Title: Special geometries associated to quaternion-Kahler 8-manifolds

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Abstract: In this talk we will discuss the (local) construction of a calibrated G_2 structure on the 7-dimensional quotient of an 8-dimensional quaternion-Kahler (QK) manifold M under the action of a group S^1 of isometries. The idea is to construct explicitly a 3-form of type G_2, using the data associated to the S^1 action and to the QK structure on M. In the same spirit, we can consider the level sets of the QK moment-map square-norm function on M, and again take the S^1 quotient: we will discuss in this case the construction of half-flat metrics in dimension 6, under suitable circumstances. This talk is based on a joint work with F. Lonegro, Y. Nagatomo and S. Salamon, still in progress.

Special Geometries associated to 8-dimensional quaternion-Kähler manifolds

Andrea Gambioli

Connections in Geometry and Physics
Perimeter Institute

5/9/2008

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$$1, \quad \imath, \quad \jmath, \quad \kappa$$

hich satisfy the relations

$$\imath^2 = \jmath^2 = \kappa^2 = -1 \quad \imath\jmath = \kappa .$$

- It is *associative* but not *commutative*;
- we can define the quaternionic vector space $\mathbb{H}^n$ with scalar multiplication on the *right*.
- We have an inclusion

$$Gl(n,\mathbb{H})\mathbb{H}^* \subset Gl(4n,\mathbb{R})$$

of invertibe matrices with quaternionic entries acting on the *left* commuting with $\mathbb{H}^*$ acting on the *right*.

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$$I_1, I_2, I_3 \quad \text{satisfying} \quad I_i^2 = -1 \quad \text{and} \quad I_i I_j = I_k$$

Riemannian manifold (M, g) endowed with a quaternionic structure *mpatible* with the metric $(g(I_i \cdot, I_i \cdot) = g(\cdot, \cdot) \quad i = 1, 2, 3)$ is called QK t satisfies one of the following (equivalent) conditions:

- the holonomy group is contained in $Sp(n)Sp(1)$;
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he projective plane $\mathbb{HP}^2$ parametrizes the quaternionic lines in $\mathbb{H}^3$. It *homogeneous* w.r.t. $Sp(3) = GL(3, \mathbb{H}) \cap SO(12)$:

$$\mathbb{HP}^2 \simeq \frac{Sp(3)}{Sp(2)Sp(1)}$$

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$$\frac{G_2}{SO(4)} \quad \text{and} \quad \mathrm{Gr}_2(\mathbb{C}^4) \simeq \frac{SU(4)}{SU(2) \times SU(2)}$$

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$_2$ Geometry

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$$\phi = (e^{12} - e^{34})e^7 - (e^{13} - e^{42})e^6 - (e^{14} - e^{23})e^5 + e^{765} \qquad (1)$$

t N be a 7 dimensional Riemannian manifold admitting a section $\in \Gamma(\Lambda^3 M)$ which can be locally expressed as 1. Then

$$G \subset G_2 \subset SO(7)$$

here G is the structure group. If moreover

$$\nabla\phi = 0 \qquad \Leftrightarrow \qquad d\phi = 0 = d * \phi$$

en $Hol_0(M,g) \subset G_2$. (Fernandez-Gray). This is relevant in M-Teory.

rst Examples of Holonomy G_2 metrics

the late '80 the first examples of complete metrics with G_2 appear in

- R.L. Bryant, S. Salamon: "On the construction of some complete metrics with exceptional holonomy", Duke Math. J. (1989)

the total spaces of vector bundles with fibre $\mathbb{R}^3$

$$\begin{array}{ccc} \Lambda^2_+ & & \Lambda^2_+ \\ \downarrow & & \downarrow \\ S^4 & & \mathbb{CP}^2 \end{array} .$$

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- D. D. Joyce: Compact Riemannian 7-manifolds with holonomy G_2 I, II. J. Differential Geom. 43 (1996)
- A. Brandhuber,J. Gomis, S. S. Gubser, S. Gukov: Gauge theory at large N and new G_2 holonomy metrics. Nuclear Phys. B 611 (2001)
- G.W. Gibbons, H. Lü, C.N. Pope, K. S. Stelle: Supersymmetric domain walls from metrics of special holonomy, Nuclear Phys. B 623 (2002)

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here X is some special holonomy 8-dimensional manifold and S^1 is a oup of isometries ?

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nfortunately, this is not the correct answer: in fact, we can find an ON
cal coframe $e^1, \cdots, e^8$ on M^8 with respect to which

$$\begin{cases} \omega_1 = e^{12} - e^{34} + e^{56} - e^{78} \\ \omega_2 = e^{13} - e^{42} + e^{57} - e^{86} \\ \omega_3 = e^{14} - e^{23} + e^{58} - e^{67}. \end{cases} \quad (2)$$

here we can identify $e^8 = \alpha_0/|\alpha_0| = K^\#/|K|$. In ths case

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hich has $G_2^* \subset SO(3,4)$ as a stabilizer.

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- a Killing vector field K
- a moment map $\mu \in \Gamma(\mathcal{G})$ defined by $c(\nabla K)^{\mathcal{G}}$
- a reduction of the structure group of $\mathcal{G}$ to $S^1 \subset SO(3)$ on $M \setminus \mu^{-1}(0)$

other words:

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ructure equations for $\mathcal{G}$

$$\nabla\omega_1 = \alpha_2 \otimes \omega_2 + \alpha_3 \otimes \omega_3$$
$$\nabla\omega_2 = -\alpha_2 \otimes \omega_1 + \beta \otimes \omega_3$$
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th $\beta, \alpha_2, \alpha_3$ local 1-forms.

-invariant forms

- $d\beta$
- $\omega_1 = \frac{\mu}{|\mu|}$
- $d\omega_1 = \alpha_2 \wedge \omega_2 + \alpha_3 \wedge \omega_3$
- $I_1 d\omega_1 = \alpha_2 \wedge \omega_3 - \alpha_3 \wedge \omega_2$

e well defined on the open set $M^8 \setminus \{\mu^{-1}(0)\}$.

alibrated G_2 Structures

et us set

$$M_0 = M^8 \setminus (\{\mu = 0\} \cup \{K = 0\}) .$$

nd let us define

$$\tau = l_1 \alpha_0 \wedge (d\beta + \omega_1)$$

heorem

onsider an S^1 action on M^8 as before. Let us consider

$$\phi = A\gamma + B\tau , \tag{3}$$

here A, B are certain functions of $\|\mu\|$ and of a parameter $t \in \mathbb{R}$. hen the form ϕ induces a 1-parameter family of calibrated G_2 ructures on the quotient $M^7 = M_0/S^1$.

xample

et us consider $\mathbb{HP}^2$ with S^1 associated to $\mathbf{w} = (1, 0, 0)$. Then

$$\{K = 0\} \cong S^4 \cup [1, 0, 0] \quad \text{and} \quad \{\mu = 0\} \cong S^4$$

oreover $C(S^1_w) = S^1 \times Sp(2)$. So

$$\mathbb{HP}^2 \setminus (S^4 \cup [1, 0, 0]) \longrightarrow \Lambda^2_+ S^4 \setminus S^4$$

urjectively and $Sp(2)$-equivariantly.

the Bryant-Salamon metric amongst the calibrated G_2-structures
tained in the theorem?

alf-flat geometry

et M^6 be a 6-dimenisonal manifold with an $SU(3)$-structure
aracterized by

$$\omega \in \Lambda^{1,1} \qquad \psi^+ + i\psi^- \in \Lambda^{3,0}.$$

en we say that M^6 is *half-flat* if the following equations are satisfied:

$$\begin{cases} d\omega \wedge \omega & = 0 \\ d\psi^- & = 0 \end{cases}$$

e *intrinsic torsion* of the $SU(3)$ structure lies in a direct sum

$$W_1^{\pm} + W_2^{\pm} + W_3 + W_4 + W_5$$

e half-flat condition implies that the projections on $W_{1,2}^-$ and $W_{4,5}$ are
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pen Questions

- find an expression for $*\phi$ in terms of the known forms, and see if in the manifold of calibrated G_2 structure there are any which are *integrable*;
- the S^1 quotients of the hypersurfaces $\{\|\mu\| = c \neq 0\}$ admit a half flat structure: does the G_2 cone over these coincide with some of the above metrics?
- study the relationships with the Bryant-Salamon examples for the weights $(1, 1, 1)$ and $(1, 0, 0)$ on $\mathbb{HP}^2$;