Title: A strong converse for classical channel coding using entangled inputs

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Abstract: A fully general strong converse for channel coding states that when the rate of sending classical information exceeds the capacity of a quantum channel, the probability of correctly decoding goes to zero exponentially in the number of channel uses, even when we allow code states which are entangled across several uses of the channel. Such a statement was previously only known for classical channels and the quantum identity channel. By relating the problem to the additivity of minimum output entropies, we show that a strong converse holds for a large class of channels, including all unital qubit channels, the d-dimensional depolarizing channel and the Werner-Holevo channel. This further justifies the interpretation of the classical capacity as a sharp threshold for information-transmission.

Joint work with Robert Koenig.

A strong converse for channel coding with entangled inputs

Stephanie Wehner

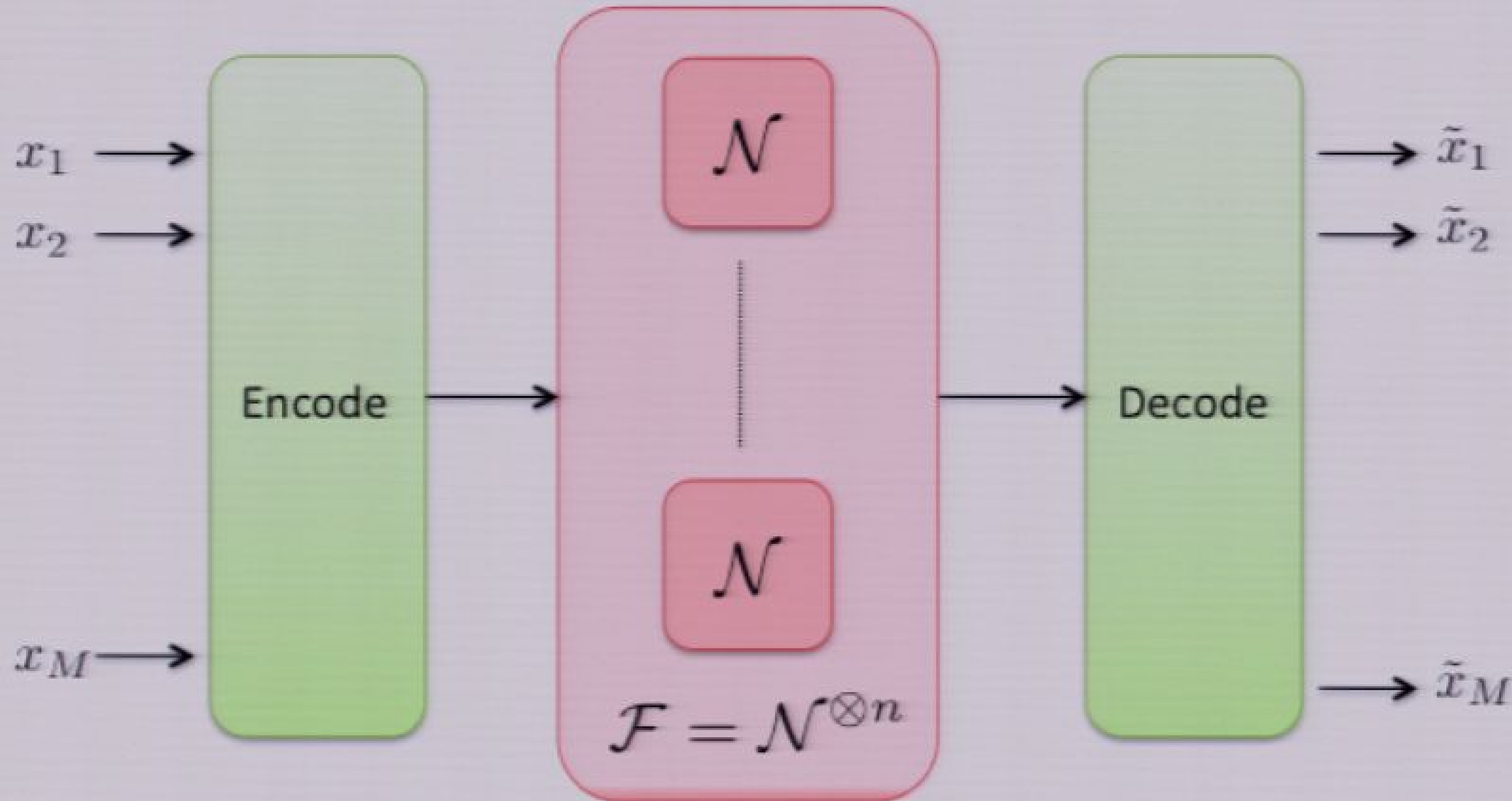
Robert Koenig

available at arXiv:0903:2838

Outline

- Transmitting classical information
  - Concepts
  - What is a strong converse?
  - Result
- Proving a general strong converse
  - Conceptual steps
  - Below and above the capacity
- Open questions

Sending classical information



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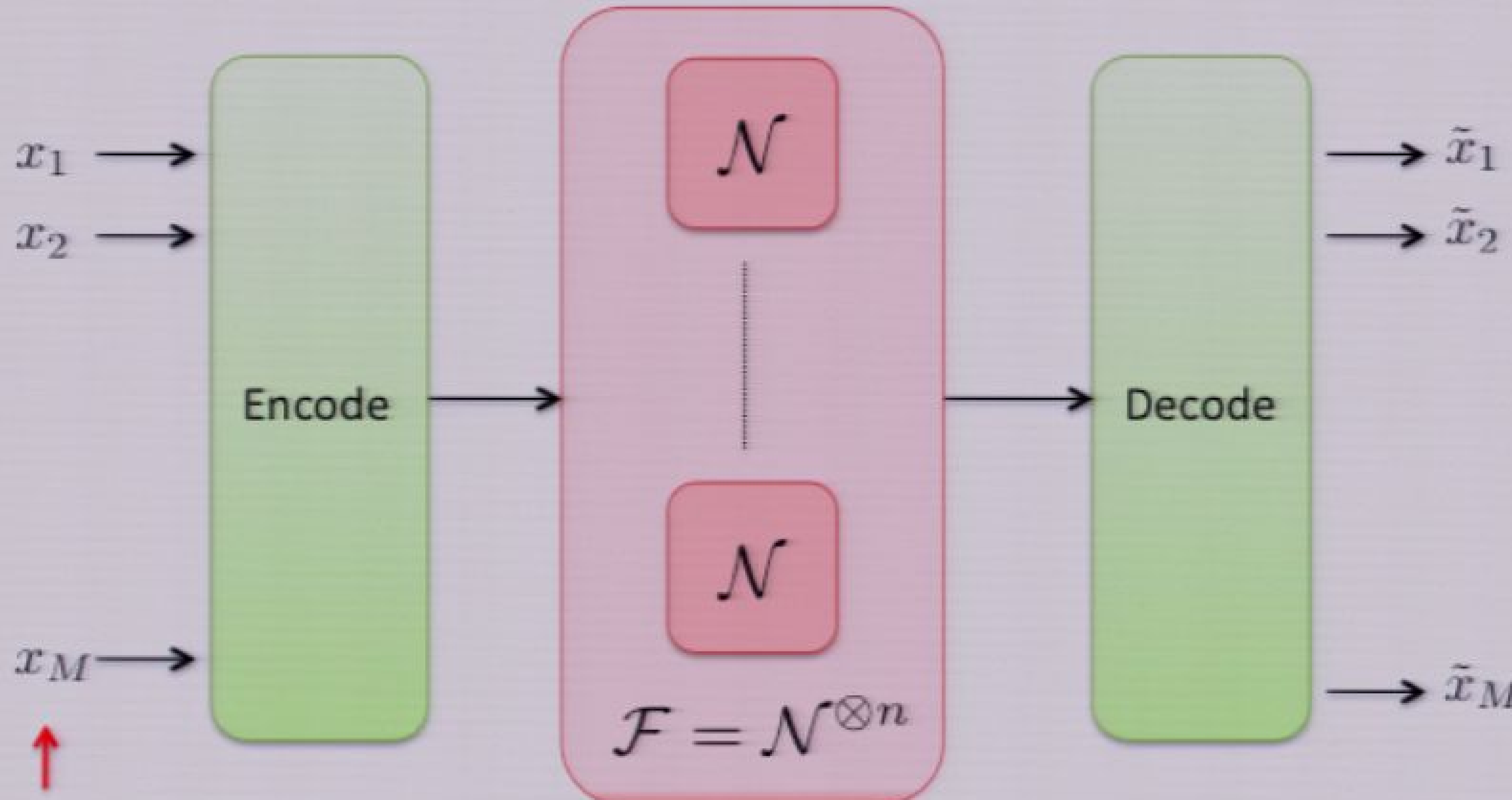
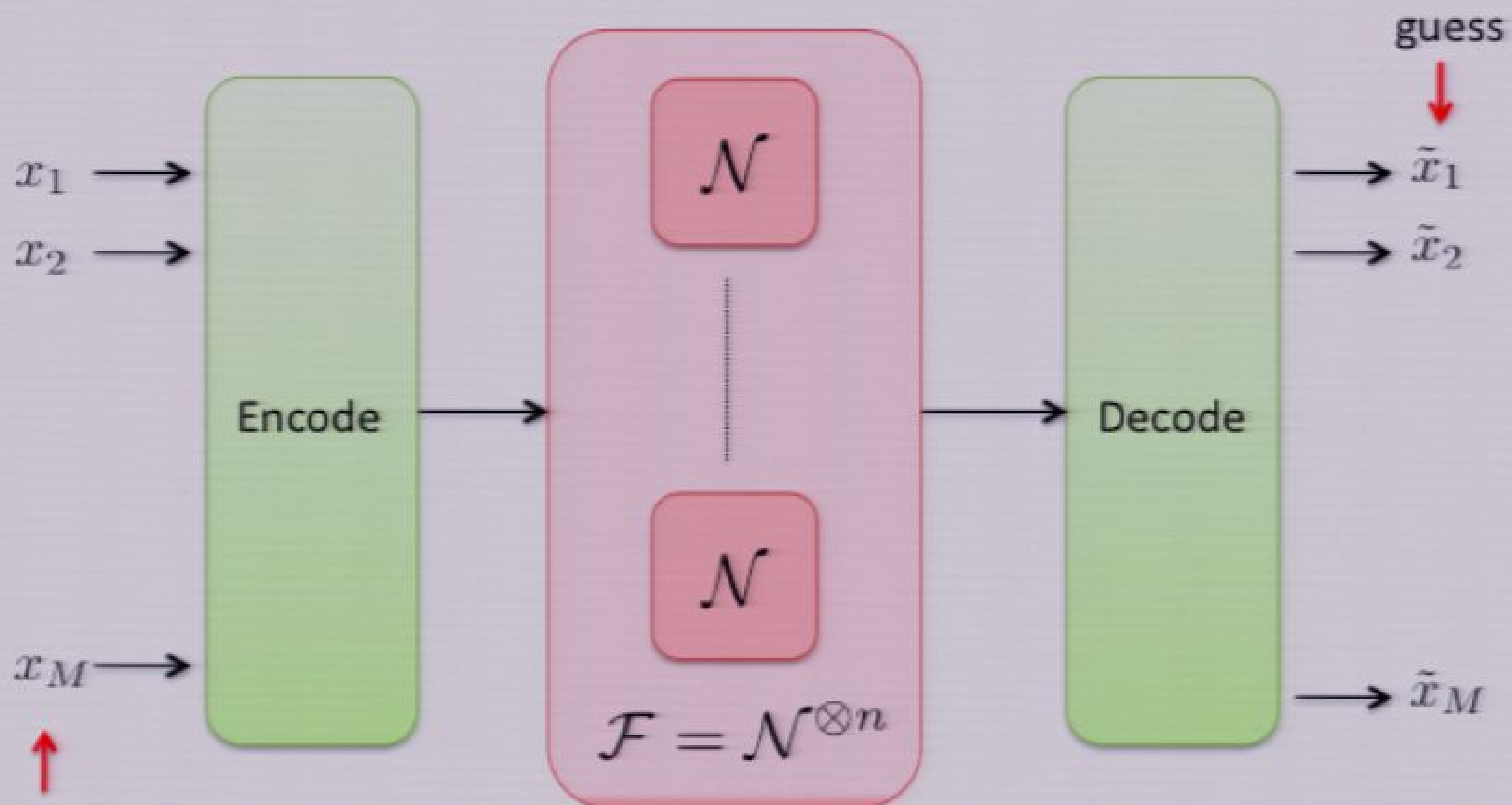


Random M-bit string

Sending classical information



Random M-bit string

Success probability $$P^{\mathcal{N}}_{\text{succ}}(M) = \max \frac{1}{2^M} \sum_{x \in \{0,1\}^M} \Pr[x = \tilde{x}]$$ $x = x_1, \ldots, x_M$

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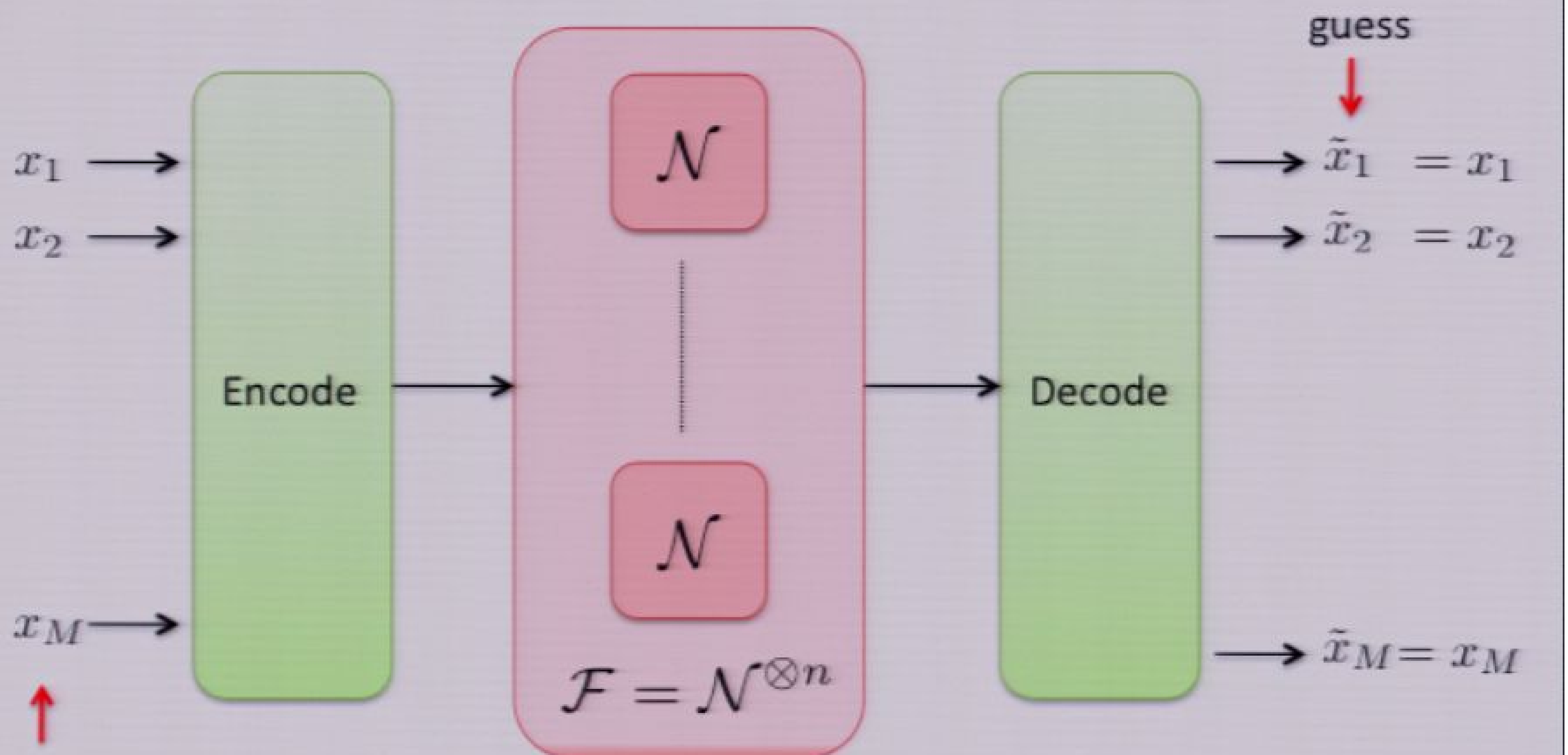

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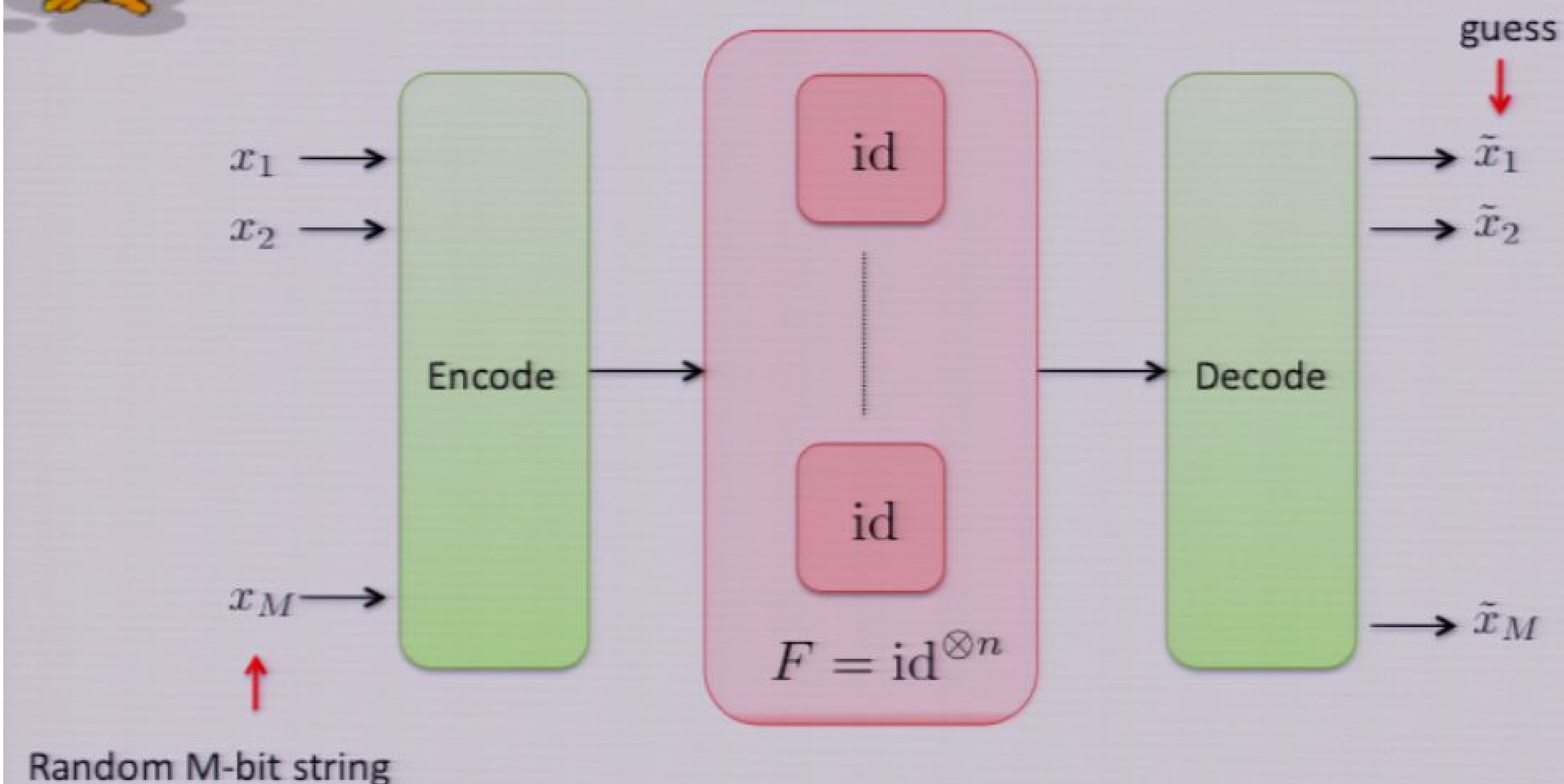
Reliable transmission $P^{\mathcal{N}}_{\text{succ}}(M) = 1$

Example: Qubit identity channel



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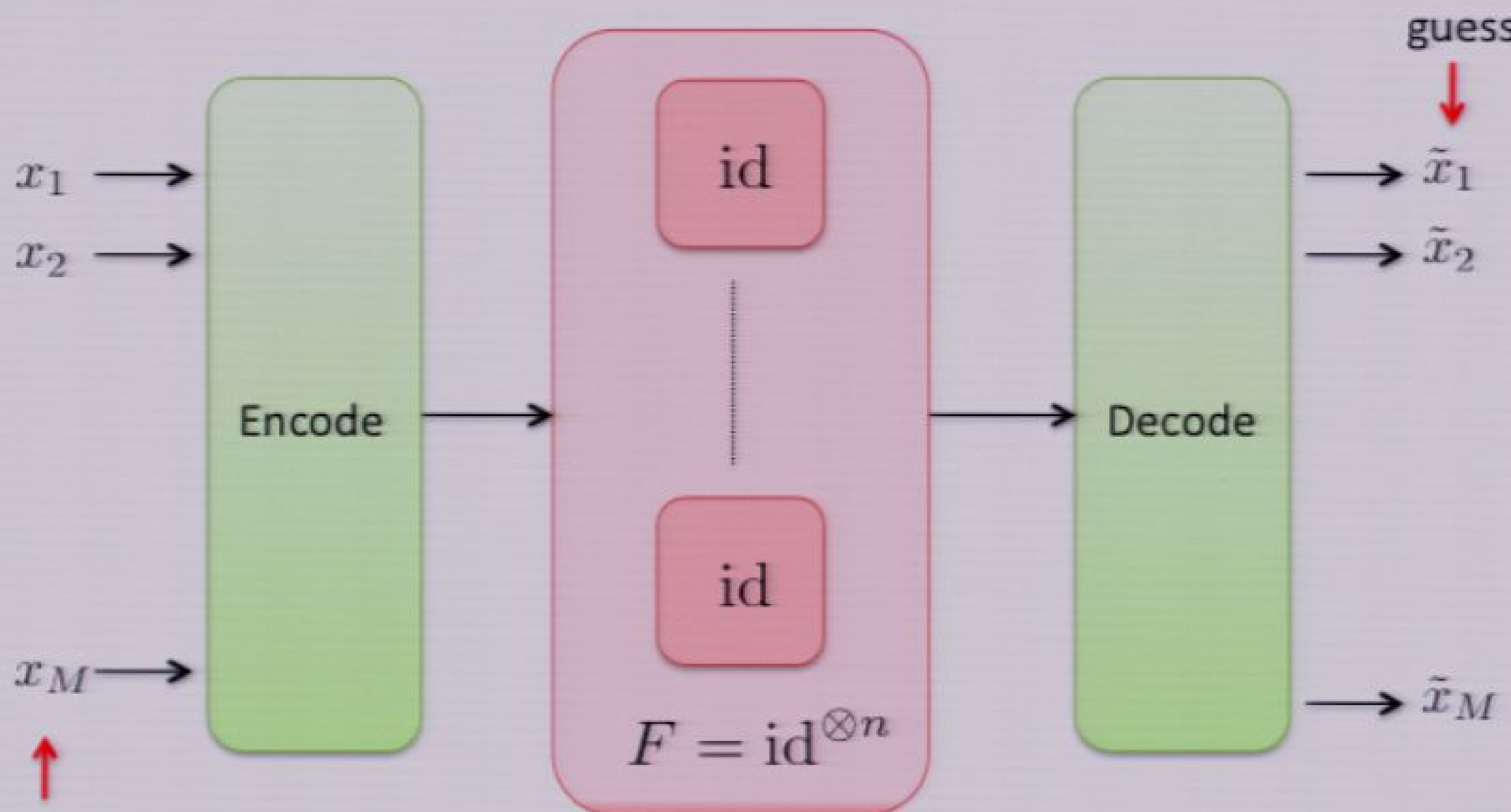

Random M-bit string

Reliable transmission $M \leq n$ $P^{\mathcal{N}}_{\mathrm{succ}}(M) = 1$

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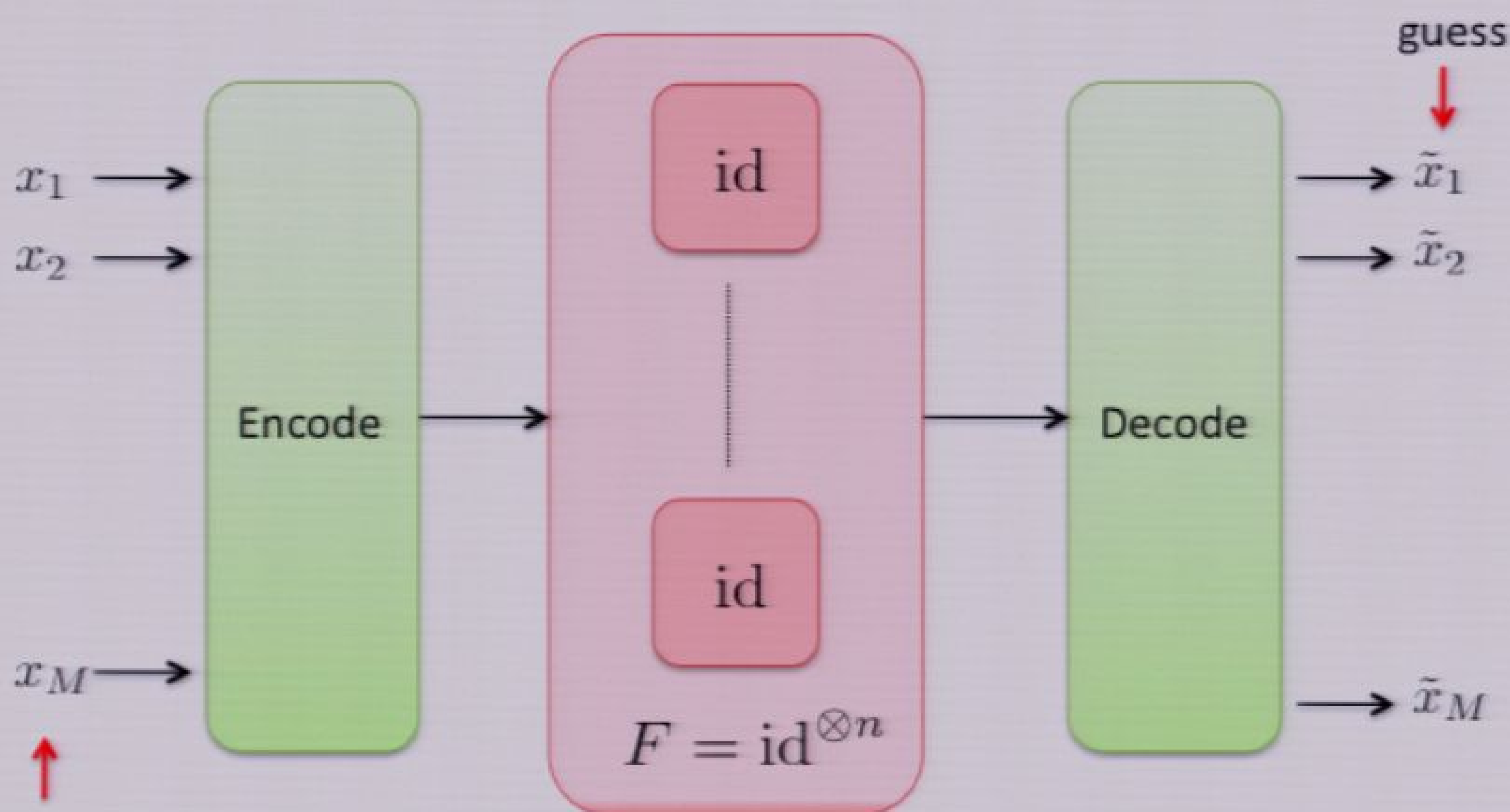

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Reliable transmission $M \leq n$ $P^{\mathcal{N}}_{\mathrm{succ}}(M) = 1$

Unreliable transmission (Holevo '73) $M > n$ $P^{\mathcal{N}}_{\mathrm{succ}}(M) < 1$

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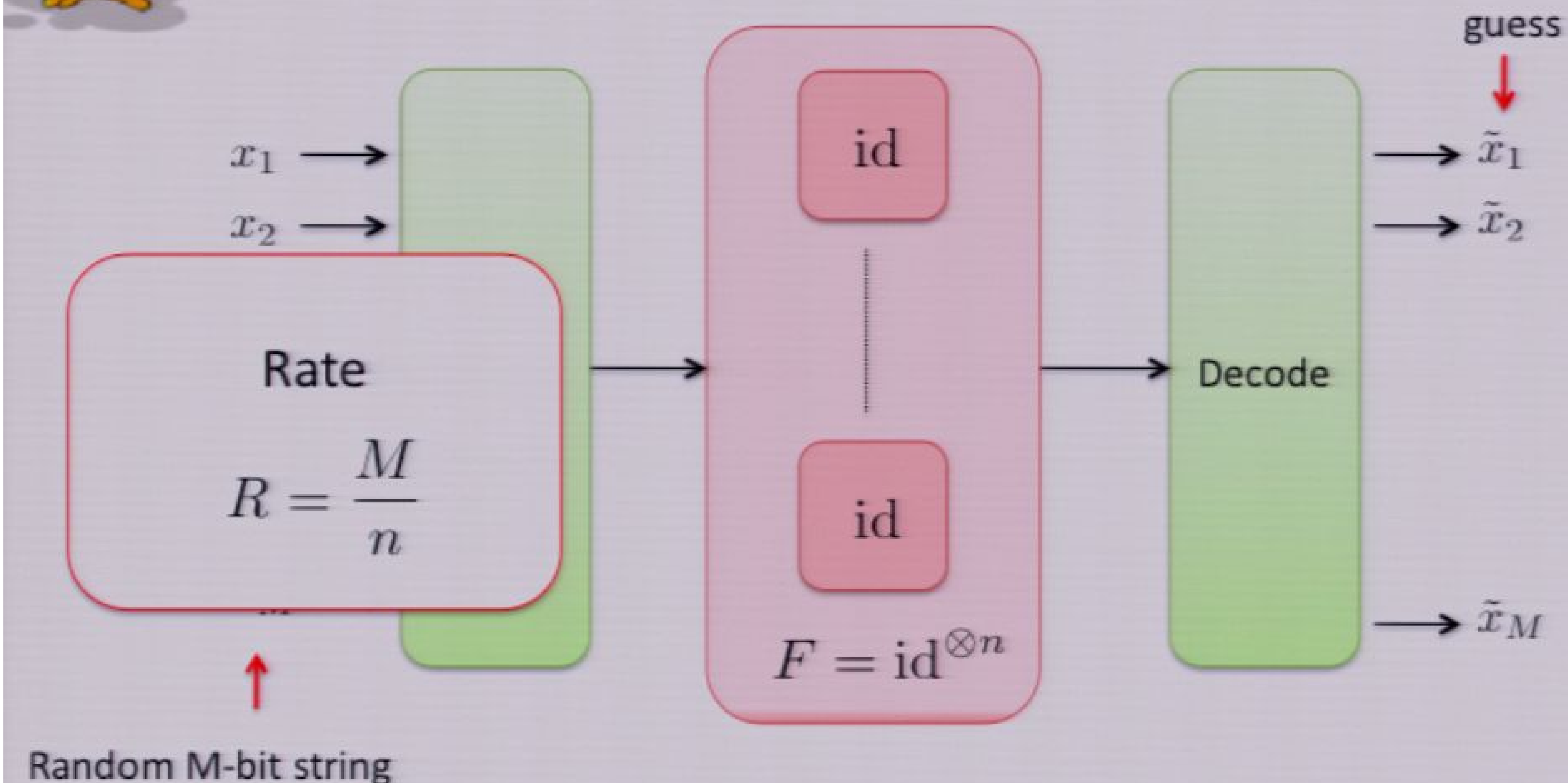

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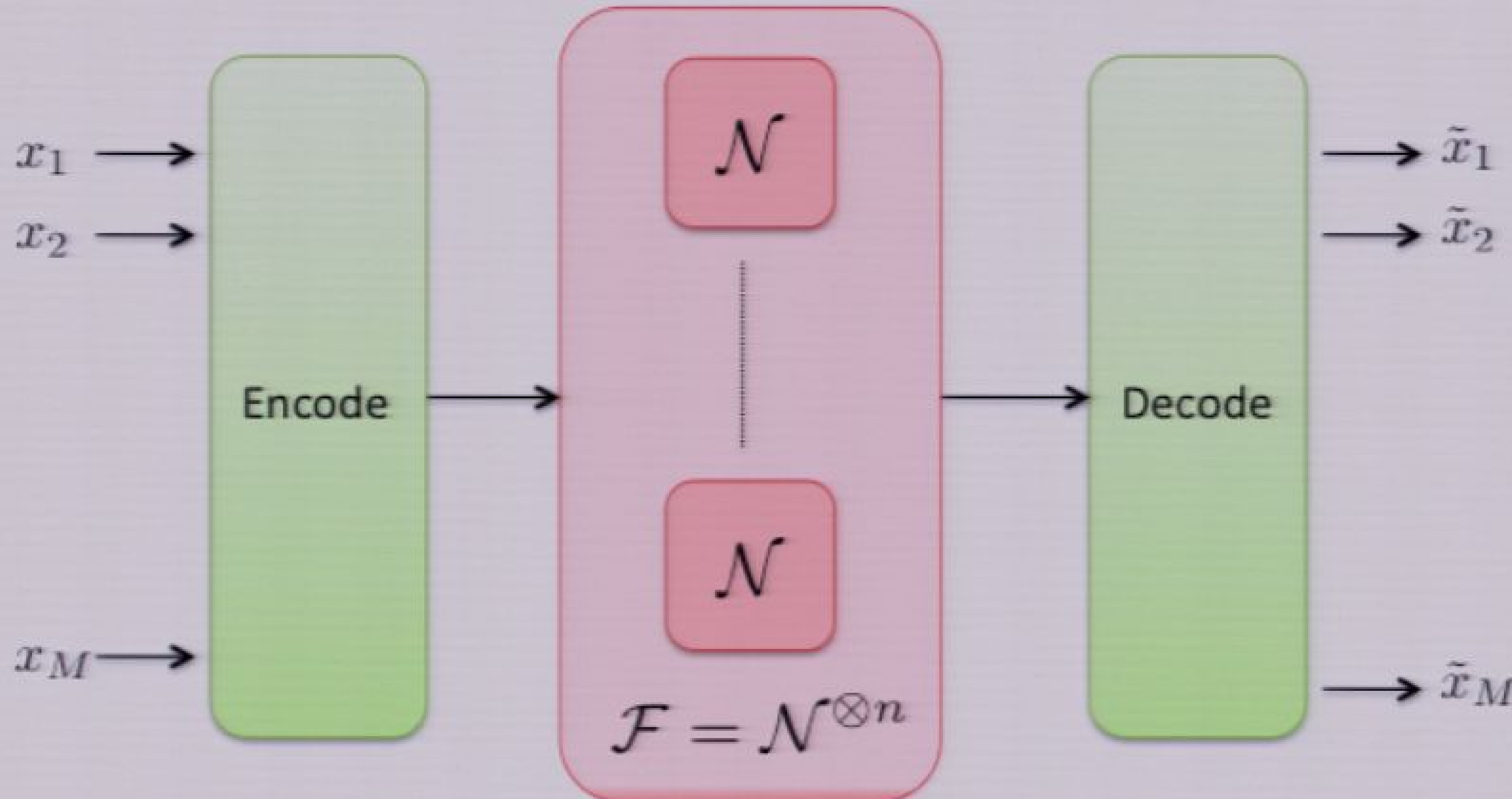
Classical channels



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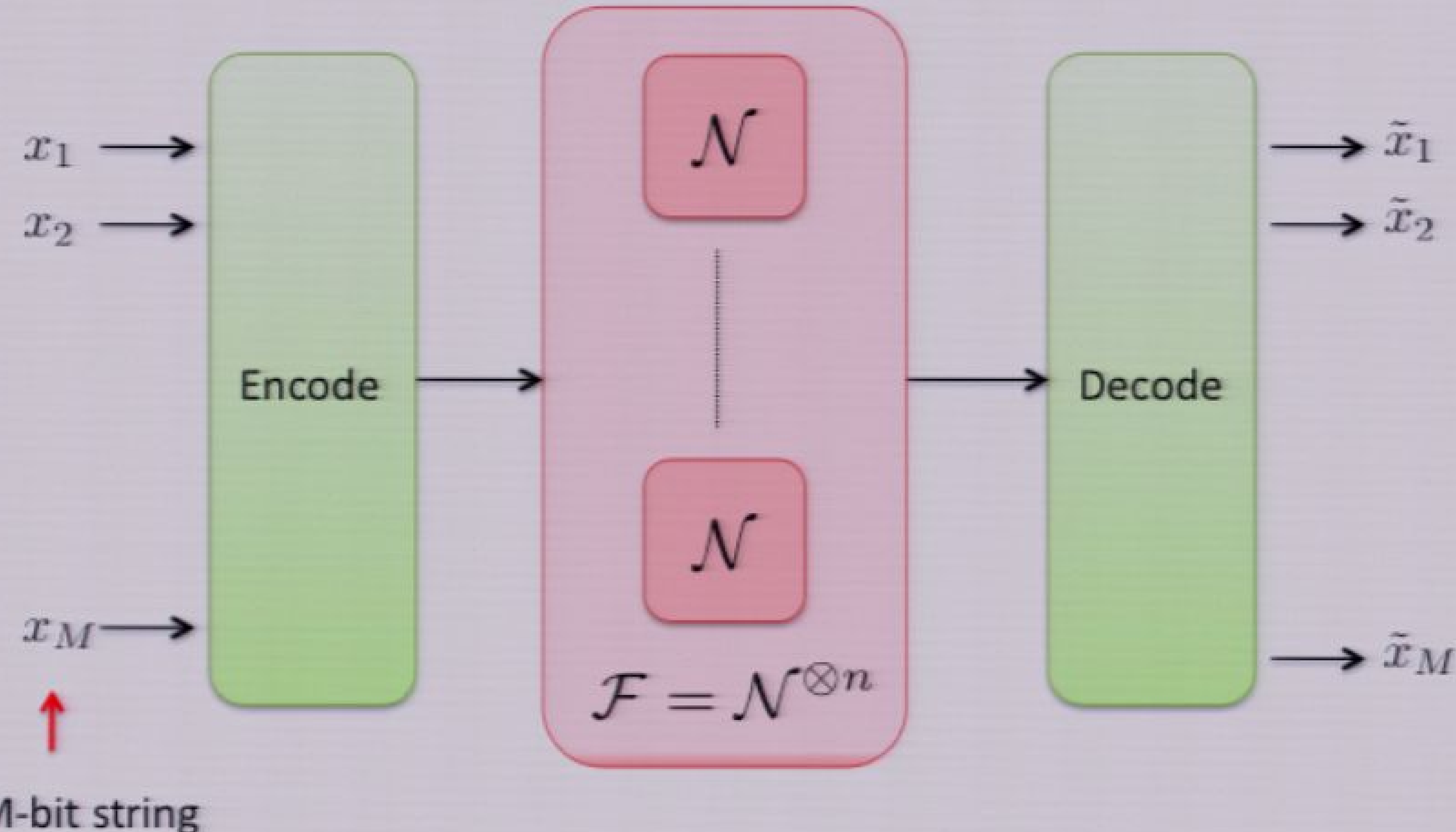

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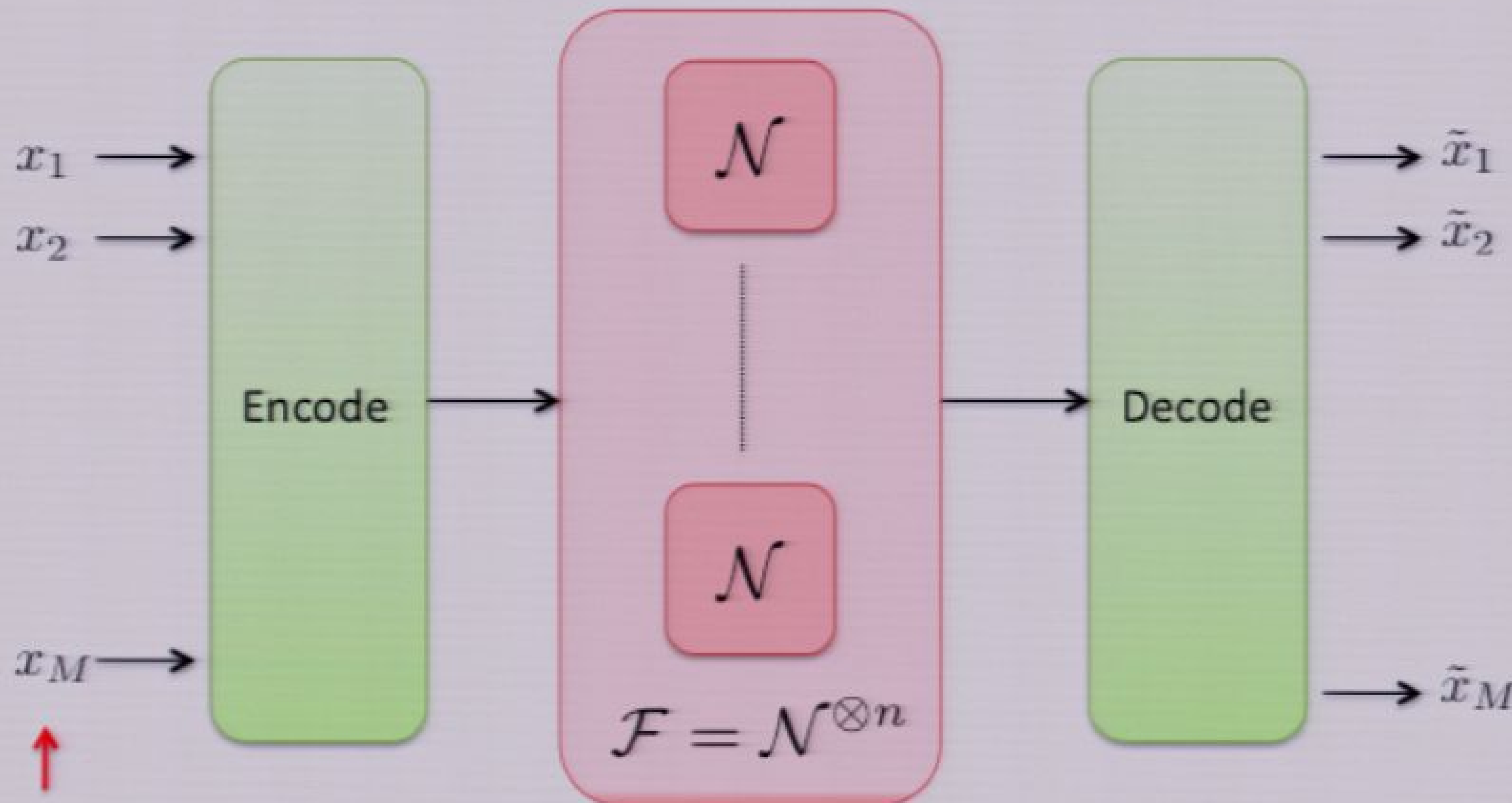

Random M-bit string

Classical capacity C:

Can reliably transmit M bits if and only if $R = \frac{M}{n} \leq C$

(Shannon's coding theorem ' 48)

Sending classical information $(n \to \infty)$

Below capacity $R \leq C$

Classical (Shannon '48)

Can send $M = nR$ bits reliably using the channel n times

$$P_{\mathrm{succ}}^{\mathcal{N}}(nR) = 1$$

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Strong converse

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Strong converse

Capacity measures the usefulness of a channel for sending information

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Quantum (Schumacher/Westmoreland '97, Holevo '98)

Can again achieve the capacity

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Quantum

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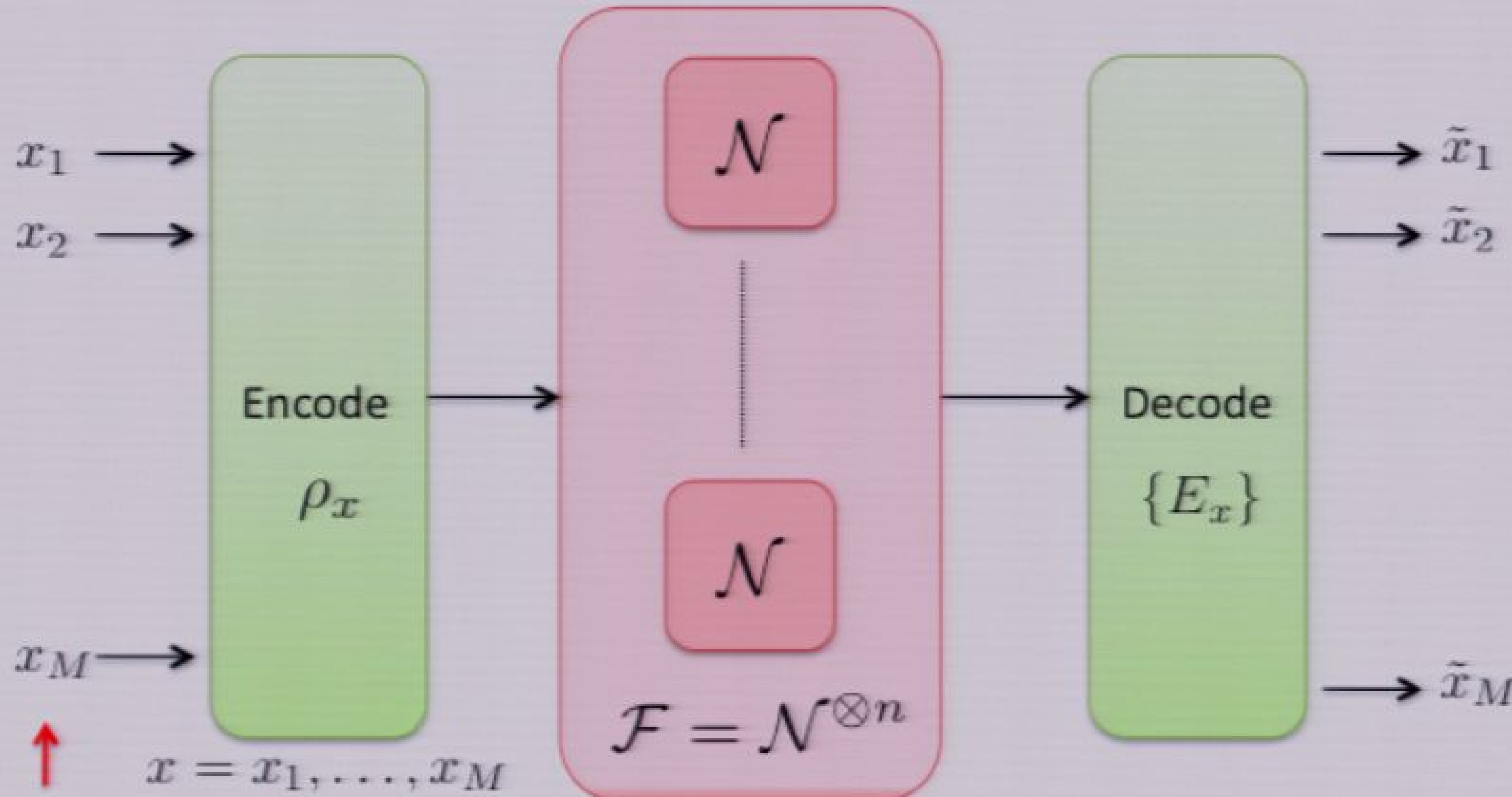
Quantum channels

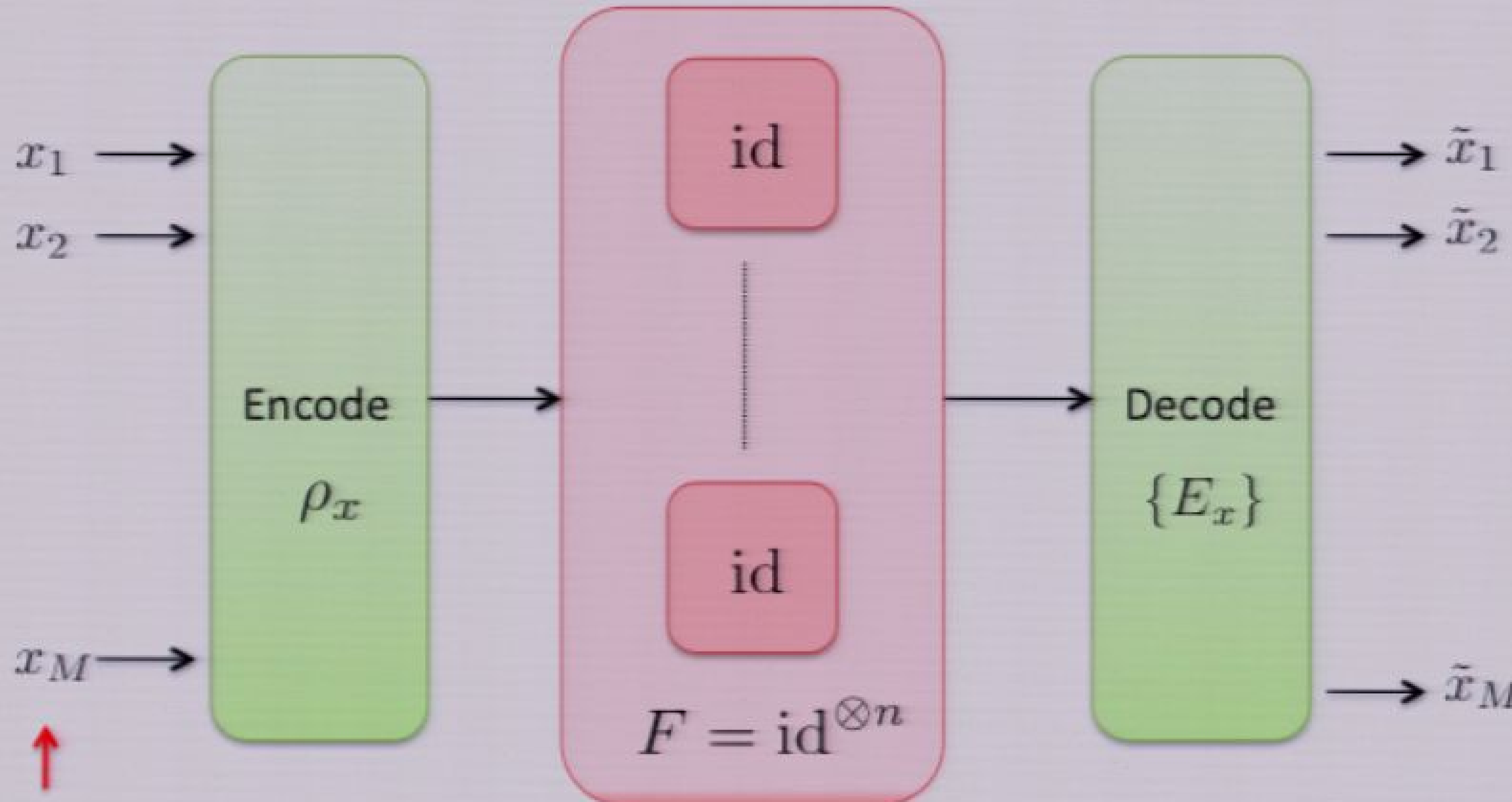


Random M-bit string

$$P_{\rm succ}^{\mathcal{N}}(M) = \max_{\{\rho_x\}_x, \{E_x\}_x} \frac{1}{2^M} \sum_{x \in \{0,1\}^M} \mathrm{tr}(E_x \rho_x)$$

$$M = nR$$

Strong converse: Qubit identity $C = 1$



Random M=nR -bit string

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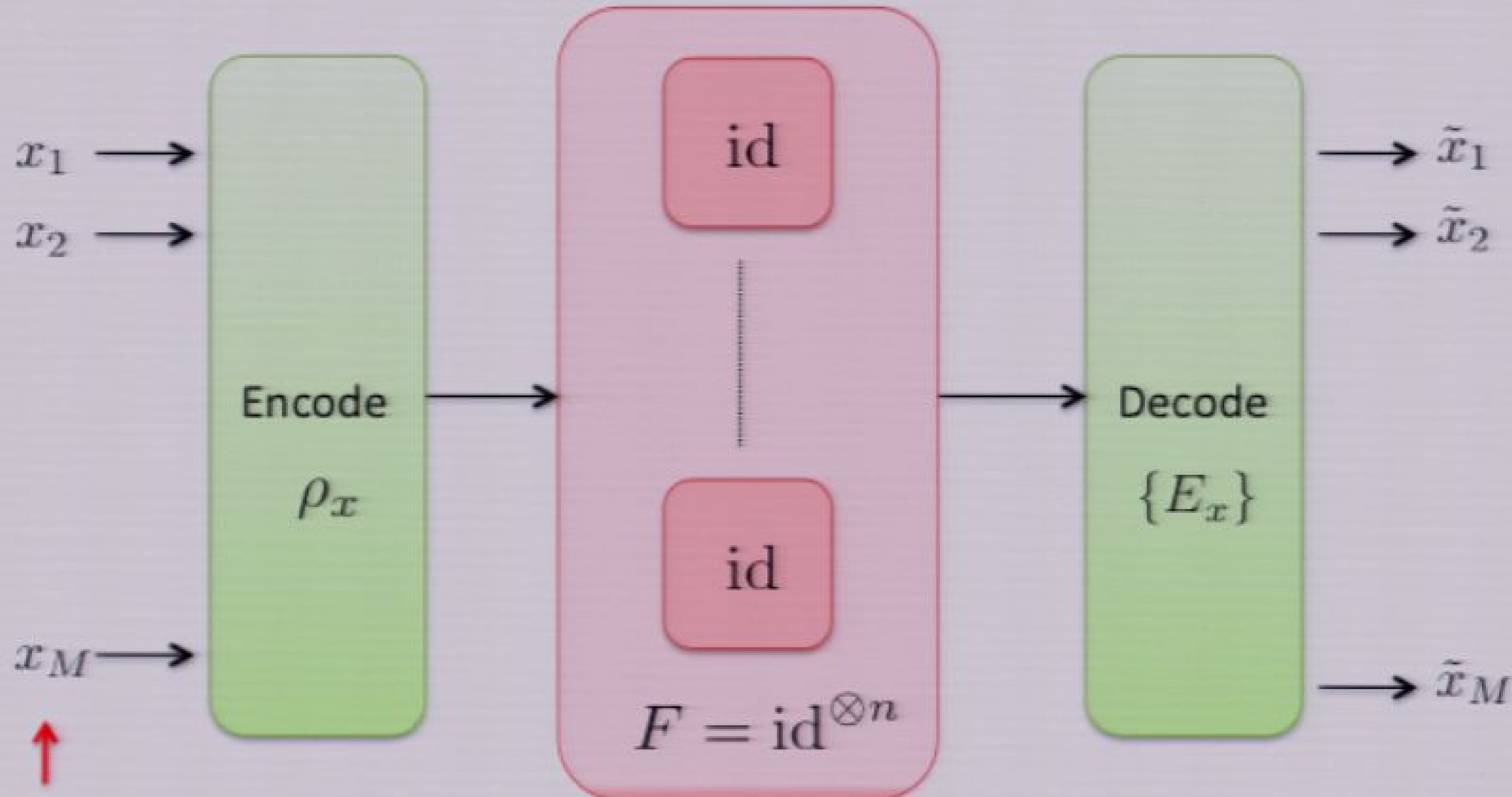
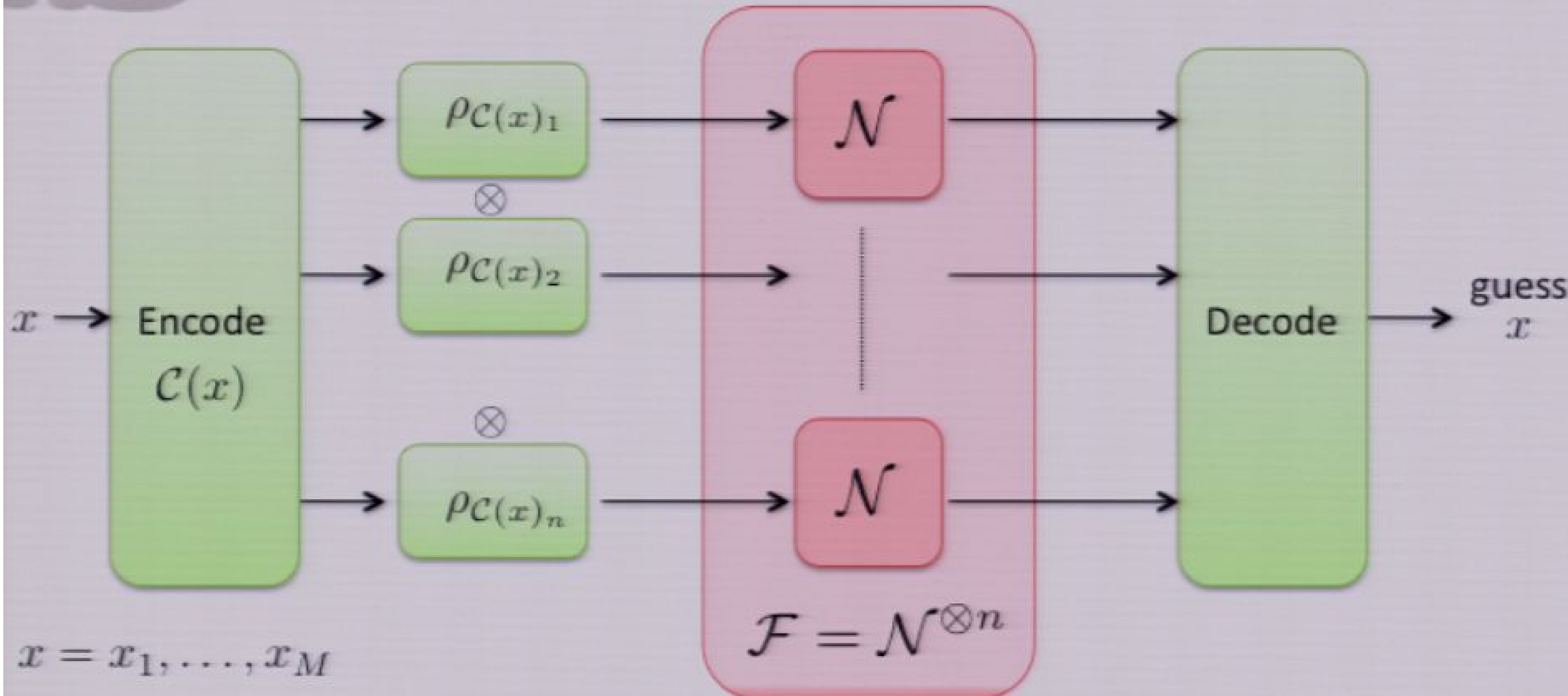


Random M=nR -bit string

$$P_{\text{succ}}^{\text{id}}(nR) = \max \frac{1}{2^{nR}} \sum_{x \in \{0,1\}^{nR}} \mathrm{tr}(E_x \rho_x) \leq \frac{1}{2^{nR}} \sum_x \mathrm{tr}(E_x) = 2^{-n(R-1)}$$

What else is known?



$x = x_1, \ldots, x_M$

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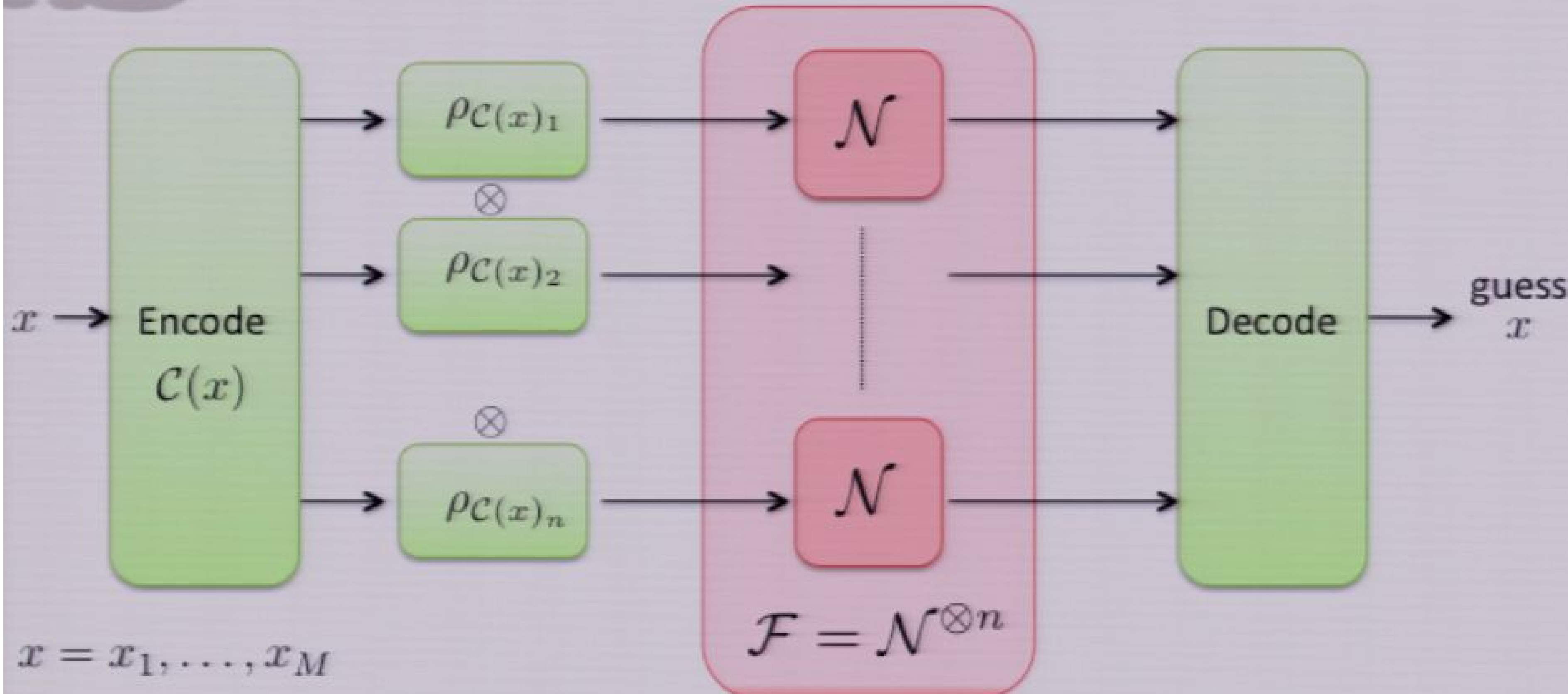

Strong converse if we assume tensor product encodings

- Andreas Winter, IEEE Transactions on I.T., 45, 2481 (1999)
- Ogawa and Nagaoka, IEEE Transactions on I.T., 45, 2486 (1999)

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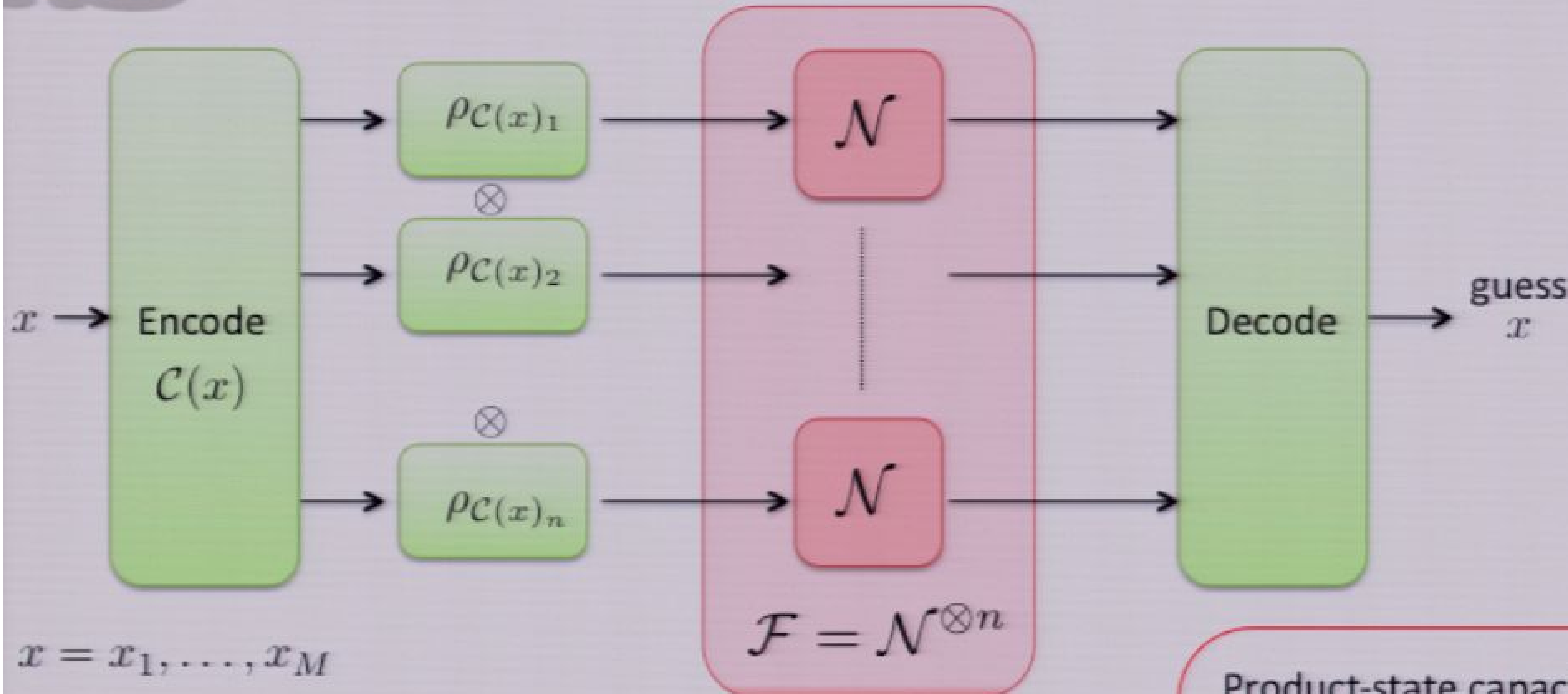

Product-state capacity

Can sent M = nR reliably using a tensor product encoding iff

$$R \leq C^{\text{prod}}$$

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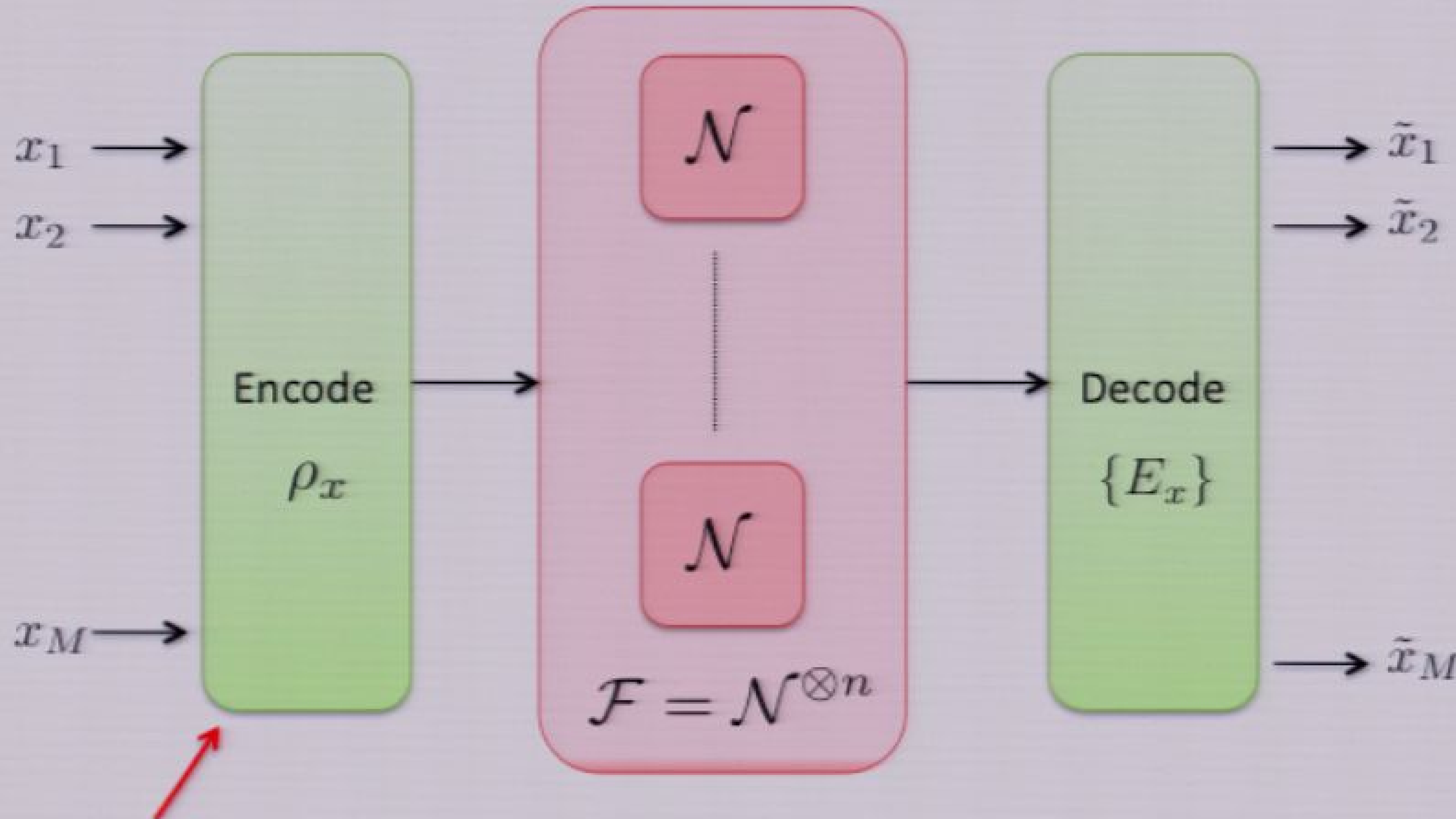
What makes this problem tricky



Best encoding could be entangled across several uses of the channel!

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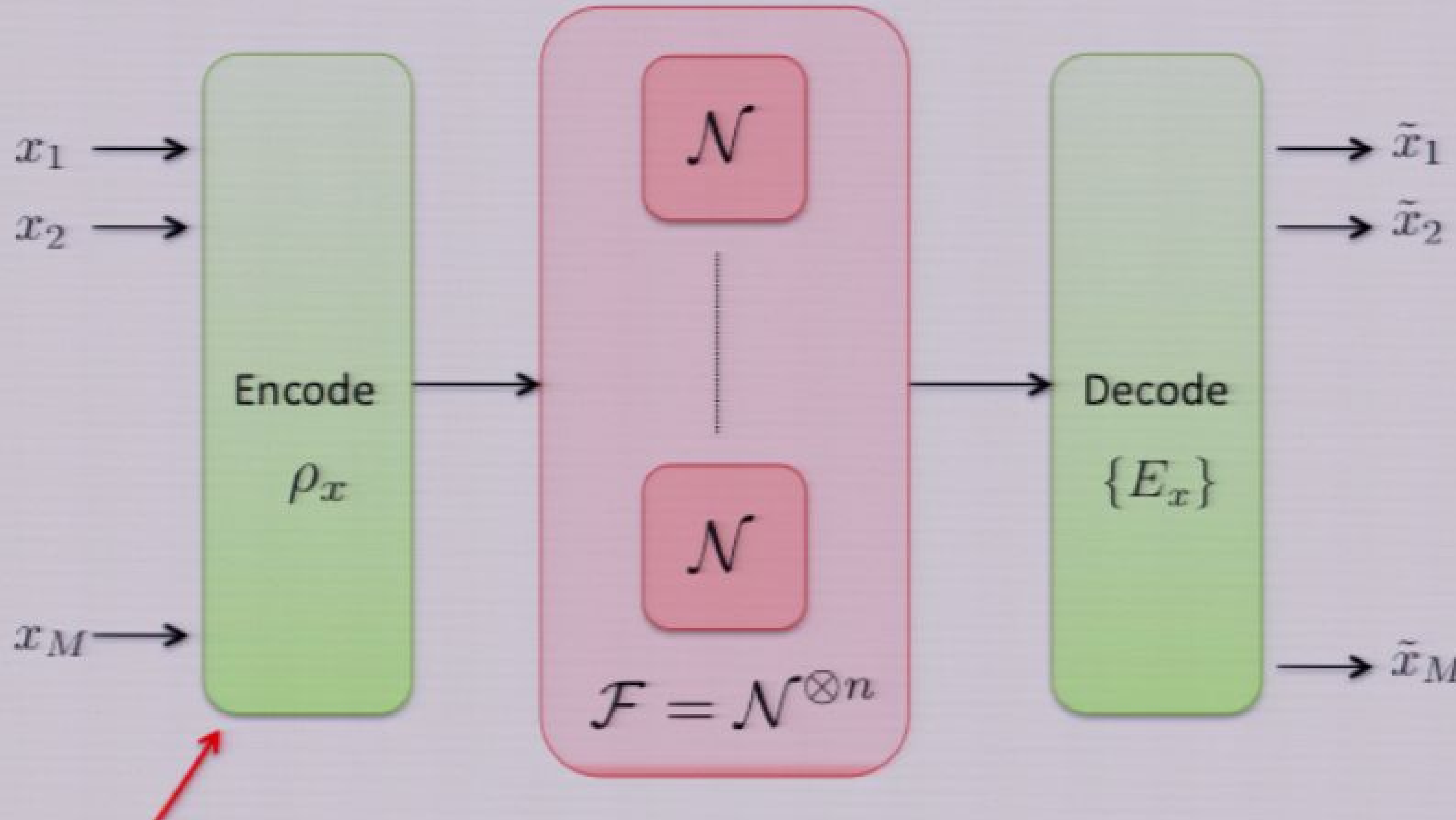

Best encoding could be entangled across several uses of the channel!

Hastings '08: entangled encodings can help at rates below the capacity

there exists channels for which $C^{\text{prod}} < C$

What makes this problem tricky

Only a weak converse was known to hold for entangled inputs (Holevo, '73)

$$P_{\text{succ}}(nR) < 1$$

Entanglement might help even for channels with $C^{\text{prod}} = C$

Best encoding could be entangled across several uses of the channel!

Hastings '08: entangled encodings can help at rates below the capacity

there exists channels for which $C^{\text{prod}} < C$

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Main Result $\mathcal{N} : \mathcal{B}(\mathcal{H}_{\text{in}}) \rightarrow \mathcal{B}(\mathcal{H}_{\text{out}})$

1. Qudit depolarizing channel

$$\Delta(\rho) = r\rho + (1-r)\mathrm{id}/d$$

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   2. Is covariant with respect to a group G where the representation on the output space is irreducible

$$g\mathcal{N}(\rho)g^\dagger = \mathcal{N}(g\rho g^\dagger) \qquad \text{for all } g \in G$$

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For any such channel, there exists a constant $\gamma > 0$ such that

$$P_{\text{succ}}^{\mathcal{N}}(nR) \leq 2^{-\gamma n(R-C)}$$

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Special case

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Conceptual steps

Relate $P_{\mathrm{succ}}^{\mathcal{N}}$ to an entropic quantity

↓

Derive some properties of this quantity

↓

Prove additivity of this quantity

↓

Strong converse

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↓

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Strong converse

Relating $P_{\text{succ}}^{\mathcal{N}}$ to an entropic quantity

Below capacity $R \leq C$

The classical capacity is given by

$$\bar{\chi}^*(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n}\chi^*(\mathcal{N}^{\otimes n})$$

Regularized version of

$$\chi^*(\mathcal{N}^{\otimes n}) = \max_{\{p_x,\sigma_x\}} \chi(\{p_x, \mathcal{N}^{\otimes n}(\sigma_x)\})$$

$$\chi(\{p_x,\rho_x\}) = S\left(\sum_x p_x\rho_x\right) - \sum_x p_x S(\rho_x)$$

Above capacity $R > C$

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For rates below the capacity

$$P_{\text{succ}}^{\mathcal{N}}(nR) = 1 - 2^{-n\delta(\bar{\chi}^*(\mathcal{N})-R)}$$

$$\delta \geq 0$$

Above capacity $R > C$

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For rates above the capacity

$$P_{\text{succ}}^{\mathcal{N}}(nR) \lesssim 2^{-n(1-\frac{1}{\alpha})(R-\bar{\chi}_\alpha^*(\mathcal{N}))}$$

$$\alpha > 1$$

Relating $P_{\mathrm{succ}}^{\mathcal{N}}$ to an entropic quantity

Below capacity $R \leq C$

The classical capacity is given by

$$\bar{\chi}^*(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n}\chi^*(\mathcal{N}^{\otimes n})$$

Evaluating the capacity is easier for channels for which the Holevo quantity is additive

$$\chi^*(\mathcal{N}^{\otimes n-1} \otimes \mathcal{N}) = \chi^*(\mathcal{N}^{\otimes n-1}) + \chi^*(\mathcal{N})$$

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Would also be easier if the α-Holevo quantity would be additive for our channels

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Rewriting $P_{\text{succ}}^{\mathcal{N}}$

- State after the channel $\sigma_x = \mathcal{N}^{\otimes n}(\rho_x)$

Rewriting $P_{\text{succ}}^{\mathcal{N}}$

Encoding of x

- State after the channel $\sigma_x = \mathcal{N}^{\otimes n}(\rho_x)$

$$P_{\text{succ}}^{\mathcal{N}} = \max_{\{\rho_x, E_x\}} \frac{1}{2^{nR}} \sum_{x \in \{0,1\}^{nR}} \text{tr}(E_x \sigma_x)$$

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$$\leq \frac{1}{2^{nR}} \sum_x \text{tr}\left[E_x \left(\sum_{x'} \sigma_{x'}^{\alpha}\right)^{1/\alpha}\right]$$

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$\sigma_x = (\sigma_x^{\alpha})^{1/\alpha} \leq \left(\sum_{x'} \sigma_{x'}^{\alpha}\right)^{1/\alpha}$

$\alpha > 1$

$$\leq \frac{1}{2^{nR}} \sum_x \mathrm{tr}\left[\left(\sum_{x'} \sigma_{x'}^{\alpha}\right)^{1/\alpha}\right]$$

$E_x \leq \mathrm{id}$

$$= 2^{\frac{\alpha-1}{\alpha}(-nR + \chi_\alpha(\{2^{-nR}, \sigma_x\})}$$

$\chi_\alpha(\{p_x, \rho_x\}) = \frac{\alpha}{\alpha - 1} \log \mathrm{tr} \left(\sum_x p_x \rho_x^{\alpha}\right)^{1/\alpha}$

Rewriting $P^{\mathcal{N}}_{\mathrm{succ}}$

Encoding of x

- State after the channel $\sigma_x = \mathcal{N}^{\otimes n}(\rho_x)$

$$P^{\mathcal{N}}_{\mathrm{succ}} \le 2^{-\left(1-\frac{1}{\alpha}\right)(nR-\chi^*_\alpha(\mathcal{N}^{\otimes n}))}$$

$P^{\mathcal{N}}_{\mathrm{succ}} =$

$1/\alpha$

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Conceptual steps

Relate $P_{\text{succ}}^{\mathcal{N}}$ to an entropic quantity

$$P_{\text{succ}}^{\mathcal{N}} \leq 2^{-\left(1-\frac{1}{\alpha}\right)\left(nR-\chi_{\alpha}^{*}(\mathcal{N}^{\otimes n})\right)}$$

Derive some properties of this quantity

Prove additivity of this quantity

Strong converse

Investigating χ_α^*

Below capacity $R \leq C$

Above capacity $R > C$

Can we relate χ^* to other quantities?

$$\chi^*(\mathcal{N}) = \min_\sigma \max_\rho D(\mathcal{N}(\rho)||\mathcal{N}(\sigma))$$

Schumacher/Westmoreland, PRA '01

$$D(\hat{\rho}||\hat{\sigma}) = \mathrm{tr}(\hat{\rho}\log\hat{\rho} - \hat{\rho}\log\hat{\sigma})$$

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$$\min_{\sigma_{\mathrm{out}}} \max_{\rho} D_\alpha(\mathcal{N}(\rho)||\sigma_{\mathrm{out}}) \leq \chi_\alpha^*(\mathcal{N})$$

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$$D_\alpha(\hat{\rho}||\hat{\sigma}) = \frac{1}{\alpha - 1}\log \mathrm{tr}(\hat{\rho}^\alpha \hat{\sigma}^{1-\alpha})$$

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Strong converse

Towards additivity of χ_α^*

Below capacity $R \leq C$

For covariant channels it is also known that

$$\chi^*(\mathcal{N}) = \log d_{\mathrm{out}} - S^{\min}(\mathcal{N})$$

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Holevo, 02

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Known channels

- Channels need to satisfy
  - Group covariance property
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- Examples
  - d-dimensional depolarizing channel (King, IEEE IT, '03)
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Strong converse

Strong converse from additivity of χ_α^*

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Done!

Outline

- Transmitting classical information
  - Concepts
  - What is a strong converse?
  - Result
- Proving a general strong converse
  - Conceptual steps
  - Below and above the capacity
- Open questions

Summary and open questions

- Strong converse for a large class of channels
  - Allows us to interpret the classical capacity as the relevant measure in how useful such channels are to transmit classical data
  - Applications in the noisy-quantum storage model

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Thank you!

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$1/\alpha$

$$\leq \frac{1}{2^{nR}} \sum_x \operatorname{tr}\left[\left(\sum_{x'} \sigma_{x'}^{\alpha}\right)\right]$$

$E_x \leq \text{id}$

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$$\chi_\alpha(\{p_x,\rho_x\}) = \frac{\alpha}{\alpha-1}\log\operatorname{tr}\left(\sum_x p_x\rho_x^\alpha\right)^{1/\alpha}$$

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Relating $P_{\mathrm{succ}}^{\mathcal{N}}$ to an entropic quantity

Below capacity $R \leq C$

The classical capacity is given by

$$\bar{\chi}^*(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n}\chi^*(\mathcal{N}^{\otimes n})$$

Evaluating the capacity is easier for channels for which the Holevo quantity is additive

$$\chi^*(\mathcal{N}^{\otimes n-1} \otimes \mathcal{N}) = \chi^*(\mathcal{N}^{\otimes n-1}) + \chi^*(\mathcal{N})$$

For rates below t apacity

$$P_{\mathrm{succ}}^{\mathcal{N}}(nR) = 1 - 2^{-n\delta(\bar{\chi}^*(\mathcal{N})-R)}$$

$$\delta \geq 0$$

Above capacity $R > C$

Define

$$\bar{\chi}_\alpha^*(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n}\chi_\alpha^*(\mathcal{N}^{\otimes n})$$

Would also be easier if the α-Holevo quantity would be additive for our channels

$$\chi_\alpha^*(\mathcal{N}^{\otimes n-1} \otimes \mathcal{N}) = \chi_\alpha^*(\mathcal{N}^{\otimes n-1}) + \chi_\alpha^*(\mathcal{N})$$

For rates above acity

$$P_{\mathrm{succ}}^{\mathcal{N}}(nR) \lesssim 2^{-n(1-\frac{1}{\alpha})(R-\bar{\chi}_\alpha^*(\mathcal{N}))}$$

$$\alpha > 1$$

Main Result

$$\mathcal{N}: \mathcal{B}(\mathcal{H}_{\text{in}}) \to \mathcal{B}(\mathcal{H}_{\text{out}})$$

Special case

1. Qudit depolarizing channel

$$\Delta(\rho) = r\rho + (1-r)\text{id}/d$$

2. Any unital qubit channel

Maximum output α-norm is multiplicative

3. More generally, any channel that
   1. Has additive maximum output α-entropy for $\alpha > 1$

   $$S_\alpha^{\min}(\mathcal{N}^{\otimes n}) = n \cdot S_\alpha^{\min}(\mathcal{N})$$

   2. Is covariant with respect to a group G where the representation on the output space is irreducible

   $$g\mathcal{N}(\rho)g^\dagger = \mathcal{N}(g\rho g^\dagger) \qquad \text{for all } g \in G$$

For any such channel, there exists a constant $\gamma > 0$ such that

$$P_{\text{succ}}^{\mathcal{N}}(nR) \leq 2^{-\gamma n(R-C)}$$

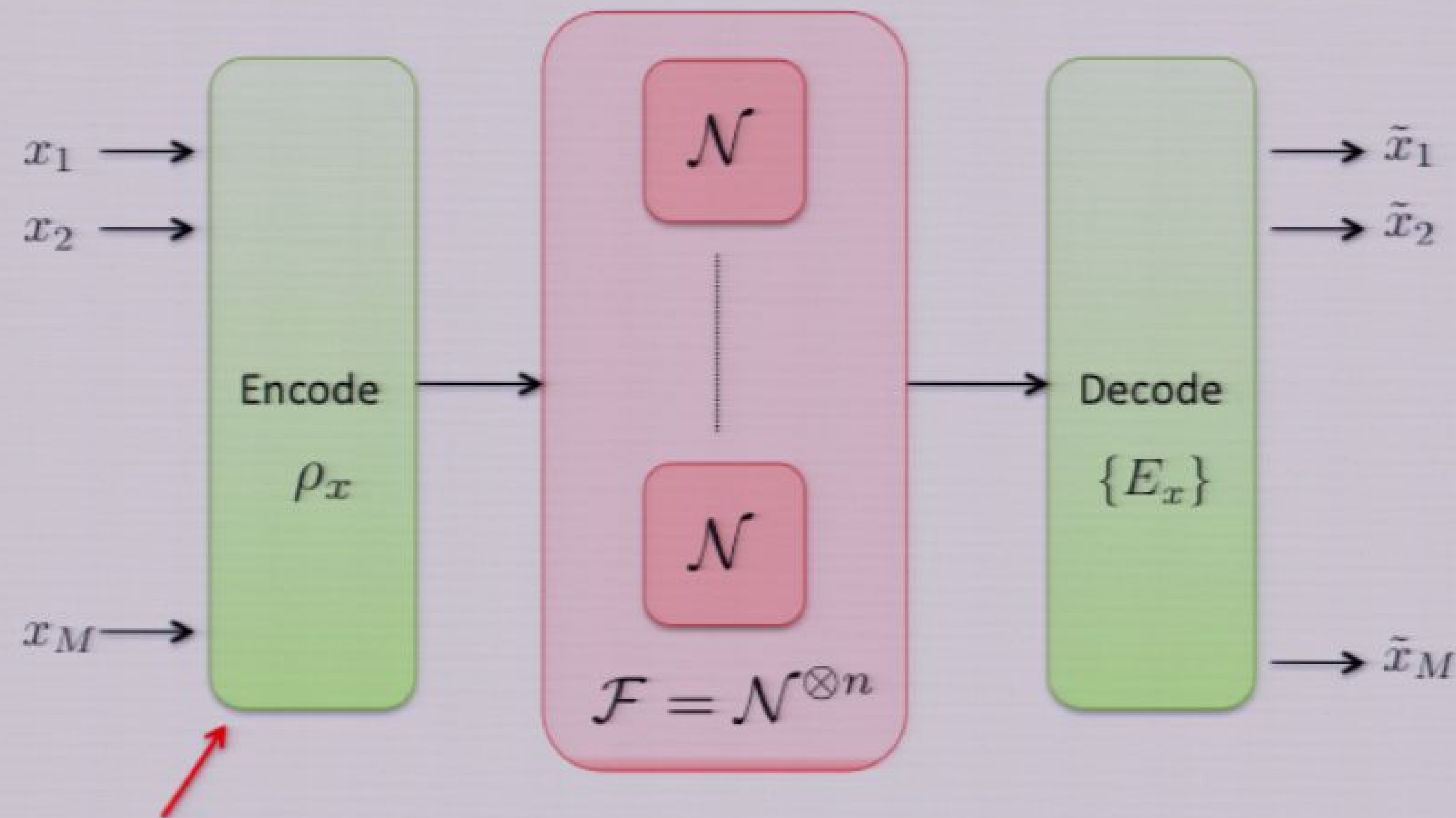
Main Result

$$\mathcal{N} : \mathcal{B}(\mathcal{H}_{\mathrm{in}}) \rightarrow \mathcal{B}(\mathcal{H}_{\mathrm{out}})$$

Outline

- Transmitting classical information
  - Concepts
  - What is a strong converse?
  - Result
- Proving a general strong converse
  - Conceptual steps
  - Below and above the capacity
- Open questions

What makes this problem tricky



Best encoding could be entangled across several uses of the channel!

Hastings '08: entangled encodings can help at rates below the capacity

there exists channels for which $C^{\text{prod}} < C$

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