Title: Extremal Kahler metrics on projective bundles over a curve

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Abstract: I will discuss the existence problem of extremal Kahler metrics (in the sense of Calabi) on the total space of a holomorphic projective bundle P(E) over a compact complex curve. The problem is not solved in full generality even in the case of a projective plane bundle over CP^1. However, I will show that sufficiently ``small'' Kahler classes admit extremal Kahler metrics if and only if the underlying vector bundle E can be decomposed as a sum of stable factors. This result can be viewed as a ``Hitchin-Kobayashi correspondence'' for projective bundles over a curve, but in the context of the search for extremal Kahler metrics. The talk will be based on a recent work with D. Calderbak, P. Gauduchon and C. Tonnesen-Friedman.

Stability and extremal Kähler metrics on projective bundles over a curve

Vestislav Apostolov[1]

May, 2009

[1] Joint work with D. Calderbank, P. Gauduchon and C. Tønnesen-Friedman

Kobayashi–Hitchin correspondence for vector bundles I

Definition (Stability)

$\pi : E \longrightarrow S$ is a holomorphic vector bundle of rank r over a compact Kähler manifold (S, J_S, ω_S) of complex dimension n:

- *slope* of E: $\mu(E) := \frac{1}{r}\int_S c_1(E) \wedge \omega_S^{n-1}$

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- 'stable' $\Rightarrow$ 'polystable' $\Rightarrow$ 'semistable'

Kobayashi–Hitchin correspondence for vector bundles II

Definition (Hermitian-Einstein metric)

A hermitian metric h on $\pi : E \to S$ is *Hermitian–Einstein* if the canonical Chern connection ∇^h has curvature $R^h(\omega_S) = c\,\mathrm{Id}_E$, where c is a constant (equal to $2\pi\mu(E)/\int_S \omega_S^n$).

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A central result in the 80's:

Theorem (Donaldson, Uhlenbeck–Yau, . . . , Lübke–Teleman)

E admits an Hermitian–Einstein metric $\Longleftrightarrow$ E is polystable.

Kobayashi–Hitchin correspondence for projective varieties I

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$(M, J, \mathcal{L})$ is a polarized (compact smooth) variety ($c_1(\mathcal{L}) > 0$):

- Kodaira embedding: $M \hookrightarrow P(H^0(M, \mathcal{L}^k)^*) \cong \mathbb{C}P^{N_k}, k \gg 1$

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- Futaki invariant: a number $\mathcal{F}(M_0, \alpha)$ (analogue of 'slope')
- $(M, J, \mathcal{L})$ is *K-stable* if for *any* $M \hookrightarrow P(H^0(M, \mathcal{L}^k)^*)$ and *any* $\mathbb{C}^\times$-subgroup α, $\mathcal{F}(M_0, \alpha) \geq 0$ with equality iff α_z preserves M.

Kobayashi–Hitchin correspondence for projective varieties II

Calabi initiated the study of the following problem in Kähler geometry:

Definition (extremal Kähler metrics)

(M, J) a compact Kähler manifold and $\Omega \in H^2_{dR}(M)$ a Kähler class:

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- $c_1(M, J) = 0$: CSC $\Leftrightarrow$ Ricci-flat or 'Calabi–Yau' (Yau)

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A central problem in Kähler geometry (today and tomorrow) is

Conjecture (Yau, Tian, Donaldson)

$(M, J, \mathcal{L})$ is a polarized variety.
$\Omega = c_1(\mathcal{L})$ admits a CSC metric $\iff (M, J, \mathcal{L})$ is K-stable.

There is a similar conjecture for the existence of extremal Kähler metric in Ω (Székelyhidi)

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Results:

- (Donaldson, 04; Chen-Tian, 08) If $\Omega = c_1(\mathcal{L})$ admits a CSC metric $\Rightarrow (M, J, \mathcal{L})$ is K-semistable $(\mathcal{F}(M_0, \alpha) \geq 0 \ \forall \ \alpha)$...

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- The other direction is very much open... and the conjecture needs modification (recent example by ACGT)

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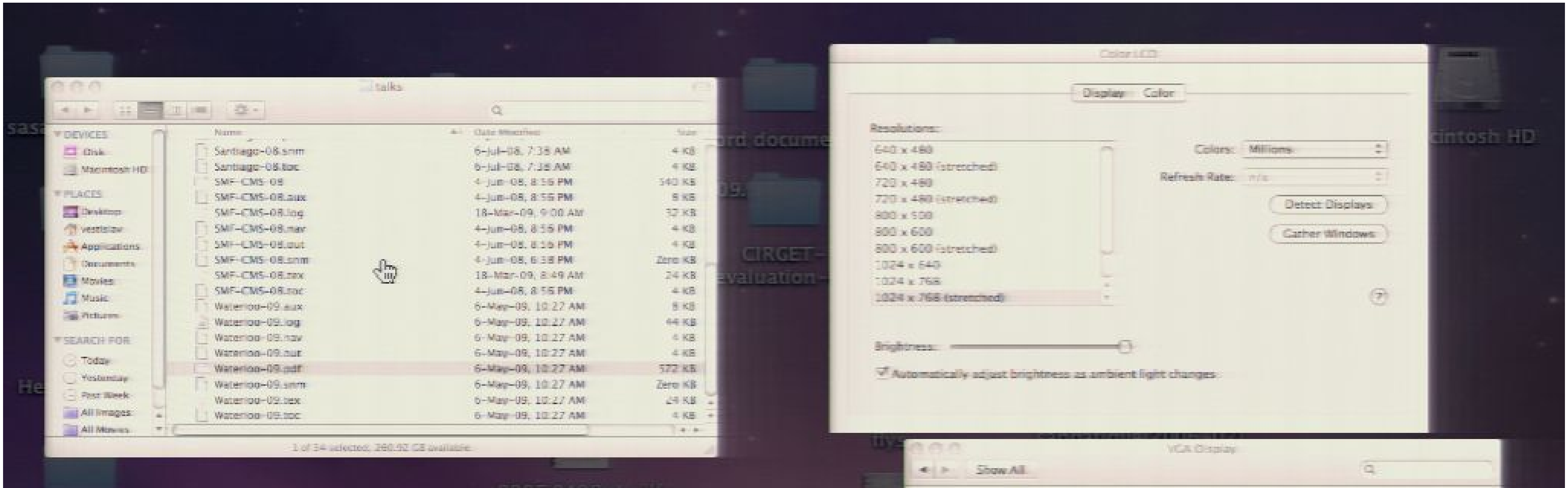

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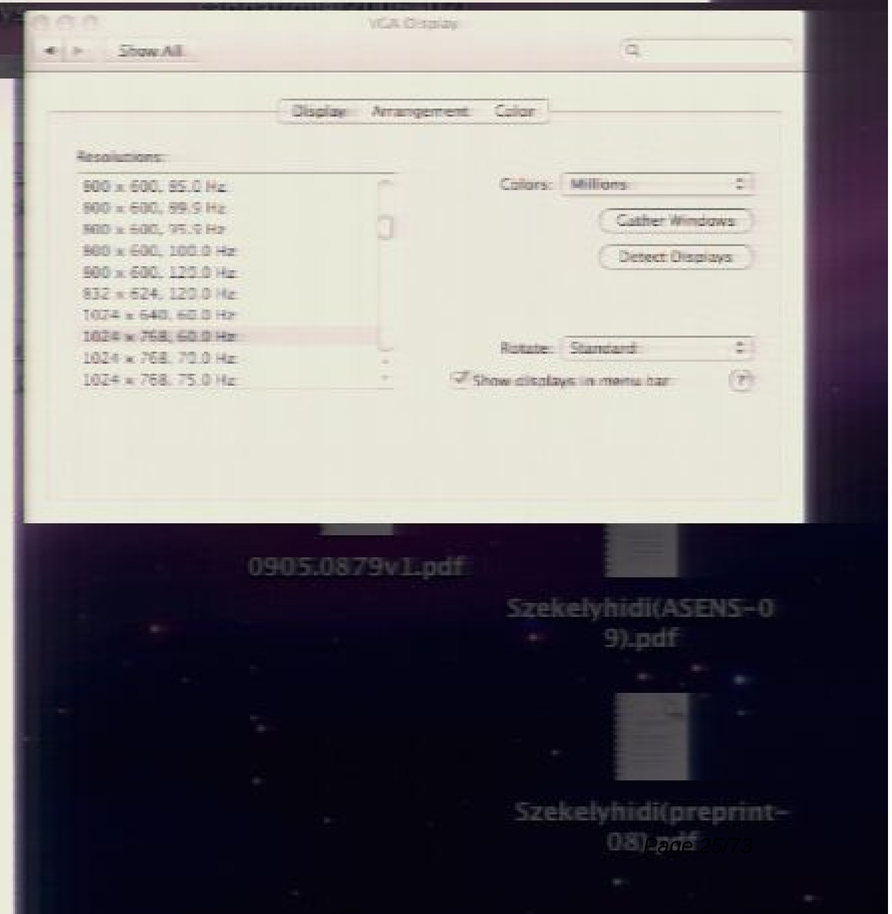

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- $(M, J) = P(E) \to \Sigma$ with genus$(\Sigma) \geq 2$ and E **simple**
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The CSC case

Theorem 1

$(M, J) = P(E) \to \Sigma$ *with genus*$(\Sigma) \geq 2$.

$\exists$ *CSC metric in some (and hence any)* $\Omega \iff E$ *is polystable.*

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- E of rank 2 and $c_1 \cdot \Omega < 0$ (LeBrun).

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- Implicit Function Theorem: $\exists\ f_t \in C_0^\infty(M)$ s.t.

$$Scal(\omega_t + dd_t^c f_t) = const$$

$(T_{(0,0)} Scal)(f) = -2\delta\delta\Big((\nabla df)^-\Big)$ is the Lichnerowicz operator and its kernel $\{f : \mathrm{grad}_{\omega_0} f$ is holomorphic$\}$ is trivial when $\mathrm{Aut}(M, J_0)$ is discrete.

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- $\tilde{\omega}_t = \omega_t + dd^c_t f_t$ is locally symmetric by the uniqueness of the CSC/extremal metric (Chen–Tian, Donaldson, Mabuchi) $\Rightarrow$ $\omega = \lim_{t\to 0} \tilde{\omega}_t$ is locally symmetric.

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- $\mathbb{T}$ induces a decomposition $E = \bigoplus_{i=0}^{\ell} E_i$ with E_i indecomposable. Consider small stable deformations $E(t) = \bigoplus_{i=0}^{\ell} E_i(t)$ and put $(M, J_t) = P(E(t))$. Then $\mathbb{T} \subset \mathrm{Aut}_0(M, J_t)$.

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- LeBrun–Simanca: work equivariantly and reduce to the Implicit Function Theorem on the Sobolev closures of the space $\overline{C_{\mathbb{T}}^{\infty}}(M)$ of $\mathbb{T}$-invariant smooth functions L^2 orthogonal the space of Killing potentials of $\mathfrak{t}$ with respect to ω.
  The point here is that $(T_{(0,0)}Scal)(f) = -2\delta\delta\Big((\nabla df)^-\Big)$ has trivial kernel on this space.

The key argument

The arguments so far are not specific to the CSC case! We have shown: Let ω be an *extremal* Kähler metric on $(M, J) = P(E)$ and $\mathbb{T}$ be an a maximal torus in $\mathrm{Isom}(M, \omega)$, then

- $\exists$ $\mathbb{T}$-invariant extremal Kähler metrics $\tilde{\omega}_t$ on $(M, J_t) = P(\bigoplus_{i=0}^{\ell} E_i(t))$ (with $E_i(t)$ stable for $t \neq 0$), such that: (1) $\lim_{t \to 0} \tilde{\omega}_t = \omega$, and (2) $[\tilde{\omega}_t] = [\omega]$

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  $\iff \mu(E_i) = \mu(E_j) \iff E(t)$ is polystable $t \neq 0$.

The key argument

The arguments so far are not specific to the CSC case! We have shown: Let ω be an *extremal* Kähler metric on $(M, J) = P(E)$ and $\mathbb{T}$ be an a maximal torus in $\mathrm{Isom}(M, \omega)$, then

- $\exists$ $\mathbb{T}$-invariant extremal Kähler metrics $\tilde{\omega}_t$ on $(M, J_t) = P(\bigoplus_{i=0}^{\ell} E_i(t))$ (with $E_i(t)$ stable for $t \neq 0$), such that: (1) $\lim_{t \to 0} \tilde{\omega}_t = \omega$, and (2) $[\tilde{\omega}_t] = [\omega]$

The CSC case:

- Futaki: $\exists$ CSC Kähler metric $\omega \Rightarrow \mathcal{F}_{[\omega]}(M, X_i) = 0$, where $X_i \in \mathfrak{t} = \mathrm{Lie}(\mathbb{T})$ are generators of S^1 subgroups. $\iff \mu(E_i) = \mu(E_j) \iff E(t)$ is polystable $t \neq 0$.
- Narasimhan–Seshadri + Chen–Tian (uniqueness): (ω, J_t) is locally symmetric. $\square$

The general case

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- The real question is how do the extremal metrics on $M = P(E_0(t) \oplus \cdots \oplus E_\ell(t))$ with $E_i(t)$ **stable** look like and to what kind of extremal metrics they can possibly converge?

The blow-up picture

Suppose $E = \bigoplus_{i=0}^{\ell} E_i$ with E_i stable. How does an extremal metric on $P(E)$ look like?

$$\begin{array}{ccc} \hat{M} = P(\mathcal{O}(-1)_{E_0} \oplus \cdots \oplus \mathcal{O}(-1)_{E_\ell}) & \longrightarrow & S = P(E_0) \times_\Sigma \cdots \times_\Sigma P(E_\ell) \\ \downarrow & & \downarrow \\ M = P(E_0 \oplus \cdots \oplus E_\ell) & \longrightarrow & \Sigma, \end{array}$$

which generalizes

$$\begin{array}{ccc} \hat{M} = P(\mathcal{O}(-1) \oplus \mathcal{O}) & \longrightarrow & S = \mathbb{C}P^1 \\ \downarrow & & \downarrow \\ M = \mathbb{C}P^2 & \longrightarrow & \{pt\}. \end{array}$$

S is the *stable* quotient of M by the complexified action $\mathbb{T}^c$.
$\hat{M}$ is a *toric* $\mathbb{C}P^\ell$-bundle over S associated to a principle $\mathbb{T}^c$ bundle.

The rigid semisimple Ansatz

We want to build a metric on M using the locally symmetric structure of S and the toric structure of $\mathbb{C}P^{\ell}$ (ACGT, JDG-04):

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- (g_S, ω_S) a locally symmetric metric on S, covered by $\mathbb{C}P^{d_0} \times \cdots \times \mathbb{C}P^{d_\ell} \times \mathbb{H}$ where the Fubini–Study metric (g_i, ω_i) on $\mathbb{C}P^{d_i}$ has scalar curvature $2d_i(d_i+1)$, and $(g_\Sigma, \omega_\Sigma)$ is CSC on Σ with $[\omega_\Sigma]$ primitive.

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- Delzant–Guillemin: g_V a toric Kähler metric on $(\mathbb{C}P^\ell, \mathbb{T})$

$$g_V = G_{rs} dz_r dz_s + G^{rs} dt_r dt_s$$

where $z : \mathbb{C}P^\ell \to \Delta \subset \mathbb{R}^\ell$ is the momentum map (Δ is a simplex), $G_{rs} = \mathrm{Hess}(U)_{rs}$ with $U(z)$ smooth of the interior of Δ.

- θ a connection 1-form on M^0 (which is the associated principal $\mathbb{T}$-bundle over S) with

$$d\theta = \sum_{i=0}^{\ell} \omega_i \otimes u_i + \omega_\Sigma \otimes u.$$

where $u_i, u \in \mathfrak{t}$ and u_i are primitive inward normals of the co-dimension 1 faces $F_i = \{z \in \mathfrak{t}^* : p_i(z) = \langle u_i, z \rangle + c_i = 0\}$.

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$$g = \sum_{i=0}^{\ell} p_i(z) g_i + (\langle u, z\rangle + k) g_\Sigma + G_{rs}(z) dz_r dz_s + G^{rs}(z)\theta_r \otimes \theta_s$$

is a Kähler metric on M in $\Omega = 2\pi(c_1(\mathcal{O}(1)_E + k[\omega_\Sigma])$ (parametrized by $U(z)$ with $G_{rs} = (\mathrm{Hess}\, U)_{rs}$).

Example

The Fubini-Study metric as a blow down:

$$\begin{array}{ccc} \hat{M} = P(\mathcal{O}(-1) \oplus \mathcal{O}) & \longrightarrow & S = \mathbb{C}P^1 \\ \downarrow & & \downarrow \\ M = \mathbb{C}P^2 & \longrightarrow & \{pt\}. \end{array}$$

$$g_{FS} = (1+z)g_{\mathbb{C}P^1} + \left(\frac{1}{1-z^2}\right)dz^2 + (1-z^2)\theta^2$$

with $G(z) = 1/(1-z^2)$ and $d\theta = \omega_{\mathbb{C}P^1}$.

Geometric properties of the Ansatz

- Observe that for any $z \in \mathrm{int}(\Delta)$, the symplectic quotient construction defines a *locally symmetric* Kähler metric $\check{g}(z) = \sum_{i=0}^{\ell} p_i(z) g_i + (\langle u, z \rangle + k) g_\Sigma$ on $S = M_{ss}/\mathbb{T}^c$.

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- The corresponding equation on $U(z)$ to get an extremal metric is

$$P_k(U) = \langle A_k, z \rangle + B_k + \sum_{i=0}^{\ell} \frac{2d_i(d_i+1)}{p_i(z)} + \frac{Scal_\Sigma}{\langle u, z \rangle + k} - \frac{1}{p_k(z)} \sum_{r,s} \frac{\partial^2}{\partial z_r \partial z_s} (p_k(z) G^{rs}(z)) = 0. \quad (1)$$

where $p_k(z) = (\langle u, z \rangle + k) \prod_{i=0}^{\ell} p_i(z)^{d_i}$, and $A_k \in \mathfrak{t}$ and $B_k \in \mathbb{R}$ are determined by $c_1(E_i)$ and Ω.

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This provides a considerable scope for extending Donaldson's theory in the toric case to this context...

Geometric properties of the Ansatz

- We believe that $2\pi(c_1(\mathcal{O}(1)_E + k[\omega_\Sigma])$ admits an extremal Kähler metric $\Longleftrightarrow$ (1) has solution (true if $\ell = \mathrm{rk}(\mathrm{Aut}(P(E)) \leq 1$, ACGT, Inv.math-08).

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Implicit Function Theorem $\implies$ existence when $k \gg 1$.

The key argument revisited

- (ω_t, J_t) is extremal on $P(E_0(t) \oplus \ldots \oplus E_\ell(t))$ with $E_i(t)$ stable for $t \neq 0$

The CSC case:

- (ω_t, J_t) is locally-symmetric on $P(E_0(t) \oplus \cdots \oplus E_\ell(t))$ for every $t \Rightarrow E_i(0)$ are stable.

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No Signal

VGA-1

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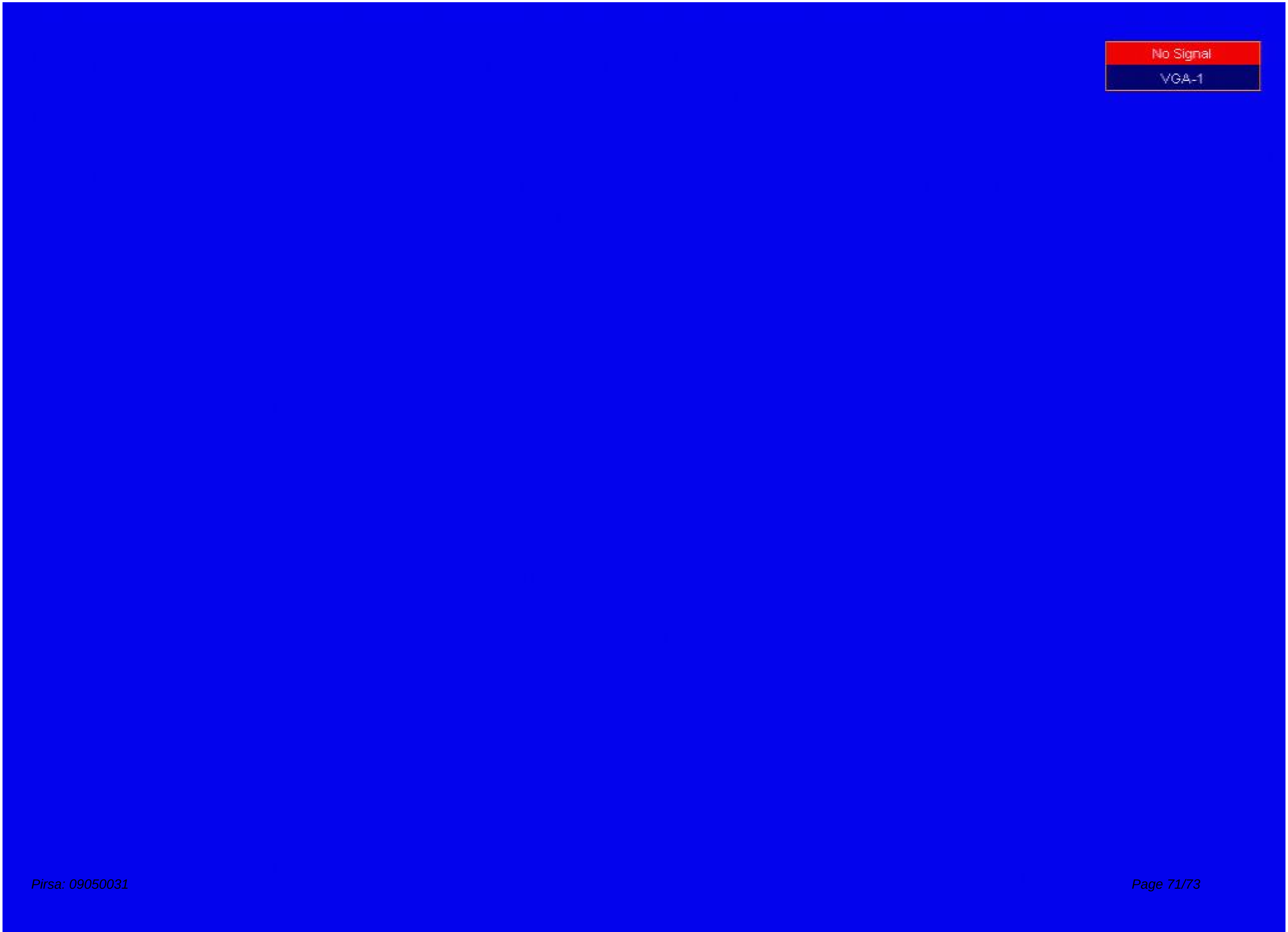
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Conclusion

Conjecture

$P(E)$ admits an extremal Kähler metric in some Kähler class Ω
$\iff E = \bigoplus_{i=0}^{\ell} E_i$ with E_i stable.

Theorem 2

There exists $k_0(E, \Sigma)$ such that $P(E)$ admits an extremal Kähler metric in a class $\Omega = 2\pi(c_1(\mathcal{O}(1)_E + k[\omega_\Sigma])$ with $k > k_0$ iff $E = \bigoplus_{i=0}^{\ell} E_i$ with E_i stable.

Theorem 1 and a recent result of ACGT confirm the conjecture in the case when $\mathrm{rk}(\mathrm{Aut}_0(P(E)) = \ell \leq 1$....