Title: Mirror Symmetry for Blow Ups

Date: May 08, 2009 10:00 AM

URL: http://pirsa.org/09050028

Abstract: This talk is a report on joint work with Mohammed Abouzaid and Ludmil Katzarkov about mirror symmetry for blowups, from the perspective of the Strominger-Yau-Zaslow conjecture. Namely, we first describe how to construct a Lagrangian torus fibration on the blowup of a toric variety X along a codimension 2 subvariety S contained in a toric hypersurface. Then we discuss the SYZ mirror and its instanton corrections, to provide an explicit description of the mirror Landau-Ginzburg model (possibly up to higher order corrections to the superpotential). This construction allows one to recover geometrically the predicted mirrors in various interesting settings: pairs of pants, curves of arbitrary genus, etc.

Mirror symmetry for blowups

Denis Auroux

MIT

May 2009 - Connections in Geometry and Physics
Perimeter Institute, Waterloo

arXiv:0706.3207 and 0902.1595 + work in progress w/ M. Abouzaid and L. Katzarkov

Mirror symmetry for Calabi-Yau manifolds

Symplectic geometry (**A**)	Complex geometry (**B**)
(X, J, ω, Ω) Calabi-Yau	$(X^\vee, J^\vee, \omega^\vee, \Omega^\vee)$ Calabi-Yau
Gromov-Witten invariants	Hodge theory
Lagrangian submanifolds	Analytic cycles
Fukaya category	Derived category of coherent sheaves

- **Calabi-Yau:** (X, J) complex manifold, ω compatible Kähler form
  $K_X = \Omega^{n,0} \simeq \mathcal{O}_X$, $\Omega \in \Omega^{n,0}(X)$ holomorphic volume form
  example: $\mathbb{C}^n$, J_0, $\omega_0 = \frac{i}{2} \sum dz_j \wedge d\bar{z}_j$, $\Omega_0 = dz_1 \wedge \cdots \wedge dz_n$
  (do not require $|\Omega|^2 \sim \omega^n$ or $Ric = 0$: "almost Calabi-Yau")
- symplectic data ω determines complex data $J^\vee, \Omega^\vee$ and vice-versa

How to construct a mirror?

The Strominger-Yau-Zaslow conjecture

SYZ conjecture

X, $X^\vee$ are dual fibrations by special Lagrangian tori over a base carrying an integral affine structure.*

* special Lagrangian fibrations are hard to come by; singularities induce "instanton corrections" on the mirror; ...

($\Rightarrow$ Joyce, Fukaya, Kontsevich-Soibelman, Gross-Siebert, ...)

The Strominger-Yau-Zaslow conjecture

SYZ conjecture

X, $X^\vee$ are dual fibrations by special Lagrangian tori over a base carrying an integral affine structure.*

* special Lagrangian fibrations are hard to come by; singularities induce "instanton corrections" on the mirror; ...

($\Rightarrow$ Joyce, Fukaya, Kontsevich-Soibelman, Gross-Siebert, ...)

SYZ from homological mirror symmetry

For each point $p \in X^\vee$, $\mathcal{O}_p \in D^b Coh(X^\vee) \longleftrightarrow \mathcal{L} \in \mathcal{F}(X)$.

$$HF(\mathcal{L}, \mathcal{L}) \simeq Ext^*(\mathcal{O}_p, \mathcal{O}_p) \simeq H^*(T^n; \mathbb{C})$$

$\Rightarrow$ points of $X^\vee \xleftrightarrow{(?)}$ Lagrangian tori in X ($+\ U(1)$ local system)

Landau-Ginzburg models

X Kähler ($c_1(X) \neq 0$), D anticanonical divisor $\Rightarrow X \setminus D$ open Calabi-Yau.

$\Rightarrow M =$ open CY mirror to $X \setminus D$. What about X?

$L \subset X \setminus D$ special Lagrangian bounds holomorphic discs in X

$\Rightarrow$ Floer homology is obstructed ($\partial^2 = m_0 \cdot \mathrm{id}$) and generically trivial

The mirror of X is a Landau-Ginzburg model $W : M \to \mathbb{C}$

(M noncompact CY mirror to $X \setminus D$, W = *superpotential*, holomorphic)

The superpotential modifies interpretation of A and B models on M, e.g.

Symplectic geometry of X $\leftrightarrow$ complex geometry of **singularities** of W

Landau-Ginzburg models

X Kähler ($c_1(X) \neq 0$), D anticanonical divisor $\Rightarrow X \setminus D$ open Calabi-Yau.

$\Rightarrow M =$ open CY mirror to $X \setminus D$. What about X?

$L \subset X \setminus D$ special Lagrangian bounds holomorphic discs in X
$\Rightarrow$ Floer homology is obstructed ($\partial^2 = \{m_0, \cdot\}$) and generically trivial.

The mirror of X is a Landau-Ginzburg model $W : M \to \mathbb{C}$
(M noncompact CY mirror to $X \setminus D$, W = *superpotential*, holomorphic)

The superpotential modifies interpretation of A and B models on M, e.g.

Symplectic geometry of X = complex geometry of **singularities** of W

Landau-Ginzburg models

X Kähler ($c_1(X) \neq 0$), D anticanonical divisor $\Rightarrow X \setminus D$ open Calabi-Yau.

$\Rightarrow M =$ open CY mirror to $X \setminus D$. What about X?

$L \subset X \setminus D$ special Lagrangian bounds holomorphic discs in X
$\Rightarrow$ Floer homology is obstructed ($\partial^2 = \{m_0, \cdot\}$) and generically trivial.

The mirror of X is a Landau-Ginzburg model $W : M \to \mathbb{C}$
(M noncompact CY mirror to $X \setminus D$; $W =$ *superpotential*, holomorphic)

The superpotential modifies interpretation of A and B models on M: e.g.,

Symplectic geometry of $X \Leftrightarrow$ complex geometry of **singularities** of W

A rough conjecture (SYZ for $-K_X$ effective)

Conjecture

(X, ω, J) compact Kähler manifold, $D \subset X$ anticanonical divisor, $\Omega \in \Omega^{n,0}(X \setminus D) \Rightarrow$ can construct a mirror as

- $M =$ moduli space of special Lagrangian tori $L \subset X \setminus D$ + flat $U(1)$ connections on trivial bundle over L
- $W : M \to \mathbb{C}$ counts holomorphic discs of Maslov index 2 in (X, L) (Fukaya-Oh-Ohta-Ono's m_0 obstruction in Floer homology)
- the fiber of W is mirror to D.

Conjecture doesn't quite hold as stated, because:

- W presents wall-crossing discontinuities caused by Maslov index 0 discs $\Rightarrow$ need "instanton corrections" to correct these discontinuities
- According to Hori-Vafa, need to enlarge M by "renormalization"

A rough conjecture (SYZ for $-K_X$ effective)

Conjecture

(X, ω, J) compact Kähler manifold, $D \subset X$ anticanonical divisor, $\Omega \in \Omega^{n,0}(X \setminus D) \Rightarrow$ can construct a mirror as

- $M =$ moduli space of special Lagrangian tori $L \subset X \setminus D$
  $+$ flat $U(1)$ connections on trivial bundle over L
- $W : M \to \mathbb{C}$ counts holomorphic discs of Maslov index 2 in (X, L)
  (Fukaya-Oh-Ohta-Ono's m_0 obstruction in Floer homology)
- the fiber of W is mirror to D.

Conjecture doesn't quite hold as stated, because:

- W presents wall-crossing discontinuities caused by Maslov index 0 discs $\Rightarrow$ need "instanton corrections" to correct these discontinuities.
- According to Hori-Vafa, need to enlarge M by "renormalization".

Special Lagrangians

(X, ω, J) compact Kähler manifold, $\dim_{\mathbb{C}} X = n$; $D = \sigma^{-1}(0) \in |K_X^{-1}|$.
$\Omega = \sigma^{-1} \in \Omega^{n,0}(X \setminus D)$, $\psi = |\Omega|_g \in C^\infty(X \setminus D, \mathbb{R}_+)$.

$L^n \subset X \setminus D$ is *special Lagrangian* if $\omega_{|L} = 0$ and $\mathrm{Im}(\Omega)_{|L} = 0$.

Special Lagrangians

(X, ω, J) compact Kähler manifold, $\dim_{\mathbb{C}} X = n$; $D = \sigma^{-1}(0) \in |K_X^{-1}|$.
$\Omega = \sigma^{-1} \in \Omega^{n,0}(X \setminus D)$, $\psi = |\Omega|_g \in C^\infty(X \setminus D, \mathbb{R}_+)$.

$L^n \subset X \setminus D$ is *special Lagrangian* if $\omega_{|L} = 0$ and $\mathrm{Im}(\Omega)_{|L} = 0$.

Proposition (McLean, Joyce)

Special Lagrangian deformations $= \mathcal{H}^1_\psi(L)$ $(\simeq H^1(L, \mathbb{R}))$, *unobstructed.*

$\mathcal{H}^1_\psi(L) = \{\theta \in \Omega^1(L, \mathbb{R}) \mid d\theta = 0,\ d^*(\psi\theta) = 0\}$ "ψ-harmonic" 1-forms

$v \in C^\infty(NL)$ is SLag iff $-\iota_v\omega = \theta$ and $\iota_v \mathrm{Im}(\Omega) = \psi * \theta$ are closed.

Special Lagrangians

(X, ω, J) compact Kähler manifold, $\dim_{\mathbb{C}} X = n$; $D = \sigma^{-1}(0) \in |K_X^{-1}|$.
$\Omega = \sigma^{-1} \in \Omega^{n,0}(X \setminus D)$, $\psi = |\Omega|_g \in C^\infty(X \setminus D, \mathbb{R}_+)$.

$L^n \subset X \setminus D$ is *special Lagrangian* if $\omega_{|L} = 0$ and $\mathrm{Im}(\Omega)_{|L} = 0$.

Proposition (McLean, Joyce)

Special Lagrangian deformations $= \mathcal{H}^1_\psi(L)$ $(\simeq H^1(L, \mathbb{R}))$, *unobstructed.*

$\mathcal{H}^1_\psi(L) = \{\theta \in \Omega^1(L, \mathbb{R}) \mid d\theta = 0,\ d^*(\psi\theta) = 0\}$ "ψ-harmonic" 1-forms

$v \in C^\infty(NL)$ is SLag iff $-\iota_v\omega = \theta$ and $\iota_v \mathrm{Im}(\Omega) = \psi * \theta$ are closed.

Example

X smooth toric variety with moment map $\phi : X \to \mathbb{R}^n$, $\Delta = \phi(X)$.
$D = \phi^{-1}(\partial\Delta)$ toric divisor, $X \setminus D \simeq (\mathbb{C}^*)^n$, $\Omega = d\log x_1 \wedge \cdots \wedge d\log x_n$.

- Toric fibers (T^n-orbits) are special Lagrangian.

The geometry of the moduli space

Definition

$M = \{(L, \nabla) \mid L \subset X \setminus D$ special Lag. torus, ∇ flat $U(1)$ conn. on $\underline{\mathbb{C}} \to L\}$.

The geometry of the moduli space

Definition

$M = \{(L, \nabla) \mid L \subset X \setminus D$ special Lag. torus, ∇ flat $U(1)$ conn. on $\underline{\mathbb{C}} \to L\}$.

Proposition

- $T_{(L,\nabla)}M = \{(v, \alpha) \in C^\infty(NL) \oplus \Omega^1(L, \mathbb{R}) \mid -\iota_v\omega + i\alpha \in \mathcal{H}^1_\psi(L) \otimes \mathbb{C}\}$.
- *Complex structure $J^\vee$ on M; local holomorphic functions:*
  given $\beta \in H_2(X, L)$, $z_\beta = \exp(-\int_\beta \omega)\,\mathrm{hol}_{\partial\beta}(\nabla) : M \to \mathbb{C}^*$.
- *Compatible Kähler form*
  $\omega^\vee((v_1, \alpha_1), (v_2, \alpha_2)) = \int_L \alpha_2 \wedge \iota_{v_1} \mathrm{Im}\,\Omega - \alpha_1 \wedge \iota_{v_2} \mathrm{Im}\,\Omega$.
- *Holom. volume form*
  $\Omega^\vee((v_1, \alpha_1), \ldots, (v_n, \alpha_n)) = \int_L (-\iota_{v_1}\omega + i\alpha_1) \wedge \cdots \wedge (-\iota_{v_n}\omega + i\alpha_n)$.

$\Rightarrow$ Assuming ψ-harmonic 1-forms on L have no zeroes, X and M are dual special Lag. torus fibrations in a nbd. of L (the projection is $(L, \nabla) \to L$)

The geometry of the moduli space

Definition

$M = \{(L, \nabla) \mid L \subset X \setminus D$ special Lag. torus, ∇ flat $U(1)$ conn. on $\underline{\mathbb{C}} \to L\}$.

Proposition

- $T_{(L,\nabla)}M = \{(v, \alpha) \in C^\infty(NL) \oplus \Omega^1(L, \mathbb{R}) \mid -\iota_v\omega + i\alpha \in \mathcal{H}^1_\psi(L) \otimes \mathbb{C}\}$.
- *Complex structure* $J^\vee$ *on* M*; local holomorphic functions:*
  given $\beta \in H_2(X, L)$, $z_\beta = \exp(-\int_\beta \omega)\, \mathrm{hol}_{\partial\beta}(\nabla) : M \to \mathbb{C}^*$.
- *Compatible Kähler form*
  $$\omega^\vee((v_1, \alpha_1), (v_2, \alpha_2)) = \int_L \alpha_2 \wedge \iota_{v_1} \mathrm{Im}\,\Omega - \alpha_1 \wedge \iota_{v_2} \mathrm{Im}\,\Omega.$$
- *Holom. volume form*
  $$\Omega^\vee((v_1, \alpha_1), \dots, (v_n, \alpha_n)) = \int_L (-\iota_{v_1}\omega + i\alpha_1) \wedge \cdots \wedge (-\iota_{v_n}\omega + i\alpha_n).$$

$\Rightarrow$ Assuming ψ-harmonic 1-forms on L have no zeroes, X and M are dual special Lag. torus fibrations in a nbd. of L (the projection is $(L, \nabla) \mapsto L$).

The superpotential

$\beta \in \pi_2(X, L) \Rightarrow$ moduli space of holom. maps $u : (D^2, \partial D^2) \to (X, L)$ in class β, of virt. dim. $n - 3 + \mu(\beta)$, where $\mu(\beta) = 2\#(\beta \cap D)$ Maslov index.

Assumption

L does not bound any nonconstant Maslov index 0 holomorphic discs; Maslov index 2 discs are *regular*.

Then for $\mu(\beta) = 2$, can count holom. discs in class β whose boundary passes through a generic given point $p \in L \Rightarrow n_\beta(L) \in \mathbb{Z}$.

The superpotential

$\beta \in \pi_2(X, L) \Rightarrow$ moduli space of holom. maps $u : (D^2, \partial D^2) \to (X, L)$ in class β, of virt. dim. $n - 3 + \mu(\beta)$, where $\mu(\beta) = 2\#(\beta \cap D)$ Maslov index.

Assumption

L does not bound any nonconstant Maslov index 0 holomorphic discs; Maslov index 2 discs are *regular*.

Then for $\mu(\beta) = 2$, can count holom. discs in class β whose boundary passes through a generic given point $p \in L \Rightarrow n_\beta(L) \in \mathbb{Z}$.

Definition

$W(L, \nabla) = \sum_{\mu(\beta)=2} n_\beta(L)\, z_\beta(L, \nabla)$, where $z_\beta = \exp(-\int_\beta \omega)\, \mathrm{hol}_{\partial\beta}(\nabla)$.

By construction $W : M \to \mathbb{C}$ is holomorphic. (Convergence OK at least if X Fano)

The toric case (see also Hori, Cho-Oh, FO^3)

X smooth toric variety with moment map $\phi : X \to \mathbb{R}^n$, $\Delta = \phi(X)$.
$D = \phi^{-1}(\partial\Delta)$ toric divisor, $X \setminus D \simeq (\mathbb{C}^*)^n$, $\Omega = d\log x_1 \wedge \cdots \wedge d\log x_n$.

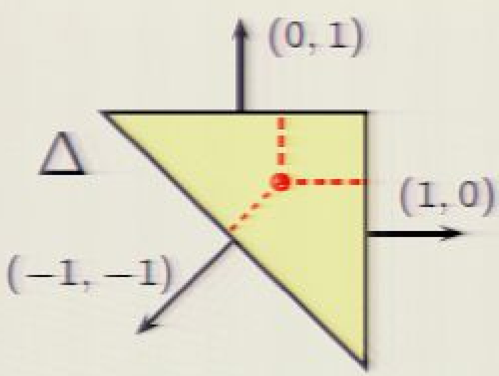
- Toric fibers (T^n-orbits) are special Lagrangian.
- $M \approx$ (domain in) $(\mathbb{C}^*)^n$.

The toric case (see also Hori, Cho-Oh, FO^3)

X smooth toric variety with moment map $\phi : X \to \mathbb{R}^n$, $\Delta = \phi(X)$.
$D = \phi^{-1}(\partial\Delta)$ toric divisor, $X \setminus D \simeq (\mathbb{C}^*)^n$, $\Omega = d\log x_1 \wedge \cdots \wedge d\log x_n$.

- Toric fibers (T^n-orbits) are special Lagrangian.
- $M \approx$ (domain in) $(\mathbb{C}^*)^n$.
- There are no Maslov index 0 discs; one family of Maslov index 2 discs for each facet F of Δ. Primitive outward normal: $\nu(F) \in \mathbb{Z}^n$.
- $W = \sum_{F \text{ facet}} e^{-2\pi\alpha(F)}\, z^{\nu(F)}$ where eqn. of F is $\langle \nu(F), \phi\rangle = \alpha(F)$.

Example: $X = \mathbb{CP}^2$

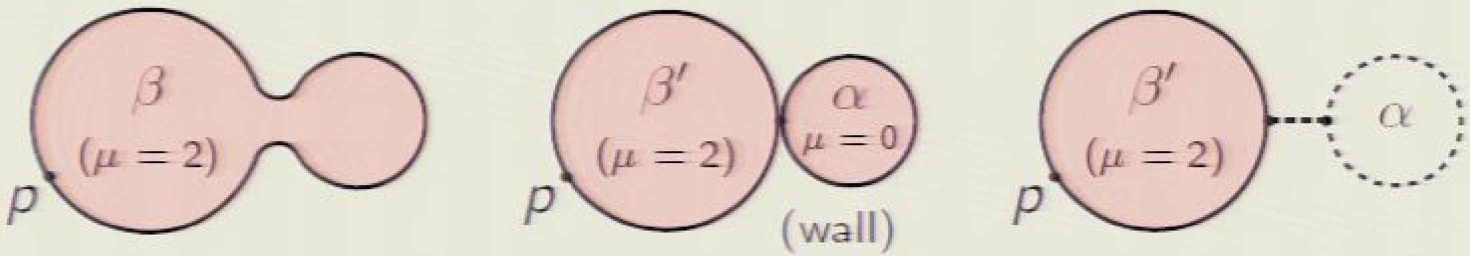


$L = S^1(r_1) \times S^1(r_2) \subset (\mathbb{C}^*)^2 \subset \mathbb{CP}^2$
bounds 3 families of $\mu = 2$ discs in $\mathbb{CP}^2$
(two of which are $D(r_1) \times \{x_2\}$ and $\{x_1\} \times D(r_2)$)

$$\Rightarrow \quad W = z_1 + z_2 + \frac{e^{-\Lambda}}{z_1 z_2} \quad (\Lambda = \textstyle\int_{\mathbb{CP}^1} \omega)$$

Maslov index 0 discs and wall-crossing

Bubbling of Maslov index 0 discs causes the disc count $n_\beta(L)$ to jump.



Typically, for $n \geq 3$ the disc count depends on $p \in L$ ($\Rightarrow W$ multivalued).
For $n = 2$ the disc count is independent of $p \in L$ but jumps where L bounds a Maslov index 0 disc ($\Rightarrow W$ discontinuous).

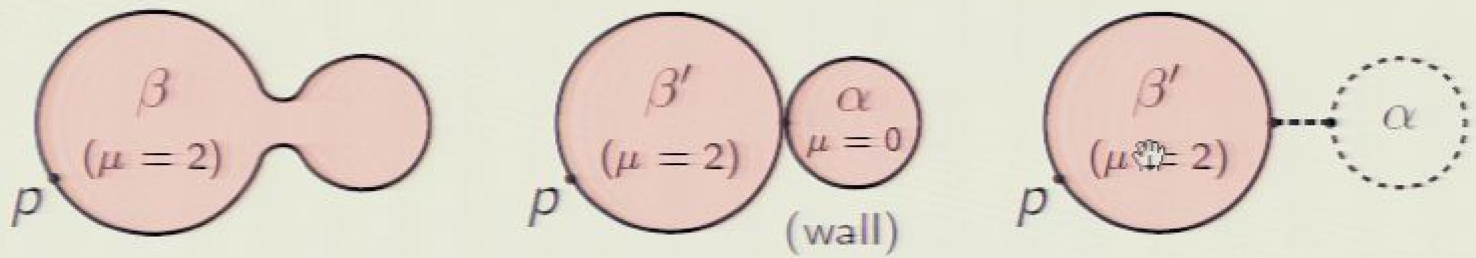
Proposition (Fukaya-Oh-Ohta-Ono + ε)

For $n = 2$, crossing a wall in which L bounds a single Maslov index 0 disc in a class α modifies W by a holomorphic substitution of variables $z_\beta \mapsto z_\beta h(z_\alpha)^{[\partial\beta]\cdot[\partial\alpha]}$ $\forall \beta \in \pi_2(X, L)$, where $h(z_\alpha) = 1 + O(z_\alpha) \in \mathbb{C}[[z_\alpha]]$

The mirror is obtained from W by gluing the various regions delimited by the walls according to these changes of variables (instanton corrections).

Maslov index 0 discs and wall-crossing

Bubbling of Maslov index 0 discs causes the disc count $n_\beta(L)$ to jump.



Typically, for $n \geq 3$ the disc count depends on $p \in L$ ($\Rightarrow$ W multivalued).
For $n = 2$ the disc count is independent of $p \in L$ but jumps where L bounds a Maslov index 0 disc ($\Rightarrow$ W discontinuous).

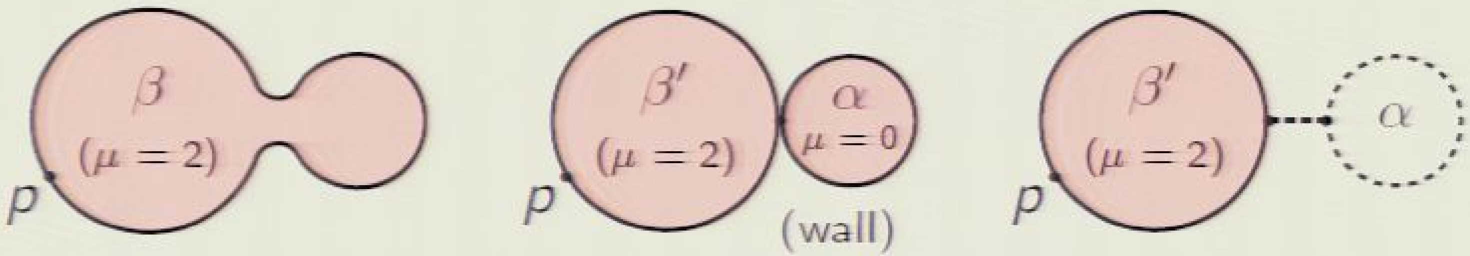
Proposition (Fukaya-Oh-Ohta-Ono + ε)

For $n = 2$, crossing a wall in which L bounds a single Maslov index 0 disc in a class α modifies W by a holomorphic substitution of variables $z_\beta \mapsto z_\beta\, h(z_\alpha)^{[\partial\beta]\cdot[\partial\alpha]}$ $\forall \beta \in \pi_2(X, L)$, where $h(z_\alpha) = 1 + O(z_\alpha) \in \mathbb{C}[[z_\alpha]]$.

The mirror is obtained from W by gluing the various regions delimited by the walls according to these changes of variables (instanton corrections).

Maslov index 0 discs and wall-crossing

Bubbling of Maslov index 0 discs causes the disc count $n_\beta(L)$ to jump.



Typically, for $n \geq 3$ the disc count depends on $p \in L$ ($\Rightarrow W$ multivalued).
For $n = 2$ the disc count is independent of $p \in L$ but jumps where L bounds a Maslov index 0 disc ($\Rightarrow W$ discontinuous).

Proposition (Fukaya-Oh-Ohta-Ono $+ \varepsilon$)

For $n = 2$, crossing a wall in which L bounds a single Maslov index 0 disc in a class α modifies W by a holomorphic substitution of variables $z_\beta \mapsto z_\beta\, h(z_\alpha)^{[\partial\beta]\cdot[\partial\alpha]}$ $\forall \beta \in \pi_2(X, L)$, where $h(z_\alpha) = 1 + O(z_\alpha) \in \mathbb{C}[[z_\alpha]]$.

The mirror is obtained from M by gluing the various regions delimited by the walls according to these changes of variables (instanton corrections).

Mirror symmetry for blow-ups

(Abouzaid-Auroux-Katzarkov, in progress)

Goal: construct mirror of $\hat{X}_Y$ = blow-up of X along a codimension 2 subvariety $Y \subset X$ (need $Y \subset D \in |-K_X|$)

Motivation: a mirror of $\hat{X}_Y$ is almost as good as a mirror of Y.

- $D^b Coh(\hat{X}_Y) \simeq \langle D^b Coh(Y), D^b Coh(X) \rangle$ (semiorthogonal decomp.) (Bondal-Orlov)
- also expect $\mathcal{F}(\hat{X}_Y)$ related to $\mathcal{F}(Y)$ (esp. if $X = D \times \mathbb{C}$ and Y fiber of a pencil in D)

Simplification: assume (X, D) **toric** (but not Y).

Motivating example: what's the mirror of a genus 2 curve Σ?
Answer: blow up $(\mathbb{P}^1)^2 \times \mathbb{C}$ along $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \{0\}$, take mirror, restrict

Mirror symmetry for blow-ups

(Abouzaid-Auroux-Katzarkov, in progress)

Goal: construct mirror of $\hat{X}_Y$ = blow-up of X along a codimension 2 subvariety $Y \subset X$ (need $Y \subset D \in |-K_X|$)

Motivation: a mirror of $\hat{X}_Y$ is almost as good as a mirror of Y.

- $D^b Coh(\hat{X}_Y) \simeq \langle D^b Coh(Y), D^b Coh(X) \rangle$ (semiorthogonal decomp.) (Bondal-Orlov)
- also expect $\mathcal{F}(\hat{X}_Y)$ related to $\mathcal{F}(Y)$ (esp. if $X = D \times \mathbb{C}$ and Y fiber of a pencil in D)

Simplification: assume (X, D) **toric** (but not Y).

Motivating example: what's the mirror of a genus 2 curve Σ?
Answer: blow up $(\mathbb{P}^1)^2 \times \mathbb{C}$ along $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \{0\}$, take mirror, restrict.

Blowing up a point

Local model in dim. 2:

$X = \mathbb{C}^* \times \mathbb{C}$, $D = \mathbb{C}^* \times \{0\}$, $\Omega = d\log x \wedge d\log y$, $\omega = \omega_0$
$\hat{X}$ = blowup at $(1, 0)$, $\hat{D}$ = proper transform, $\hat{\Omega} = \pi^*\Omega$, $\hat{\omega} = \hat{\omega}_\epsilon$ $\left(\int_E \hat{\omega} = \epsilon\right)$

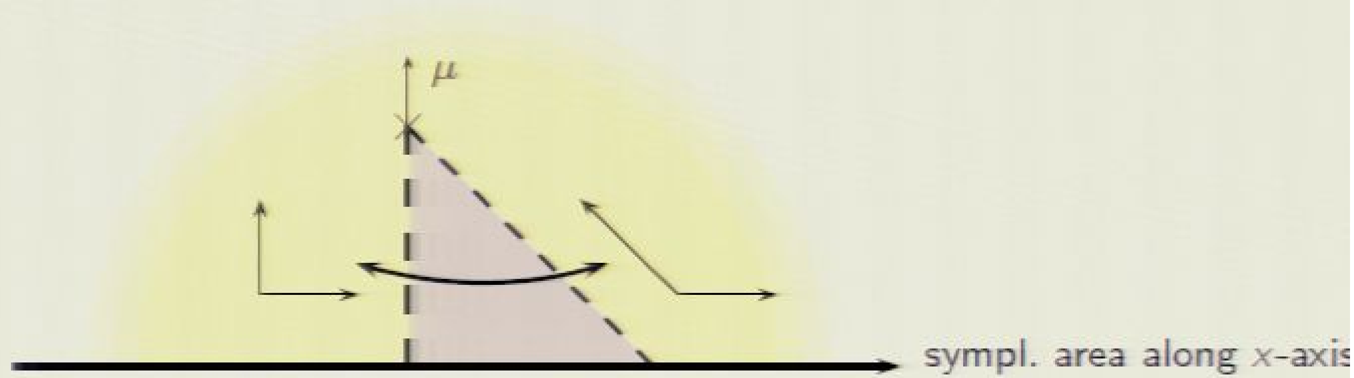
S^1 action $(y \mapsto e^{i\theta}y)$ lifts, fixed point set $\hat{D} \cup \{pt\}$, μ = moment map
S^1-invariant SLag. fibration on $\hat{X} \setminus \hat{D}$: $L_{t_1,t_2} = \{\log|\pi^* x| = t_1, \mu = t_2\}$

Blowing up a point

Local model in dim. 2:

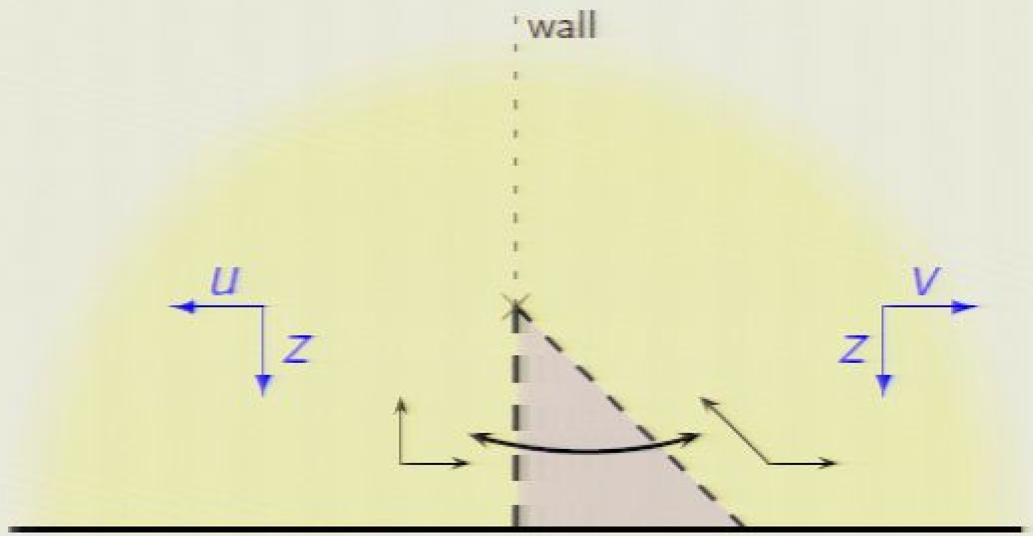
$X = \mathbb{C}^* \times \mathbb{C}$, $D = \mathbb{C}^* \times \{0\}$, $\Omega = d\log x \wedge d\log y$, $\omega = \omega_0$
$\hat{X} =$ blowup at $(1,0)$, $\hat{D} =$ proper transform, $\hat{\Omega} = \pi^*\Omega$, $\hat{\omega} = \hat{\omega}_\epsilon$ $(\int_E \hat{\omega} = \epsilon)$

S^1 action $(y \mapsto e^{i\theta}y)$ lifts, fixed point set $\hat{D} \cup \{pt\}$. $\mu :=$ moment map.
S^1-invariant S.Lag. fibration on $\hat{X} \setminus \hat{D}$: $L_{t_1,t_2} = \{\log|\pi^* x| = t_1,\ \mu = t_2\}$.



Blowing up a point (continued)

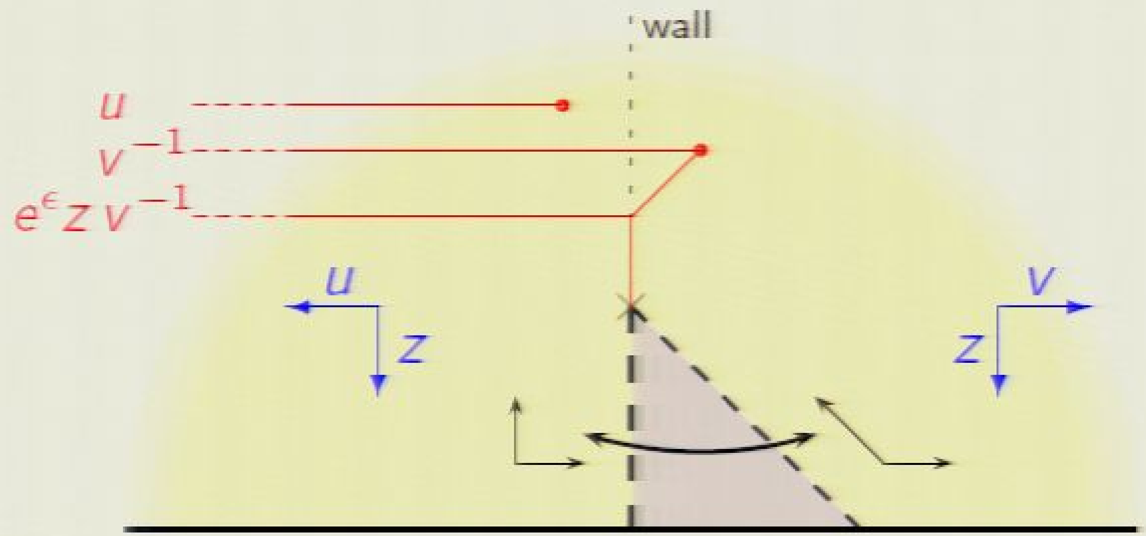
$X = \mathbb{C}^* \times \mathbb{C}$, $D = \mathbb{C}^* \times \{0\}$, $\Omega = d\log x \wedge d\log y$, $\omega = \omega_0$
$\hat{X}$ = blowup at $(1, 0)$, $\hat{D}$ = proper transform, $\hat{\Omega} = \pi^*\Omega$, $\hat{\omega} = \hat{\omega}_\epsilon$ $(\int_E \hat{\omega} = \epsilon)$
S^1-invariant S.Lag. fibration on $\hat{X} \setminus \hat{D}$: $L_{t_1,t_2} = \{\log|\pi^* x| = t_1,\ \mu = t_2\}$.



Classical: $uv = 1$ for $|z| < e^{-\epsilon}$ (above $\times$); $uv = e^{\epsilon} z$ for $|z| > e^{-\epsilon}$ (below $\times$).

Blowing up a point (continued)

$X = \mathbb{C}^* \times \mathbb{C}$, $D = \mathbb{C}^* \times \{0\}$, $\Omega = d\log x \wedge d\log y$, $\omega = \omega_0$
$\hat{X}$ = blowup at $(1, 0)$, $\hat{D}$ = proper transform, $\hat{\Omega} = \pi^*\Omega$, $\hat{\omega} = \hat{\omega}_\epsilon$ $(\int_E \hat{\omega} = \epsilon)$
S^1-invariant S.Lag. fibration on $\hat{X} \setminus \hat{D}$: $L_{t_1,t_2} = \{\log|\pi^* x| = t_1,\ \mu = t_2\}$.



Classical: $uv = 1$ for $|z| < e^{-\epsilon}$ (above $\times$); $uv = e^\epsilon z$ for $|z| > e^{-\epsilon}$ (below $\times$).

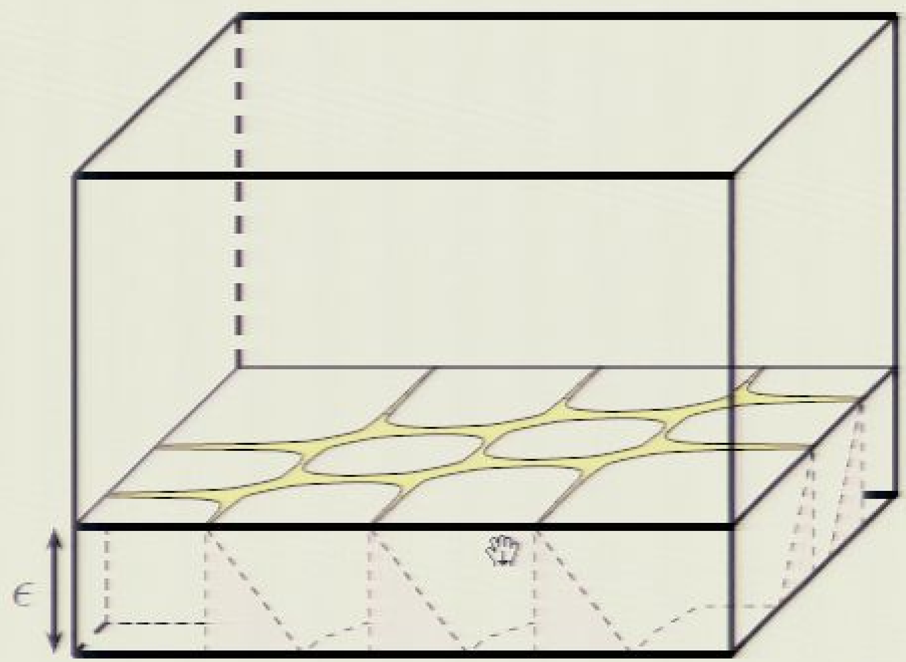
Corrected: $\{(u, v, z) \in \mathbb{C}^2 \times \mathbb{C}^*,\ uv = 1 + e^\epsilon z\}$ Superpotential: $W = z$

($\Rightarrow$ blowup of $\mathbb{P}^1 \times \mathbb{P}^1$: $W = z + e^{-A} z^{-1} + u + e^{-B} v$)

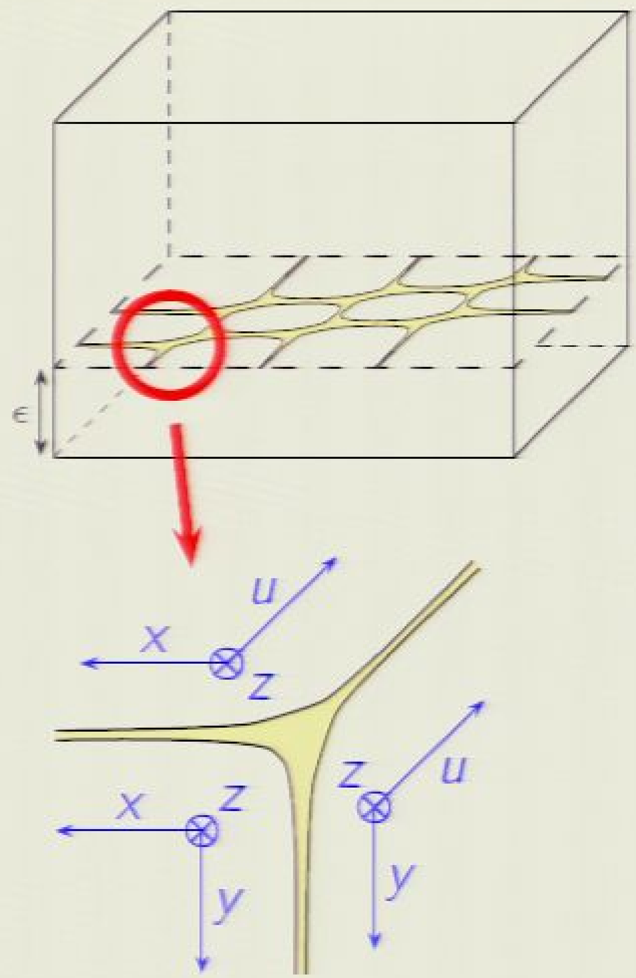
Blowing up a curve (A.-A.-K., in progress)

$X = (\mathbb{CP}^1)^3$, $D = \bigcup$ toric strata, $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \{0\} \subset D \subset X$.
$\hat{X} =$ blowup along Σ, $\hat{D} =$ proper transform, $\hat{\Omega} = \pi^*\Omega$, $\int_E \hat{\omega} = \epsilon$.

- S^1-action (3^{rd} factor) lifts; new fixed point stratum $\simeq \Sigma$ at $\mu = \epsilon$. All reduced spaces $\simeq \mathbb{CP}^1 \times \mathbb{CP}^1$, carry (S.??) Lag. torus fibrations.
- This gives a Lagr. T^3 fibration on $\hat{X} \setminus \hat{D}$, with discriminant locus $\simeq$ amoeba of Σ.



Blowing up a curve (continued)

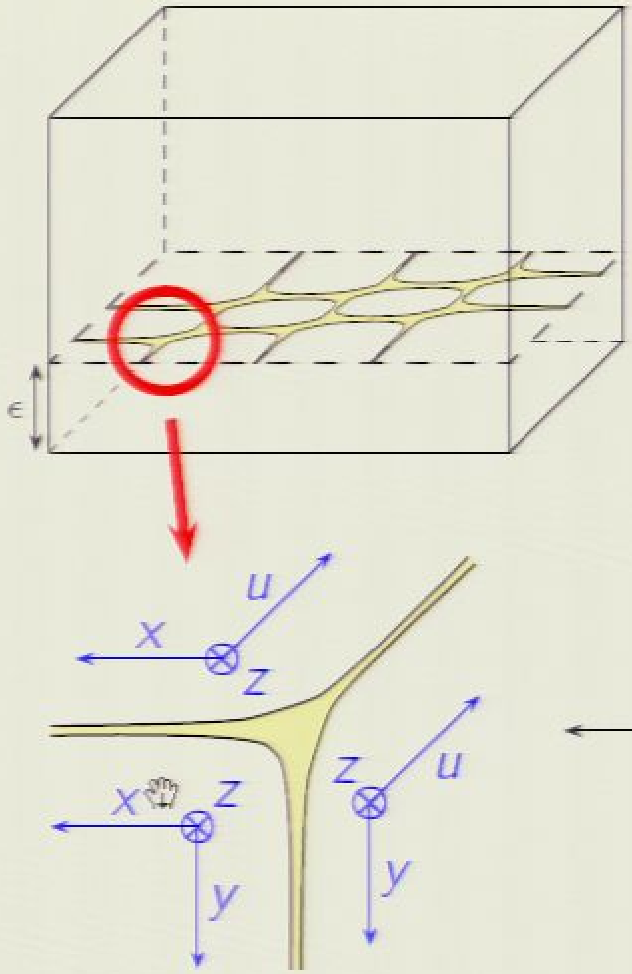


Walls propagate "vertically" from the amoeba.
Chambers $\leftrightarrow$ components in its complement.

Need: instanton-corrected gluing across walls.

Work out local models (and glue them together)

Blowing up a curve (continued)



Walls propagate "vertically" from the amoeba.
Chambers $\leftrightarrow$ components in its complement.

Need: instanton-corrected gluing across walls.

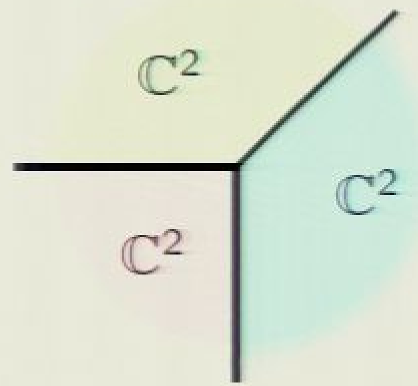
Work out local models (and glue them together)

Mirror: $\{xyu = 1 + e^{\epsilon} z\}$

$$\longleftrightarrow$$

$\{z = -e^{-\epsilon}\}$
$= \{xyu = 0\}$

three $\mathbb{C}^2$'s glued along coord. axes



Local mirror: $\{(x, y, u, z) \in \mathbb{C}^3 \times \mathbb{C}^*,\ xyu = 1 + e^{\epsilon} z\}$
Superpotential: $W = z +$ other terms

Gluing pieces...

For open curve in $(\mathbb{C}^*)^2 \times \mathbb{C}$, superpotential $W = z$

$\Rightarrow$ singular fiber = union of toric surfaces glued along $\mathbb{P}^1$'s and $\mathbb{C}$'s

(combinatorics governed by tropicalization of Σ)

crit. locus of W