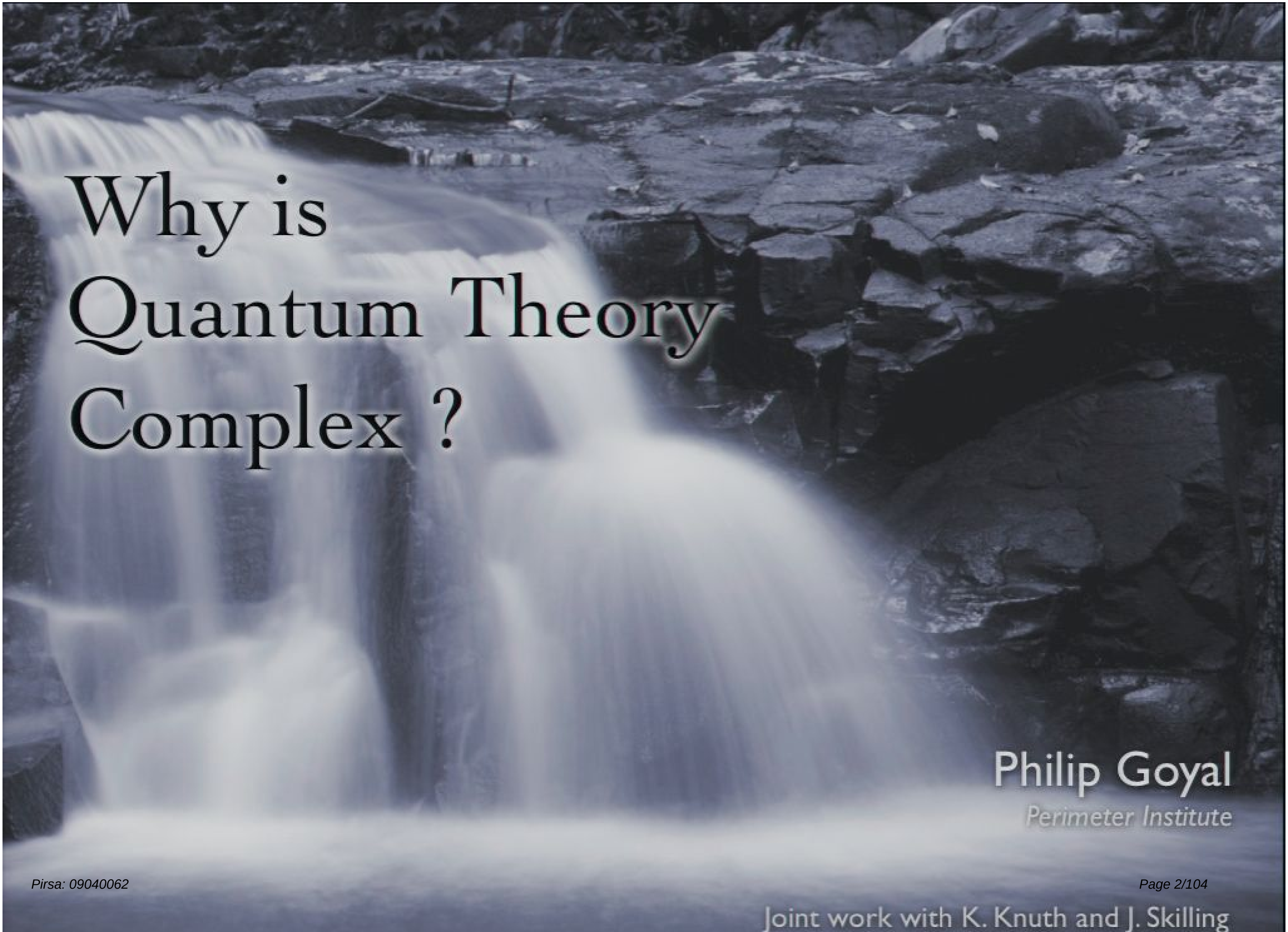
Title: Why Quantum Theory is Complex

Date: Aug 11, 2009 11:00 AM

URL: http://pirsa.org/09040062

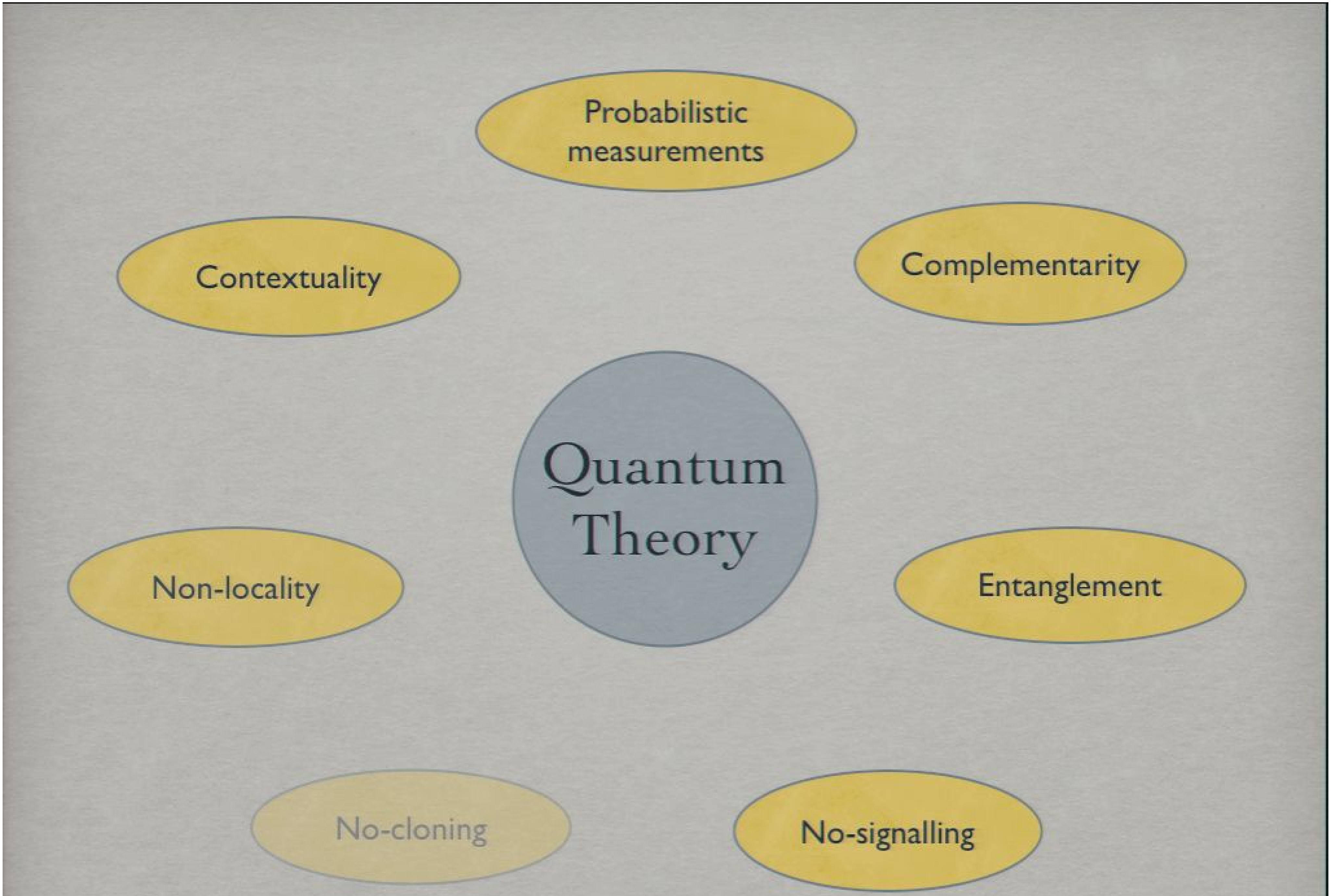
Abstract: Complex numbers are an intrinsic part of the mathematical formalism of quantum theory, and are perhaps its most mysterious feature. We show that it is possible to derive the complex nature of the quantum formalism directly from the assumption that a pair of real numbers is associated to each sequence of measurement outcomes, and that the probability of this sequence is a real-valued function of this number pair. By making use of elementary symmetry and consistency conditions, and without assuming that these real number pairs have any other algebraic structure, we show that these pairs must be manipulated according to the rules of complex arithmetic. We demonstrate that these complex numbers combine according to Feynman's sum and product rules, with the modulus-squared yielding the probability of a sequence of outcomes.

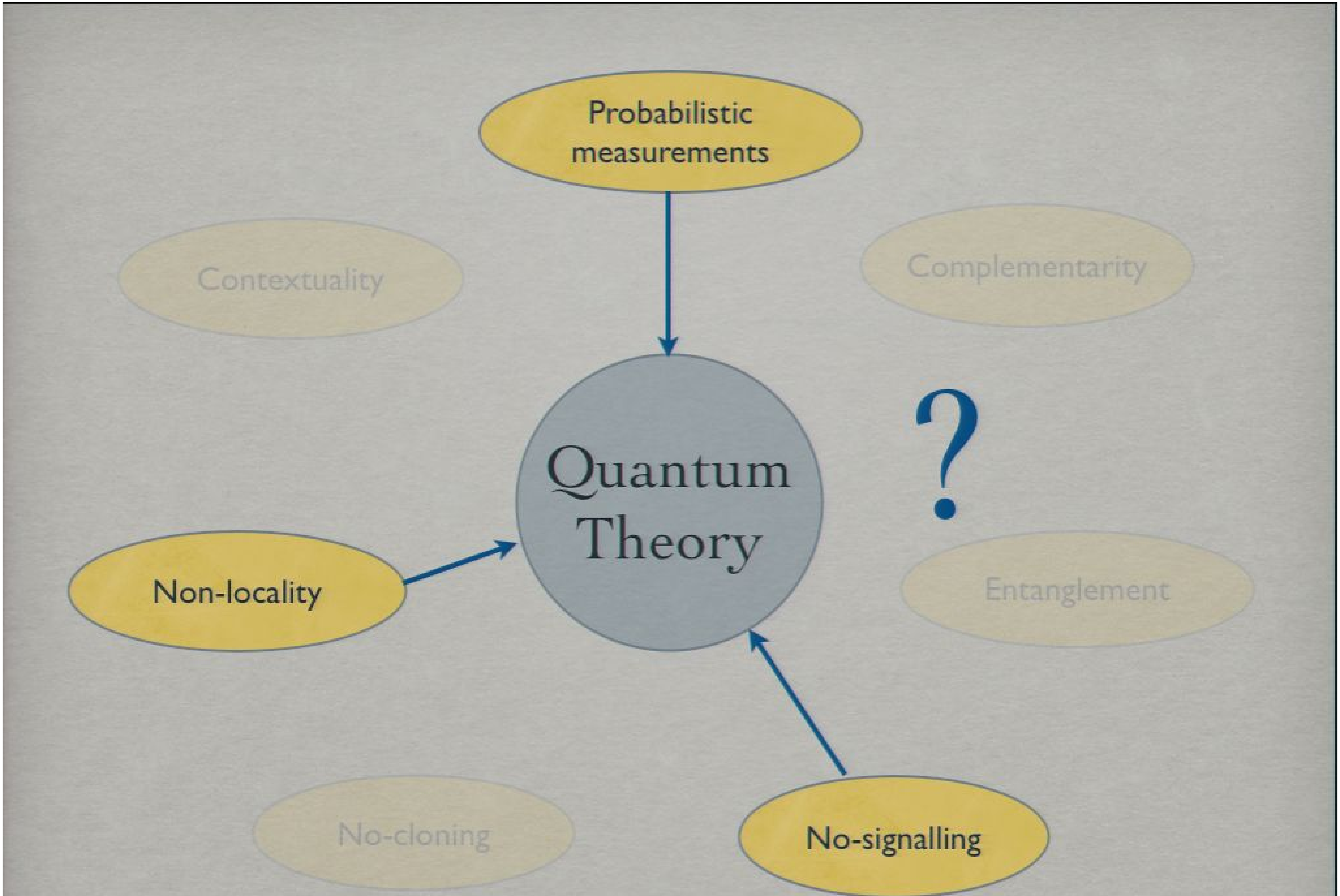
Reference: arXiv:0907.0909 (http://arxiv.org/abs/0907.0909)


Why is
Quantum Theory
Complex ?
Philip Goyal
Perimeter Institute
Joint work with K. Knuth and J. Skilling

Probabilistic measurements

Quantum Theory


Probabilistic measurements
Contextuality
Complementarity
Quantum Theory
Non-locality
Entanglement
No-cloning
No-signalling


Probabilistic measurements
Contextuality
Complementarity
Quantum Theory
?
Non-locality
Entanglement
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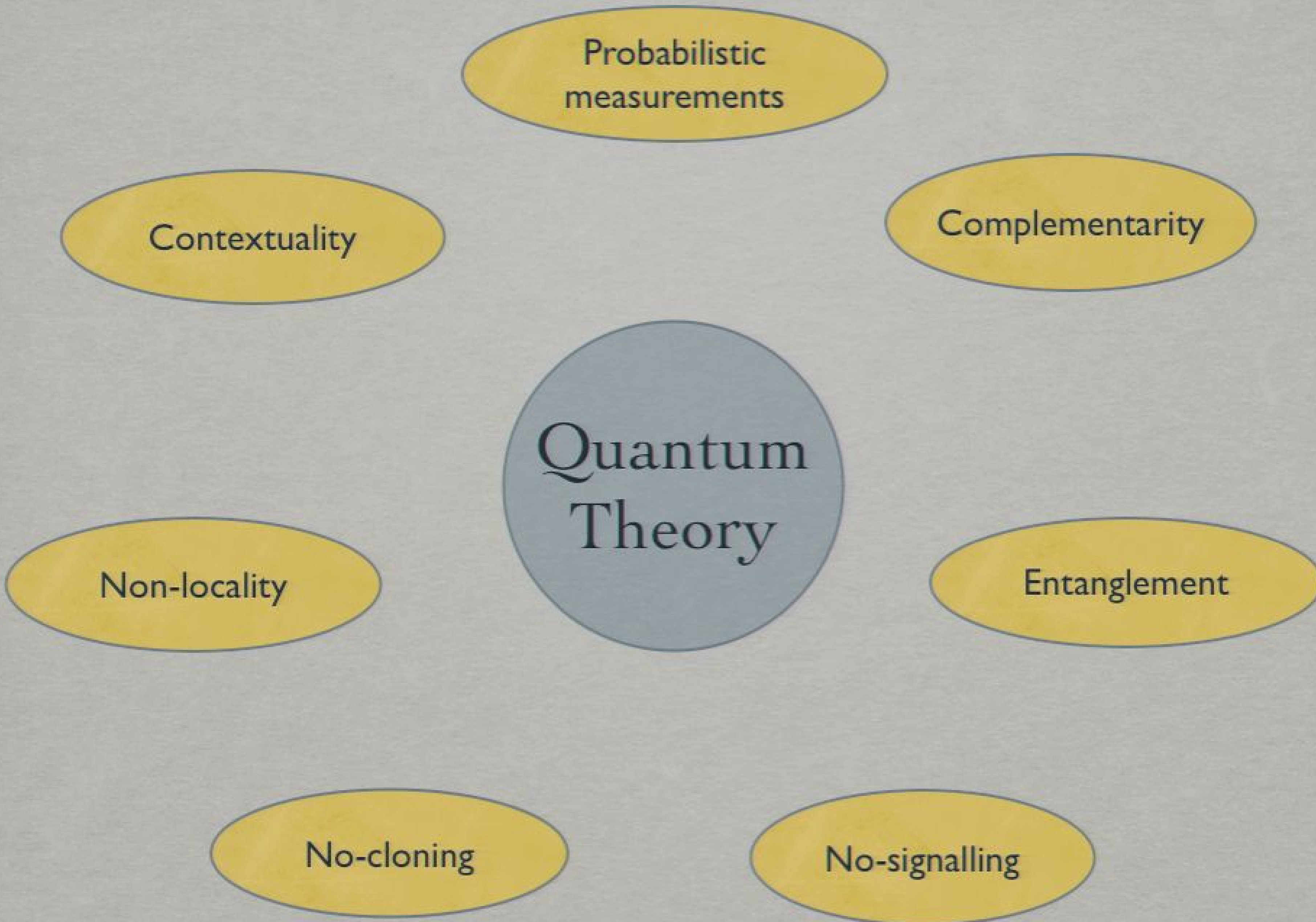
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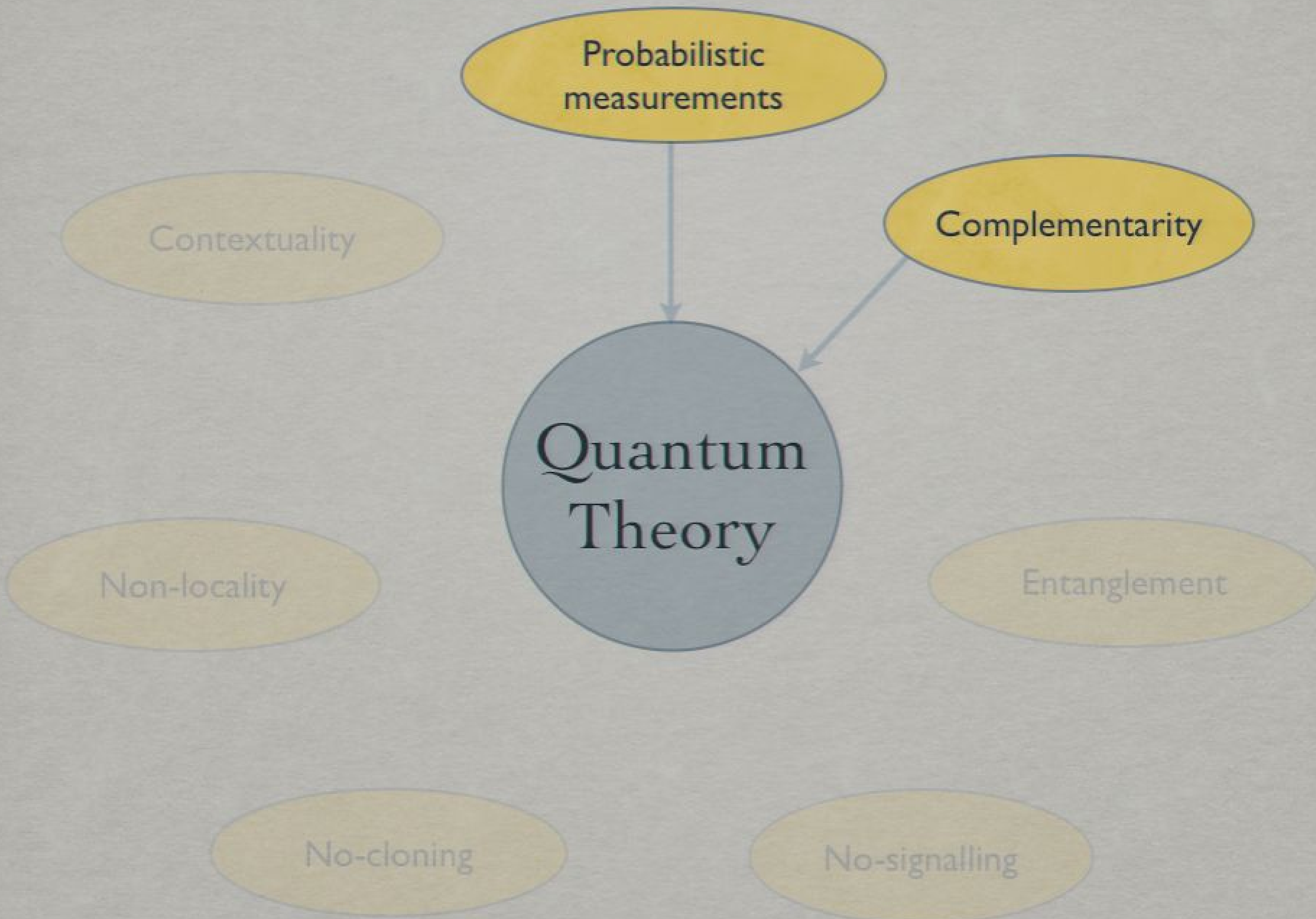
Probabilistic measurements
Complementarity
Contextuality
Quantum Theory
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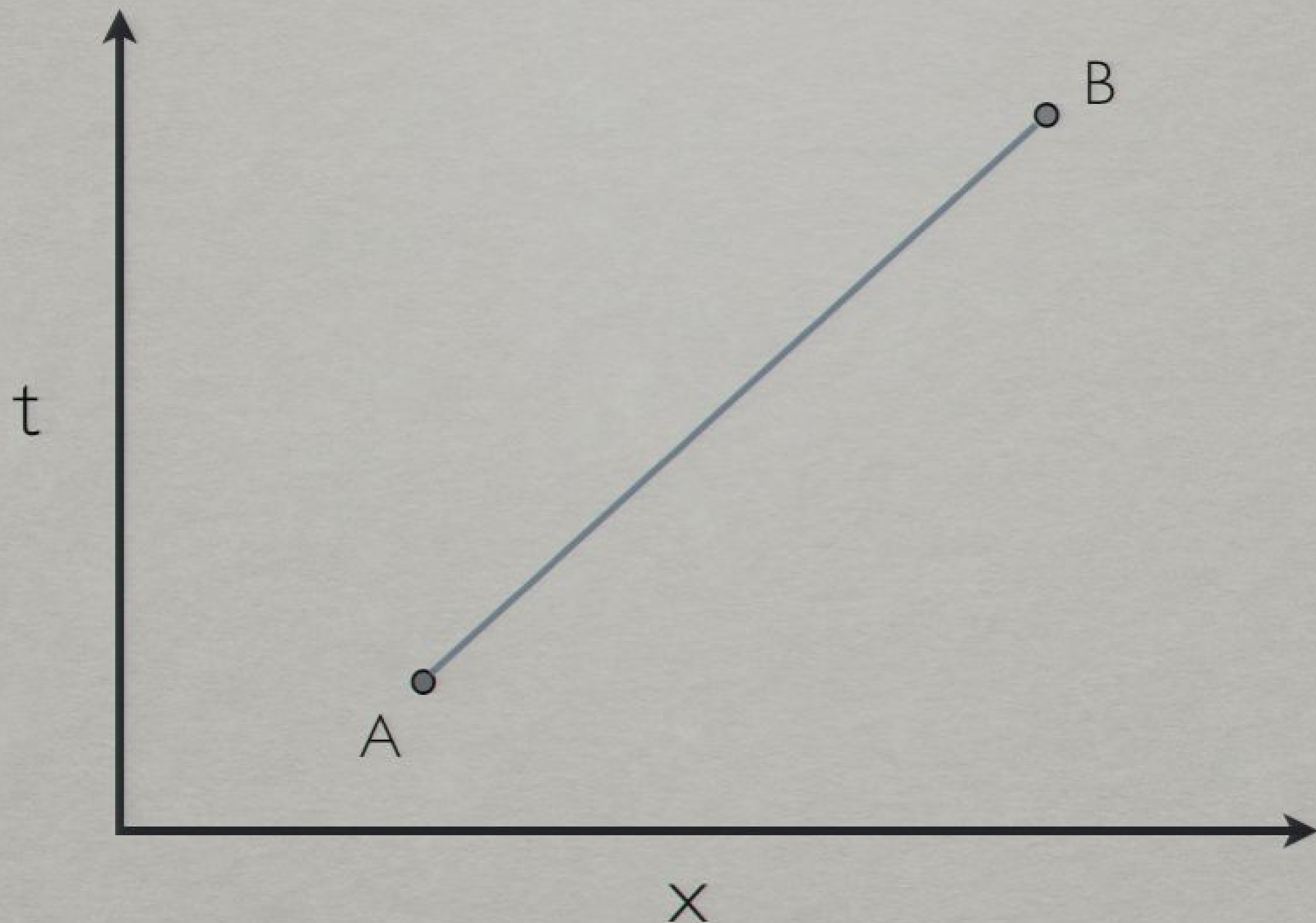
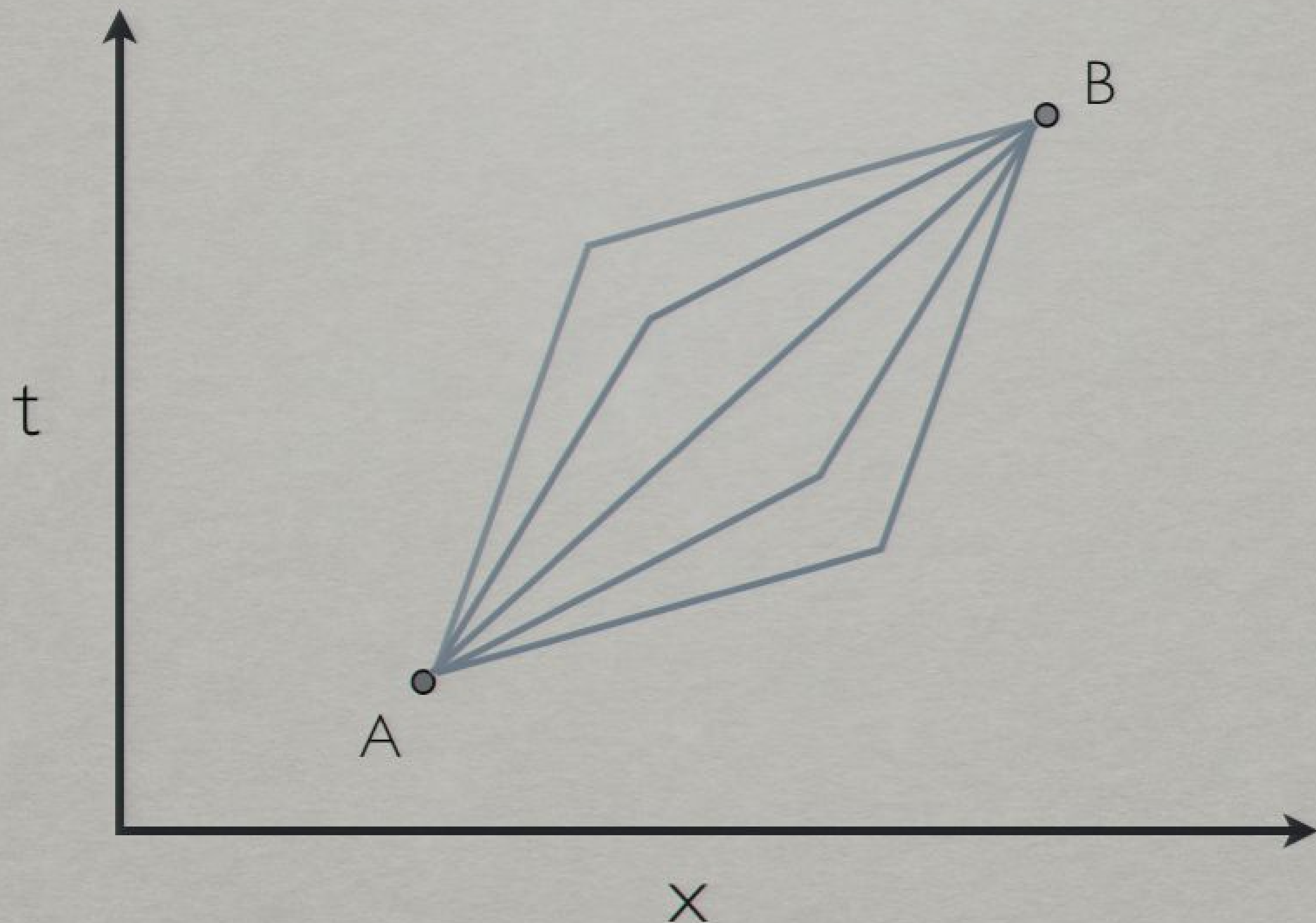
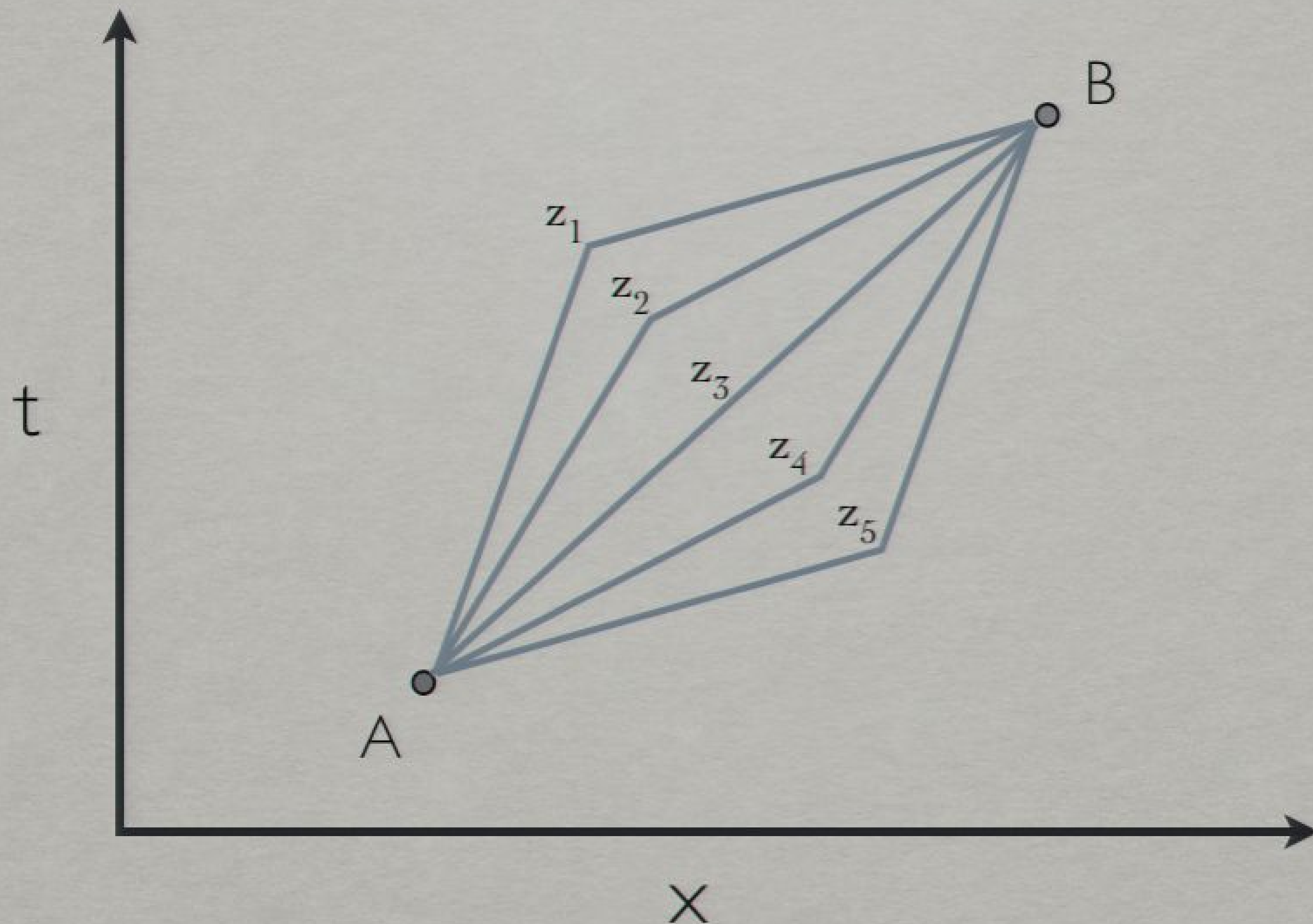
Entanglement
No-cloning
No-signalling

Feynman's Rules of Quantum Theory

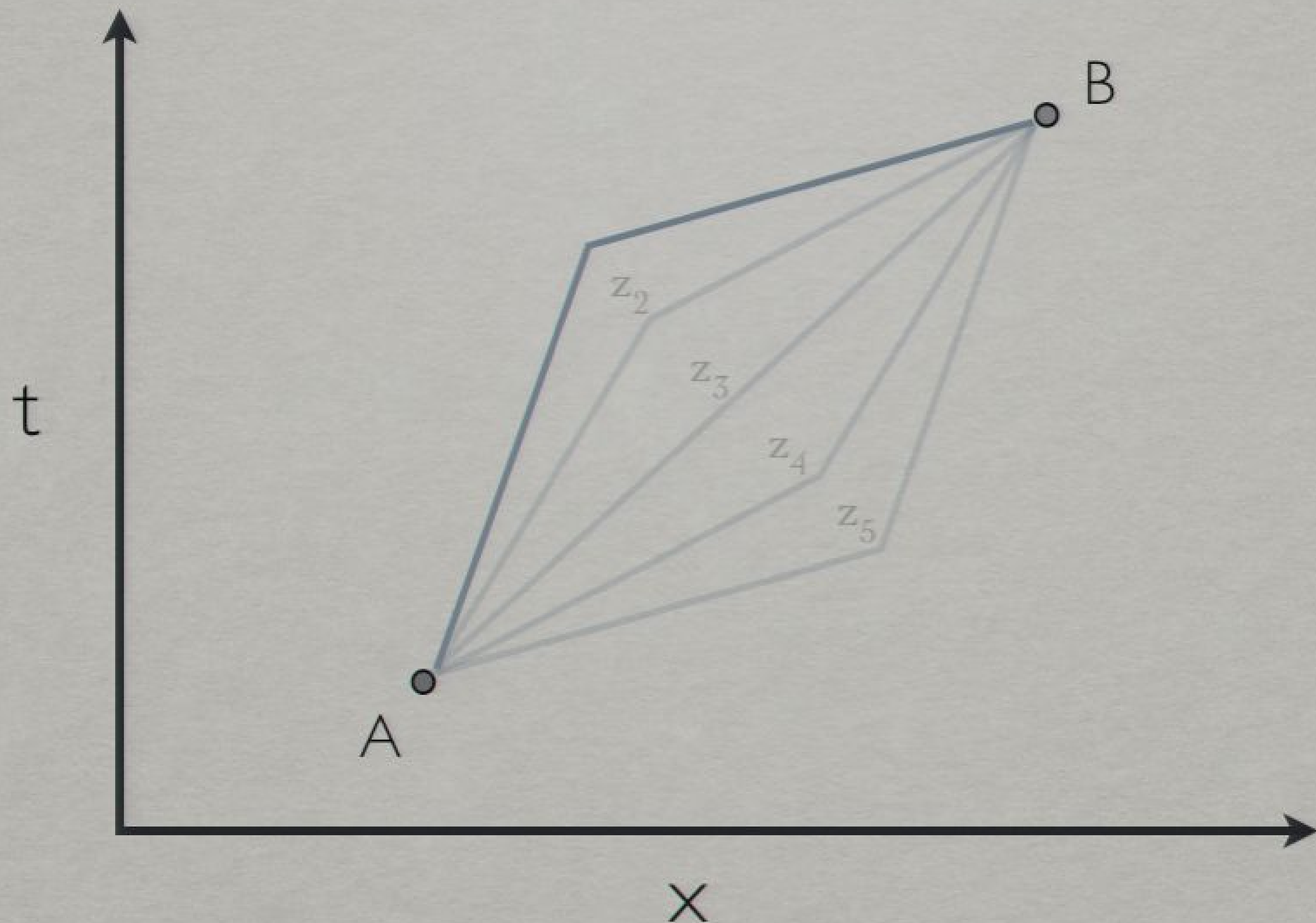

B
t
A
x

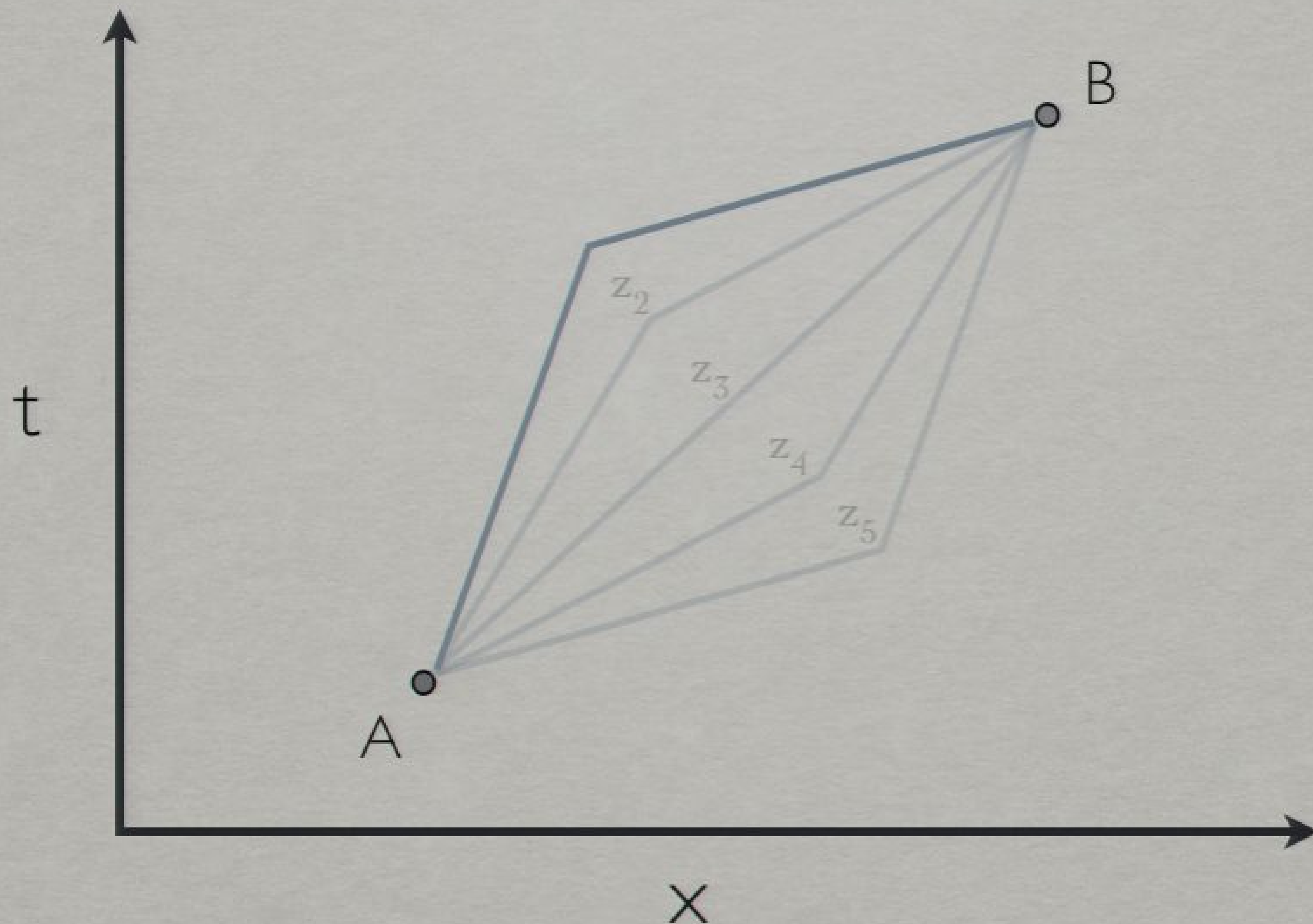

B
t
A
x

$$z = z_1 + \ldots + z_5$$

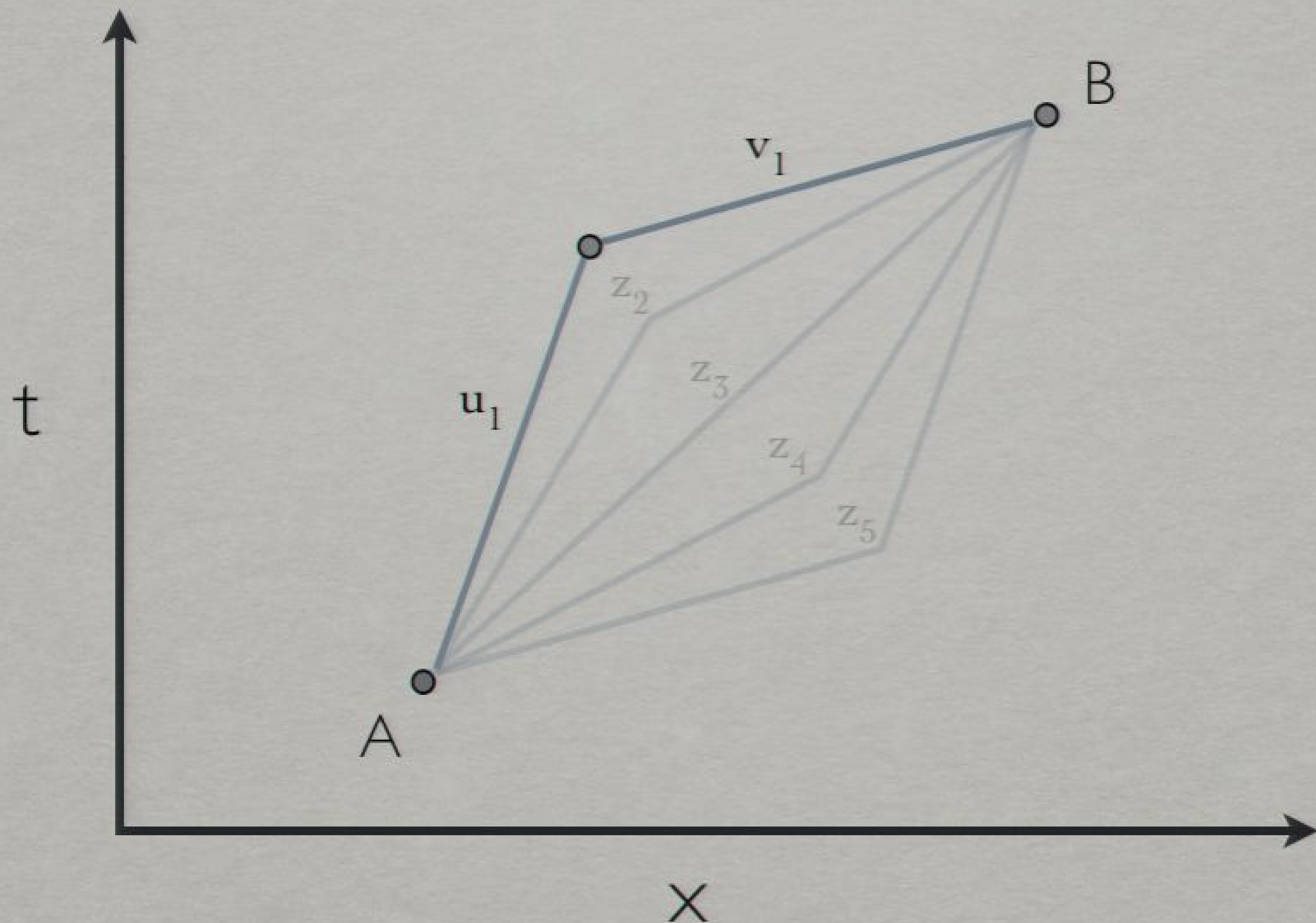


$$z = z_1 + \ldots + z_5$$

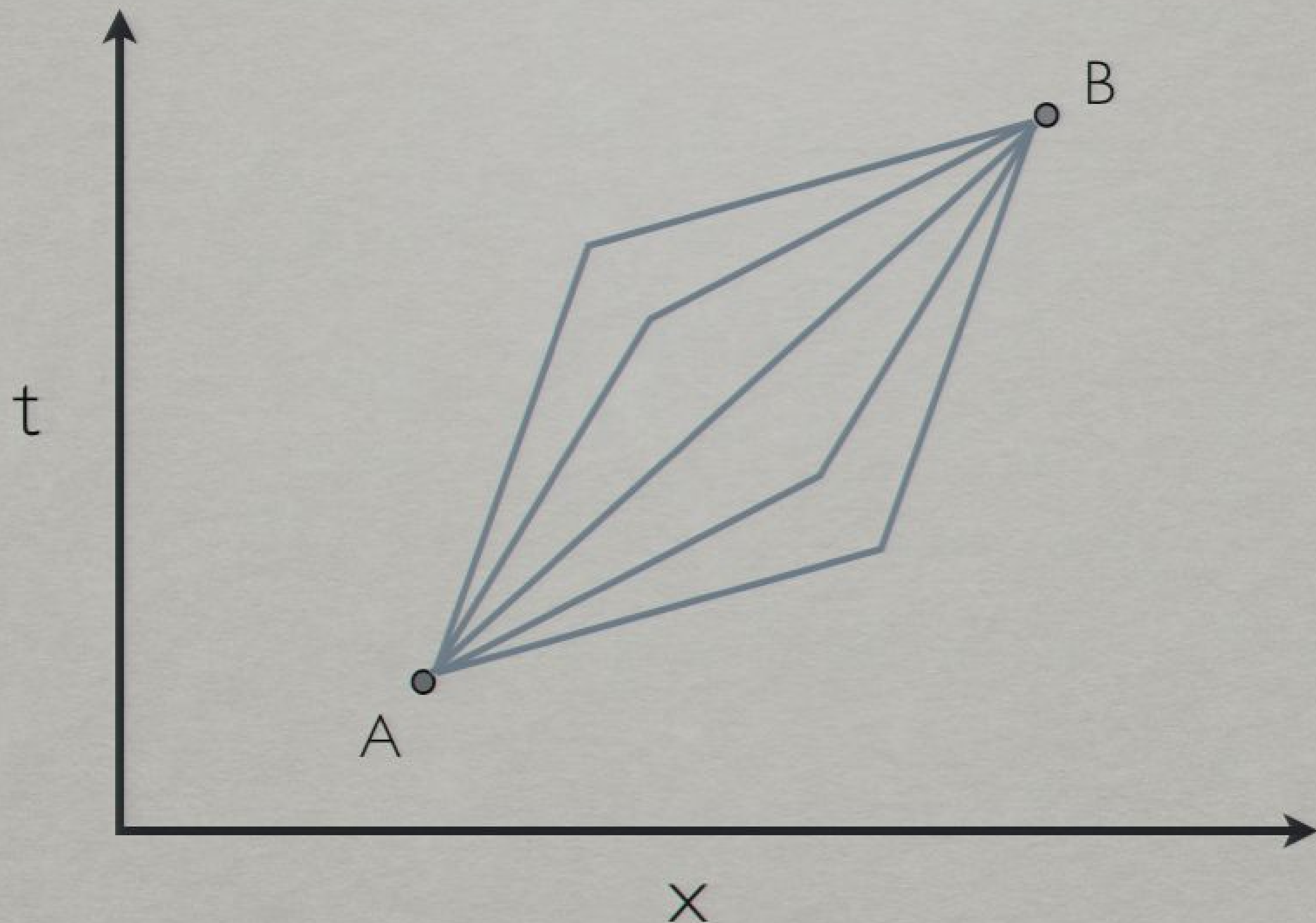



B
z_2
z_3
t
z_4
z_5
A
x

$$z_1 = u_1 v_1$$



$$\Pr(B|A) \propto |z|^2$$



Feynman's Rules of Quantum Theory

- Sum Rule

$$z = z_1 + z_2 + z_3 + \ldots$$

- Product Rule

$$z_1 = u_1 \; v_1$$

Feynman's Rules of Quantum Theory

- Sum Rule

$$z = z_1 + z_2 + z_3 + \ldots$$

- Product Rule

$$z_1 = u_1 v_1$$

- Probability Rule

$$p = |z|^2$$

Complementarity

Pair Postulate

- Each ***sequence of experimental outcomes*** has an associated ***pair of real numbers.***

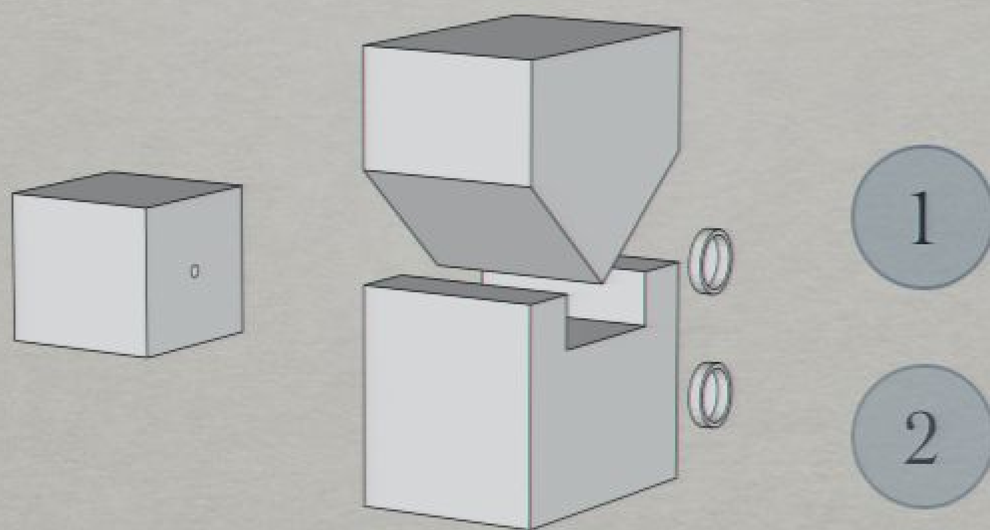
Pair Postulate

- Each ***sequence of experimental outcomes*** has an associated ***pair of real numbers.***

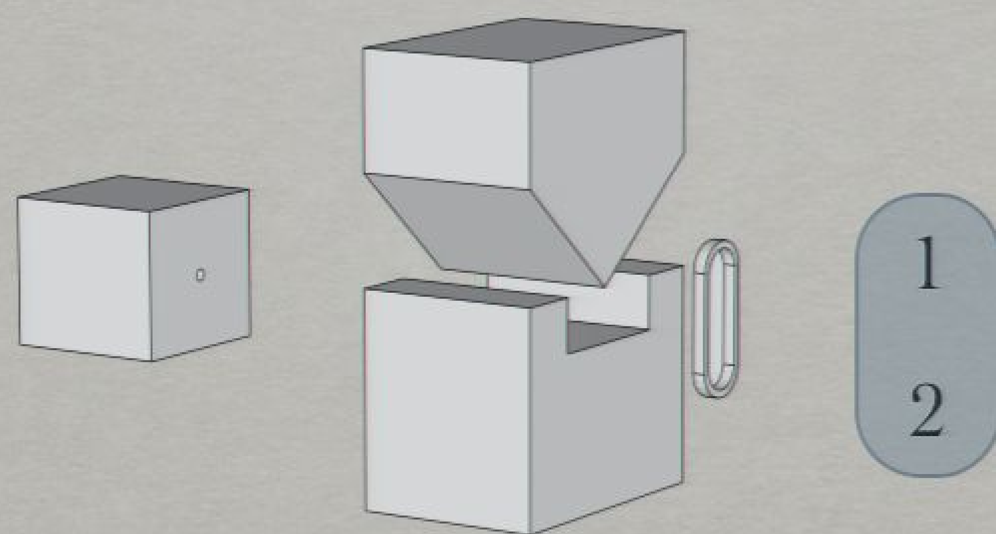
- The ***probability*** of each sequence is a continuous, non-trivial ***function*** of this pair.

Experimental Framework

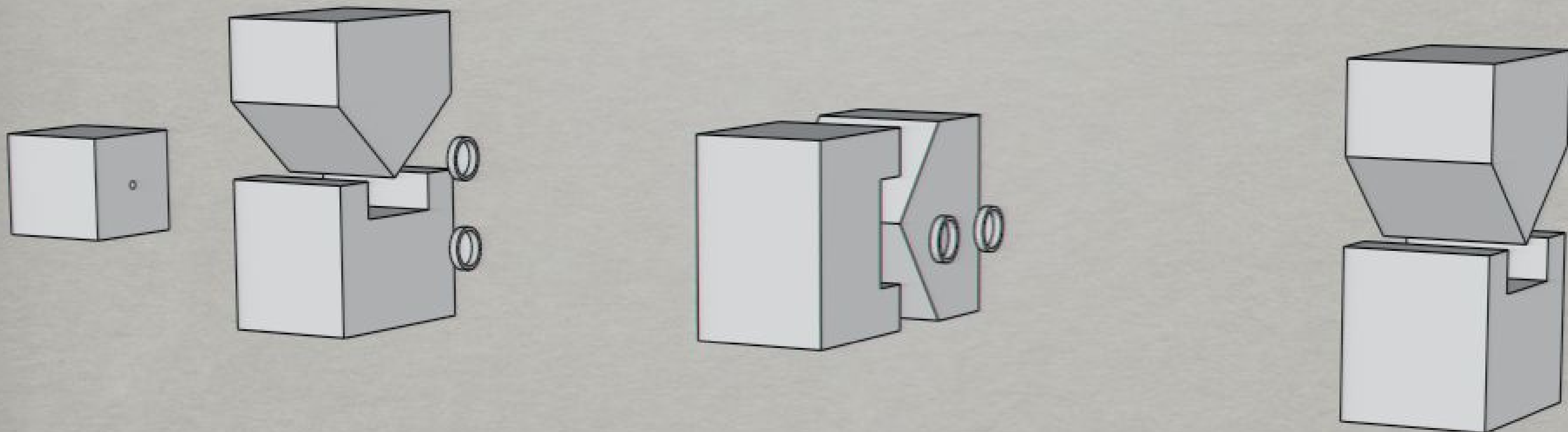
Stern Gerlach Measurement



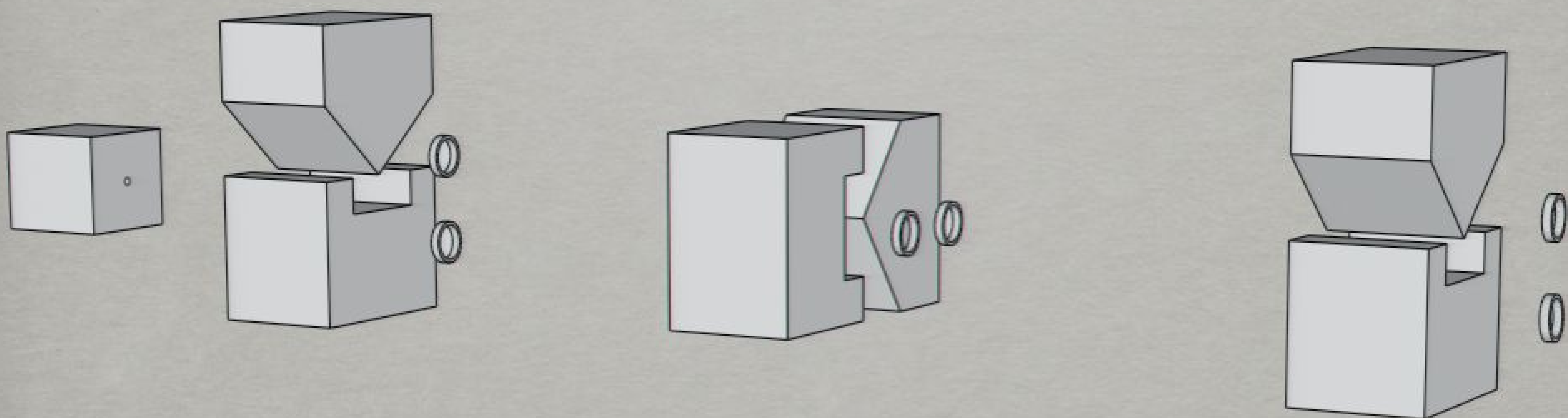
Stern Gerlach Measurement



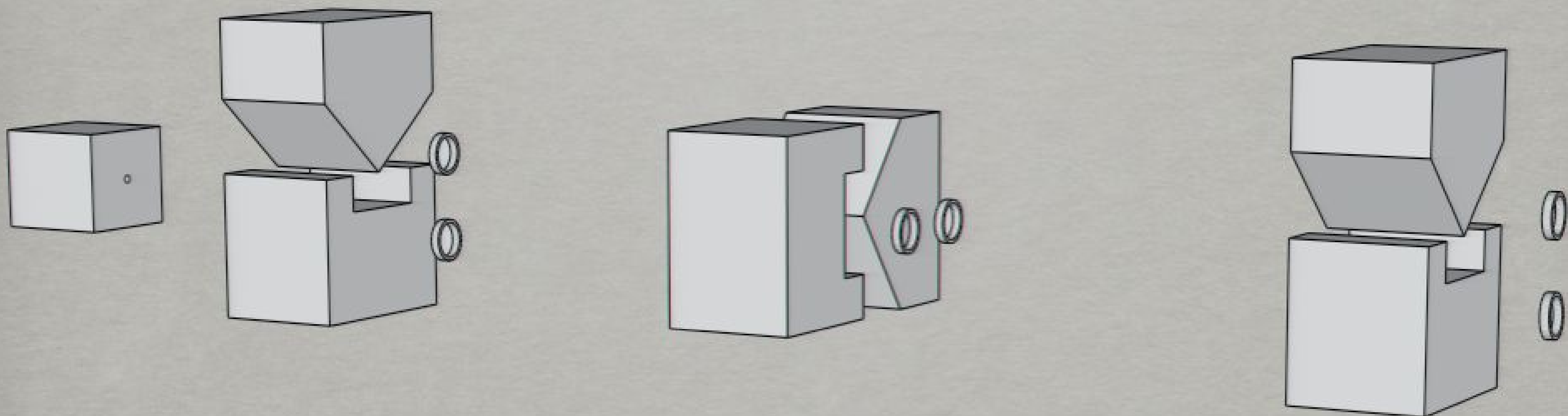
Experimental Setups



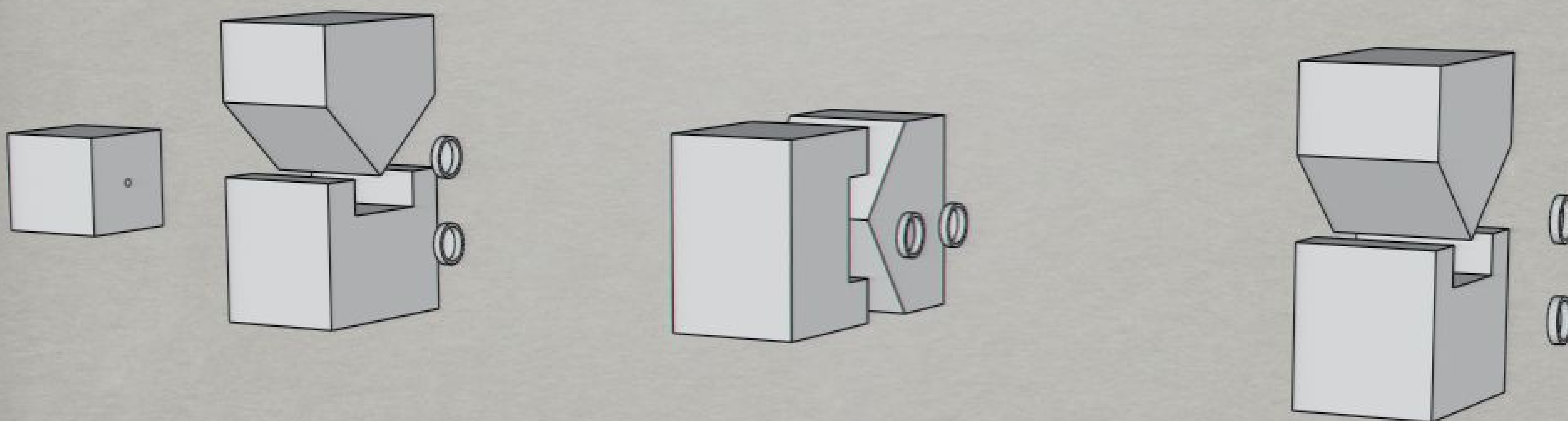
Experimental Setups



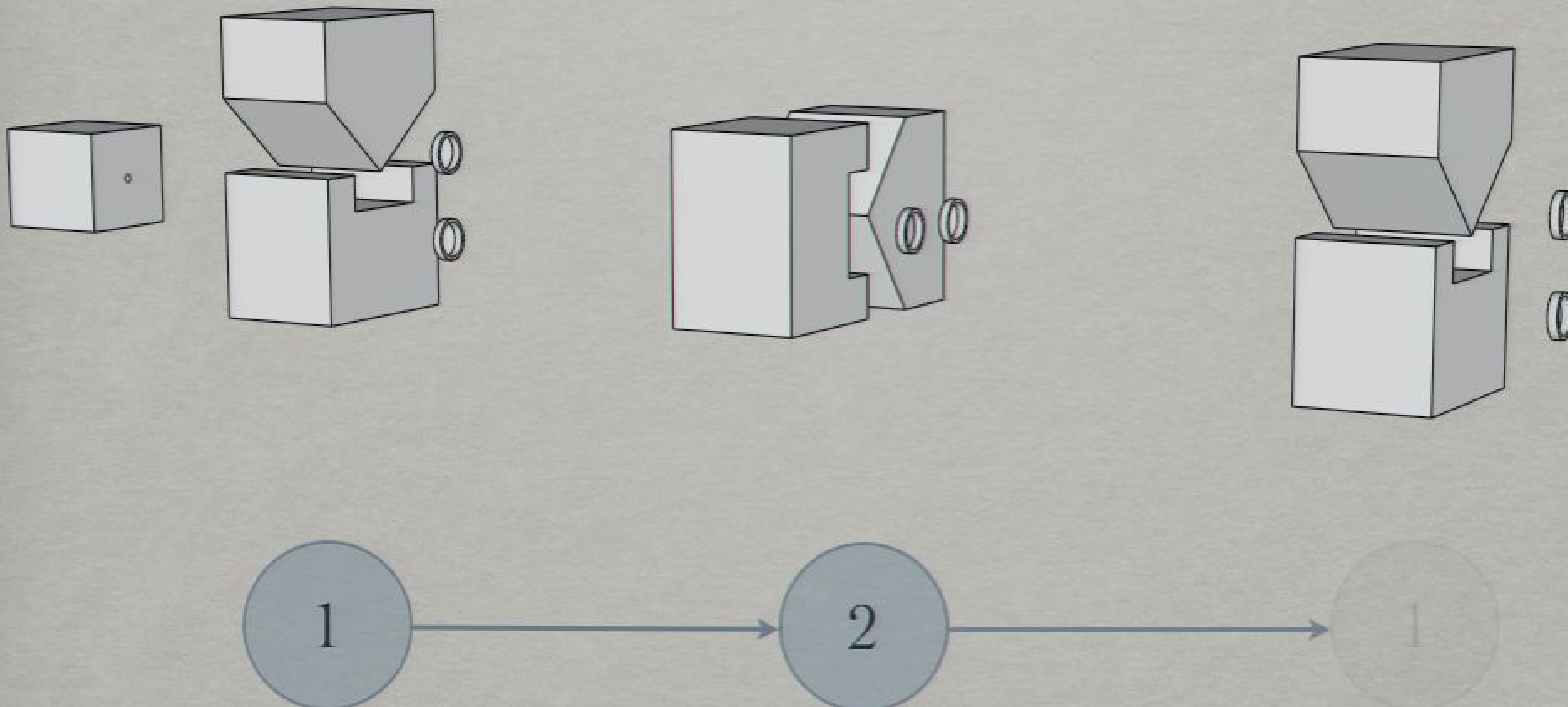
Experimental Setups



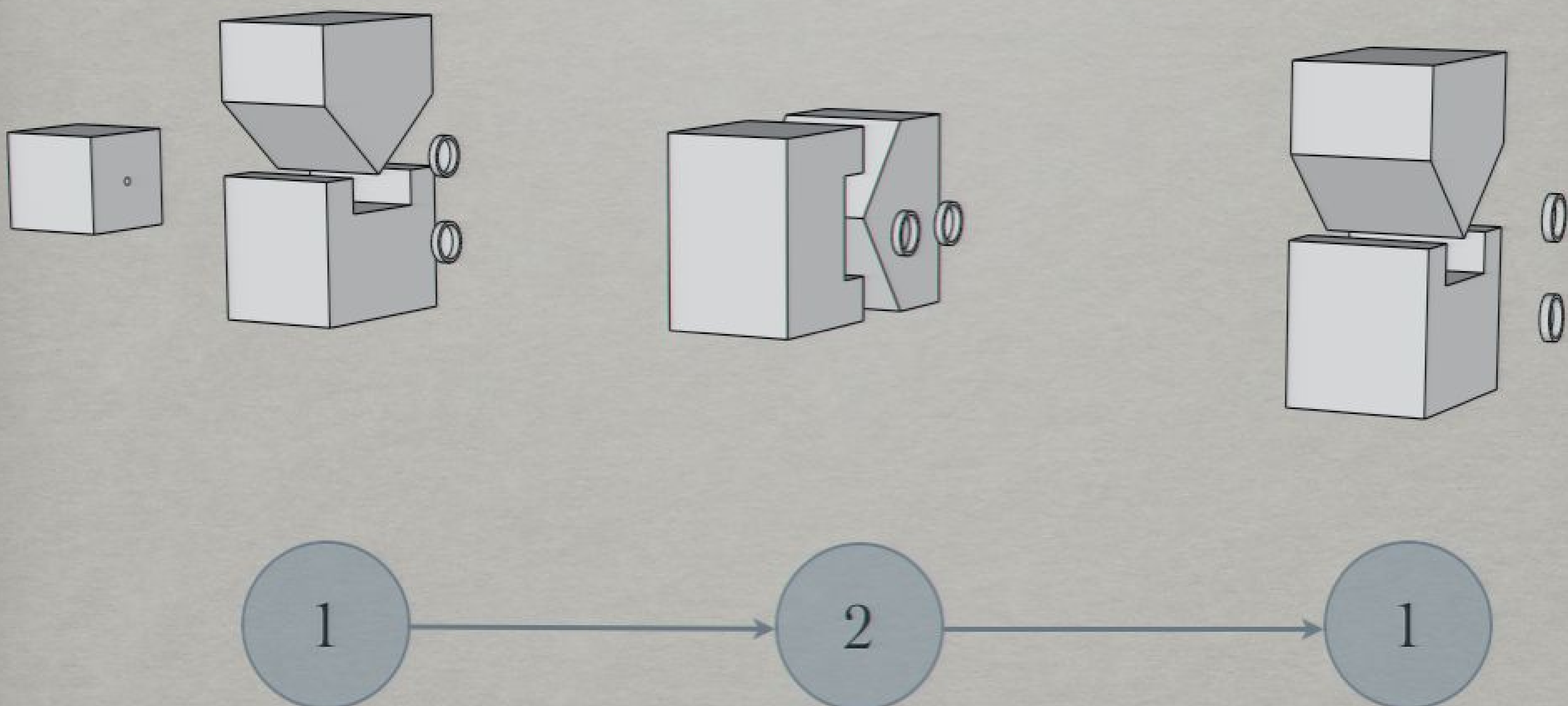
Experimental Setups



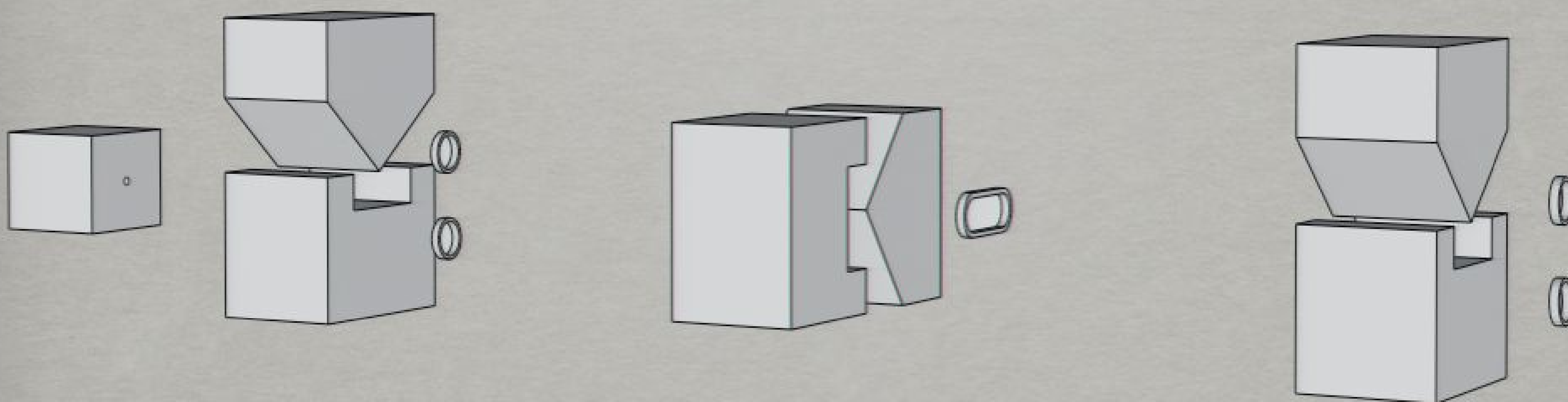
Experimental Setups



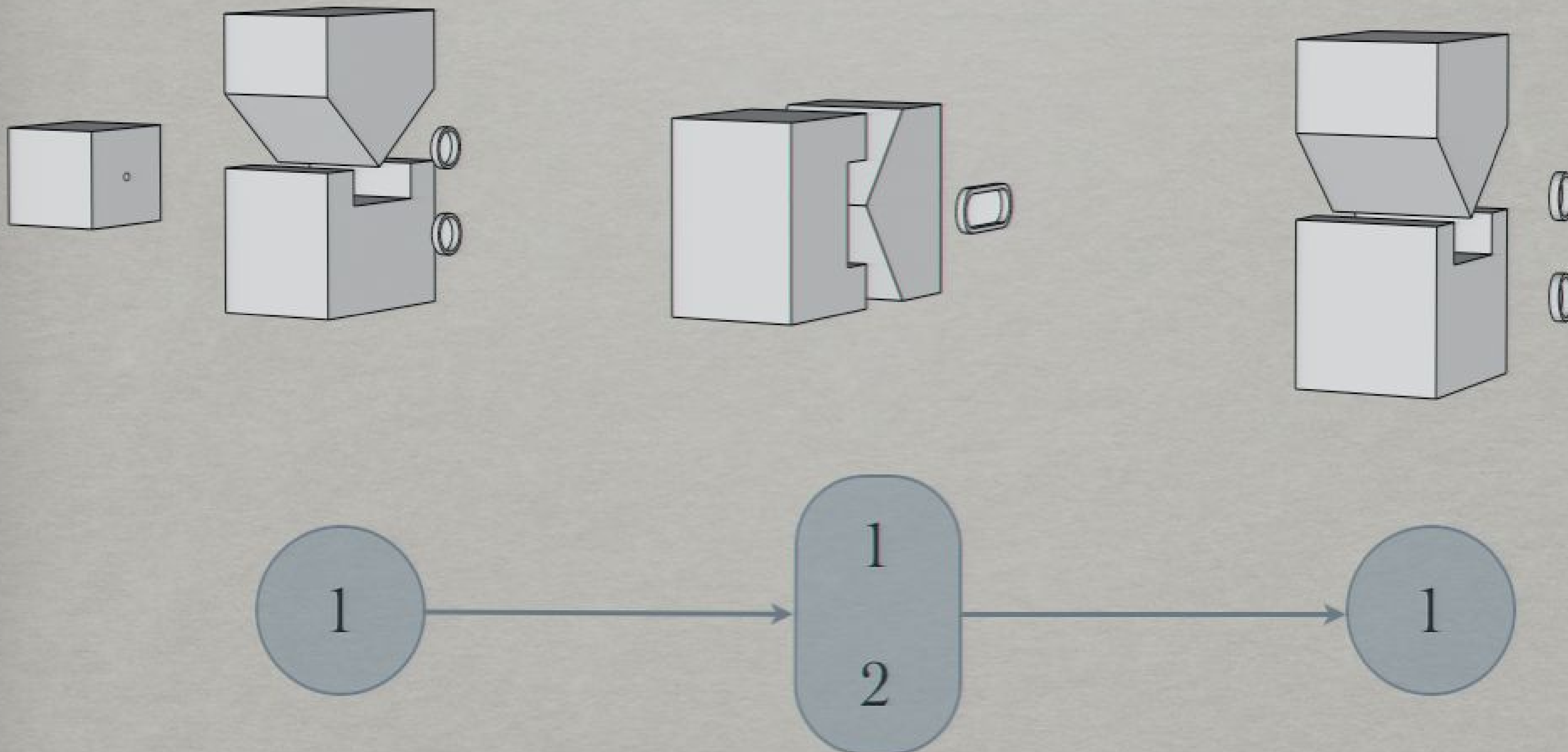
Experimental Setups



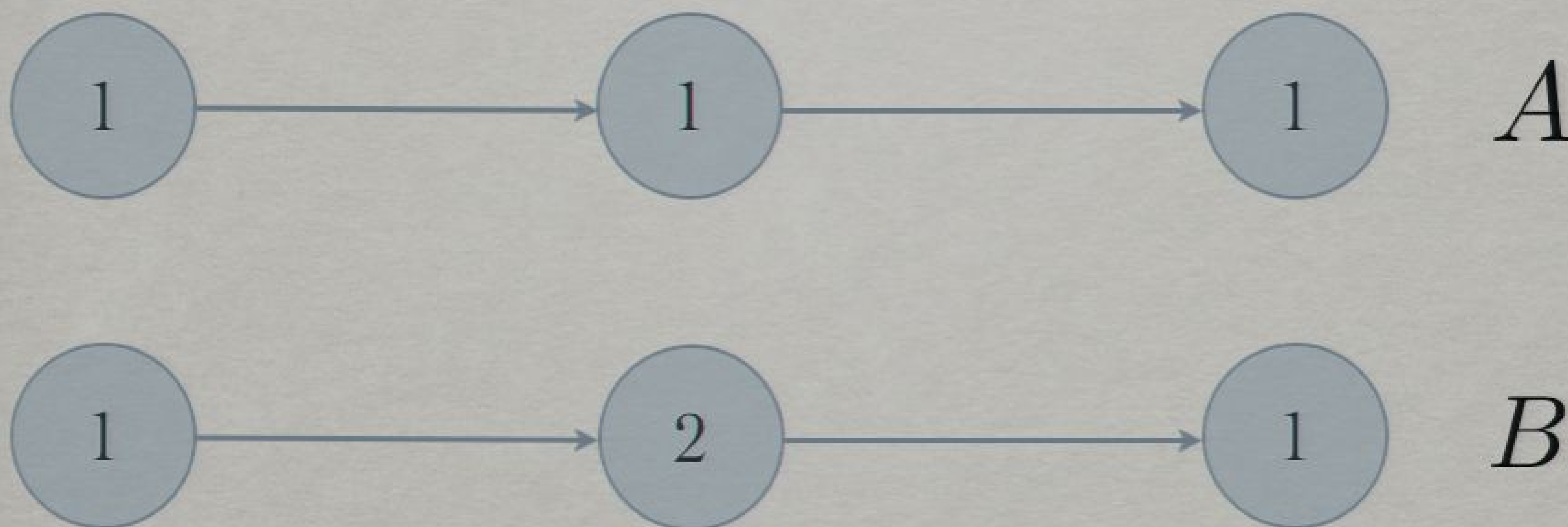
Experimental Setups



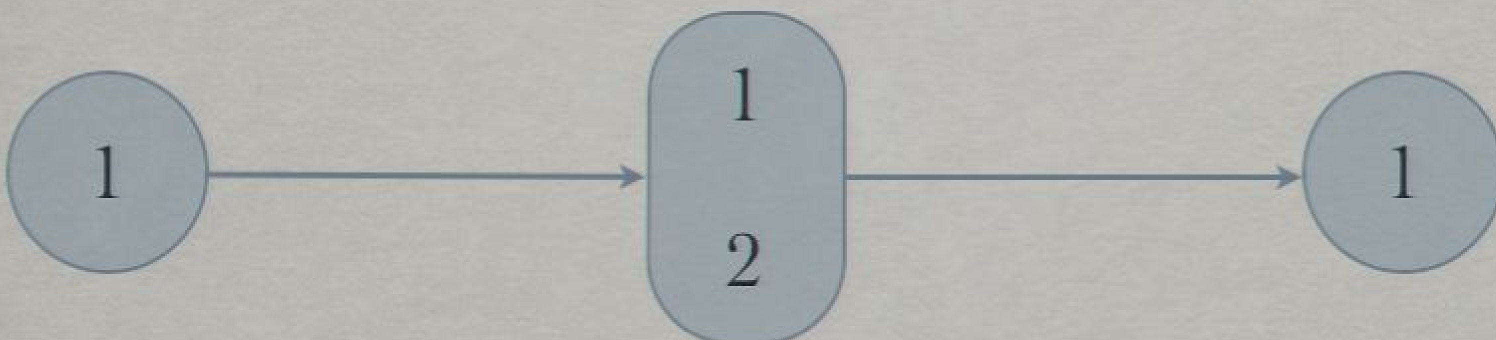
Experimental Setups



Combining Sequences


1
1
1
A
1
2
1
B

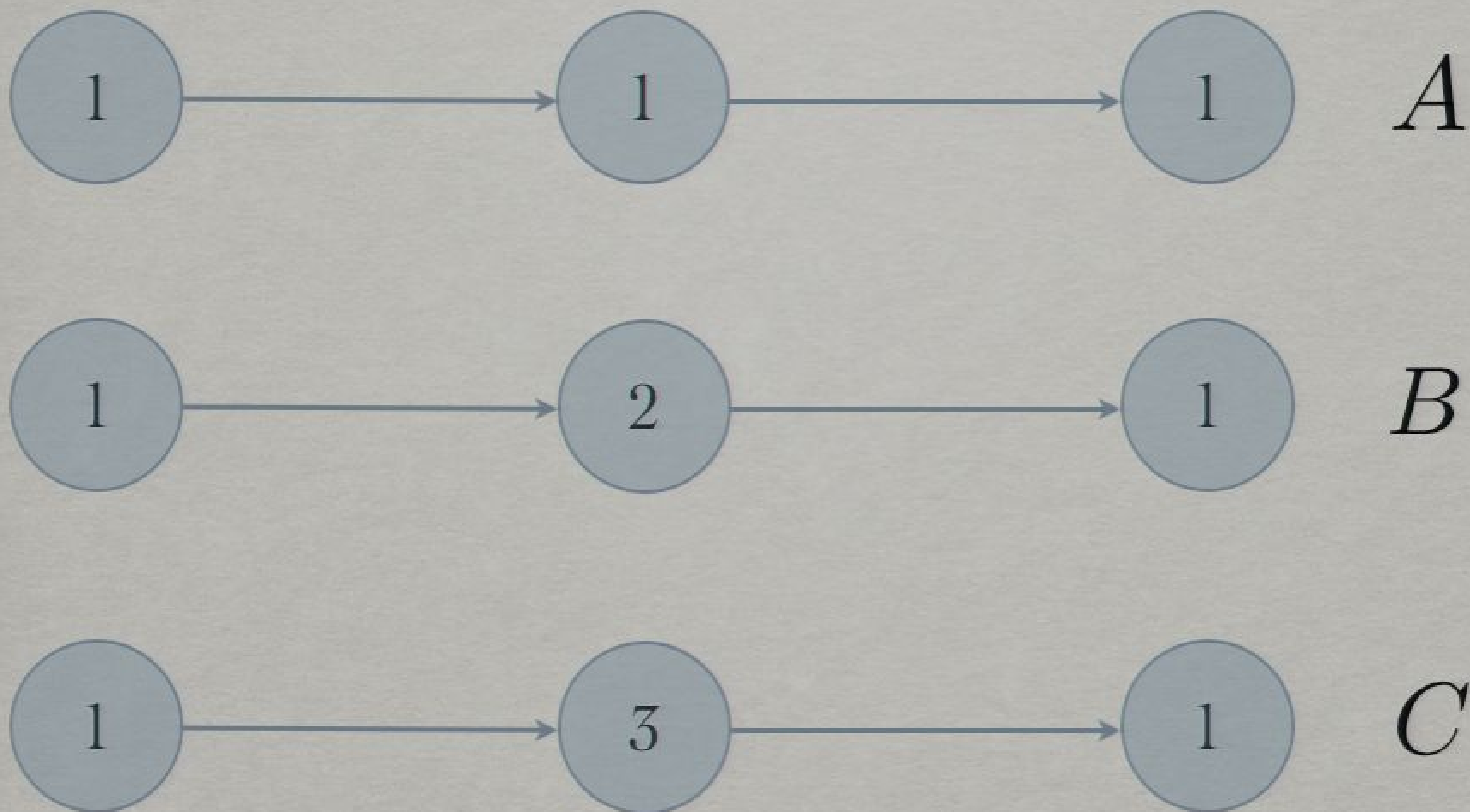
$$C = A \vee B$$

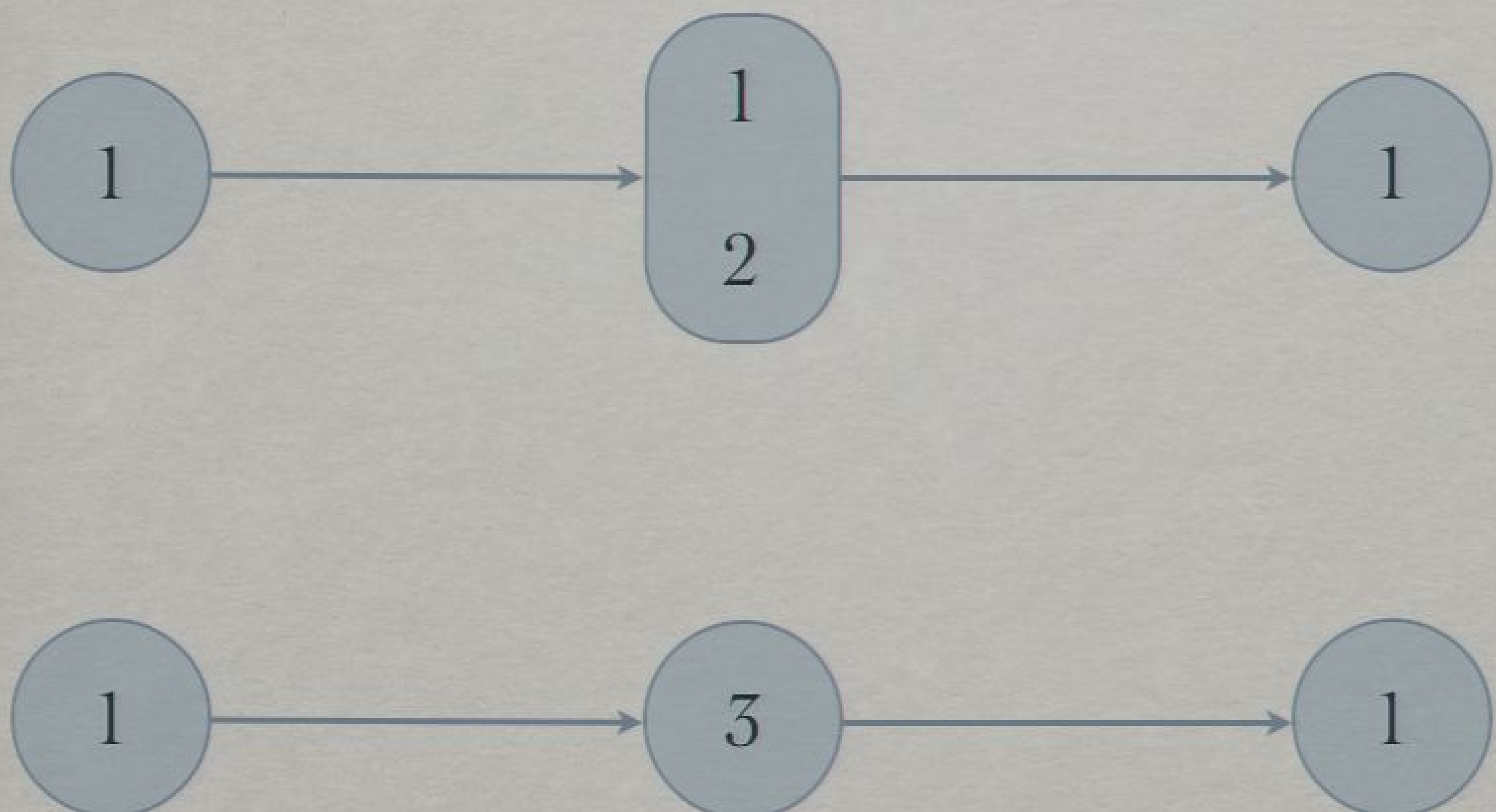


C

Commutativity of Parallel Combination

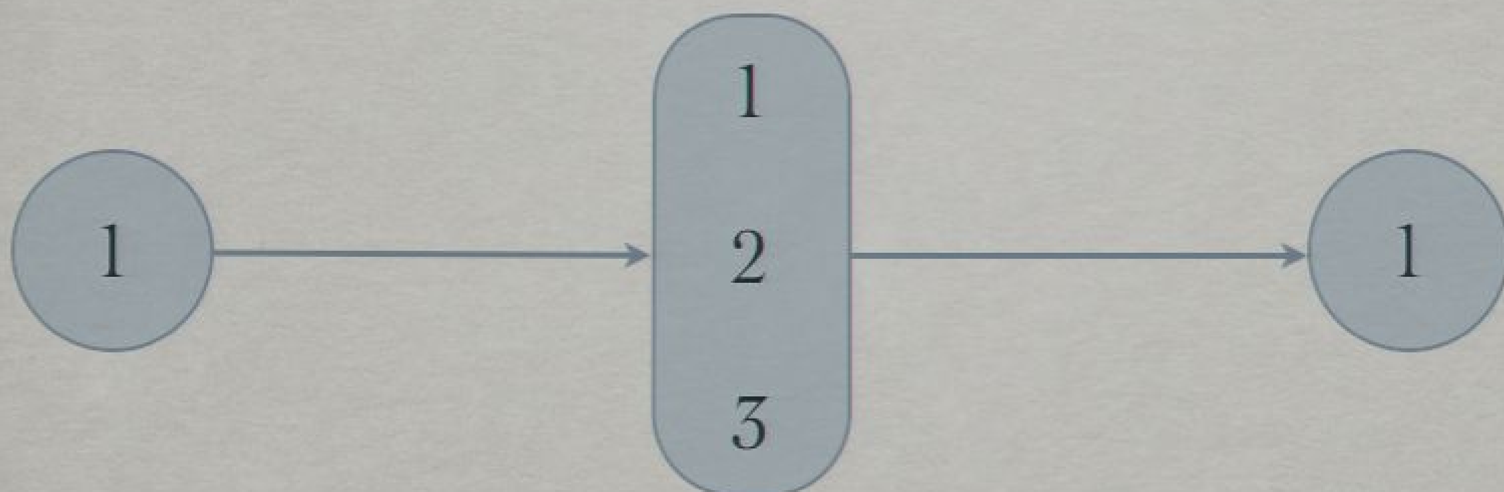
$$A \vee B = B \vee A$$

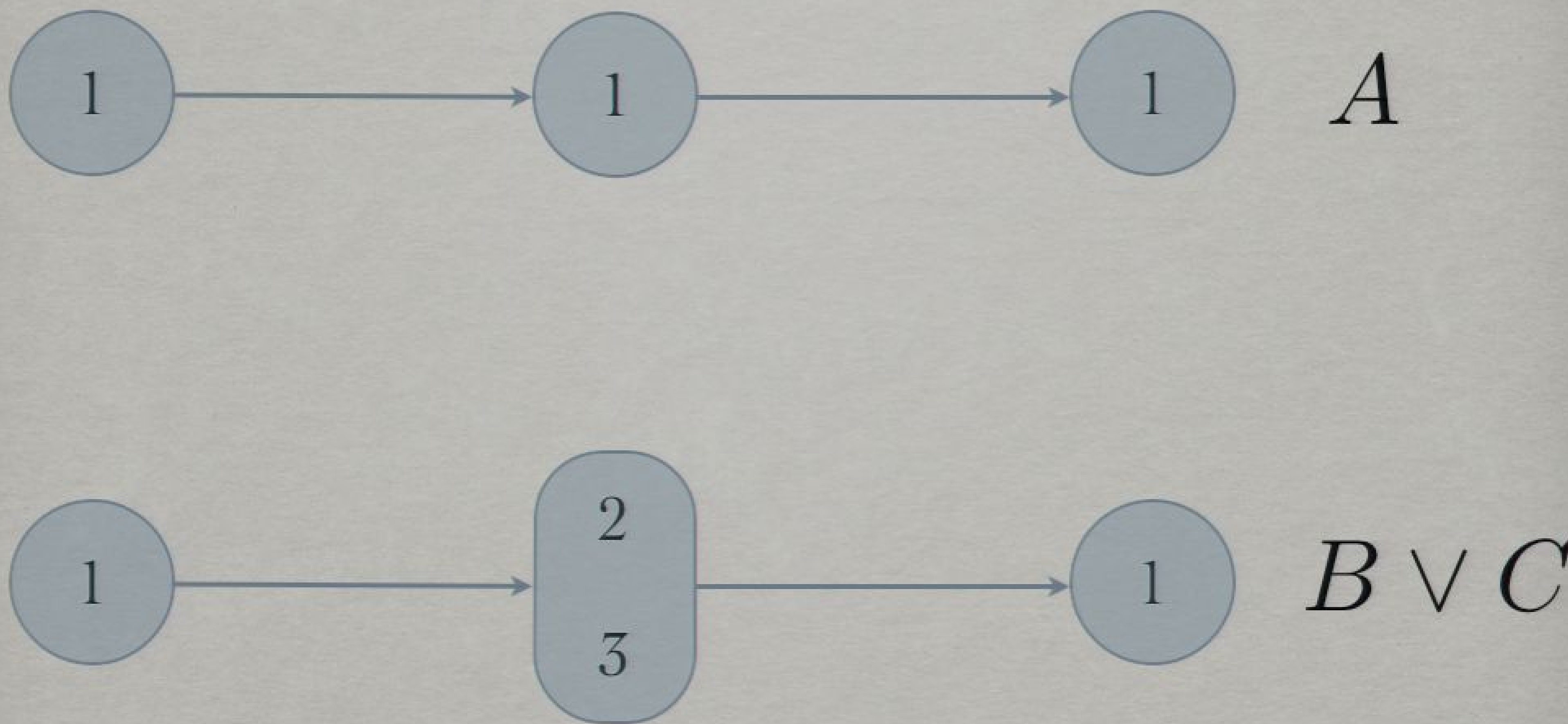

1
1
1
A
1
2
1
B
1
3
1
C



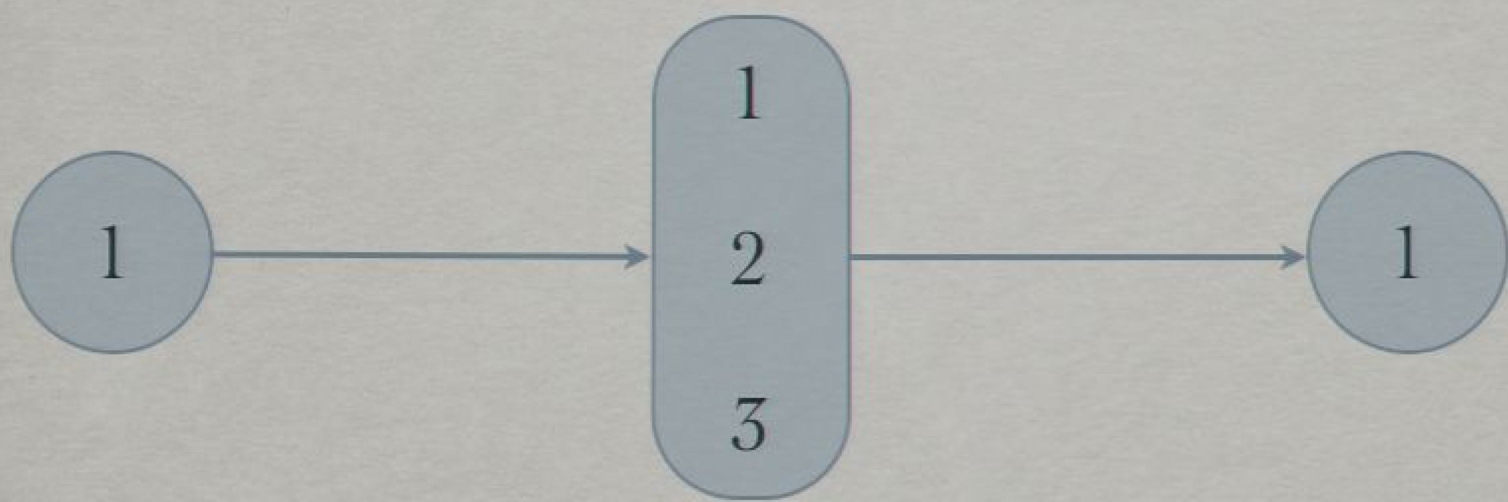
C

$$(A \vee B) \vee C$$



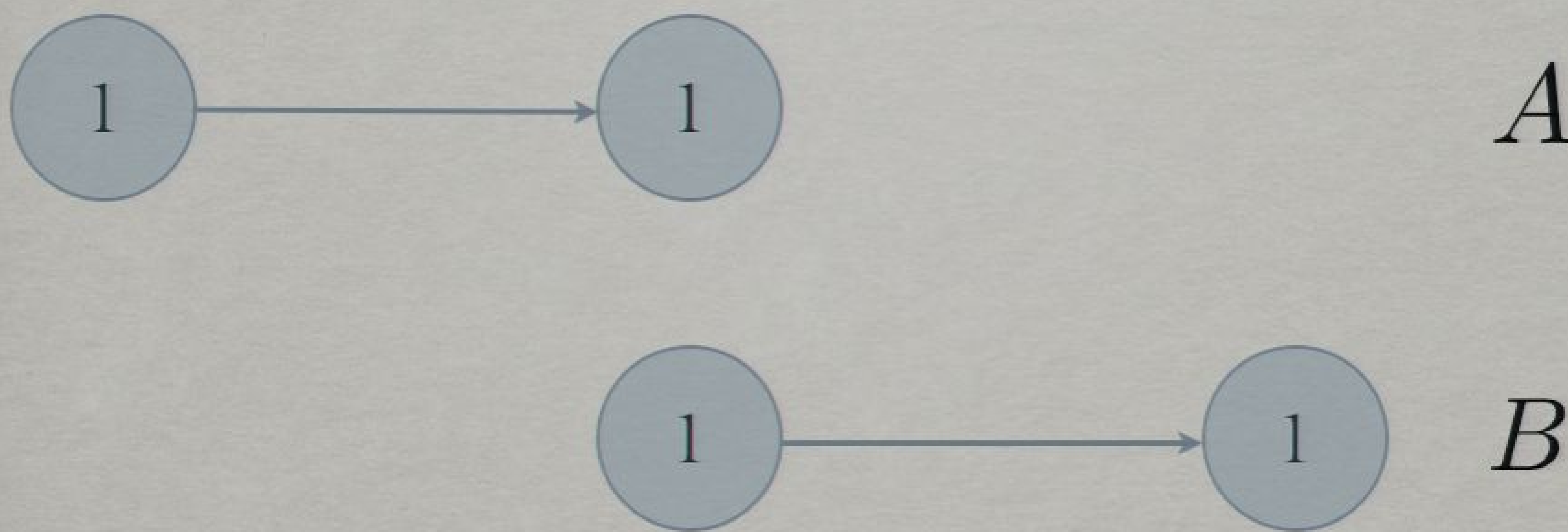

1
1
1
A
1
2
3
1
B ∨ C

$$A \vee (B \vee C)$$



Associativity of Parallel Combination

$$(A \vee B) \vee C = A \vee (B \vee C)$$


1
1
A
1
1
B

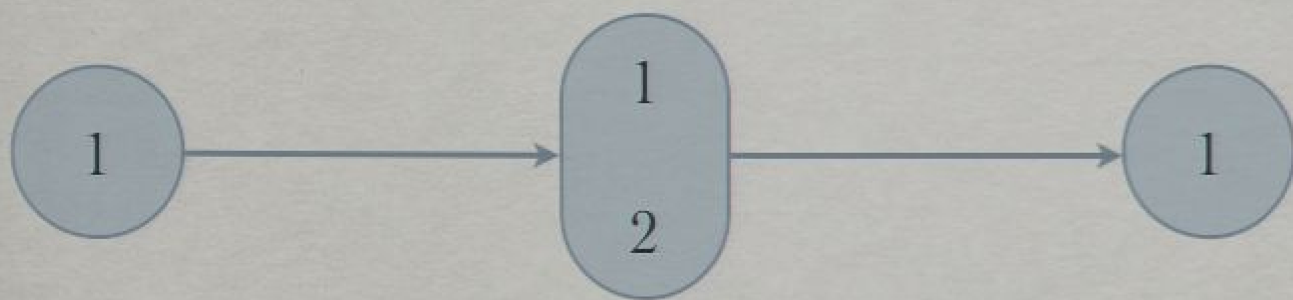
$$C = A \cdot B$$

C

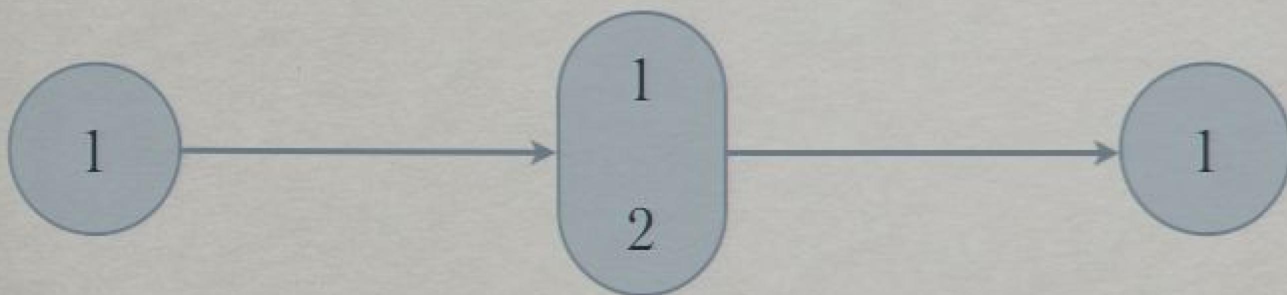
Associativity of Series Combination

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

$$A \vee B$$

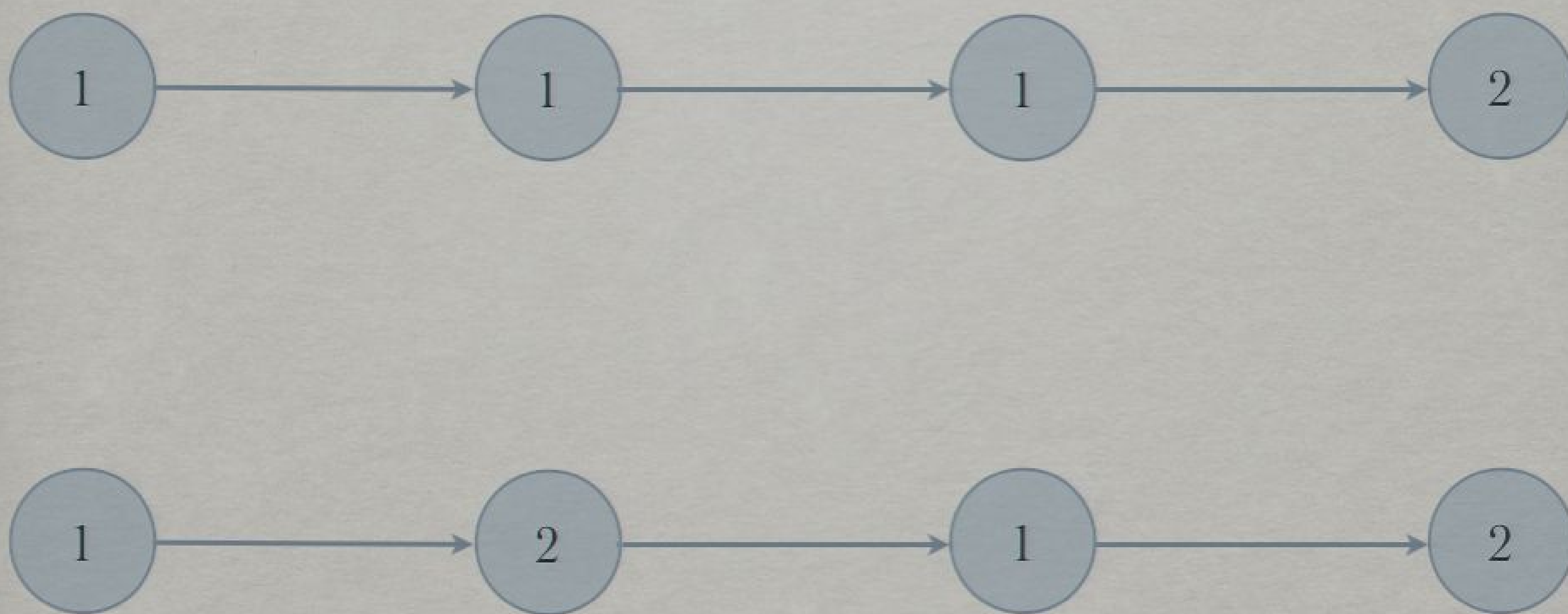


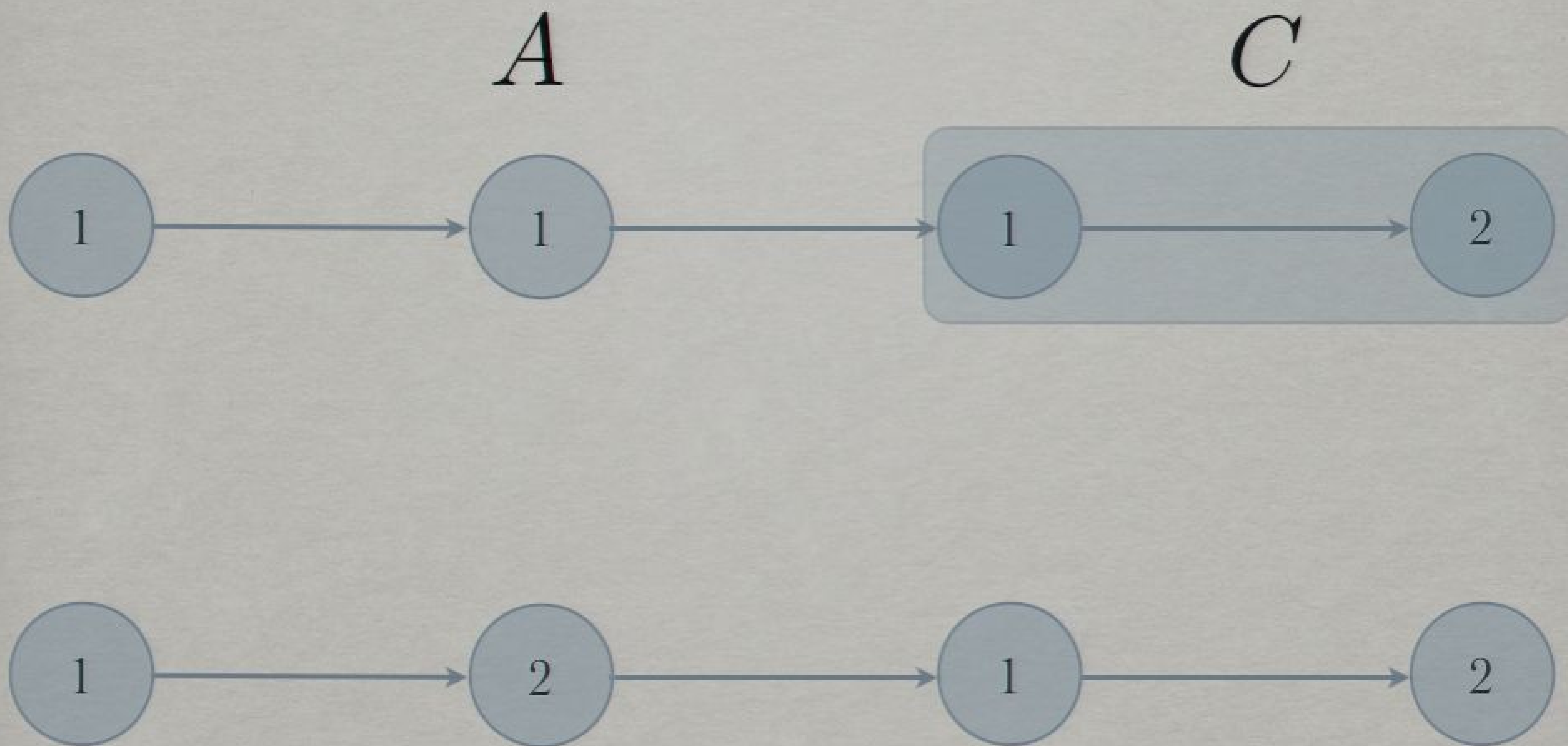
$A \vee B$

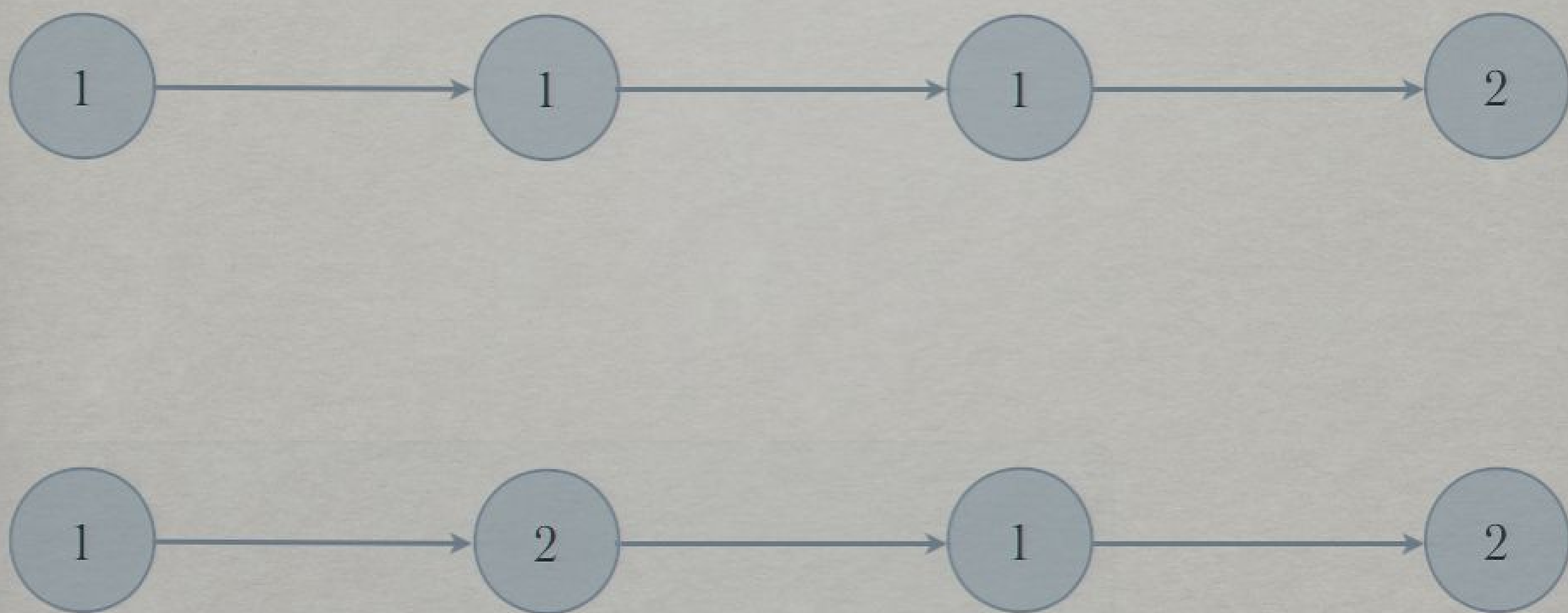


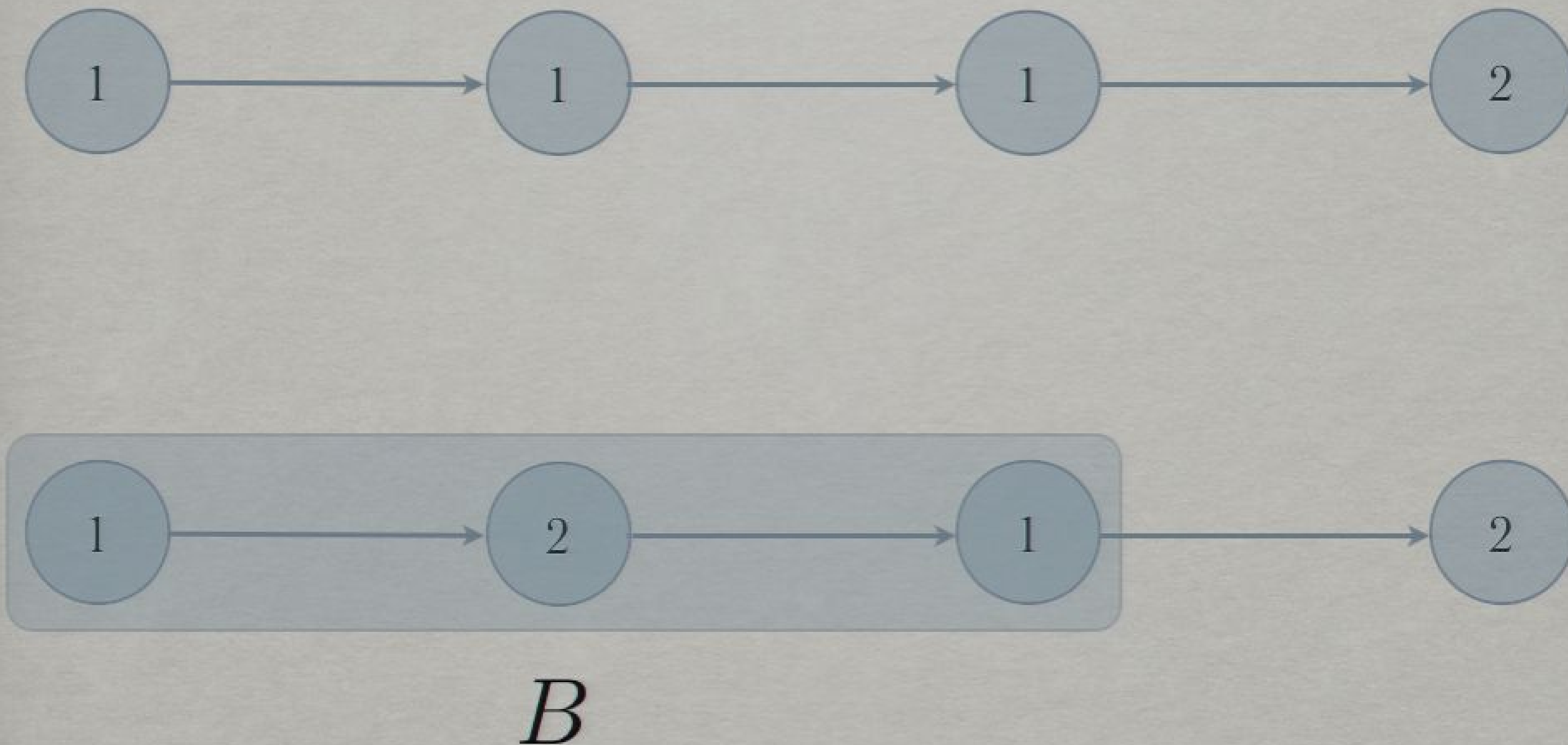
C

$$(A \vee B) \cdot C$$

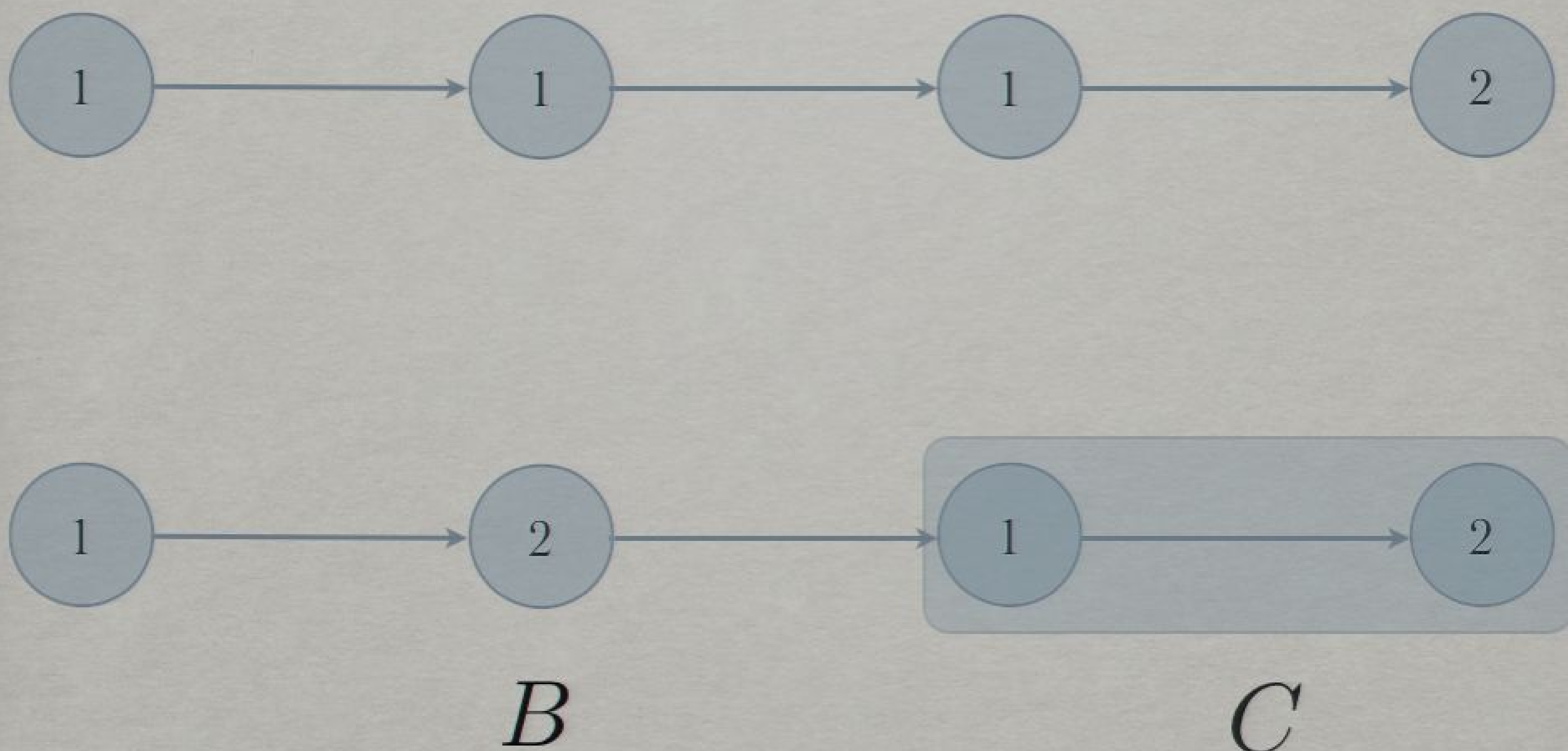

1
1
1
2
1
2
1
2

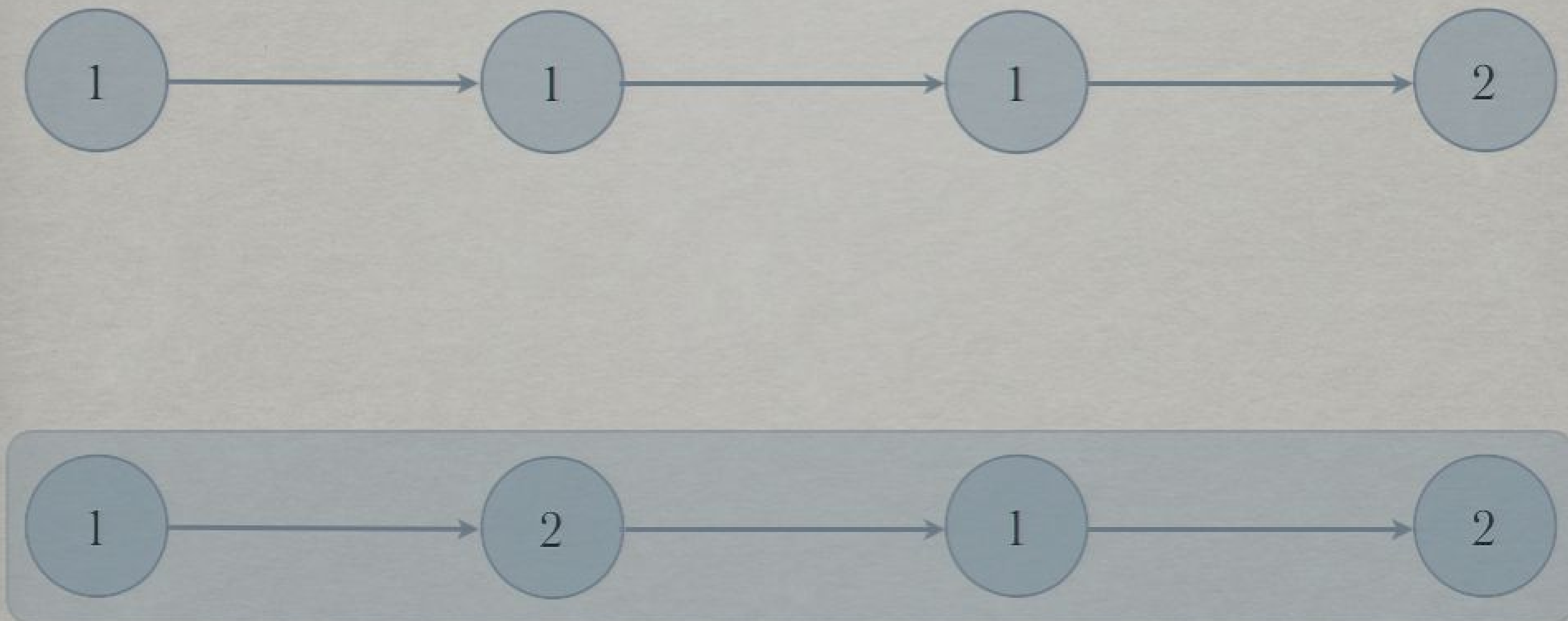

A
C
1
1
1
2
1
2
1
2


1
1
1
2
1
2
1
2



1
1
1
2
1
2
1
2
B


1
1
1
2
1
2
1
2
B
C



$$B \cdot C$$

$$(A \cdot C) \vee (B \cdot C)$$

Distributivity

$$(A \vee B) \cdot C = (A \cdot C) \vee (B \cdot C)$$

Distributivity

$$(A \vee B) \cdot C = (A \cdot C) \vee (B \cdot C)$$

$$A \cdot (B \vee C) = (A \cdot B) \vee (A \cdot C)$$

Symmetry Properties of **Sequence** Combination Operators

$$(A \vee B) \vee C = A \vee (B \vee C)$$

$$A \vee B = B \vee A$$

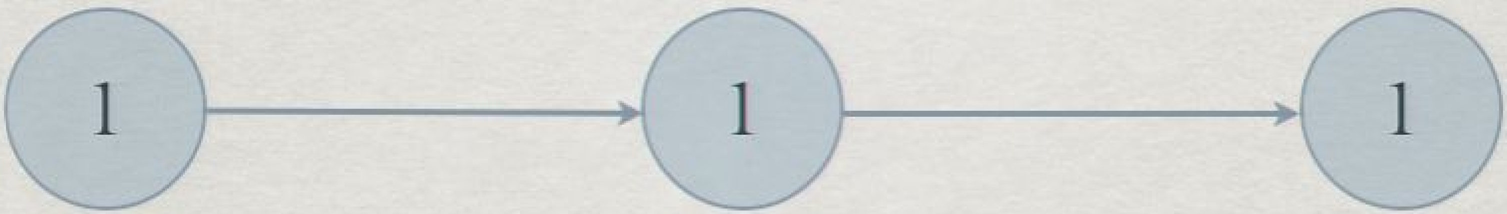
$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

$$(A \vee B) \cdot C = (A \cdot C) \vee (B \cdot C)$$

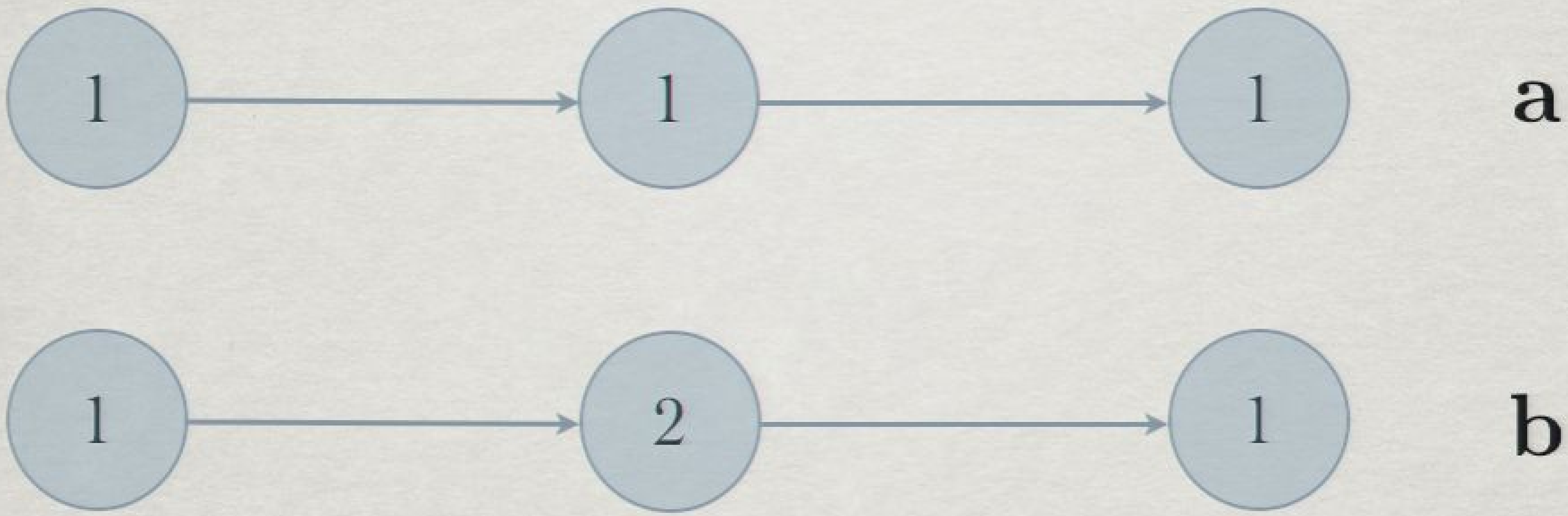
$$A \cdot (B \vee C) = (A \cdot B) \vee (A \cdot C)$$

Quantifying Sequences

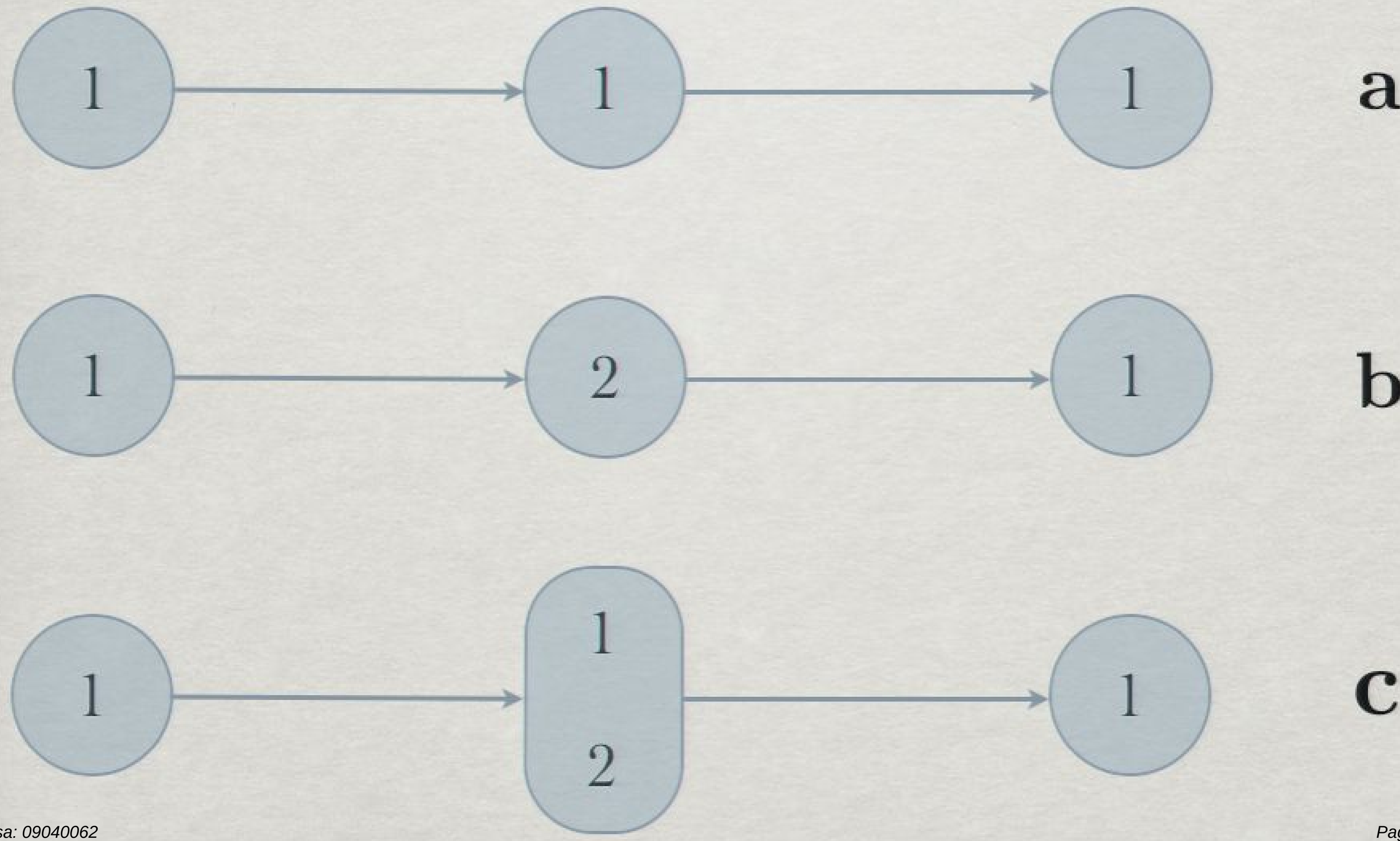
Sequence **Pair**



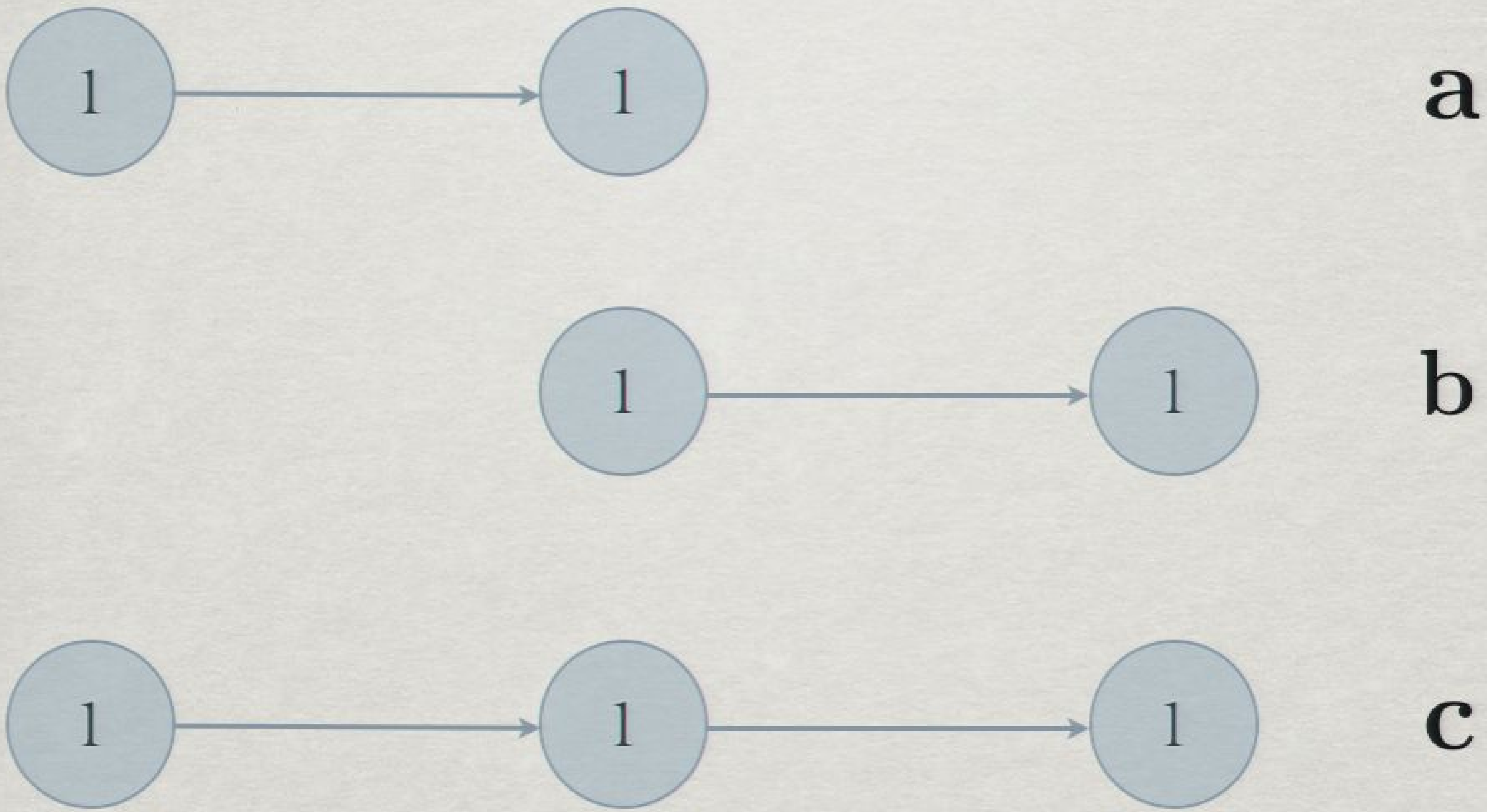
$$\mathbf{a} = (a_1, a_2)$$


1
1
1
a
1
2
1
b

$$\mathbf{c} = \mathbf{a} \oplus \mathbf{b}$$



$$\mathbf{c} = \mathbf{a} \odot \mathbf{b}$$



Symmetry Properties of **Pair** Combination Operators

$$\mathbf{a} \oplus \mathbf{b} = \mathbf{b} \oplus \mathbf{a}$$

$$(\mathbf{a} \oplus \mathbf{b}) \oplus \mathbf{c} = \mathbf{a} \oplus (\mathbf{b} \oplus \mathbf{c})$$

$$(\mathbf{a} \odot \mathbf{b}) \odot \mathbf{c} = \mathbf{a} \odot (\mathbf{b} \odot \mathbf{c})$$

$$(\mathbf{a} \oplus \mathbf{b}) \odot \mathbf{c} = (\mathbf{a} \odot \mathbf{c}) \oplus (\mathbf{b} \odot \mathbf{c})$$

$$\mathbf{a} \odot (\mathbf{b} \oplus \mathbf{c}) = (\mathbf{a} \odot \mathbf{b}) \oplus (\mathbf{a} \odot \mathbf{c})$$

$$\mathbf{a} \oplus \mathbf{b} = \mathbf{b} \oplus \mathbf{a}$$

$$(\mathbf{a} \oplus \mathbf{b}) \oplus \mathbf{c} = \mathbf{a} \oplus (\mathbf{b} \oplus \mathbf{c})$$

↓ Aczél and Hosszú (1956)

$$\mathbf{f}(\mathbf{a} \oplus \mathbf{b}) = \mathbf{f}(\mathbf{a}) + \mathbf{f}(\mathbf{b})$$

$$\mathbf{f}(\mathbf{a} \oplus \mathbf{b}) = \mathbf{f}(\mathbf{a}) + \mathbf{f}(\mathbf{b})$$

One-to-one transformation of pair-space

$$\mathbf{a} \oplus \mathbf{b} = (a_1 + b_1, a_2 + b_2)$$

$$\mathbf{a} \oplus \mathbf{b} = (a_1 + b_1, a_2 + b_2)$$

Distributivity of $\odot$ over $\oplus$

$$\mathbf{a} \odot \mathbf{b} = (\gamma_1 a_1 b_1 + \gamma_2 a_1 b_2 + \gamma_3 a_2 b_1 + \gamma_4 a_2 b_2, \\ \gamma_5 a_1 b_1 + \gamma_6 a_1 b_2 + \gamma_7 a_2 b_1 + \gamma_8 a_2 b_2)$$

$$\mathbf{a} \odot \mathbf{b} = (\gamma_1 a_1 b_1 + \gamma_2 a_1 b_2 + \gamma_3 a_2 b_1 + \gamma_4 a_2 b_2, \\ \gamma_5 a_1 b_1 + \gamma_6 a_1 b_2 + \gamma_7 a_2 b_1 + \gamma_8 a_2 b_2)$$

Associativity of $\odot$

$$\mathbf{a} \odot \mathbf{b} =$$

Commutative

Non-Commutative

$$\mathbf{a} \odot \mathbf{b} = (\gamma_1 a_1 b_1 + \gamma_2 a_1 b_2 + \gamma_3 a_2 b_1 + \gamma_4 a_2 b_2, \\ \gamma_5 a_1 b_1 + \gamma_6 a_1 b_2 + \gamma_7 a_2 b_1 + \gamma_8 a_2 b_2)$$

Associativity of $\odot$

$$\mathbf{a} \odot \mathbf{b} = \begin{cases} (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1) \\ (a_1 b_1 \quad , a_1 b_2 + a_2 b_1) \\ (a_1 b_1 + a_2 b_2, a_1 b_2 + a_2 b_1) \\ (a_1 b_1 \quad , a_1 b_2 \quad) \\ (a_1 b_1 \quad , \quad a_2 b_1) \end{cases}$$

Commutative

Non-Commutative

$$\mathbf{a} \odot \mathbf{b} = (\gamma_1 a_1 b_1 + \gamma_2 a_1 b_2 + \gamma_3 a_2 b_1 + \gamma_4 a_2 b_2,$$
$$\gamma_5 a_1 b_1 + \gamma_6 a_1 b_2 + \gamma_7 a_2 b_1 + \gamma_8 a_2 b_2)$$

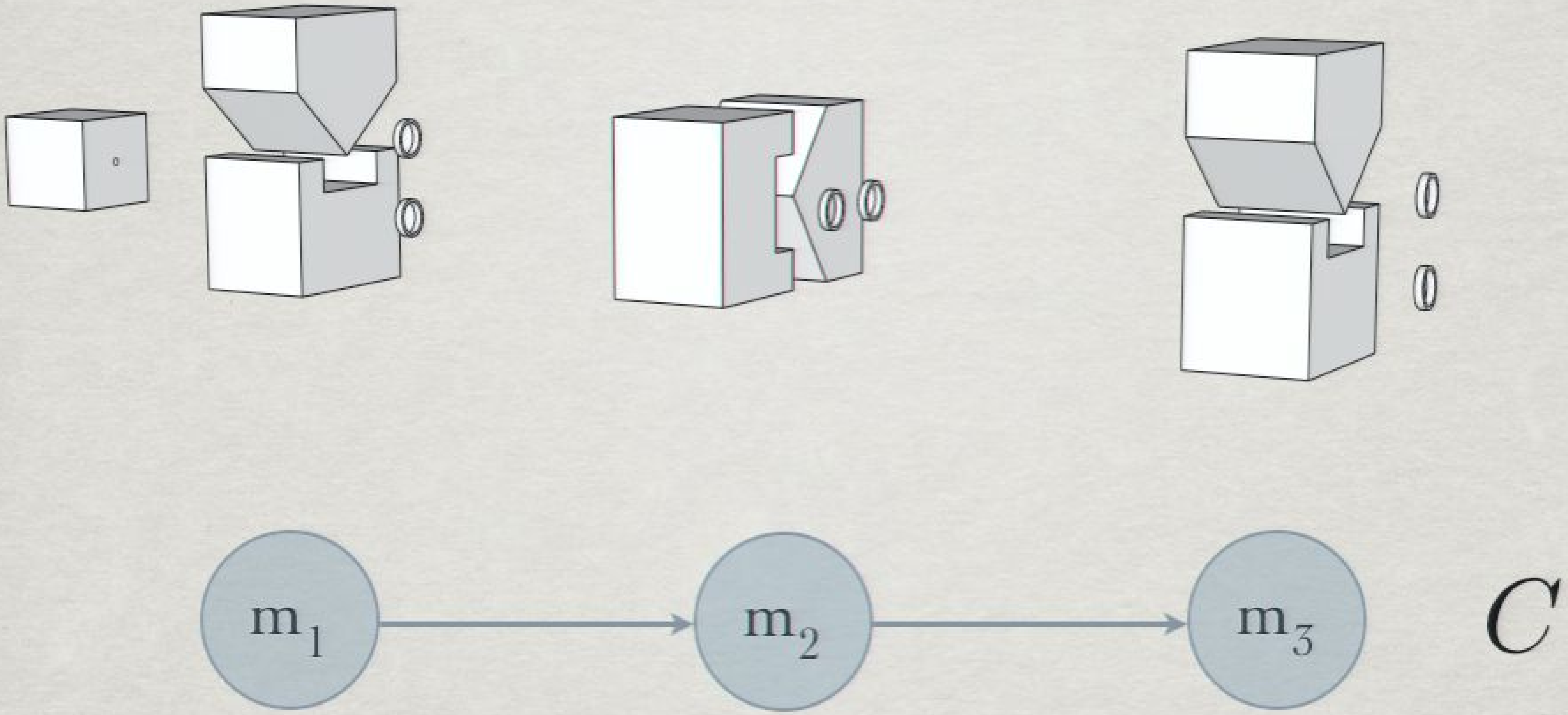
Associativity of $\odot$

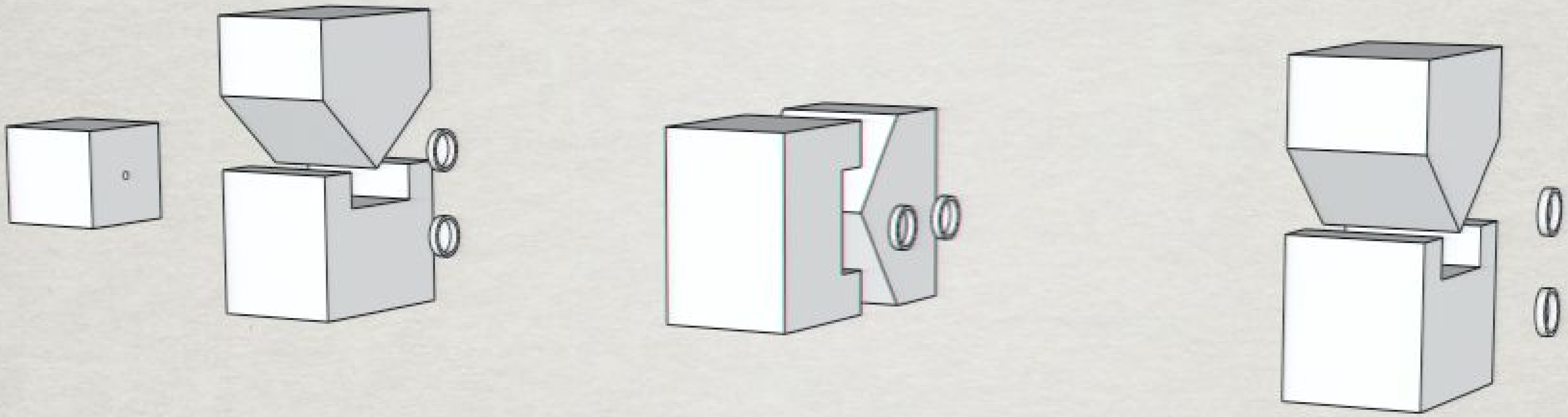
Commutative

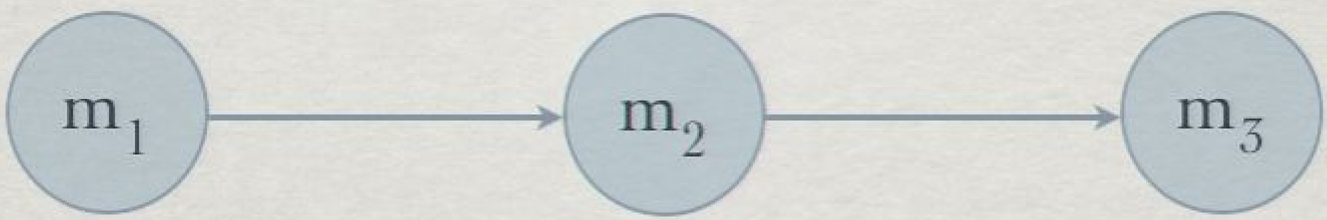
$$\mathbf{a} \odot \mathbf{b} = \begin{cases} (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1) \\ (a_1 b_1, a_1 b_2 + a_2 b_1) \\ (a_1 b_1, a_2 b_2) \end{cases}$$

Non-Commutative

$$\begin{cases} (a_1 b_1, a_1 b_2) \\ (a_1 b_1, a_2 b_1) \end{cases}$$

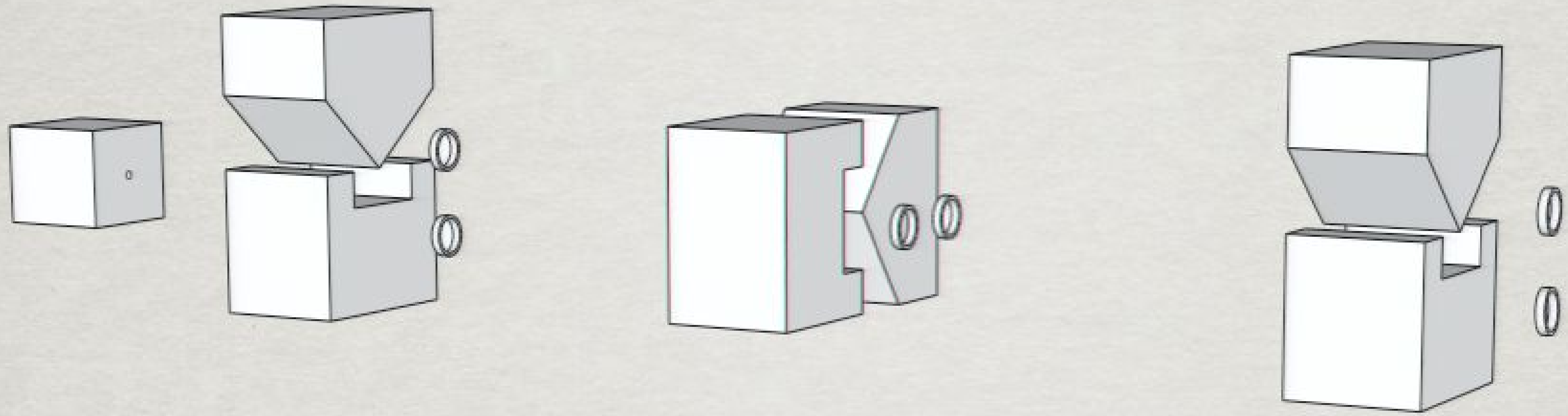

m1
m2
m3
C

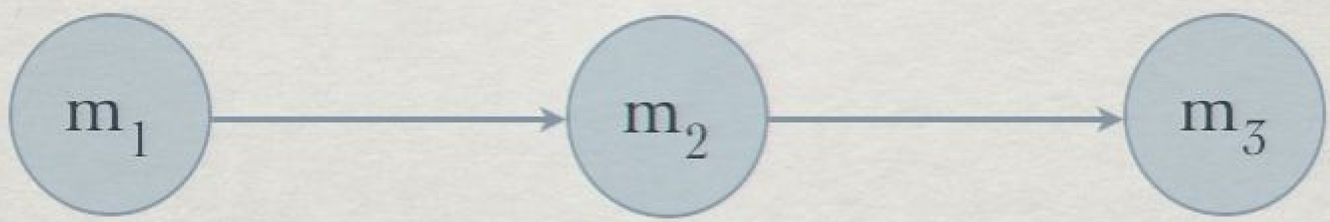




C

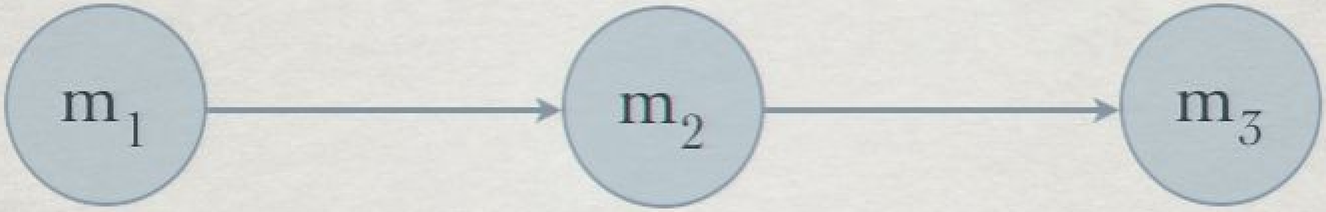
$$P(C) = \Pr(m_2, m_3 \,|\, m_1)$$



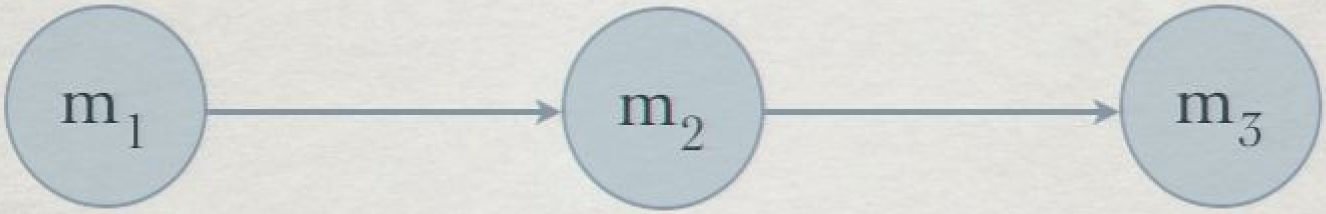


C

$$P(C) = \Pr(m_2, m_3 \mid m_1) = p(\mathbf{c})$$

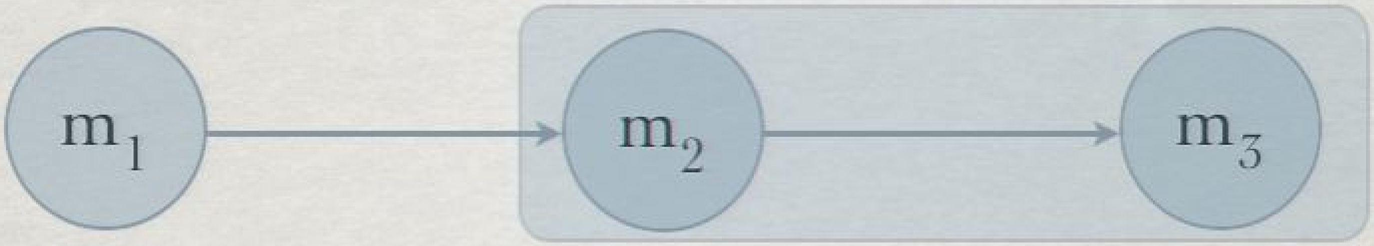


$$P(C) = \Pr(m_2, m_3 \,|\, m_1)$$



$$P(C) = \Pr(m_2, m_3 \,|\, m_1)$$
$$= \Pr(m_2 \,|\, m_1)\ \Pr(m_3 \,|\, m_2, m_1)$$

$$P(B) = \Pr(m_3 \,|\, m_2)$$



$$P(C) = \Pr(m_2, m_3 \,|\, m_1)$$

$$= \Pr(m_2 \,|\, m_1)\ \Pr(m_3 \,|\, m_2)$$

$$= \quad P(A) \quad \cdot$$

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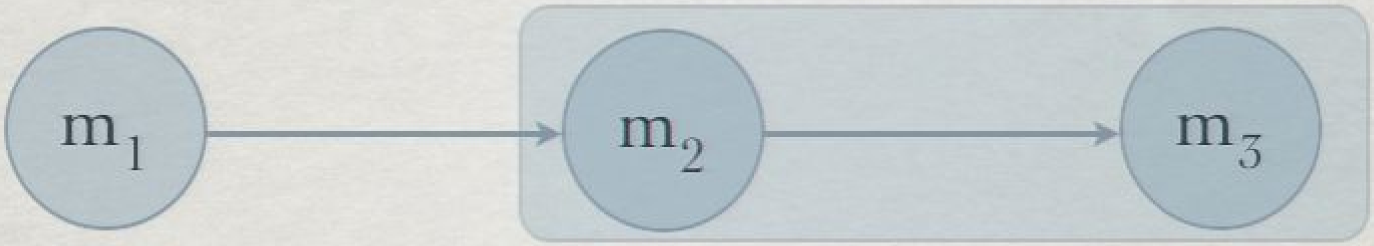

$$\begin{aligned} P(C) &= \Pr(m_2, m_3 \,|\, m_1) \\ &= \Pr(m_2 \,|\, m_1)\ \Pr(m_3 \,|\, m_2) \\ &= \quad P(A) \quad \cdot \quad P(B) \end{aligned}$$

$$P(A \cdot B) = P(A)P(B)$$

$$\mathbf{a} \odot \mathbf{b} = \left[\begin{array}{l} (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1) \\ (a_1 b_1 \quad , a_1 b_2 + a_2 b_1) \\ (a_1 b_1 \quad , \quad a_2 b_2) \\ (a_1 b_1 \quad , a_1 b_2 \quad) \\ (a_1 b_1 \quad , \quad a_2 b_1) \end{array} \right.$$

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$$p(\mathbf{a}) = \left(a_1^2 + a_2^2\right)^{\alpha/2}$$

$$p(\mathbf{a}) = |a_1|^{\alpha} e^{\beta a_2 / a_1}$$

$$p(\mathbf{a}) = |a_1|^{\alpha} |a_2|^{\beta}$$

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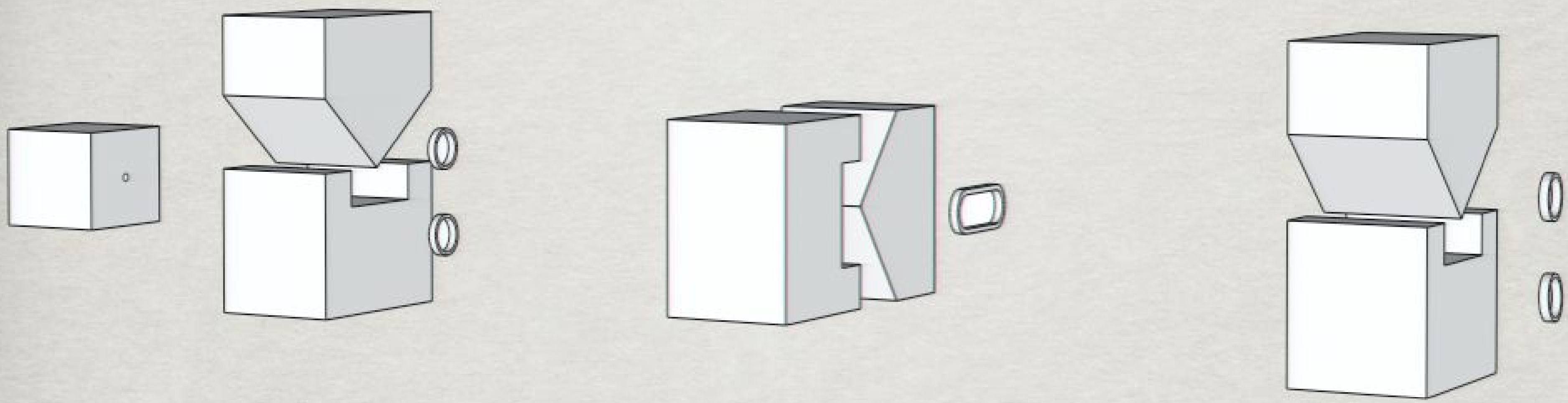
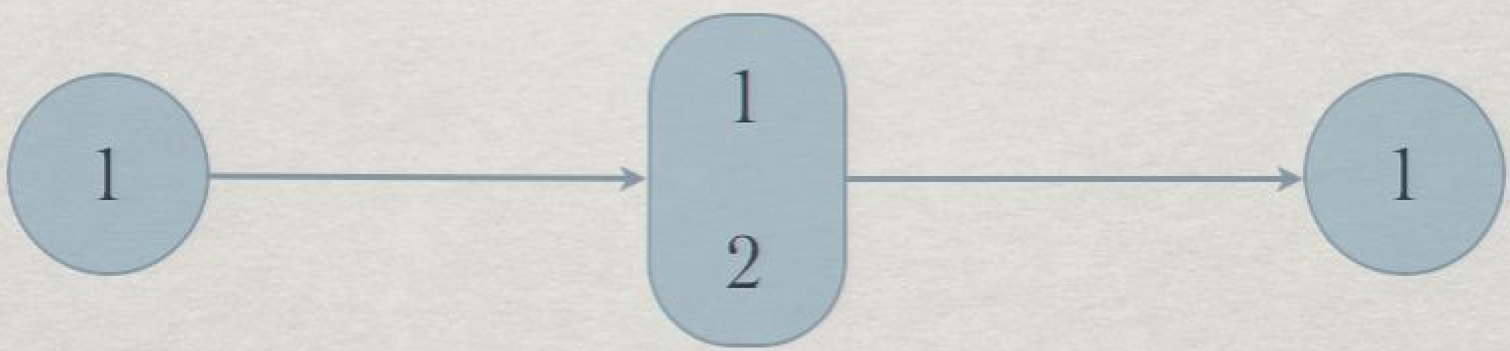
Repeated Measurements





Repeated Measurements

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Summary

- Sum Rule

$$\mathbf{a} \oplus \mathbf{b} = (a_1 + b_1, a_2 + b_2)$$

- Product Rule

$$\mathbf{a} \odot \mathbf{b} = (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1)$$

- Probability Rule

$$P(A) = a_1^2 + a_2^2$$

Associativity and Commutativity of $\oplus$

$$\mathbf{a} \oplus \mathbf{b} = (a_1 + b_1, a_2 + b_2)$$

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$$\mathbf{a} \odot \mathbf{b} = \begin{cases} (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1) \\ (a_1 b_1 \quad , a_1 b_2 + a_2 b_1) \\ (a_1 b_1 + a_2 b_2, a_1 b_2 + a_2 b_1) \\ (a_1 b_1 \quad , a_1 b_2 \quad) \\ (a_1 b_1 \quad , \quad a_2 b_1) \end{cases}$$

Associativity and Commutativity of ⊕

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p(A · B) = p(A) · p(B)

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Repeated Measurements

$$\mathbf{a} \odot \mathbf{b} = (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1)$$

$$p(A) = a_1^2 + a_2^2$$

Further Steps

- Tensor Product Rule

$$\mathbf{a} \circ \mathbf{b} = (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1)$$

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$$\mathsf{U}_t(dt) = \exp(i\mathsf{H}\, dt/\hbar)$$

- Correspondence Rules

$$[\mathsf{L}_x, \mathsf{L}_y] = i\hbar\mathsf{L}_z, \text{etc.}$$

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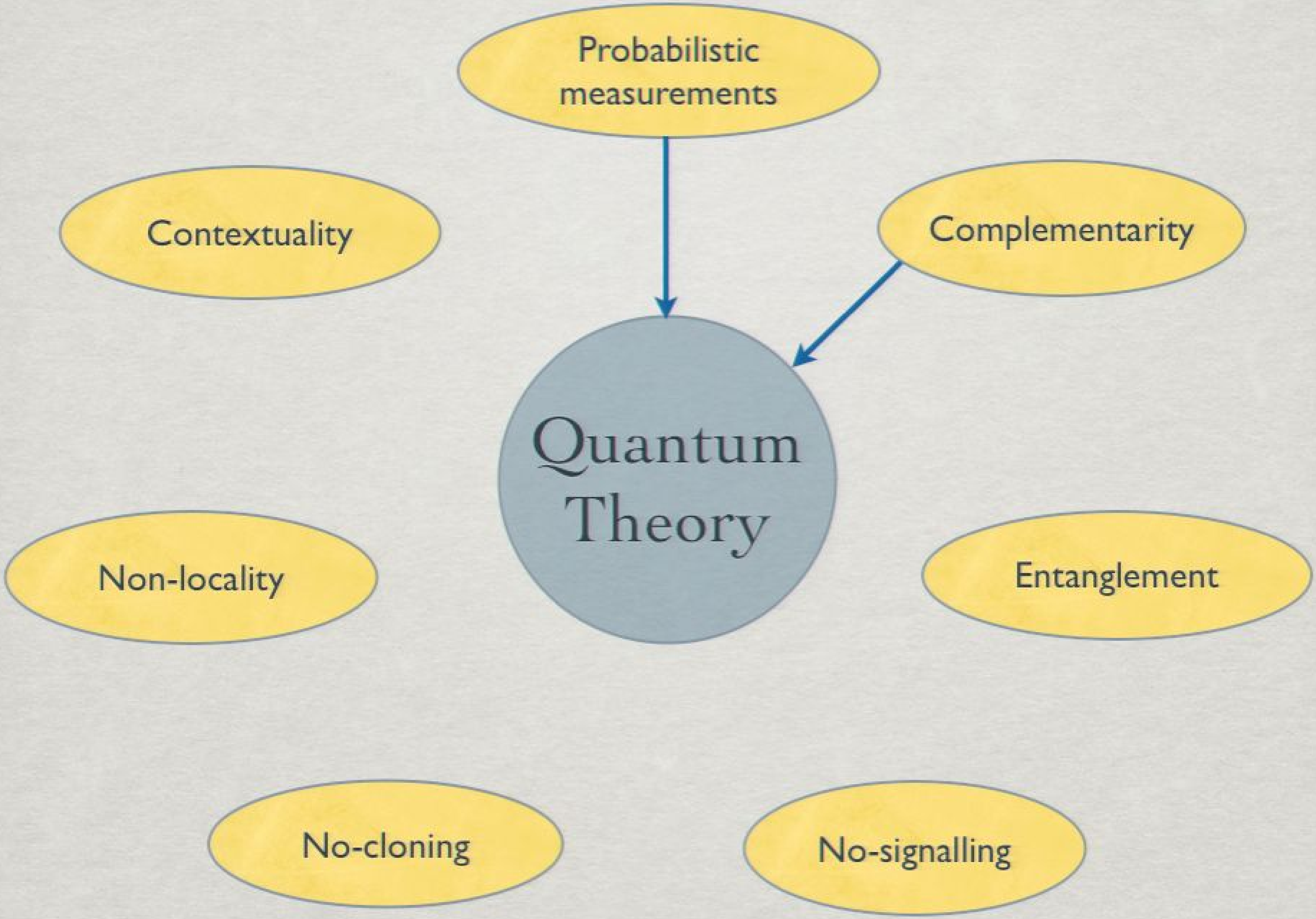
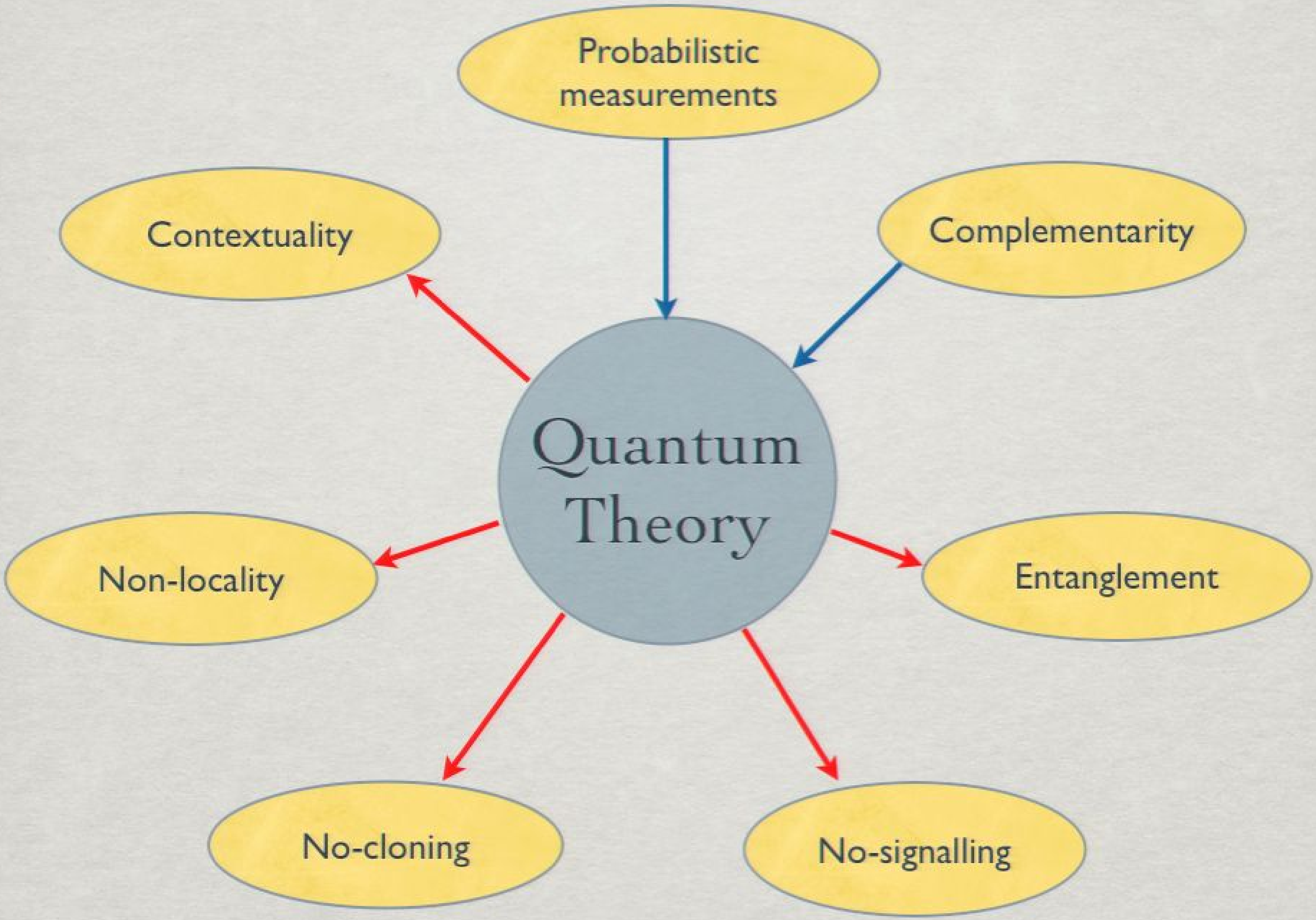
arXiv: 0805.2765


Probabilistic measurements
Contextuality
Complementarity
Quantum Theory
Non-locality
Entanglement
No-cloning
No-signalling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Additional and Multiplication Operators for Real Number Pairs

$$\mathbf{a} \oplus \mathbf{b} = \mathbf{b} \oplus \mathbf{a}$$

$$(\mathbf{a} \oplus \mathbf{b}) \oplus \mathbf{c} = \mathbf{a} \oplus (\mathbf{b} \oplus \mathbf{c})$$

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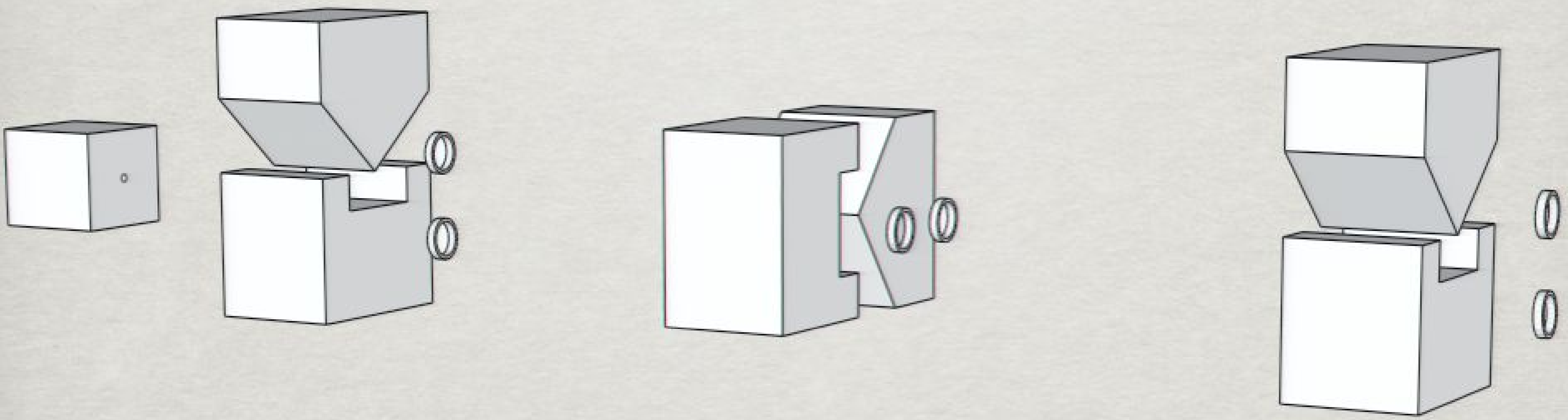
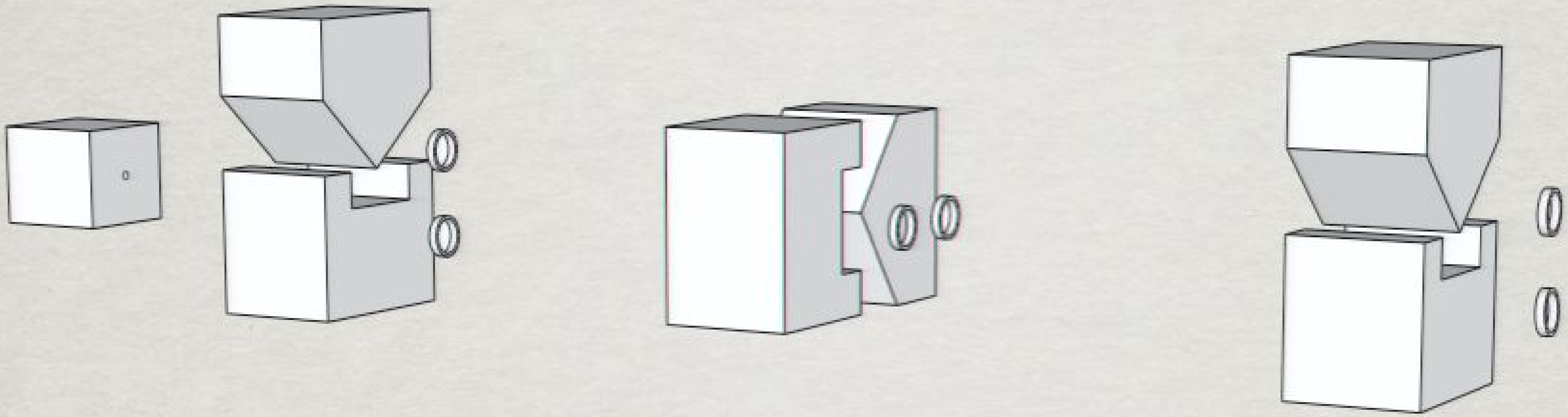
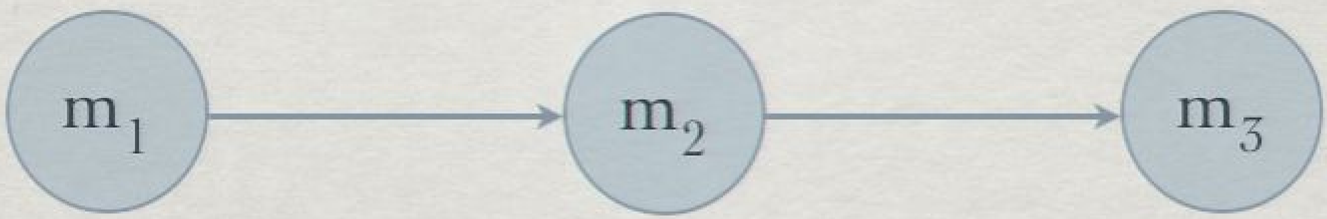
Experimental Setups

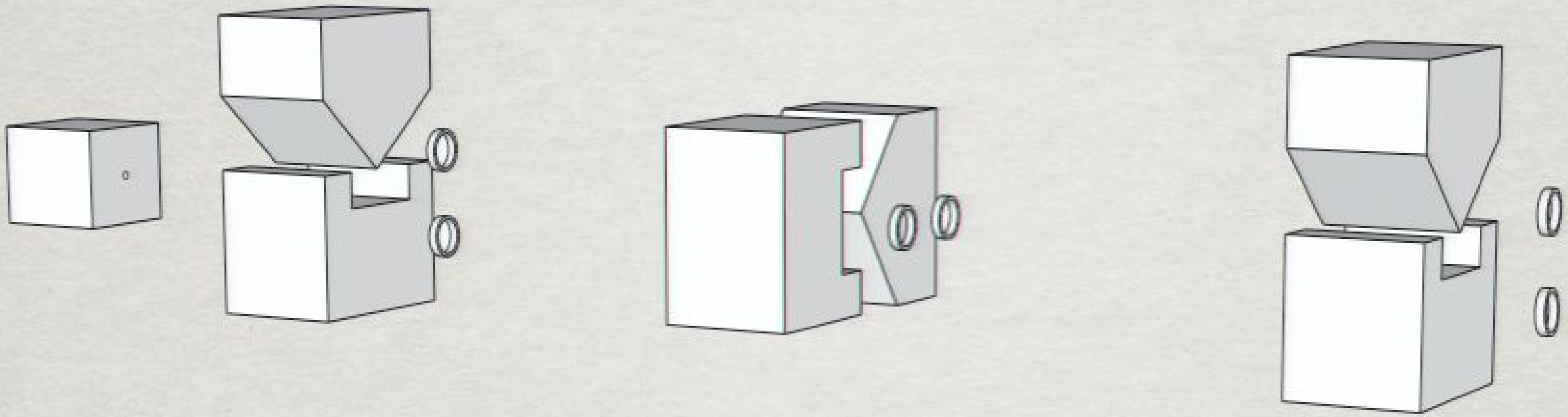


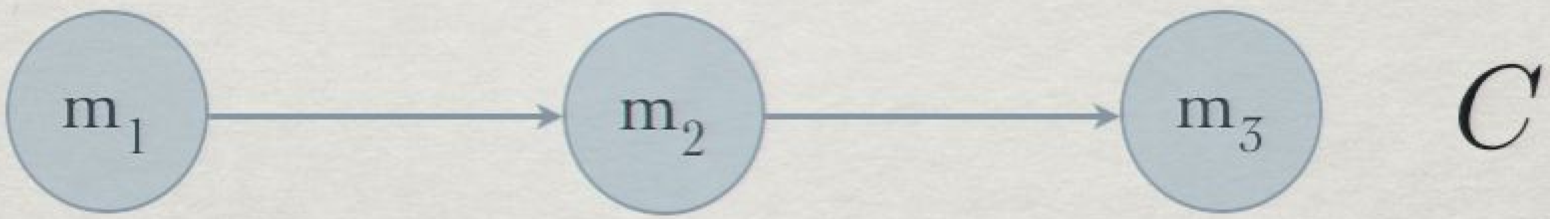




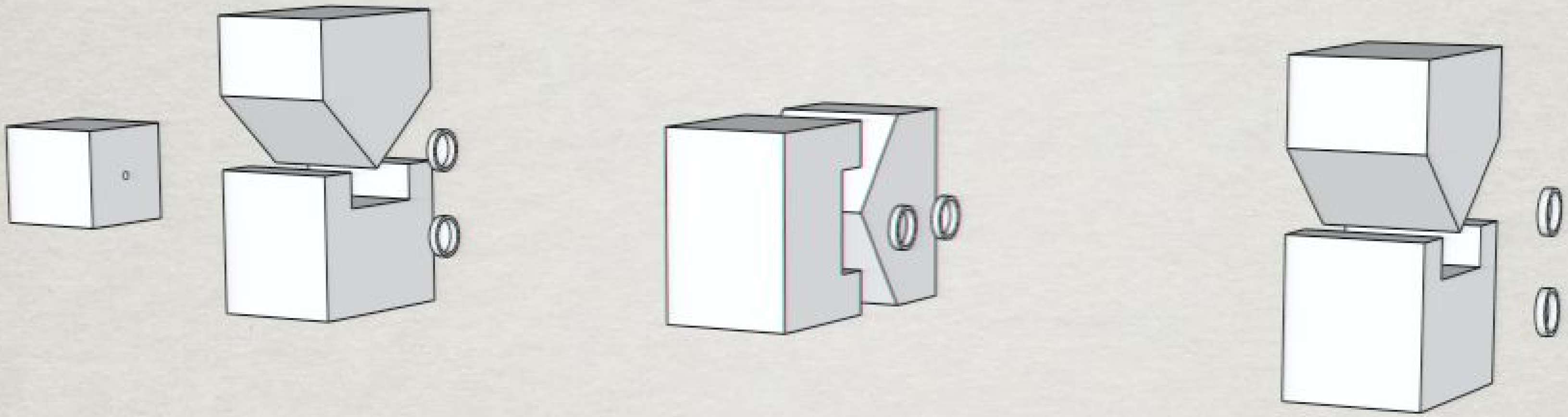
C

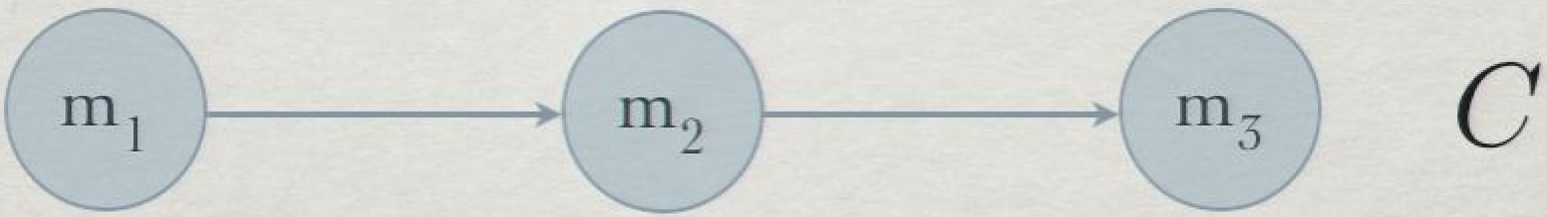
$$P(C)$$





$$P(C) = \Pr(m_2, m_3 \,|\, m_1)$$





$$P(C) = \Pr(m_2, m_3 \,|\, m_1) = p(\mathbf{c})$$

Repeated Measurements

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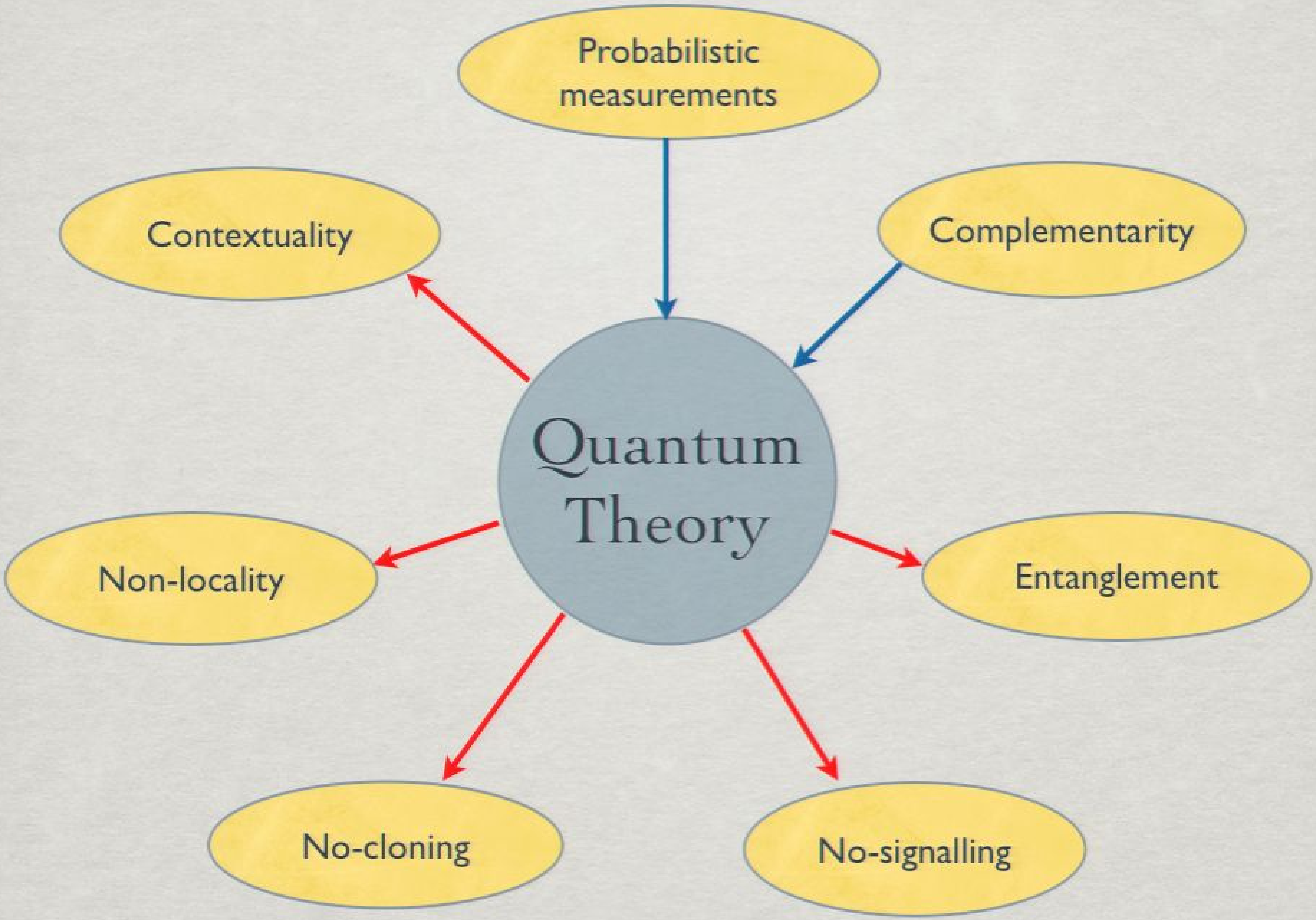
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