

Title: Quantum Gravitation and the Renormalization Group

Date: May 13, 2009 04:00 PM

URL: <http://pirsa.org/09050006>

Abstract: In my talk I will provide an overview of the applications of Wilson's modern renormalization group (RG) to problems in quantum gravity. I will first discuss the development of the RG for continuum gravity within the framework of Feynman's covariant path integral approach. Then I will discuss a number of issues that arise when implementing the path integral approach with an explicit lattice UV regulator, and later how non-perturbative RG flows and universal non-trivial scaling dimensions can in principle be extracted from these calculations. Towards the end I will discuss recent attempts at formulating RG flows for gravitational couplings within the framework of a set of manifestly covariant, but non-local, effective field equations suitable for quantum cosmology.



Quantum Gravity
and the
Renormalization Group

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QG and the RG



- *Motivation:* Perturbative Quantum Gravity
 - Failure of perturbative renormalization in $d=4$
 - Quantum Gravity in $2+\epsilon$ dimensions
 - Non-renormalizable theories: Sigma Model
- *Methods:* Formulation of Lattice Quantum Gravity
 - Simplicial lattice-regularized formulation
 - Matter fields, Observables
 - Methods for determining non-trivial scaling dimensions
- *Outlook:* Possible non-perturbative ground state scenarios
 - RG running of Newton's G
 - Effective non-local, covariant relativistic field equations

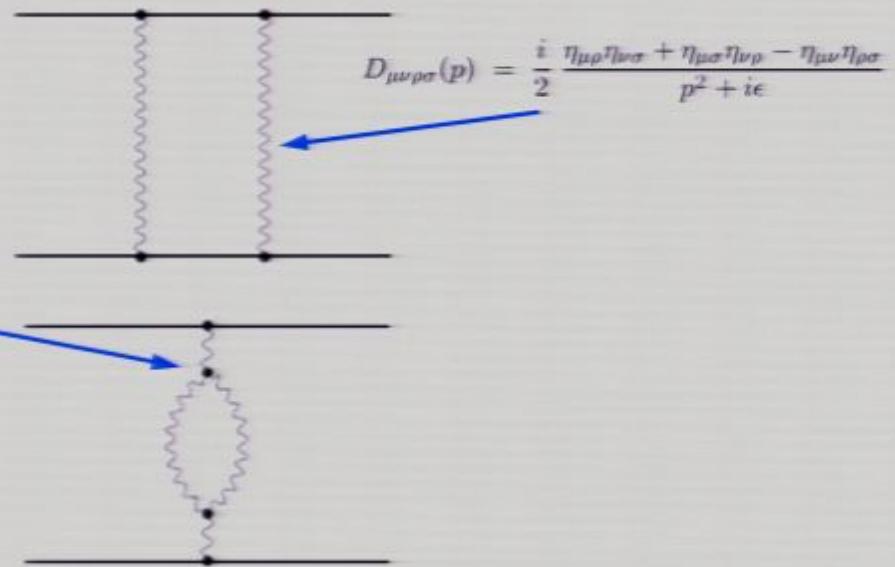
Perturbative Quantum Gravity

UV Divergences: Compute QM amplitudes by
Feynman diagram perturbation theory:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$U(q_1, q_2, q_3)_{\alpha_1\beta_1, \alpha_2\beta_2, \alpha_3\beta_3} =$$

$$\begin{aligned} & -i\frac{\kappa}{2} \left[q_{(\alpha_1}^2 q_{\beta_1)}^3 \left(2\eta_{\alpha_2(\alpha_3} \eta_{\beta_3)\beta_2} - \frac{2}{d-2} \eta_{\alpha_2\beta_2} \eta_{\alpha_3\beta_3} \right) \right. \\ & + q_{(\alpha_2}^1 q_{\beta_2)}^3 \left(2\eta_{\alpha_1(\alpha_3} \eta_{\beta_3)\beta_1} - \frac{2}{d-2} \eta_{\alpha_1\beta_1} \eta_{\alpha_3\beta_3} \right) \\ & + q_{(\alpha_3}^1 q_{\beta_3)}^2 \left(2\eta_{\alpha_1(\alpha_2} \eta_{\beta_2)\beta_1} - \frac{2}{d-2} \eta_{\alpha_1\beta_1} \eta_{\alpha_2\beta_2} \right) \\ & + 2q_{(\alpha_2}^3 \eta_{\beta_2)(\alpha_1} \eta_{\beta_1)} (\alpha_3 q_{\beta_3}^2) + 2q_{(\alpha_3}^1 \eta_{\beta_3)(\alpha_2} \eta_{\beta_2)} (\alpha_1 q_{\beta_1}^3) + 2q_{(\alpha_1}^2 \eta_{\beta_1)(\alpha_3} \eta_{\beta_3)} (\alpha_2 q_{\beta_2}^1) \\ & + q^2 \cdot q^3 \left(\frac{2}{d-2} \eta_{\alpha_1(\alpha_2} \eta_{\beta_2)\beta_1} \eta_{\alpha_3\beta_3} + \frac{2}{d-2} \eta_{\alpha_1(\alpha_3} \eta_{\beta_3)\beta_1} \eta_{\alpha_2\beta_2} - 2\eta_{\alpha_1(\alpha_2} \eta_{\beta_2)(\alpha_3} \eta_{\beta_3)\beta_1} \right) \\ & + q^1 \cdot q^3 \left(\frac{2}{d-2} \eta_{\alpha_2(\alpha_1} \eta_{\beta_1)\beta_2} \eta_{\alpha_3\beta_3} + \frac{2}{d-2} \eta_{\alpha_2(\alpha_3} \eta_{\beta_3)\beta_2} \eta_{\alpha_1\beta_1} - 2\eta_{\alpha_2(\alpha_1} \eta_{\beta_1)(\alpha_3} \eta_{\beta_3)\beta_2} \right) \\ & \left. + q^1 \cdot q^2 \left(\frac{2}{d-2} \eta_{\alpha_3(\alpha_1} \eta_{\beta_1)\beta_3} \eta_{\alpha_2\beta_2} + \frac{2}{d-2} \eta_{\alpha_3(\alpha_2} \eta_{\beta_2)\beta_3} \eta_{\alpha_1\beta_1} - 2\eta_{\alpha_3(\alpha_1} \eta_{\beta_1)(\alpha_2} \eta_{\beta_2)\beta_3} \right) \right] \end{aligned}$$



Bad high energy behavior:

$$\frac{G(k^2)}{G} = 1 + \text{const. } G k^2 + \dots$$

Non-Renormalizability in Four Dimensions

$$I = \lambda \int d^d x \sqrt{g} - \frac{1}{16\pi G} \int d^d x \sqrt{g} R$$

$$\Gamma_{div}^{(1)} = \frac{1}{4-d} \frac{\hbar}{16\pi^2} \int d^4 x \sqrt{g} \left(\frac{7}{20} R_{\mu\nu} R^{\mu\nu} + \frac{1}{120} R^2 \right)$$

$$\Gamma_{div}^{(2)} = \frac{1}{4-d} \frac{209}{2880} \frac{\hbar^2 G}{(16\pi^2)^2} \int d^4 x \sqrt{g} R_{\mu\nu}^{\rho\sigma} R_{\rho\sigma}^{\kappa\lambda} R_{\kappa\lambda}^{\mu\nu}$$

$$I \xrightarrow{\textcolor{blue}{\rightarrow}} \lambda \int d^d x \sqrt{g} - \frac{1}{16\pi G} \int d^d x \sqrt{g} R + \frac{\alpha_0}{\Lambda^{4-d}} \int d^d x \sqrt{g} R_{\mu\nu} R^{\mu\nu} + \frac{\beta_0}{\Lambda^{4-d}} \int d^d x \sqrt{g} R^2 + \dots$$

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Also, radiative corrections generate a host of new interactions...

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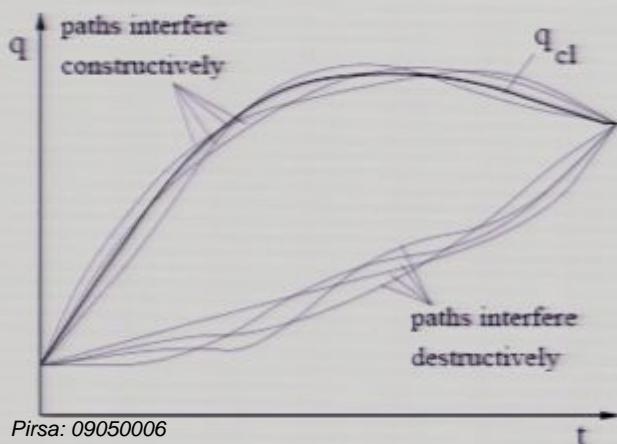
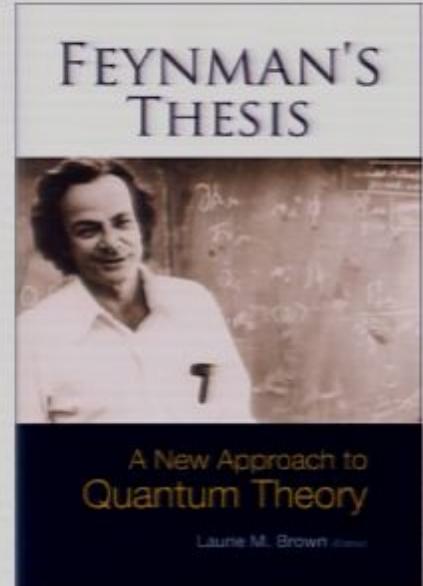
- 4-d perturbation theory in (ordinary) gravity seemingly leads to a **dead end**...
- Non-perturbative methods \Rightarrow non-perturbative regularization, search for a new vacuum.

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Feynman Path Integral

Reformulate QM amplitudes in terms of discrete
Sum over Paths

- non-commuting operators P, Q replaced by random Wiener paths.
- In complex time $t = -i\tau$ probabilities are real (as in statistical mechanics: $KT \rightarrow \hbar$).



$$K(q'', q'; T) = \sum_{\text{all paths}} A e^{iS(q'', q'; T)/\hbar}$$

$$\longrightarrow K = \int \mathcal{D}q(t) e^{iS[q(t)]}$$

Path Integral for Quantum Gravitation

$$\|\delta g\|^2 \equiv \int d^d x G^{\mu\nu,\alpha\beta}[g(x)] \delta g_{\mu\nu}(x) \delta g_{\alpha\beta}(x)$$

DeWitt approach to measure: Volume element in function space obtained from super-metric over metric deformations.

$$G^{\mu\nu,\alpha\beta}[g(x)] = \tfrac{1}{2} \sqrt{g(x)} [g^{\mu\alpha}(x)g^{\nu\beta}(x) + g^{\mu\beta}(x)g^{\nu\alpha}(x) + \lambda g^{\mu\nu}(x)g^{\alpha\beta}(x)]$$

$$\int d\mu[g] = \int \prod_x (\det[G(g(x))])^{\frac{1}{2}} \prod dg_{\mu\nu}(x)$$

$$\int d\mu[g] = \int \prod_x [g(x)]^{(d-4)(d+1)/8} \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \xrightarrow{d=4} \int \prod_x \prod_{\mu \geq \nu} dg_{\mu\nu}(x)$$

$$Z_{cont} = \int [d g_{\mu\nu}] e^{-\lambda \int dx \sqrt{g} + \frac{1}{16\pi G} \int dx \sqrt{g} R}$$

Euclidean E-H action unbounded below (conformal instability).

Only One Coupling

Pure gravity path integral:

$$Z = \int [dg_{\mu\nu}] e^{-I_E[g]}$$

$$I_E[g] = \lambda_0 \Lambda^d \int dx \sqrt{g} - \frac{1}{16\pi G_0} \Lambda^{d-2} \int dx \sqrt{g} R$$

Rescale metric (or edge lengths):

$$g'_{\mu\nu} = \lambda_0^{2/d} g_{\mu\nu} \quad g'^{\mu\nu} = \lambda_0^{-2/d} g^{\mu\nu}$$

- In the absence of matter, only one dimensionless coupling:

$$\tilde{G} \equiv G_0 \lambda_0^{(d-2)/d}$$

Similar to the g of Y.M. !

$$I_E[g] = \Lambda^d \int dx \sqrt{g'} - \frac{1}{16\pi G_0 \lambda_0^{\frac{d-2}{d}}} \Lambda^{d-2} \int dx \sqrt{g'} R'$$

H. H. & R. Williams, NPB 1984

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Functional Measure cont'd

Add volume term to functional measure (Misner 1955) ;
under coordinate transformation: $x^\mu + \epsilon^\mu(x)$

$$\prod_x [g(x)]^{\sigma/2} \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \rightarrow \prod_x \left(\det \frac{\partial x'^\beta}{\partial x^\alpha} \right)^\gamma [g(x)]^{\sigma/2} \prod_{\mu \geq \nu} dg_{\mu\nu}(x)$$

$$\prod_x \left(\det \frac{\partial x'^\beta}{\partial x^\alpha} \right)^\gamma = \prod_x [\det(\delta_\alpha^\beta + \partial_\alpha \epsilon^\beta)]^\gamma = \exp \left\{ \gamma \delta^d(0) \int d^d x \partial_\alpha \epsilon^\alpha \right\} = 1 \quad [\text{eg Faddeev \& Popov, 1973}]$$

Skeptics should systematically investigate (on the lattice) effects due to the addition of an ultra-local term of the type

$$\prod_x [g(x)]^{\sigma/2} = \exp \left\{ \frac{1}{2} \sigma \delta^d(0) \int d^d x \ln g(x) \right\}$$

Such a term would not be expected to affect the propagation properties of gravitons (which are determined by R-term).

* Perturbatively Non-Renorm. Interactions

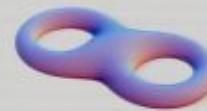
Some early work :

- K.G. Wilson, *Quantum Field Theory Models in $D < 4$* , PRD 1973.
- K. Symanzik, *Renormalization of Nonrenormalizable Massless φ^4 Theory*, CMP 1975.
- G. Parisi, *Renormalizability of not Renormalizable Theories*, LNC 1973.
- G. Parisi, *Theory Of Nonrenormalizable Interactions - Large N* , NPB 1975.
- E. Brézin and J. Zinn-Justin, *Nonlinear σ Model in $2+\epsilon$ Dimensions*, PRL 1976.
- D. Gross and A. Neveu ...

Gravity in 2.000001 Dimensions



- **Wilson expansion:** formulate in $2+\varepsilon$ dimensions...



G is dim-less,
theory is now perturbatively renormalizable :

$$\mu \frac{\partial}{\partial \mu} G(\mu) = \beta(G(\mu))$$

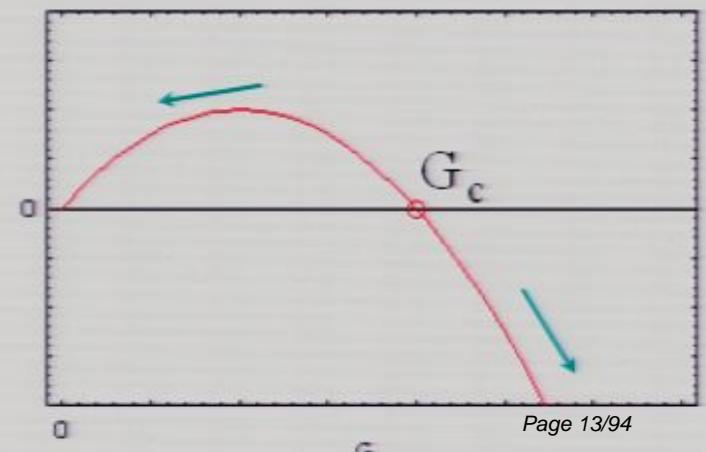
$$\beta(G) = (d-2)G - \frac{2}{3}(25-n_s)G^2 - \frac{20}{3}(25-n_s)G^3 + \dots$$

$$\left\{ \begin{array}{l} G_c = \frac{3}{2(25-n_s)}(d-2) - \frac{45}{2(25-n_s)^2}(d-2)^2 + \dots \\ \nu^{-1} = -\beta'(G_c) = (d-2) + \frac{15}{25-n_s}(d-2)^2 + \dots \end{array} \right.$$

$$D_{\mu\nu\rho\sigma}(p) = \frac{i}{2} \frac{\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \frac{2}{d-2}\eta_{\mu\nu}\eta_{\rho\sigma}}{p^2 + i\epsilon}$$

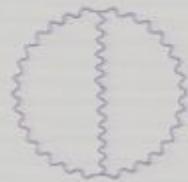
Weinberg 1977
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$$\beta(G)$$

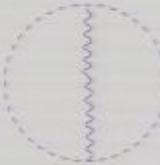
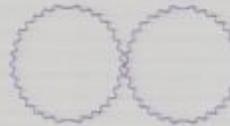


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More on 2.000001 dim's ...



Graviton loops



Graviton-ghost loops



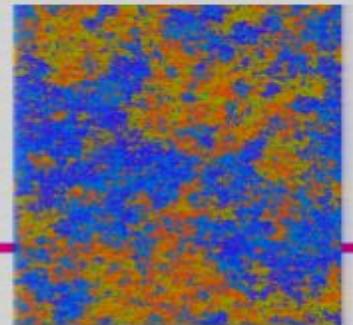
- Singularity structure in $d > 2$ unclear (Borel)...
- But, analytical control of UV fixed point at G_c .

$$G(k^2) = \frac{G_c}{1 \pm \textcircled{(} (m^2/k^2)^{(d-2)/2} \textcircled{)}}$$

Nontrivial scaling determined by UV FP:

$$m \sim \Lambda \exp \left(- \int^G \frac{dG'}{\beta(G')} \right) \underset{G \rightarrow G_c}{\sim} \Lambda | G - G_c |^{V/G'(G_c)}$$

Detour: Non-linear Sigma model



- Field theory description [$O(N)$ Heisenberg model] :

$$Z = \int [d\sigma] \prod_x \delta(\sigma^a(x) \sigma^a(x) - 1) \exp \left(-\frac{\Lambda^{d-2}}{g^2} \int d^d x \partial_\mu \sigma^a(x) \partial_\mu \sigma^a(x) + \int d^d x j^a(x) \sigma^a(x) \right)$$

Coupling g becomes dimensionless in $d = 2$.

For $d > 2$ theory is not perturbatively renormalizable,
but in the $2 + \epsilon$ expansion one finds:

E. Brezin J. Zinn-Justin 1975
F. Wegner, 1989
N.A. Kivel et al, 1994
E. Brezin and S. Hikami, 1996

$$\Lambda \frac{\partial g^2}{\partial \Lambda} \equiv \beta(g^2) = (d-2)g^2 - \frac{N-2}{2\pi} g^4 + O(g^6, (d-2)g^4)$$

Phase Transition = non-trivial UV fixed point; new non-perturbative mass scale.

$$\nu^{-1}(\epsilon) = \epsilon + \frac{\epsilon^2}{n-2} + \frac{\epsilon^3}{2(n-2)} - [30 - 14 + n^2 + (54 - 18n)\zeta(3)] \frac{\epsilon^4}{4(n-2)^3} + \dots$$

$$\xi(g^2) \equiv m^{-1}(g^2) \simeq c_d \Lambda \left(\frac{1}{g_c^2} - \frac{1}{g^2} \right)^{\nu}$$

$$\langle \vec{S}(\mathbf{x}) \cdot \vec{S}(\mathbf{0}) \rangle \sim \exp\{-|\mathbf{x}|/\xi\}$$

*

Renormalization Group Equations

In the framework of the *double* (g and $2+\varepsilon$) expansion the model looks just like any other renormalizable theory, to every order...

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \frac{n}{2} \zeta(g) + \rho(g) h \frac{\partial}{\partial h} \right] \Gamma^{(n)}(p_i, g, h, \Lambda) = 0$$

$$\Lambda \frac{\partial}{\partial \Lambda}|_{\text{ren.fixed } g} = \beta(g)$$

Callan-Symanzik Eq.

$$\Lambda \frac{\partial}{\partial \Lambda}|_{\text{ren.fixed}} (-\ln Z) = \zeta(g)$$

$$g = (\Lambda/\mu)^{d-2} Z_g g_r \quad \pi(x) = Z^{-1/2} \pi(x)$$

$$2-d + \frac{1}{2}\zeta(g) - \frac{\beta(g)}{g} = \rho(g)$$

$$h = Z_h h_r \quad Z_h = Z_g / \sqrt{Z}$$

$$\Gamma_F^{(n)}(p_i, g_r, h_r, \mu) = Z^{n/2}(\Lambda/\mu, g) \Gamma^{(n)}(p_i, g, h, \Lambda)$$

... but the price one pays is that now one needs $\varepsilon \rightarrow 1$!

Similar result are obtained in large N limit...

Example of Dimensional Transmutation:

The structure of the renormalization group equations implies the existence of a new non-perturbative scale :

$$\xi^{-1}(g) = m(g) \propto \Lambda \exp \left(- \int^g \frac{dg'}{\beta(g')} \right)$$

such that the *Callan-Symanzik* equation for m is satisfied :

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} \right] m[\Lambda, g(\Lambda)] = 0$$

But are the QFT predictions correct ?

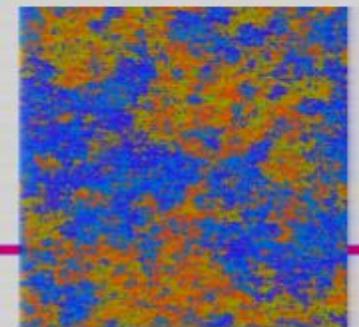
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↗

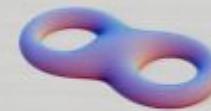
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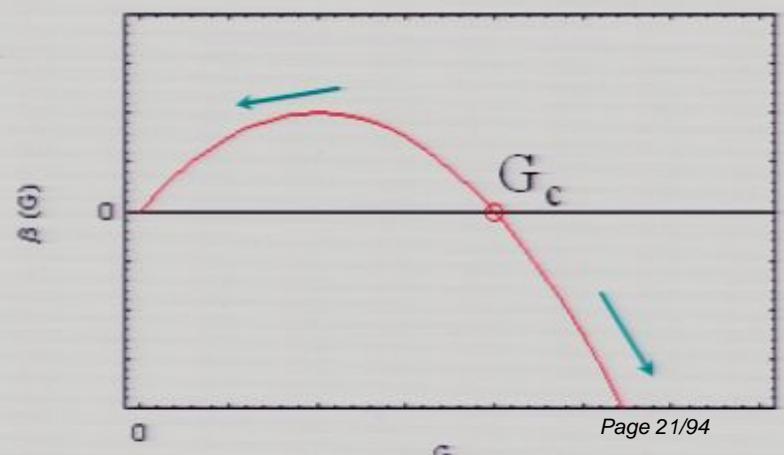
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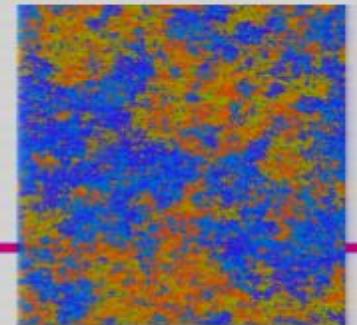
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$$\Gamma_r^{(n)}(p_i, g_r, h_r, \mu) = Z_r^{n/2} (\Lambda/\mu, g) \Gamma^{(n)}(p_i, g, h, \Lambda)$$

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$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} \right] m[\Lambda, g(\Lambda)] = 0$$

But are the QFT predictions correct ?

But are the QFT predictions correct ?

Experimental test: $O(2)$ non-linear sigma model describes the phase transition of *superfluid Helium*

Space Shuttle experiment (2003)

High precision measurement of specific heat of superfluid Helium He4
(zero momentum energy-energy correlation at UV FP)

J.A. Lipa et al, Phys Rev 2003:

$$\alpha = 2 - 3\nu = -0.0127(3)$$

MC, 4- ε expansion to 4 loops, & to 6 loops in d=3: $\alpha = 2 - 3\nu \approx -0.0125(4)$

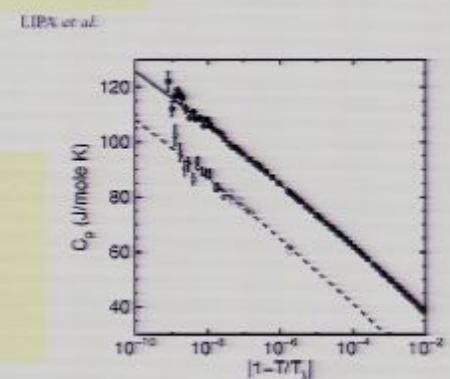


FIG. 15. Semilogarithmic plot of the specific heat vs reduced temperature over the full range measured. Below the transition the data (closed symbols) were binned with a density of 10 bins per decade, and above (open symbols) with a density of 8 bins per decade. Lines show best fits to the data.

One of the most accurate predictions of QFT !

But are the QFT predictions correct ?

Example of Dimensional Transmutation:

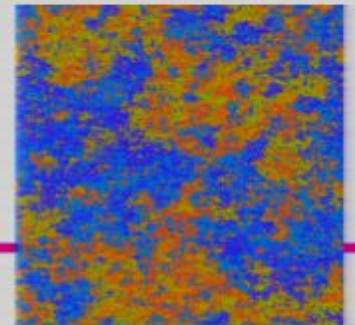
The structure of the renormalization group equations implies the existence of a new non-perturbative scale :

$$\xi^{-1}(g) = m(g) \propto \Lambda \exp \left(- \int^g \frac{dg'}{\beta(g')} \right)$$

such that the Callan-Symanzik equation for m is satisfied :

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} \right] m[\Lambda, g(\Lambda)] = 0$$

Detour: Non-linear Sigma model



- Field theory description [$O(N)$ Heisenberg model] :

$$Z = \int [d\sigma] \prod_x \delta(\sigma^a(x) \sigma^a(x) - 1) \exp \left(-\frac{\Lambda^{d-2}}{g^2} \int d^d x \partial_\mu \sigma^a(x) \partial_\mu \sigma^a(x) + \int d^d x j^a(x) \sigma^a(x) \right)$$

Coupling g becomes dimensionless in $d = 2$.

For $d > 2$ theory is not perturbatively renormalizable,
but in the $2 + \epsilon$ expansion one finds:

E. Brezin J. Zinn-Justin 1975
F. Wegner, 1989
N.A. Kivel et al, 1994
E. Brezin and S. Hikami, 1996

$$\Lambda \frac{\partial g^2}{\partial \Lambda} \equiv \beta(g^2) = (d-2)g^2 - \frac{N-2}{2\pi} g^4 + O(g^6, (d-2)g^4)$$

Phase Transition = non-trivial UV fixed point; new non-perturbative mass scale.

$$\nu^{-1}(\epsilon) = \epsilon + \frac{\epsilon^2}{n-2} + \frac{\epsilon^3}{2(n-2)} - [30 - 14 + n^2 + (54 - 18n)\zeta(3)] \frac{\epsilon^4}{4(n-2)^3} + \dots$$

$$\xi(g^2) \equiv m^{-1}(g^2) \simeq c_d \Lambda \left(\frac{1}{g_c^2} - \frac{1}{g^2} \right)^p$$

$$\langle \vec{S}(\mathbf{x}) \cdot \vec{S}(\mathbf{0}) \rangle \sim \exp\{-|\mathbf{x}|/\xi\}$$

*

Renormalization Group Equations

In the framework of the *double* (g and $2+\varepsilon$) expansion the model looks just like any other renormalizable theory, to every order...

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \frac{n}{2} \zeta(g) + \rho(g) h \frac{\partial}{\partial h} \right] \Gamma^{(n)}(p_i, g, h, \Lambda) = 0$$

$$\Lambda \frac{\partial}{\partial \Lambda}|_{\text{ren.fixed } g} = \beta(g)$$

Callan-Symanzik Eq.

$$\Lambda \frac{\partial}{\partial \Lambda}|_{\text{renormalized}} (-\ln Z) = \zeta(g)$$

$$g = (\Lambda/\mu)^{d-2} Z_g g_r \quad \pi(x) = Z^{-1/2} \pi(x)$$

$$2-d + \frac{1}{2}\zeta(g) - \frac{\beta(g)}{g} = \rho(g)$$

$$h = Z_h h_r \quad Z_h = Z_g / \sqrt{Z}$$

$$\Gamma_r^{(n)}(p_i, g_r, h_r, \mu) = Z^{n/2}(\Lambda/\mu, g) \Gamma^{(n)}(p_i, g, h, \Lambda)$$

... but the price one pays is that now one needs $\varepsilon \rightarrow 1$!

Similar result are obtained in large N limit...

Example of Dimensional Transmutation:

The structure of the renormalization group equations implies the existence of a new non-perturbative scale :

$$\xi^{-1}(g) = m(g) \propto \Lambda \exp \left(- \int^g \frac{dg'}{\beta(g')} \right)$$

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Lipa et al.

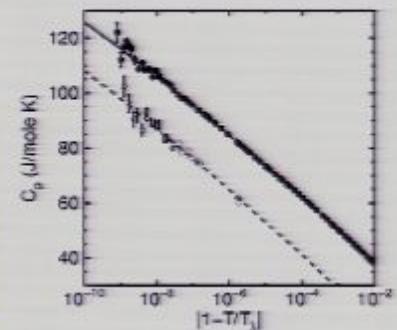


FIG. 15. Semilogarithmic plot of the specific heat vs reduced temperature over the full range measured. Below the transition the data (closed symbols) were binned with a density of 10 bins per decade, and above (open symbols) with a density of 8 bins per decade. Lines show best fits to the data.

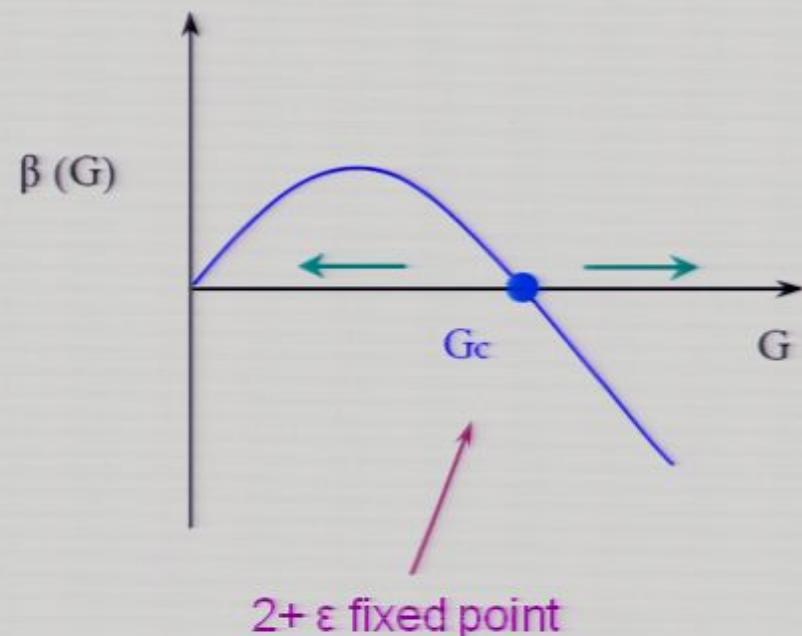
One of the most accurate predictions of QFT !

*The non-linear sigma model in 3d provides **an explicit example of a field theory** which :*

- ✓ Is not perturbatively renormalizable in $d=3$.
- ✓ Nevertheless leads to detailed, calculable predictions in the scaling limit $r \gg a$ ($q^2 \ll \Lambda^2$).
- ✓ Involves a new non-perturbative scale ξ , essential in determining the scaling behavior in the vicinity of the FP.
- ✓ Whose non-trivial, universal predictions agree with experiments.

Key question - Back to Gravity :

What is left of the above q. gravity scenario in 4 dimensions?



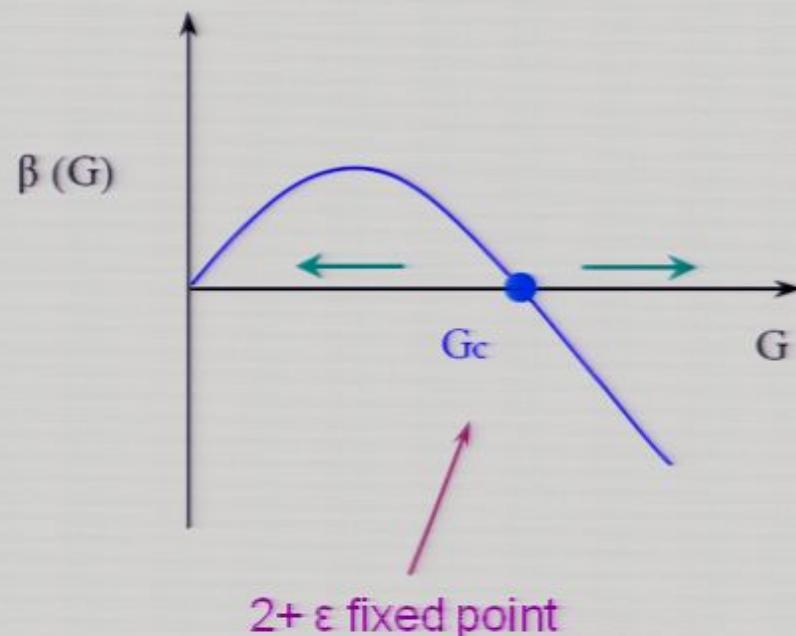
Strongly coupled gravity



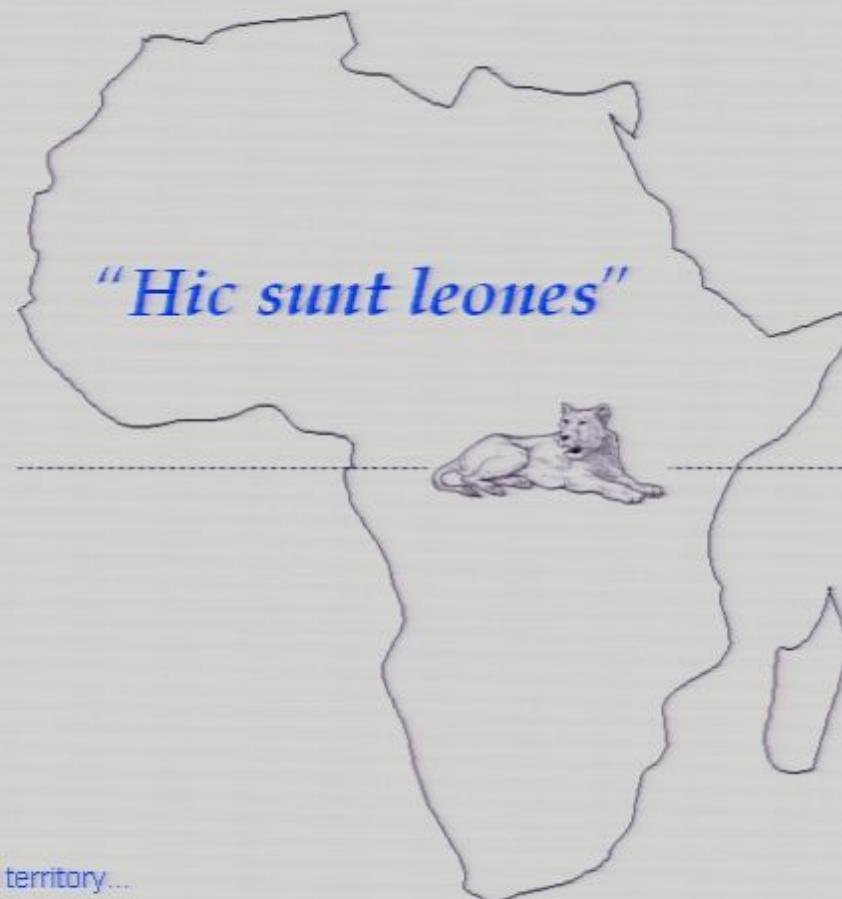
The Roman's description of unknown territory...

Key question - Back to Gravity :

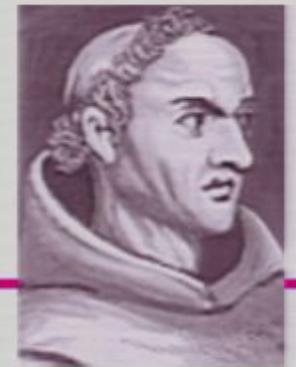
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Strongly coupled gravity



The Roman's description of unknown territory...

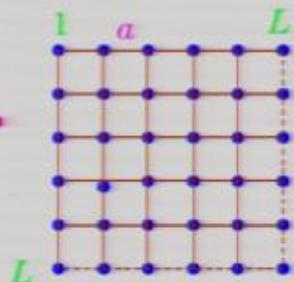


The Lattice Theory

*

Lattice Quantum Gravity

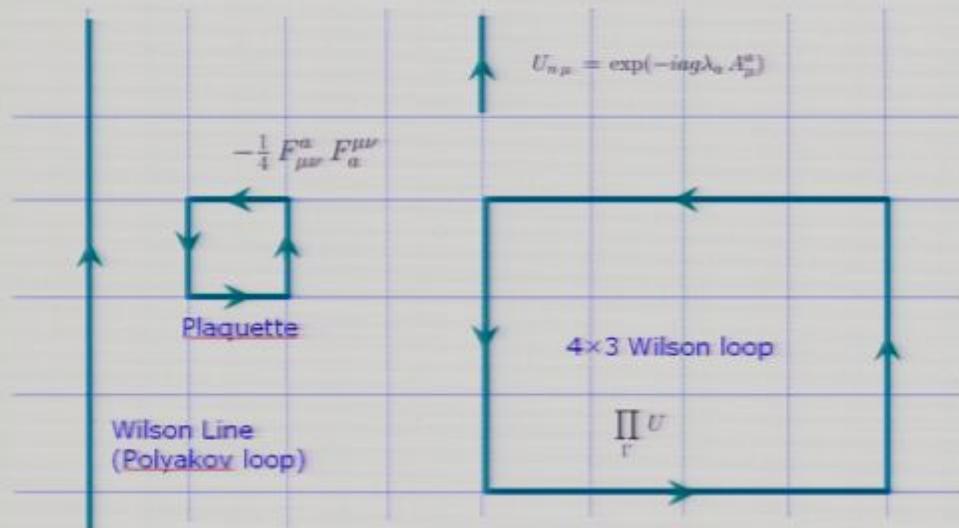
Lattice regularization provides explicit short distance cutoff.



- Regularized theory is finite, allows non-perturbative treatment
- Methods of statistical field theory
- Multi-year experience with lattice QCD
- Numerical evaluation feasible
- Continuum limit requires UV fixed point

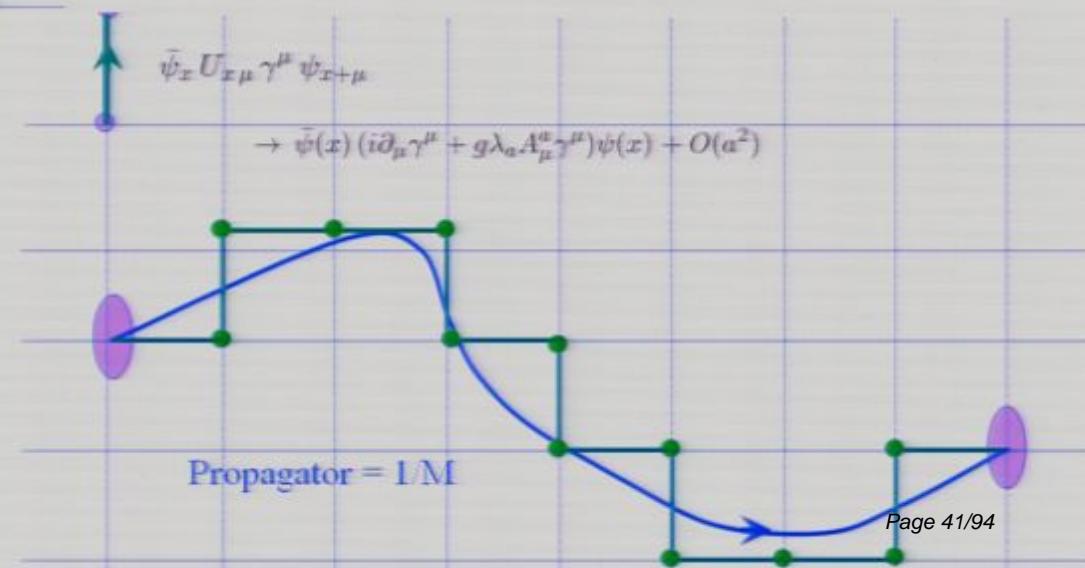
statistical mechanics	↔	quantum field theory
ensemble	↔	phase space
ensemble average	↔	path integral
$\exp\{-\beta_k H\}$	↔	$\exp\{-S^E\}$
$\beta_k \int d^3x \mathcal{H}$	↔	$\int dx_4 \int d^3x \mathcal{L}$
finite β_k	↔	finite $\int dx_4 = T$
zero temperature	↔	infinite time extent T

* Proto: Wilson' Lattice Gauge Theory



Local gauge invariance

→ exact lattice Ward identities



Lattice Gauge Theory Works

$$L_{\text{QCD}} = -\frac{1}{4} F_{\mu\nu}^{(a)} F^{(a)\mu\nu} + i \sum_q \bar{\psi}_q^i \gamma^\mu (D_\mu)_{ij} \psi_q^j - \sum_a m_q \bar{\psi}_q^i \psi_{qi},$$

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln(\mu^2/\Lambda^2)} \left[1 - \frac{2\beta_1}{\beta_0^2} \frac{\ln[\ln(\mu^2/\Lambda^2)]}{\ln(\mu^2/\Lambda^2)} + \frac{4\beta_1^2}{\beta_0^4 \ln^2(\mu^2/\Lambda^2)} \right. \\ \times \left. \left(\left(\ln[\ln(\mu^2/\Lambda^2)] - \frac{1}{2} \right)^2 + \frac{\beta_2 \beta_0}{8\beta_1^2} - \frac{5}{4} \right) \right].$$

Lattice gauge theory provides so far the only convincing evidence for *confinement* and *chiral symmetry breaking* in QCD.

Only formulation that allows computation of *proton mass* from *first principles*.

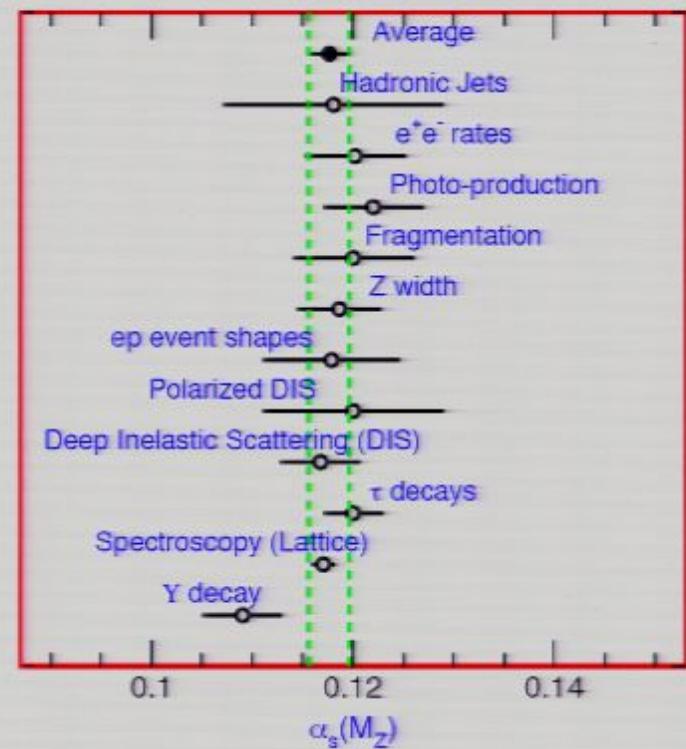


Figure 9.1: Summary of the value of $\alpha_s(M_Z)$ from various processes. The values shown indicate the process and the measured value of α_s extrapolated to $\mu = M_Z$. The error shown is the *total* error including theoretical uncertainties. The average quoted in this report which comes from these measurements is also shown. See text for discussion of errors.

Quantum Continuum Limit

- Naïve continuum limit : $a \rightarrow 0 \quad (\Lambda = \pi/a \rightarrow \infty)$
- Quantum continuum limit
(based on RG) : $a \rightarrow 0 \quad g(a) \rightarrow 0$

$$\xi = \frac{1}{m_{\text{phys}}} = \text{const.} \times a \exp \left\{ \frac{1}{2\beta_0 g^2(a)} \right\} \quad \text{fixed}$$

or simply:

$$\boxed{\xi/a \rightarrow \infty}$$

Generally a *phase transition* (UV fixed point) is required for the existence of a non-trivial continuum limit [Wilson, 1974].

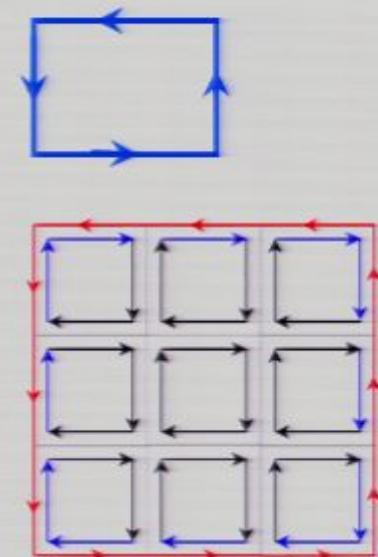
* Wilson Loop in $SU(N)$ Gauge Theories

- Wilson loop in Lattice Gauge Theories,

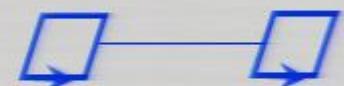
$$W(C) = \left\langle \text{tr} \mathcal{P} \exp \left\{ ig \oint_C A_\mu(x) dx^\mu \right\} \right\rangle, \quad \sim_{A \rightarrow \infty} \exp(-A(C)/\xi^2),$$

Gives linear confinement [eg Peskin & Schroeder p. 783]

ξ = gauge correlation length



$$G_{\square}(x) = \left\langle \text{tr} \mathcal{P} \exp \left\{ ig \oint_{C_\epsilon} A_\mu(x') dx'^\mu \right\} \times (x) \text{tr} \mathcal{P} \exp \left\{ ig \oint_{C'_\epsilon} A_\mu(x'') dx''^\mu \right\} (0) \right\rangle_c \sim_{|x| \rightarrow \infty} \exp(-|x|/\xi).$$

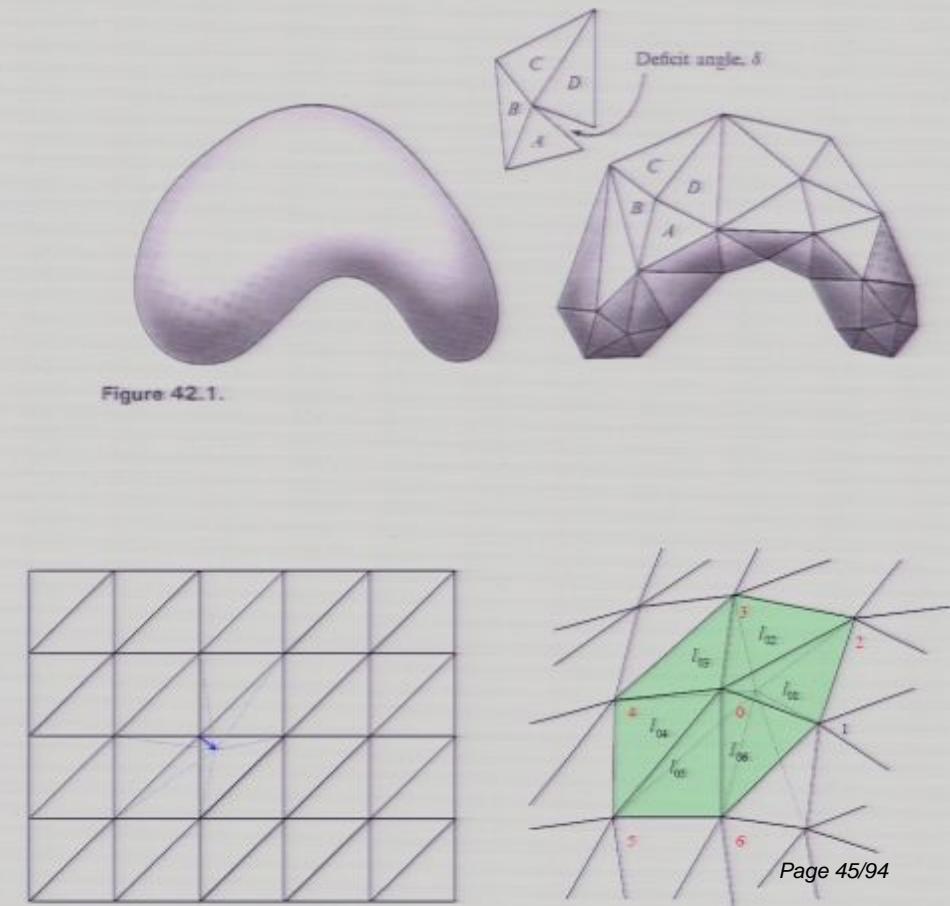


... Both results are essentially geometric in nature:
they follow (almost trivially) from the use of the $SU(N)$ Haar measure.

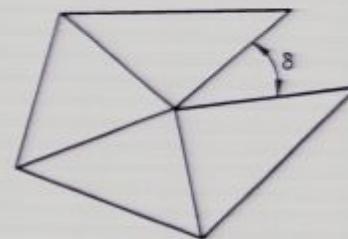
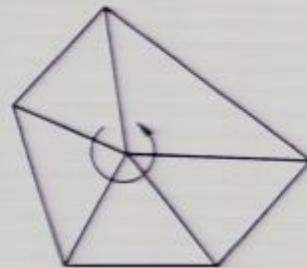
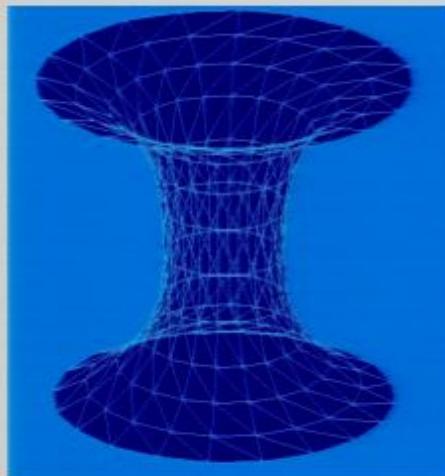
Simplicial Lattice Formulation

“General Relativity without coordinates” (T.Regge)

- Based on a dynamical lattice.
- Incorporates continuous local invariance.
- Puts within the reach of computation problems which in practical terms are beyond the power of normal analytical methods.
- It affords any desired level of accuracy by a sufficiently fine subdivision of the space-time region under consideration.



Curvature - Described by Angles



$$d = 2$$

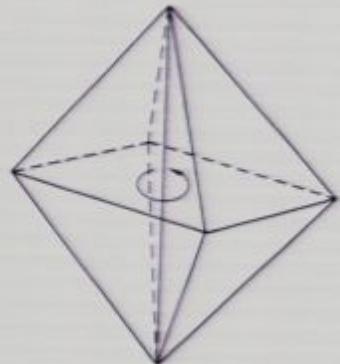
$$\chi = \frac{1}{2\pi} \sum_h \delta_h$$

$$g_{ij} = \frac{1}{2} (l_{1,i+1}^2 + l_{1,j+1}^2 - l_{i+1,j+1}^2)$$

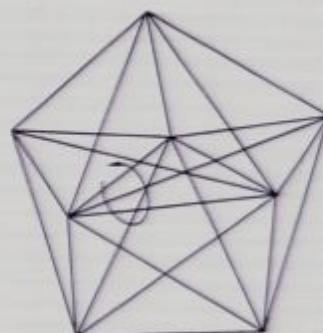
$$V_d = \frac{1}{d!} \sqrt{\det g_{ij}}$$

$$\sin \theta_d = \frac{d}{d-1} \frac{V_d V_{d-2}}{V_{d-1} V'_{d-1}}$$

$$\delta_h = 2\pi - \sum_{\text{d-simplices meeting on h}} \theta_d$$



$$d = 3$$



$$d = 4$$

Curvature det. by edge lengths.

Lattice Rotations

$$\phi^\mu(s_{n+1}) = R_\nu^\mu(P) \phi^\nu(s_1) \quad R_\nu^\mu = \left[P e^{\int_{\text{path between simplices}} \Gamma_\lambda dx^\lambda} \right]_\nu^\mu$$

$$\mathbf{R}(C) = \mathbf{R}(s_1, s_n) \cdots \mathbf{R}(s_2, s_1)$$

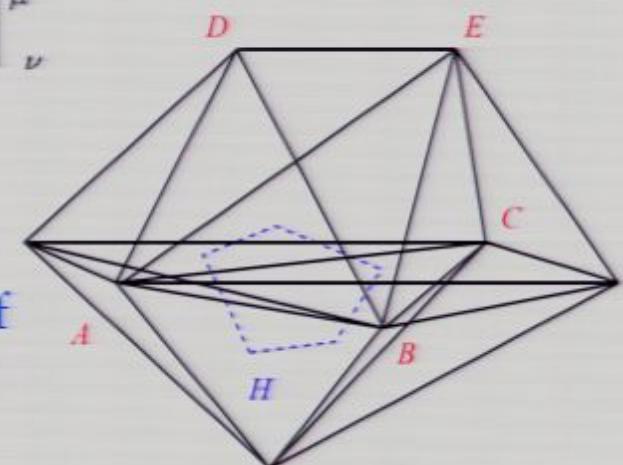
Due to the hinge's intrinsic orientation, only components of the vector in the plane *perpendicular to the hinge* are rotated:

$$U_{\mu\nu}(h) = N \epsilon_{\mu\nu\alpha_1\alpha_{d-2}} l_{(1)}^{\alpha_1} \cdots l_{(d-2)}^{\alpha_{d-2}}$$

$$R_\nu^\mu(C) = \left(e^{\delta U} \right)_\nu^\mu$$

$$R_{\mu\nu\lambda\sigma}(h) = \frac{\delta(h)}{A_C(h)} U_{\mu\nu}(h) U_{\lambda\sigma}(h)$$

$$R(h) = 2 \frac{\delta(h)}{A_C(h)}$$

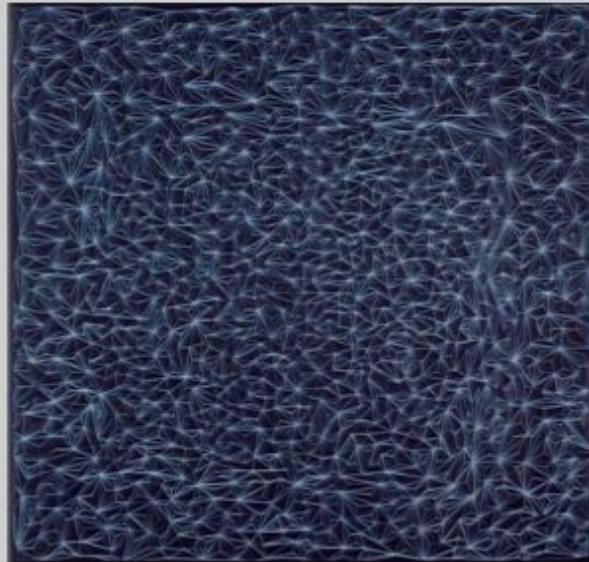


Elementary polygonal path around a hinge (triangle) in four dimensions

Exact lattice Bianchi identity,

$$\prod_{\substack{\text{hinges } h \\ \text{meeting on edge } p}} \left[e^{\delta(h) U(h)} \right]_\nu^\mu = 1$$

Choice of Lattice Structure

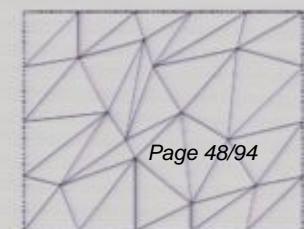
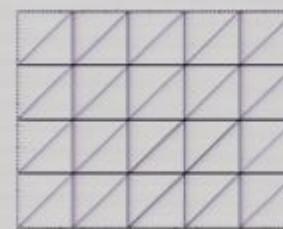
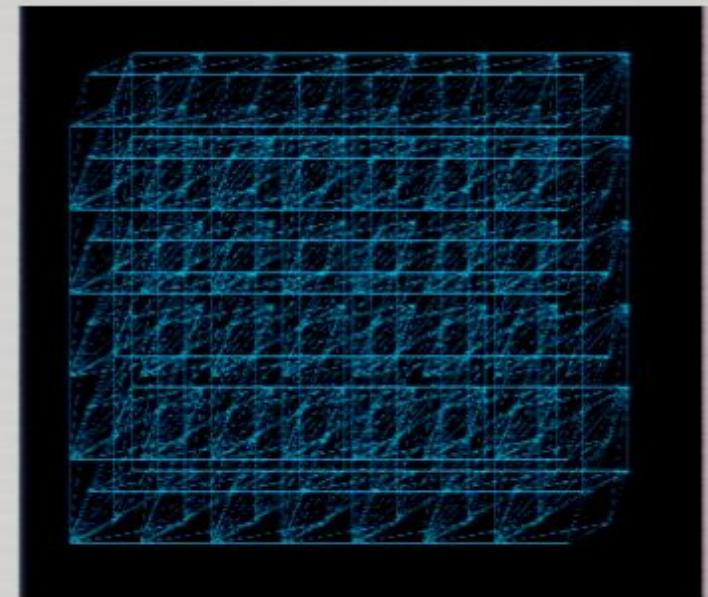
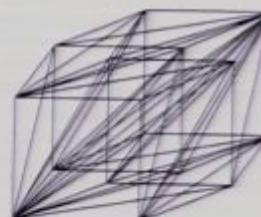


A not so regular lattice ...

Timothy Nolan,
Carl Berg Gallery, Los Angeles

... and a more regular one:

Regular geometric objects can be stacked - to form a coordinated lattice of infinite extent.



*

Lattice Measure

Metric deformations linearly related to
squared edge lengths

$$\delta g_{ij}(l^2) = \frac{1}{2} (\delta l_{0i}^2 + \delta l_{0j}^2 - \delta l_{ij}^2)$$

Jacobian from g 's to l 's is constant within a simplex,

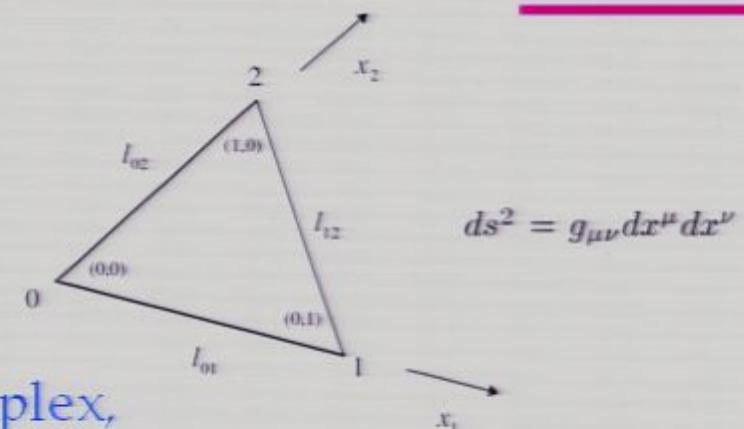
$$\left(\frac{1}{d!} \sqrt{\det g_{ij}(s)} \right)^\sigma \prod_{i \geq j} dg_{ij}(s) = (-\tfrac{1}{2})^{\frac{d(d-1)}{2}} [V_d(l^2)]^\sigma \prod_{k=1}^{d(d+1)/2} dl_k^2$$

—————> $\int [dl^2] = \int_0^\infty \prod_s (V_d(s))^\sigma \prod_{ij} dl_{ij}^2 \Theta[l_{ij}^2]$

R. Schrader et al, 1982 ;
J. Hartle, 1984 ;
H. & Williams, 1984 ; 1998

Alternatively, can construct the discrete analog of DeWitt's (super) metric [R. Schrader]...

$$\| \delta g(s) \|^2 = \sum_s G^{ijkl}(g(s)) \delta g_{ij}(s) \delta g_{kl}(s)$$



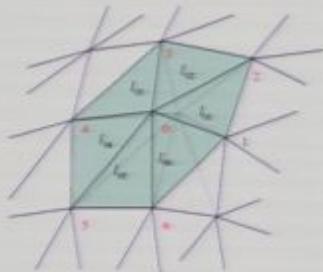
*

Lattice Measure is Non-Trivial

There are important nontrivial constraints on the lattice gravitational measure,

$$\int [dl^2] = \int_0^\infty \prod_s (V_d(s))^\sigma \prod_{ij} dl_{ij}^2 \Theta[l_{ij}^2]$$

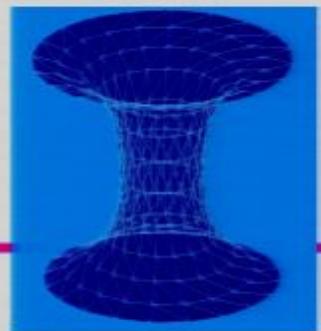
which is generally subject to the "triangle inequality constraints:



$$\left\{ \begin{array}{l} l_{ij}^2 > 0 \\ V_k^2 = \left(\frac{1}{k!}\right)^2 \det g_{ij}^{(k)}(s) > 0 \quad k=2\dots d \end{array} \right.$$

Generally these are implied in the *continuum* functional measure as well, but are normally not spelled out in detail ...

Lattice Path Integral



General structure follows from *edge to metric* correspondence,

$$g_{ij} = \frac{1}{2} (l_{1,i+1}^2 + l_{1,j+1}^2 - l_{i+1,j+1}^2) \quad V_d = \frac{1}{d!} \sqrt{\det g_{ij}}$$

$$I_E[g] = \lambda_0 \Lambda^d \int dx \sqrt{g} - \frac{1}{16\pi G_0} \Lambda^{d-2} \int dx \sqrt{g} R \longrightarrow I_L = \lambda_0 \sum_h V_h(l^2) - 2\kappa_0 \sum_h \delta_h(l^2) A_h(l^2)$$

$$Z = \int [dg_{\mu\nu}] e^{-\lambda_0 \int d^d x \sqrt{g} + \frac{1}{16\pi G} \int d^d x \sqrt{g} R} \longrightarrow Z_{\text{latt}}(\lambda_0, k) = \int [dl^2] \exp(-I_{\text{latt}}(l^2))$$

$$\int [dg_{\mu\nu}] = \int \prod_x [g(x)]^{\frac{(d-4)(d+1)}{8}} \prod_{\mu \geq \nu} dg_{\mu\nu}(x) \longrightarrow \int [dl^2] \equiv \int_0^\infty \prod_{ij} dl_{ij}^2 \prod_s [V_d(s)]^\sigma \Theta(l_{ij}^2)$$

Without loss of generality, one can set bare $\lambda_0 = 1$ (fixed volume);

Besides the cutoff, the only relevant coupling is κ (or G).

Lattice Weak Field Expansion

- Exhibits correct nature of gravitational degrees of freedoms in the *lattice weak field limit*.
- Allows *clear connection between lattice and continuum operators*.

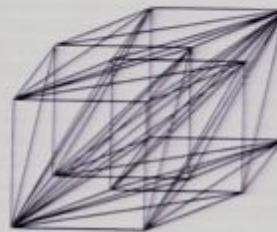
... start from Regge lattice action

Roček and Williams, PLB 1981
CMS 1983, T.D. Lee 1984

$$- 2 \kappa_0 \sum_h \delta_h(l^2) A_h(l^2)$$

... call small edge fluctuations “*e*” :

$$I_R = \frac{1}{2} \sum_{ij} e_i M_{ij} e_j$$



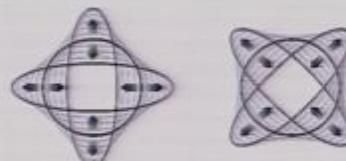
... then Fourier transform, and express result in terms of metric

deformations :

$$\delta g_{ij}(l^2) = \frac{1}{2} (\delta l_{0i}^2 + \delta l_{0j}^2 - \delta l_{ij}^2)$$

... obtaining in the vacuum gauge precisely the familiar *TT form* in $k \rightarrow 0$ limit:

$$\frac{1}{4} \mathbf{k}^2 \bar{h}_{ij}^{TT}(\mathbf{k}) h_{ij}^{TT}(\mathbf{k})$$



* Wilson Loop vs. Loop correlations

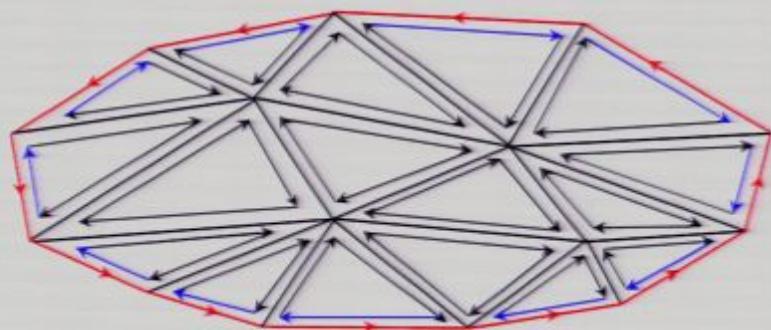


FIG. 2 (color online). Gravitational analog of the Wilson loop.

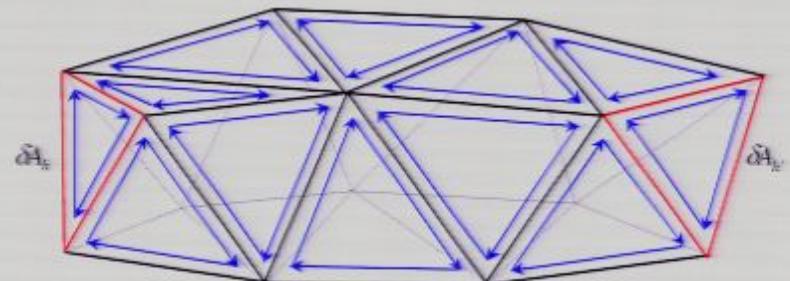


FIG. 6 (color online). Correlations between action contributions on hinge h and hinge h' arise to lowest order in the strong

G. Modanese PRD, NPB 1995

$$R^\alpha{}_\beta(C) = \left[\mathcal{P} \exp \left\{ \oint_{\text{path } C} \Gamma_\lambda dx^\lambda \right\} \right]^\alpha{}_\beta.$$

* Wilson Loop does *not* give Potential

In ordinary LGT, Wilson loop gives $V(r)$

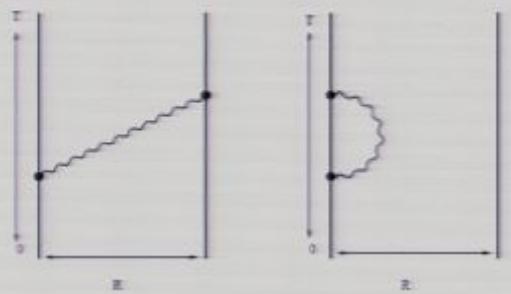
$$W(\Gamma) = \langle \exp \left\{ ie \oint_{\Gamma} A_{\mu}(x) dx^{\mu} \right\} \rangle$$

$$V(R) = - \lim_{T \rightarrow \infty} \frac{1}{T} \log \langle \exp \left\{ ie \oint_{\Gamma} A_{\mu} dx^{\mu} \right\} \rangle$$

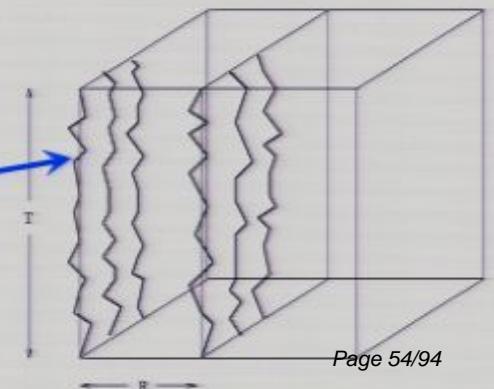
In lattice regularized gravity, potential is computed from the correlation of geodesic *line segments*, associated with the particle's world line:

$$\mu \int_{\tau(a)}^{\tau(b)} d\tau \sqrt{g_{\mu\nu}(x) \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau}}$$

$$L(0; \mu_1) = \exp \left\{ -\mu_1 \int d\tau \sqrt{g_{\mu\nu}(x) \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau}} \right\}$$



G. Modanese, PRD 1994;
NPB 1995



*

Correlations

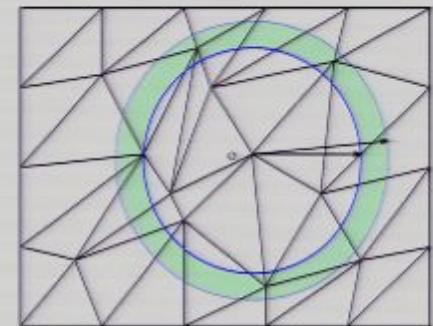
... of invariant operators at fixed geodesic distance.

But distance is a function of metric, which fluctuates:

$$d(x, y | g) = \min_{\xi} \int_{\tau(x)}^{\tau(y)} d\tau \sqrt{g_{\mu\nu}(\xi) \frac{d\xi^\mu}{d\tau} \frac{d\xi^\nu}{d\tau}}$$

$$< \int dx \int dy \sqrt{g} R(x) \sqrt{g} R(y) \delta(|x - y| - d) >$$

$$\longrightarrow G_R(d) \equiv < \sum_{h \supset x} \delta_h A_h \sum_{h' \supset y} \delta_{h'} A_{h'} \delta(|x - y| - d) >_c$$



$$G(x, y | g) = < x | \frac{1}{-\Delta(g) + m^2} | y >$$
$$\sim_{d(x,y) \rightarrow \infty} d^{-(d-1)/2}(x, y) \exp\{-m d(x, y)\}$$

*

Dynamical Triangulations

- Simplified version of Regge Gravity
- Edge lengths *fixed to unity*, vary incidence matrix [David 1984, ...]



$$V_d = \frac{1}{d!} \sqrt{\frac{d+1}{2^d}} \quad \cos \theta_d = \frac{1}{d} \quad \delta(h) = 2\pi - n_s(h) \theta_d$$

an integer

- No immediate notion of continuous metric, or continuous diffeomorphisms
- Curvature varies in discrete steps.
- No continuous metric deformations – hence no w.f.e., and no gravitons (at least not in an explicit way).

Constraints on functional measure unclear, since theory contains no metric.

Pathological behavior of Euclidean theory [Loll et al] → explore

numerically *Lorentzian* path integral (with unresolved convergence issues).

*

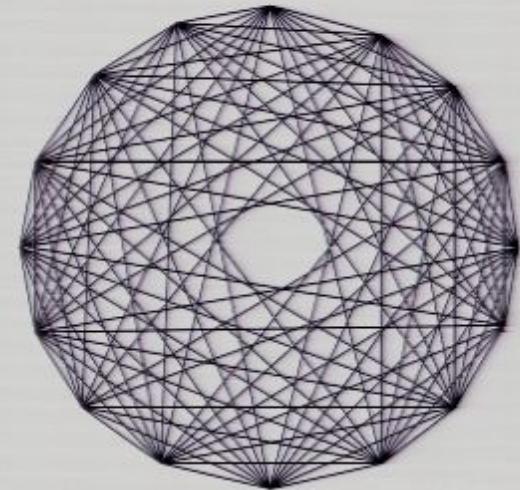
Large D Limit



Early work in *continuum* by A. Strominger (1984, $\lambda=0$), ...

On the lattice, phase transition persists at $d = \infty$:

$$\left\{ \begin{array}{l} k_c = \frac{\lambda_0^{\frac{d-2}{d}}}{d^3} \left[\frac{2}{d} \frac{d! 2^{d/2}}{\sqrt{d+1}} \right]^{2/d} \\ l_0^2 = \frac{1}{\lambda_0^{2/d}} \left[\frac{2}{d} \frac{d! 2^{d/2}}{\sqrt{d+1}} \right]^{2/d} \end{array} \right.$$



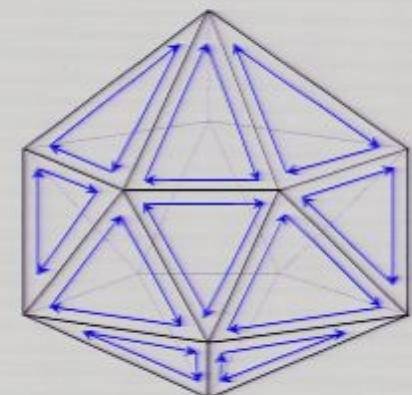
- Conformal mode instability disappears, $O(1/d)$.
- At large d , partition function at large G dominated by *closed surfaces*, tiled with elementary parallel transport polygonal loops.
Large surfaces dominant as $k \rightarrow k_c$.

N-cross polytope, homeomorphic to a sphere

H H & R Williams, PRD 2006

Large D Limit - Exponent ν

- At large d , characteristic size ξ of random surface diverges logarithmically as $G \rightarrow G_c$ (D. Gross PLB 1984).
- Suggests universal correlation length exponent $\nu = 0$.



Known results from random surface theory then imply:

$$\xi \sim \sqrt{\log T} \underset{k \rightarrow k_c}{\sim} |\log(k_c - k)|^{1/2}$$

$$\nu = 1/(d-1)$$



$$\nu = 1/2d$$

Lattice

D. Litim PRL 2004,
PLB 2007

$$\text{scalar field} \quad \nu = \frac{1}{2}$$

$$\text{lattice gauge field} \quad \nu = \frac{1}{4}$$

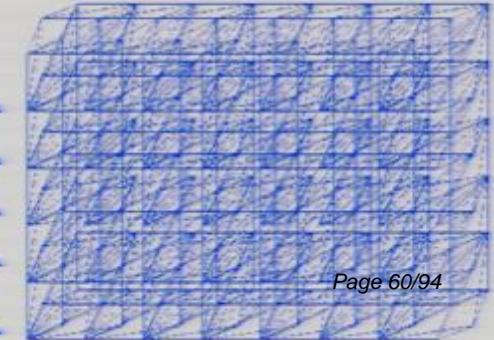
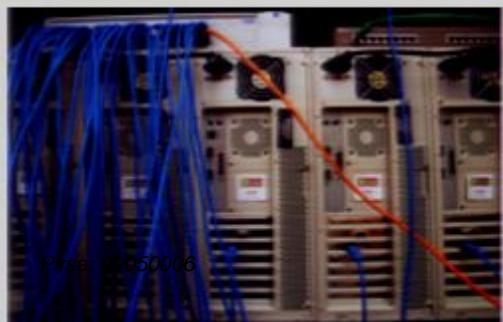
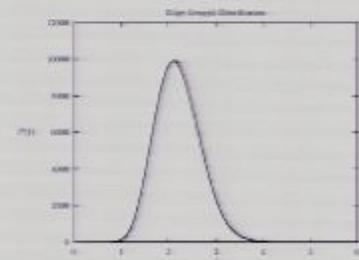
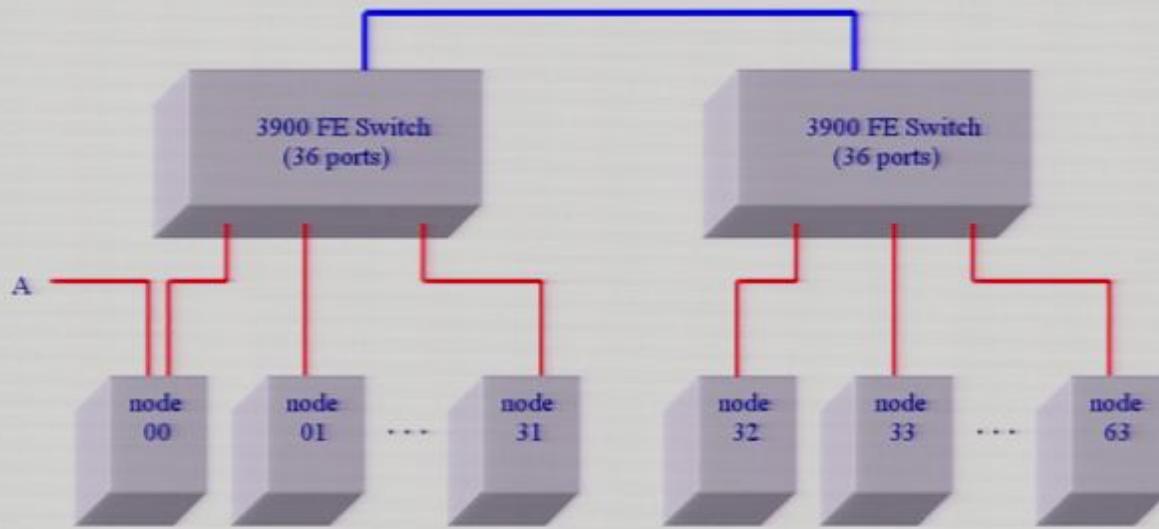
$$\text{lattice gravity} \quad \nu = 0$$

Numerical Evaluation of Z

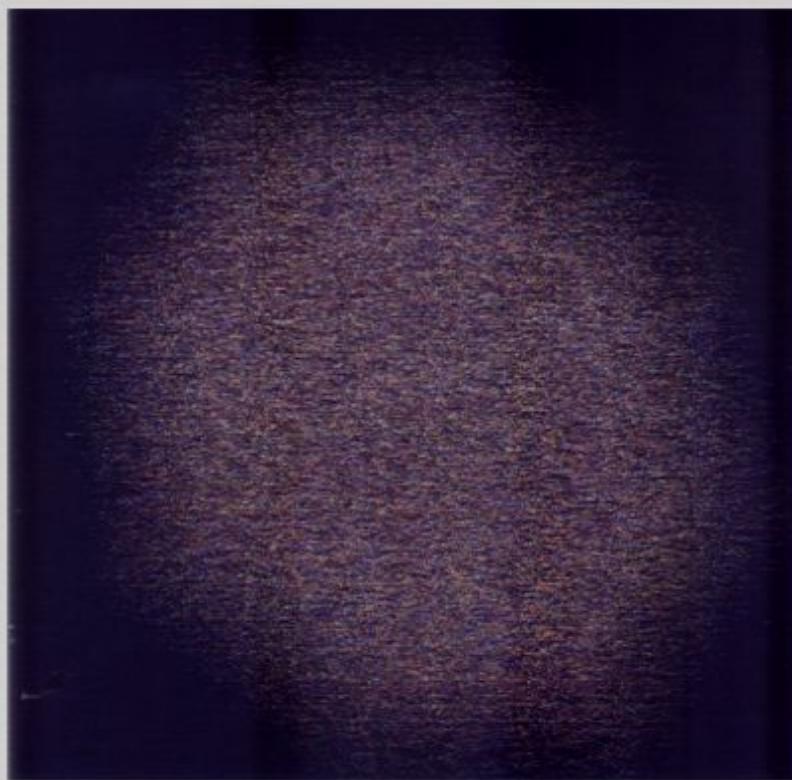


CM5 at NCSA, 512 processors

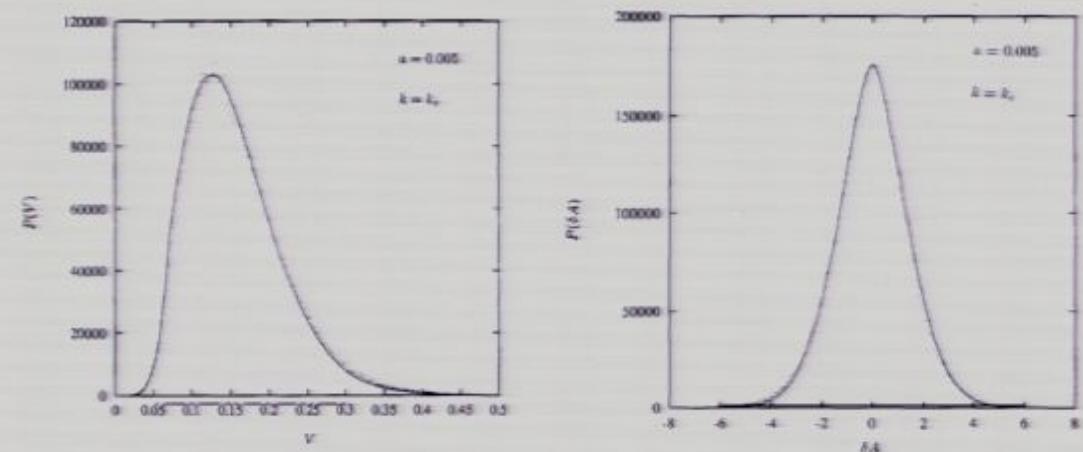
Dedicated Parallel Supercomputer



Edge length/metric distributions



- $L=4 \rightarrow 6,144$ simplices
- $L=8 \rightarrow 98,304$ simplices
- $L=16 \rightarrow 1,572,864$ simplices
- $L=32 \rightarrow 25,165,824$ simplices

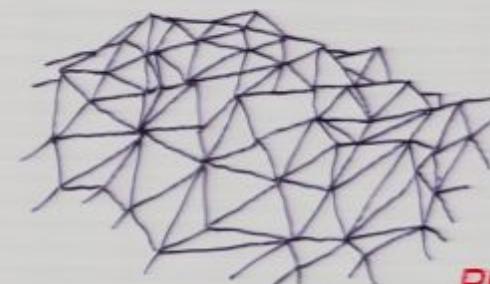


Two Phases of L. Quantum Gravity

Earliest studies of Regge lattice theories found evidence for:

$G > G_c$ Smooth phase: $R \approx 0$

$$\langle g_{\mu\nu} \rangle \approx c \eta_{\mu\nu}$$

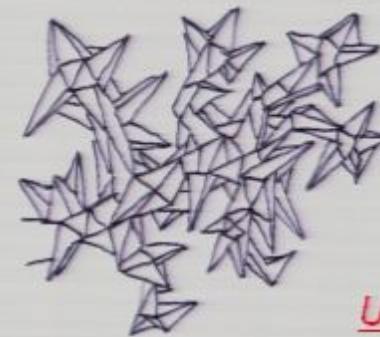


Physical

$$N(\tau) \underset{\tau \rightarrow \infty}{\sim} \tau^{d_v}$$

$G < G_c$. Rough phase:
branched polymer, $d \approx 2$

$$\langle g_{\mu\nu} \rangle = 0$$



Unphysical

Similar two-phase structure also found later in some $d=4$ DTRS models [Migdal et al ...]

Lattice manifestation of conformal instability

Invariant Averages

$$\mathcal{R}(k) \sim \frac{\langle \int d^4x \sqrt{g} R(x) \rangle}{\langle \int d^4x \sqrt{g} \rangle}$$

$$\chi_{\mathcal{R}}(k) \sim \frac{\langle (\int \sqrt{g} R)^2 \rangle - \langle \int \sqrt{g} R \rangle^2}{\langle \int \sqrt{g} \rangle}$$

$$\mathcal{R}(k) \sim \frac{1}{V} \frac{\partial}{\partial k} \ln Z_L$$

$$\chi_{\mathcal{R}}(k) \sim \frac{1}{V} \frac{\partial^2}{\partial k^2} \ln Z_L$$

Singularities in the free energy F are determined from non-analiticities in invariant local averages.

- Divergent local averages provide information about non-trivial *exponents*.
- Finite Size Scaling (FSS) theory useful.

$$O(L, t) = L^{x_O/\nu} [\tilde{f}_O(L t^\nu) + \mathcal{O}(L^{-\omega})]$$

- Correlations* are harder to compute directly (geodesic distance).

$$F_{sing}(G) \sim \xi^{-d}$$

$$\xi \sim t^{-\nu}$$

"Scaling assumption" for $F = \ln Z$

Determination of Scaling Exponents

$$\mathcal{R}(k) \underset{k \rightarrow k_c}{\sim} -A_{\mathcal{R}} (k_c - k)^{\delta} \quad \nu = \frac{1 + \delta}{d}$$

$$\chi_{\mathcal{R}}(k) \underset{k \rightarrow k_c}{\sim} -A_{\mathcal{R}} (k_c - k)^{-(1-\delta)}$$

Scaling assumption:

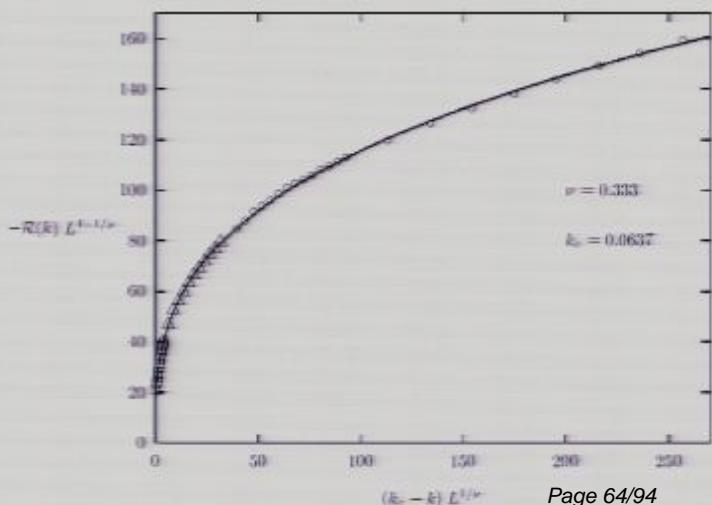
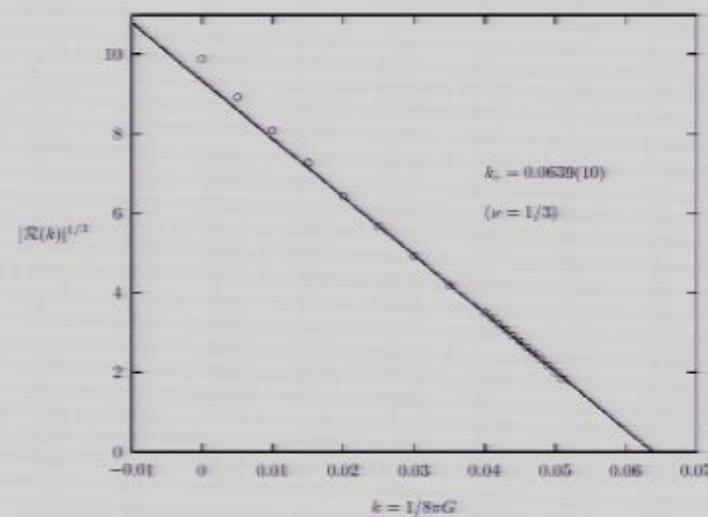
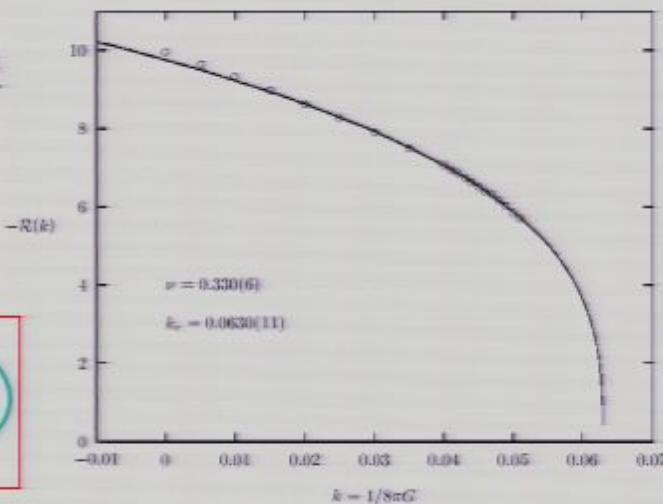
$$F_{sing}(G) \sim \xi^{-d}$$

$$\xi(k) \equiv m(k)^{-1} \underset{k \rightarrow k_c}{\sim} A_{\xi} (k_c - k)^{-\nu}$$

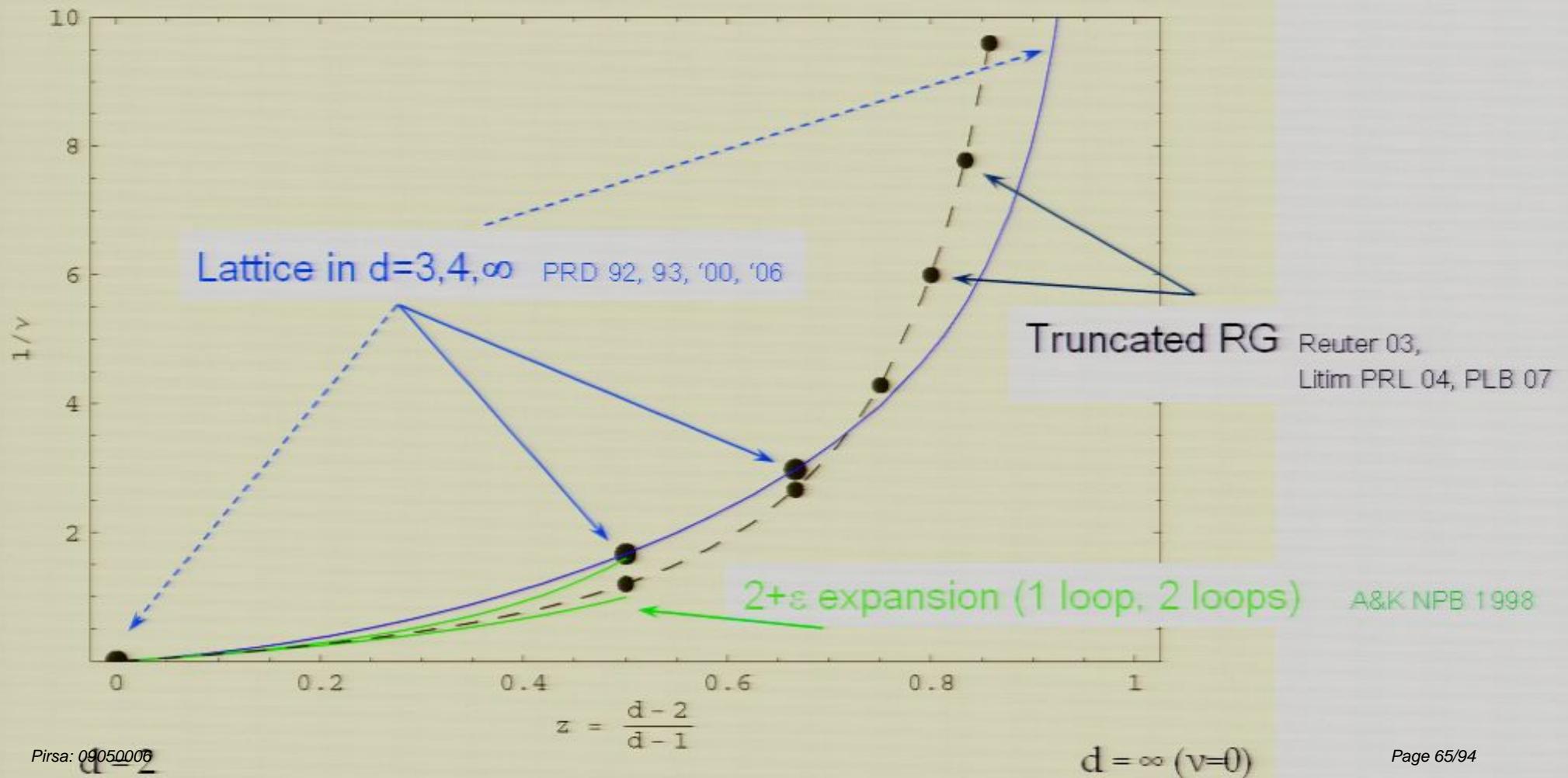
Find value close to 1/3:

$$k_c = 0.0636(11) \quad \nu = 0.335(9)$$

$$\nu \approx 1/3$$

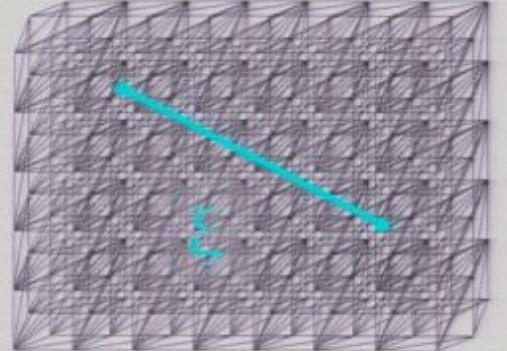


Gravitational exponent ν compared



(Lattice) Continuum Limit $\Lambda \rightarrow \infty$

Standard (Wilson) procedure in cutoff field theory:

$$\xi = 1/m \quad m \underset{G(\Lambda) \rightarrow G_c}{\sim} \Lambda \left[\frac{G(\Lambda) - G_c}{a_0 G_c} \right]^\nu$$


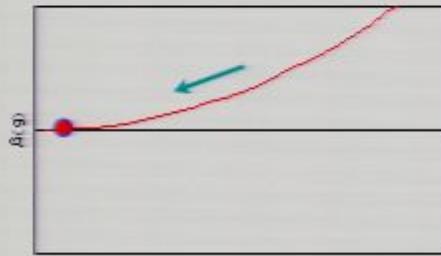
RG invariant correlation length ξ is kept fixed

UV cutoff $\Lambda \rightarrow \infty$
(average lattice spacing $\rightarrow 0$)

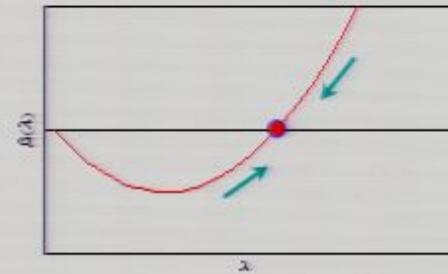
Bare G must approach UV fixed point at G_c .

The *very same* relation gives the RG running of $G(\mu)$ close to the FP.

RG Running Scenarios

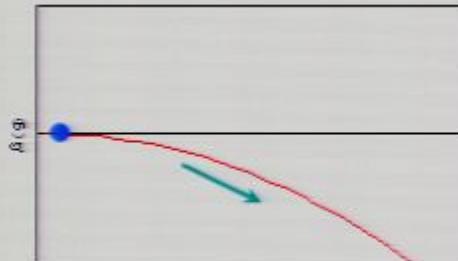


"Triviality" of lambda phi 4

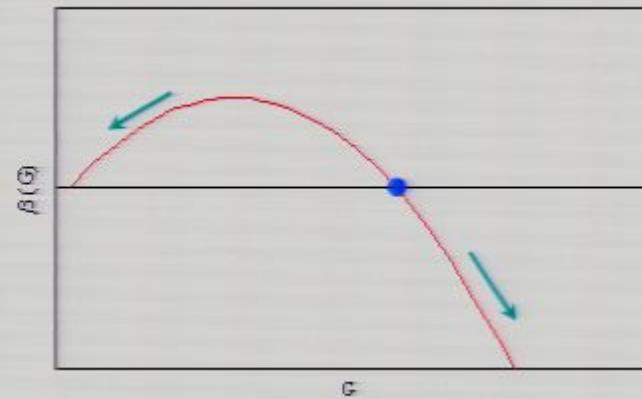


Wilson-Fisher FP in d<4

- Coupling gets weaker at *large r*
- ... approaches an IR FP at large r.
- ... gets weaker at *small r*: UV FP
- Both possibilities can coexist: nontrivial UV fixed point.



Asymptotic freedom of YM



Ising model, sigma-model, Gravity (2+epsilon, lattice)

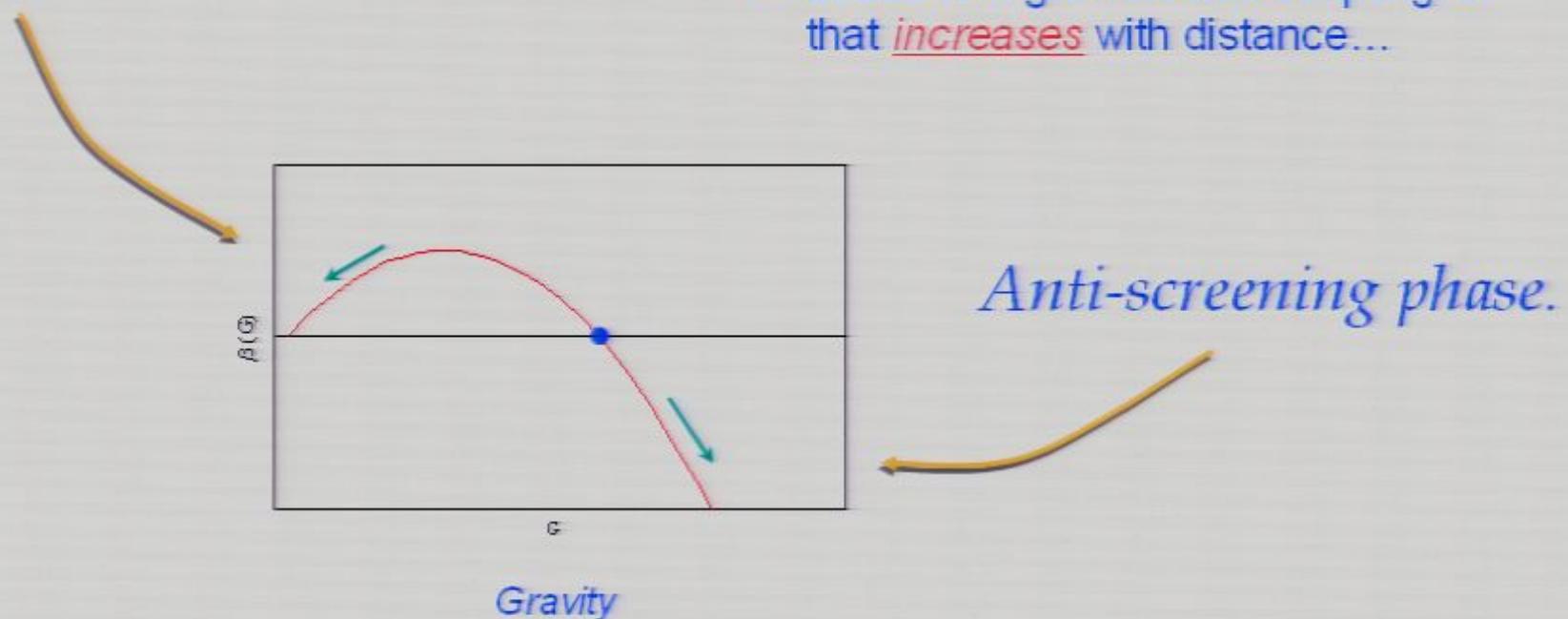
Callan-Symanzik. beta function(s):

$$\mu \frac{\partial}{\partial \mu} G(\mu) = \beta(G(\mu))$$

Only One Phase?

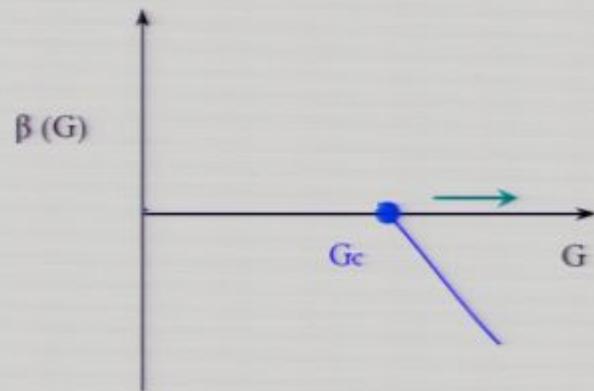
Weak coupling phase is seemingly unphysical (branched polymer).

- ✓ Lattice results appear to exclude the weak coupling phase as physically relevant...
- ✓ Leads to a gravitational coupling G that increases with distance...



Next question then :

Is this new scenario physically acceptable?



Running Newton's G

- ξ is a new invariant scale of gravity.
- Newton's constant G must run (as in $2+\varepsilon$).
- *Cutoff* dependence determines β -function :

$$\Lambda \frac{d}{d\Lambda} m(\Lambda, G(\Lambda)) = 0 \quad \text{and} \quad \Lambda \frac{\partial}{\partial \Lambda} G(\Lambda) = \beta(G(\Lambda)) \quad \rightarrow \quad \beta(G) = -\frac{F(G)}{\partial F(G)/\partial G}$$

[In fact, one can be quite specific ...

$$\beta(G) \underset{G \rightarrow G_c}{\sim} -\frac{1}{\nu}(G - G_c) \quad \leftrightarrow \quad m \sim \Lambda \exp \left(- \int^G \frac{dG'}{\beta(G')} \right) \underset{G \rightarrow G_c}{\sim} \Lambda |G - G_c|^{-1/\beta(G_c)} \quad]$$

Running of G det. largely by ξ and ν :

H H & R Williams, NPB 1994

$$\mu \frac{\partial}{\partial \mu} G(\mu) = \beta(G(\mu)) \quad \rightarrow$$

$$G(k^2) = G_c \left[1 + a_0 \left(\frac{m^2}{k^2} \right)^{\frac{1}{2\nu}} + O((m^2/k^2)^{\frac{1}{\nu}}) \right]$$

So, what value to take for ξ ?

- ξ is an RG invariant.
- $m=1/\xi$ has dimensions of a mass.

In Yang-Mills : $m = \text{mass of spin-0 glueball}$

*

Three Theories Compared

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\partial^\mu F_{\mu\nu} + \mu^2 A_\nu = 4\pi e j_\nu$$

$$\partial^\mu \partial_\mu \phi + m^2 \phi = \frac{g}{3!} \phi^3$$

$$\lambda_{phys} \simeq \frac{1}{\xi^2}$$

$$m = 1/\xi$$

*

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RG invariants

$$m = 1/\xi$$

*

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$$\partial^\mu F_{\mu\nu} + \mu^2 A_\nu = 4\pi e j_\nu$$

$$\partial^\mu \partial_\mu \phi + m^2 \phi = \frac{g}{3!} \phi^3$$

Suggests $\lambda_{phys} \simeq \frac{1}{\xi^2}$



RG invariants



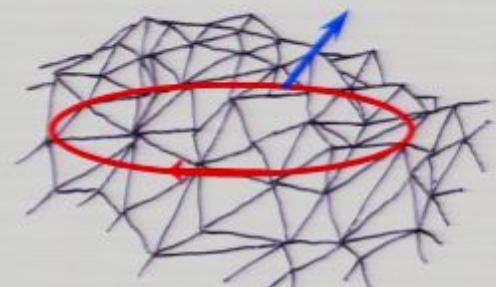
Running couplings

$$m = 1/\xi$$

Better: Gravitational Wilson Loops

- In gravity, Wilson loop not related to static potential
[G. Modanese PRD 1993; 1994]
- Parallel transport of a vector done via lattice rotation matrix

$$R^\alpha_\beta = \left[\mathcal{P} e^{\int_{\substack{\text{path} \\ \text{between simplices}}} \Gamma^\lambda dx_\lambda} \right]^\alpha_\beta$$



For a *large* closed circuit obtain *Wilson loop* - which can be computed at strong coupling using a first order formulation of Regge gravity [Caselle, d'Adda, Magnea PLB 1989]

$$W(\Gamma) \sim \text{Tr } \mathcal{P} \exp \left[\int_C \Gamma^\lambda .. dx_\lambda \right] \sim \exp \left[\int_{S(C)} R^\cdot_{.. \mu\nu} A_C^{\mu\nu} \right] \sim \exp(-A/\xi^2)$$

- Stokes theorem -

- ξ related to curvature.
- ξ RG invariant.
- prediction of positive cosmological constant.

$$\lambda_{phys} \simeq \frac{1}{\xi^2}$$

“Area law”
would follows from loop tiling ...

*

Vacuum Condensate Picture of QG

- Lattice Quantum Gravity: Curvature condensate

See also J.D.Bjorken, PRD '05

$$\mathcal{R} \simeq (10^{-30} eV)^2 \sim \xi^{-2} \quad \lambda_{phys} \simeq \frac{1}{\xi^2}$$

- Quantum Chromodynamics: Gluon and Fermion condensate

$$\alpha_S \langle F_{\mu\nu} \cdot F^{\mu\nu} \rangle \simeq (250 MeV)^4 \sim \xi^{-4}$$

$$\xi_{QCD}^{-1} \sim \Lambda_{\overline{MS}}$$

$$(\alpha_S)^{4/\beta_0} \langle \bar{\psi} \psi \rangle \simeq -(230 MeV)^3 \sim \xi^{-3}$$

- Electroweak Theory: Higgs condensate

Effective Theory

Graviton Vacuum Polarization Cloud

Require a covariant running of G :

Effective field equations

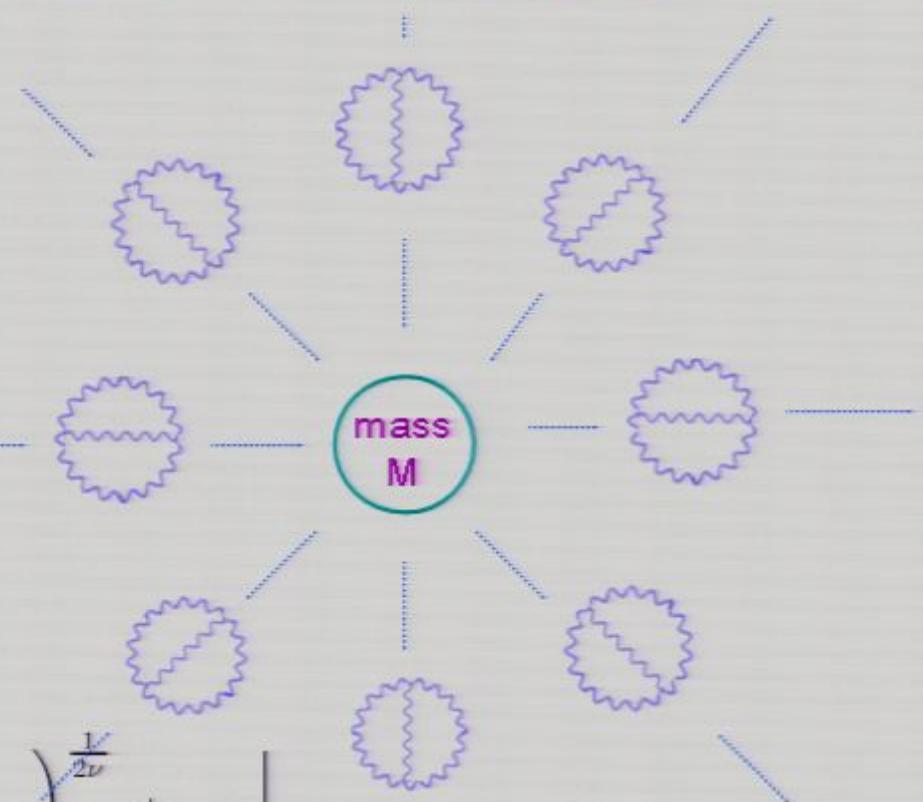
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G(\square) T_{\mu\nu}$$

$$\lambda \simeq 1/\xi^2$$

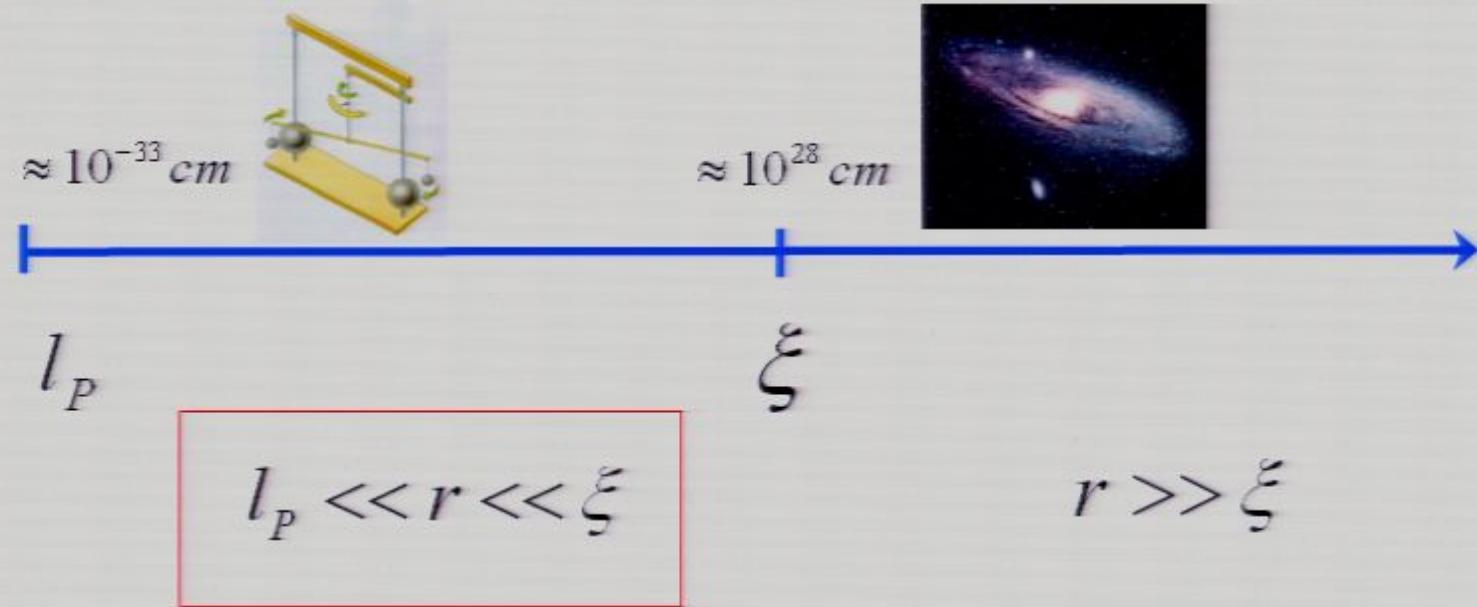
$$G \rightarrow G(\square) = G \left[1 + a_0 \left(\frac{m^2}{-\square + m^2} \right)^{\frac{1}{2\nu}} + \dots \right]$$

$$\square \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$$

Source mass M surrounded by virtual graviton cloud

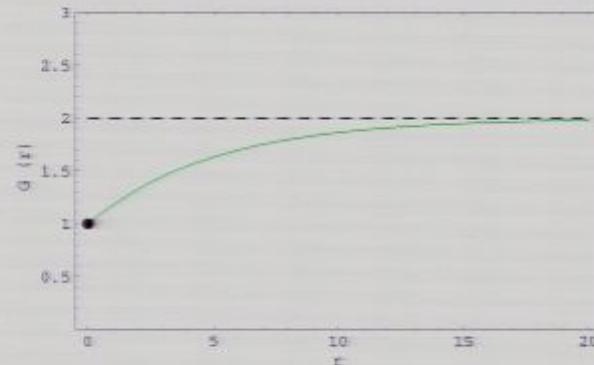


Relative Scales in the Cutoff Theory



At the Planck length new terms appear:

higher derivative terms, string corrections, conformal anomaly contributions...



Cosmological Solutions



Explore generally covariant effective field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G (1 + A(\square)) T_{\mu\nu}$$

G. Veneziano; G.A. Vilkovisky
HH & R Williams PRD 2007
Deser & Woodard 2008

$$A(\square) = c_\square \left(\frac{1}{\xi^2 \square} \right)^{1/2\nu}$$

... for RW metric

$\Lambda = 0$ initially for simplicity

$$ds^2 = -dt^2 + a^2(t) \left\{ \frac{dr^2}{1 - k r^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right\}$$

... and perfect fluid $p(t) = 0$

Consistency condition:

$$\nabla^\mu \tilde{T}_{\mu\nu} \equiv \nabla^\mu [(1 + A(\square)) T_{\mu\nu}] = 0$$

$$\square = g^{\mu\nu} \nabla_\mu \nabla_\nu \quad \square T^{\alpha\beta\dots}_{\gamma\delta\dots} = g^{\mu\nu} \nabla_\mu (\nabla_\nu T^{\alpha\beta\dots}_{\gamma\delta\dots})$$

Specific form of D'Alembertian depends on object it acts on.

*

Trace Equation

- Trace equation is a *lot* simpler ...

$$R = -8\pi G (1 + A(\square)) T_\mu^\mu$$

$$\square S(x) = \frac{1}{\sqrt{g}} \partial_\mu g^{\mu\nu} \sqrt{g} \partial_\nu S(x)$$
$$T_\mu^\mu = -\rho(t) \quad \Lambda = 0$$

Assume power law $\rho(t) = \rho_0 t^\beta$ and act repeatedly on source

$$\square^n (-\rho(t)) \rightarrow 4^n (-1)^{n+1} \frac{\Gamma(\frac{\beta}{2} + 1) \Gamma(\frac{\beta+3\alpha+1}{2})}{\Gamma(\frac{\beta}{2} + 1 - n) \Gamma(\frac{\beta+3\alpha+1}{2} - n)} \rho_0 t^{\beta-2n}$$

Analytically continue in n : $n = -1/2\nu$

→ existence of solution for larger t requires: $\beta = -2 - 1/\nu$

Cosmological Solutions



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Analytically continue in n : $n = -1/2\nu$

→ existence of solution for larger t requires: $\beta = -2 - 1/\nu$

*

Full Effective Field Equation

- Full effective field equation involves D'Alembertian on tensor

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G (1 + A(\square)) T_{\mu\nu}$$

$$A(\square) = c_\square \left(\frac{1}{\xi^2 \square} \right)^{1/2\nu}$$

Repeated action of D'Alembertian \square^n , $n = -1/2\nu$

$$\Lambda = 0$$

$$(\square T_{\mu\nu})_{tt} = 6 [\rho(t) + p(t)] \left(\frac{\dot{R}(t)}{R(t)} \right)^2 - 3 \dot{\rho}(t) \frac{\dot{R}(t)}{R(t)} - \ddot{\rho}(t)$$

$$(\square T_{\mu\nu})_{rr} = \frac{1}{1 - k r^2} \left\{ 2 [\rho(t) + p(t)] \dot{R}(t)^2 - 3 \dot{p}(t) R(t) \dot{R}(t) - \ddot{p}(t) R(t)^2 \right\}$$

$$(\square T_{\mu\nu})_{\theta\theta} = r^2 (1 - k r^2) (\square T_{\mu\nu})_{rr}$$

$$(\square T_{\mu\nu})_{\varphi\varphi} = r^2 (1 - k r^2) \sin^2 \theta (\square T_{\mu\nu})_{rr}$$

→ existence of solution requires $\beta = -2 - 1/\nu$ as before,

and $R(t) \sim t^\alpha \sim t^{1/2}$ $\rho(t) \sim t^{-2-1/\nu} \sim (R(t))^{-2(2+1/\nu)}$

Cosmological Solutions – Cont'd

- Solution acquires a significant *radiation-like* (vac. pol.) component at large times.

$$\begin{aligned}\frac{k}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)} &= \frac{8\pi G(t)}{3} \rho(t) + \frac{1}{3\xi^2} && t-t \text{ eq.} \\ &= \frac{8\pi G}{3} \left[1 + c_\xi (t/\xi)^{1/\nu} + \dots \right] \rho(t) + \frac{1}{3\xi^2} && \rho_{eff}(t) = \frac{G(t)}{G} \rho(t)\end{aligned}$$

$$\frac{k}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)} + \frac{2\ddot{a}(t)}{a(t)} = -\frac{8\pi G}{3} \left[c_\xi (t/\xi)^{1/\nu} + \dots \right] \rho(t) + \frac{1}{\xi^2} \quad r-r \text{ eq.} \quad p_{eff}(t) = \frac{1}{3} \left(\frac{G(t)}{G} - 1 \right) \rho(t)$$

Effective pressure term

Similarities to:

$$\begin{aligned}p(t) &= \omega \rho(t) \\ \omega &= 1/3\end{aligned}$$

Cosmological Solutions – Cont'd

- Solution acquires a significant *radiation-like* (vac. pol.) component at large times.

$$\begin{aligned} \frac{k}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)} &= \frac{8\pi G(t)}{3} \rho(t) + \frac{1}{3\xi^2} && t-t \text{ eq.} \\ &= \frac{8\pi G}{3} \left[1 + c_\xi (t/\xi)^{1/\nu} + \dots \right] \rho(t) + \frac{1}{3\xi^2} && \rho_{eff}(t) = \frac{G(t)}{G} \rho(t) \end{aligned}$$

$$\frac{k}{a^2(t)} + \frac{\dot{a}^2(t)}{a^2(t)} + \frac{2\ddot{a}(t)}{a(t)} = -\frac{8\pi G}{3} \left[c_\xi (t/\xi)^{1/\nu} + \dots \right] \rho(t) + \frac{1}{\xi^2} \quad r-r \text{ eq.} \quad p_{eff}(t) = \frac{1}{3} \left(\frac{G(t)}{G} - 1 \right) \rho(t)$$

Effective pressure term

Similarities to:

$$p(t) = \omega \rho(t)$$

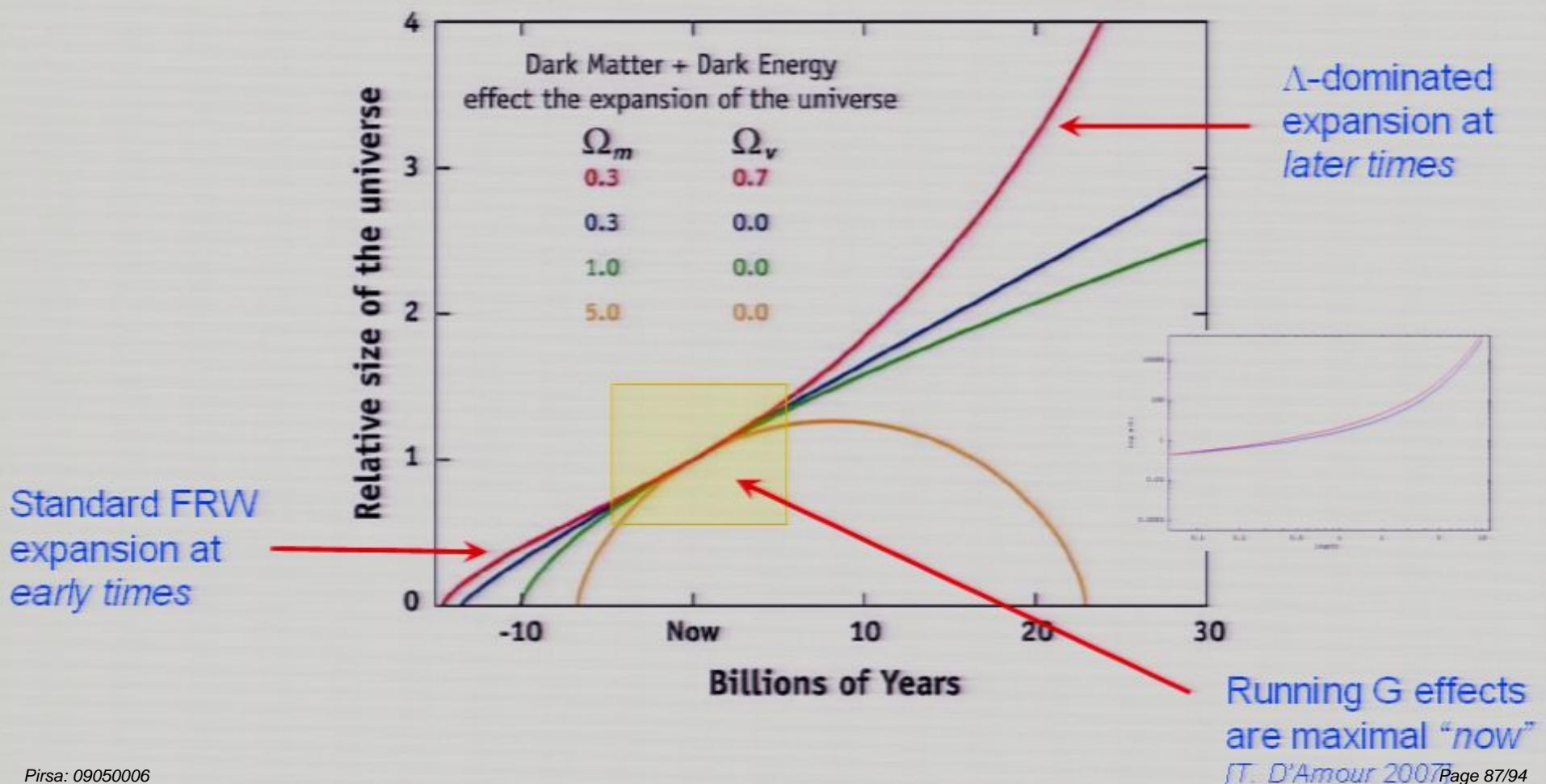
$$\omega = 1/3$$

At (very) large times, G is further modified to:

$$G(t) = G \left[1 + c_\xi \left(\frac{t}{\xi} \right)^{1/\nu} + \dots \right] \longrightarrow G(t) \simeq G \left[1 + c_\xi \left(\frac{t^2}{t^2 + \xi^2} \right)^{\frac{1}{2\nu}} + \dots \right]$$

IR regulator

Modified Cosmological Expansion Rate



Static Isotropic Solution

Start again from *fully covariant* effective field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G (1 + A(\square)) T_{\mu\nu}$$

$$A(\square) = a_0 \left(\frac{m^2}{-\square + m^2} \right)^{1/2\nu}$$

General static isotropic metric

$$\lambda \simeq 1/\xi^2 \longrightarrow 0$$

$$ds^2 = -B(r) dt^2 + A(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

$$A(r)^{-1} = 1 - \frac{2MG}{r} + \frac{\sigma(r)}{r} \quad a_0 \text{ small}$$

$$B(r) = 1 - \frac{2MG}{r} + \frac{\theta(r)}{r} \quad r \gg 2MG$$

Search solution for a point source, or vacuum solution for $r \neq 0$.

$$T_{\mu\nu} = \text{diag}[B(r)\rho(r), A(r)p(r), r^2 p(r), r^2 \sin^2 \theta p(r)]$$

H. & Williams, PLB 2006;
PRD 2007

*

Non-Relativistic Case

- Non-relativistic case first (simpler)

$$G(k^2) \simeq G_c \left[1 + a_0 \left(\frac{m^2}{k^2 + m^2} \right)^{\frac{1}{2\nu}} + \dots \right] \quad m = 1/\xi$$

IR regulator

Static potential obtained, as usual, by FT:

$$\phi(r) = (-M) \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} G(\mathbf{k}^2) \frac{4\pi}{\mathbf{k}^2}$$

$$\left\{ \begin{array}{l} \phi(r) \underset{r \rightarrow \infty}{\sim} -\frac{MG}{r} \left[1 + a_0 \left(1 - c_l (mr)^{\frac{1}{2\nu}-1} e^{-mr} \right) \right] \quad c_l = 1/(\nu 2^{\frac{1}{2\nu}} \Gamma(\frac{1}{2\nu})) \\ \phi(r) \underset{r \rightarrow 0}{\sim} -\frac{MG}{r} + a_0 M G c_s m^{\frac{1}{\nu}} r^{\frac{1}{\nu}-1} + \dots \quad c_s = \nu |\sec(\frac{\pi}{2\nu})| / \Gamma(\frac{1}{\nu}) \end{array} \right.$$

$\nu = 1/3$ special case:

$$(-MG) a_0 m^2 \frac{r^2}{3\pi} \left[\ln\left(\frac{mr}{2}\right) + \gamma - \frac{5}{6} \right] + O(r^3)$$

*

Poisson's Equation



Same result from (radial) Poisson's equation

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = 4\pi G \rho_m(r)$$

with (normalized) vac. pol. density:

$$\rho_m(r) = \frac{1}{8\pi} c_\nu a_0 M m^3 (m r)^{-\frac{1}{2}(3-\frac{1}{\nu})} K_{\frac{1}{2}(3-\frac{1}{\nu})}(m r)$$

$$4\pi \int_0^\infty r^2 dr \rho_m(r) = a_0 M \quad m = 1/\xi$$

*

Poisson's Equation



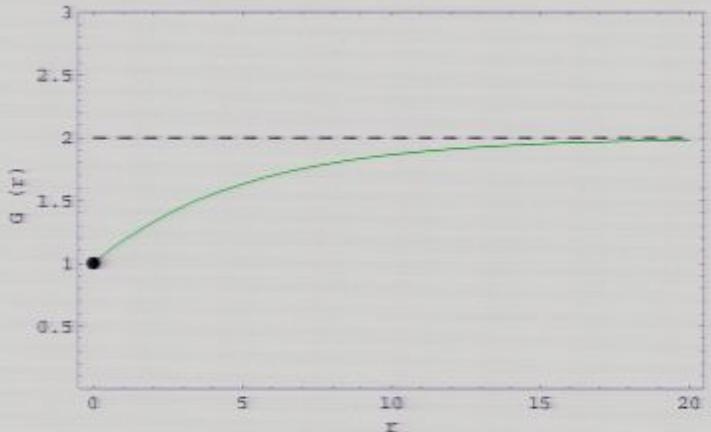
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Closed form solution for $\nu = 1/2$

$$G(r) = G_\infty \left(1 - \frac{a_0}{1 + a_0} e^{-mr} \right)$$

$$\begin{cases} G \equiv G(0) \\ G_\infty \equiv (1 + a_0) G \end{cases}$$

Relativistic Fluid cont'd

Solution of Full Cov. Equation (for $\nu=1/3$)

$$B(r) = 1 - \frac{2MG}{r} + \frac{4a_0 MG m^3}{3\pi} r^2 \ln(mr) + \dots$$

$$A^{-1}(r) = 1 - \frac{2MG}{r} + \frac{4a_0 MG m^3}{3\pi} r^2 \ln(mr) + \dots$$

...which can be consistently interpreted as a $G(r)$:

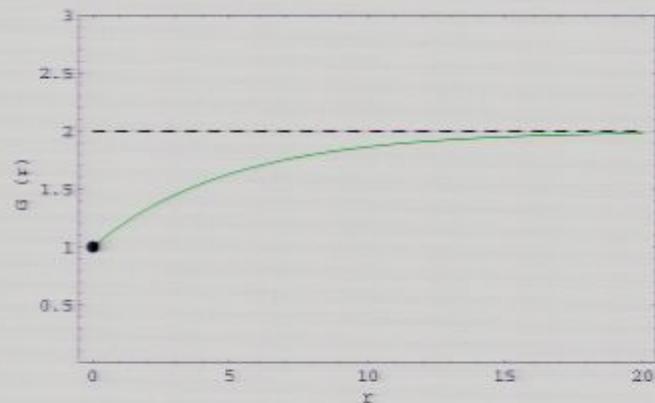
$$G \rightarrow G(r) = G \left(1 + \frac{a_0}{3\pi} m^3 r^3 \ln \frac{1}{m^2 r^2} + \dots \right)$$

$$m = 1/\xi$$

Reminiscent of QED (Uehling) answer:

$$a_0 \approx 42.$$

$$Q(r) = 1 + \frac{\alpha}{3\pi} \ln \frac{1}{m^2 r^2} + \dots \quad mr \ll 1$$



Outlook



- More Work is Needed in Lattice Theory
 - $2 + \varepsilon$ expansion to three loops is a clear, perhaps feasible goal.
 - Careful investigation of 4d latt. gravity should be continued.
 - Status of weak coupling phase unclear.
 - Connection with other lattice models, eg hypercubic.
- Covariant Effective Field Equations
 - Formulation of fractional operators for non-local terms
 - Further investigation on nature of solutions (horizons).
 - Possible Cosmological observable ramifications.
 - Nature of density perturbations with running $G(\square)$

The End