Title: Special geometries associated to quaternion-Kahler 8-manifolds

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Abstract: In this talk we will discuss the (local) construction of a calibrated G\_2 structure on the 7-dimensional quotient of an 8-dimensional quaternion-Kahler (QK) manifold M under the action of a group S^1 of isometries. The idea is to construct explicitly a 3-form of type G\_2, using the data associated to the S^1 action and to the QK structure on M. In the same spirit, we can consider the level sets of the QK moment-map square-norm function on M, and again take the S^1 quotient: we will discuss in this case the construction of half-flat metrics in dimension 6, under suitable circumstances. This talk is based on a joint work with F. Lonegro, Y. Nagatomo and S. Salamon, still in progress.

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# Special Geometries associated to 8-dimensional quaternion-Kähler manifolds

Andrea Gambioli

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5/9/2008

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$$i^2 = j^2 = \kappa^2 = -1$$
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$$l_1, l_2, l_3$$
 satisfying  $l_i^2 = -1$  and  $l_i l_j = l_k$ 

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the holonomy group is contained in Sp(n)Sp(1)

the bundle G, satisfies  $\nabla_X G \subseteq G \cap X \in TM$ :

if  $\omega_i = g(I_{i-1})$  span  $G \subset \bigwedge^2 TM$  then the 4-form  $\Omega = \sum_{i=1}^3 z_i$  is parallel.

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## 2 Geometry

the compact Lie group  $G_2$  can be defined as the subgroup of  $GI(7, \mathbb{R})$  eserving the 3-form

$$\phi = (e^{12} - e^{34})e^7 - (e^{13} - e^{42})e^6 - (e^{14} - e^{23})e^5 + e^{765}$$
 (1)

et N be a 7 dimensional Riemannian manifold admitting a section  $\in \Gamma(\Lambda^3 M)$  which can be locally expressed as 1. Then

$$G \subset G_2 \subset SO(7)$$

here G is the structure group. If moreover

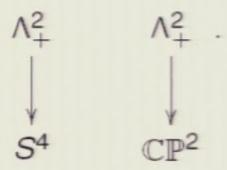
$$\nabla \phi = 0 \quad \Leftrightarrow \quad d\phi = 0 = d * \phi$$

en  $Hol_0(M,g) \subset G_2$ . (Fernandez-Gray). This is relevant in M-Teory.

### rst Examples of Holonomy G2 metrics

the late '80 the first examples of complete metrics with  $G_2$  appear in

 R.L. Bryant, S. Salamon: "On the construction of some complete metrics with exceptional holonomy", Duke Math. J. (1989)
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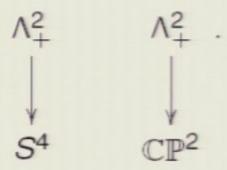
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## ther Examples

- D. D. Joyce: Compact Riemannian 7-manifolds with holonomy G<sub>2</sub>
   I, II. J. Differential Geom. 43 (1996)
- A. Brandhuber, J. Gomis, S. S. Gubser, S. Gukov: Gauge theory at large N and new G<sub>2</sub> holonomy metrics. Nuclear Phys. B 611 (2001)
- G.W. Gibbons, H. Lü, C.N. Pope, K. S. Stelle: Supersymmetric domain walls from metrics of special holonomy, Nuclear Phys. B 623 (2002)

#### uestion

it possible to construct G2-holonomy metrics on quotients

 $X/S^1$ 

here X is some special holonomy 8-dimensional manifold and  $S^1$  is a oup of isometries ?

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$$\begin{cases}
\omega_1 = e^{12} - e^{34} + e^{56} - e^{78} \\
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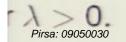
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$$-3e^{765} \longrightarrow +\lambda e^{765}$$



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the presence of an  $S^1$  action on a QK manifold provides the following lated objects:

- a Killing vector field K
- a moment map  $\mu \in \Gamma(\mathcal{G})$  defined by  $c(\nabla K)$
- a reduction of the structure group of G to  $S^1 \equiv SO(3)$  on

other words:

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other words:

$$G = \mu \oplus N$$

here N is spanned locally by  $\omega_2$ ,  $\omega_3$ .

#### ructure equations for $\mathcal{G}$

$$\nabla \omega_1 = \alpha_2 \otimes \omega_2 + \alpha_3 \otimes \omega_3$$
$$\nabla \omega_2 = -\alpha_2 \otimes \omega_1 + \beta \otimes \omega_3$$
$$\nabla \omega_1 = -\beta \otimes \omega_2 + \alpha_3 \otimes \omega_1$$

th  $\beta$ ,  $\alpha_2$ ,  $\alpha_3$  local 1-forms.

#### -invariant forms

- dβ
- $\bullet \ \omega_1 = \frac{\mu}{|\mu|}$
- $d\omega_1 = \alpha_2 \wedge \omega_2 + \alpha_3 \wedge \omega_3$
- $I_1 d\omega_1 = \alpha_2 \wedge \omega_3 \alpha_3 \wedge \omega_2$
- e well defined on the open set  $M^8 \setminus \{\mu^{-1}(0)\}$ .

### alibrated G<sub>2</sub> Structures

t us set

$$M_0 = M^8 \setminus (\{\mu = 0\} \cup \{K = 0\}),$$

id let us define

$$\tau = I_1 \alpha_0 \wedge (d\beta + \omega_1)$$

#### neorem

onsider an S1 action on M8 as before. Let us consider

$$\phi = A\gamma + B\tau, \tag{3}$$

here A, B are certain functions of  $\|\mu\|$  and of a parameter  $t \in \mathbb{R}$ . nen the form  $\phi$  induces a 1-parameter family of calibrated  $G_2$  ructures on the quotient  $M^7 = M_0/S^1$ .

# cample

t us consider  $\mathbb{HP}^2$  with  $S^1$  associated to  $\mathbf{w} = (1,0,0)$ . Then

$$\{K = 0\} \cong S^4 \cup [1, 0, 0] \text{ and } \{\mu = 0\} \cong S^4$$

preover 
$$C(S_w^1) = S^1 \times Sp(2)$$
. So

$$\mathbb{HP}^2 \setminus (S^4 \cup [1,0,0]) \longrightarrow \Lambda^2_+ S^4 \setminus S^4$$

rjectively and Sp(2)-equivariantly.

the Bryant-Salamon metric amongst the calibrated  $G_2$ -structures tained in the theorem?

# alf-flat geometry

et  $M^6$  be a 6-dimenisonal manifold with an SU(3)-structure aracterized by

$$\omega \in \Lambda^{1,1}$$
  $\psi^+ + \imath \psi^- \in \Lambda^{3,0}$ .

en we say that  $M^6$  is half the following equations are satisfied:

$$\begin{cases} d\omega & \omega & = 0 \\ d\omega^+ & = 0 \end{cases}$$

e intrinsic torsion of the SU(3) structure lies in a direct sum

e half-flat condition implies that the projections on  $W_{1,2}^-$  and  $W_{4,5}$  are

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$$\{K = 0\} \cong S^4 \cup [1, 0, 0] \text{ and } \{\mu = 0\} \cong S^4$$

preover  $C(S_w^1) = S^1 \times Sp(2)$ . So

$$\mathbb{HP}^2\setminus (S^4\cup [1, \overset{\mathfrak{h}}{0}, 0])\longrightarrow \Lambda^2_+S^4\setminus S^4$$

rjectively and Sp(2)-equivariantly.

the Bryant-Salamon metric amongst the calibrated  $G_2$ -structures tained in the theorem?



# alf-flat geometry

t  $M^6$  be a 6-dimenisonal manifold with an SU(3)-structure aracterized by

$$\omega \in \Lambda^{1,1}$$
  $\psi^+ + \imath \psi^- \in \Lambda^{3,0}$ .

ien we say that  $M^6$  is half-flat if the following equations are satisfied:

$$\begin{cases} d\omega \wedge \omega = 0 \\ d\psi^{+} = 0 \end{cases}$$

e intrinsic torsion of the SU(3) structure lies in a direct sum

e half-flat condition implies that the projections on  $W_{1,2}^{\pm}$  and  $W_{4,5}$  are

### pen Questions

- find an expression for \*φ in terms of the known forms, and see if in the manifold of calibrated G<sub>2</sub> structure there are any which are integrable;
- the S¹ quotients of the hypersurfaces {||μ|| = c ≠ 0} admit a half flat structure: does the G₂ cone over these coincide with some of the above metrics?
- study the relationships with the Bryant-Salamon examples for the weights (1, 1, 1) and (1, 0, 0) on HP<sup>2</sup>;