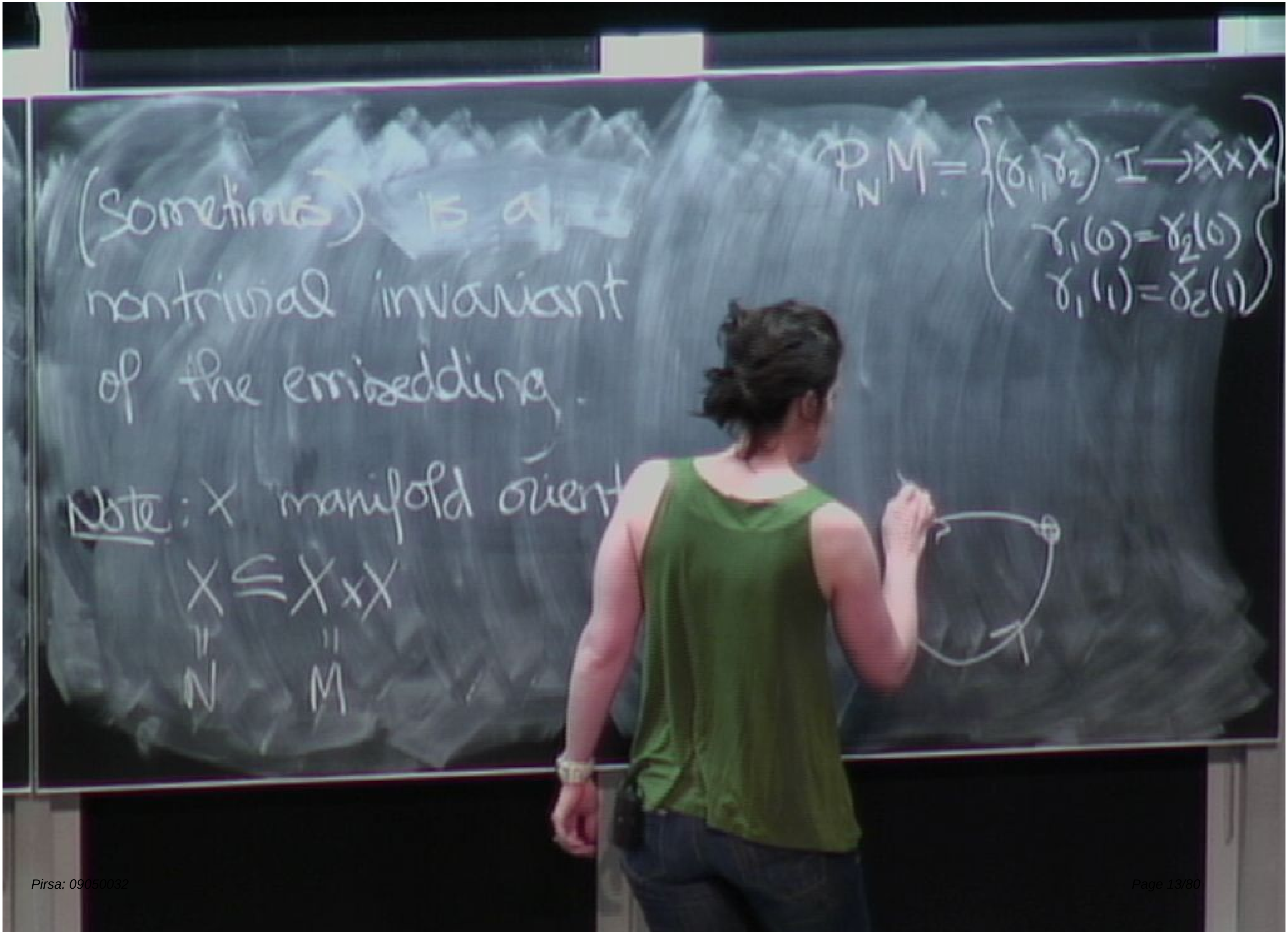
Note: X manifold oriented
X ⊆ X×X
N = X, M = X×X


(sometimes) is a
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Note: X manifold orient
$X \subseteq X \times X$
N M
$P_N M = \{(\gamma_1, \gamma_2): I \to X \times X$
$\gamma_1(0) = \gamma_2(0)$
$\gamma_1(1) = \gamma_2(1)\}$

(sometimes) is a nontrivial invariant of the embedding.

$$P_N M = \left\{ (\gamma_1, \gamma_2) : I \to X \times X \;\middle|\; \begin{array}{l} \gamma_1(0) = \gamma_2(0) \\ \gamma_1(1) = \gamma_2(1) \end{array} \right\}$$

<u>Note</u>: X manifold oriented.

$$\begin{array}{ccc} X & \subseteq & X \times X \\ \| & & \| \\ N & & M \end{array}$$

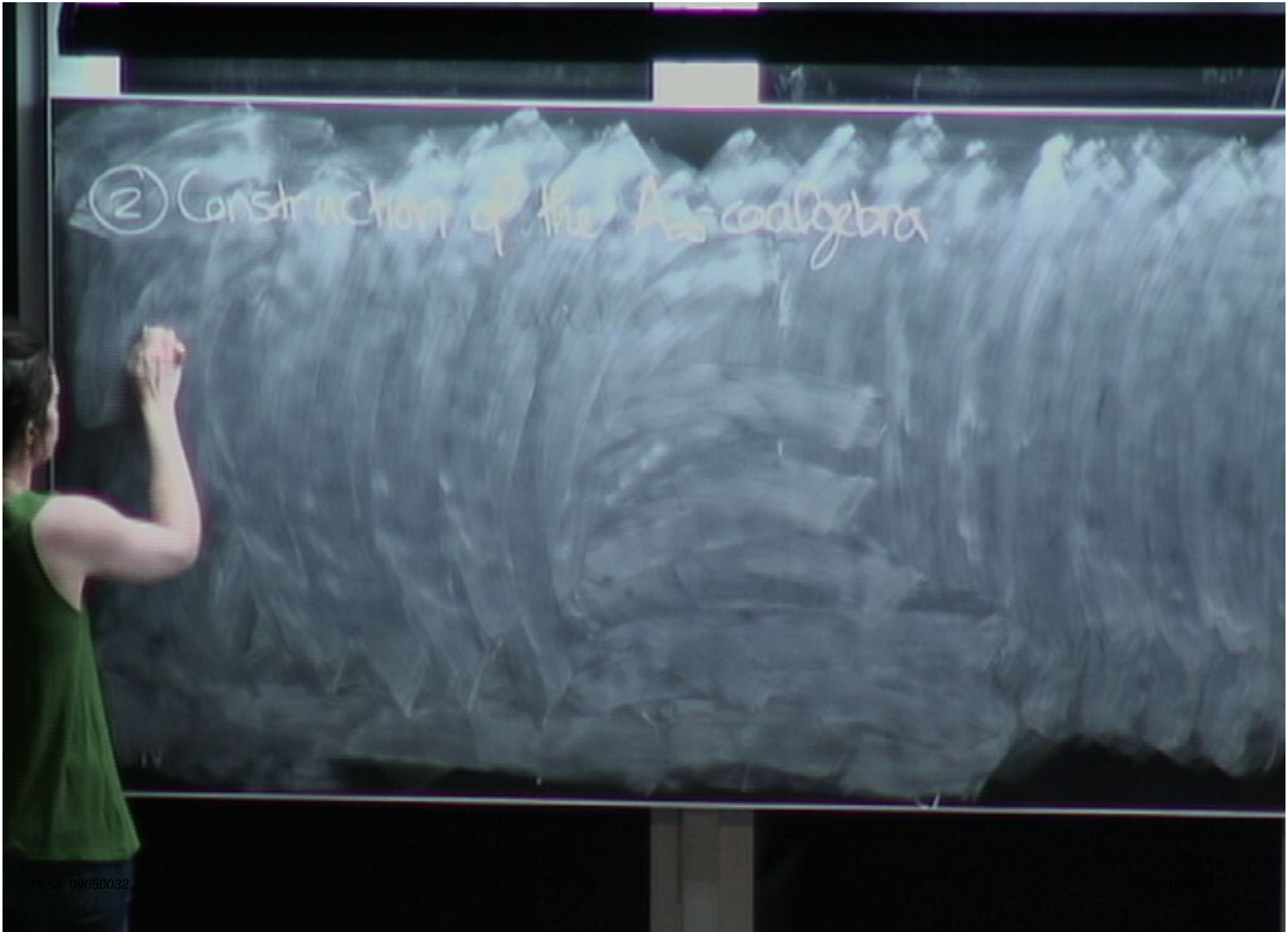
(sometimes) is a nontrivial invariant of the embedding.

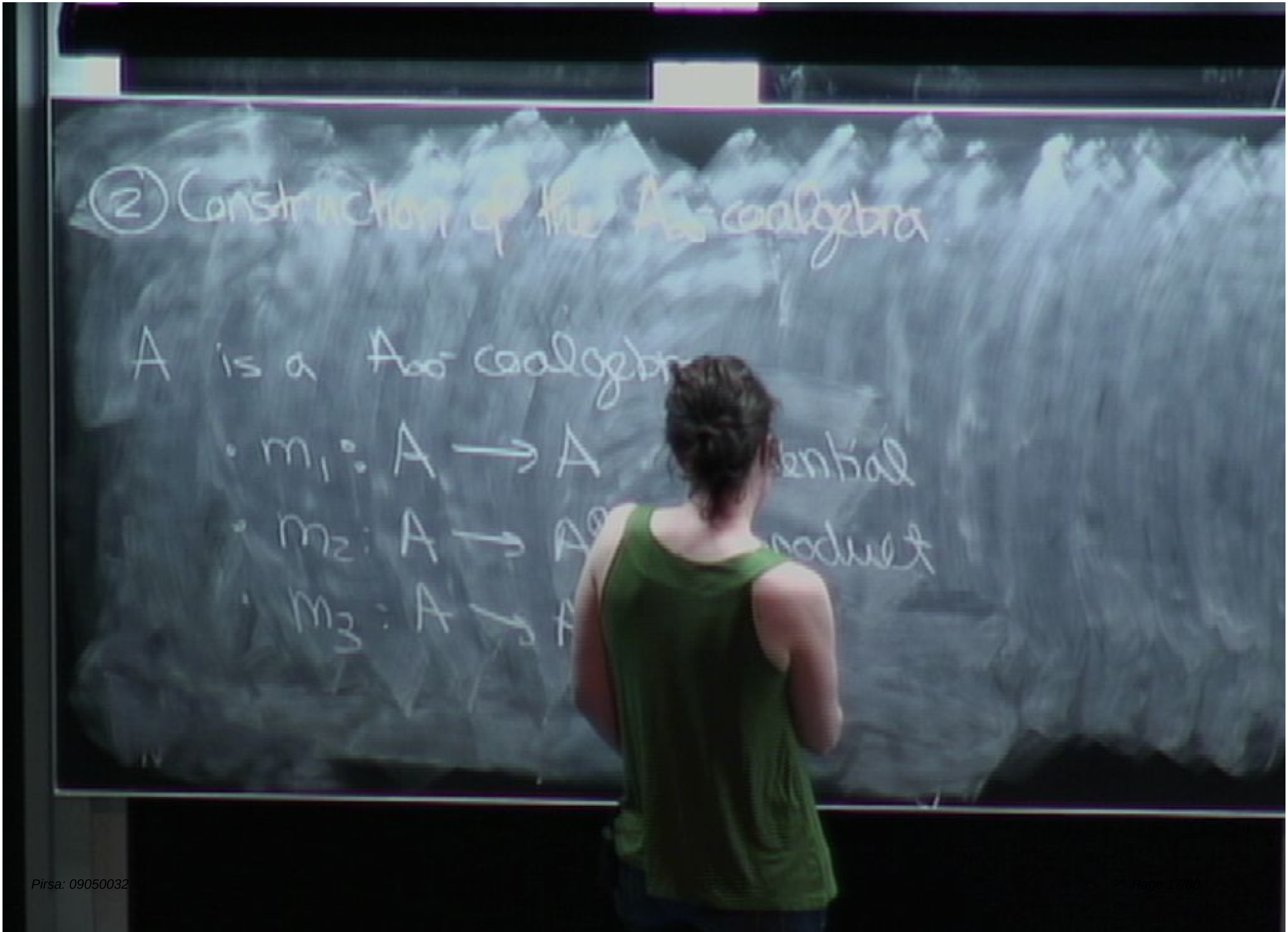
$$P_N M = \left\{ (\gamma_1, \gamma_2) : I \to X \times X \;\middle|\; \begin{array}{l} \gamma_1(0) = \gamma_2(0) \\ \gamma_1(1) = \gamma_2(1) \end{array} \right\}$$

$$\cong LM.$$

<u>Note</u>: X manifold oriented.

$$\underset{\displaystyle N}{\underset{\|}{X}} \subseteq \underset{\displaystyle M}{\underset{\|}{X \times X}}$$


② Construction of the A_∞ coalgebra


② Construction of the A∞ coalgebra
A is a A∞ coalgebra
• m1 : A → A
ential
• m2 : A → A
oduct
• m3 : A → A

② Construction of the A_∞-coalgebra

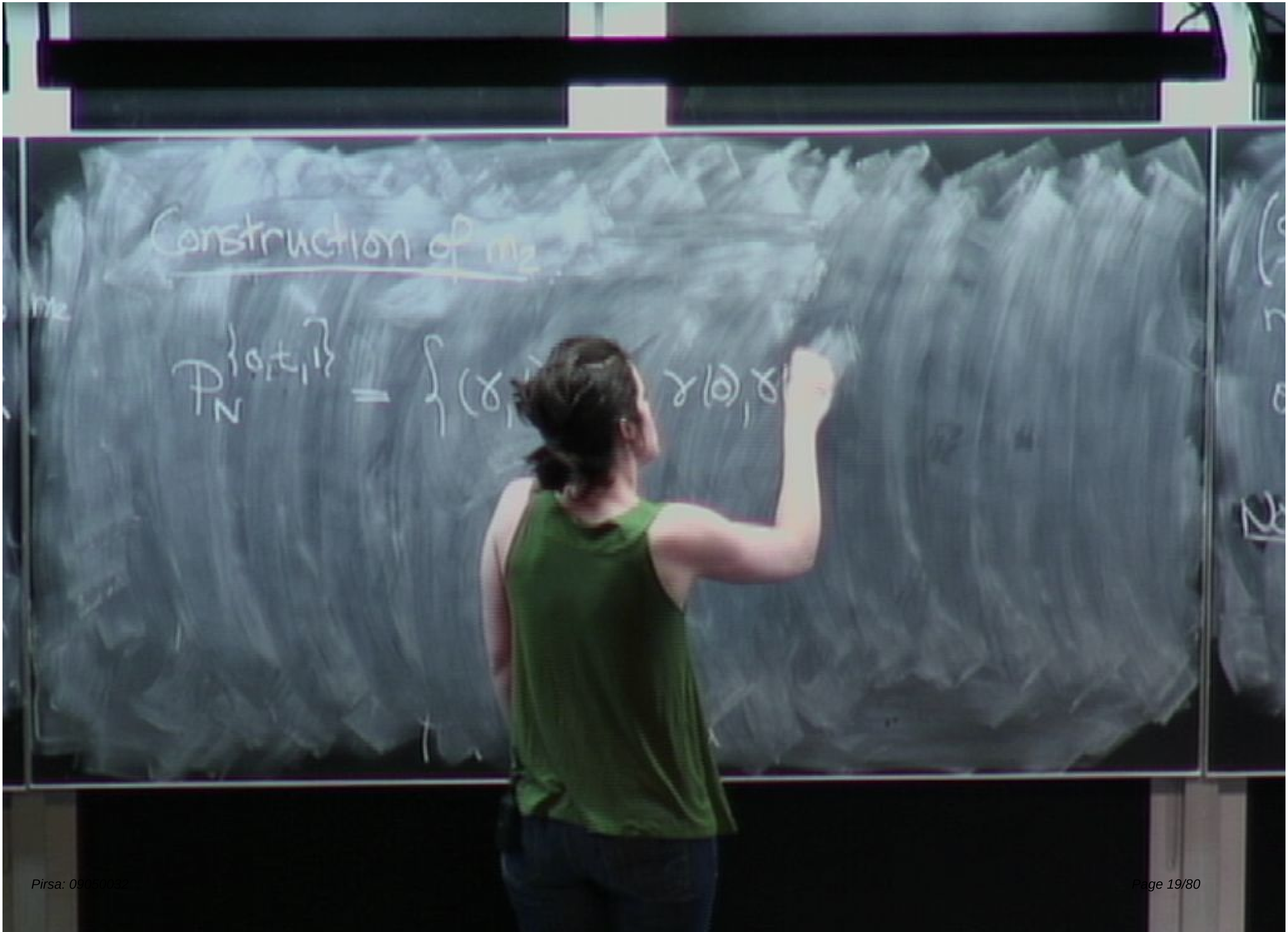
$$\begin{tikzcd} A \arrow[r] \arrow[d] & A\otimes A \arrow[d, "Id\otimes m_2"] \\ A^{\otimes 2} \arrow[r, "m_2\otimes Id"'] & A\otimes A \otimes A \end{tikzcd}$$

A is a A_∞-coalgebra

- $m_1 : A \longrightarrow A$ differential
- $m_2 : A \longrightarrow A^{\otimes 2}$ coproduct
- $m_3 : A \longrightarrow A^{\otimes 3}$ homotopy

Higher homol

$A \xrightarrow{m_n} A^{\otimes n}$

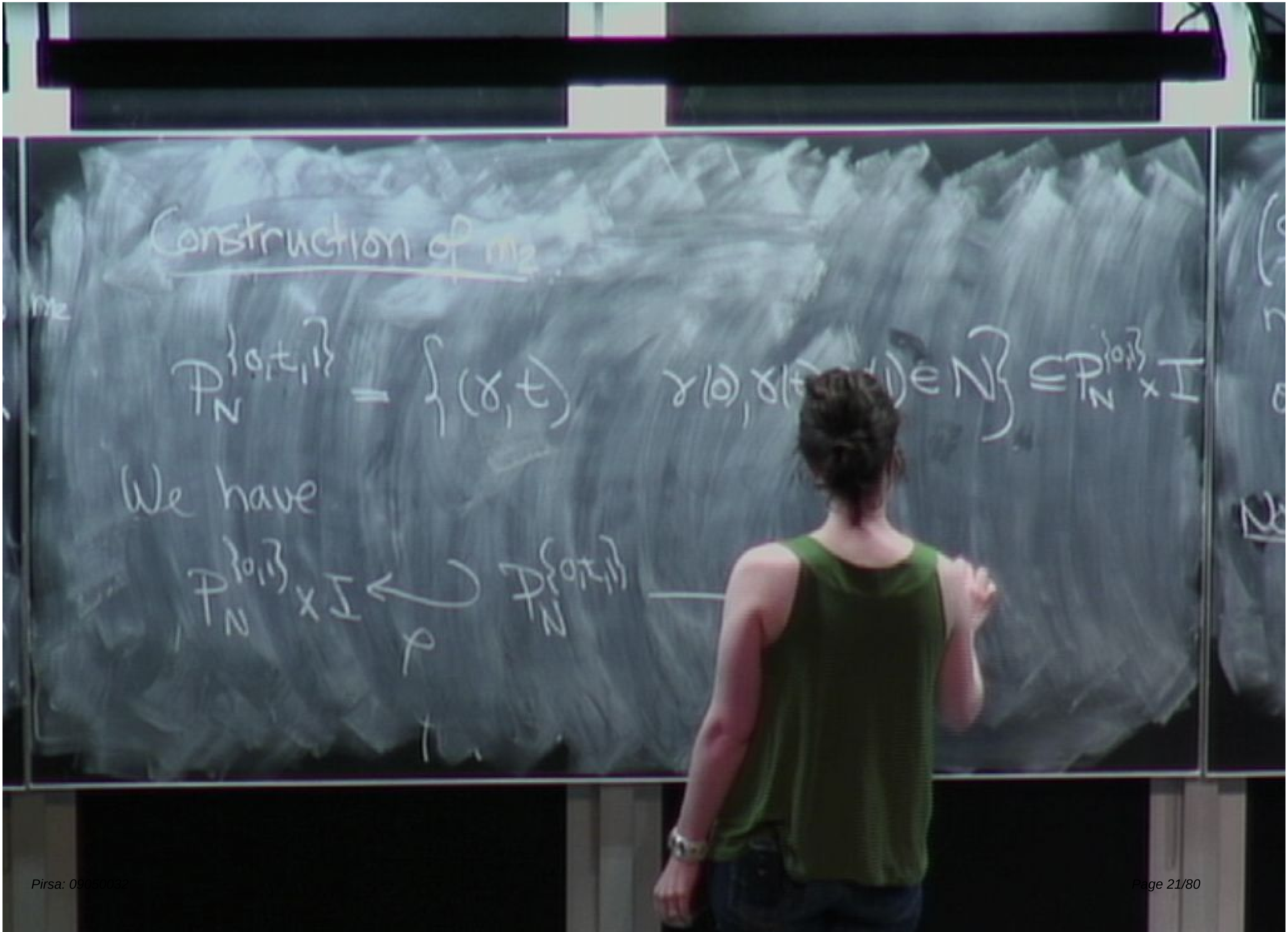

Construction of m_2
$P_N^{\{0,t,1\}} = \{(\gamma,$ … $\gamma(0), \gamma$

Construction of m_2

$$P_N^{\{0,t,1\}} = \{(\gamma,t) \mid \gamma(0),\gamma(t),\gamma(1)\in N\} \subseteq P_N^{\{0,1\}} \times I$$

We have

P


Construction of m2
P_N^{0,t,1} = {(γ,t) γ(0), γ(t), γ(1) ∈ N} ⊆ P_N^{0,1} × I
We have
P_N^{0,1} × I ↩ P_N^{0,t,1}
ρ

Construction of m_2

$$P_N^{\{0,t,1\}} = \{(\gamma, t) \quad \gamma(0), \gamma(t), \gamma(1) \in N\} \subseteq P_N^{\{0,1\}} \times I$$

We have

$$P_N^{\{0,1\}} \times I \xhookleftarrow{\rho} P_N^{\{0,t,1\}} \xrightarrow[\text{split}]{} P_N^{\{0,1\}} \times P_N^{\{0,1\}}$$

split the path at t.

Construction of m_2

$$P_N^{\{0,t,1\}} = \{(\gamma,t) \quad \gamma(0),\gamma(t),\gamma(1)\in N\} \subseteq P_N^{\{0,1\}}\times I$$

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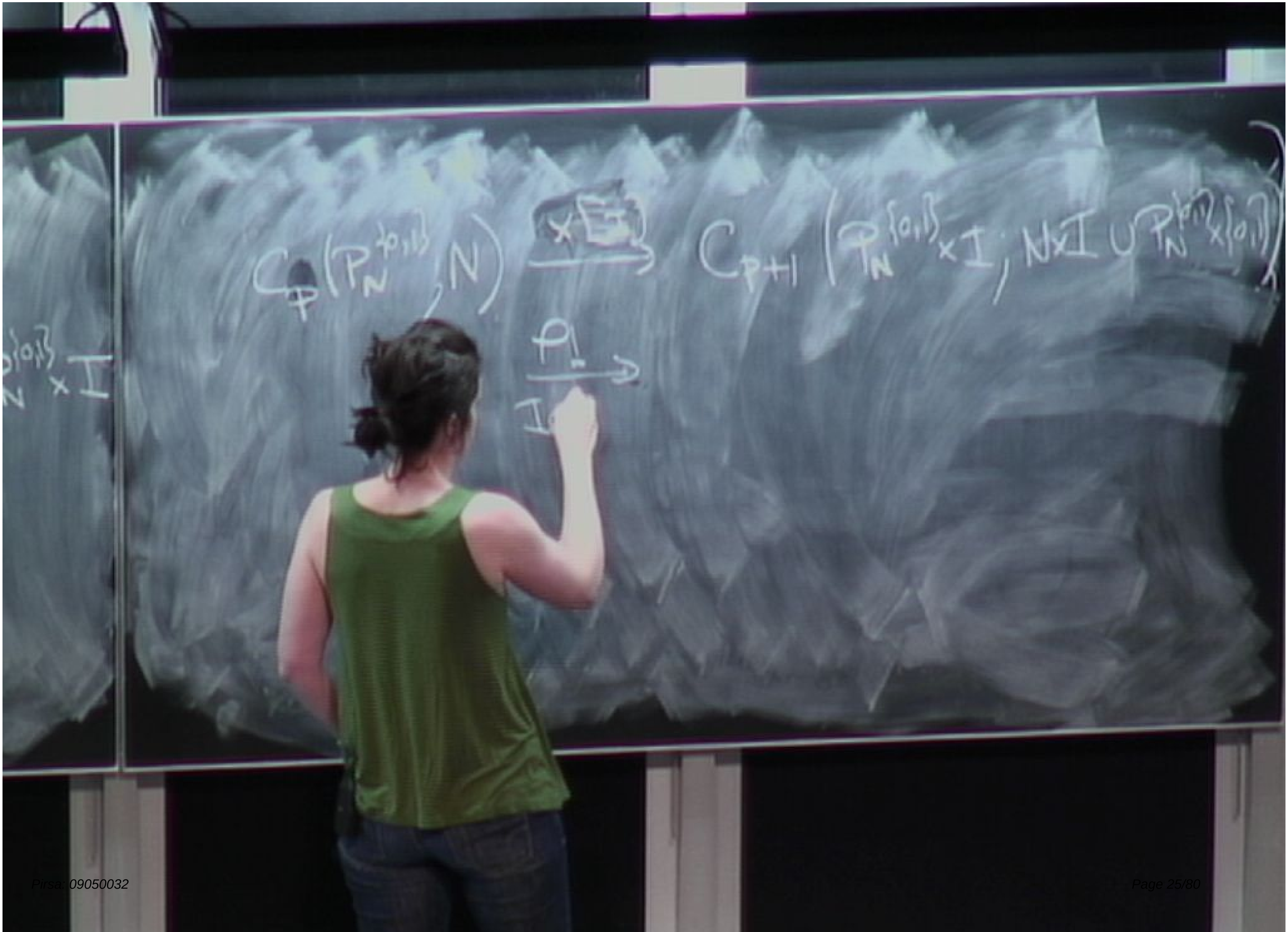
$$P_N^{\{0,1\}}\times I \xleftarrow{\rho} P_N^{\{0,t,1\}} \xrightarrow[\text{split the path at } t.]{} P_N^{\{0,1\}}\times P_N^{\{0,1\}}$$

$C_\Phi(P_N^{\{0,1\}}, N)$

$^{\{0,1\}}_{N} \times I$


$C_p(P_N^{\{0,1\}}, N) \xrightarrow{\times I} C_{p+1}(P_N^{\{0,1\}} \times I, N \times I \cup P_N^{\{0,1\}} \times \{0,1\})$

$P_N^{\{0,1\}} \times I$

$$C_p(P_N^{\{0,1\}}, N) \xrightarrow{\times [illegible]} C_{p+1}(P_N^{\{0,1\}} \times I, N \times I \cup P_N^{\{1\}} \times \{0,1\})$$

$$\xrightarrow[\text{Intersecting}]{\rho_1} C_{p+1-k}(P_N^{\{0,+,1\}}, N \times I \cup P_N^{\{0,1\}} \times \{0,1\})$$

$$\xrightarrow[\text{Split}]{} (C_*(P_N M))^{\otimes 2}, \; p+1-k$$

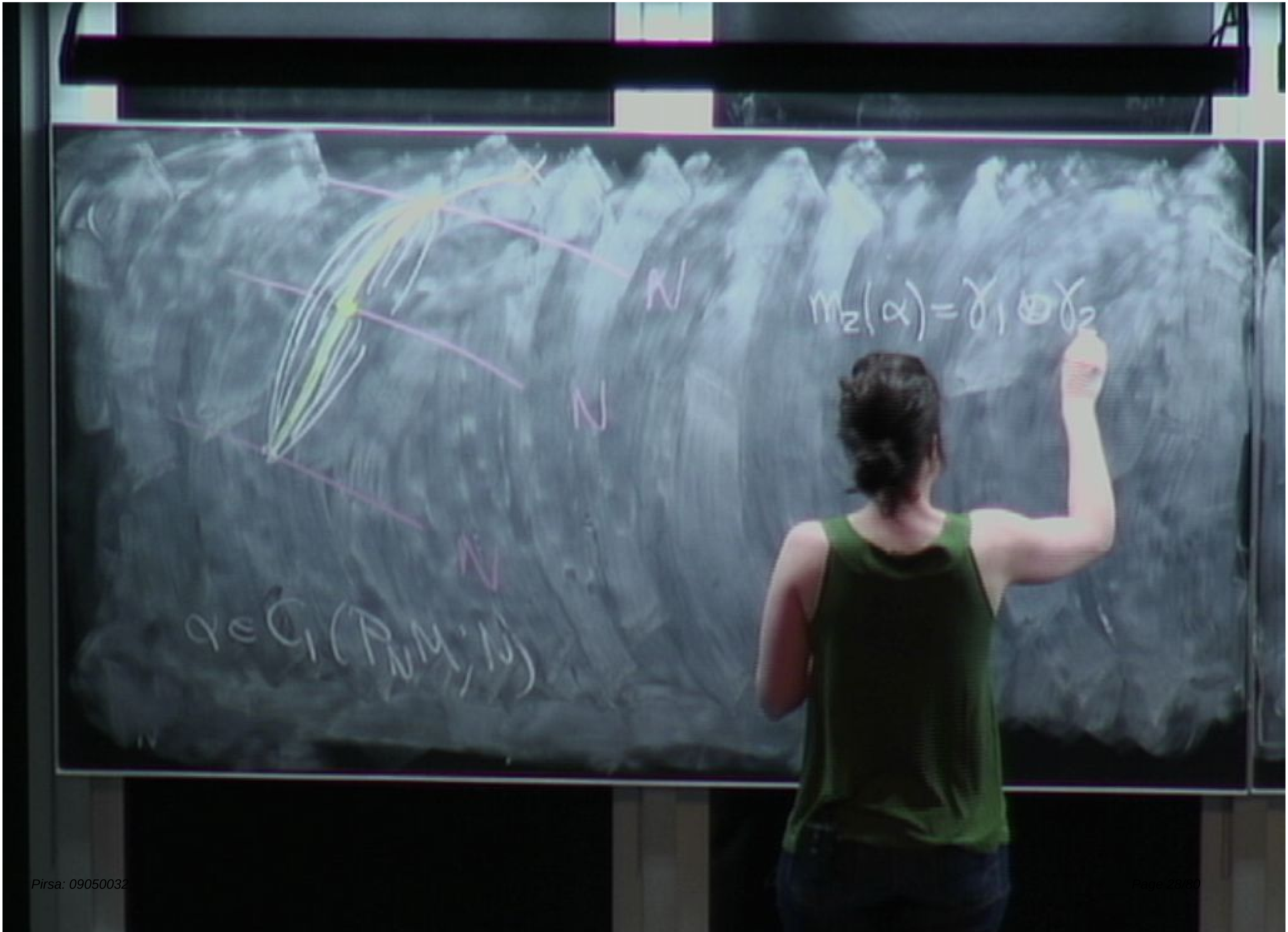
$\alpha \in C_1(P_N M; \mathbb{N})$

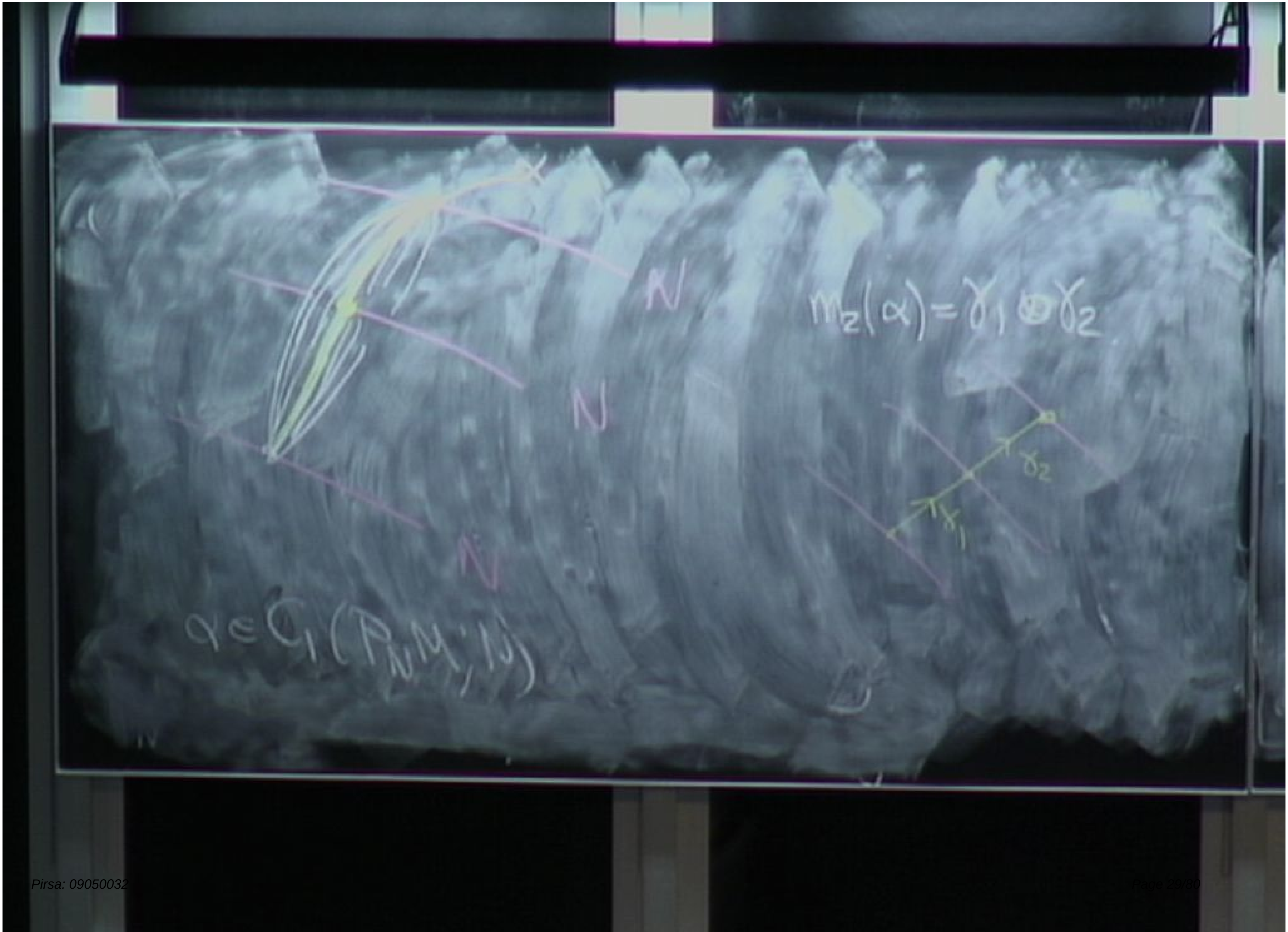
$$\begin{tikzcd} A \arrow[r] \arrow[d] & A \otimes A \arrow[d, "\mathrm{Id} \otimes"] \\ A^{\otimes 2} \arrow[r] & A \otimes A \end{tikzcd}$$

$m \otimes \mathrm{Id} \quad \otimes A$

Higher normal

$A \xrightarrow{m_n} A^{\otimes n}$


N
$m_2(\alpha) = \gamma_1 \otimes \gamma_2$
N
N
$\alpha \in C_1(P_N M; \Lambda)$


N
N
N
m2(α) = γ1 ⊗ γ2
γ2
γ1
α ∈ C1(PN M; N)

Construction of m_2

$$P_N^{\{0,t,1\}} = \{(\gamma,t) \mid \gamma(0),\gamma(t),\gamma(1)\in N\} \subseteq P_N^{\{0,1\}}\times I$$

We have

$$\begin{tikzcd}
P_N^{\{0,1\}}\times I & P_N^{\{0,t,1\}} \arrow[l, hook', "\rho"'] \arrow[r, "\text{split the path at } t"] \arrow[d, "ev_t"] & P_N^{\{0,1\}}\times P_N^{\{0,1\}} \\
M & N \arrow[l, "f"'] &
\end{tikzcd}$$

Construction of m_2

$$P_N^{\{0,t,1\}} = \{(\gamma,t) \mid \gamma(0),\gamma(t),\gamma(1)\in N\} \subseteq P_N^{\{0,1\}} \times I$$

We have

$$\begin{tikzcd}
P_N^{\{0,1\}} \times I \arrow[d, "ev_0"'] \arrow[dr, phantom, "\varphi"] & P_N^{\{0,t,1\}} \arrow[l, hook'] \arrow[r] \arrow[d, "ev_t"] & P_N^{\{0,1\}} \times P_N^{\{0,1\}} \\
M & N \arrow[l] &
\end{tikzcd}$$

split the path at t.

example: $M = S^n$ sphere

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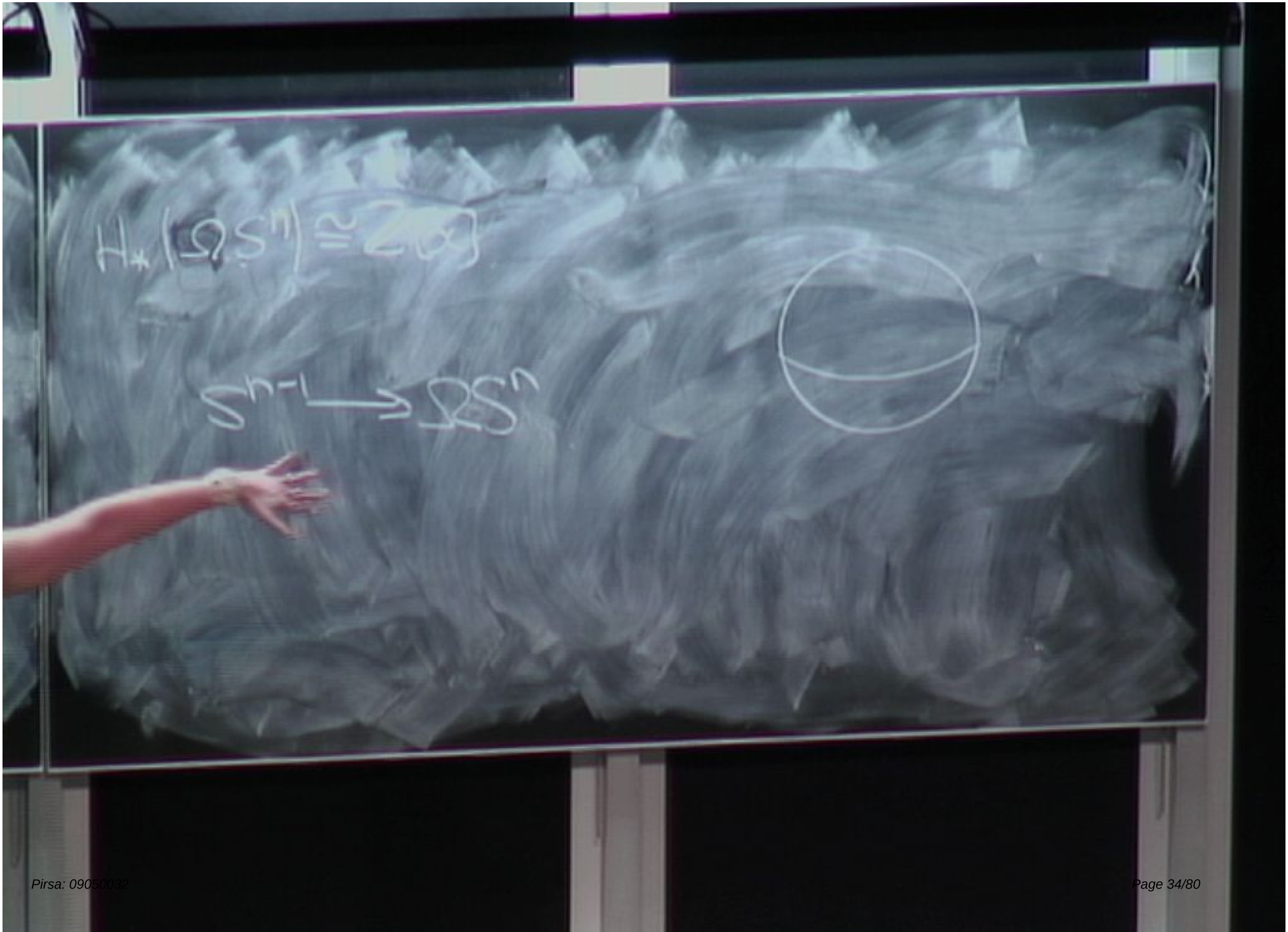
$N = pt$

$$P_N M = \{ \gamma : I \to S^n \quad \gamma(0) = \gamma(1) = pt \}$$

$$= \Omega_{pt} M$$



$H_*(\Omega S^n) \cong \mathbb{Z}[\alpha]$
$S^{n-1} \hookrightarrow \Omega S^n$

$$H_*(\Omega S^n) \cong \mathbb{Z}[a]$$

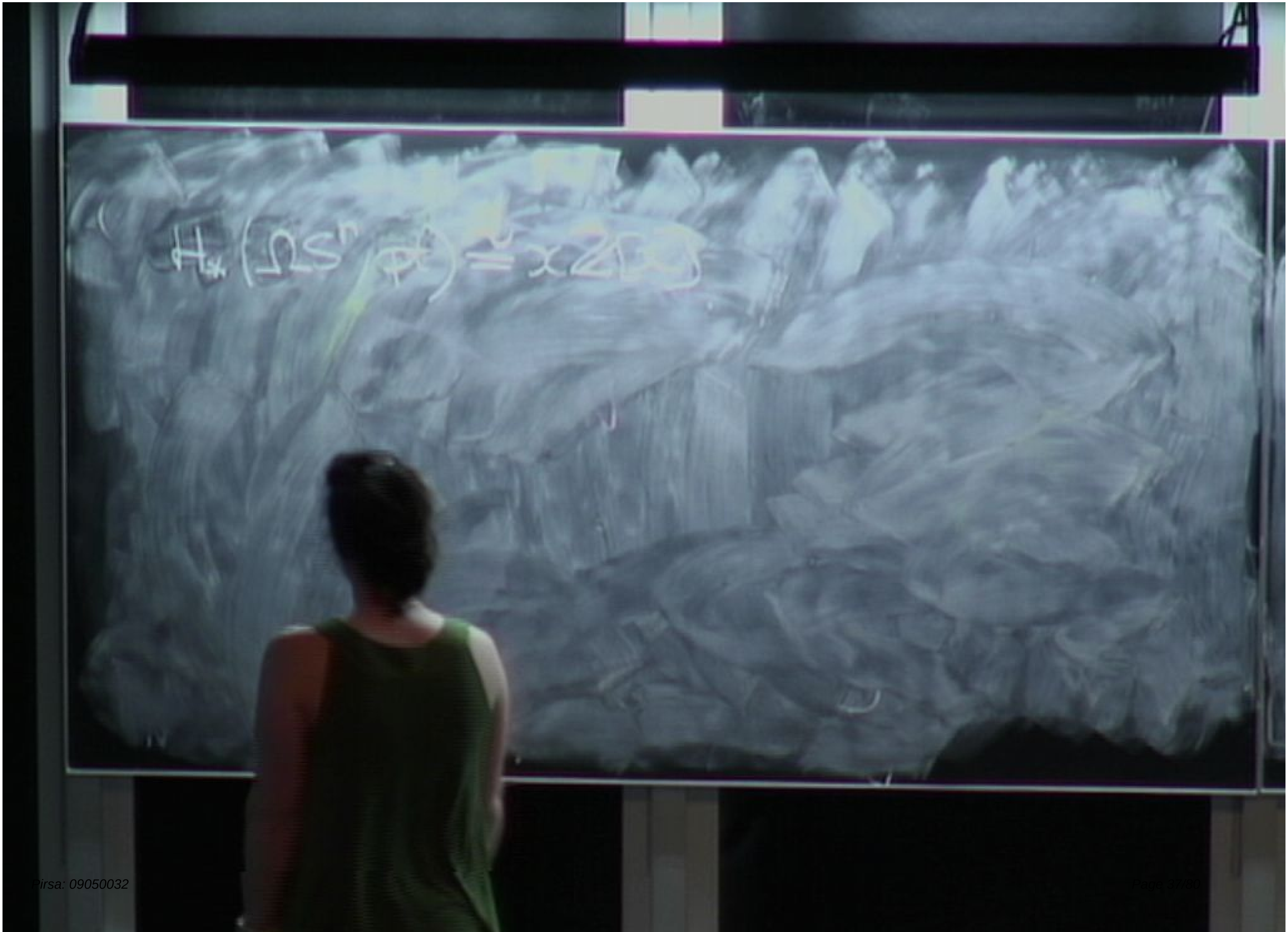
$$S^{n-1} \longrightarrow \Omega S^n$$

$H_*(\Omega S^n) \cong \mathbb{Z}[x]$

$\psi \quad S^{n-1} \longrightarrow \Omega S^n$

$z \longmapsto \gamma_z$

$x = \psi_*([S^{n-1}])$



$$H_*(\Omega S^n; \phi^t) \cong x\mathbb{Z}[\ldots]$$

$$m_2(x^k) = \sum_{i+j=k-1} x^i \otimes x^j$$

$$H_*(\Omega S^n, \phi) = x\mathbb{Z}[x]$$

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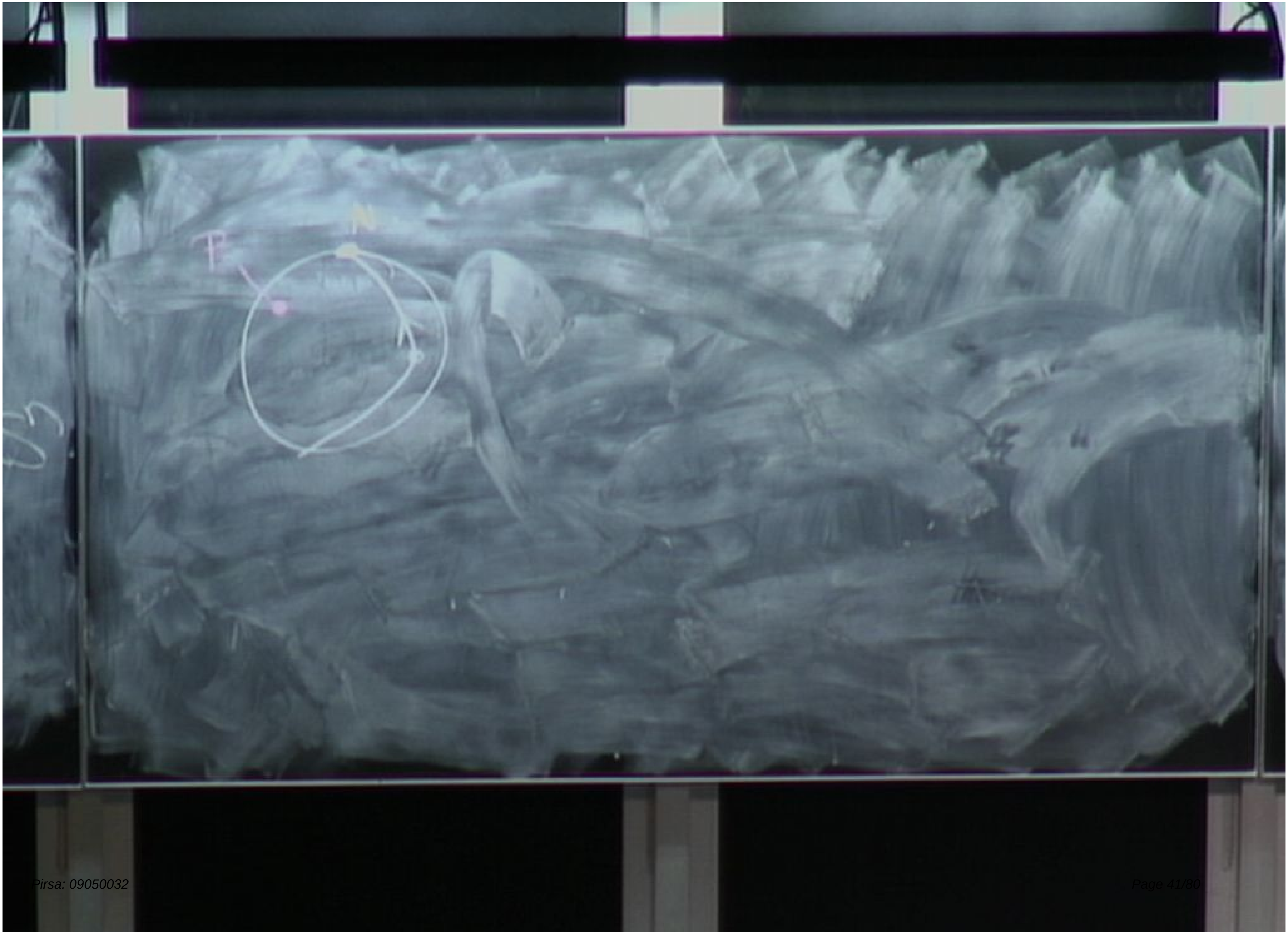
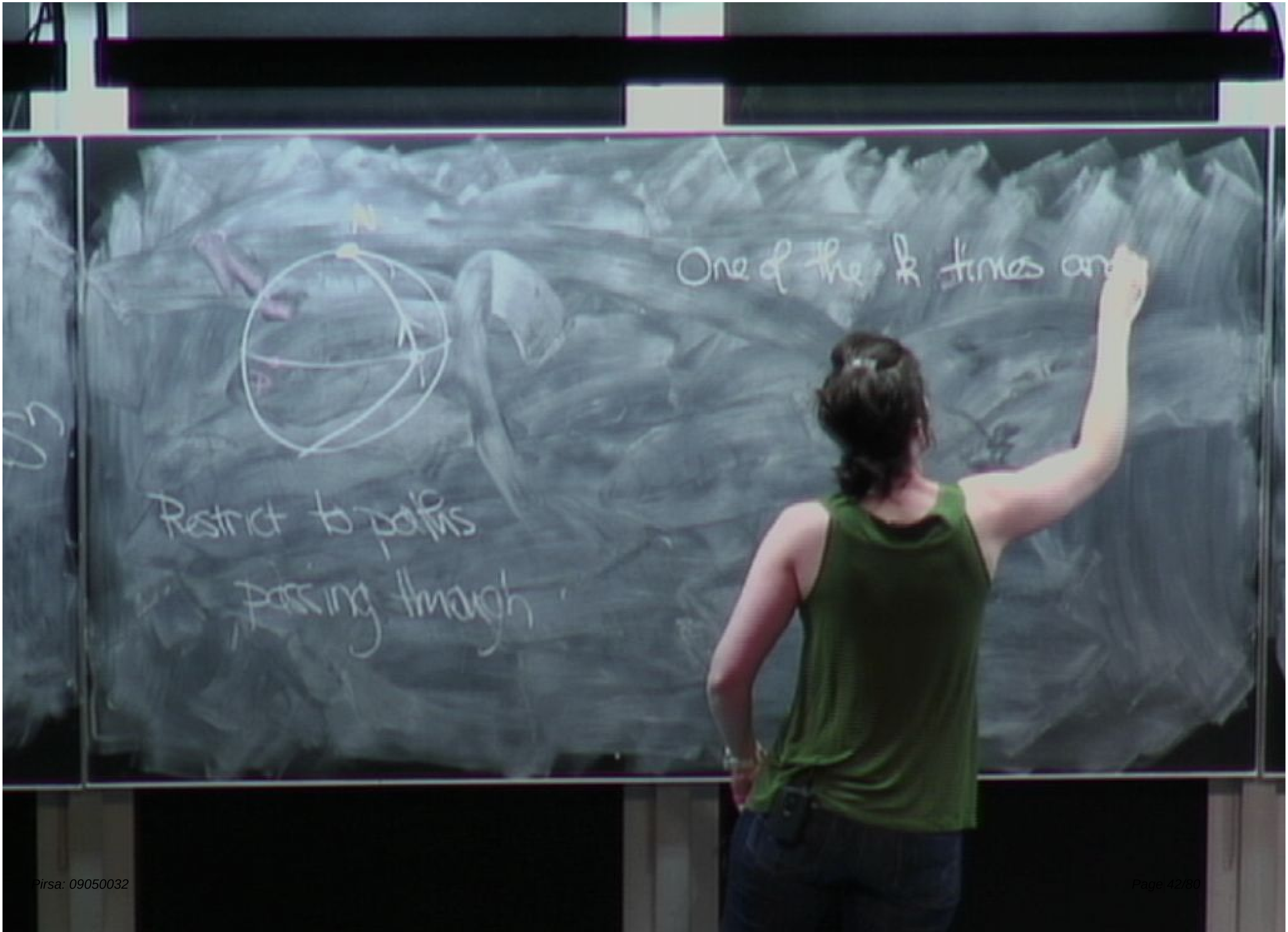
$$m_2(x^k) = \sum_{i+j=k-1} x^i \otimes x^j$$

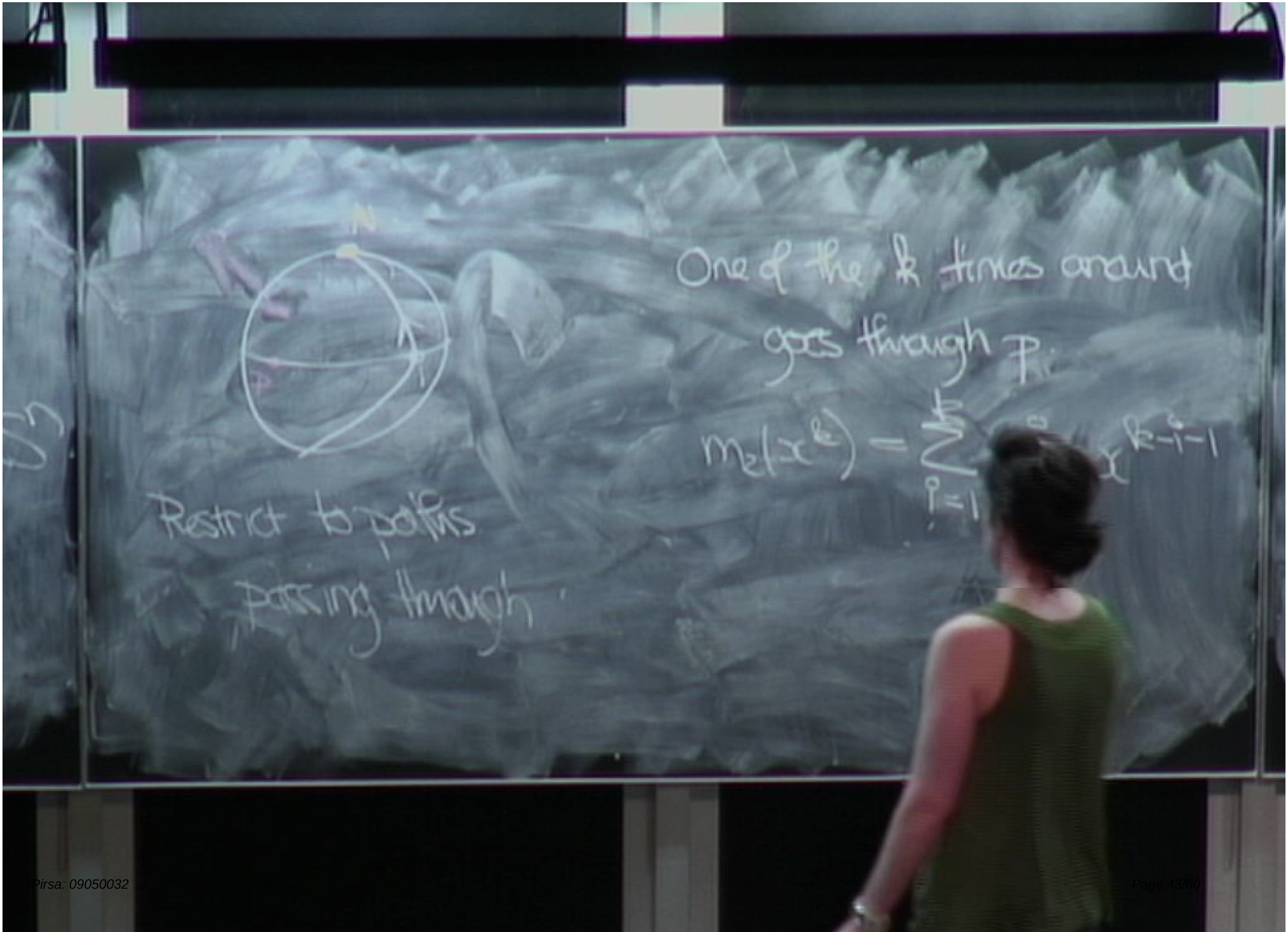
$\Phi: x^k$ is given by

by

$$(S^{n-1})^k \longrightarrow \Omega S^n$$




N
One of the k times are
Restrict to paths
passing through



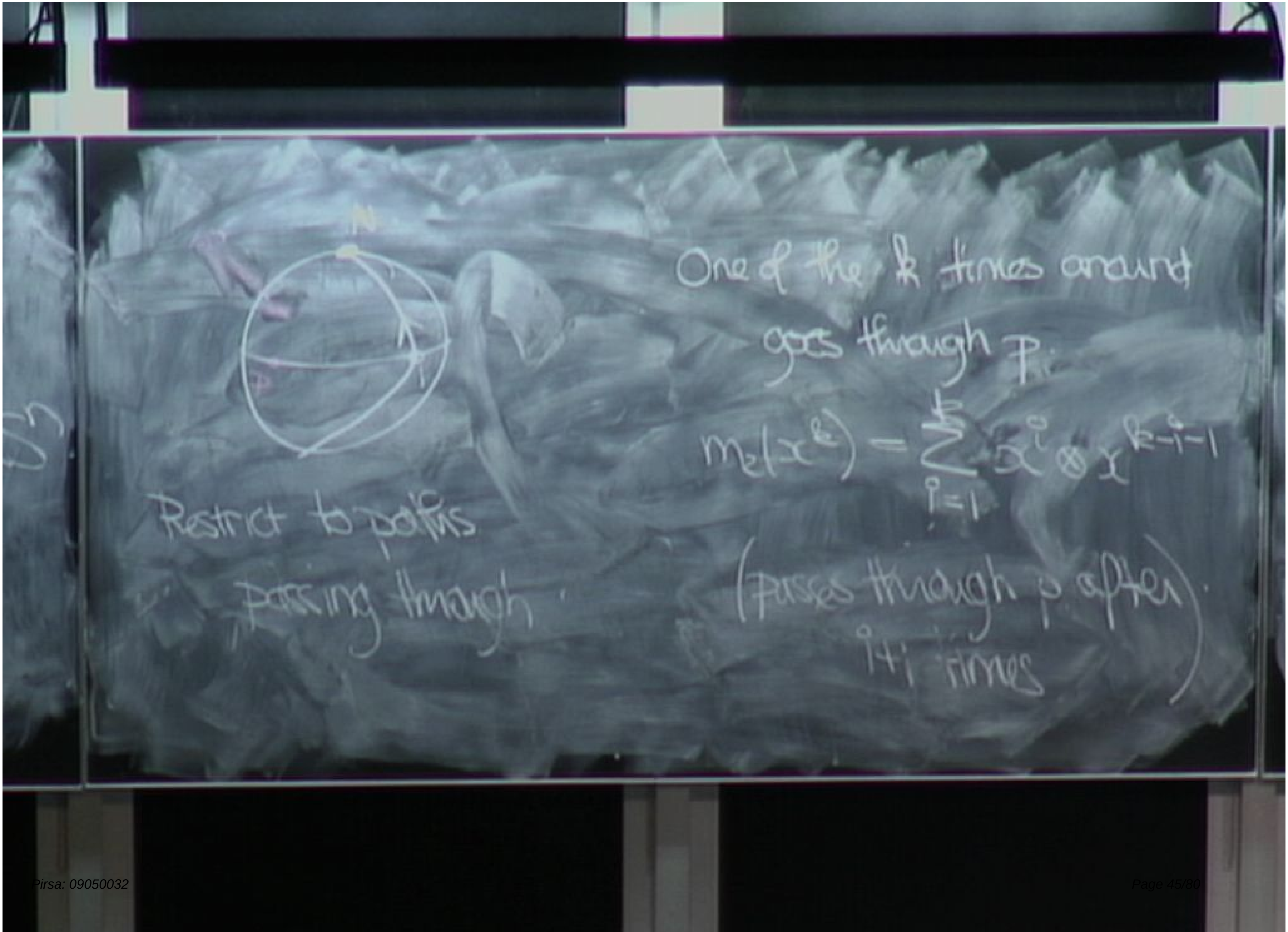
One of the k times around
goes through p_i
Restrict to paths
passing through

One of the k times around goes through p_i

$$m_2(x^k) = \sum_{i=1}^{k} \alpha^i \otimes x^{k-i-1}$$

(passes through p after $i+1$ times).

Restrict to paths passing through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goes through p_i
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(passes through p after
i+1 times)
Restrict to paths
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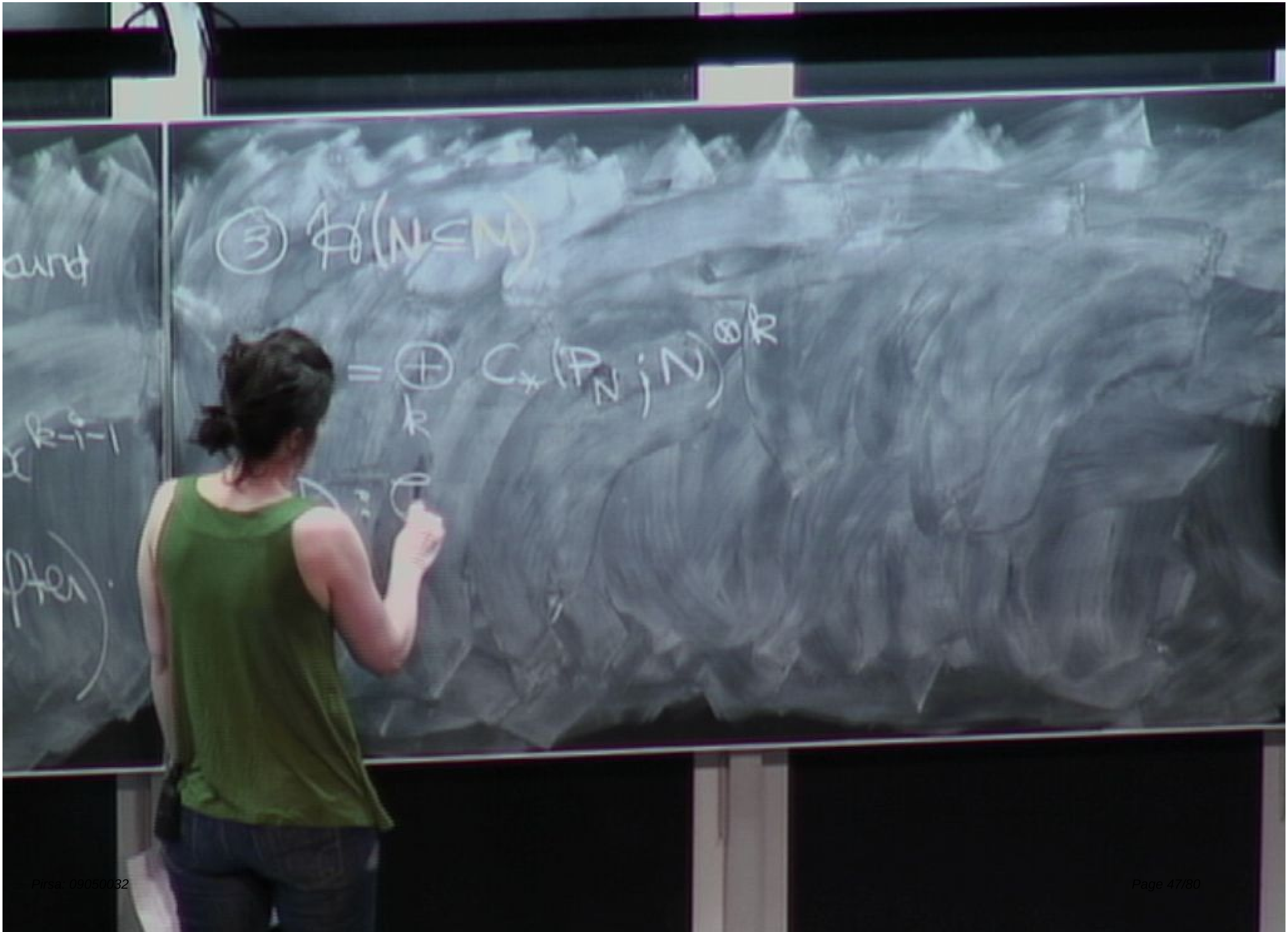
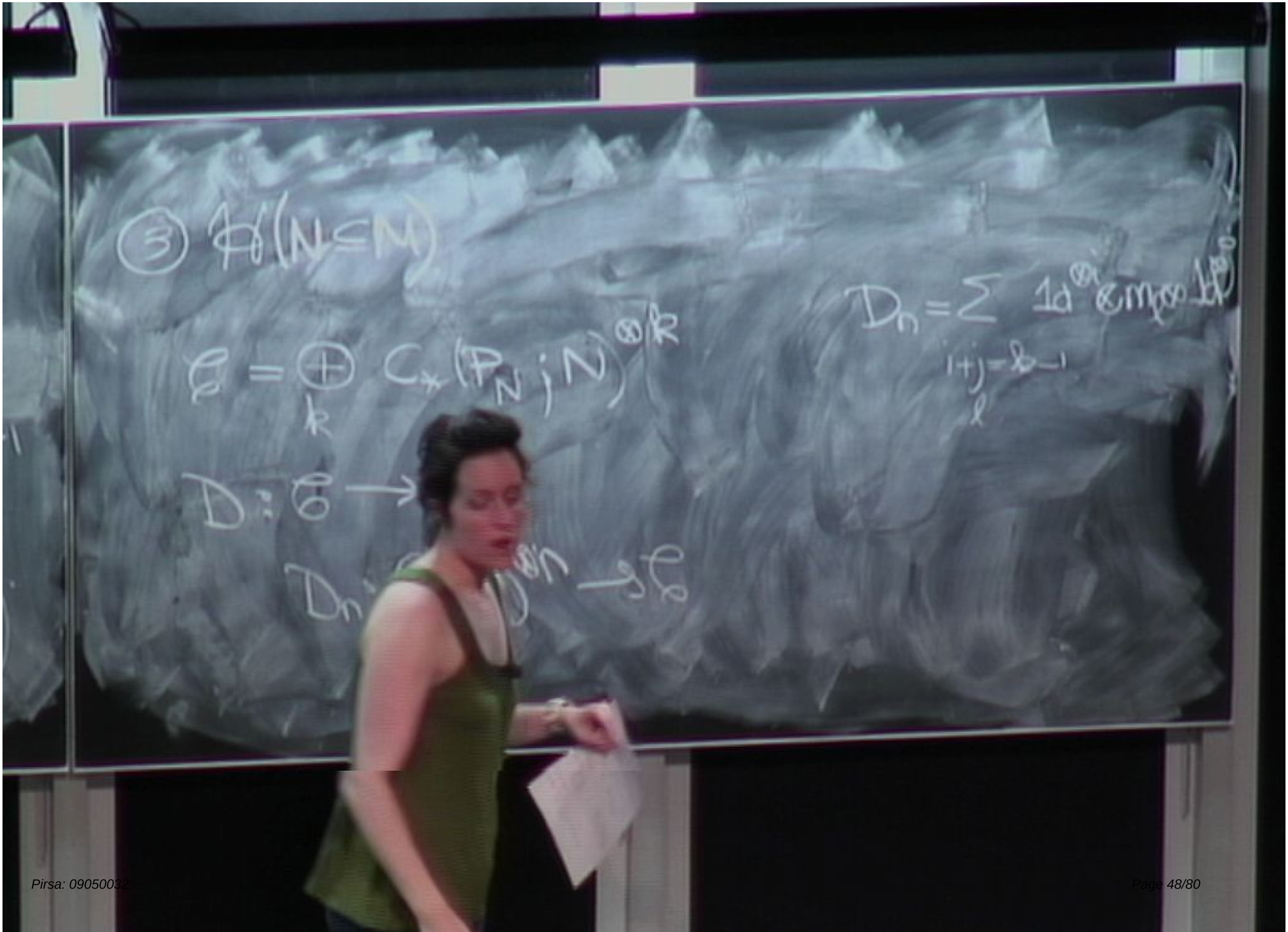
$$m_2(x^k) = \sum_{i=1}^{k} x^i \otimes x^{k-i-1}$$

(passes through p after $i+1$ times)


③ $\mathcal{H}(N \subseteq M)$
$= \bigoplus_k C_*(P_N; N)^{\otimes k}$


③ H(N⊂M)
$\mathcal{C} = \bigoplus_k C_*(P_N; N)^{\otimes k}$
$D : \mathcal{C} \rightarrow$
$D_n = \sum_{i+j=k-1} 1d^{\otimes i} \otimes m_\ell \otimes 1d^{\otimes j}$

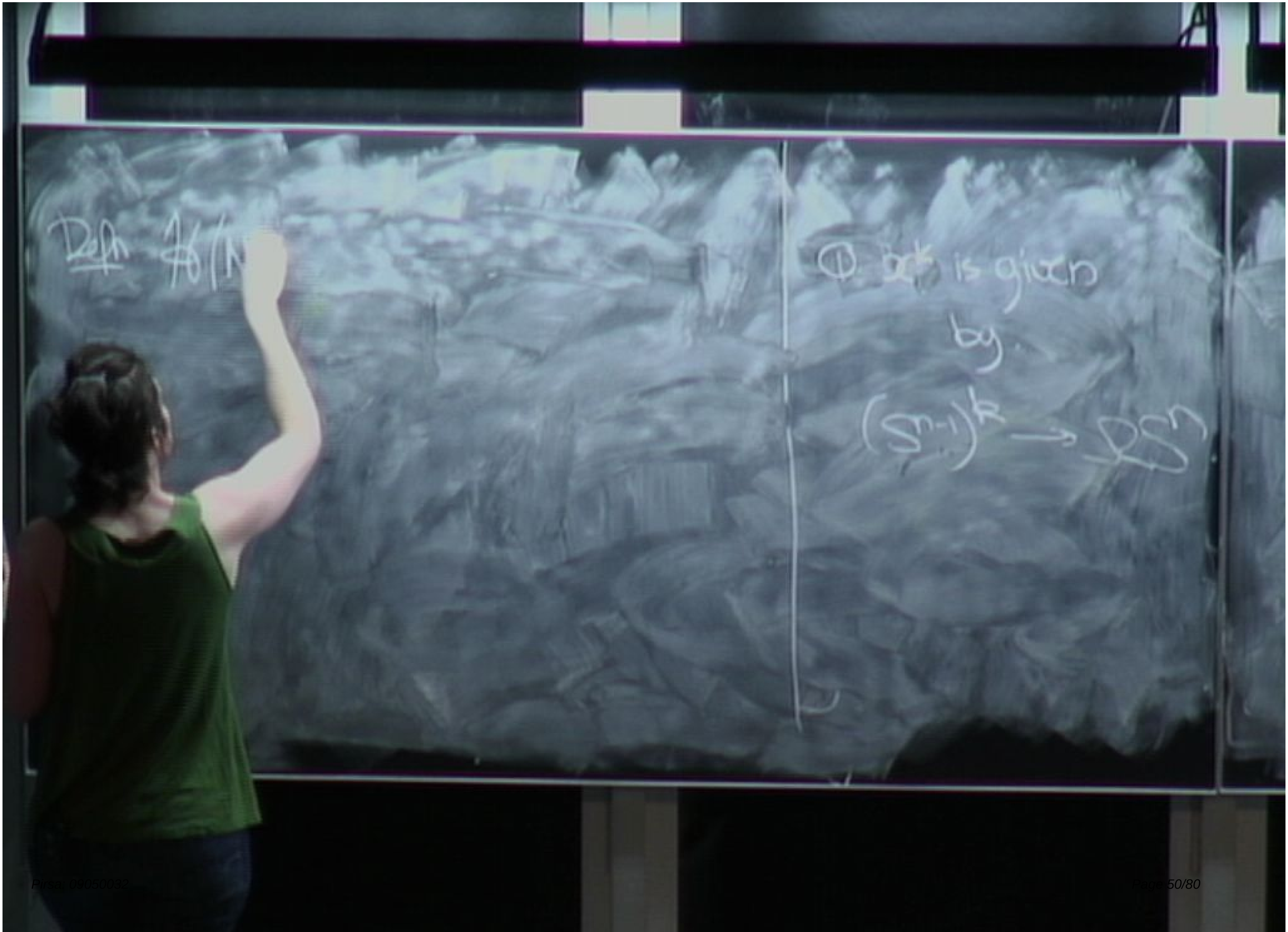
③ $\mathcal{H}(N \subseteq M)$

$$\mathcal{C} = \bigoplus_{k} C_*(P_N; N)^{\otimes k}$$

$$D : \mathcal{C} \longrightarrow \mathcal{C}$$

$$D_n : C_*(-;)^{\otimes n} \longrightarrow \mathcal{C}$$

$$D_n = \sum_{\substack{i+j=k-1 \\ \ell}} 1d^{\otimes i} \otimes m_\ell \otimes 1d^{\otimes j}$$


Defn
is given
by

<u>Defn</u> $\mathcal{H}(N \in M) = H_*(\mathcal{C}, D)$

Defn $\mathcal{H}(N \subseteq M) = H_*(\mathcal{C}, D)$

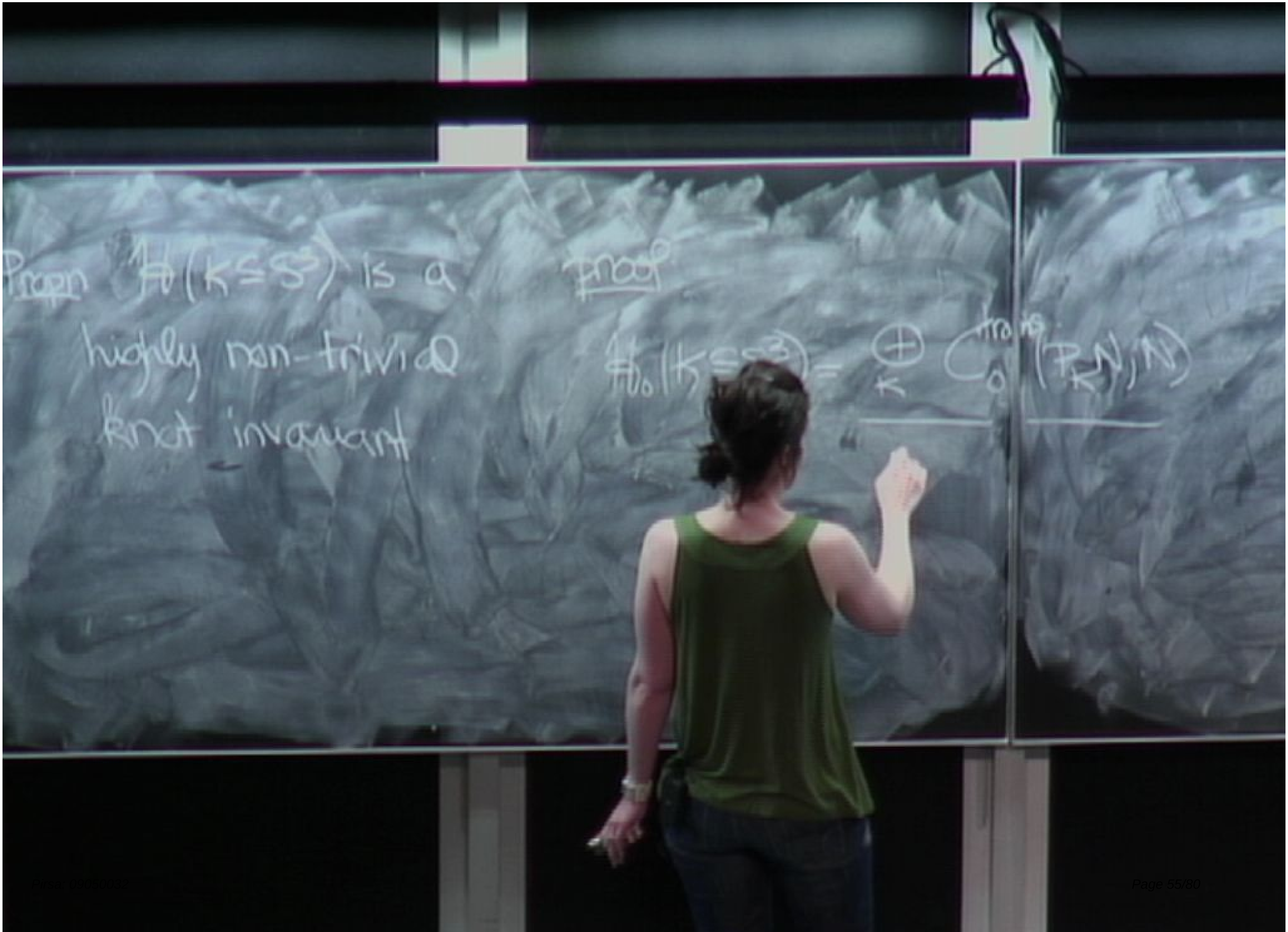
example $K \subseteq S^3$

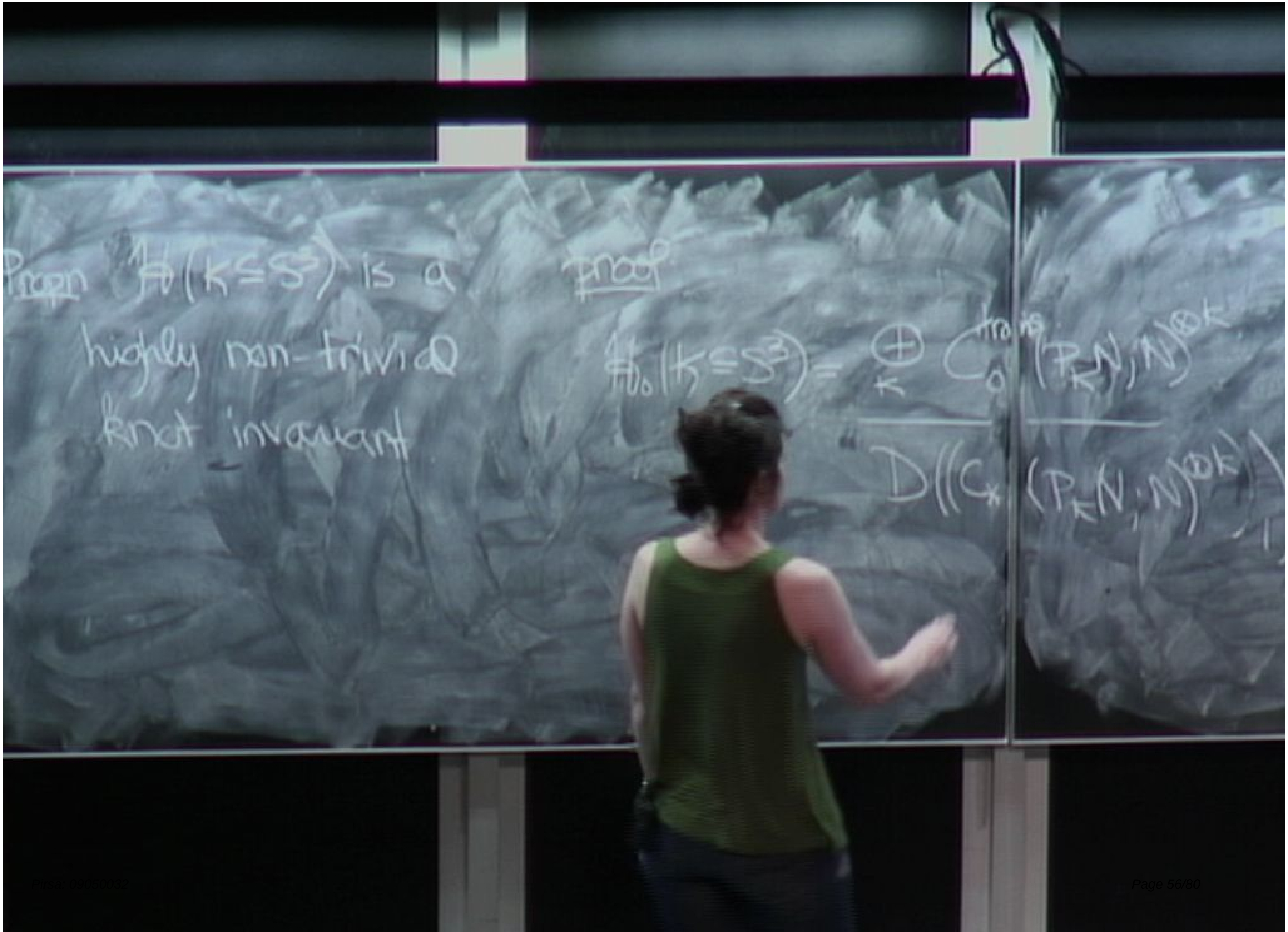
$$m_2 : C_p(P_K S^3; S^3) \longrightarrow \left(C_*(P_K S^3, S^3)^{\otimes 2}\right)_{p-1}$$

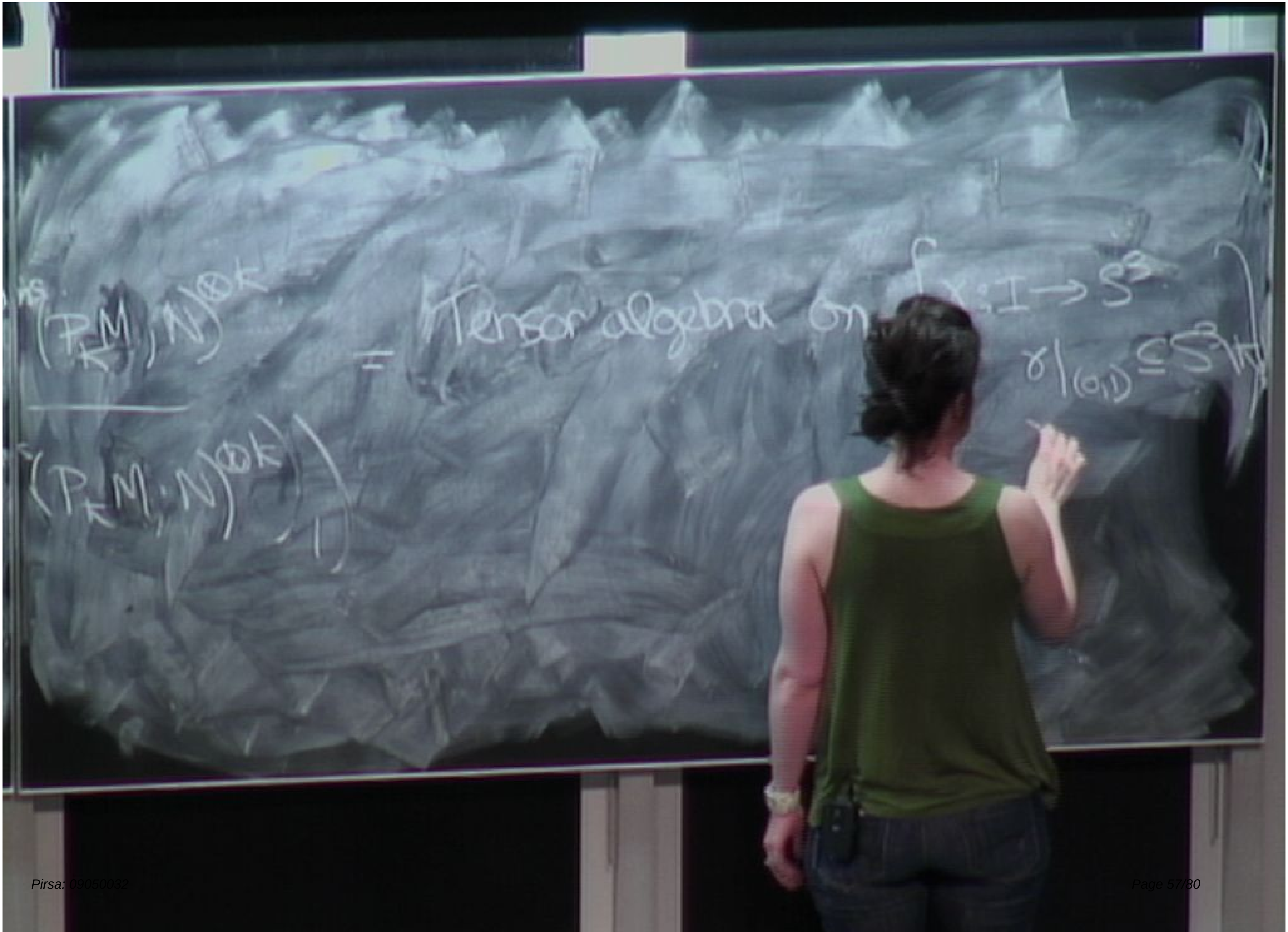
<u>Propn</u> $\mathcal{H}(K \subseteq S^3)$ is a highly non-trivial knot invariant

Propn $\mathcal{H}(K \subseteq S^3)$ is a highly non-trivial knot invariant

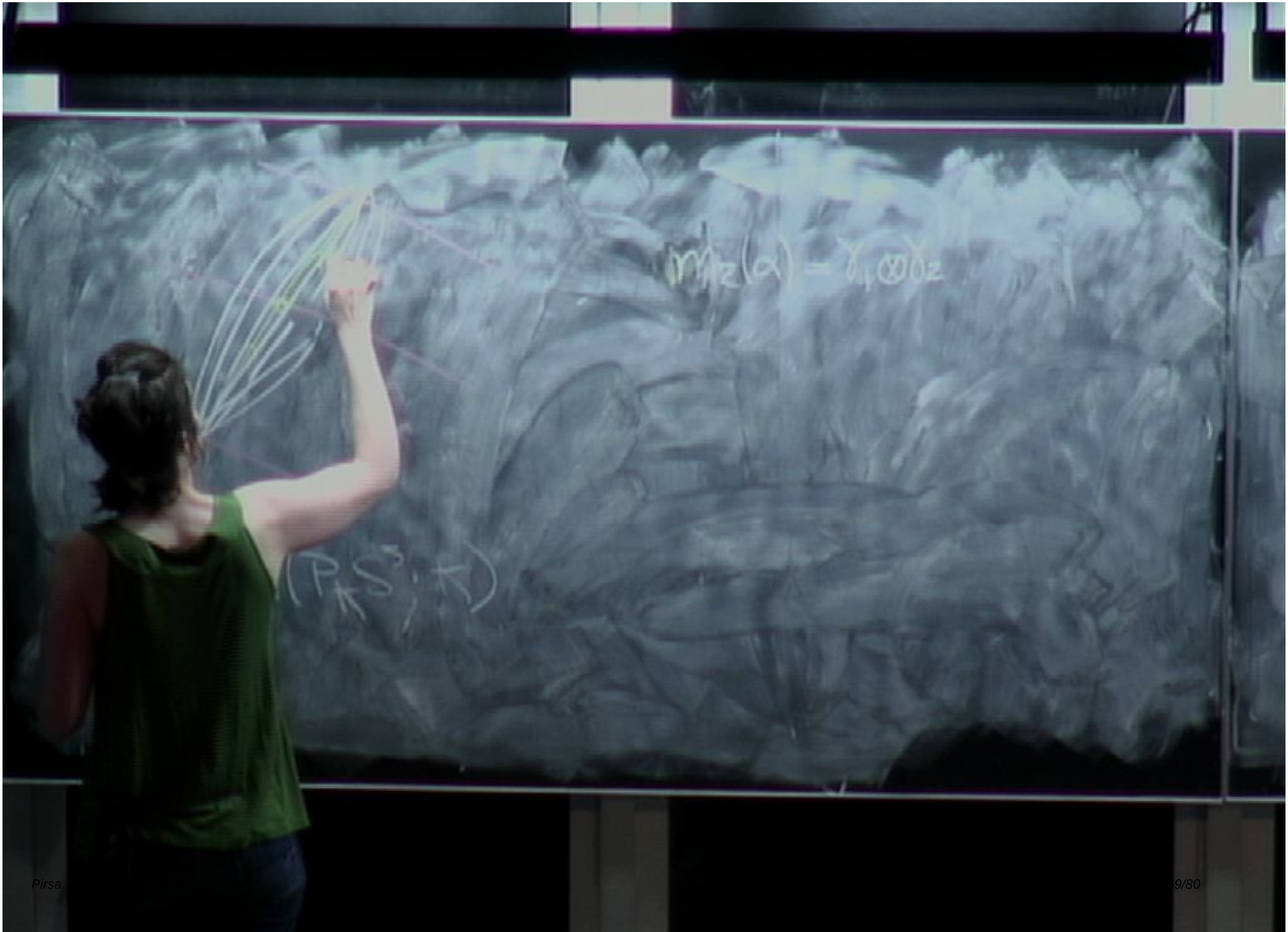
Proof

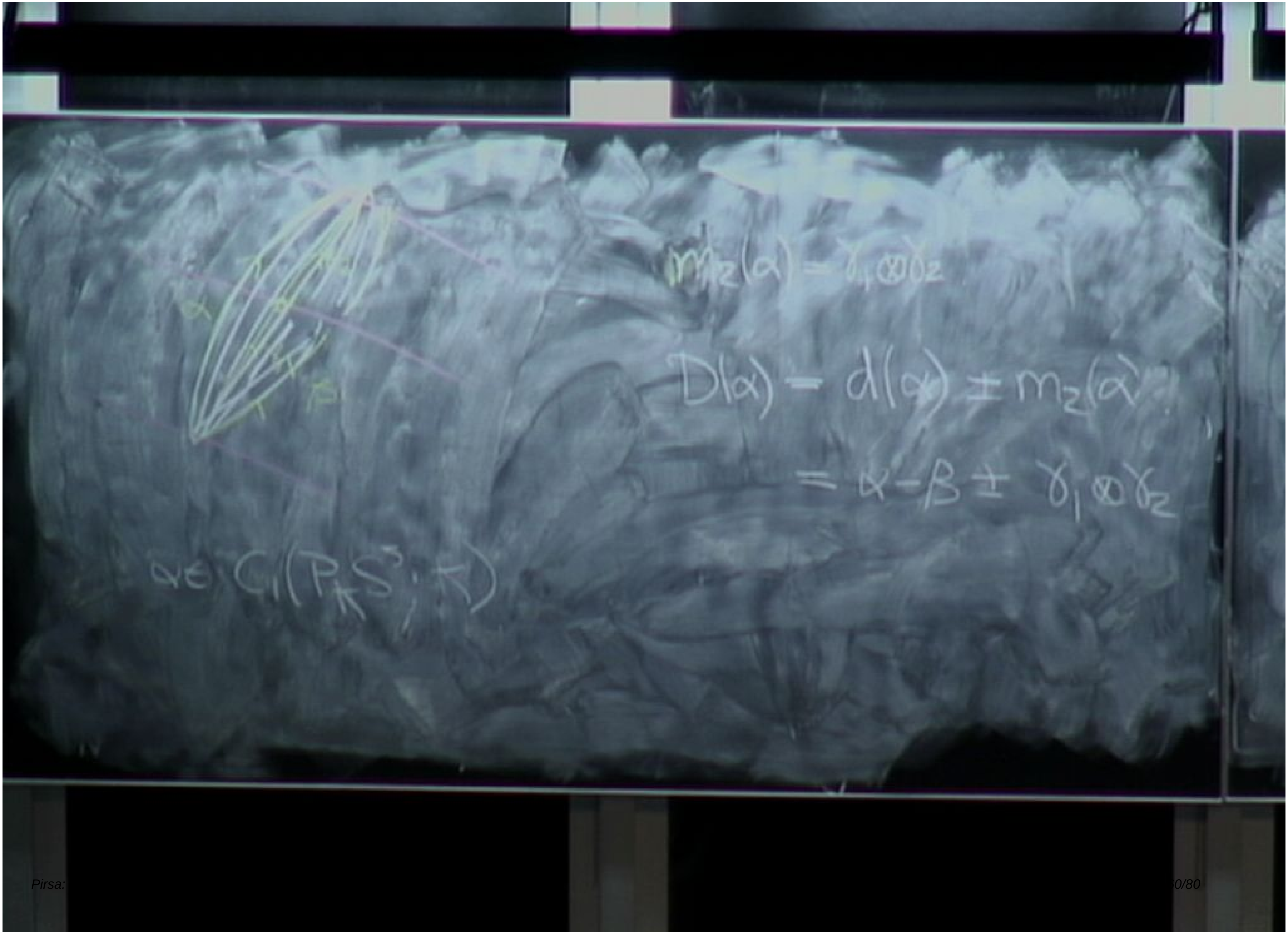

Propn $\mathcal{H}(K \subseteq S^3)$ is a
highly non-trivial
knot invariant
Proof
$\mathcal{H}_0(K \subseteq S^3) = \bigoplus_K C_0^{trans}(P_K N, N)$


Propn $\mathcal{H}_0(K \subseteq S^3)$ is a
highly non-trivial
knot invariant
proof
$\mathcal{H}_0(K \subseteq S^3) = \bigoplus_k C_0^{trans}(P_k N, N)^{\otimes k}$
$D((C_*(P_k N, N)^{\otimes k}))$


Tensor algebra on

$$\frac{(P_K M, N)^{\otimes k}}{(P_K M, N)^{\otimes k}_{1}} = \frac{\text{Tensor algebra on } \left\{ \begin{array}{l} \gamma : I \to S^3 \\ \gamma|_{(0,1)} \subseteq S^3 \setminus K \\ \gamma(0), \gamma(1) \in K \end{array} \right\}}{\text{Relation}}$$

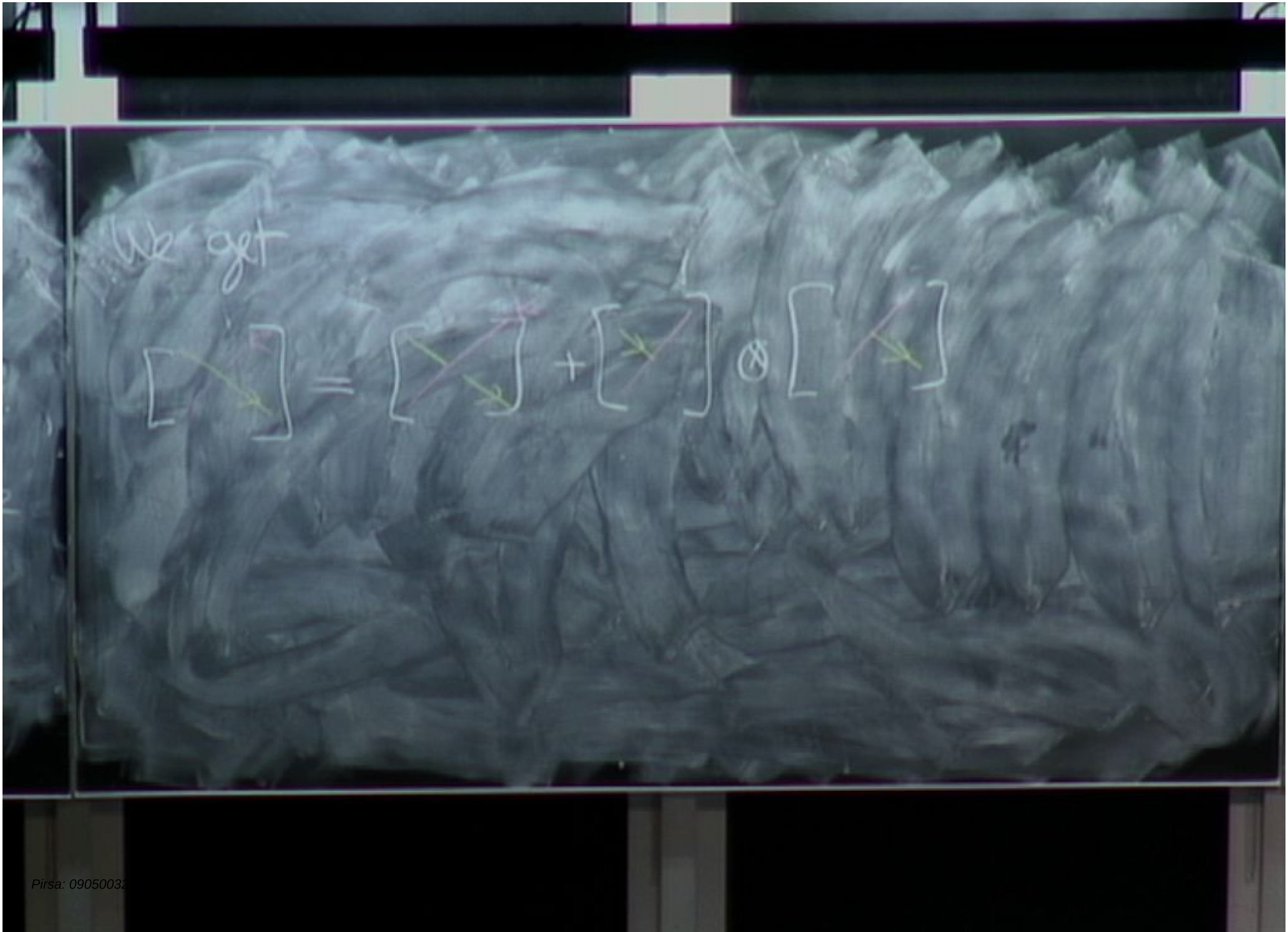



$m_2(\alpha) = \gamma_1 \otimes \gamma_2$
$D(\alpha) = d(\alpha) \pm m_2(\alpha)$
$= \alpha - \beta \pm \gamma_1 \otimes \gamma_2$
$\alpha \in C_1(P_K S^3; K)$

We get

Proof

$$H_0(K \subseteq S^3) = \frac{\bigoplus_k C_0^{\text{tra}\ldots}}{D((C_*^{\text{tra}\ldots}}$$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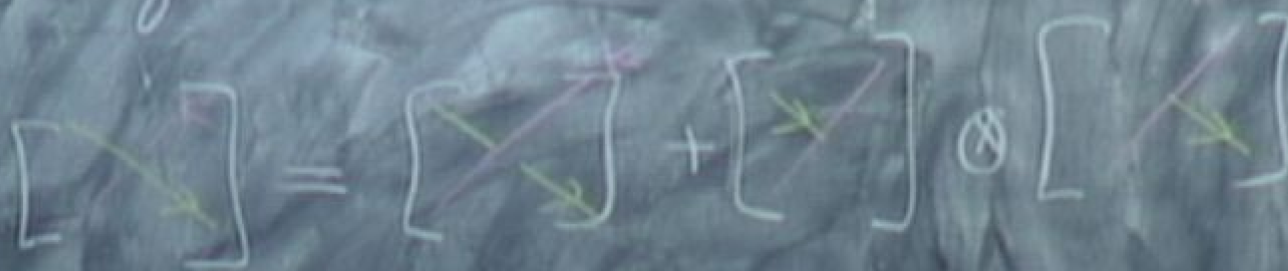
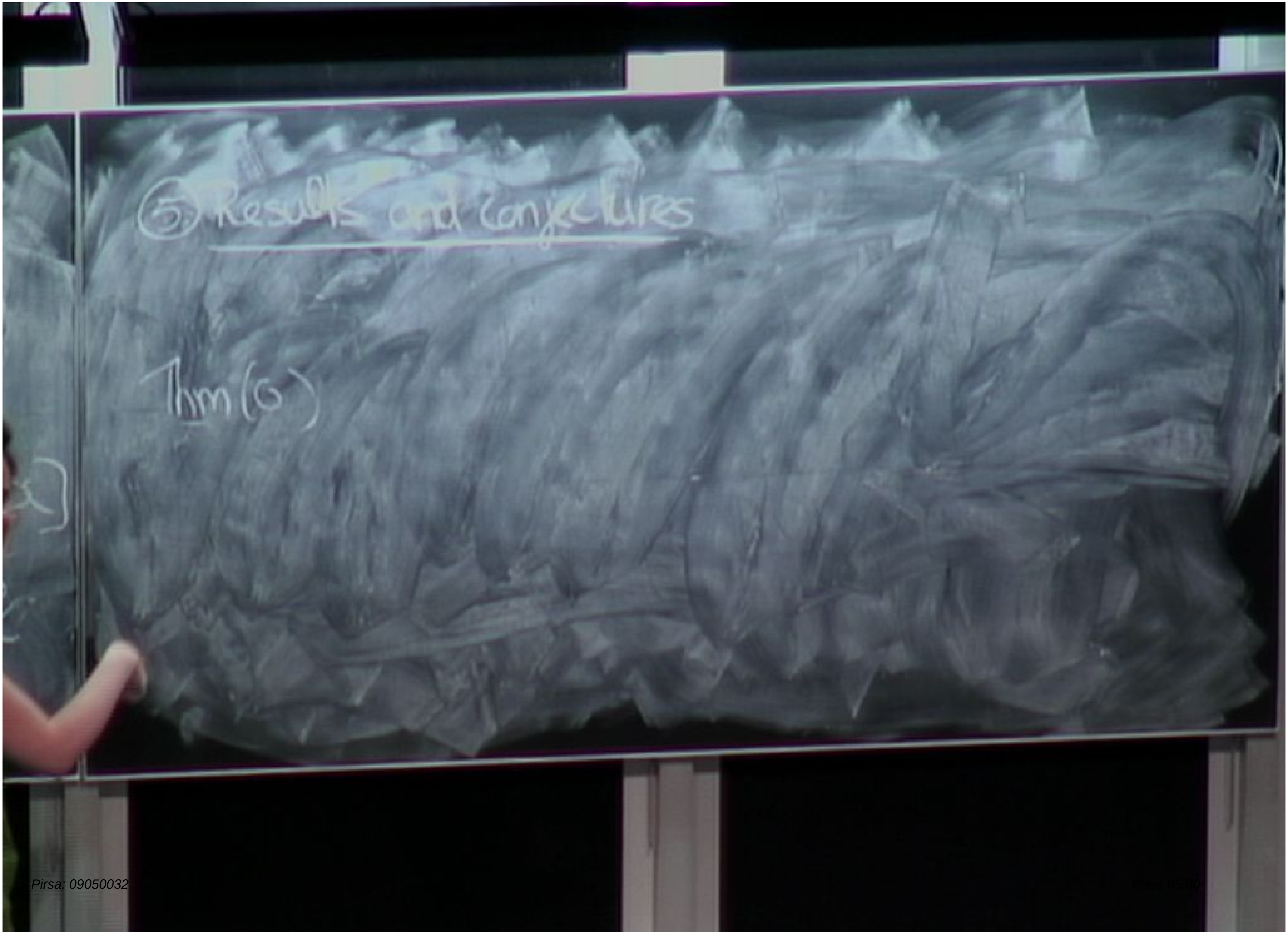


<u>example</u>: K = trefoil knot, $\mathcal{H}_0(K \subseteq S^3) \cong \mathbb{Z}[x]$

K = unknot $\qquad \mathcal{H}_0(K \subseteq S^3) \cong \mathbb{Z}$

5 Results and conjectures


5) Results and conjectures
Thm (G)

5 Results and conjectures

Thm (G) Rationnally, there is a similar structure which only sees the homotopy type

Conjecture

1. It is the one introduced today
2. The same thing can be done over $\mathbb{Z}$

$$m_2(\alpha) = \gamma_1 \otimes \gamma_2$$

$$D(\alpha) = d(\alpha) \pm m_2(\alpha)$$

$$= \alpha - \beta \pm \gamma_1 \otimes \gamma_2$$

Conjecture

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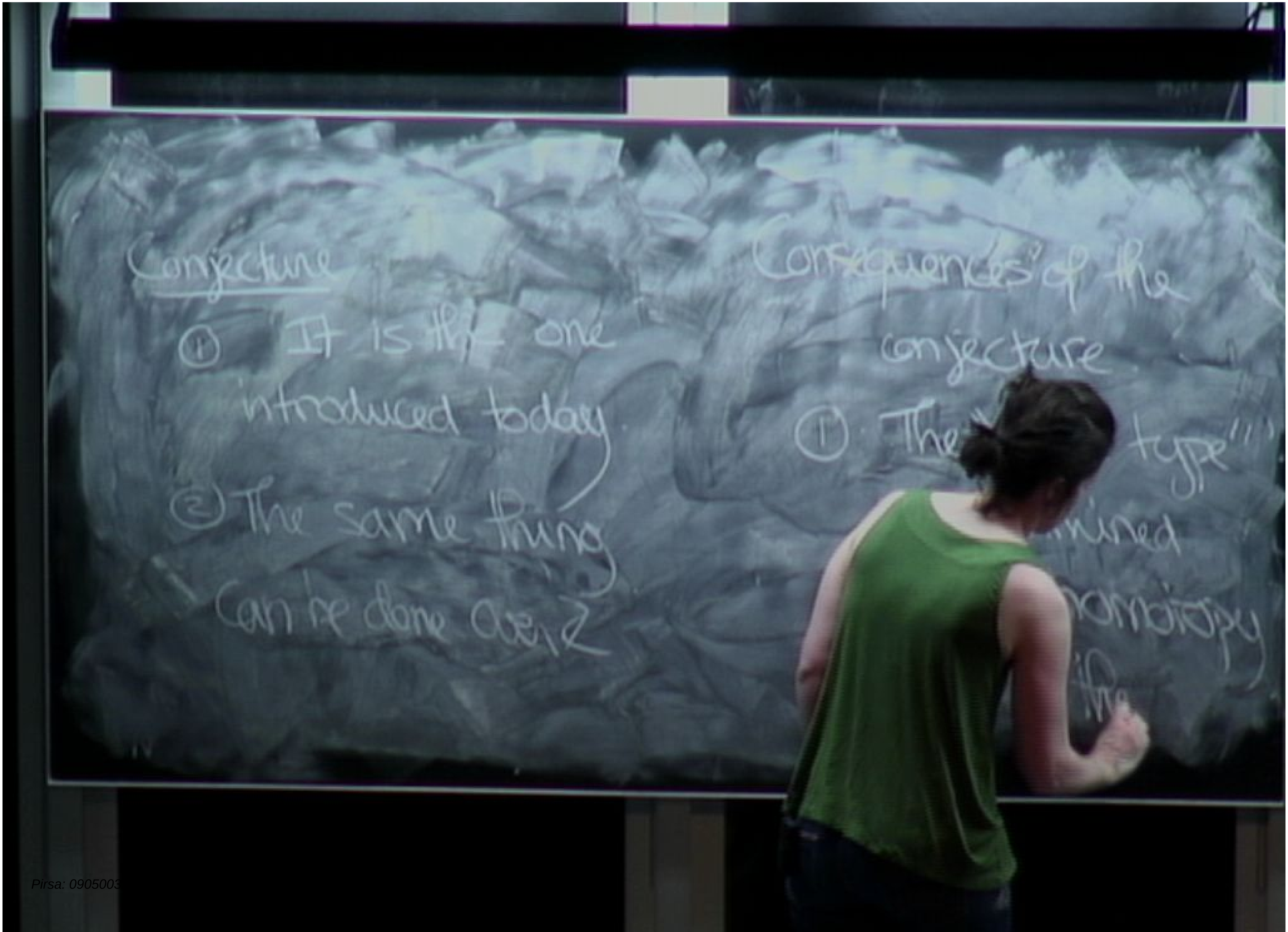

Conjecture
① It is the one
introduced today.
② The same thing
can be done over Z
Consequences of the
conjecture
① The
type"

Conjecture

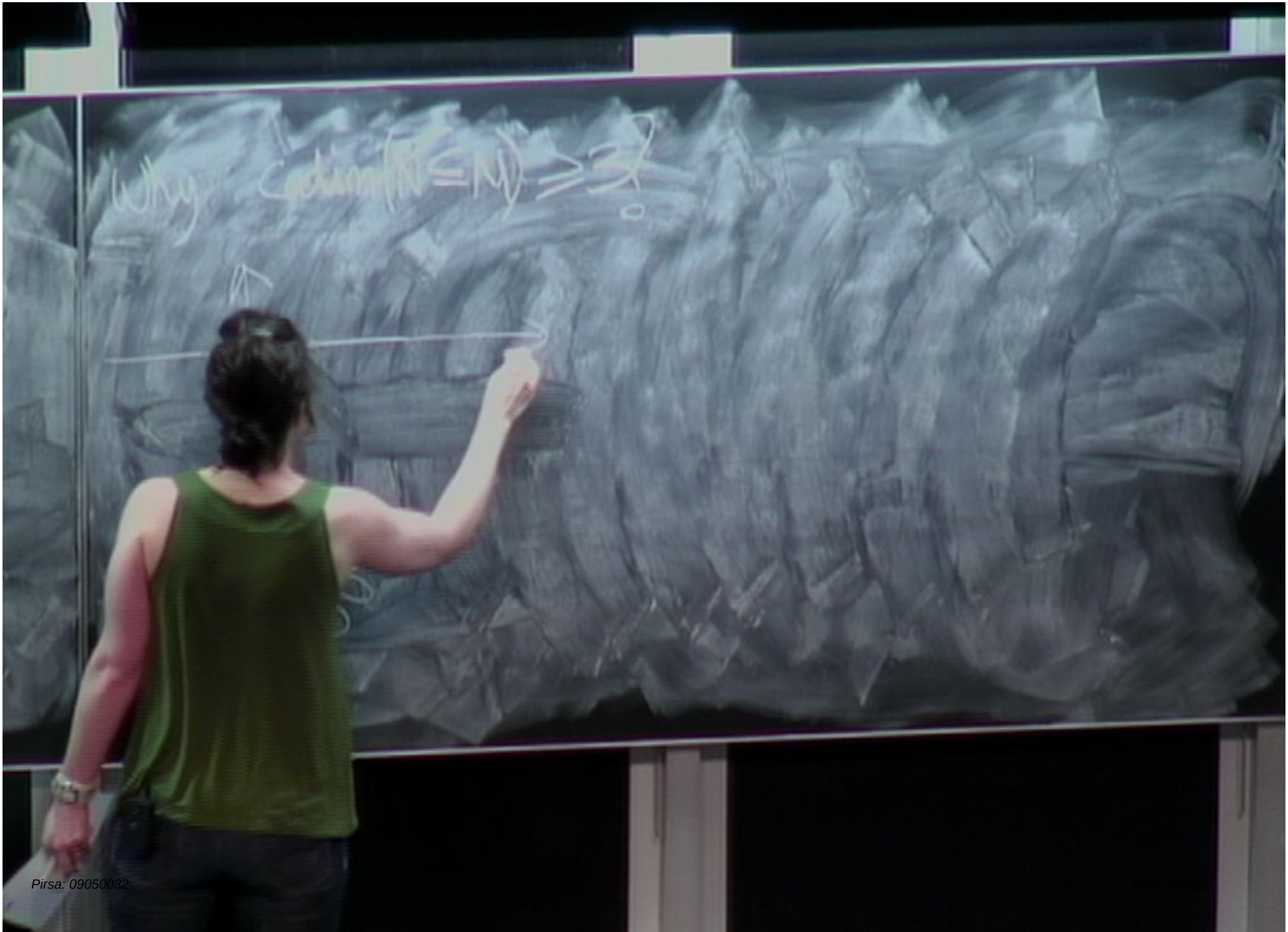
1. It is the one introduced today
2. The same thing can be done over $\mathbb{Z}$

Consequences of the conjecture.

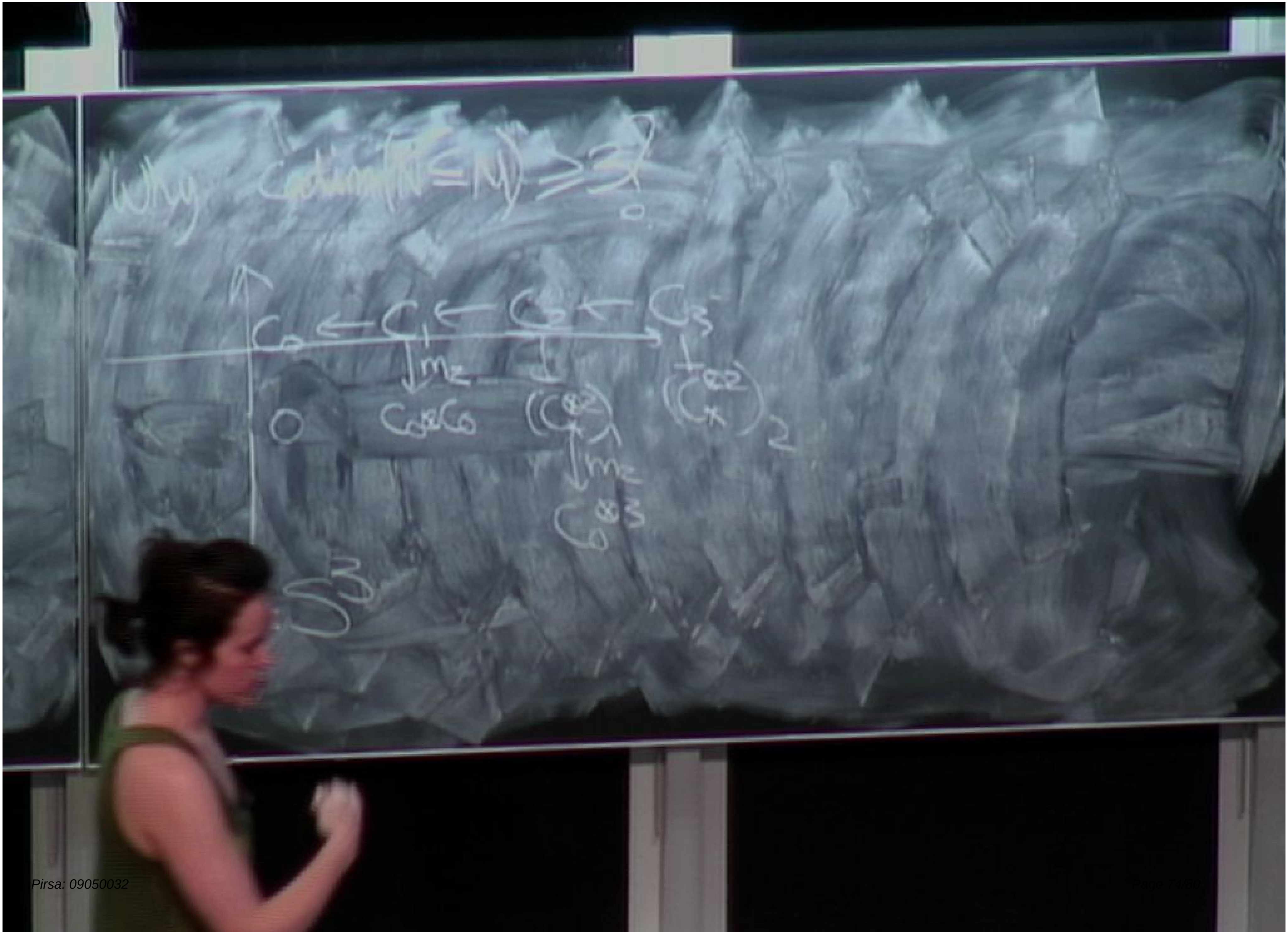
1. The "weak type" is determined by the homotopy type of the manifold

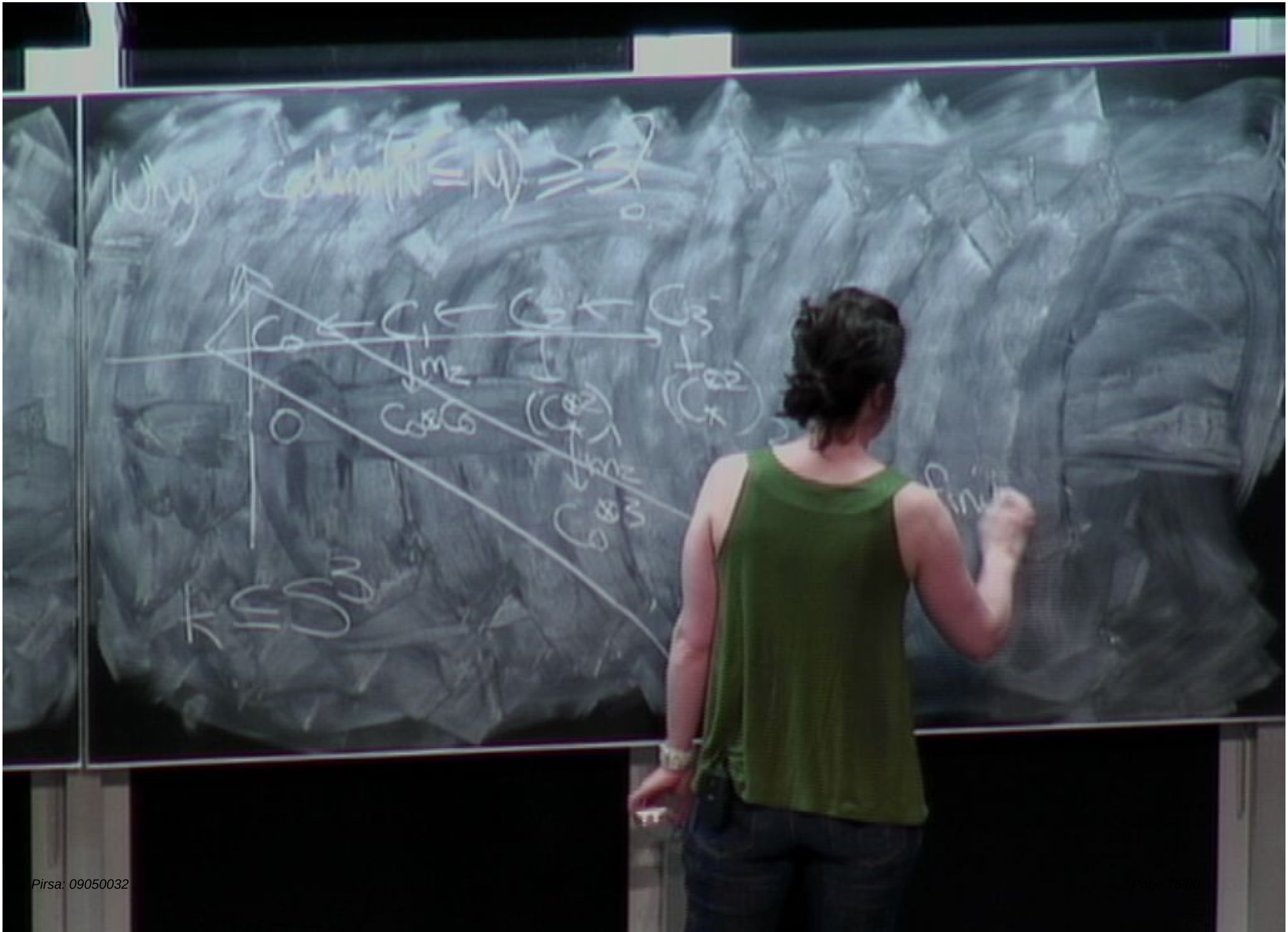
(2) If $\mathrm{codim}(N \subseteq M) \geq 3$, then $\mathcal{F}(N \subseteq M)$ is an invariant of the homotopy type of (N, M)


Why codim(Π ≤ N) ≥ 3?



Why codim$(\tilde{N} \subseteq M) \geq 3$?




Why
$C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow C_3$
$C_0 \otimes C_0$
$C_0^{\otimes 3}$
$K \subseteq S^3$

Why $\operatorname{Codim}(\tilde{K} \subseteq M) \geq 3$?

$C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow C_3$

$\downarrow m_2$

$C_0 \otimes C_0$ $\quad (C_*^{\otimes 2})_1$ $\quad (C_*^{\otimes 2})_2$

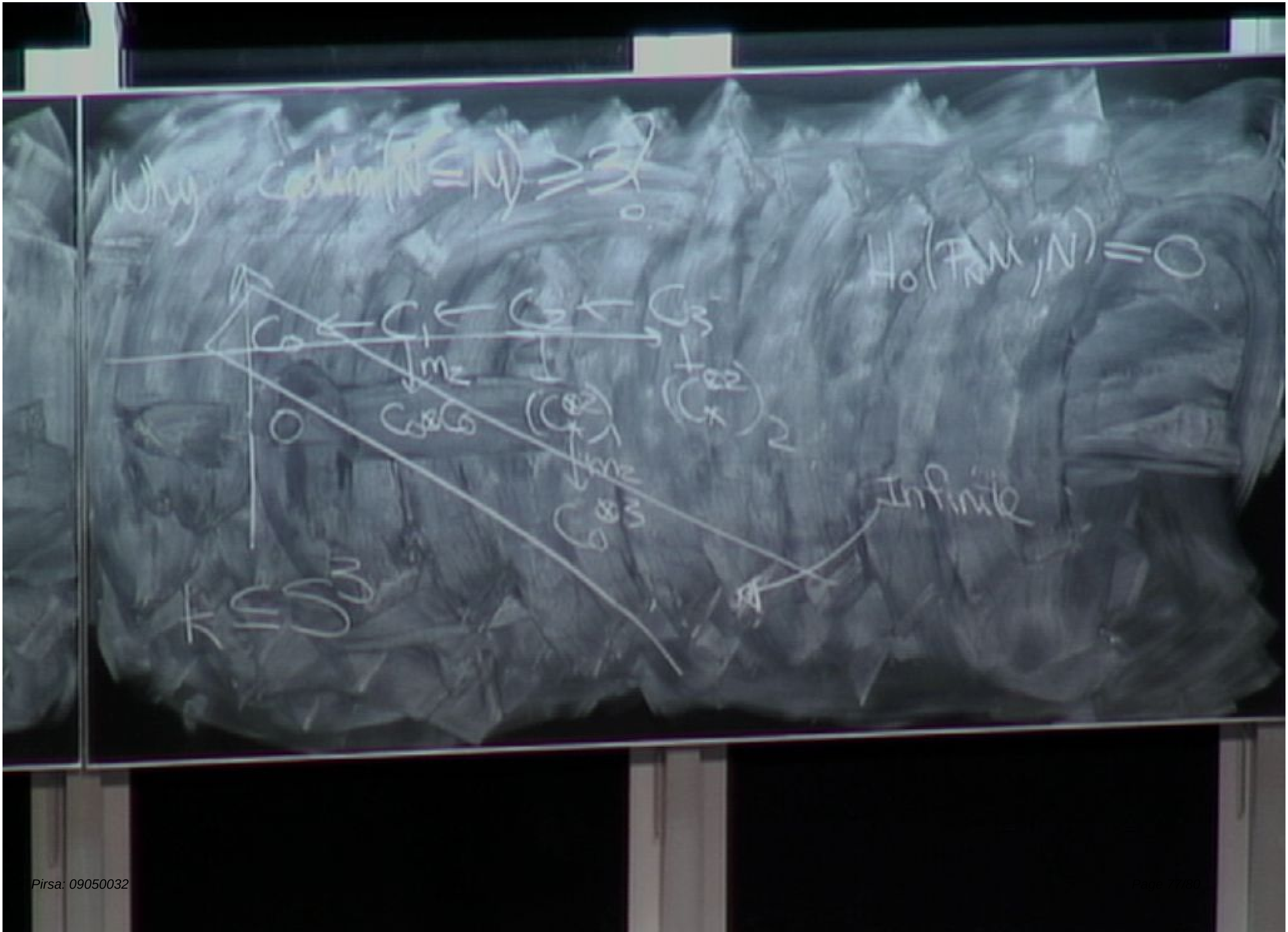
$\downarrow m_2$

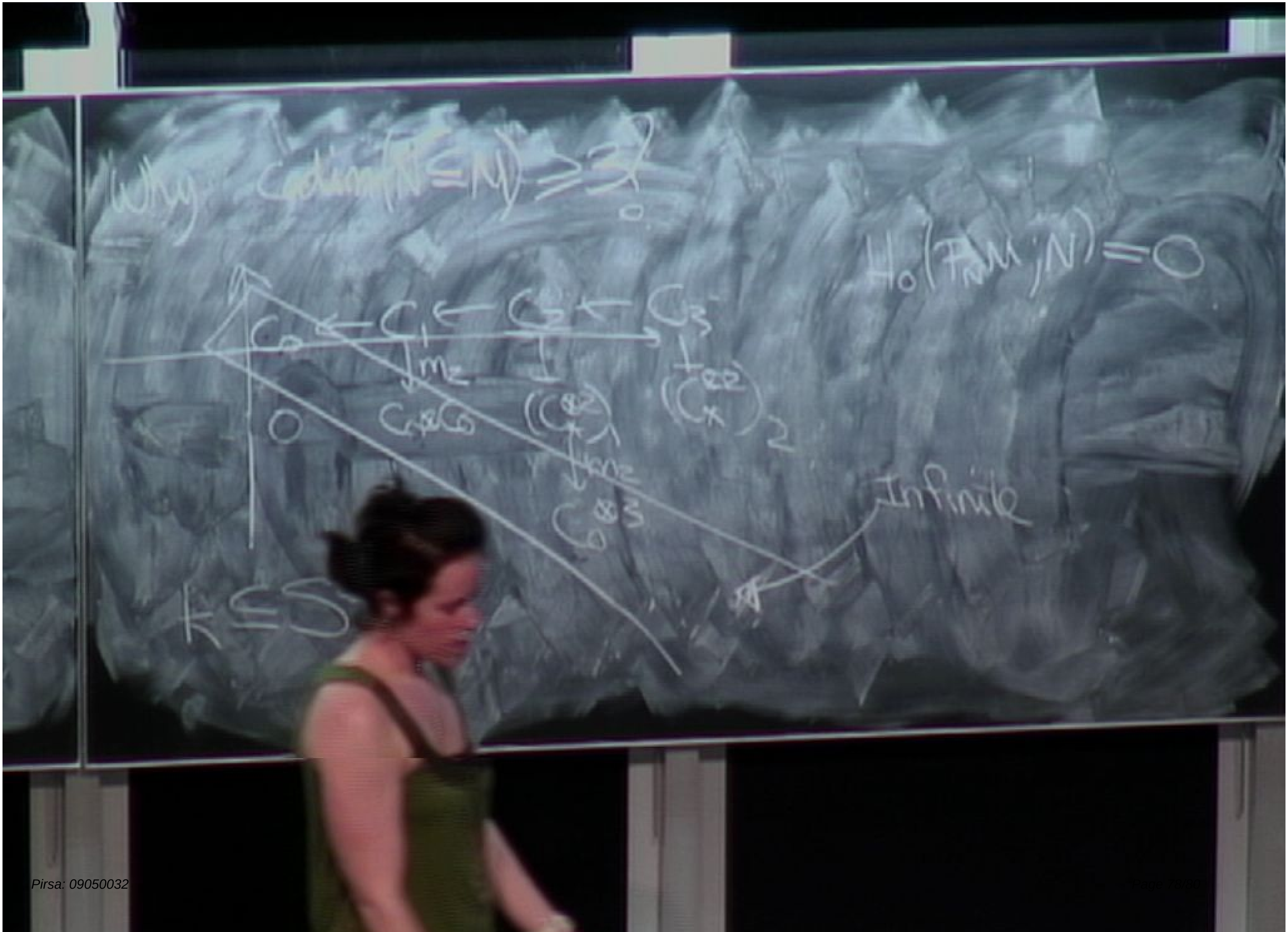
$C_0^{\otimes 3}$

0

Infinite

$K \subseteq S^3$


Why codim(N ⊆ M) ≥ 3?
H₀(P_N M; N) = 0
C₀ ← C₁ ← C₂ ← C₃
m₂
C₀⊗C₀
C₀^⊗3
Infinite
K ⊆ S³


Why
H0(P_N M ; N) = 0
Infinite

For codim

$$
\begin{array}{ccccccccc}
C_0 & \leftarrow & C_1 & \leftarrow & C_2 & & & & \\
 & & & & \downarrow & & & & \\
0 & & 0 & & C_0 \otimes C_0 & \leftarrow & (C_*^{\otimes 2})_1 & \leftarrow & (C_*^{\otimes 2})_2 \\
 & & & & & & & & \downarrow \\
0 & & 0 & & 0 & & 0 & & (C^{\otimes 3})_0
\end{array}
$$

For codim

$$
\begin{array}{ccccc}
C_0 \leftarrow C_1 \leftarrow C_2 & & & & \\
0 & 0 & C_0 \otimes C_0 \leftarrow (C_*^{\otimes 2})_1 & \leftarrow & (C_*^{\otimes 2})_2 \\
0 & 0 & 0 & 0 & (C_0^{\otimes 3})
\end{array}
$$

Spectral sequence converges