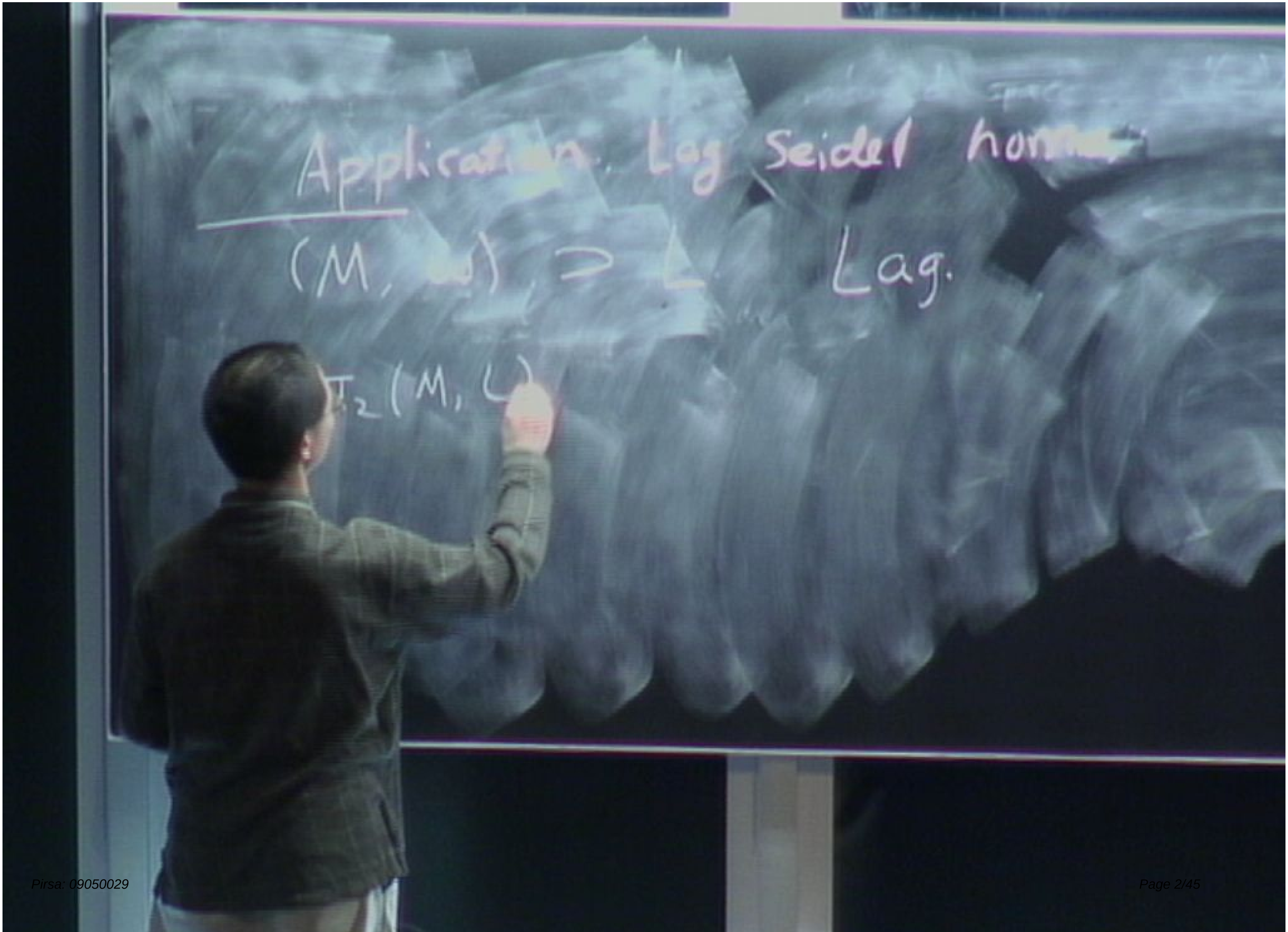
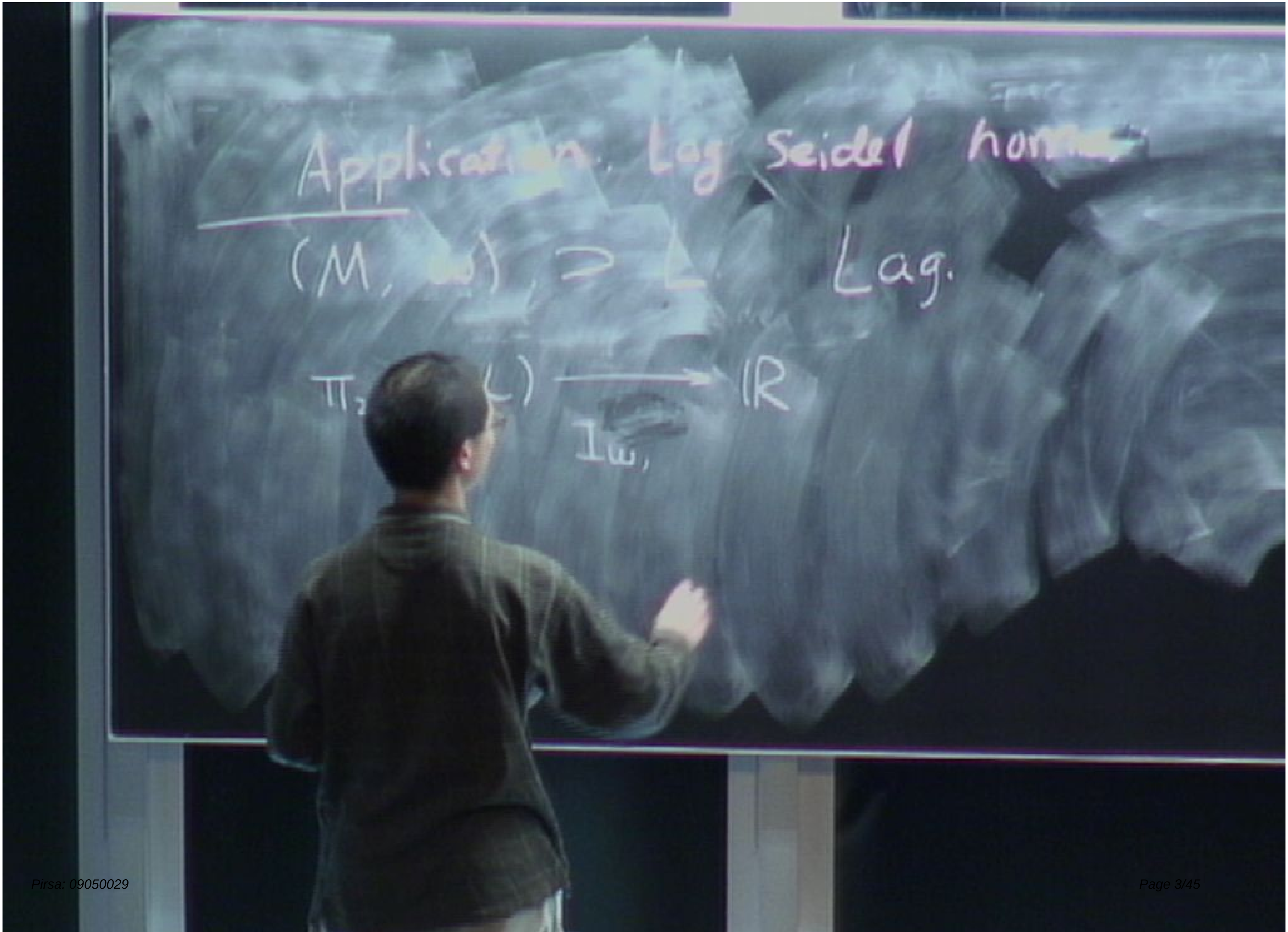
Title: Lagrangian Seidel homomorphism and an application

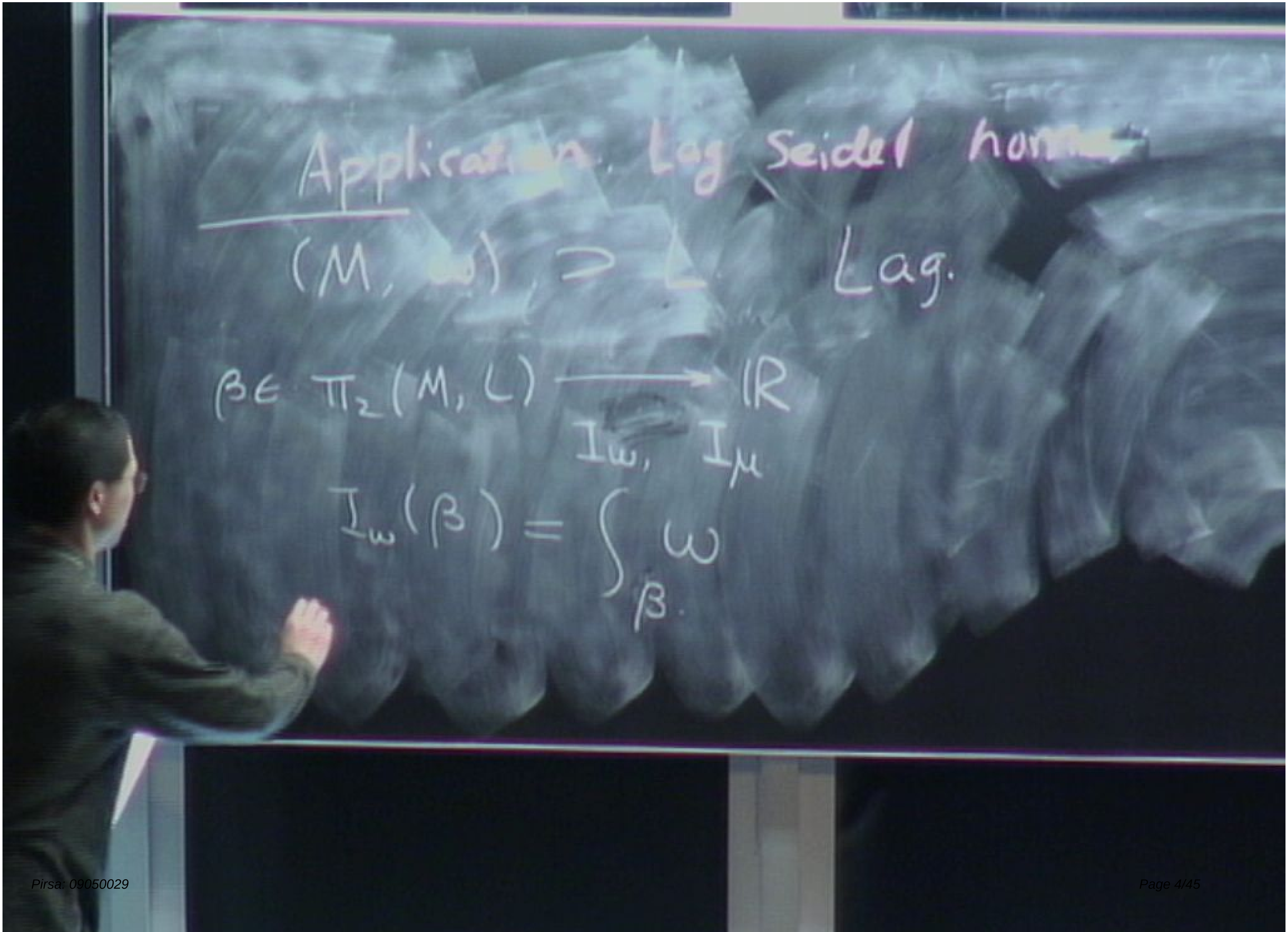
Date: May 08, 2009 11:15 AM

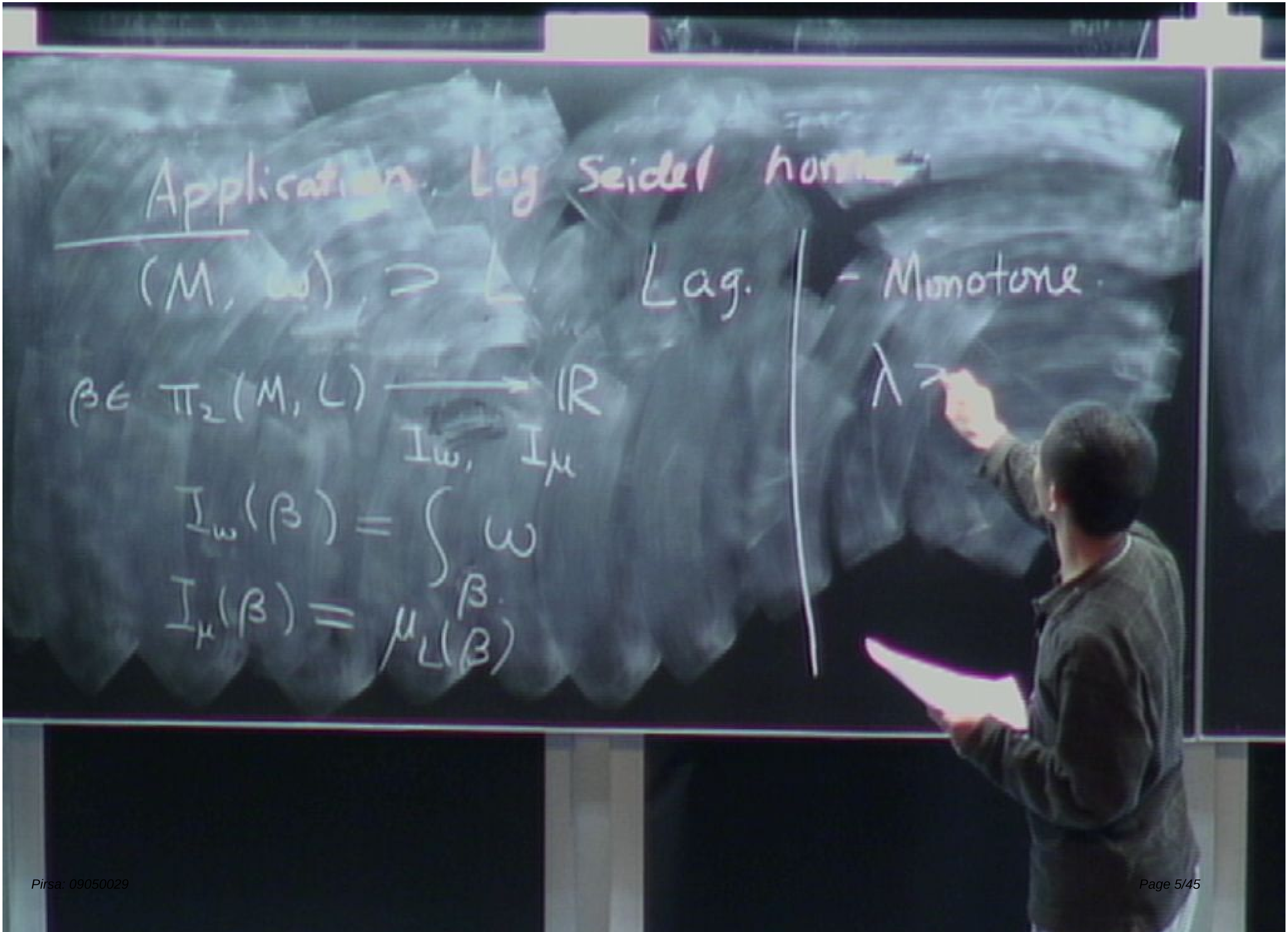
URL: http://pirsa.org/09050029

Abstract: This is joint work with Francois Lalonde. Using an analogue of Seidel's homomorphism in Lagrangian Floer homology for one Lagrangian, we give a condition for a diffeomorphism on a Lagrangian to extend to a Hamiltonian diffeomorphism on the whole symplectic manifold.


Application. Lag Seidel
(M, ω) ⊃ L Lag.
π2(M, L


Application. Lag Seidel hom
(M, ω) ⊃ L Lag.
ℝ
Iω


Application. Lag Seidel hom
$(M, \omega) \supset L$ Lag.
$\beta \in \pi_2(M, L) \longrightarrow \mathbb{R}$
$I_\omega, \; I_\mu$
$I_\omega(\beta) = \int_\beta \omega$


Application: Lag Seidel hom
$(M, \omega) \supset L$ Lag.
- Monotone.
$\lambda >$
$\beta \in \pi_2(M, L) \longrightarrow \mathbb{R}$
$I_\omega, \; I_\mu$
$I_\omega(\beta) = \int_\beta \omega$
$I_\mu(\beta) = \mu_L(\beta)$

Application: Lag Seidel hom

$(M, \omega) \supset L$ Lag. — Monotone.

$$\beta \in \pi_2(M, L) \longrightarrow \mathbb{R}$$

$I_\omega, \ I_\mu$

$$I_\omega(\beta) = \int_\beta \omega$$

$$I_\mu(\beta) = \mu_L(\beta)$$

$\exists\, \lambda \geq 0$

$$I_\omega = \lambda I_\mu.$$

Application: Lag Seidel homo

$(M, \omega) \supset L$ Lag.

$\beta \in \pi_2(M, L) \xrightarrow{I_\omega,\ I_\mu} \mathbb{R}$

$I_\omega(\beta) = \int_\beta \omega$

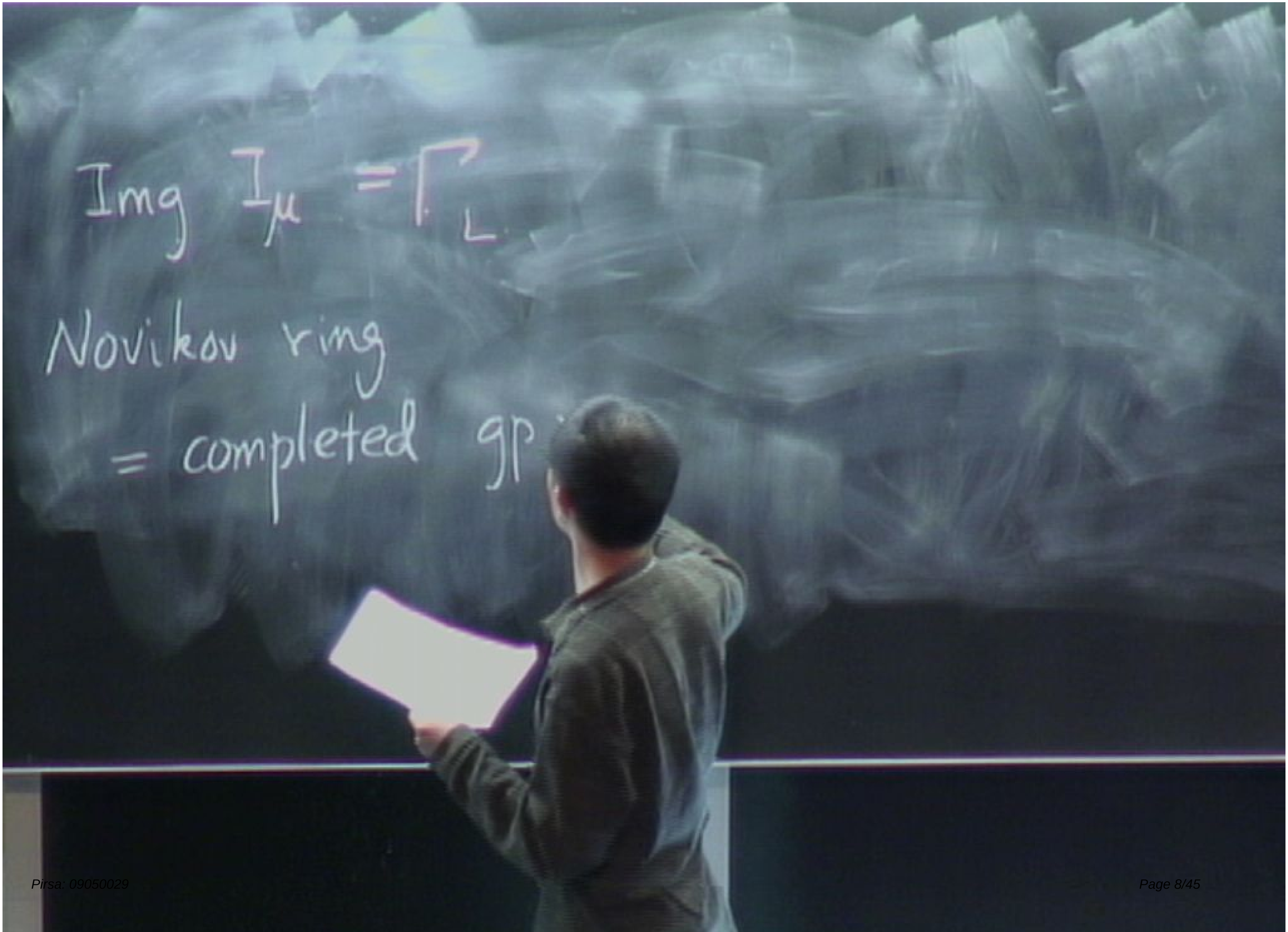
$I_\mu(\beta) = \mu_L(\beta)$

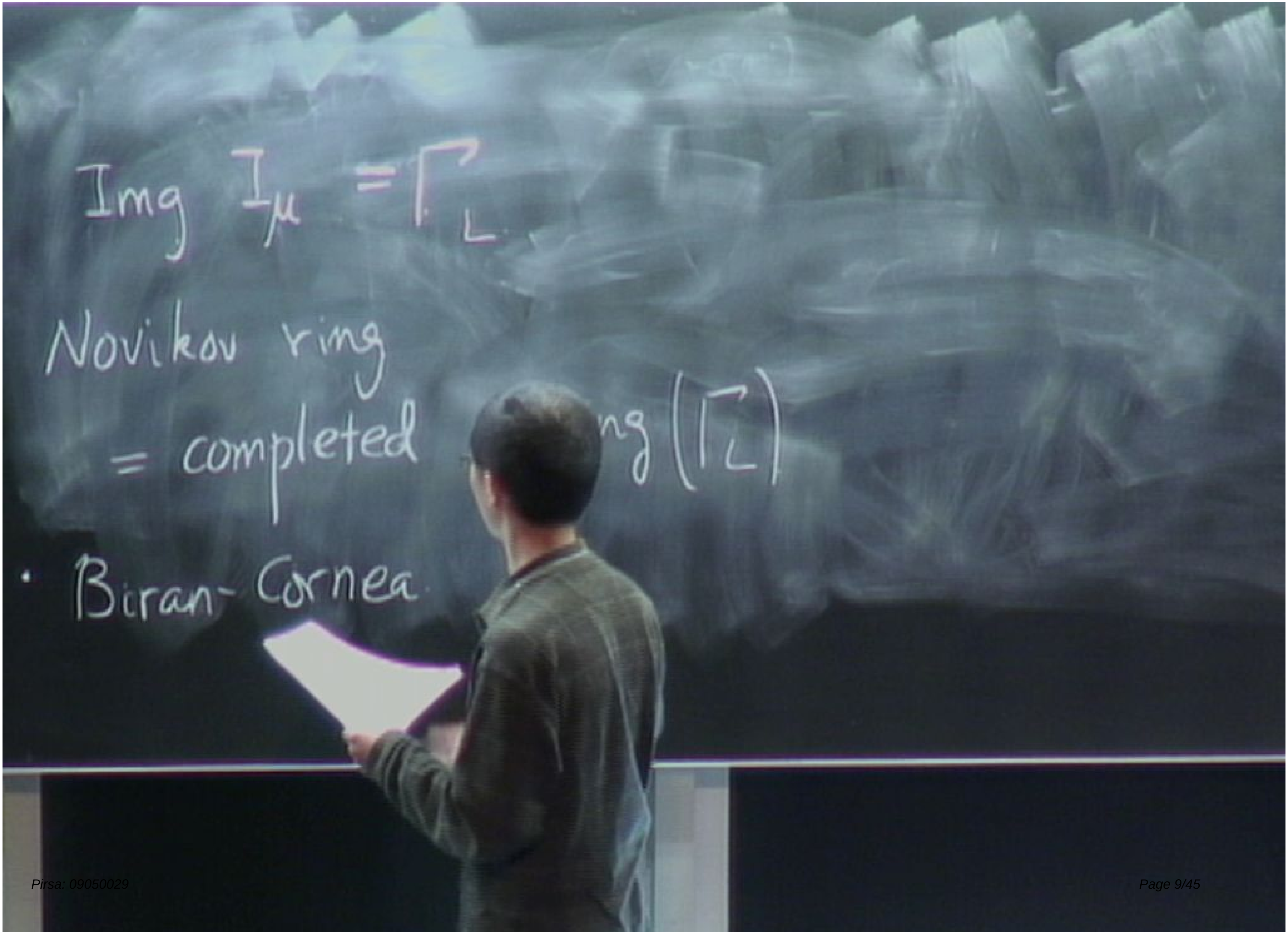
- Monotone.

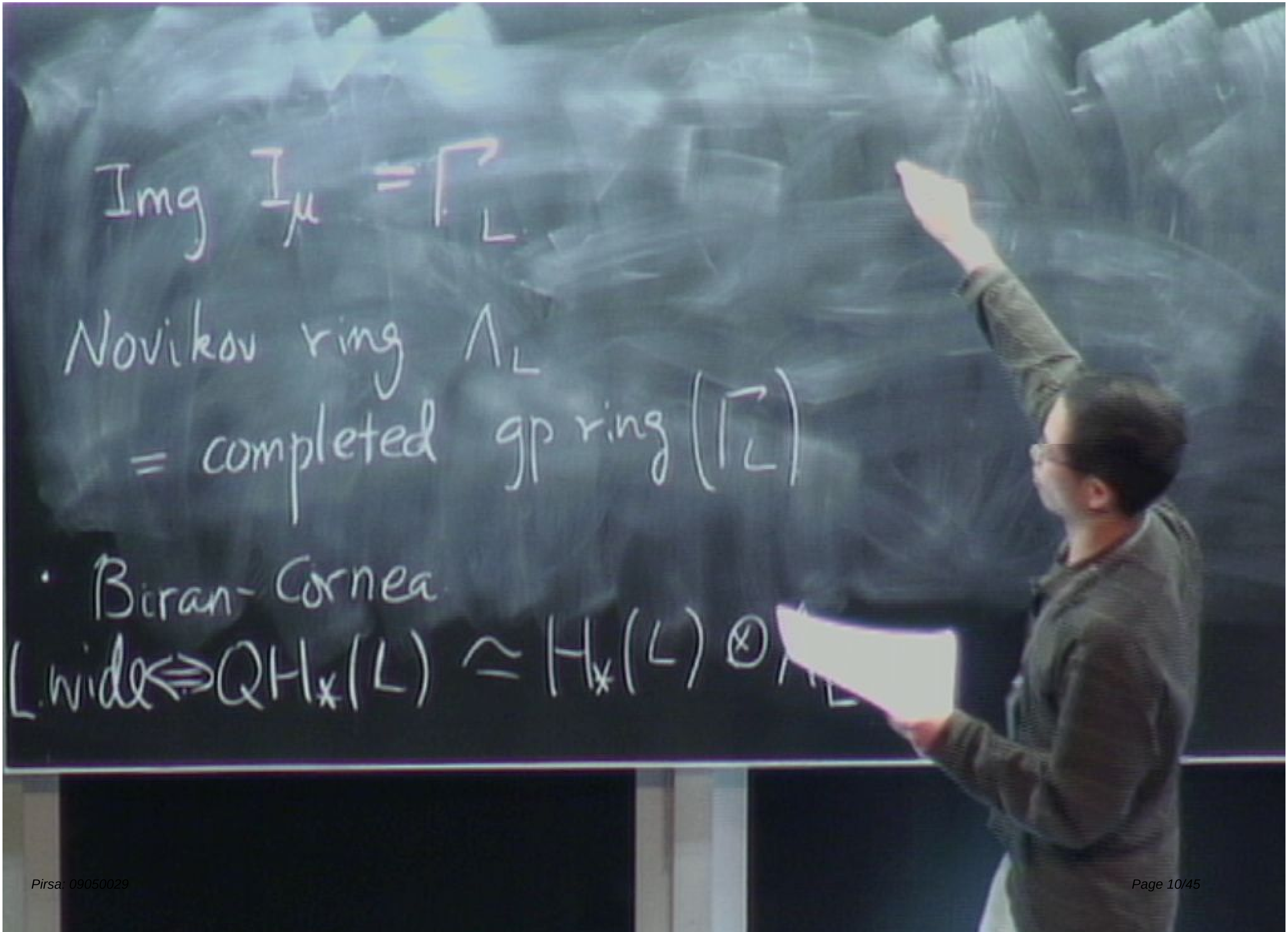
$\exists \lambda \geq 0$

$I_\omega = \lambda I_\mu$

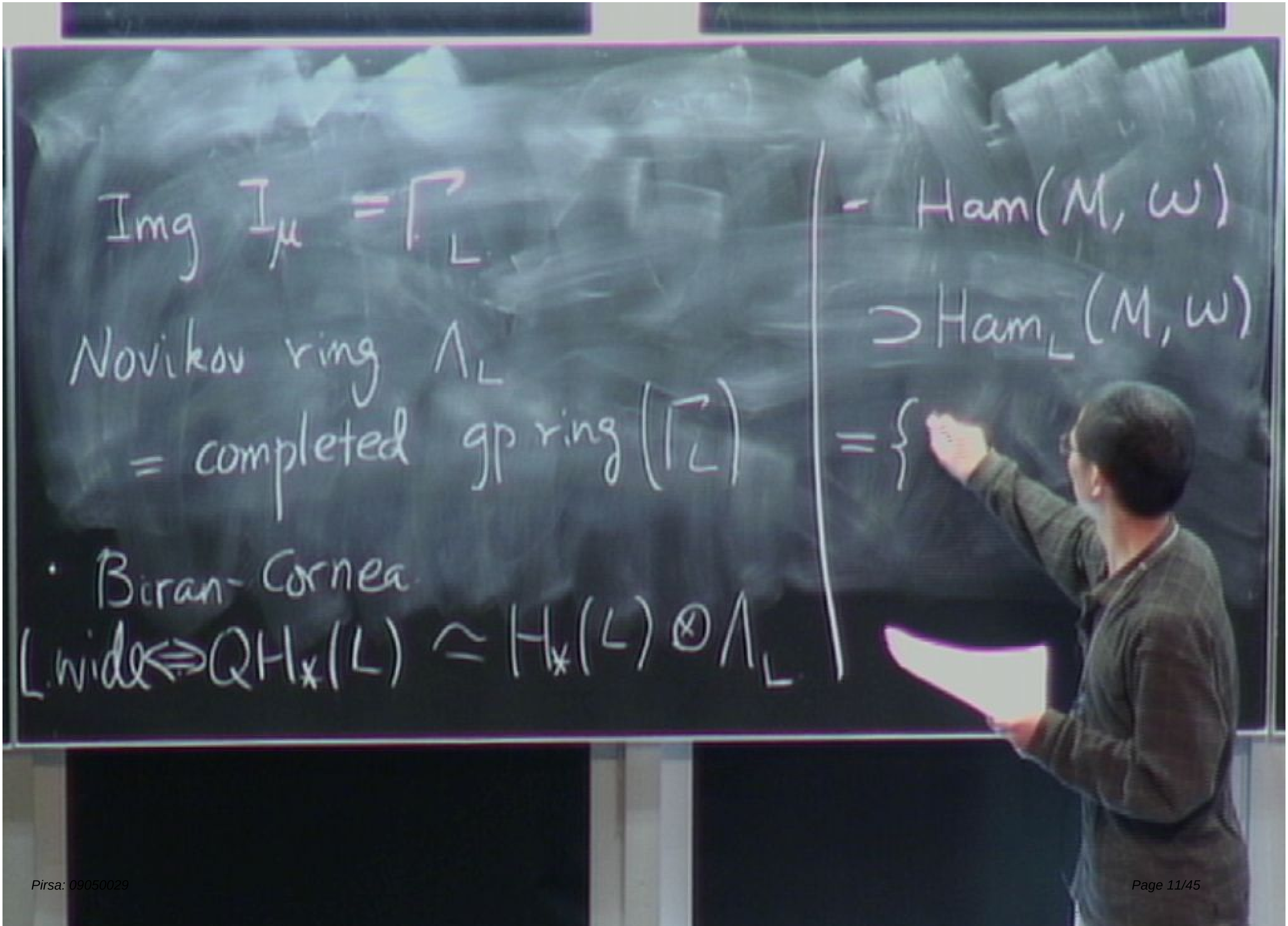
- $I_\mu \geq 2$.


Img $I_\mu = \Gamma_L$
Novikov ring
= completed gp


Img $I_\mu = \Gamma_L$
Novikov ring
= completed
ng (Γ_L)
• Biran-Cornea



$\mathrm{Img}\ I_\mu = \Gamma_L$
Novikov ring Λ_L
$=$ completed gp ring (Γ_L)
• Biran-Cornea
L wide $\Leftrightarrow QH_*(L) \simeq H_*(L) \otimes \Lambda_L$


$\mathrm{Img}\ I_\mu = \Gamma_L$
Novikov ring Λ_L
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• Biran-Cornea
L wide $\Leftrightarrow QH_*(L) \simeq H_*(L) \otimes \Lambda_L$
- $\mathrm{Ham}(M,\omega)$
$\supset \mathrm{Ham}_L(M,\omega)$
$= \{$

$\mathrm{Img}\ I_\mu = \Gamma_L$

Novikov ring Λ_L

= completed gp ring (Γ_L)

- Biran-Cornea

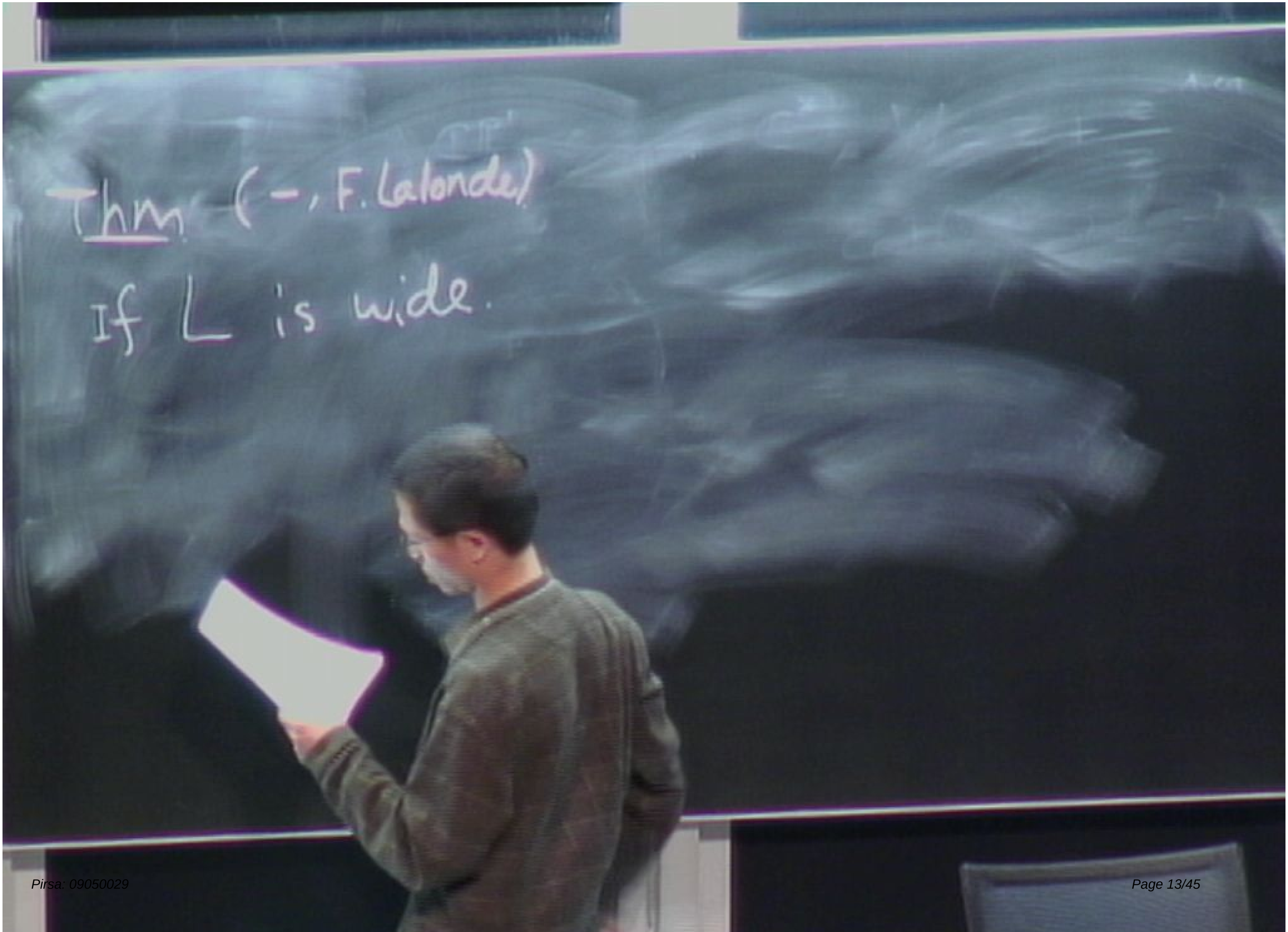
L wide $\Leftrightarrow QH_*(L) \simeq H_*(L) \otimes \Lambda_L$

- $\mathrm{Ham}(M,\omega)$

$\supset \mathrm{Ham}_L(M,\omega)$

$= \{\varphi \mid \varphi(L) = L\}$

$\varphi|_L : L \to L$


Thm (–, F. Lalonde)
If L is wide.

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If L is wide then

$(\varphi|_L)_* : H_*(L) \longrightarrow H_*(L)$

$= id.$

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If L is wide then

$$(\varphi|_L)_* : H_*(L) \to H_*(L)$$

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$g \in \mathrm{Diff}(L)$. $\quad g_* = id$

Application: Lag Seidel homo

Rmk $\mathrm{Ham}_L(M,\omega)$ needed

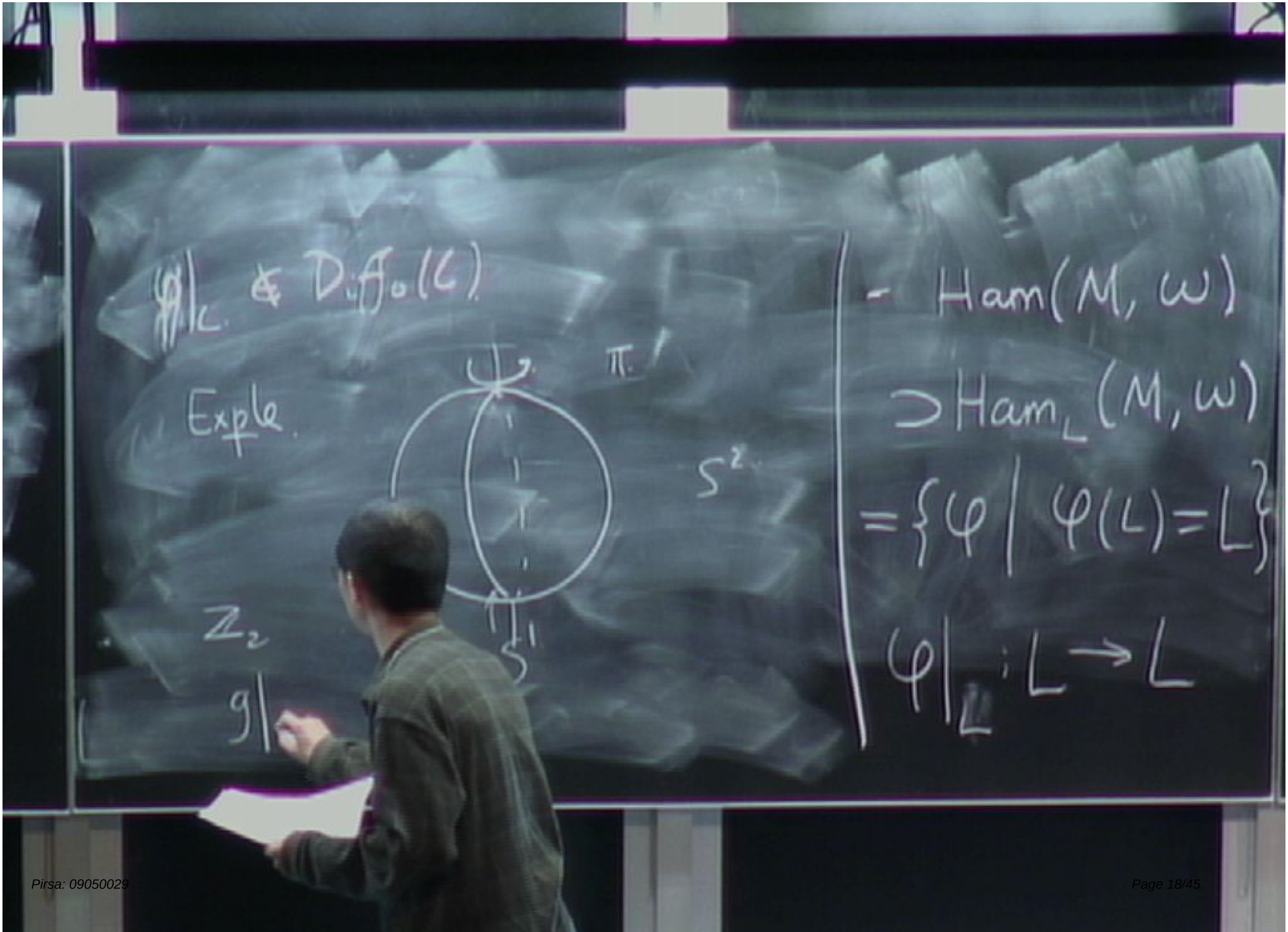
Application. Lag Seidel homs

Rmk $\mathrm{Ham}_L(M,\omega)$ needed

Exple. $\Big(M \times M, \ \omega \oplus (-\omega) \Big) \supset \Delta$

$\varphi \in \mathrm{Symp}(M,\omega)$, $\tilde{\varphi} = (\varphi, \varphi)$

$\varphi_* \neq id.$


φ|L ∈ Diff0(L)
Exple.
π
S²
Z₂
g|
S¹
Ham(M, ω)
⊃ HamL(M, ω)
= {φ | φ(L) = L}
φ|L : L → L

$\varphi|_L \notin \mathrm{Diff}_0(L)$

Exple.

S^2

$(\varphi|_L)_* = id$ in $\mathbb{Z}_2$

$\mathbb{Z}_2$

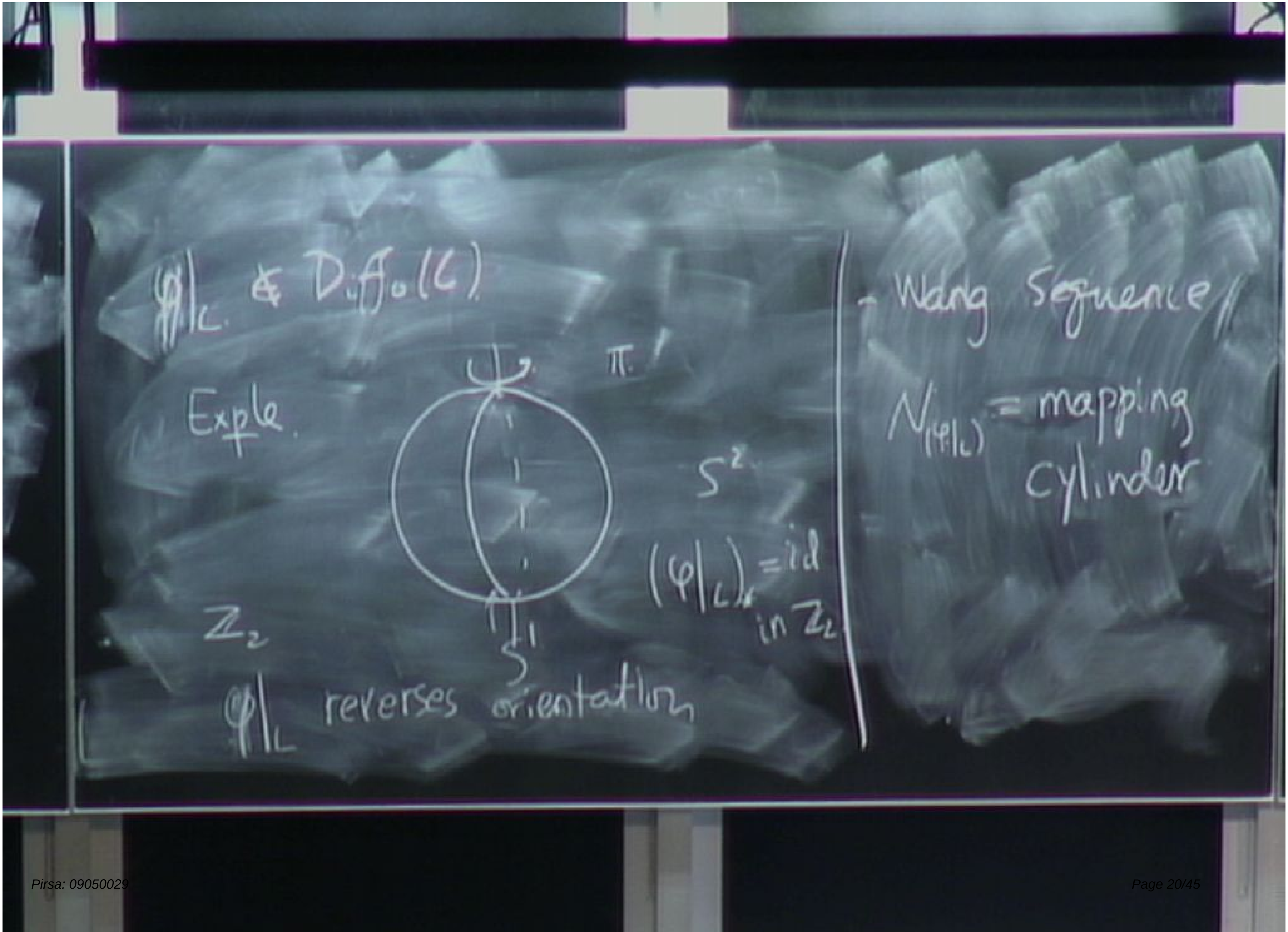
$\varphi|_L$ reverses orientation

$-\ \mathrm{Ham}(M, \omega)$

$\supset \mathrm{Ham}_L(M, \omega)$

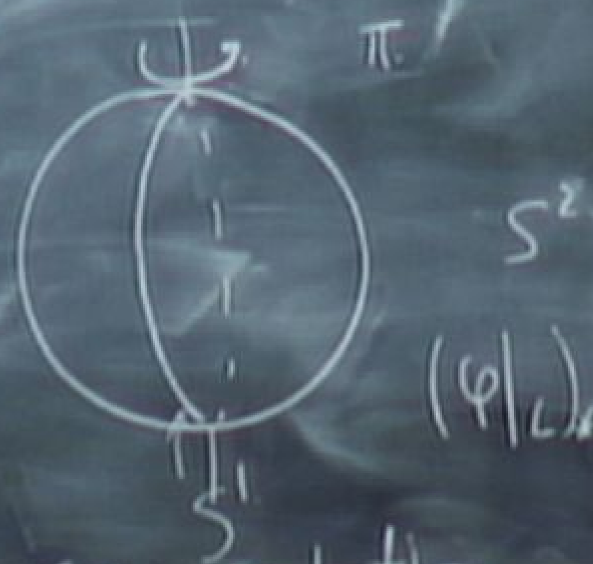
$= \{\varphi \mid \varphi(L) = L\}$

$\varphi|_L : L \to L$


$\varphi|_L \in \mathrm{Diff}_0(L)$.
Exple.
π.
S^2
$(\varphi|_L)_* = id$ in $\mathbb{Z}_2$
$\mathbb{Z}_2$
S^1
$\varphi|_L$ reverses orientation
Wang Sequence
$N_{(\varphi|_L)}$ = mapping cylinder

$\varphi|_L \notin \mathrm{Diff}_0(L)$

Exple.



$\mathbb{Z}_2$

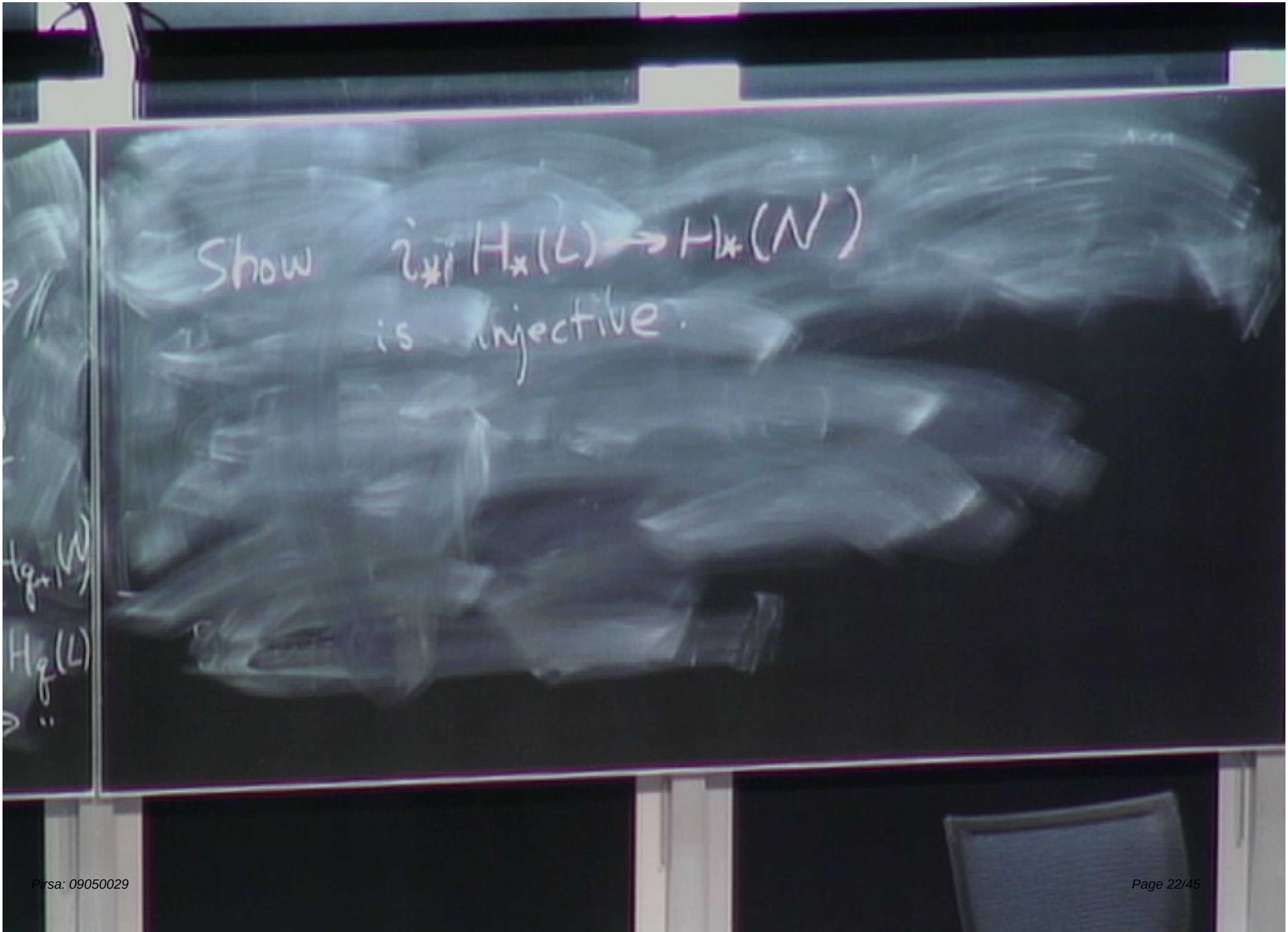
$(\varphi|_L)_* = id$ in $\mathbb{Z}_2$

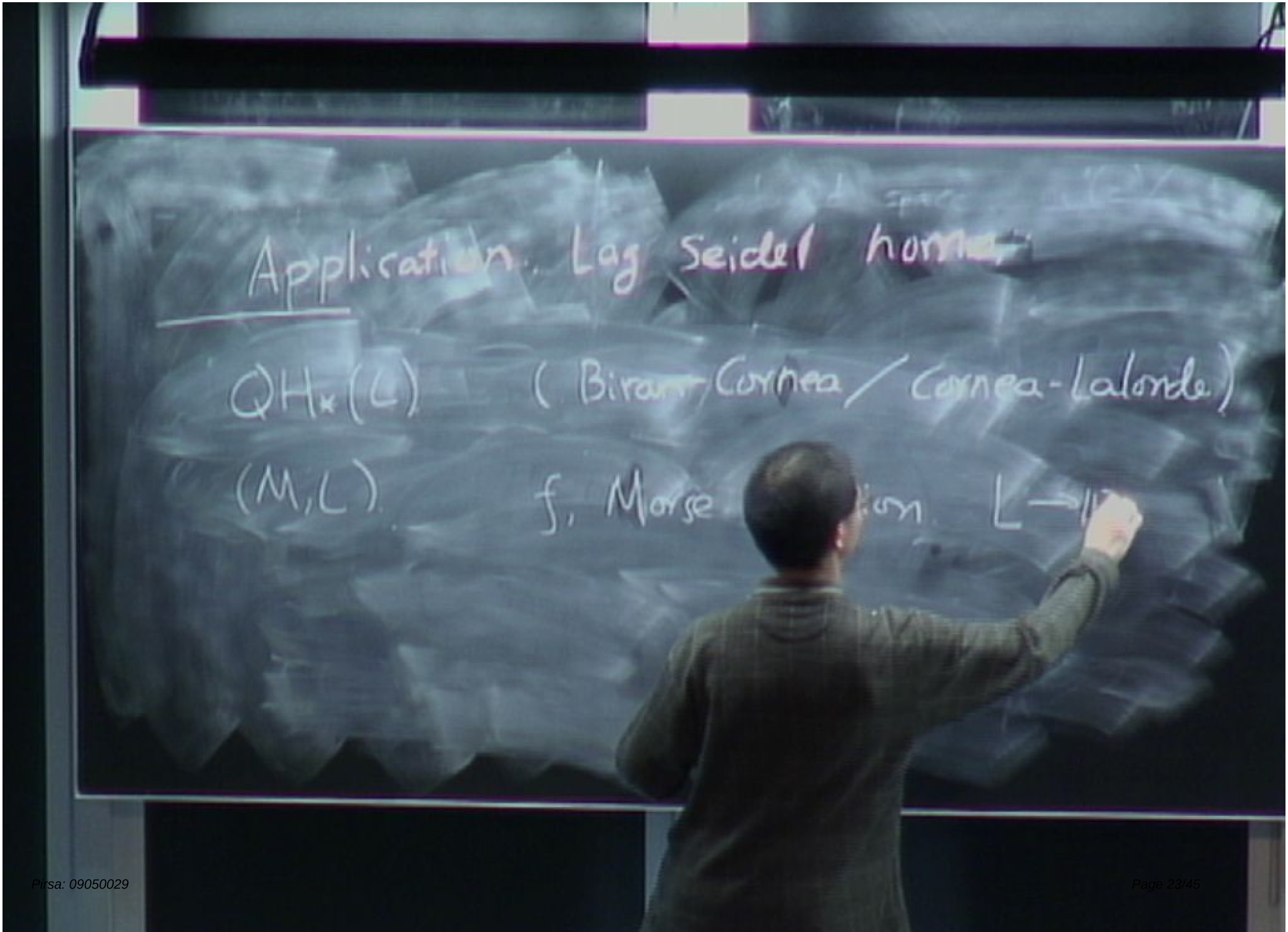
$\varphi|_L$ reverses orientation

- Wang Sequence

$N_{(\varphi|_L)}$ = mapping cylinder

$$\to H_{q+1}(L) \xrightarrow{i_*} H_{q+1}(N) \to H_q(L) \xrightarrow{(\varphi|_L)_* - id} H_q(L) \boxed{\xrightarrow{i_*}} H_q(N) \to \cdots$$


Show $i_*: H_*(L) \to H_*(N)$
is injective.


Application. Lag Seidel homo
QH*(L) (Biran-Cornea / Cornea-Lalonde)
(M,L).
f. Morse
L

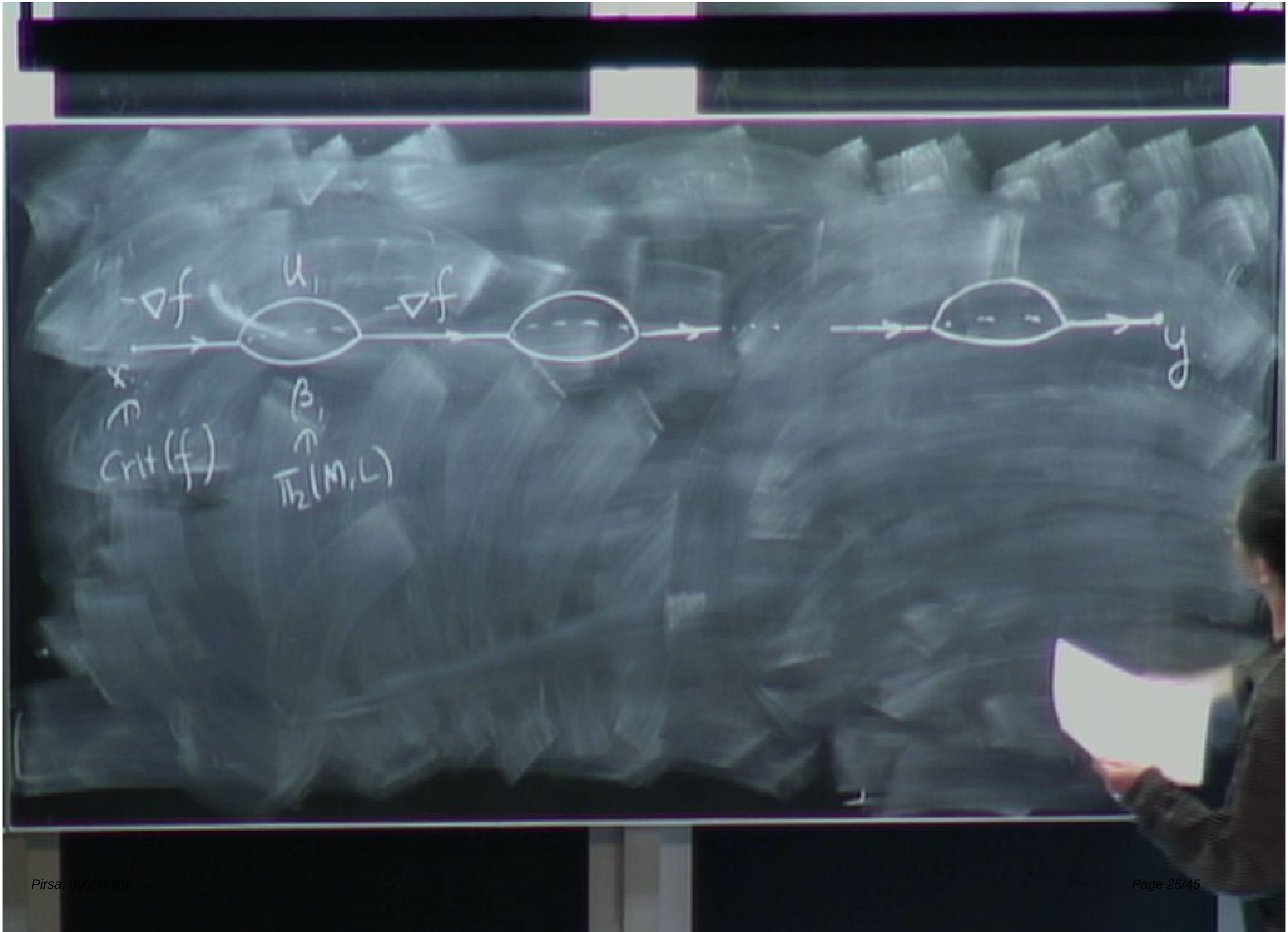
Application. Lag Seidel homom.

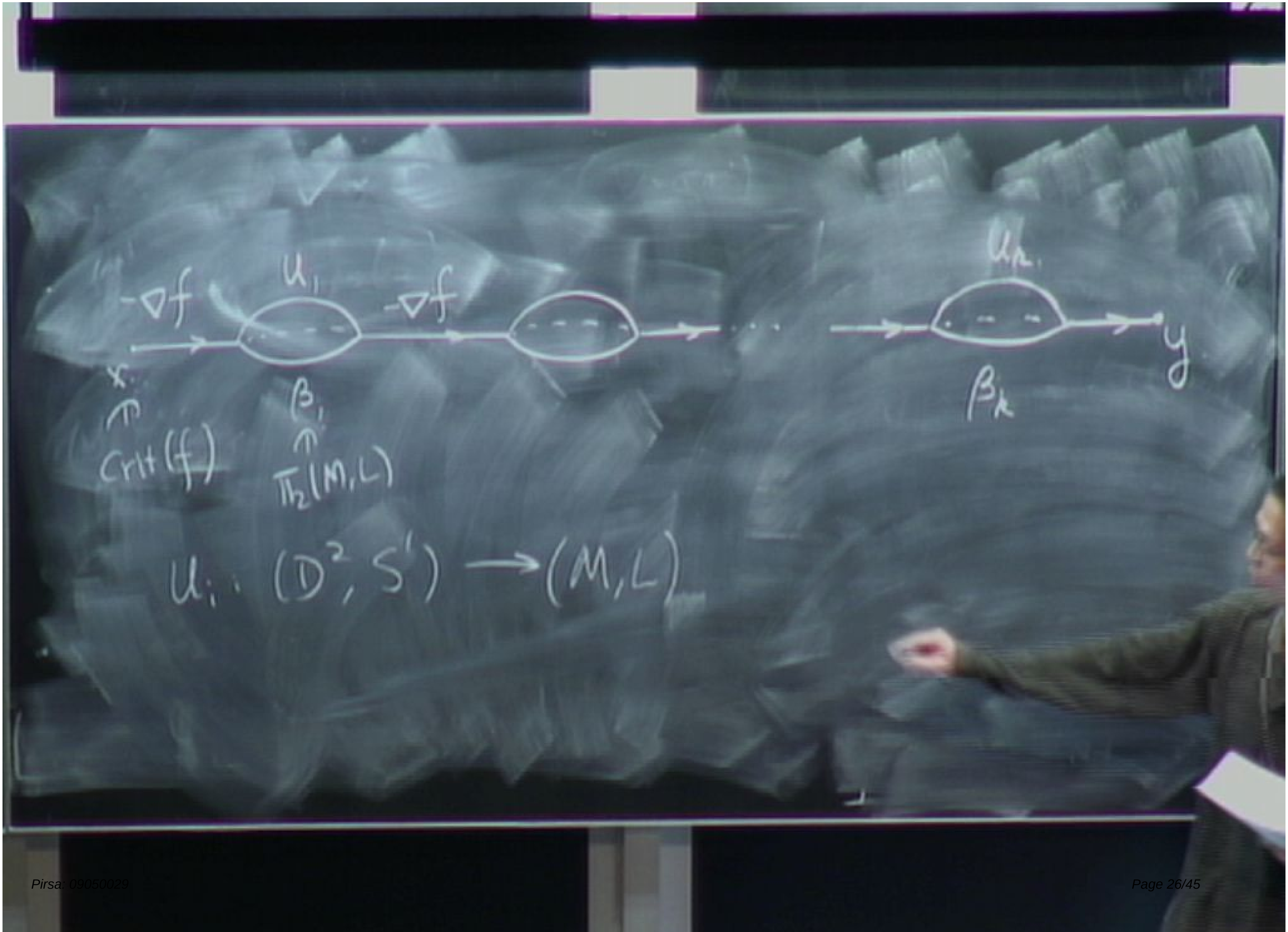
$QH_*(L)$ (Biran-Cornea / Cornea-Lalonde)

(M, L). f: Morse function $L \to \mathbb{R}$.

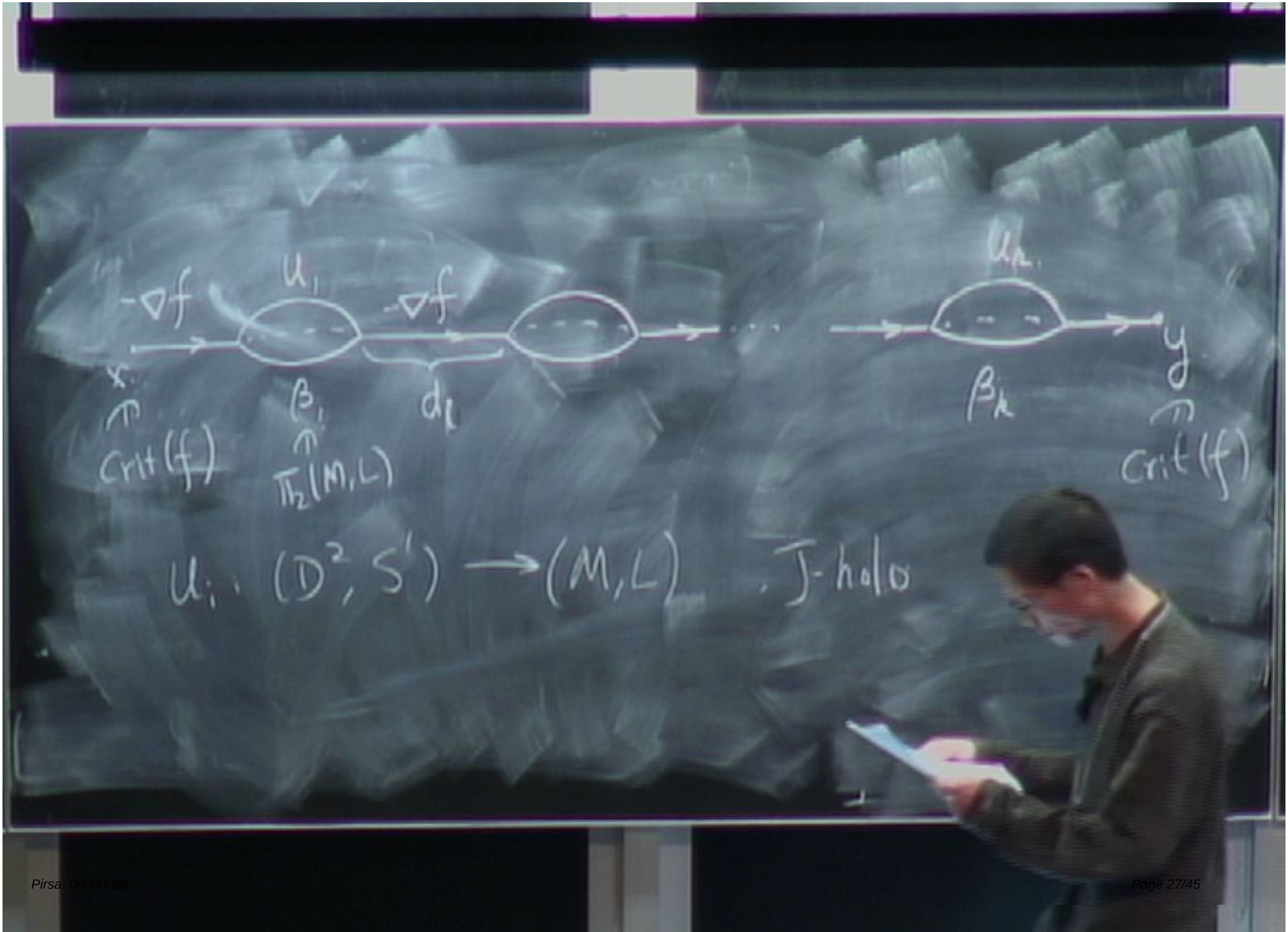
ρ: metric on L

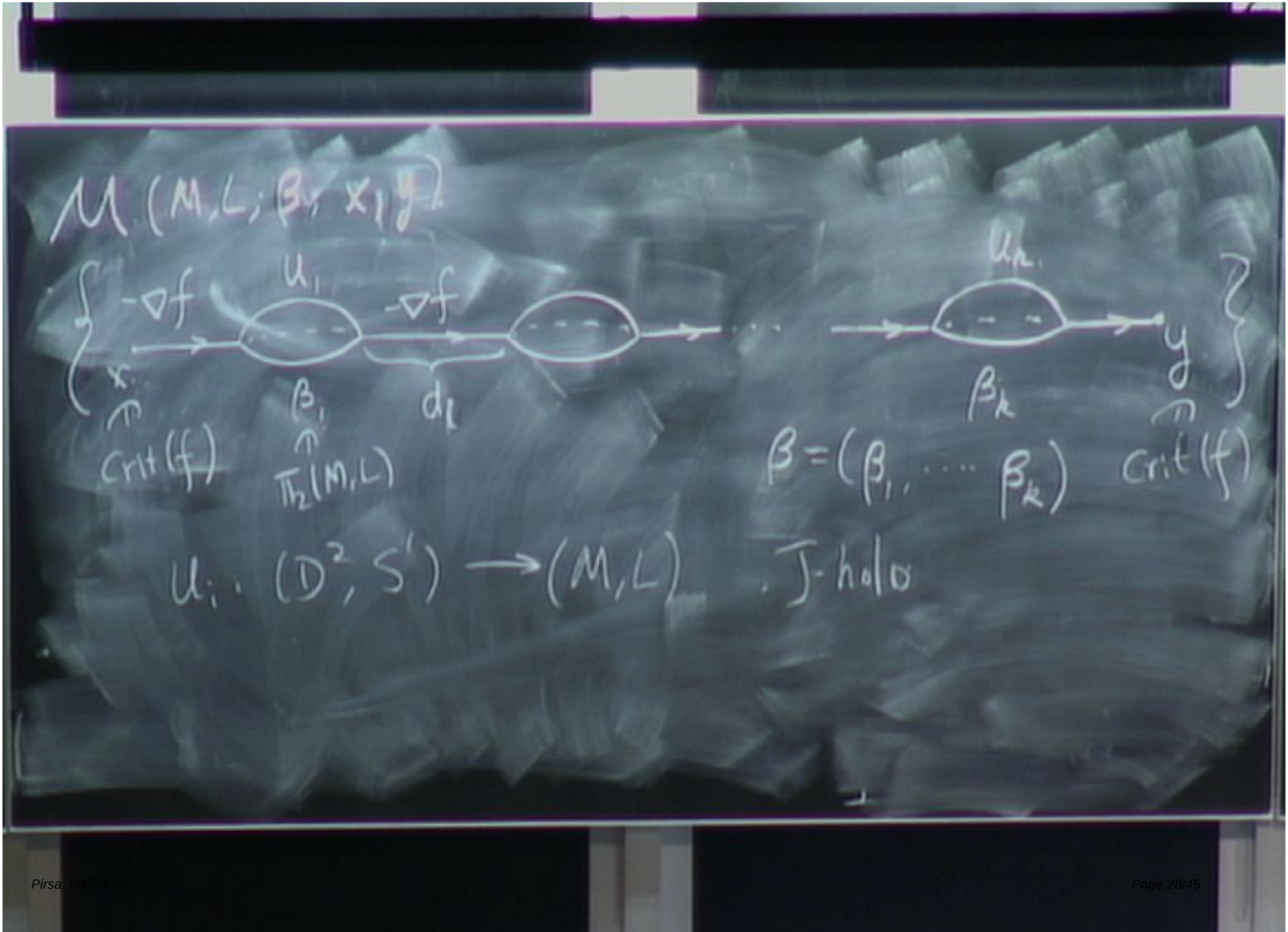
J: ω-compatible a.c.s. on M.

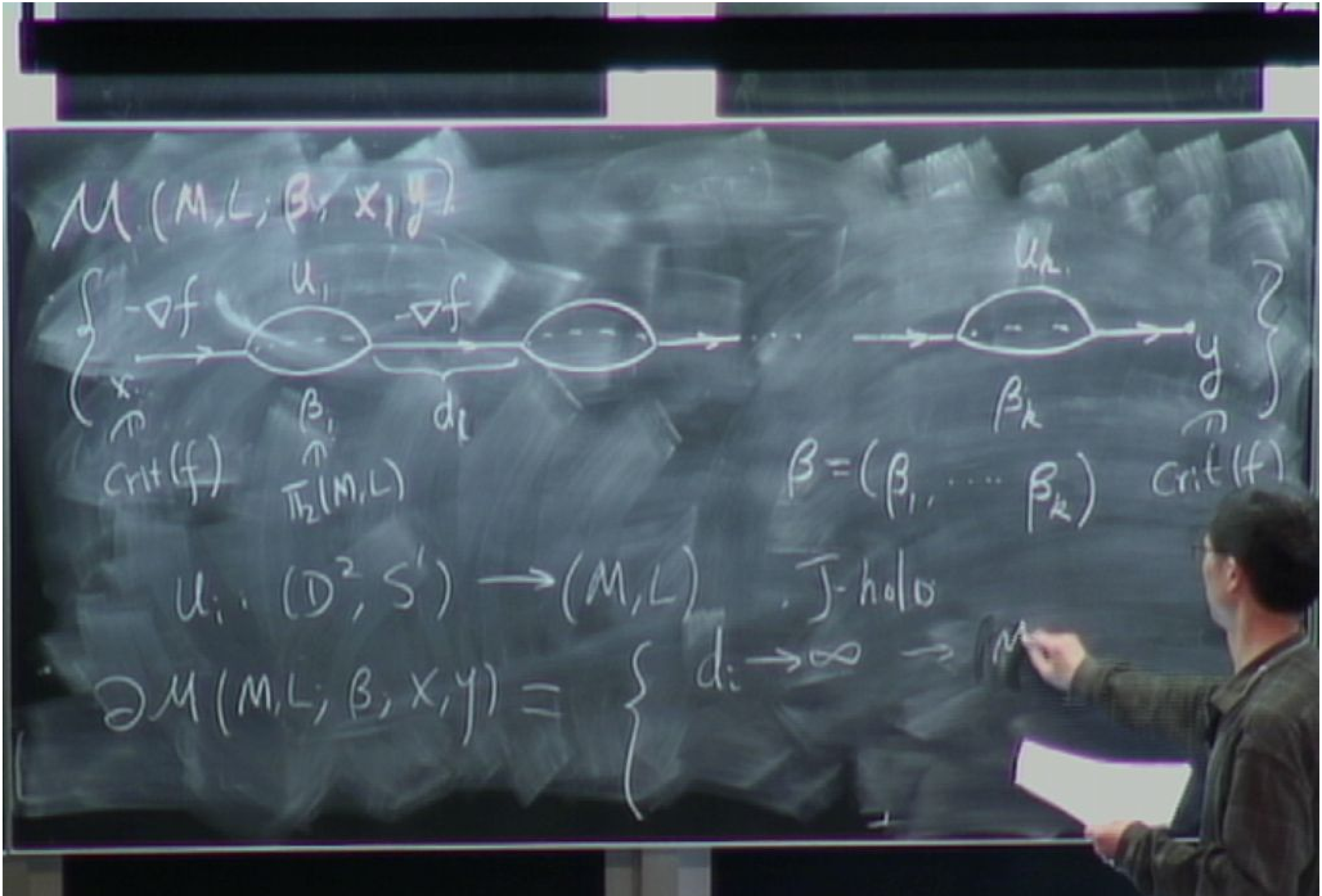

-∇f
u₁
-∇f
x
Crit(f)
β₁
π₂(M,L)
y



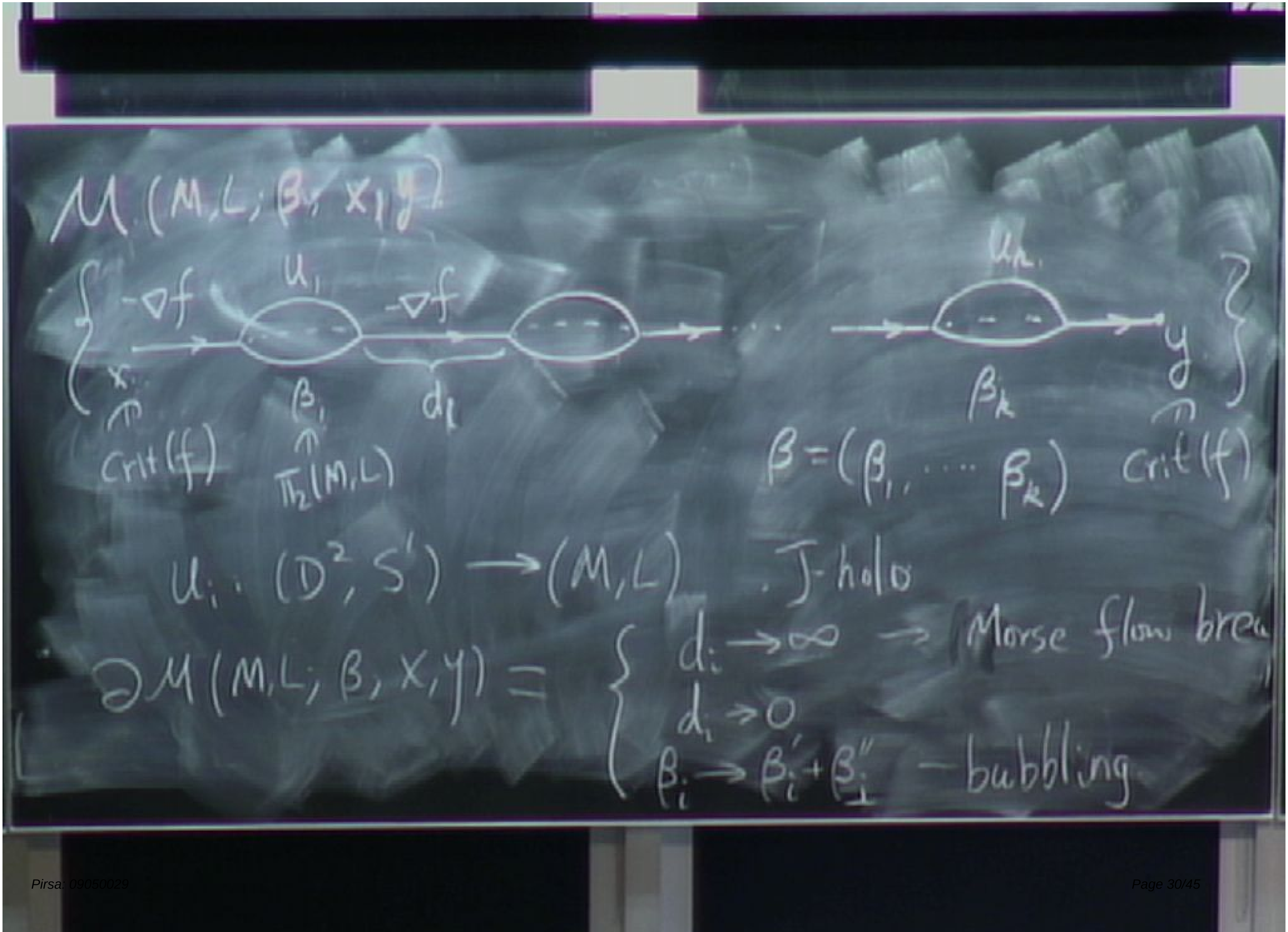
$-\nabla f$
u_1
$-\nabla f$
u_k
x
$\Uparrow$
Crit(f)
β_1
$\Uparrow$
$\pi_2(M,L)$
β_k
y
$u_i : (D^2, S^1) \longrightarrow (M, L)$


-∇f
u₁
-∇f
u_k
x
y
β₁
d_ℓ
β_k
Crit(f)
π₂(M,L)
Crit(f)
uᵢ : (D², S¹) → (M,L) J-holo


$\mathcal{M}(M,L;\beta;x,y)$
$-\nabla f$
u_1
$-\nabla f$
u_k
x
y
β_1
d_1
β_k
Crit(f)
$\pi_2(M,L)$
$\beta=(\beta_1,\dots,\beta_k)$
Crit(f)
$u_i:(D^2,S^1)\longrightarrow(M,L)$ J-holo



M (M, L; β, x, y)
−∇f
u1
−∇f
uk
x
β1
d1
βk
y
Crit(f)
π2(M, L)
β = (β1, ..., βk)
Crit(f)
ui : (D², S¹) → (M, L)
J-holo
∂M (M, L; β, x, y) = {
di → ∞ →

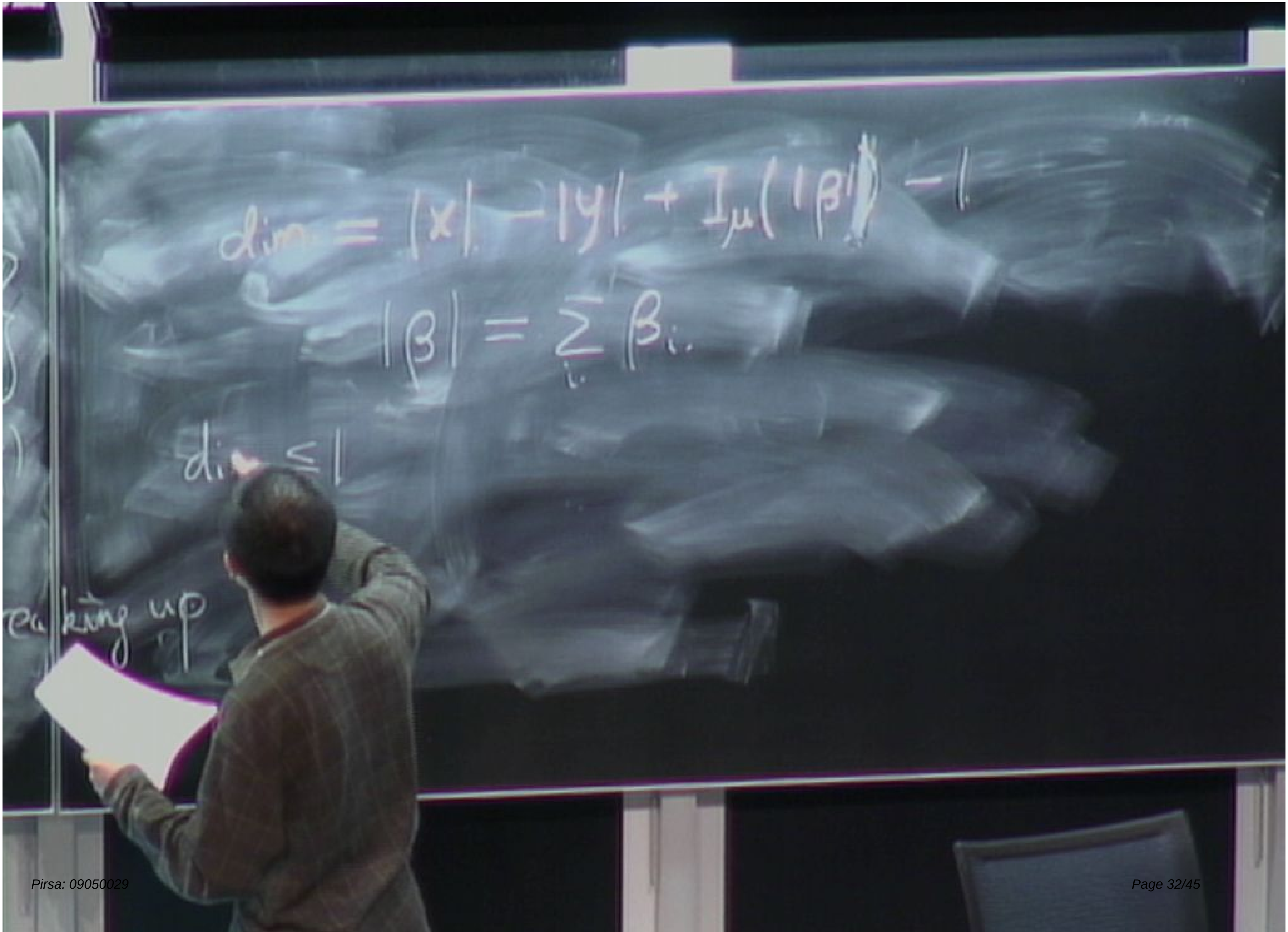

$\mathcal{M}(M,L;\beta,x,y)$
$\{ \; -\nabla f \quad u_1 \quad -\nabla f \quad \cdots \quad u_k \; \}$
$x \quad \beta_1 \quad d_1 \quad \beta_k \quad y$
$x \in \mathrm{Crit}(f)$, $\beta_1 \in \pi_2(M,L)$, $y \in \mathrm{Crit}(f)$
$\beta = (\beta_1, \cdots \beta_k)$
$u_i : (D^2, S^1) \longrightarrow (M,L)$ J-holo
$\partial \mathcal{M}(M,L;\beta,x,y) = \begin{cases} d_i \to \infty \to \text{Morse flow brea} \\ d_i \to 0 \\ \beta_i \to \beta_i' + \beta_i'' \text{ — bubbling} \end{cases}$

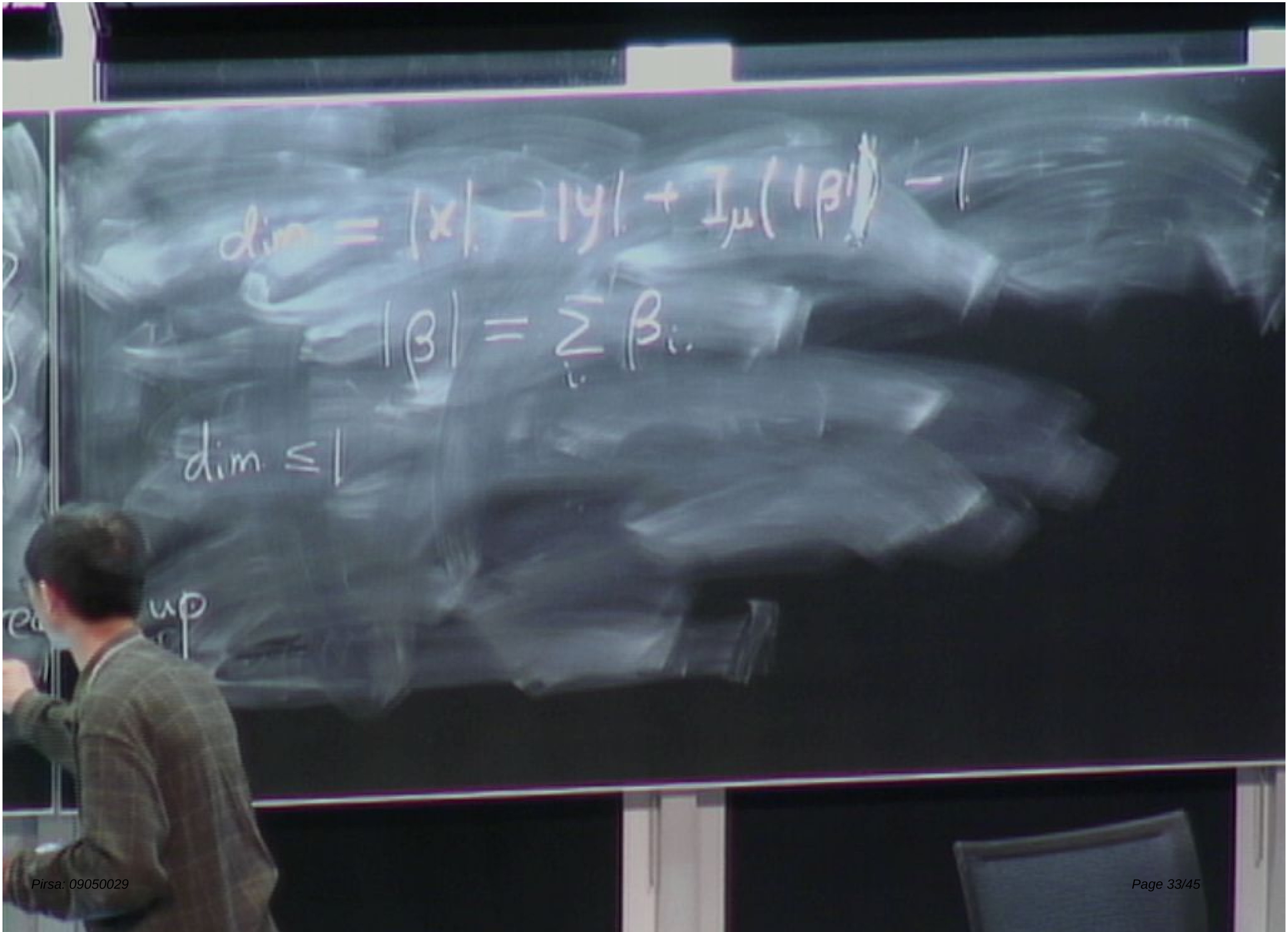
$\mathcal{M}(M, L; \beta, x, y)$

$\beta = (\beta_1, \dots, \beta_k)$

$u_i : (D^2, S^1) \longrightarrow (M, L)$ J-holo

$$\partial\mathcal{M}(M,L;\beta,x,y) = \begin{cases} d_i \to \infty & \to \text{Morse flow brea} \\ d_i \to 0 & \\ \beta_i \to \beta_i' + \beta_i'' & - \text{bubbling} \end{cases}$$


dim = |x| - |y| + I_μ(|β|) - 1
|β| = Σ_i β_i
dim ≤ 1
aking up


dim = |x| − |y| + I_μ(|β|) − 1
|β| = Σ_i β_i
dim ≤ 1
up

$$\dim = |x| - |y| + I_\mu(|\beta|) - 1$$

$$|\beta| = \sum_i \beta_i$$

$$\dim \leq 1$$

$$\mathcal{M}(M, L; B; x, y)$$

ea king up

$$= \bigcup_{|\beta| = B} \mathcal{M}(M, L; \beta; x, y)$$

Application: Lag Seidel homo...

$QH_*(L)$ (Biran-Cornea / Cornea-Lalonde)

(M,L). $\langle Crit(f) \rangle \otimes \Lambda_L$.

$$\partial_{pearl}\, x = \sum_{B,y} \# \mathcal{M}(M,L;B;x,y)\, e^{B} y$$

$$\partial^2_{pearl} = 0$$

Application: Lag Seidel homm

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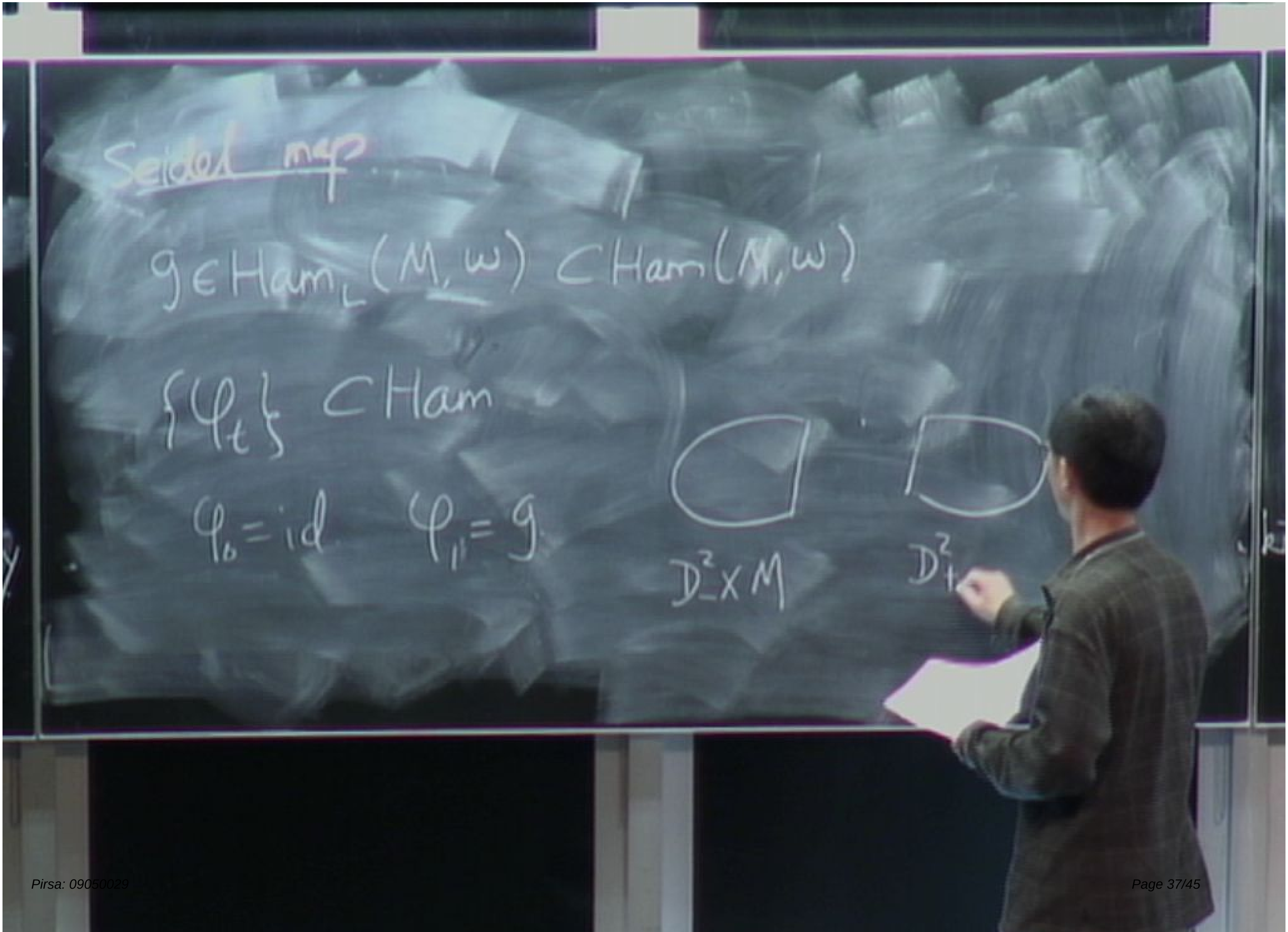
Seidel map
$g \in \mathrm{Ham}_L(M, \omega) \subset \mathrm{Ham}(M, \omega)$
$\{\varphi_t\} \subset \mathrm{Ham}$
$\varphi_0 = id$
$\varphi_1 = g$
$D^2_- \times M$
D^2_+

Seidel map.

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$\{\varphi_t\} \subset \mathrm{Ham}$

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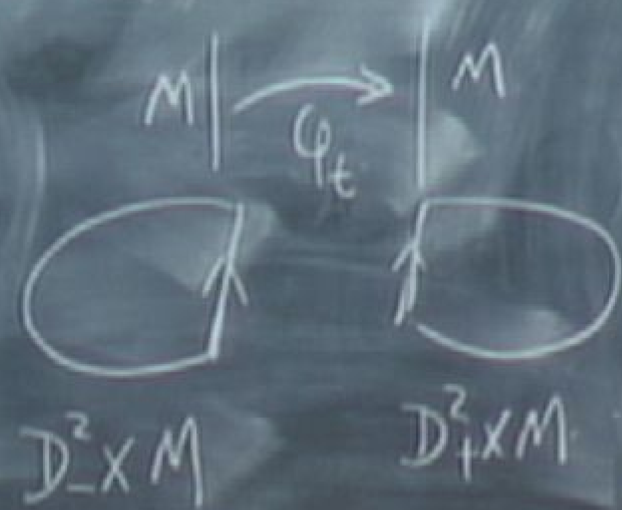
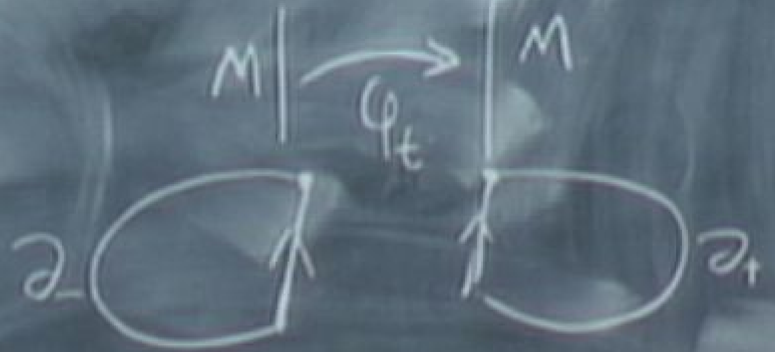


Seidel map

$g \in Ham_L(M,\omega) \subset Ham(M,\omega)$

$\{\varphi_t\} \subset Ham$

$\varphi_0 = id \qquad \varphi_1 = g$



$D^2_- \times M \qquad D^2_+ \times M$

$\partial_- \times L \qquad \partial_+ \times L$

M

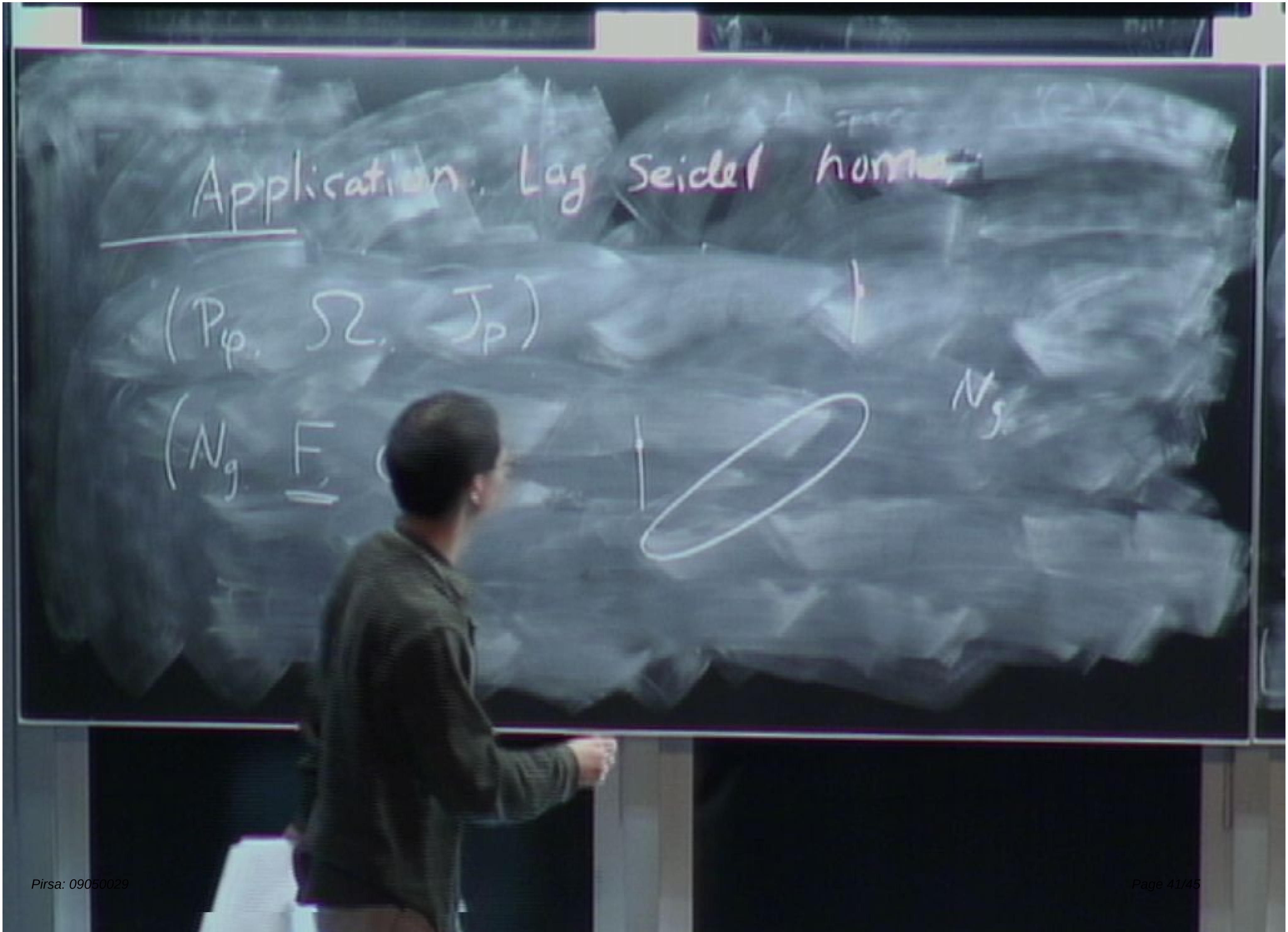
D^2 S^1

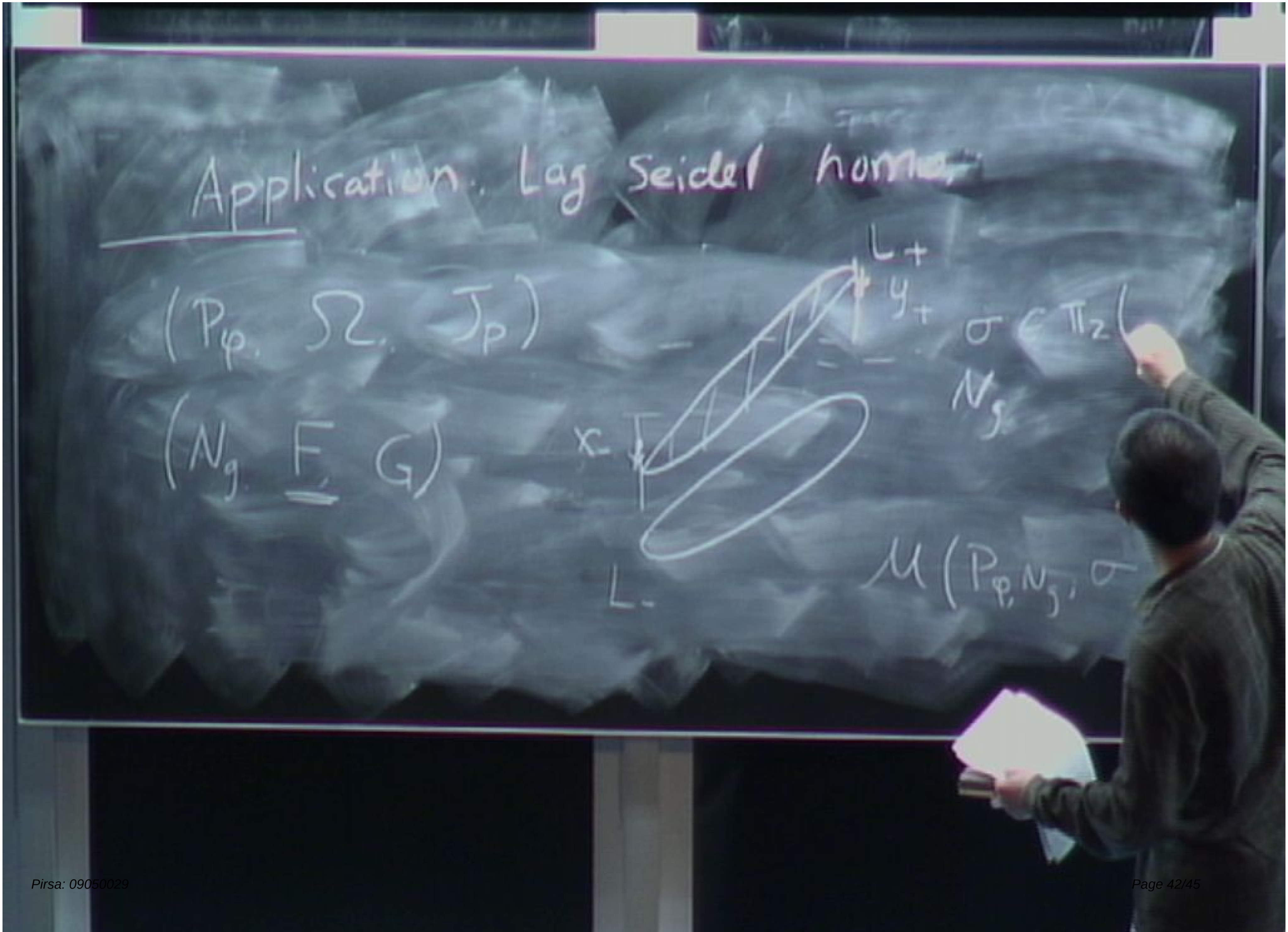
P_φ

$N_g \to S^1$

Look at section classes.

in $\pi_2(P_\varphi, N_g)$

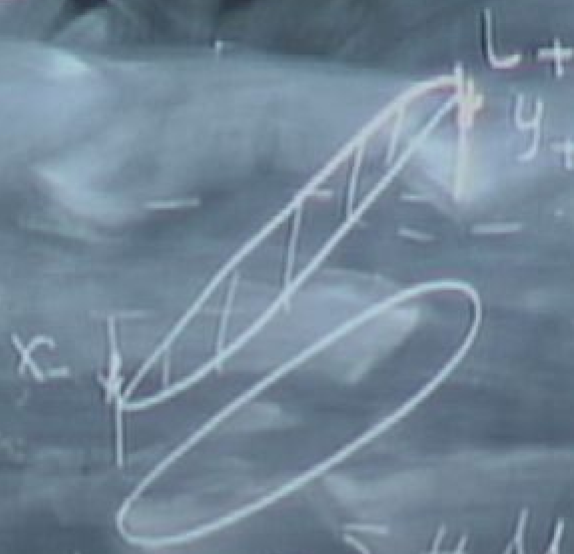

Application. Lag Seidel
(P_φ, Ω, J_p)
(N_g, F,
N_g


Application: Lag Seidel homo
(P_φ, Ω, J_P)
(N_g, F, G)
L_+
y_+
x_-
L_-
σ ∈ π_2(
N_g
M(P_φ, N_g, σ

Application: Lag Seidel homo...

(P_φ, Ω, J_P)

(N_g, F, G)



$\sigma \in \pi_2(P_\varphi, N_g)$

N_g

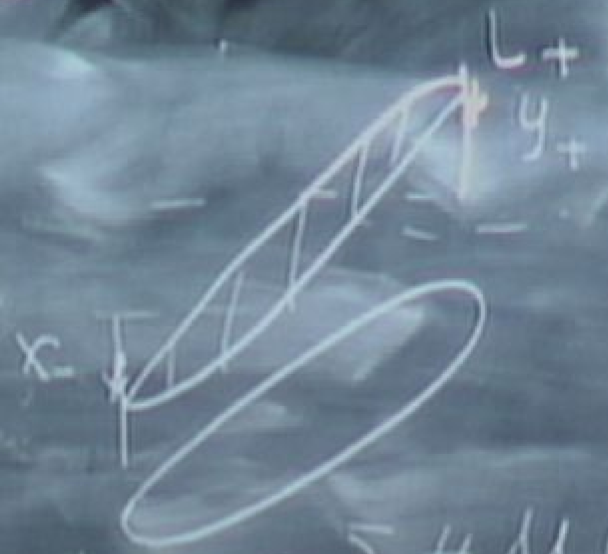
$\sigma_0 + B$

$L_- \quad \Phi_1(x_-) = \sum \# \mathcal{M}(P_\varphi, N_g, \overset{\sigma_0+B}{\overset{\|}{\sigma}}; x_-, y)$

Application: Lag Seidel homo.

(P_φ, Ω, J_p)

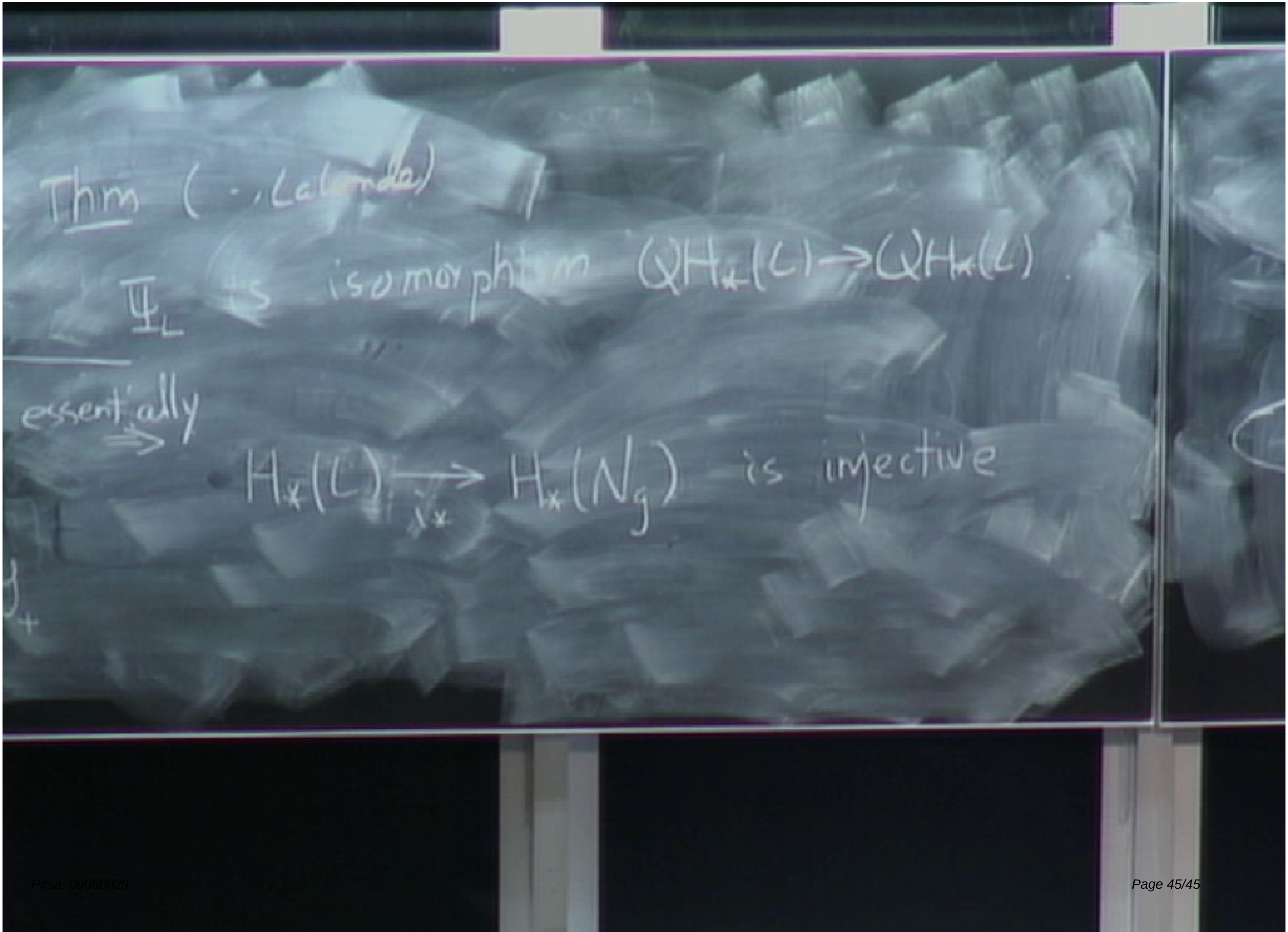
$(N_g, \underline{F}, G)$



$\sigma \in \pi_2(P_\varphi, N_g)$

N_g

$\Phi_L(x_-) = \sum \# \mathcal{M}(P_\varphi, N_g, \overset{\sigma_0 + B}{\overset{\parallel}{\sigma}}, x_-, y_+) e$


Thm (· , Lalonde)
Ψ_L is isomorphism $QH_*(L) \to QH_*(L)$
essentially $\Rightarrow$
$H_*(L) \xrightarrow[i_*]{} H_*(N_g)$ is injective