Title: Untangling entanglement: An observer-dependent perspective

Date: Apr 08, 2009 02:00 PM

URL: http://pirsa.org/09040008

Abstract: Entanglement is one of the most fundamental and yet most elusive properties of quantum mechanics. Not only does entanglement play a central role in quantum information science, it also provides an increasingly prominent bridging notion across different subfields of Physics --- including quantum foundations, quantum gravity, quantum statistical mechanics, and beyond. Arguably, the property of a state being entangled or not is by no means unambiguously defined. Rather, it depends strongly on how we decide to regard the whole as composed of its part or, more generally, on the restricted ways in which we are able to observe and control the system at hand. Acknowledging the implications of such an operationally constrained point of view naturally has led to a notion of 'generalized entanglement,' which is directly based on quantum observables and offers added flexibility in a variety of contexts. In this talk, I will survey some of the main accomplishments of the generalized entanglement program to date, with an eye toward recent developments and open problems.

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

Physics Colloquium
April 8, 2009

Untangling Entanglement:
An observer-dependent perspective

Lorenza Viola

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Prelude

The 'coherent tangles' (over the years...)

Howard Barnum (LANL)
Manny Knill (NIST Boulder)
Gerardo Ortiz (Indiana)
⋮
Rolando Somma (LANL/PI)
⋮
Simone Montangero (Pisa/Ulm)
Vladimir Akulin (Orsay)
Yaakov Weinstein (MITRE)
Sergio Boixo (UNM/Caltech)
Hui Khoon Ng (Caltech)
⋮

At Dartmouth:
Lea F. Dos Santos (now at Yeshiva)
Wenxian Zhang (now at Fudan)
⋮
Aikaterina Mandilara (now in Paris)
David Starling (now in Rochester)
⋮
Winton G. Brown
Shusa Deng
⋮

This talk:

Question: Do we have an entanglement theory which is as *general and flexible* as desirable?...

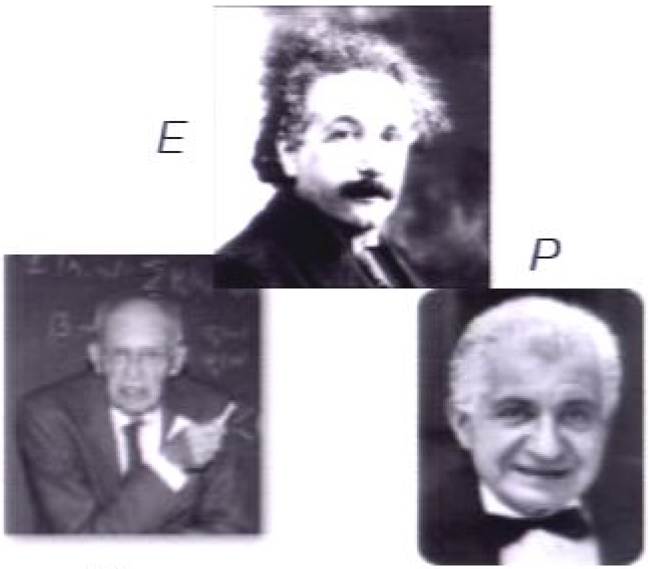
Proposal: 'Generalized entanglement' (GE)

Entanglement: From a **spooky** action...

E

P

R



The EPR dilemma:

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?
Phys. Rev. **47**, 777 (1935).

"If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality that corresponds to this physical quantity."

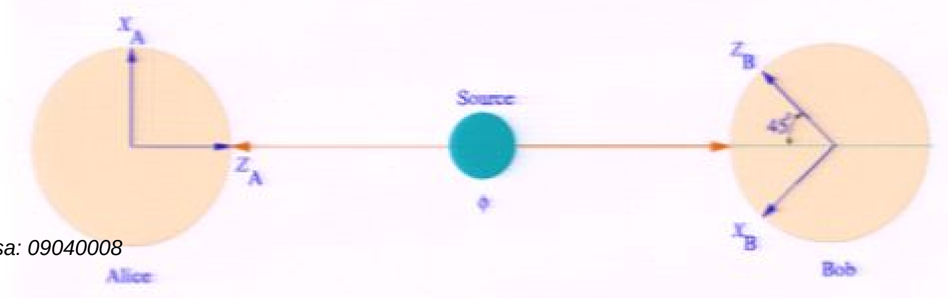
But then...

"... either (1) the description of reality given by the wave function is not complete or...

No longer true that...

(2) ... the real factual situation of the system S_2 is independent of what is done with the system S_1, which is spatially separated from the former."

$$|\Phi^-\rangle = \frac{|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B}{\sqrt{2}}$$



?

What is the point:

Discussion of Probability Relations Between Separate Systems
Proc. Cambridge Philos. Soc. **31**, 555 (1935).

"...*This is the point.* Whenever one has a complete expectation catalog - a maximum total knowledge – ψ-function – for two completely separated bodies,... then one obviously has it also for the two bodies together...

But the converse is *not* true. Maximal knowledge of a total system does not necessarily include total knowledge of all its parts, not even when they are fully separated from each other and at the moment are not influencing each other at all."

S

"Verschränkung"

"I would not call that *one* but *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought."

From an **experimental** reality...

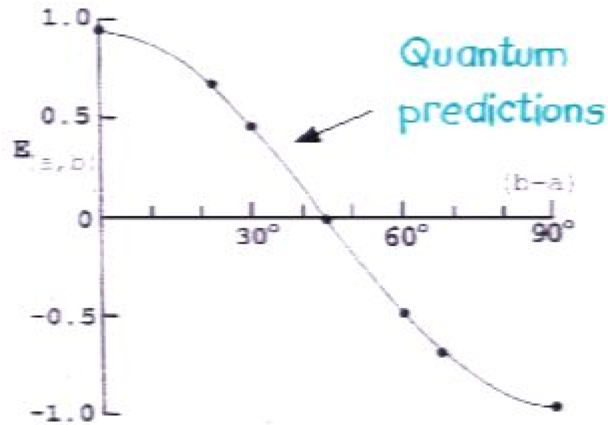
Entanglement today:
(1) A fact of life –

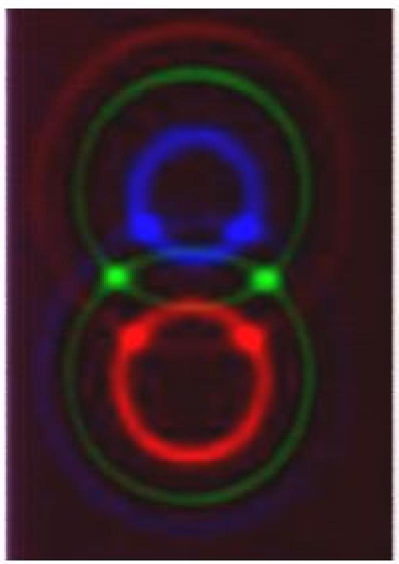
- Intrinsically *nonclassical correlations* that can violate Bell's inequalities (by a LOT!...)

Experimental Realization of EPR Gedankenexperiment, Aspect, Grangier & Gerard, *Phys. Rev. Lett.* ***49***, 91 (1982).

Violation of Bell's Inequality under Strict Einstein Locality Conditions, Weihs *et al*, *Phys. Rev. Lett.* ***81***, *5039* (1998).

⋮

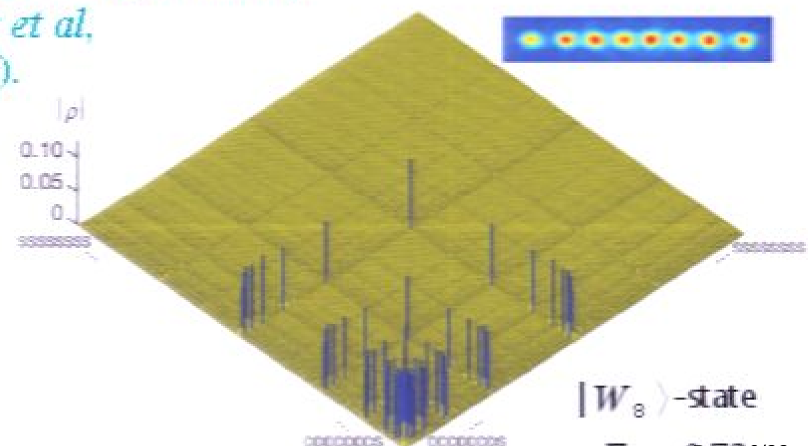




New High-Intensity Source of Polarization-Entangled Photon Pairs, Kwiat *et al*, *Phys. Rev. Lett.* ***75***, 4337 (1995).

- *Controlled* generation and characterization: Growing list of quantum technologies...

Scalable Multiparticle Entanglement of Trapped Ions, Häffner *et al*, *Nature* ***438***, *643* (2005).

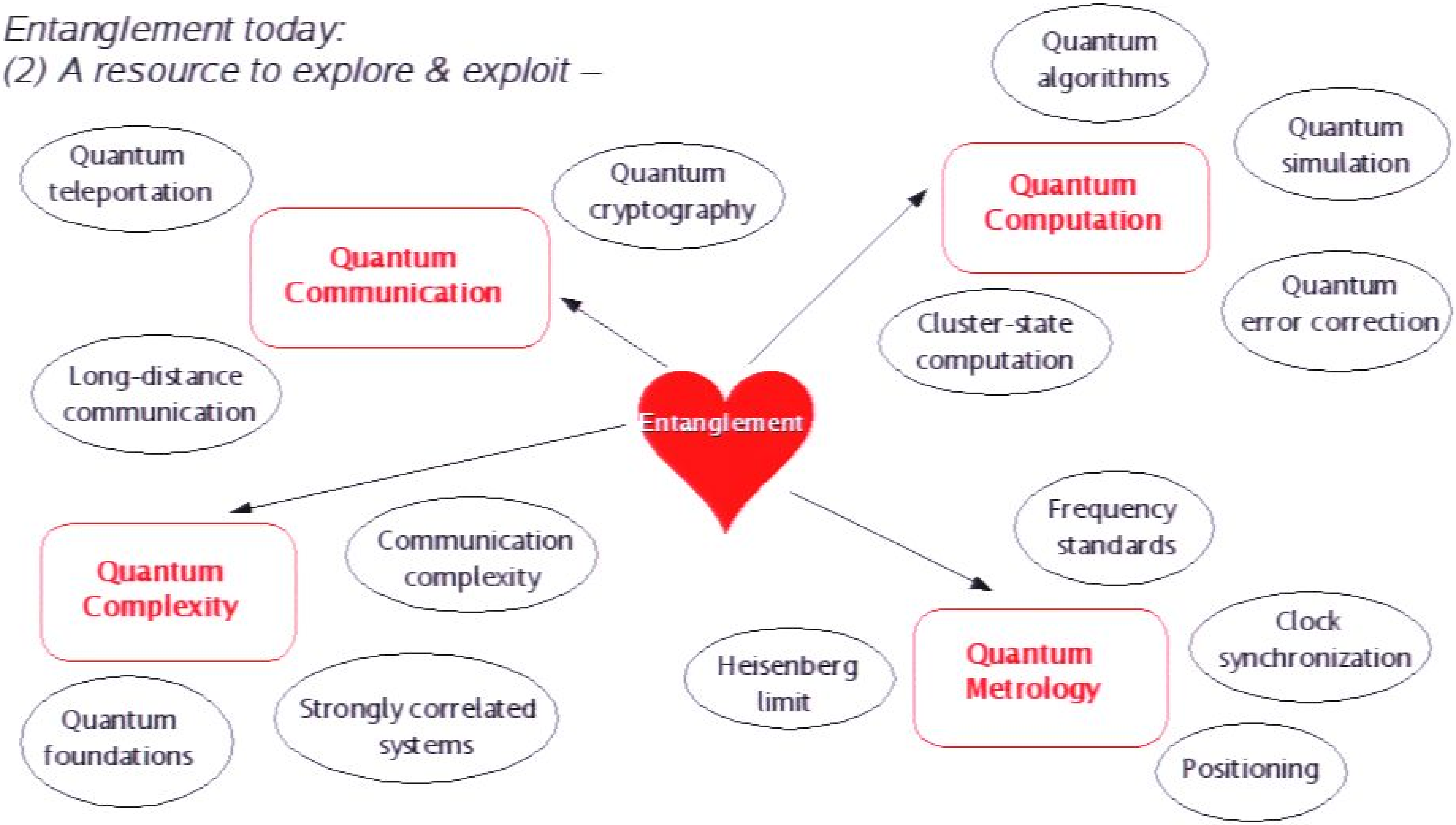


$|W_8\rangle$-state

$F_8 = 0.72$

...To a quantum **resource** at the heart of QIS...

Entanglement today:
(2) A resource to explore & exploit –



Quantum Entanglement,
(Ryszard, Pawel, Michal & Karol) Horodecki, *Rev. Mod. Phys.* (in press); arXiv:quant-ph/0702225.

Entanglement in Many-Body Systems,
Amico, Fazio, Osterloh & Vedral, *Rev. Mod. Phys.* **80**, 517 (2008).

So what **is** entanglement?

Entanglement is...

John Bell: ...A correlation that is stronger than any classical correlation.

David Mermin: ...A correlation that contradicts the theory of elements of reality.

Asher Peres: ...A trick that quantum magicians use and cannot be imitated by classical ones.

Charlie Bennett: ...A resource that enables quantum teleportation.

Peter Shor: ...A global structure of the wavefunction that allows for faster algorithms.

Artur Eckert: ...A tool for secure communications.

Horodecki family: ...The need for first applications for positive maps in physics.

Paul Kwiat: ...The crown jewel of quantum mechanics.

Disclaimer:

Our view and understanding of entanglement may continue to be modified during the coming years!

That said...

LV: ...A relative property of quantum states that arises out of operational constraints.

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Entanglement is relative...

The standard definition:

Entangled states: Joint states of a composite quantum system that *cannot* be expressed as (a mixture of) products of states of the constituent subsystems.

$\mathcal{H}_\mathrm{A} \times \mathcal{H}_\mathrm{B}$ not a vector space $\Rightarrow$ $\mathcal{H} \simeq \mathcal{H}_\mathrm{A} \otimes \mathcal{H}_\mathrm{B}$, $\quad \mathcal{H}_A = \mathrm{span}\{|0\rangle_A, |1\rangle_A\}$, $\mathcal{H}_B = \mathrm{span}\{|0\rangle_B, |1\rangle_B\}$

$$|\Phi^\pm\rangle = \frac{|0\rangle_A \otimes |1\rangle_B \pm |1\rangle_A \otimes |0\rangle_B}{\sqrt{2}}$$

$$|\Psi^\pm\rangle = \frac{|0\rangle_A \otimes |0\rangle_B \pm |1\rangle_A \otimes |1\rangle_B}{\sqrt{2}}$$

Bell states – max entangled

- The way to partition a system into subsystems need NOT be unique! *e.g.:*
  - Hydrogen atom:

    Eigenstates = Tensor product of free CM + relative particle, BUT: non-factorizable in terms of electron and proton degrees of freedom

  - Alice & Bob with a twist:

$$\mathcal{H} \simeq \mathcal{H}_C \otimes \mathcal{H}_D, \quad \mathcal{H}_C = \mathrm{span}\{|X\rangle_C, X = \Phi, \Psi\} \quad \mathcal{H}_D = \mathrm{span}\{|\delta\rangle_D, \delta = +, -\}$$

$$|\Psi^\pm\rangle = |\Psi\rangle_C \otimes |\pm\rangle_D, \qquad |\Phi^\pm\rangle = |\Phi\rangle_C \otimes |\pm\rangle_D,$$

$$\frac{1}{2}(|0\rangle + i|1\rangle)_A \otimes (|0\rangle + i|1\rangle)_B = \frac{1}{\sqrt{2}}(|\Psi\rangle_C \otimes |-\rangle_D + i|\Phi\rangle_C \otimes |+\rangle_D)$$

- Entanglement is *un*ambiguous only relative to a preferred subsystem decomposition.

...To **what**, really?

What if identifying a preferred subsystem decomposition is problematic because...

- No unique tensor product notion exists...

↳ *Quantum foundations* – Entanglement in general probabilistic theories:

[Spekkens, 2004; Barrett, 2005; Barnum *et al*, 2006; Barnum *et al*, 2008...]

Subsystems may still be defined, but mathematical framework of 'convex sets' needed:

Abstract state space = Vector space V equipped with a distinguished linear functional u_V

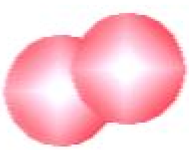
→ Quantum theory: $V = \mathrm{Herm}(\mathcal{H})$, $u_V = \mathrm{Tr}(\cdot)$ $\Rightarrow$ Set of density operators is a 'cone'

- No natural tensor product notion exists...

↳ *Condensed-matter physics* – Entanglement in systems of indistinguishable particles:

[Schliemann *et al.*, 2001; Eckert *et al.*, 2002; Zanardi 2002; Kindermann, 2006; Banuls *et al*, 2007...]

Accessible state space is a *subspace* of the full tensor product space *e.g.*, two fermions in two modes



$$\langle \vec{r}_1, \vec{r}_2 | \Psi_F \rangle = \frac{1}{\sqrt{2}} \begin{vmatrix} \phi_\alpha(\vec{r}_1) & \phi_\alpha(\vec{r}_2) \\ \phi_\beta(\vec{r}_1) & \phi_\beta(\vec{r}_2) \end{vmatrix} = -\langle \vec{r}_2, \vec{r}_1 | \Psi_F \rangle$$

Slater determinant

→ Entanglement of particles or modes? Which set of modes (if any)? Which algebraic language?

(More later...)

Towards generalized entanglement...

Desiderata for a generalized theory of entanglement:

- ➡ Consistent with *existing theory and results* in well-characterized, limiting conditions
- ➡ Applicable to *arbitrary operator languages* (fermions, bosons, spins, anyons...)
- ➡ Capable to incorporate *physical constraints* (*e.g.*, limited means to access/control system, superselection rules...)

Hint:

A pure unentangled state remains pure to a 'local' observer, whereas an entangled state can be 'locally' indistinguishable from a mixture...

$$Tr_B\left(|Bell\rangle_{AB}\langle Bell|\right)=\frac{I_A}{2} \quad \text{– fully mixed...}$$

Strategy:

Bypass the need for preferred subsystems by redefining entanglement as a property that depends directly on a set of preferred physical observables...

Observables → Unrestricted ⇔ (Generalized) 'global observer'

Observables → Restricted ⇔ (Generalized) 'local observer' → (Generalized) Entanglement

Defining generalized entanglement

Goal: Translate 'mixedness' into lack of extremality relative to distinguished set Ω.

- Steps toward GE:

(1) Introduce a notion of 'reduced state' which avoids the need of a tensor product:
Consider states as positive *linear functionals* on operators:

$$\mathcal{H}\text{-state } |\psi\rangle: \quad \lambda : \mathrm{End}(\mathcal{H}) \rightarrow \mathbb{C}, \quad \lambda(X) = \mathrm{Tr}(|\psi\rangle\langle\psi| X)$$

A *reduced state* relative to Ω is defined only by *expectation values* of observables in Ω:

$$\Omega\text{-state:} \quad \omega : \Omega \rightarrow \mathrm{R}, \quad \omega = \lambda\,|\Omega$$

(2) Observe that a Ω-reduced state is pure iff it is *extremal i.e.*, it cannot be written as a convex combination of other reduced states.

$$x, y \in C \;\Rightarrow\; px + (1-p)\,y \in C, \quad p \in [0,1]$$

Degree of entanglement determined by expectations of distinguished observables:

A pure state $|\psi\rangle$ is *generalized unentangled relative to Ω* if its reduced state is pure (extremal), generalized entangled otherwise.

Quantifying generalized entanglement

Simplest setting: Ω is a Lie algebra $\mathfrak{h}$, irreducibly represented in $\mathcal{H}$.

- Natural GE measure: Let $\{x_i\}$ be a Hermitian, orthogonal basis for $\mathfrak{h}$. Define

$$P_{\mathfrak{h}}(|\psi\rangle) = K \sum_i |\langle\psi|x_i|\psi\rangle|^2$$

$\mathfrak{h}$-purity

K is a global normalization factor chosen such that $P_{\mathfrak{h}}^{max} = 1$ for all G-unentangled $|\psi\rangle$

- Complete characterization of set of generalized-unentangled states:

A pure state is *generalized unentangled relative to* $\mathfrak{h}$ iff it is a Generalized Coherent State (GCS) of the Lie group generated by $\mathfrak{h}$.

$$|GCS(\vec{\alpha})\rangle = \exp(\sum_k \alpha_k A_k - \alpha_k^* A_{-k})|REF\rangle, \quad \alpha_l \in \mathbb{C}$$

Most classical states!...

GCSs $\Leftrightarrow$ Generalized unentanglement $\Leftrightarrow$ Extremal projection $\Leftrightarrow$ Maximal purity, $P(|GCS\rangle) = 1$

Standard entanglement revisited

Standard locality constraint:

- Means for manipulating and observing system are restricted to arbitrary local observables acting on each subsystem:

  → GCSs of SU(m) × SU(n): States reachable from $|0\rangle_A \otimes |0\rangle_B$ via local transformations...

$$|\Phi^+\rangle = \frac{|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B}{\sqrt{2}}$$

$$\rho_A = Tr_B\, |\Phi^+\rangle\langle\Phi^+| = I_A/2$$

$$\rho_B = Tr_A\, |\Phi^+\rangle\langle\Phi^+| = I_B/2$$

Standard entanglement ≡
GE relative to *all* local observables

- Extend to multipartite setting:

  → Algebra of *all* local observables, $h_{loc} = su(2)_1 \oplus \ldots \oplus su(2)_N = \text{span}\{\sigma_\alpha^i \mid i = 1, \ldots, N;\ \alpha = x, y, z\}$

$$P_{loc}(|\psi\rangle) = \frac{1}{N}\sum_{i,\alpha} \langle\psi|\sigma_\alpha^i|\psi\rangle^2$$

Local purity

- The local purity is proportional to the average subsystem purity (*global entanglement*).
- *Every* pure state is unentangled relative to the *full* algebra, $g = su(2^N) = \text{span}\{\sigma_\alpha^1 \otimes \ldots \otimes \sigma_\alpha^N\}$.

So far so good ☺

Entanglement **without** subsystems...

System: A single spin-1 particle

→ State space $\mathcal{H} \simeq \mathbb{C}^3$:

- Carries the spin-1 irrep of $su(2)$, generated by operators J_x, J_y, J_z.

$$J_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- $|1,-1\rangle$ may be chosen as a reference state.

→ Assume that distinguished observables are *linear in angular momentum* J_a: $\mathfrak{h} = su(2)$

- The reduced states can be identified with *vectors of expectations* of the generators:

$$\Omega\text{-state} \Leftrightarrow \begin{pmatrix} \langle J_x \rangle \\ \langle J_y \rangle \\ \langle J_z \rangle \end{pmatrix} \in \mathbb{R}^3, \quad \text{with} \quad \langle J_x \rangle^2 + \langle J_y \rangle^2 + \langle J_z \rangle^2 \le 1$$

- Pure states are those on the surface = SU(2) spin coherent states:

$$\{ |1, \xi\rangle = e^{\xi J_+ - \bar{\xi} J_-} |1, -1\rangle \}, \quad \xi \in \mathbb{C}$$

- $|1,-1\rangle, |1,1\rangle$ are GCSs, $|1,0\rangle$ is NOT: $|1,0\rangle$ is G-entangled relative to $su(2)$.

Generalized entanglement may exist for indecomposable systems!

Separability **with** entanglement...

System: Two spin-1 particles.

- State space $\mathcal{H} \simeq \mathbb{C}^3 \otimes \mathbb{C}^3$:
- GE reduces to conventional entanglement if the set of all local observables is distinguished:

$$h_{loc} = su(3) \oplus su(3) = \text{span}\{\, \lambda_\alpha^i \mid i=1,2;\ \alpha=1,\dots,8 \,\}$$

$$P_h(|\psi\rangle) = \frac{3}{4}\sum_{i,\alpha} \langle\psi|\lambda_\alpha^i|\psi\rangle^2 \quad \Rightarrow \quad \begin{array}{ll} P_h = 1 & \text{product states} \\ \dots & \dots \\ P_h = 0 & \text{max entangled states} \end{array}$$

- Different results are found if a *sub*algebra of local observable is distinguished, *e.g.* If only angular momentum observables are accessible:

$$h = su(2) \oplus su(2) = \text{span}\{\, J_\alpha^i \mid i=1,2;\ \alpha=x,y,z \,\}$$

- GCSs are states with maximal total spin along some direction, *i.e.* $|1,1\rangle_\alpha \otimes |1,1\rangle_\alpha$.
- A state like $|1,0\rangle_\alpha \otimes |1,0\rangle_\alpha$ is G-entangled relative to the local subalgebra: *No* operation in the available set can connect $|1,0\rangle_\alpha \otimes |1,0\rangle_\alpha \;\Leftrightarrow\; |1,1\rangle_\alpha \otimes |1,1\rangle_\alpha$.

Separability $\neq$ Generalized unentanglement!

Intermezzo

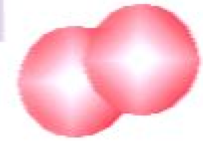
What added understanding has generalized entanglement allowed so far?

Sample problems...

- **GE and aspects of 'complexity'.**
  - How to define entanglement in indistinguishable particles?
    ↳ Unifying, (operator) language-independent notion.
  - What properties and role does entanglement have in many-body systems?
    ↳ Diagnostic tools for quantum correlations → Ground states – *e.g.* Static/dynamic QPTs; → Typical states – *e.g.* Quantum chaos/Randomness
  - What is the role of entanglement in computational speed-up?
    ↳ A 'Lie-algebraic' QC can be efficiently simulated by CC if it does not generate enough GE...
    Somma *et al*, Phys. Rev. Lett. **97**, 190501 (2006).
- **GE and aspects of 'classicality'.**
  - How does GE relate to cloning/broadcasting/teleportation in probabilistic theories?
    ↳ Clonable/broadcastable sets are G-unentangled relative to appropriate observer...
    Barnum *et al*, arXiv: quant-ph/0611295.
  - How does GE tie in with the decoherence program?
    ↳ G-unentangled states emerge as 'pointer states' for large class of Markovian dynamics...
    Boixo, Viola & Ortiz, Europhys. Lett. **79**, 40003 (2007).

Problem 1: Fermionic entanglement

Notion of locality in 'real space' lost due to quantum statistics constraint...



System: K (spinless) fermions in N single-particle states (modes), $N \geq K$.

- **Approach 1: Slater rank criterion, *K=2*** – Think of entanglement between particles, but 'subtract' correlations arising from antisymmetry constraint...

  Schliemann *et al.* Phys. Rev. A **64**, 022303 (2001).

$$|\Psi_F\rangle = \sum_{i,j=1}^{N} w_{ij}\, c_i^\dagger c_j^\dagger |vac\rangle \quad \Rightarrow \quad W' = U W U^\dagger = diag[Z_1, \ldots, Z_r, Z_0], \quad Z_0 \equiv 0$$

'Entanglement' iff Slater rank $r > 1$. r = Slater rank

→ How to extend criterion and quantify 'entanglement' beyond two fermions?

- **Approach 2: Mode entanglement** – Think of entanglement between modes, by mapping 2^N-dim fermionic Fock space to state space of N fictitious qubits (Jordan-Wigner mapping)...

  Zanardi, Phys. Rev. A **65**, 042101 (2002).

→ Modes are distinguishable: Apply standard entanglement definition, except...

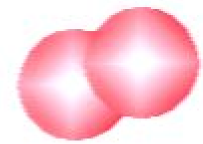
→ Particle-number conservation imposed by hand to avoid *unphysical* states!

Can GE help?...

Fermionic GE

Somma *et al*, *Phys. Rev.* A **70**, 042311 (2004); Ng & LV, forthcoming.

$$\{c_i, c_j^\dagger\} = \delta_{ij},\ \{c_i, c_j\} = 0,\ \{c_i^\dagger, c_j^\dagger\} = 0, \quad c_j \to \sum_i U_{ij} c_i,\ U \in Mat(N \times N)$$



- Associate 'local' resources to particle-preserving fermionic operators:

$$h_F = u(N) = \text{span}\left\{ c_i^\dagger c_i - \frac{1}{2}, \frac{c_i^\dagger c_j + c_j^\dagger c_i}{\sqrt{2}}, \frac{c_i^\dagger c_j - c_j^\dagger c_i}{i\sqrt{2}} \right\}, \qquad 1 \le i < j \le N$$

1st quantization

$$\text{span}\{A \otimes I \ldots \otimes I + I \otimes \ldots A \otimes \ldots \otimes I \ldots + cyclic\}, \quad \forall A = \text{One-particle observables}$$

The GCSs of $u(N)$ are the fermionic product states = Slater determinants

$$|\Psi\rangle = \prod_i c_i^\dagger |VAC\rangle = |CGS\rangle_{U(N)}$$

Extension of Slater rank criterion to arbitrary K

- Natural measure of fermionic entanglement:

$$P_{u(N)}(|\psi\rangle) = \frac{2}{N} \sum_{j<j'=1}^{N} \left[\langle c_j^\dagger c_{j'} + c_{j'}^\dagger c_j \rangle^2 - \langle c_j^\dagger c_{j'} - c_{j'}^\dagger c_j \rangle^2 \right] + \frac{4}{N} \sum_{j=1}^{N} \langle c_j^\dagger c_j - 1/2 \rangle^2$$

→ $P_{u(N)} = 1$ for *any* Slater determinant (for any K)

→ $P_{u(N)} < 1$ for any other (non-extremal) pure fermionic state

GE depends only upon observer – not upon set of modes and language chosen!

Problem 2: Entanglement in quantum critical systems

QPT: A change in the ground-state structure of a many-body quantum system at T=0.

- Conceptual significance:
  - → Central challenge of condensed matter theory, atomic physics, quantum statistical mechanics (coexistence/competition between multiple interactions and quantum orders...)
  - → Engineering of novel foms of matter...

- Practical significance:
  - → Material science and device technology;
  - → Experimental quantum computation and simulation (ultracold atoms in optical lattices...)

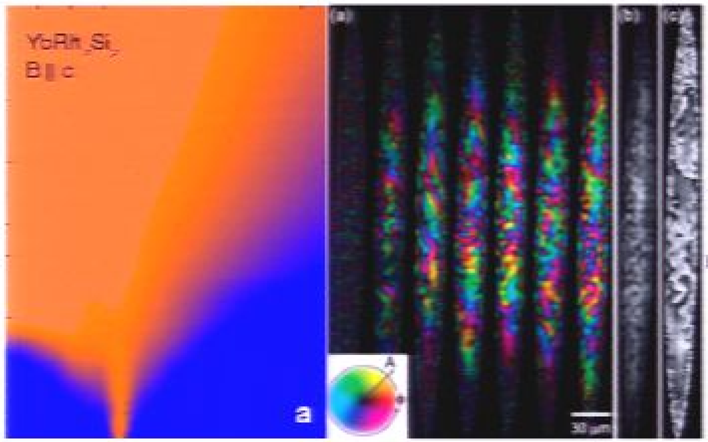
Growing body of experimental work yet theoretical understanding far from complete...

Chief difficulty: complexity of quantum correlations in many-body states and dynamical evolutions.

Can entanglement theory help?...

Flurry of investigations, see RMP08...

[Greiner *et al*, Nature 2002]



[Gegenwart *et al*, PRL 2002][Sadler *et al*, Nature 2006]

Case study: The alternating XY spin chain

Deng, Ortiz & LV, *Recent Progr. Many-Body Theories* **11**, 387 (2008), arXiv:0802.3941.

$$H = -\sum_{i=1}^{N} \left[(1+\gamma)\sigma_x^i \sigma_x^{i+1} + (1-\gamma)\sigma_y^i \sigma_y^{i+1} \right] + \sum_{i=1}^{N} \left(h - (-)^i \delta \right) \sigma_z^i$$

Exactly solvable via (even-odd) Jordan-Wigner mapping to N non-interacting fermion modes.

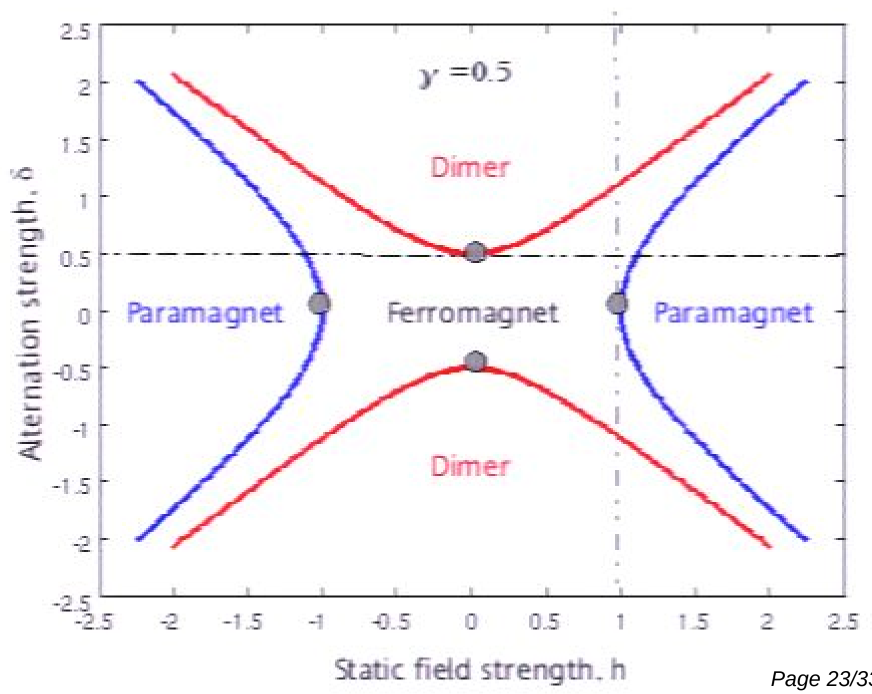
→ Quantum phase boundaries (2nd order QCPs):

$$h^2 = \delta^2 + 1$$
$$\delta^2 = h^2 + \gamma^2$$

→ Ground state may develop weak singularities $(h, \delta) = (0, \delta = \pm\gamma)$ & $(\pm 1, \delta = 0)$ corresponding to 4th order QCPs.

→ Universality classes, anisotropic case:

$\nu = 1,\ z = 1$	Ising
$\nu = 2,\ z = 1$	Alternating



GE as a QPT indicator: Static phase diagram

- Relevant (Lie) algebras of observables acting on the 2^N-dimensional spin space:

  $u(N)$ = { number-conserving quadratic fermionic operators } $\subset so(2N)$

  → The GS is always a GCS of $so(2N)$, GE relative to $so(2N)$ gives no information about QCPs.

  → The GS becomes a GCS of $u(N)$ in the fully PM and DM limits.

$$P_{u(N)}(|GS\rangle) = \frac{4}{N}\sum_k \langle a_k^\dagger a_k - 1/2\rangle^2 + \langle a_{-k}^\dagger a_{-k} - 1/2\rangle^2 + \langle b_k^\dagger b_k - 1/2\rangle^2 + \langle b_{-k}^\dagger b_{-k} - 1/2\rangle^2 + $$
$$+ 2|\langle a_k^\dagger b_k\rangle|^2 + 2|\langle a_{-k}^\dagger b_{-k}\rangle|^2$$

- Ground-state fermionic GE faithfullly portraits underlying phase diagram:

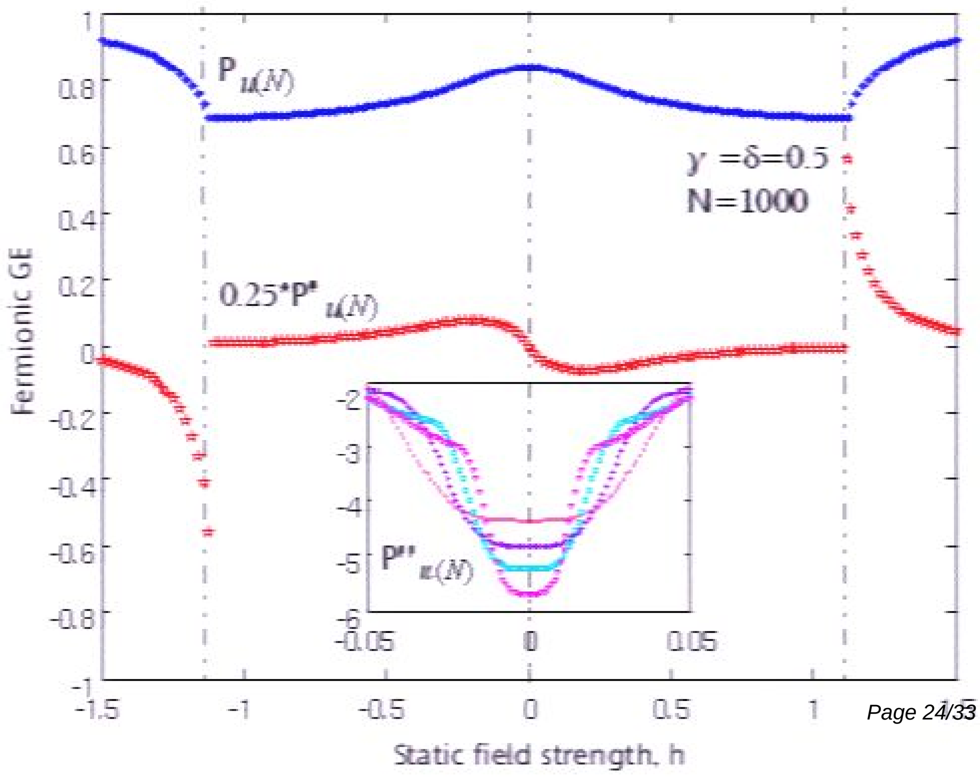
  → Analytical result available for δ= 0.

  → GE sharply detects PM-FM QPTs.

  [Somma *et al*, PRA 2004]

  Distinctive feature:

  → Derivatives of GE (or GE itself) develop singular behavior at (and only at) QCPs.



GE as a QPT indicator: Static scaling properties

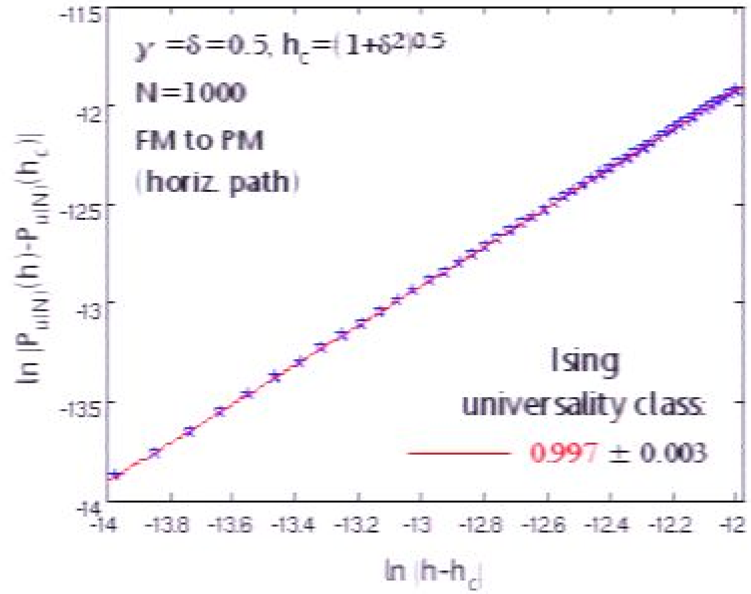
- Ground-state fermionic GE contains complete information about static critical exponents:
  - ➡ Taylor-expand purity near QCP.

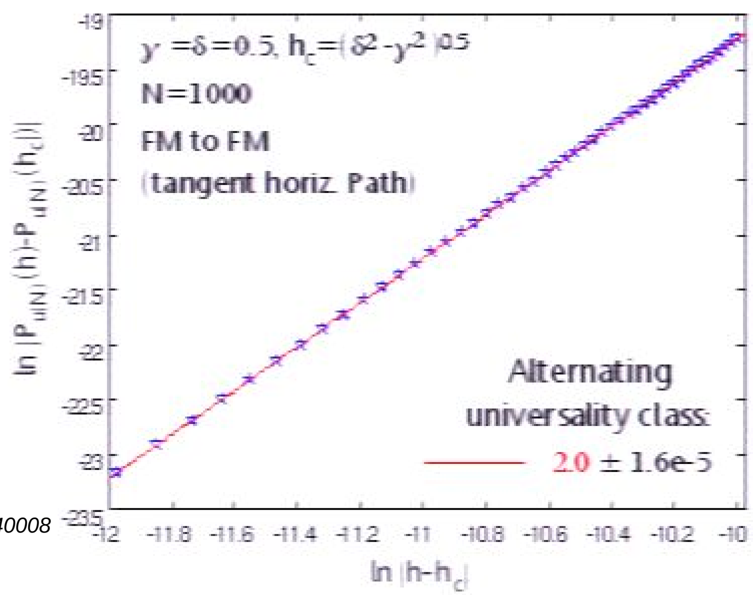
Distinctive feature:

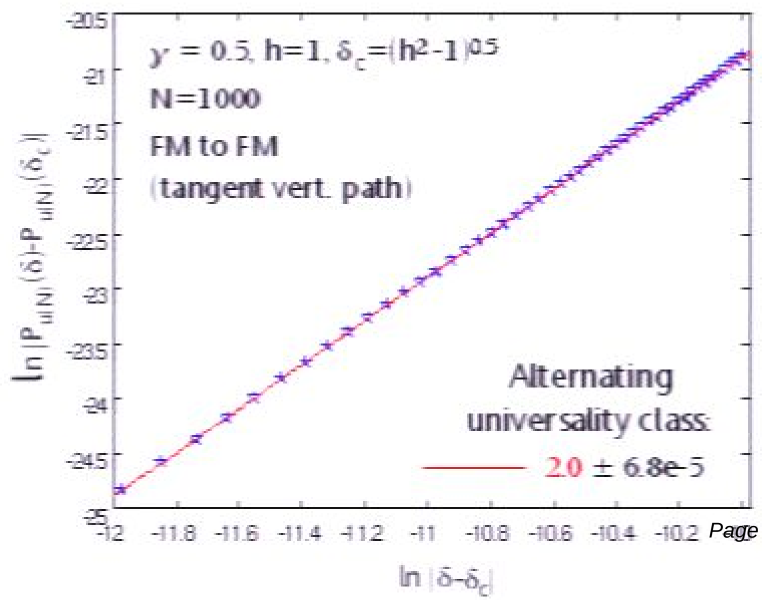
➡ Ground-state GE is related to the variance of the total fermionic number operator...

$$P_{so(2N)}(|GS\rangle) - P_{su(N)}(|GS\rangle)$$

quantifies 'fermionic pairing'...





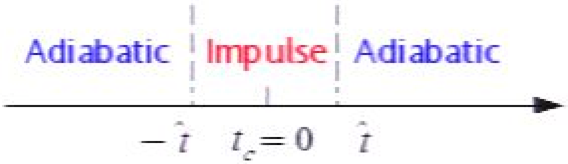


Non-equilibrium universal GE scaling

Deng, Ortiz & LV, *Europhys. Lett.* **87**, 64008 (2008).

- Simple dynamical scenario: Adiabatic quench across an isolated QCP

$$g(t)-g_c=\frac{t-t_c}{\tau_Q}, \qquad \tau(\hat{t})=\left|\frac{g(\hat{t})-g_c}{g'(\hat{t})}\right|$$



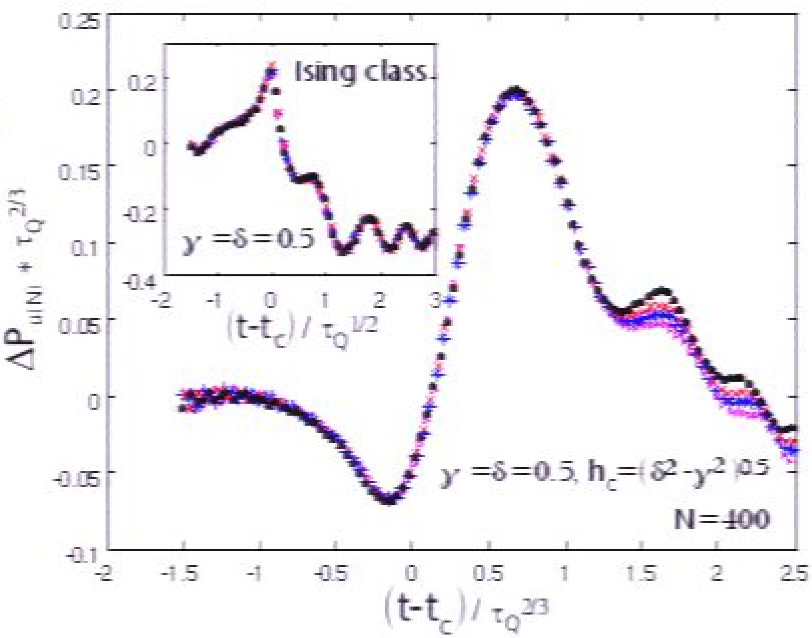
→ Kibble-Zurek mechanism: Scaling of 'freeze-out' time determines scaling of final excitation density

$$\hat{t}\sim\tau_Q^{\nu z/(\nu z+1)} \quad\Rightarrow\quad n(t_F)\sim\tau_Q^{-\nu/(\nu z+1)}$$

- GE (and arbitrary observables) also obeys scaling behavior across the entire quench duration:

$$P_{u(N)}(|\Psi(t)\rangle)-P_{u(N)}(|\Psi_0(t)\rangle)=$$
$$=\tau_Q^{-\nu/(\nu z+1)}G\left(\frac{t-t_c}{\hat{t}}\right)$$

→ Captured by adiabatic renormalization argument.



Details of low-lying excitation spectrum become essential for gapless regions...

Problem 3: Entanglement of random pure states

LV & W.G. Brown, *J. Phys.* A **40**, 8109 (2007).

Complex quantum states closely resemble random quantum states...

- Characterization of 'typical' entanglement has both a long history and broad relevance to QIP...

$|\psi\rangle \in \mathcal{H}$, $\dim(\mathcal{H}) = N$: Uniformly sampled from unitarily invariant measure on unit vectors
↳ induced by *Haar measure* on SU(N)

[Lubkin, 1978; Page, 1993; Scott & Caves, 2003; Hayden *et al*, 2004...]

- **Theorem:** Let $\mathfrak{h}$ be any Hermitian-closed d-dim subspace of traceless observables on $\mathcal{H}$. Then

$$\overline{P_{\mathfrak{h}}} = E^{(Haar)}\{P_{\mathfrak{h}}(|\psi\rangle)\} = k_{\mathfrak{h}} \frac{d}{N+1}$$

(Ex. 1) n-qubit system, $N=2^n$, standard entanglement: $\overline{P_{loc}} = k_{loc} \frac{3n}{N+1} = \frac{3}{N+1}$

(Ex. 2) Spin-J system, $N=2J+1$, GE relative to SU(2): $\overline{P_{su(2)}} = k_{su(2)} \frac{3}{N+1} = \frac{1}{2J}$

(Ex. 3) Natural extension to *typical subspace GE, e.g.* Average entanglement of n-qubit states in

$S_0 = \text{span}\{ |\psi\rangle \mid \sum_i \sigma_z^i = 0 \}$, $N_0 = n!/[(n/2)!]^2$

$$\overline{P_{loc}}|_{S_0} = k_{loc} \frac{2n}{N_0+2} = \frac{2}{N_0+2}$$

Entanglement in non-integrable spin systems

W.G. Brown *et al*, *Phys. Rev. E* **77**, 021106 (2008).

- Transition from integrability to chaos in many-body systems:

$$H = H_0 + \lambda H_1$$

Interaction strength ≈ Average energy spacing between directly coupled states

→ Crossover from Poisson to Wigner-Dyson statistics;

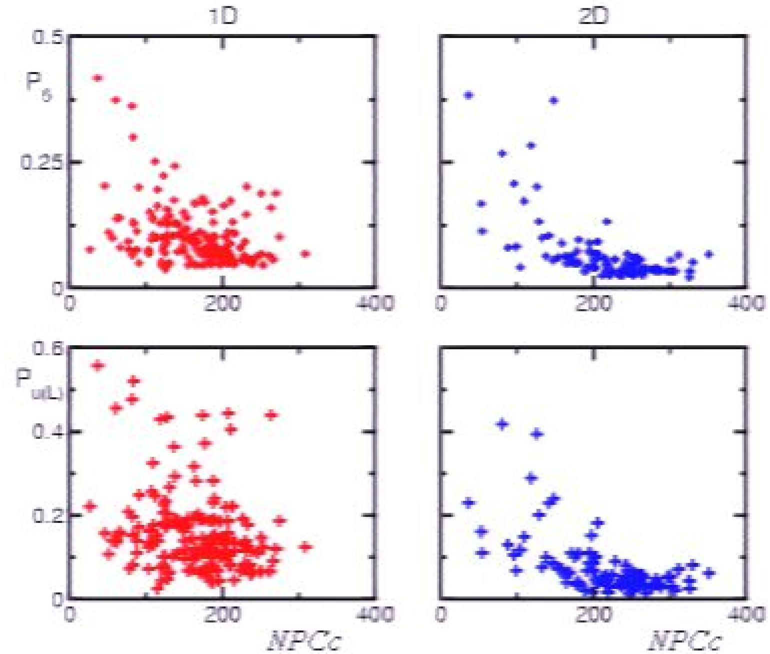
→ Structural change/*massive delocalization* in any *typical* many-body eigenstate wrto unperturbed basis $\{|k\rangle\}$,

$$NPC(|\psi\rangle) = \left(\sum_k |\langle k|\psi\rangle|^4\right)^{-1} \equiv \left(\sum_k |a_k|^4\right)^{-1}$$

Can GE shed further light on quantum chaos?

$$H = \sum_{\langle i,j\rangle} \frac{J}{4} \vec{\sigma}^{(i)} \cdot \vec{\sigma}^{(j)},$$

N=12
$S_z=0$; S=1



→ Dependence of GE upon NPC is much stronger in the chaotic 2D system than in 1D integrable system.

→ For the chaotic system, all states near in energy are strongly coupled, resulting in random mixing...

Related to eigenstate thermalization hypothesis...

[Srednicki, 1994; Zelevinsky *et al*, 1996; Olshanii, Nature **452**, 2008]

Summary and outlook

- GE provides a **unifying conceptual framework** for defining entanglement relative to any physically relevant, distinguished **subspace of observables**.
- For Lie-algebraic observable sets, generalized-*un*entangled states are naturally identified with **generalized coherent states**.
- GE ties together different aspects of **'complexity'** and **'classicality'** at the quantum level.

Some (of many) open problems...

- What about other aspects of classicality?
  - Does GE imply violation of Bell-type inequalities?
    - ↳ Violation of 'pentagram ineq' for all but GCSs in spin-1systems...

Klyachko et al, Phys. Rev. Lett. **101**, 020403 (2008).

- What about info-theoretic and resource-based characterizations of GE?
  - How to define 'GLOCC maps' and GE monotones?
  - Can GE be efficiently detected? GE witnesses?
  - Can GE be a resource for quantum estimation? Quantum simulation? Frame transmission?
    - ↳ Max G-entangled states maximize QFI in estimating Lie-algebraic channel strength...

Boixo & LV, forthcoming.

- What about relativistic extensions?
  - Can GE be useful in non-inertial frames?

Epilogue

Entanglement is – inevitably – a relative concept.

Is generalized entanglement capturing the right *relativity? Only time will tell...*

Essential references:

- *Generalizations of entanglement based on coherent states and convex sets* –
  H. Barnum, E. Knill, G. Ortiz & L. Viola, *Phys. Rev. A* **68**, 032308 (2003).
- *A subsystem-independent generalization of entanglement* –
  H. Barnum, E. Knill, G. Ortiz, R. Somma & L. Viola, *Phys. Rev. Lett.* **92**, 107902 (2004).
- *Nature and measure of entanglement in quantum phase transitions* –
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- *Entanglement beyond subsystems* –
  L. Viola, H. Barnum, E. Knill, G. Ortiz & R. Somma, *Contemp. Math.* **381**, 117 (2005).
- *Entanglement and subsystems, entanglement beyond subsystems, and all that* –
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