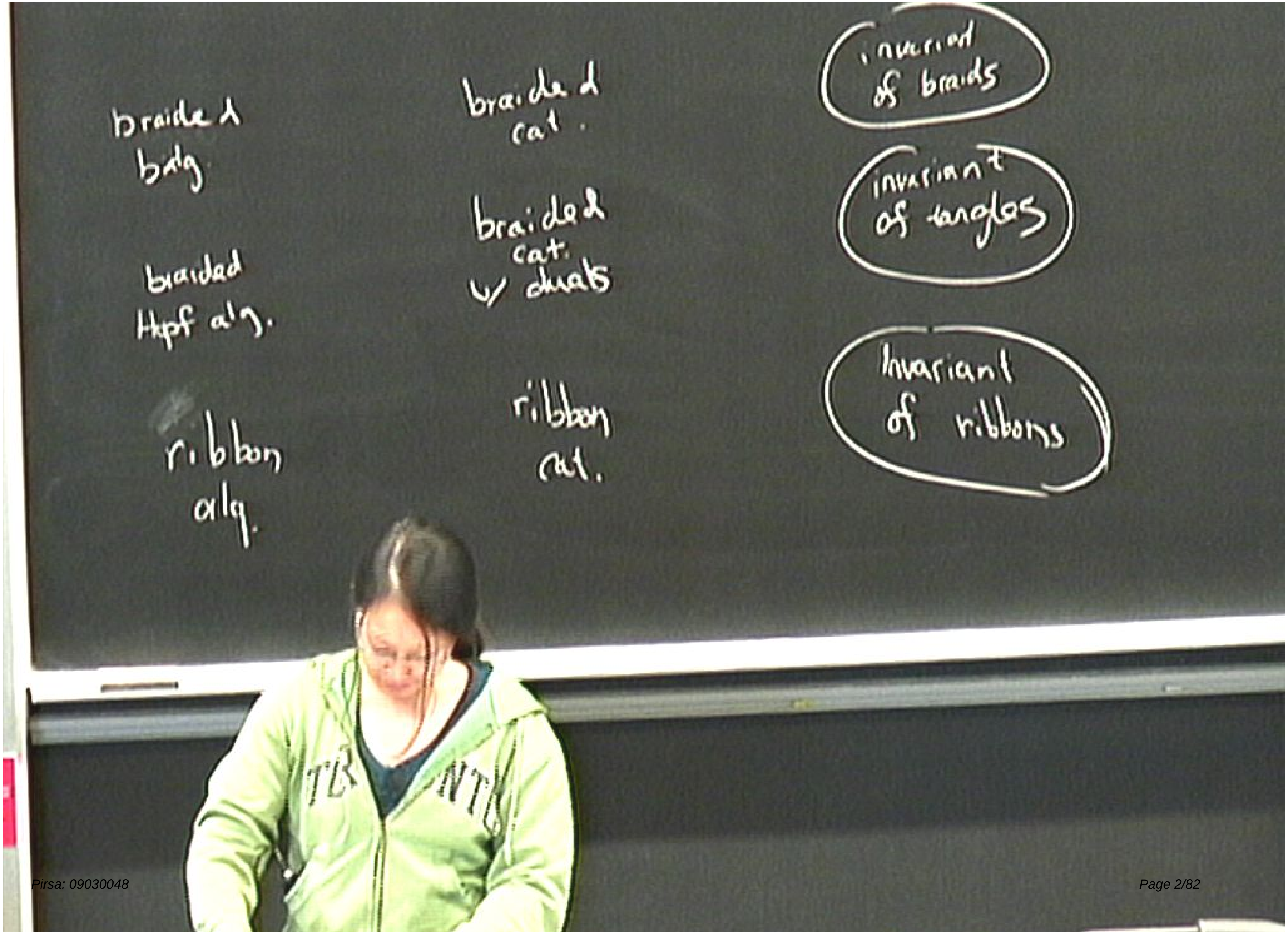
Title: Grad Talks 6

Date: Mar 23, 2009 11:00 AM

URL: http://pirsa.org/09030048

Abstract: Lecture on Quantum Groups by Lucy Zhang


braided
bialg.
braided
Hopf alg.
ribbon
alg.
braided
cat.
braided
cat.
w/ duals
ribbon
cat.
invariant
of braids
invariant
of tangles
Invariant
of ribbons

THM:

Let A be an alg. with "comultiplication" Δ

and "counit" ε

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Let A be an alg. with "comultiplication" Δ
and "counit" ε

a) (A, Δ, ε) bialgebra $\iff$ A-Mod is a tensor subcat. of $\mathrm{Vect}(k)$

Proof a) $\mathbb{I}_n$ Vect k,

Proof a) I_n Vect k,

$\otimes = \otimes$

$\mathrm{I} = k$

$a, \ell, r = \mathrm{id}$

<u>Proof</u> a) ~~In Vect k~~,

$\otimes = \otimes$ (with tensor A-module structure)

$\underline{I} = k$

$a, \ell, r = \mathrm{id}$

$x.(u \otimes v) = \Delta(x)(u \otimes v)$

<u>Proof</u> a) ~~In Vect k~~,

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<u>Proof</u> a) ~~$\underline{I}_n$ Vect k~~,

$x.(u \otimes v) = \Delta(x)(u \otimes v)$

$\otimes = \otimes$ (with tensor A-module structure)

$\underline{I} = K$ (with trivial A-module structure)

$x.\lambda = \varepsilon(x)\lambda$

$a, l, r = id$

$x.(u \otimes v) = \Delta(x)(u \otimes v)$

Proof a) ~~In~~ ~~Vect k~~,

$(\Leftarrow)$

$\otimes = \otimes$ (with tensor A-module structure)

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Proof a) ~~In Vect k~~,

($\Leftarrow$)

$\otimes = \otimes$ (with tensor A-module structure)

$x.(u\otimes v) = \Delta(x)(u\otimes v)$

$I = k$ (with trivial A-module structure) $x.\lambda = \varepsilon(x)\lambda$

$a, l, r = \mathrm{id}$

$\in \mathrm{Hom}(A\text{-Mod})$

$x.(u \otimes v) = \Delta(x)(u \otimes v)$

Proof a) ~~In~~ Vect k, (with tensor A-module structure)

($\Leftarrow$) $\otimes = \otimes$ (with trivial A-module structure)

$\underline{I} = K$ (with trivial A-module structure)

$x.\lambda = \varepsilon(x)\lambda$

$a, l, r = id$

$\in$ Hom(A-Mod)

$a_{u,v,w}$

$x.(u\otimes v) = \Delta(x)(u \otimes v)$

<u>Proof</u> a) ~~In~~ Vect_k, (with tensor A-module structure)

($\Leftarrow$) $\otimes = \otimes$ (with trivial A-module structure)

$\underline{I} = k$ (with trivial A-module structure)

$x.\lambda = \varepsilon(x)\lambda$

$a, l, r = \mathrm{id}$

$\in \mathrm{Hom}(A\text{-}\mathrm{Mod})$

$a_{U,V,W} : (U\otimes V)\otimes W \xrightarrow{\mathrm{id}} U\otimes(V\otimes W)$

(with tensor ...

$(\Leftarrow)$ $\otimes = \otimes$ (with tensor ... module structure)

$I = k$ (with trivial Λ-module structure

$x.\lambda = \varepsilon(x)\lambda$

$a,\ \ell,\ r = \mathrm{id}$

$\in \mathrm{Hom}(A\text{-Mod})$

$$a_{U,V,W} : (U\otimes V)\otimes W \xrightarrow{\mathrm{id}} U\otimes(V\otimes W)$$

$(\Leftarrow)$ $\quad \otimes = \otimes \quad$ (with tensor Λ-module structure)

$I = k \quad$ (with trivial Λ-module structure $x.\lambda = \varepsilon(x)\lambda$

$a, \ell, r = \mathrm{id}$

$\in \mathrm{Hom}(A\text{-Mod})$

$$a_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{\mathrm{id}} U \otimes (V \otimes W)$$

$$a_{U,V,W} \circ x = x \circ a_{U,V,W} \quad \forall x \in A$$

$(\Leftarrow)$ $\quad \otimes = \otimes$ (with tensor A-mod... structure)

$I = k$ (with trivial A-module structure $x.\lambda = \varepsilon(x)\lambda$

$a, l, r = \mathrm{id}$

$\in \mathrm{Hom}(A\text{-Mod})$

$$a_{u,v,w} : (U \otimes V) \otimes W \xrightarrow{\mathrm{id}} U \otimes (V \otimes W)$$

$\circ\, x(u \otimes v \otimes w) = x \circ a_{u,v,w} \quad \forall x \in A$

$u \in U,\ v \in V,\ w \in W$

$(\Leftarrow)$ $\otimes = \otimes$ (with tensor A-module structure)

$\underline{I} = k$ (with trivial A-module structure) $x.\lambda = \varepsilon(x)\lambda$

$a, l, r = id$

$\in \mathrm{Hom}(A.\mathrm{Mod})$

$$a_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{id} U \otimes (V \otimes W)$$

$$a_{U,V,W} \circ x(u \otimes v \otimes w) = x \circ a_{U,V,W}(u \otimes v \otimes w) \quad \forall x \in A$$

$u \in U, v \in V, w \in W$

Proof (⇐)

$\otimes = \otimes$ (with tensor A-module structure)

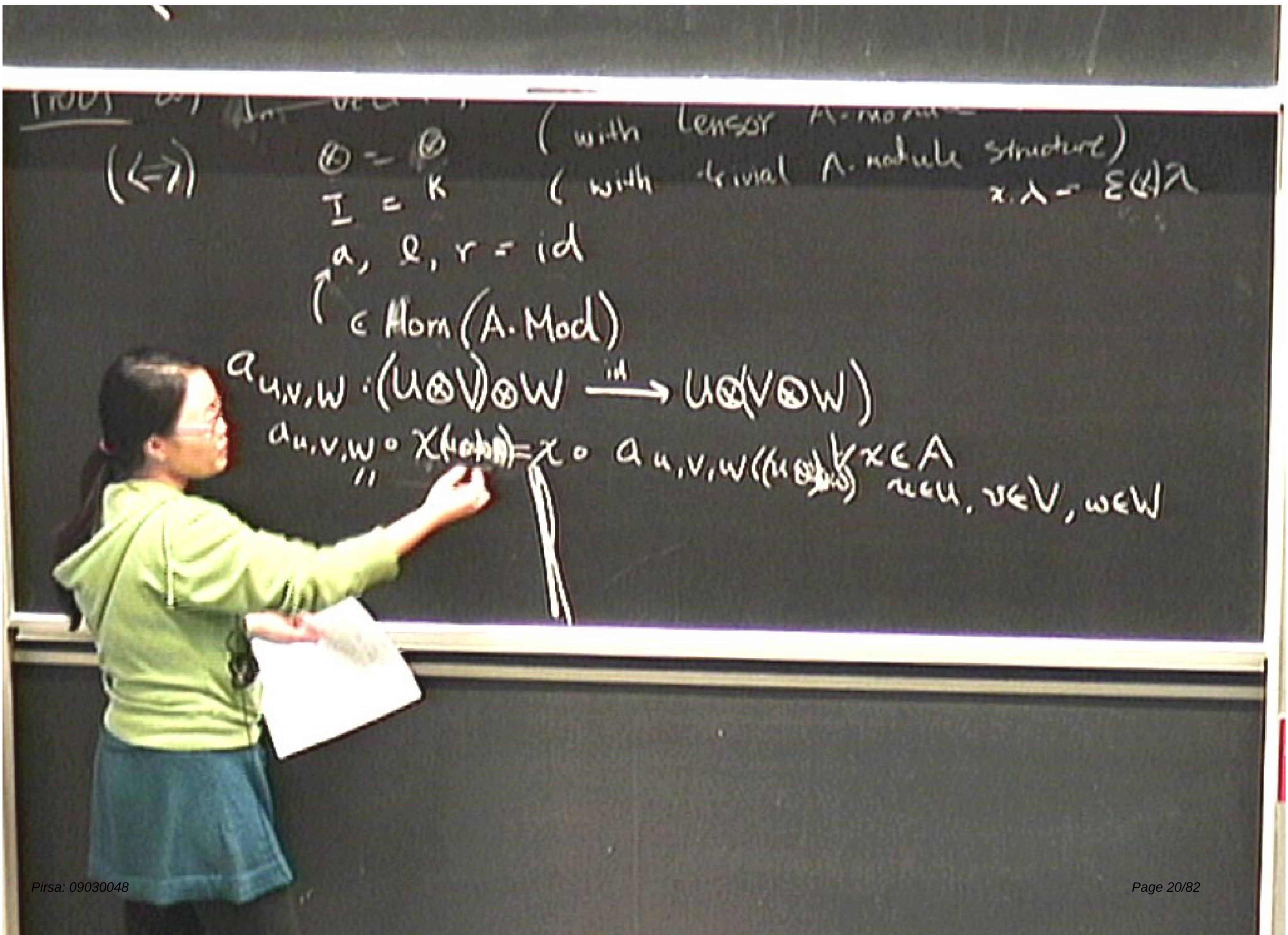
$\underline{I} = k$ (with trivial A-module structure) $x.\lambda = \varepsilon(x)\lambda$

$a, l, r = \mathrm{id}$

$\in \mathrm{Hom}(A.\mathrm{Mod})$

$$a_{U,V,W} : (U\otimes V)\otimes W \xrightarrow{\mathrm{id}} U\otimes(V\otimes W)$$

$$a_{U,V,W} \circ x(u\otimes v\otimes w) = x \circ a_{U,V,W}(u\otimes v\otimes w) \quad \forall x \in A, \ u\in U, v\in V, w\in W$$


(⇐) ⊗ = ⊗ (with tensor A-module structure)
I = K (with trivial A-module structure x.λ = ε(x)λ
a, l, r = id
∈ Hom (A.Mod)
a_{U,V,W} : (U⊗V)⊗W → U⊗(V⊗W)
id
a_{U,V,W} ∘ x(u⊗v⊗w) = x ∘ a_{U,V,W}((u⊗v)⊗w) ∀x∈A
u∈U, v∈V, w∈W

Hopf ... (with tensor A-module ...)

$(\Leftarrow)$ $\quad \otimes = \otimes$ (with tensor A-module structure)

$\underline{I} = K$ (with trivial A-module structure $x.\lambda = \varepsilon(x)\lambda$

$a, l, r = id$

$\in \mathrm{Hom}(A.\mathrm{Mod})$

$$a_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{id} U \otimes (V \otimes W)$$

$$a_{U,V,W} \circ x(u \otimes v \otimes w) = x \circ a_{U,V,W}(u \otimes v \otimes w) \quad \forall x \in A$$

$$(\Delta \otimes id)(\Delta x)(u \otimes v \otimes w) \qquad u \in U,\ v \in V,\ w \in W$$

Proof

$(\Leftarrow)$ $\quad \otimes = \otimes$ (with tensor A-module structure)

$I = K$ (with trivial A-module structure) $x.\lambda = \varepsilon(x)\lambda$

$a, l, r = id$

$\in Hom(A.Mod)$

$a_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{id} U \otimes (V \otimes W)$

$a_{u,v,w} \circ x(u \otimes v \otimes w) = x \circ a_{u,v,w}(u \otimes v \otimes w)$ $\forall x \in A$, $u \in U, v \in V, w \in W$

$(\Delta \otimes id)(\Delta x)(u \otimes v \otimes w)$

(⟸) $\otimes = \otimes$ (with tensor A-module structure)

$\underline{I} = K$ (with trivial A-module structure) $x.\lambda = \varepsilon(x)\lambda$

$a, \ell, r = id$

$\in \mathrm{Hom}(A.\mathrm{Mod})$

$$a_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{id} U \otimes (V \otimes W)$$

$a_{u,v,w} \circ x(u \otimes v \otimes w) = a_{u,v,w}(x(u \otimes v \otimes w))$ $\forall x \in A$, $u \in U$, $v \in V$, $w \in W$

$(\Delta \otimes id)(\Delta x)(u \otimes v \otimes w)$ … $x.(u \otimes (v \otimes w))$

Proof of ... (with tensor A-module ...

$(\Leftarrow)$ $\quad \otimes = \otimes$ (with tensor A-module structure)

$I = K$ (with trivial A-module structure) $\quad x.\lambda = \varepsilon(x)\lambda$

$a, l, r = id$

$\in \mathrm{Hom}(A.\mathrm{Mod})$

$$a_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{id} U \otimes (V \otimes W)$$

$$a_{U,V,W} \circ x((u\otimes v)\otimes w) = x \circ a_{U,V,W}((u\otimes v)\otimes w) \quad \forall x \in A,\ u\in U,\ v \in V,\ w \in W$$

$$(\Delta \otimes id)(\Delta x)(u \otimes v \otimes w) \qquad x.(u \otimes (v \otimes w)) = (id \otimes \Delta)(\Delta x)(u \otimes v \otimes w)$$

($\Leftarrow$) $\quad \otimes = \otimes \quad$ (with tensor A-module structure)

$\underline{I} = K \quad$ (with trivial A-module structure $x.\lambda = \varepsilon(x)\lambda$

$a, l, r = \mathrm{id}$

$\in \mathrm{Hom}(A\text{-}Mod)$

$$a_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{\mathrm{id}} U \otimes (V \otimes W)$$

$$a_{U,V,W} \circ \underline{x(u \otimes v \otimes w)} = x \circ a_{U,V,W}((u \otimes v) \otimes w) \quad \forall x \in A \quad u \in U, v \in V, w \in W$$

$$(\Delta \otimes \mathrm{id})(\Delta x)(u \otimes v \otimes w) \qquad x.(u \otimes (v \otimes w)) \qquad (\mathrm{id} \otimes \Delta)(\Delta x)(u \otimes v \otimes w)$$

$(\Leftarrow)$ $\quad \otimes = \otimes$ (with tensor A-module structure)

$\underline{I} = K$ (with trivial A-module structure) $\quad x.\lambda = \varepsilon(x)\lambda$

$a, l, r = id$

$\in$ Hom(A-Mod)

Δ coassociative

$(id \otimes \Delta)\cdot\Delta = (\Delta \otimes id)\cdot\Delta$

$a_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{id} U \otimes (V \otimes W)$

$a_{U,V,W} \circ \underline{x(u \otimes v \otimes w)} = x \circ a_{U,V,W}((u \otimes v) \otimes w) \quad \forall x \in A$

$(\Delta \otimes id)(\Delta x)(u \otimes v \otimes w) \qquad x.(u \otimes (v \otimes w)) \quad u \in U, v \in V, w \in W$

$(id \otimes \Delta)(\Delta x)(u \otimes v \otimes w)$

$A \longrightarrow A \otimes A$

$A \qquad A \otimes A \otimes A$

$(\Leftarrow)$ $\quad \otimes = \otimes \quad$ (with tensor A-module structure)

$I = K \quad$ (with trivial A-module structure) $\quad x.\lambda = \varepsilon(x)\lambda$

$a, \ell, r = \mathrm{id}$

$\in \mathrm{Hom}(A\text{-Mod})$

Δ coassociative

$(\mathrm{id}\otimes\Delta)\cdot\Delta = (\Delta\otimes\mathrm{id})\cdot\Delta$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A\otimes A \arrow[d] \\ A\otimes A \arrow[r] & A\otimes A\otimes A \end{tikzcd}$$

$a_{U,V,W} : (U\otimes V)\otimes W \xrightarrow{\mathrm{id}} U\otimes(V\otimes W)$

$a_{u,v,w} \circ x((u\otimes v)\otimes w) = x \circ a_{u,v,w}((u\otimes v)\otimes w) \quad \forall x\in A$

$(\Delta\otimes\mathrm{id})(\Delta x)((u\otimes v)\otimes w) \quad x.(u\otimes(v\otimes w)) \quad u\in U,\ v\in V,\ w\in W$

$(\mathrm{id}\otimes\Delta)(\Delta x)(u\otimes v\otimes w)$

Proof

$(\Leftarrow)$

$\otimes = \otimes$ (with tensor A-module structure)

$I = K$ (with trivial A-module structure $x.\lambda = \varepsilon(x)\lambda$

$a, l, r = id$

$\in Hom(A\text{-}Mod)$

Δ coassociative

$(id\otimes\Delta)\cdot\Delta = (\Delta\otimes id)\cdot\Delta$

$$\begin{array}{ccc} A & \longrightarrow & A\otimes A \\ \downarrow & & \downarrow \\ A\otimes A & \longrightarrow & A\otimes A\otimes A \end{array}$$

$a_{U,V,W} : (U\otimes V)\otimes W \xrightarrow{id} U\otimes(V\otimes W)$

$a_{U,V,W}\circ x((u\otimes v)\otimes w) = x\circ a_{U,V,W}((u\otimes v)\otimes w)$ $\forall x\in A$, $u\in U, v\in V, w\in W$

$(\Delta\otimes id)(\Delta x)(u\otimes v\otimes w)$ $\quad$ $x.(u\otimes(v\otimes w))$

$(id\otimes\Delta)(\Delta x)(u\otimes v\otimes w)$

Proof (⟸)

$\otimes = \otimes$ (with tensor A-module structure)

$\underline{I} = K$ (with trivial A-module structure) $x.\lambda = \varepsilon(x)\lambda$

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$(\Delta\otimes\mathrm{id})(\Delta x)(u\otimes v\otimes w) \qquad x.(u\otimes(v\otimes w))$

$(\mathrm{id}\otimes\Delta)(\Delta x)(u\otimes v\otimes w)$

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$\underbrace{(\Delta\otimes id)(\Delta x)}_{A\otimes A\otimes A}(u\otimes v\otimes w) \qquad x.(u\otimes(v\otimes w))$

$\underbrace{(id\otimes\Delta)}(\Delta x)(u\otimes v\otimes w)$

$\otimes = \otimes$ (with trivial A-module structure)

$I = K$ (with trivial A-module structure $x \cdot \lambda = \varepsilon(x)\lambda$)

$a, l, r = id$

$\in \mathrm{Hom}(A\text{-Mod})$

$a_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{id} U \otimes (V \otimes W)$

$a_{U,V,W} \circ x((u \otimes v) \otimes w) = x \circ a_{U,V,W}((u \otimes v) \otimes w) \quad \forall x \in A, \ u \in U, v \in V, w \in W$

$\underbrace{(\Delta \otimes id)}_{A \otimes A \otimes A}(\Delta x)(u \otimes v \otimes w) \qquad x.(u \otimes (v \otimes w))$

$(id \otimes \Delta)(\Delta x)(u \otimes v \otimes w)$

Δ coassociative

A

$(\Leftarrow)$

$\otimes = \otimes$ (with trivial A-module structure)

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$x.\lambda = \varepsilon(x)\lambda$

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$(\mathrm{id}\otimes\Delta)\cdot\Delta = (\Delta\otimes\mathrm{id})\cdot\Delta$

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$$a_{u,v,w}\circ x((u\otimes v)\otimes w) = x\circ a_{u,v,w}((u\otimes v)\otimes w) \quad \forall x\in A,\ u\in U,\ v\in V,\ w\in W$$

$(\Delta\otimes\mathrm{id})(\Delta x)(u\otimes v\otimes w)$

$\| \quad x.(u\otimes(v\otimes w))$

$\| \quad (\mathrm{id}\otimes\Delta)(\Delta x)(u\otimes v\otimes w)$

Proof a) ~~In Vect k~~,

$(\Leftarrow)$

$\otimes = \otimes$ (with tensor A-module structure)

$I = k$ (with trivial Λ-module structure)

$x.\lambda = \varepsilon(x)\lambda$

$a, l, r = \mathrm{id}$

$\in \mathrm{Hom}(A.\mathrm{Mod})$

Δ coassociative

$(\mathrm{id}\otimes\Delta)\cdot\Delta = (\Delta\otimes\mathrm{id})\cdot\Delta$

$$\begin{array}{ccc} A & \longrightarrow & \Lambda\otimes\Lambda \\ \downarrow & & \downarrow \\ A\otimes A & \longrightarrow & \Lambda\otimes\Lambda\otimes\Lambda \end{array}$$

$a_{u,v,w} : (U\otimes V)\otimes W \xrightarrow{\mathrm{id}} U\otimes(V \ldots$

$a_{u,v,w}\circ x(u\otimes v\otimes w) = x \circ a_{u,v,w} \ldots$

$(\Delta\otimes\mathrm{id})(\Delta x)(u\otimes v\otimes w)$

$\Lambda\otimes\Lambda\otimes\Lambda$

$\ldots u, v\in V, w\in W$

$x. \ldots$

$(\mathrm{id}\otimes\Delta) \ldots$

Δ coassoc	$a \in \mathrm{Hom}(\Lambda\ldots)$
ε	$l \in \mathrm{Hom}(\Lambda\ldots)$
	$r \in \mathrm{Hom}(\Lambda\ldots)$

THM:

Let A be an alg. with "comultiplication" Δ and "counit" ε

a) (A, Δ, ε) bialgebra $\iff$ A-Mod is a tensor subcat. of $\mathrm{Vect}(k)$

b) $(\ \ \Delta, \varepsilon, \ \ )$

THM:

Let A be an alg. with "comultiplication" Δ
and "counit" ε

a) (A, Δ, ε) bialgebra $\iff$ A-Mod is a tensor subcat. of $\mathrm{Vect}(\Bbbk)$

$(A, \Delta, \varepsilon, R)$ braided bialg. $\iff$ A-Mod is braided $c^R_{U,V} = \tau \circ R$

THM:

Let A be an alg. with "comultiplication" Δ
and "counit" ε

a) (A, Δ, ε) bialgebra $\Longleftrightarrow$ A-Mod is a tensor subcat. of Vect(k)

b) $(A, \Delta, \varepsilon, R)$ braided bialg. $\Longleftrightarrow$ A-Mod is braided $c^R_{U,V} = \tau \circ R$

Proof (b):

$R \in H \otimes H$

Proof (b).

$R \in H \otimes H$

C natural

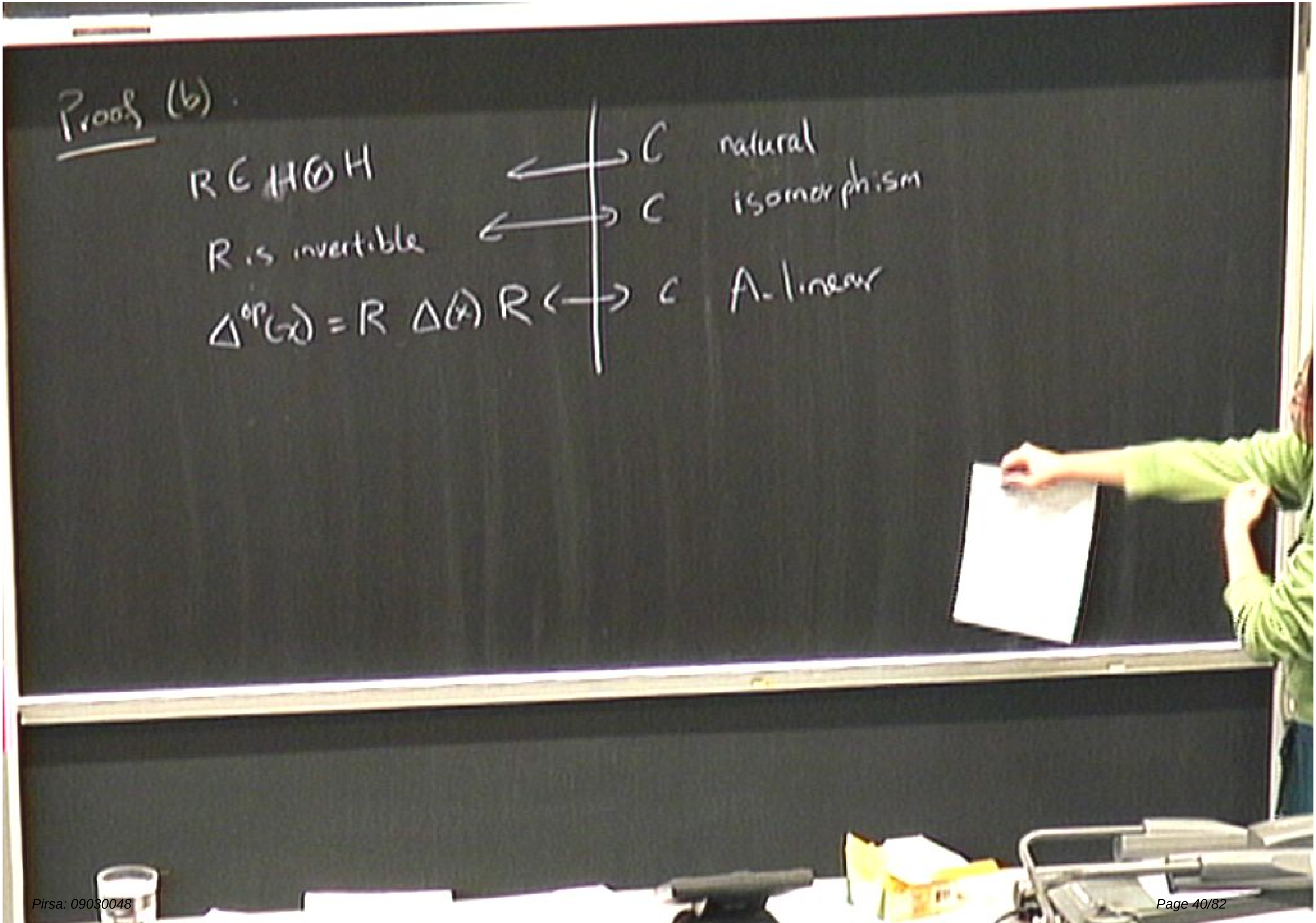
Proof (b).

$R \in H \otimes H$

R is invertible

C natural

C isomorphism


Proof (b).
$R \in H \otimes H$ ⟷ C natural
⟷ C isomorphism
R is invertible ⟷ C
$\Delta^{op}(x) = R\,\Delta(x)\,R$ ⟷ C A-linear

Proof (b):

$R \in H \otimes H \longleftrightarrow C$ natural

R is invertible $\longleftrightarrow C$ isomorphism

$\Delta^{op}(x) = R\,\Delta(x)\,R^{-1} \longleftrightarrow C$ A-linear

$(\Delta \otimes id_H) R = R_{13} R_{23} \longleftrightarrow C$ satisfies (H1)

$(1 \otimes \Delta) R = \ldots \longleftrightarrow C$ satisfies (H2)

Proof (b).

$R \in H \otimes H \longleftrightarrow C$ natural

R is invertible $\longleftrightarrow C$ isomorphism

$\Delta^{op}(x) = R\ \Delta(x)\ R \longleftrightarrow C$ A-linear

$(\Delta \otimes id_H)\, R = R_{13} R_{23} \longleftrightarrow C$ satisfies (H1)

$(1 \otimes \Delta)\, R = R_{13} R_{12} \longleftrightarrow C$ satisfies (H2)

Proof (b):

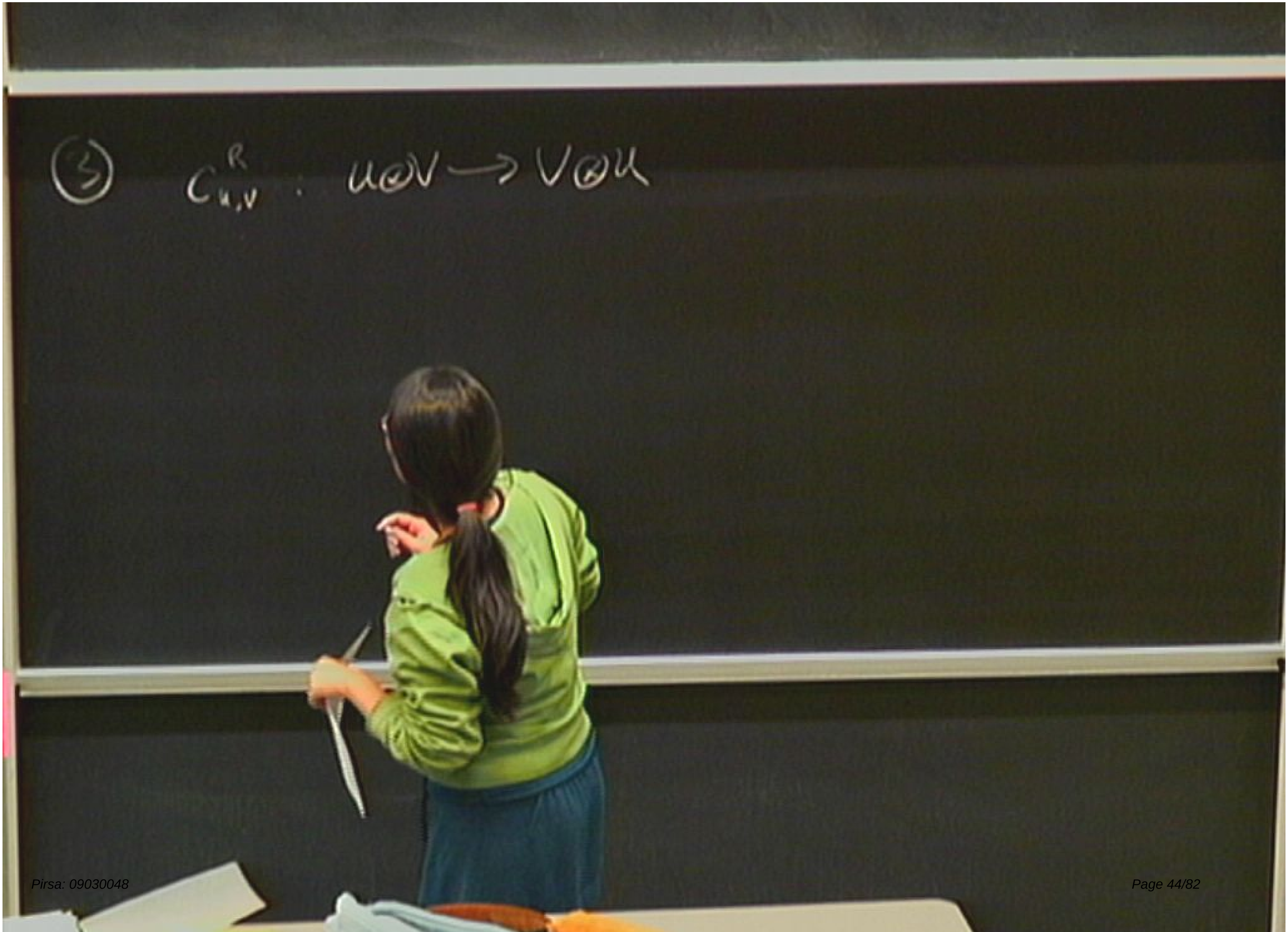
$R \in H \otimes H \longleftrightarrow C$ natural

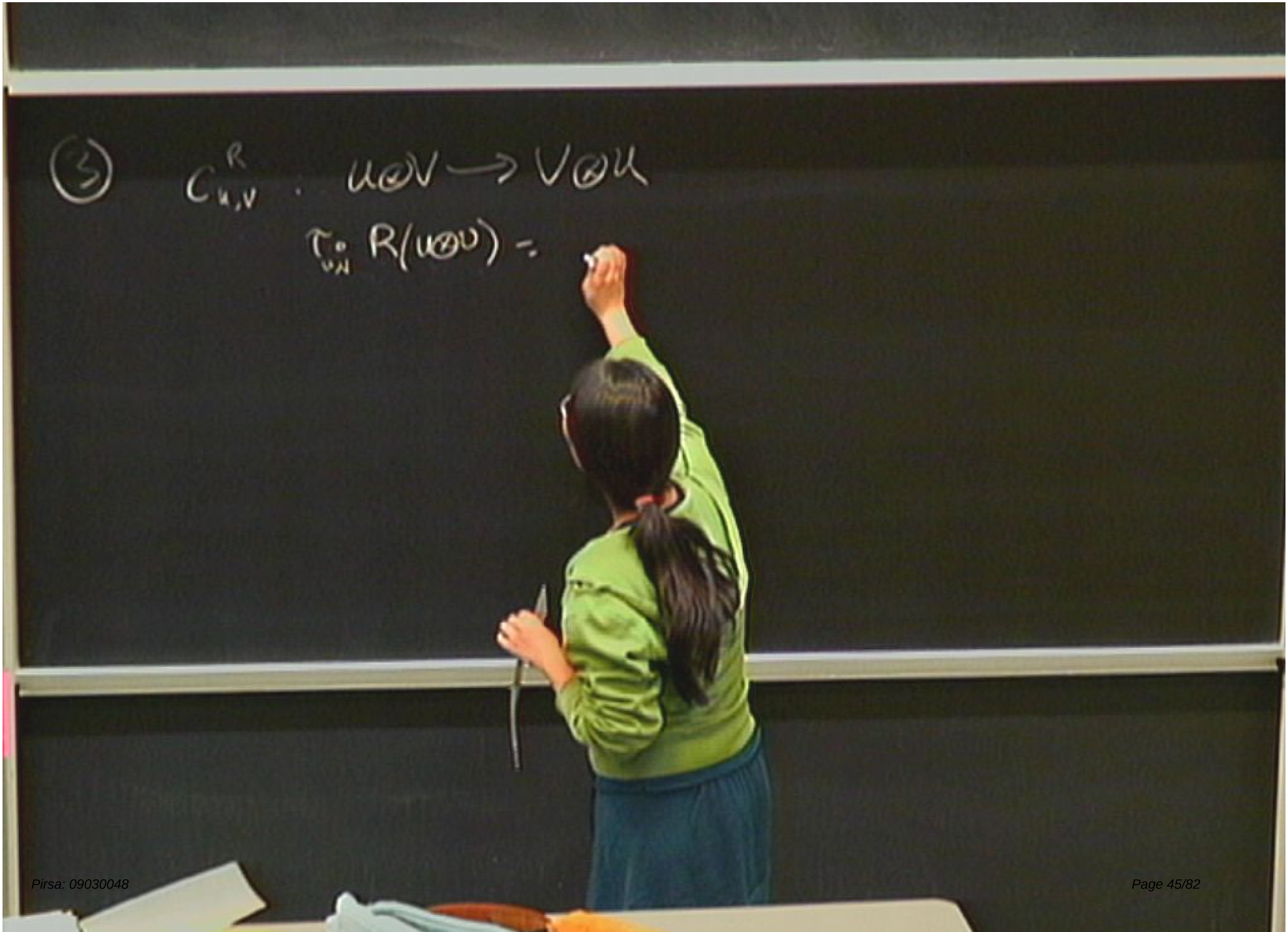
R is invertible $\longleftrightarrow C$ isomorphism

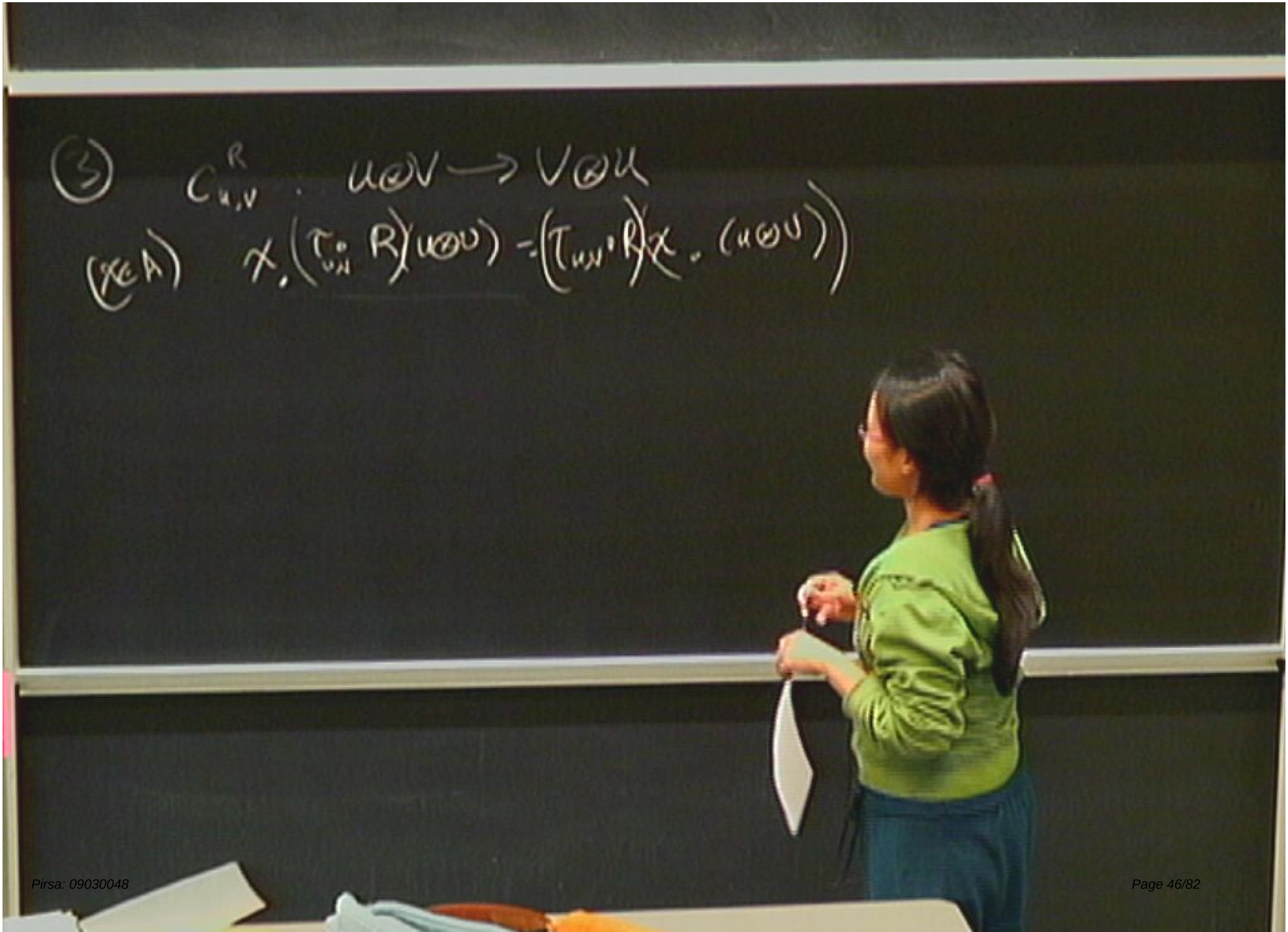
$\Delta^{op}(x) = R\,\Delta(x)\,R \longleftrightarrow C$ A-linear (3)

$(\Delta \otimes id_H)R = R_{13}R_{23} \longleftrightarrow C$ satisfies (H1)

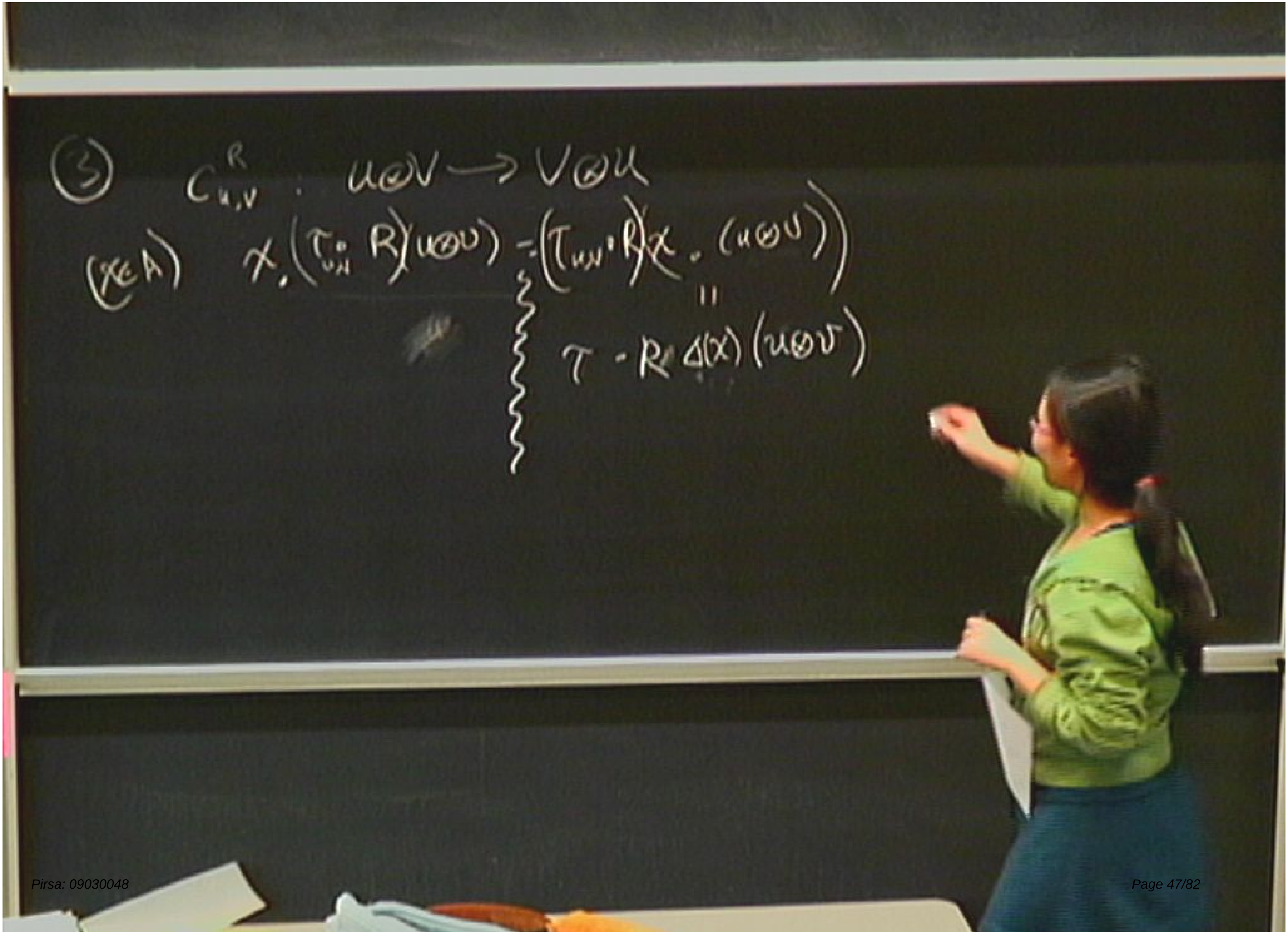
$(1 \otimes \Delta)R = R_{13}R_{12} \longleftrightarrow C$ satisfies (H2)


3 $C^{R}_{u,v}$: $u \otimes V \to V \otimes u$


5
$C^{R}_{U,V} : U \otimes V \to V \otimes U$
$\tau_{U,V}\, R(U \otimes V) =$



(5) $c^R_{U,V} : U \otimes V \to V \otimes U$
$(x \in A)$ $x.\left((\tau_{V,U} \circ R)(u \otimes v)\right) = \left((\tau_{U,V} \circ R)(x.(u \otimes v))\right)$


5 $c^{R}_{u,v} : u \otimes v \to v \otimes u$
$(x \in A)$ $x \cdot (\tau_{v,u} \cdot R)(u \otimes v) = (\tau_{u,v} \cdot R)(x \cdot (u \otimes v))$
$=$
$\tau \cdot R \cdot \Delta(x)(u \otimes v)$

(5) $\quad C^{R}_{U,V} : \; U \otimes V \longrightarrow V \otimes U$

$(x \in A) \quad x \cdot \left((\tau_{U,V} \cdot R)(u \otimes v) \right) = (\tau_{U,V} \cdot R)\left(x \cdot (u \otimes v) \right)$

$$\Delta(x) \; \tau \cdot R(u \otimes v) \qquad\qquad \tau \cdot R \, \Delta(x)(u \otimes v)$$

Proof (b).

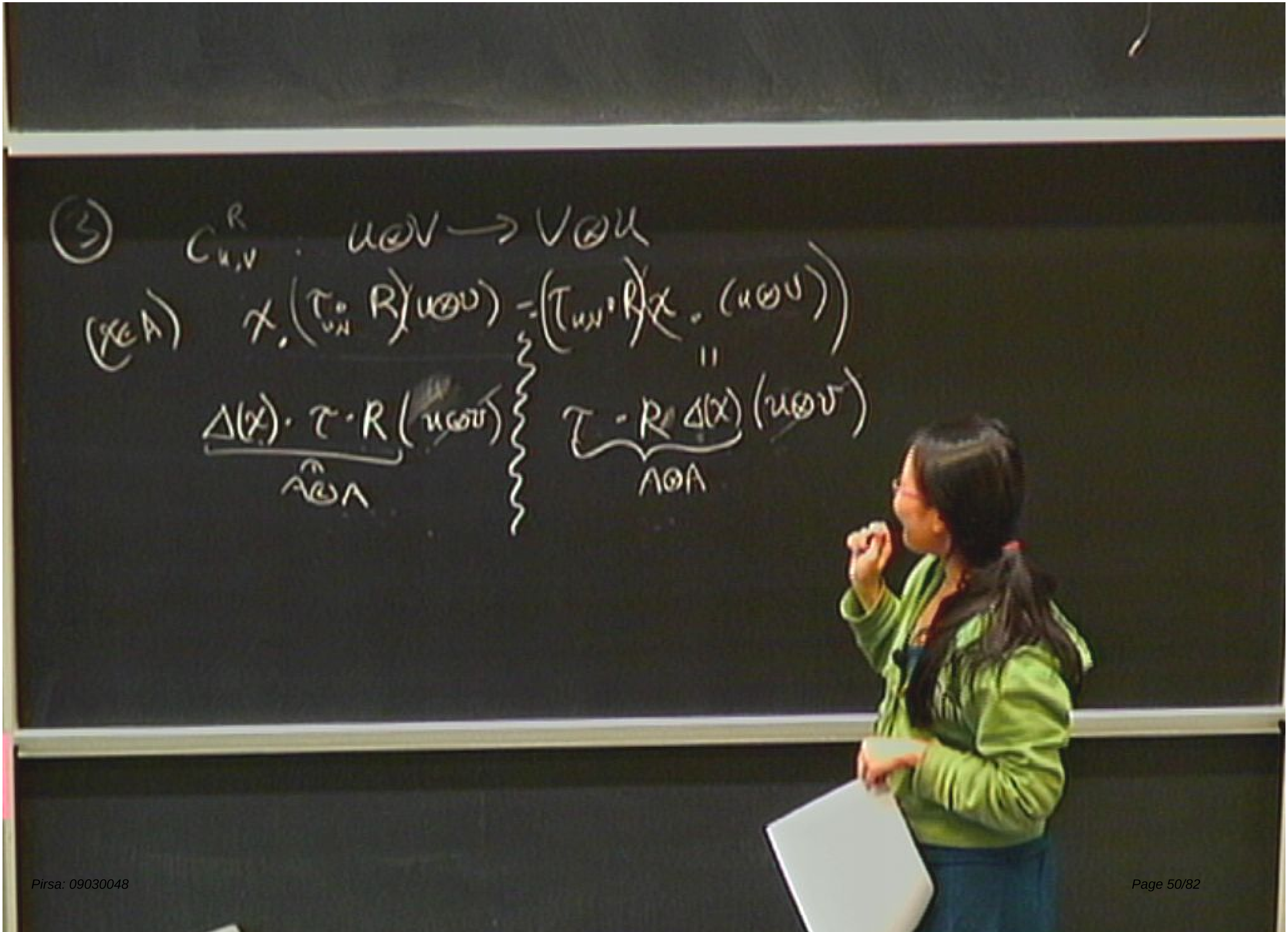
$R \in H \otimes H \longleftrightarrow C$ natural

R is invertible $\longleftrightarrow C$ isomorphism

$\Delta^{op}(x) = R\,\Delta(x)\,R^{-1} \longleftrightarrow C$ A-linear ③

$(\Delta \otimes id_H)\,R = R_{13} R_{23} \longleftrightarrow C$ satisfies (H1)

$(id \otimes \Delta)\,R = R_{13} R_{12} \longleftrightarrow C$ satisfies (H2)



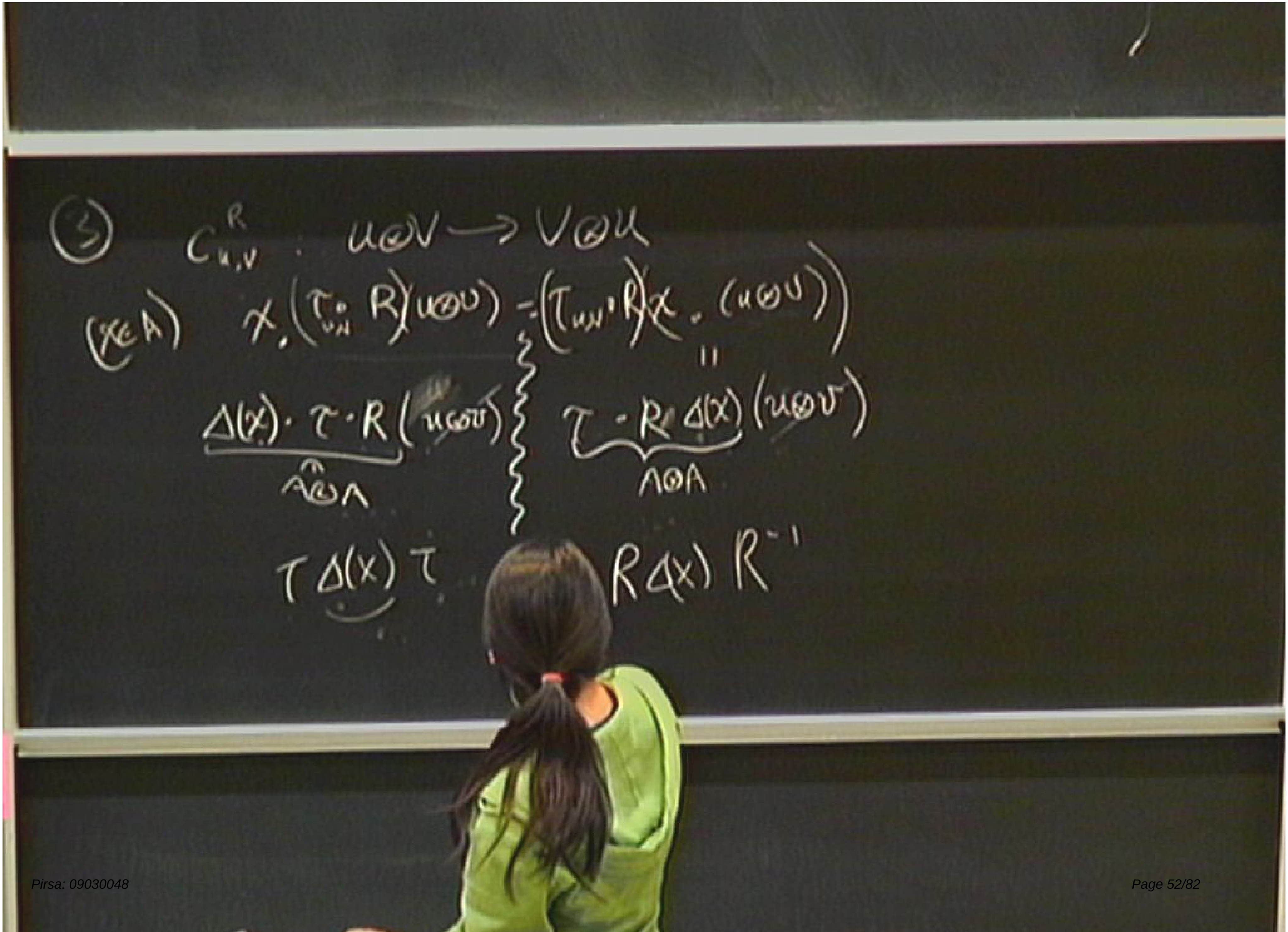
(5) $c^R_{U,V} : U\otimes V \to V\otimes U$
$(x\in A)$ $x.(\tau_{V,U}\cdot R)(u\otimes v) = (\tau_{U,V}\cdot R)(x.(u\otimes v))$
$\underbrace{\Delta(x)\cdot\tau\cdot R}_{\hat{A}\otimes A}(u\otimes v)$
$\underbrace{\tau\cdot R\,\Delta(x)}_{A\otimes A}(u\otimes v)$

(5) $c^{R}_{U,V} : U\otimes V \longrightarrow V\otimes U$

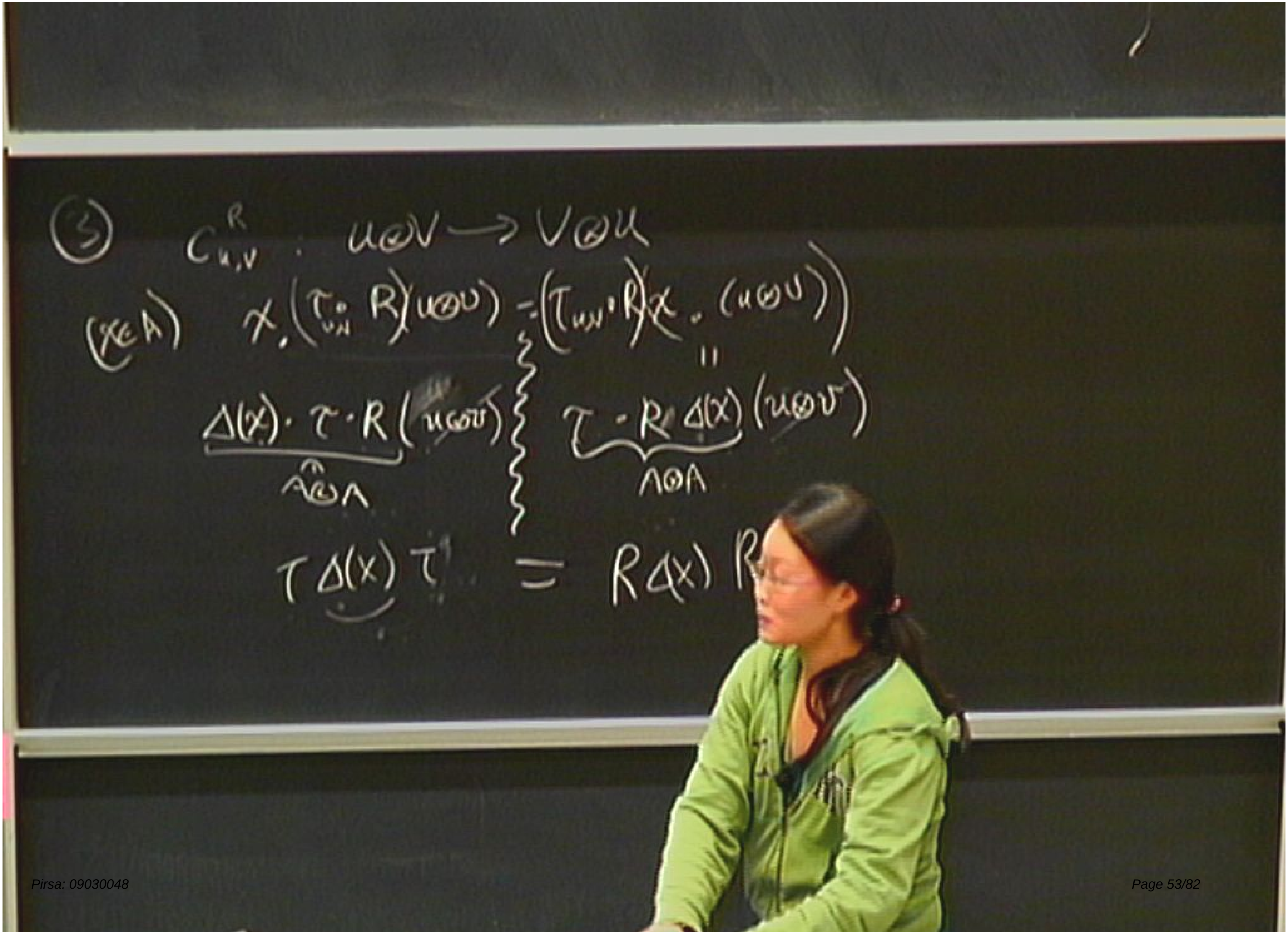
$(x\in A)$ $\quad x\cdot\left(\tau_{V,U}\cdot R\right)(u\otimes v) = \left(\tau_{U,V}\cdot R\right)\left(x\cdot(u\otimes v)\right)$

$$\underbrace{\Delta(x)\cdot\tau\cdot R}_{A\otimes A}(u\otimes v) \qquad \tau\cdot\underbrace{R\,\Delta(x)}_{A\otimes A}(u\otimes v)$$

$$\tau\,\Delta(x)\,\tau = R\,\Delta(x)\,R^{-1}$$



③ $c^{R}_{u,v} : u\otimes v \longrightarrow v\otimes u$
$(x\in A)$ $x.(\tau_{v,u}\cdot R)(u\otimes v) = (\tau_{u,v}\cdot R)(x.(u\otimes v))$
$\underbrace{\Delta(x)\cdot\tau\cdot R}_{\hat{A}\otimes A}(u\otimes v)$
$\underbrace{\tau\cdot R\,\Delta(x)}_{A\otimes A}(u\otimes v)$
$\tau\,\Delta(x)\,\tau$
$R\,\Delta(x)\,R^{-1}$

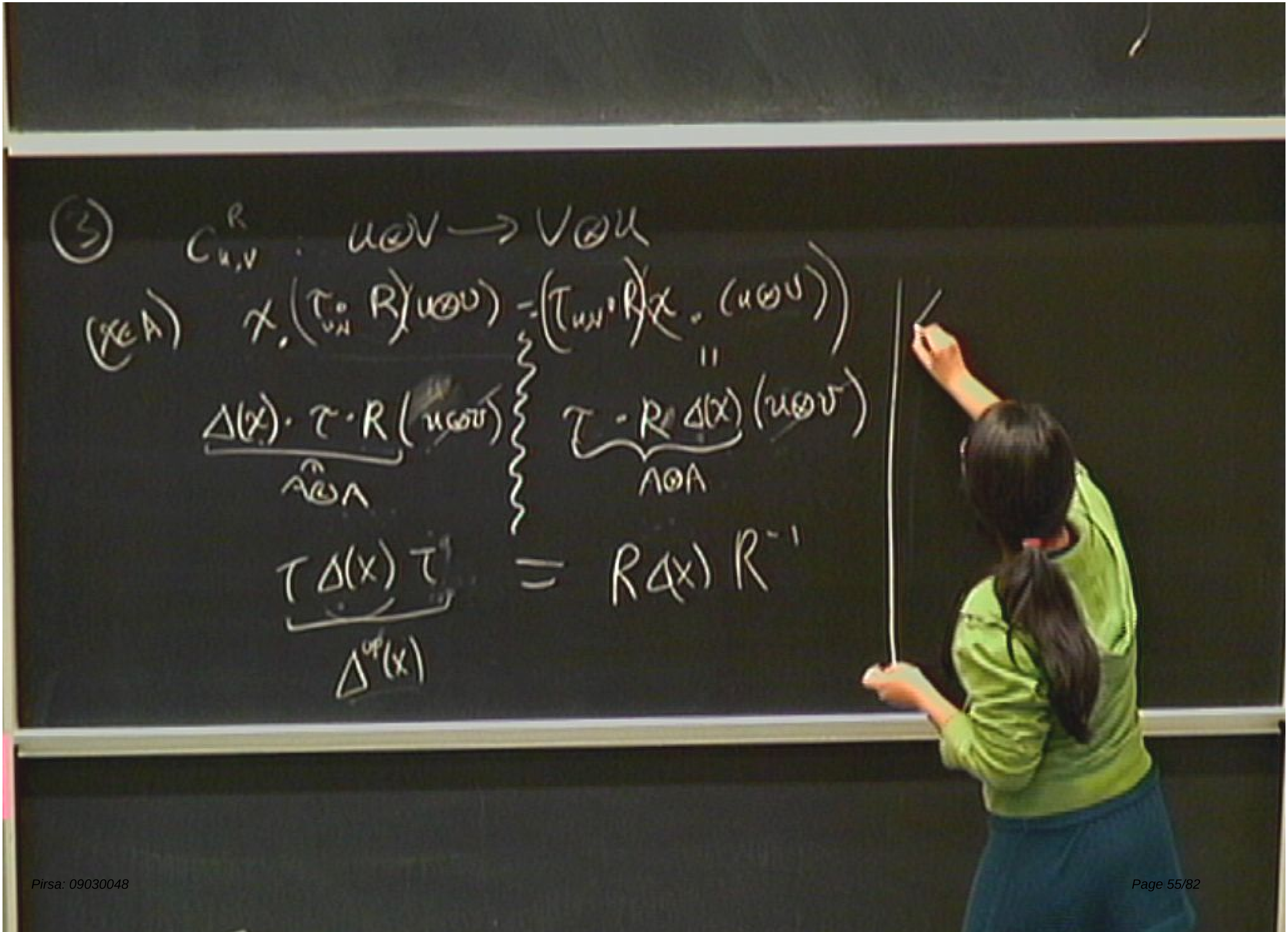

(5) $c^{R}_{u,v} : u\otimes v \to v\otimes u$
$(x\in A)$ $x.(\tau_{v,u}\cdot R)(u\otimes v) = (\tau_{u,v}\cdot R)(x.(u\otimes v))$
$\Delta(x)\cdot\tau\cdot R(u\otimes v)$
$\tau\cdot R\,\Delta(x)(u\otimes v)$
$A\otimes A$
$A\otimes A$
$\tau\,\Delta(x)\,\tau^{-1} = R\,\Delta(x)\,R$

(3) $c^R_{U,V} : U \otimes V \longrightarrow V \otimes U$

$(x \in A)$ $\quad x \cdot (\tau_{U,V} \circ R)(u \otimes v) = (\tau_{U,V} \circ R)(x \cdot (u \otimes v))$

$$\underbrace{\Delta(x) \cdot \tau \cdot R}_{A \hat{\otimes} A}(u \otimes v) \quad \wr \quad \tau \cdot \underbrace{R\, \Delta(x)}_{A \otimes A}(u \otimes v)$$

$$\tau \Delta(x) \tau^{-1} = R \Delta(x) R^{-1}$$


(3) $C^{R}_{u,v} : u\otimes v \to v\otimes u$
$(x\in A)$ $x.(\tau_{u,v}\cdot R)(u\otimes v) = (\tau_{u,v}\cdot R)(x.(u\otimes v))$
$\underbrace{\Delta(x)\cdot\tau\cdot R}_{\hat{A}\otimes A}(u\otimes v)$
$\tau\cdot \underbrace{R\,\Delta(x)}_{A\otimes A}(u\otimes v)$
$\underbrace{\tau\,\Delta(x)\,\tau^{-1}}_{\Delta^{op}(x)} = R\,\Delta(x)\,R^{-1}$

③ $c^R_{u,v}: u\otimes v \longrightarrow v\otimes u$

$(x\in A)$ $x_\cdot\left((\tau_{v,u}\circ R)(u\otimes v)\right) = (\tau_{u,v}\circ R)\left(x_\cdot(u\otimes v)\right)$

$\underbrace{\Delta(x)\cdot\tau\cdot R}_{A\hat{\otimes}A}(u\otimes v)$ $\quad$ $\underbrace{\tau\cdot R\,\Delta(x)}_{A\otimes A}(u\otimes v)$

$\underbrace{\tau\,\Delta(x)}_{\Delta}\ ($ $\quad$ $R\,\Delta(x)\,R^{-1}$

$\Delta^{op}(x)(u\otimes v)$

$= \sum_{(x)} x''\otimes x'\,(u\otimes v)$

$= \sum_{(x)} x''(u)\otimes x'(v)$

$= \tau_{u,v}$

③ $C^R_{U,V} : U \otimes V \to V \otimes U$

$(x \in A)$ $\quad x \cdot (\tau_{V,U} \circ R)(u \otimes v) = (\tau_{U,V} \circ R)(x \cdot (u \otimes v))$

$\underbrace{\Delta(x) \cdot \tau \cdot R}_{A \hat{\otimes} A}(u \otimes v) \quad \| \quad \underbrace{\tau \cdot R\, \Delta(x)}_{A \otimes A}(u \otimes v)$

$\underbrace{\tau \Delta(x) \tau}_{\Delta^{op}(x)} = R \Delta(x) R^{-1}$

$\Delta^{op}(x)(u \otimes v)$

$= \sum_{(x)} x'' \otimes x' (u \otimes v)$

$= \sum_{(x)} x''(u) \otimes x'(v)$

$= \tau_{V,U} \cdot \Delta(x) \cdot \tau_{U,V} (u \otimes v)$

(5) $c^R_{u,v} : u \otimes V \to V \otimes u$

$(x \in A)$ $x \cdot (\tau_{u,v} \circ R)(u \otimes v) = (\tau_{u,v} \circ R)(x \cdot (u \otimes v))$

$\underbrace{\Delta(x) \cdot \tau \cdot R}_{\hat{A} \otimes A}(u \otimes v)$ $\quad$ $\underbrace{\tau \cdot R \, \Delta(x)}_{A \otimes A}(u \otimes v)$

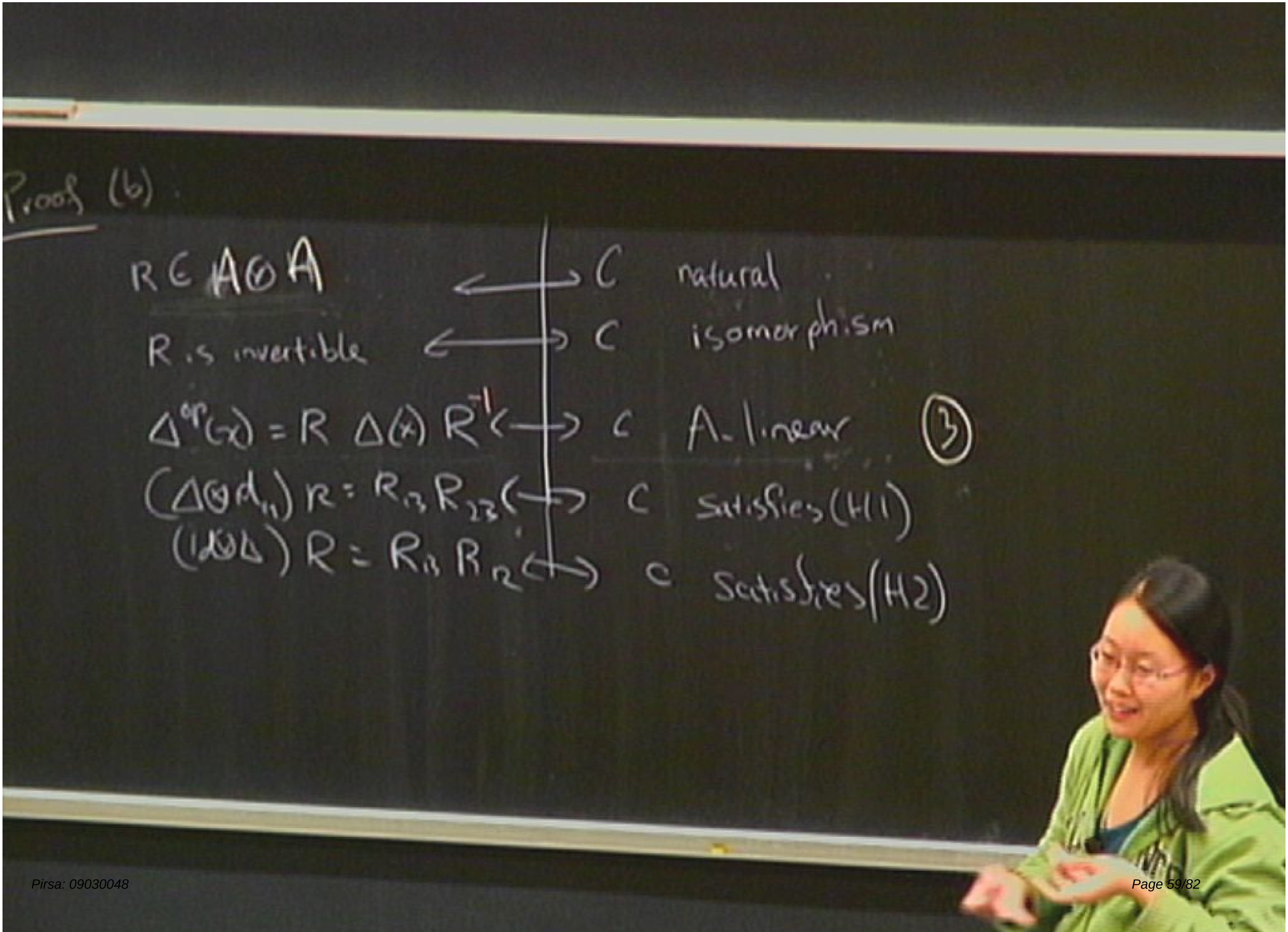
$\underbrace{\tau_{v,u} \Delta(x) \tau_{u,v}}_{\Delta^{op}(x)} = R \Delta(x) R^{-1}$

$\Delta^{op}(x)(u \otimes v)$

$= \sum_{(x)} x'' \otimes x' \, (u \otimes v)$

$= \sum_{(x)} x''(u) \otimes x'(v)$

$= \tau_{v,u} \circ \Delta(x) \circ \tau_{u,v}(u \otimes v)$


Proof (b)
$R \in A \otimes A \longleftrightarrow C$ natural
R is invertible $\longleftrightarrow C$ isomorphism
$\Delta^{op}(x) = R\,\Delta(x)\,R^{-1} \longleftrightarrow C$ A-linear
3
$(\Delta \otimes id)R = R_{13}R_{23} \longleftrightarrow C$ satisfies (H1)
$(1 \otimes \Delta)R = R_{13}R_{12} \longleftrightarrow C$ satisfies (H2)

Proof (b):

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R is invertible $\longleftrightarrow C$ isomorphism

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$(1 \otimes \Delta) R = R_{13} R_{12} \longleftrightarrow C$ satisfies (H2)

$$\begin{array}{ccc} C_{U,V} : U \otimes V & \longrightarrow & V \otimes U \\ \downarrow f \otimes g & & \downarrow g \otimes f \end{array}$$

Proof (b):

$$
\begin{tikzcd}
U\otimes V \arrow[r, "c_{U,V}"] \arrow[d, "f\otimes g"'] & V\otimes U \arrow[d, "g\otimes f"] \\
{} & {}
\end{tikzcd}
$$

$R \in A\otimes A \longleftrightarrow c$ natural

R is invertible $\longleftrightarrow c$ isomorphism

$\Delta^{op}(x) = R\,\Delta(x)\,R^{-1} \longleftrightarrow c$ A-linear (3)

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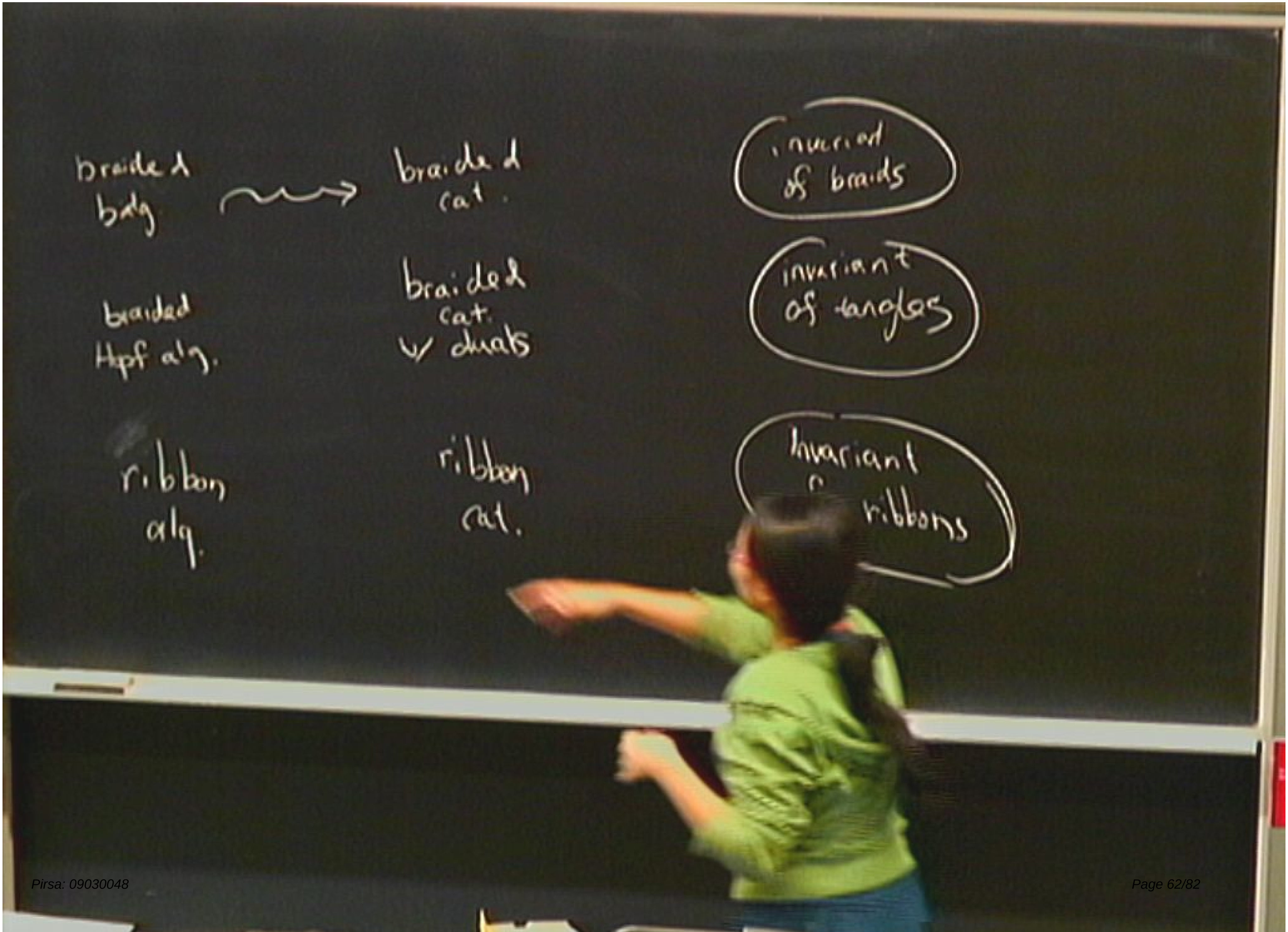
braided
b'alg
braided
cat.
invariant
of braids
braided
Hopf alg.
braided
cat.
w/ duals
invariant
of tangles
ribbon
alg.
ribbon
cat.
invariant
ribbons

braided
balg. ⇝ braided
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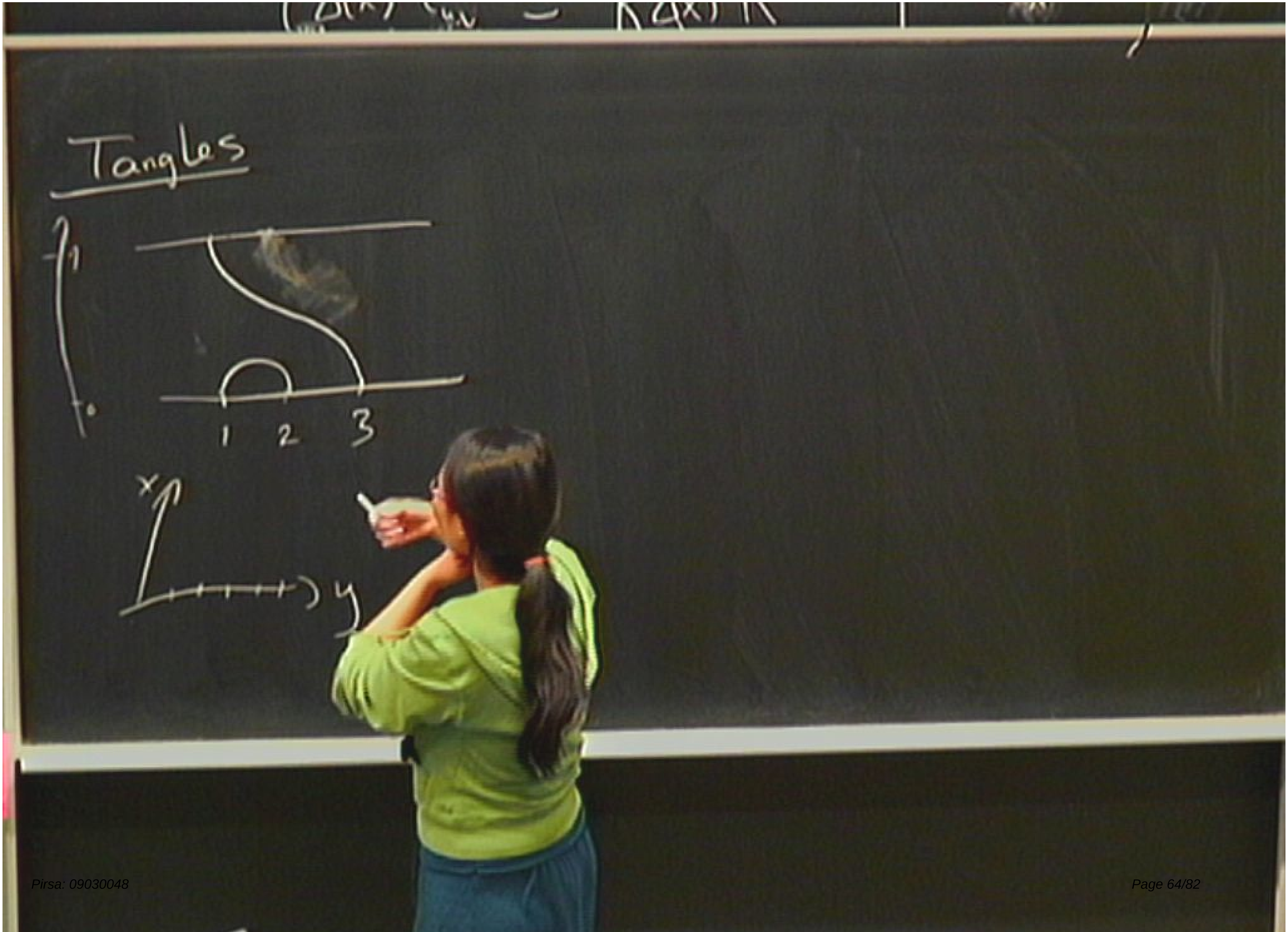
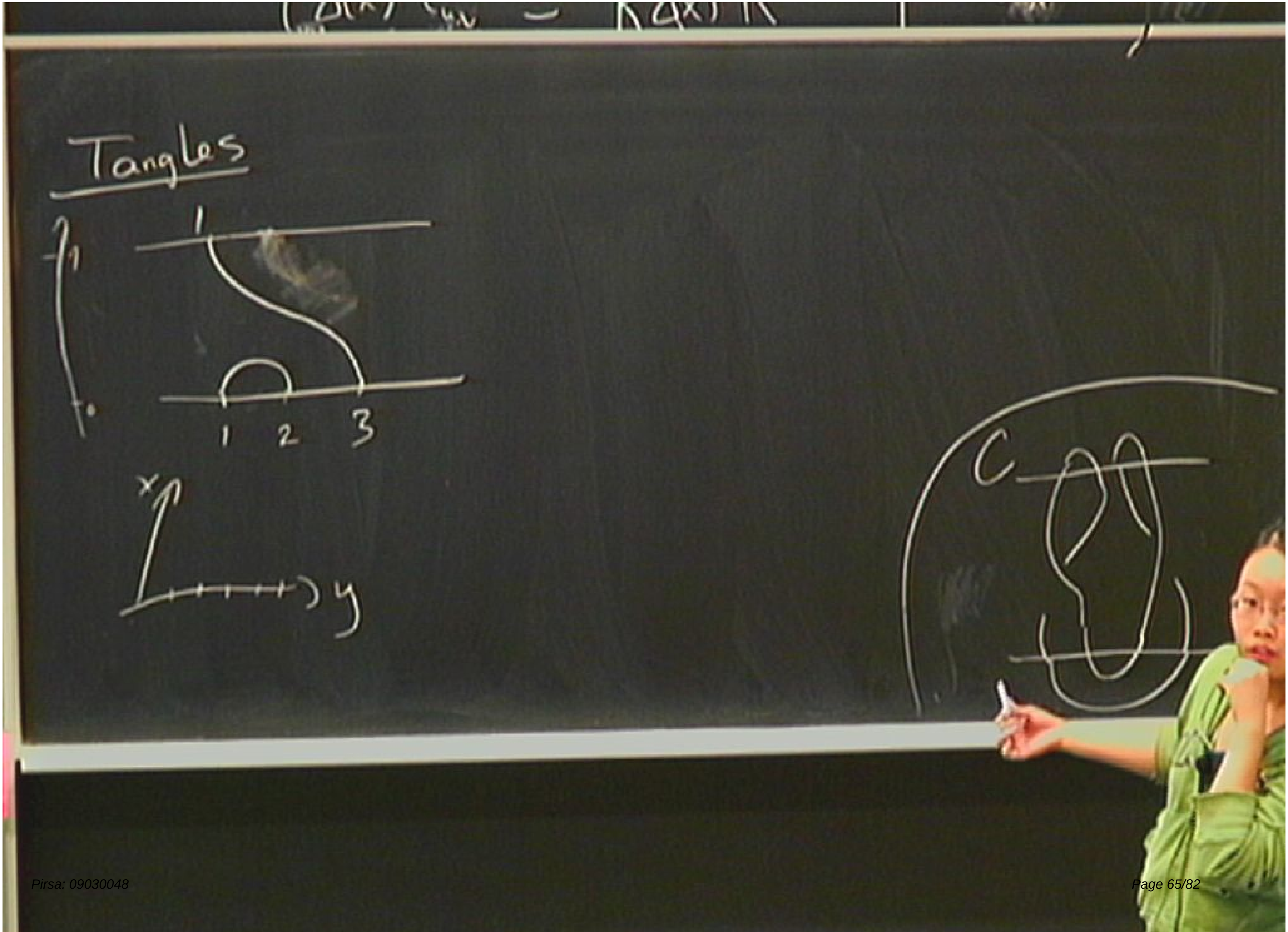
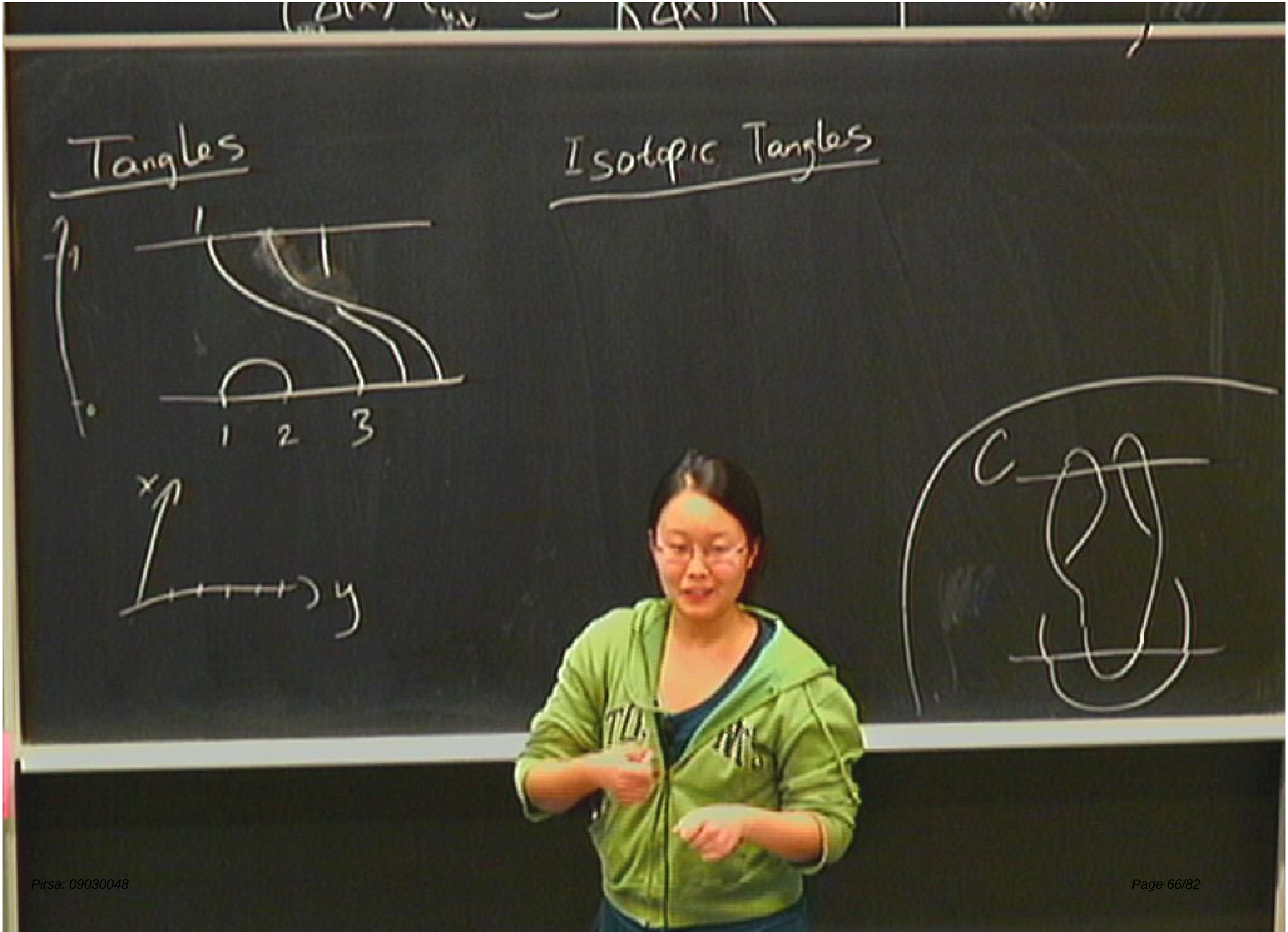
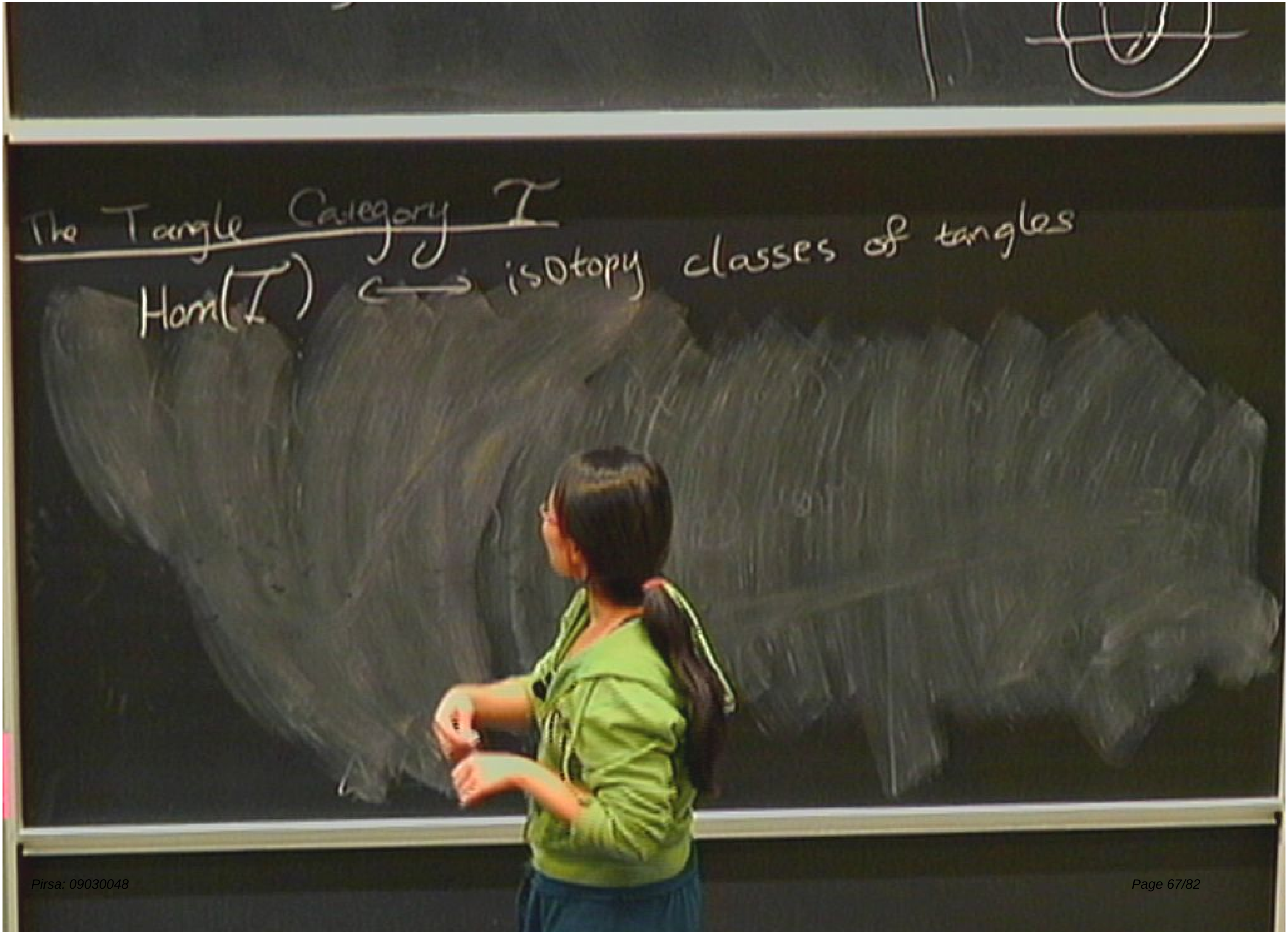
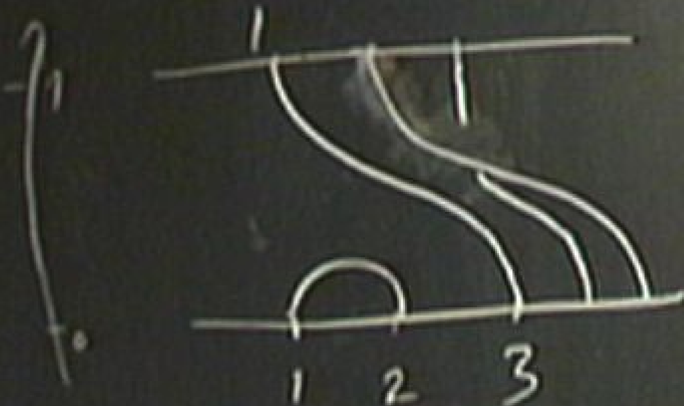

Tangles
1
0
1 2 3
x
y

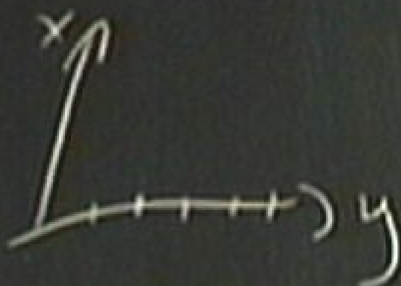

Tangles
1
1 2 3
x
y
C


Tangles
Isotopic Tangles
1 2 3
x
y
C


The Tangle Category T
Hom(T) ⟷ isotopy classes of tangles

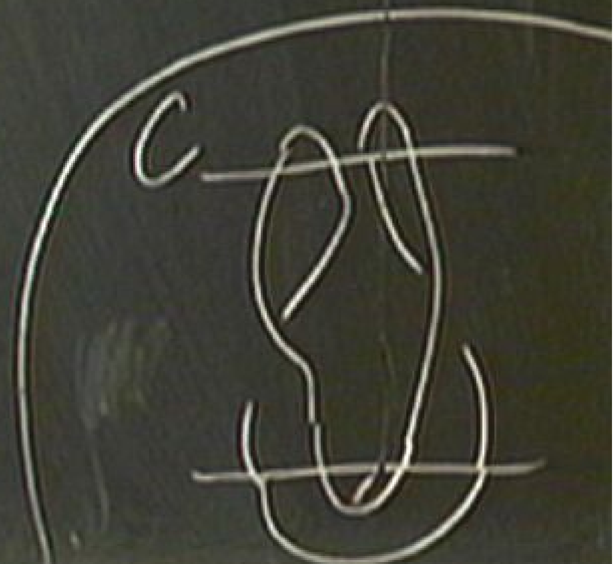
Tangles





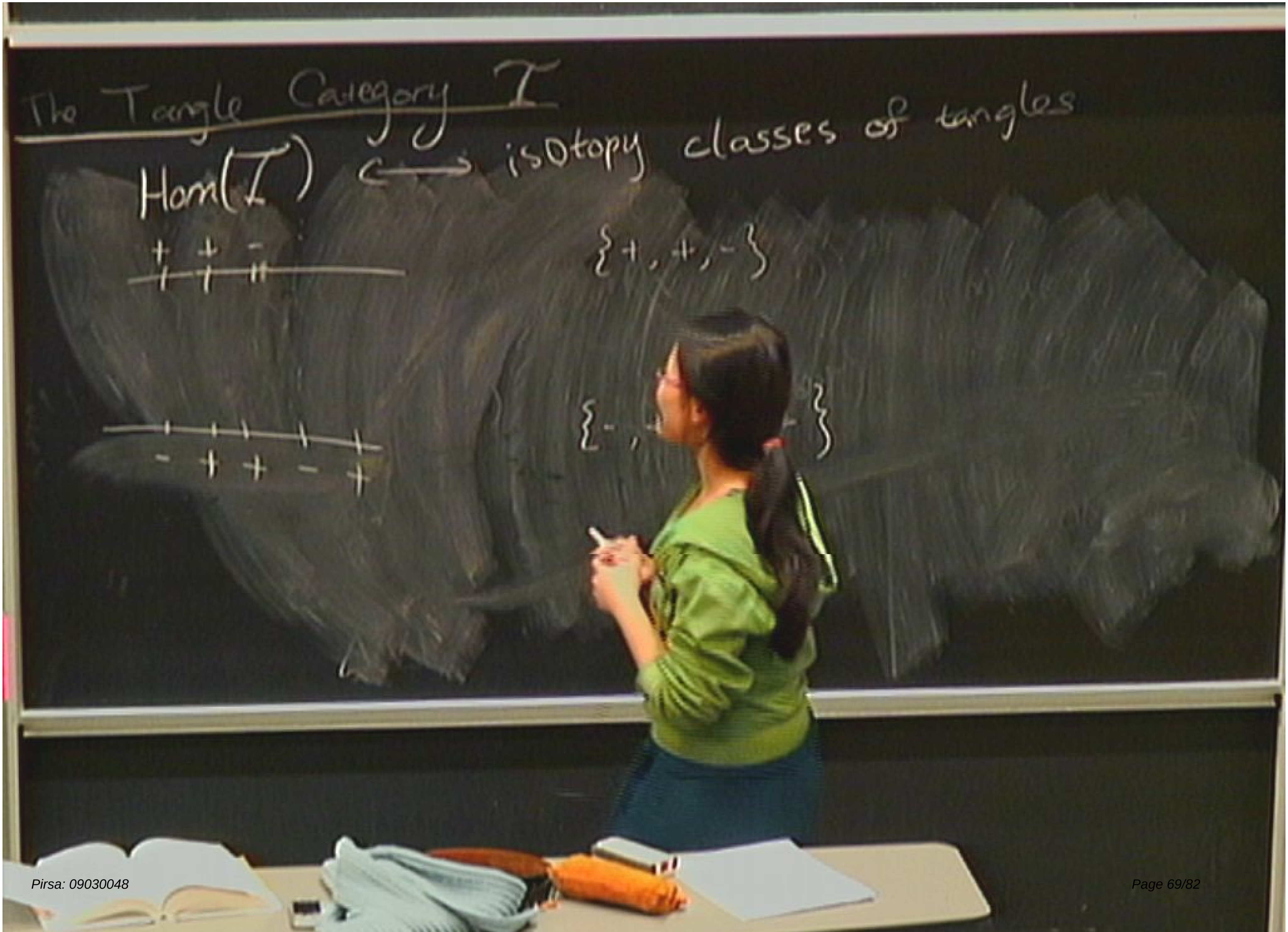
Isotopic Tangles

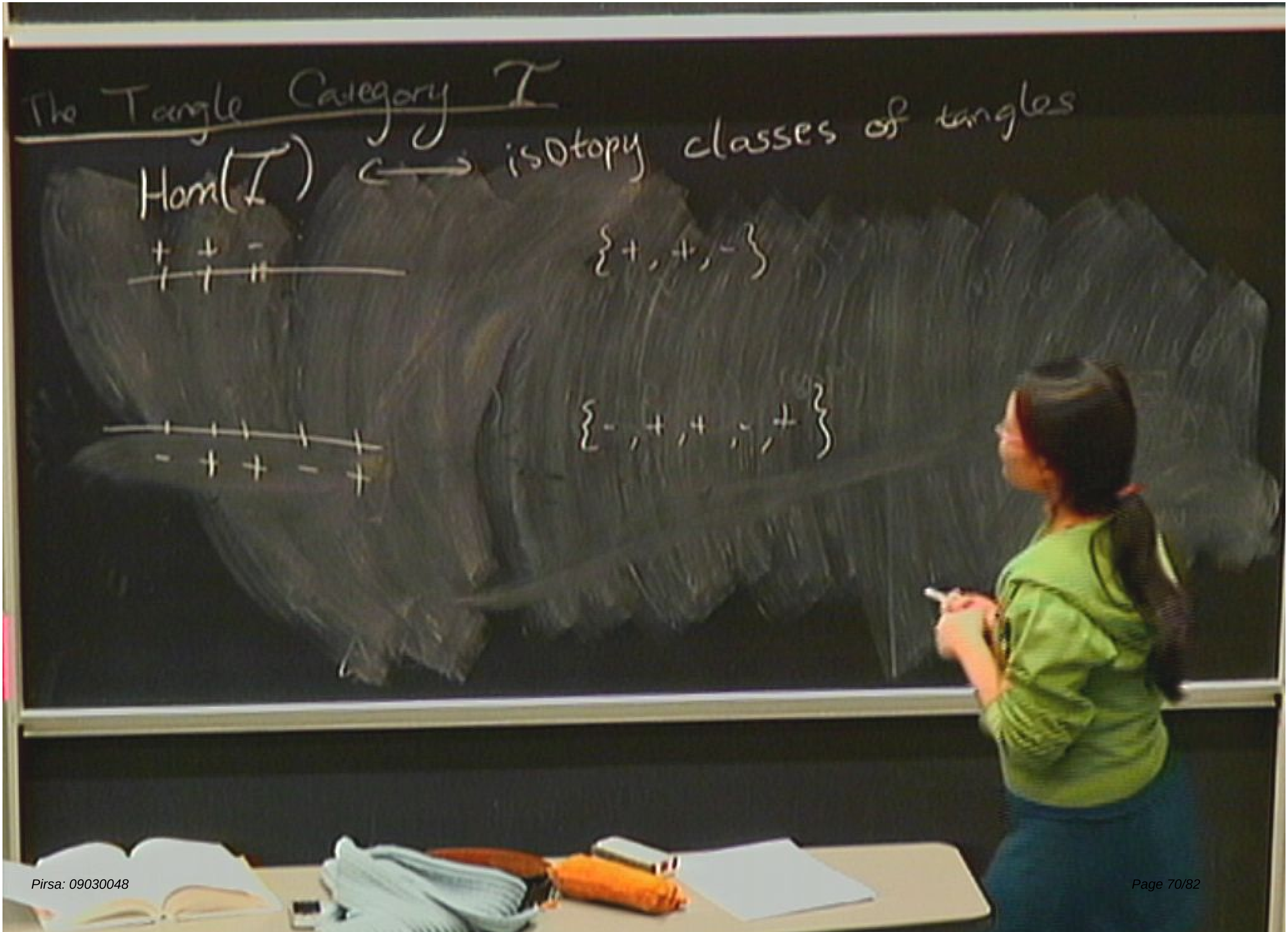
intuitive

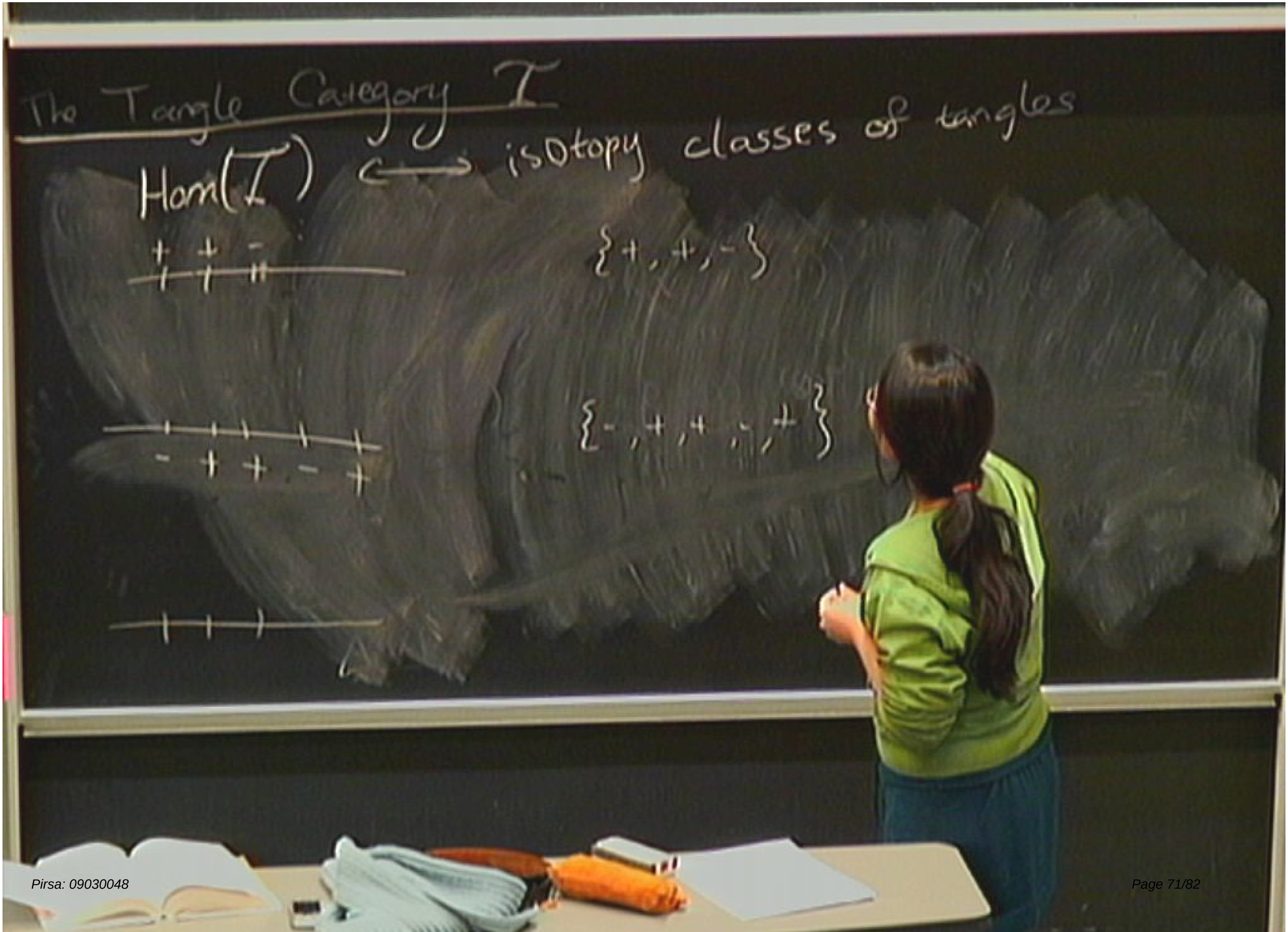


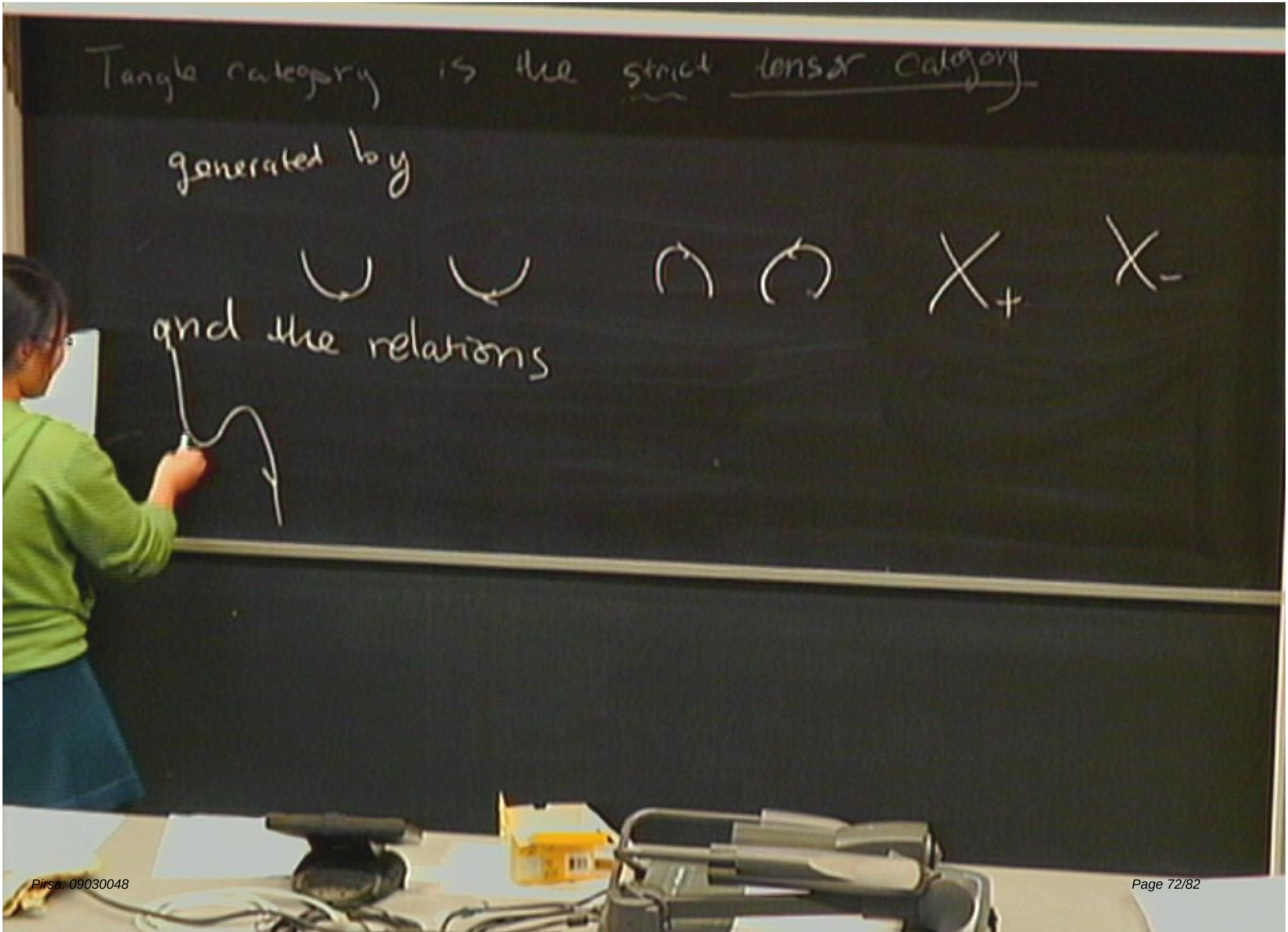
The Tangle Category $\mathcal{T}$

$\mathrm{Hom}(\mathcal{T}) \longleftrightarrow$ isotopy classes of tangles


The Tangle Category T
Hom(T) ⟷ isotopy classes of tangles
{+, +, -}


The Tangle Category T
Hom(T) ⟷ isotopy classes of tangles
+ + −
{+, +, −}
− + + − +
{−, +, +, −, +}


The Tangle Category T
Hom(T) ⟷ isotopy classes of tangles
{+, +, -}
{-, +, +, -, +}

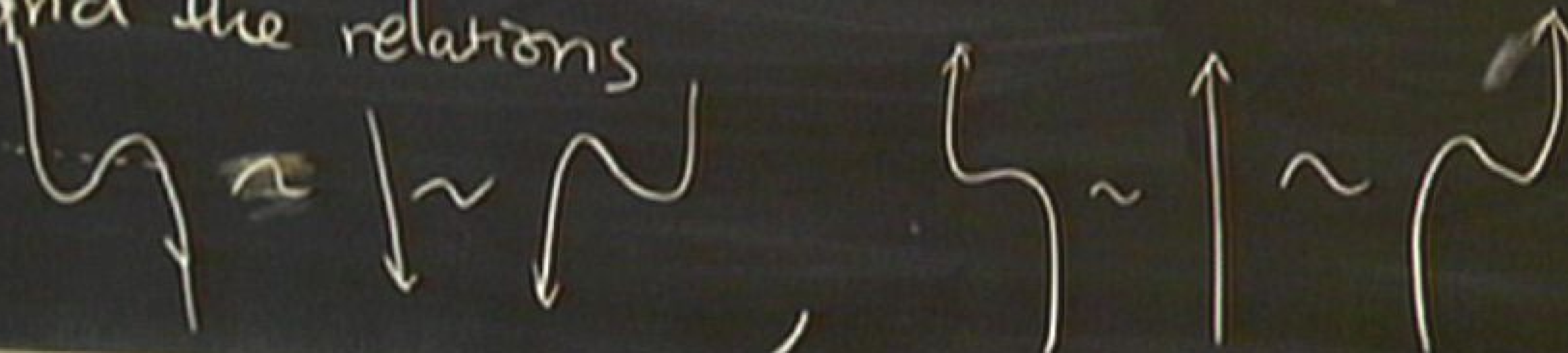

Tangle category is the strict tensor category
generated by
X+ X-
and the relations

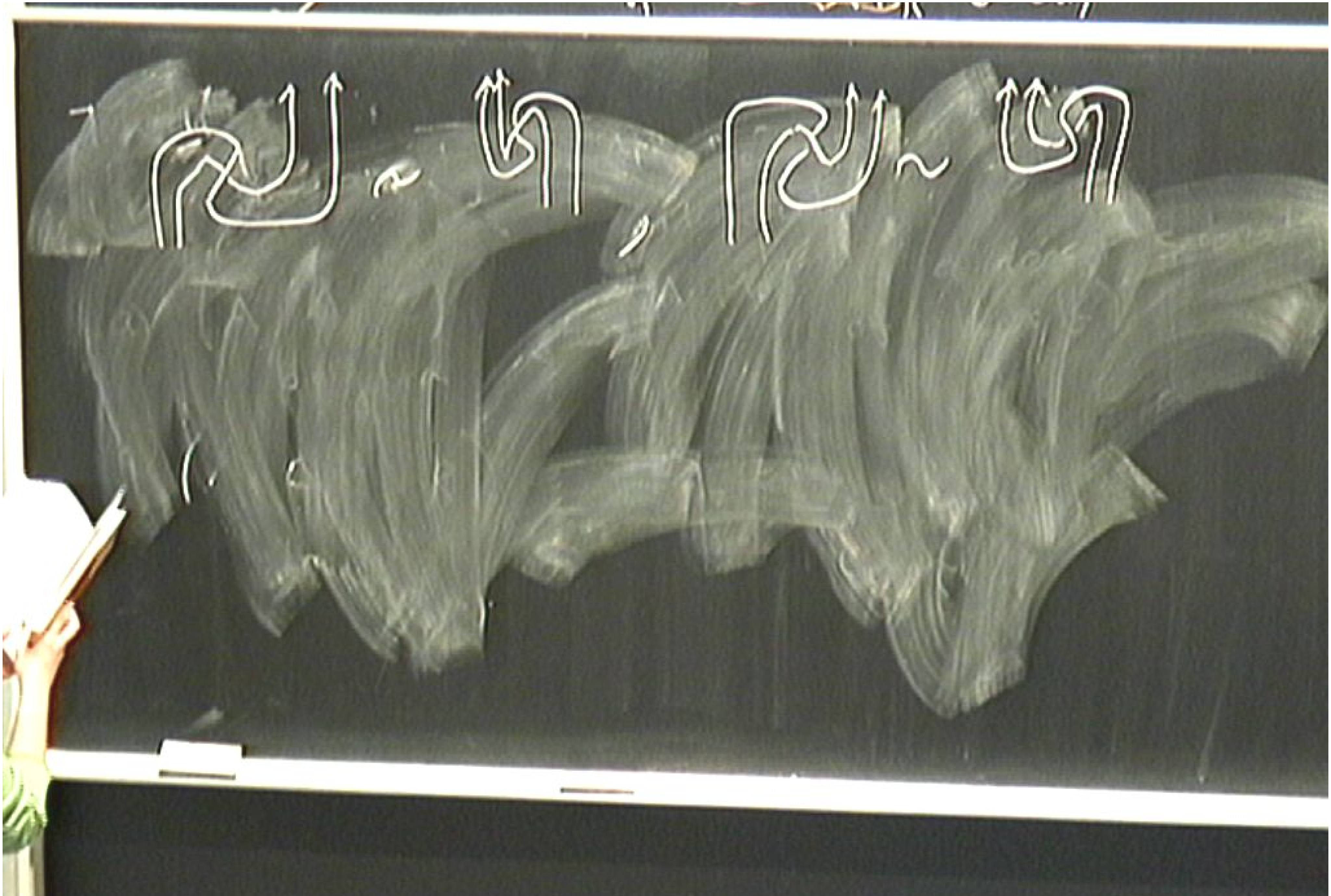
Tangle category is the strict tensor category

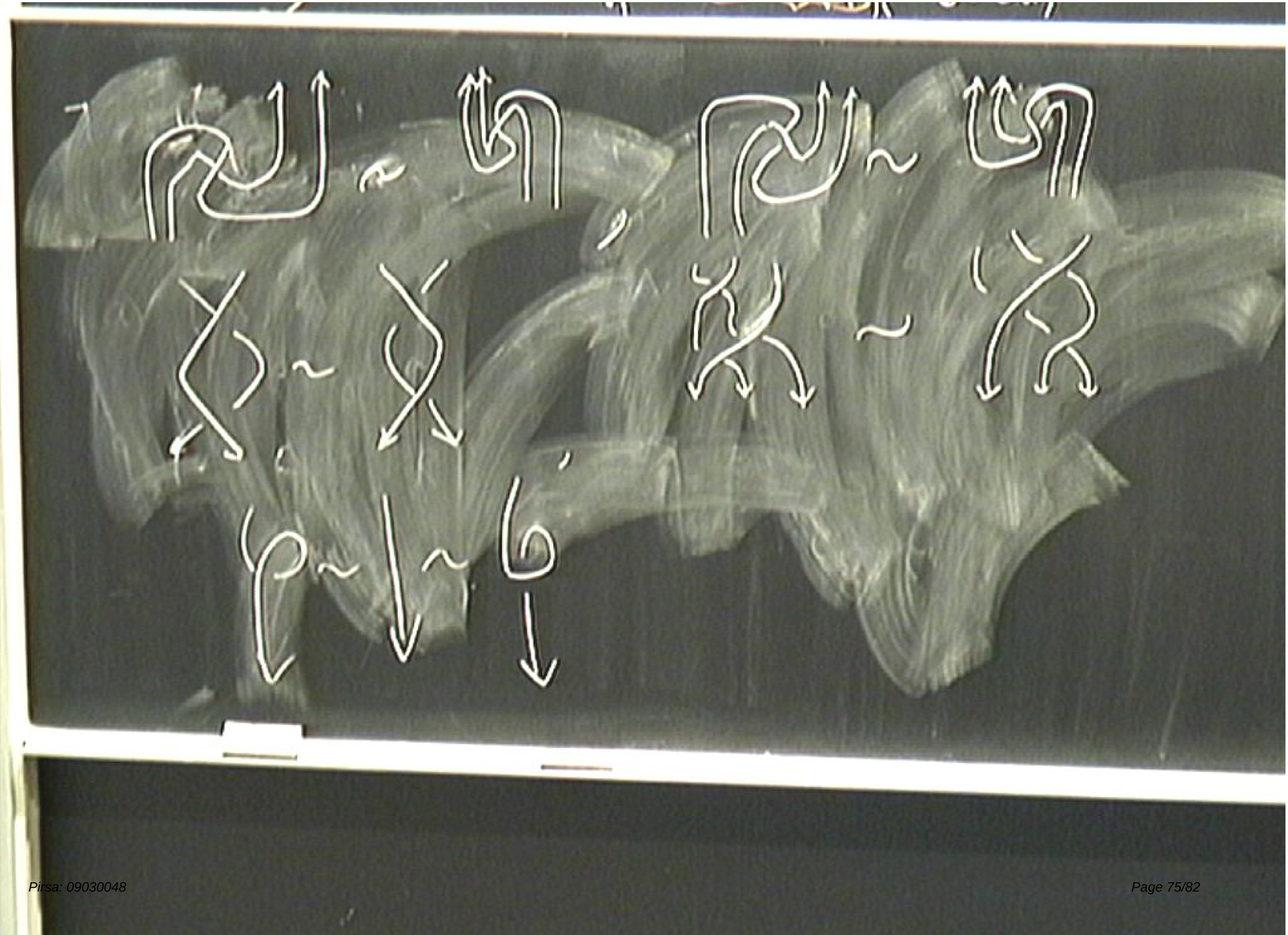
generated by

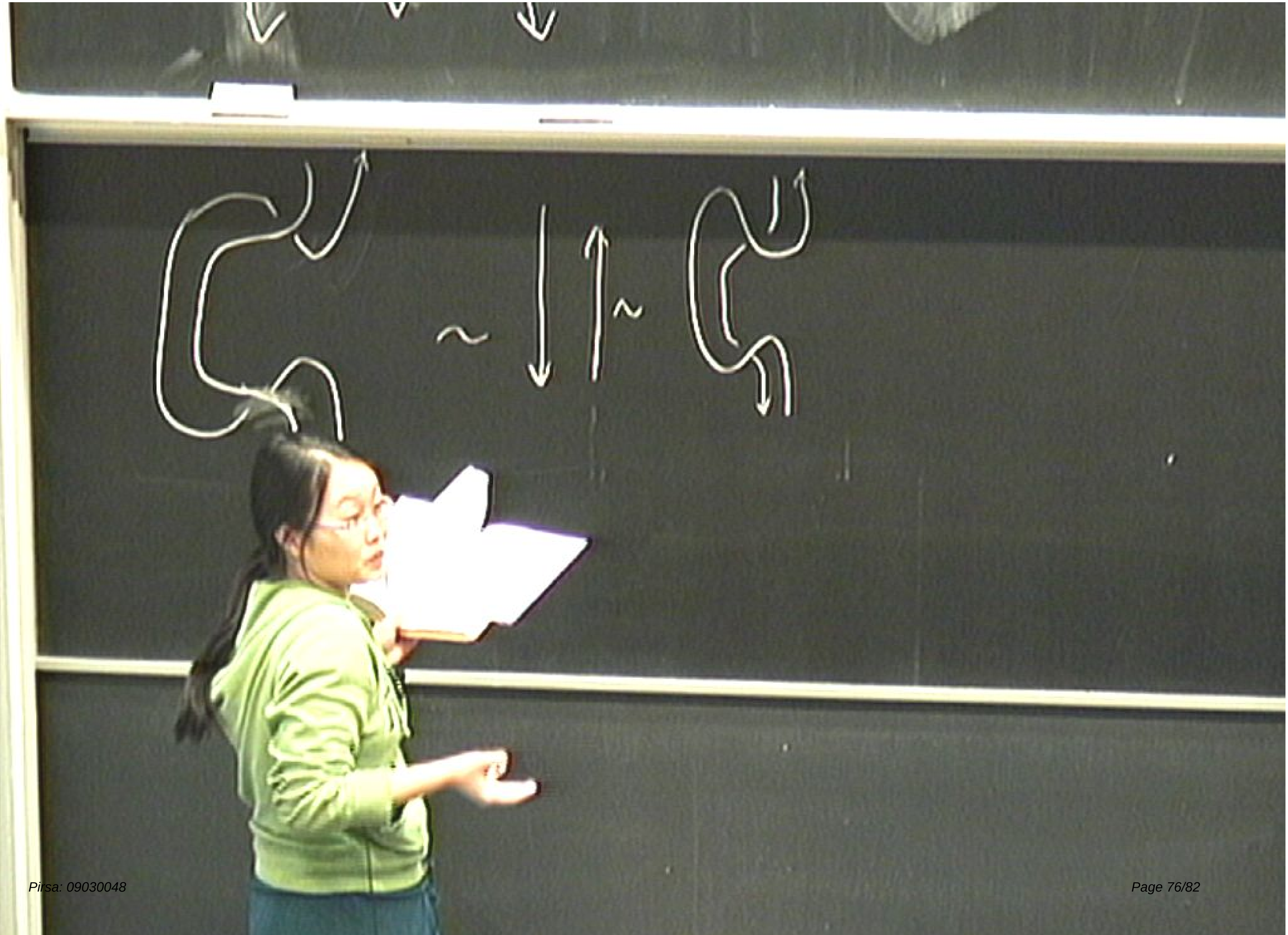
X_+ X_-

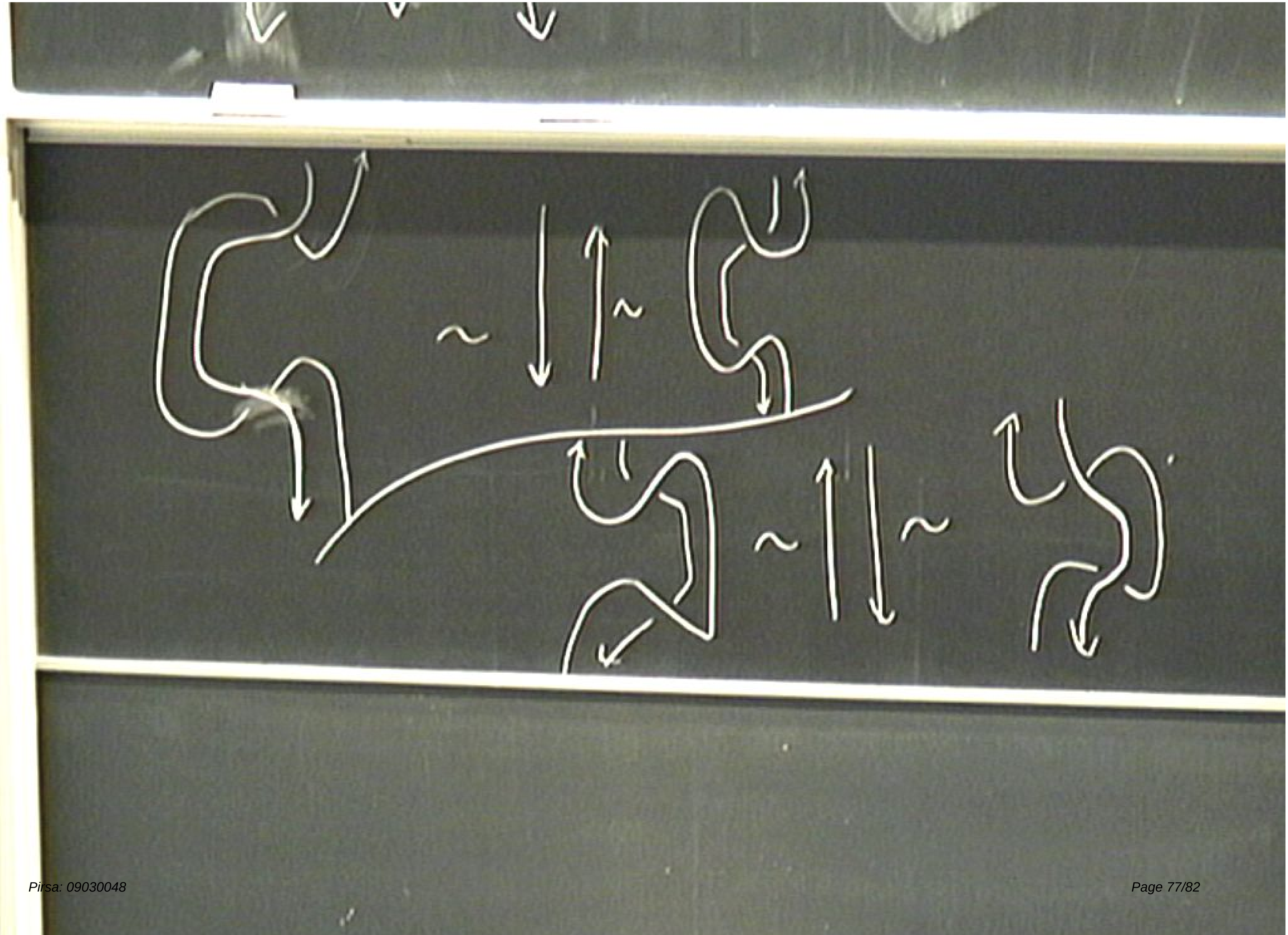
and the relations

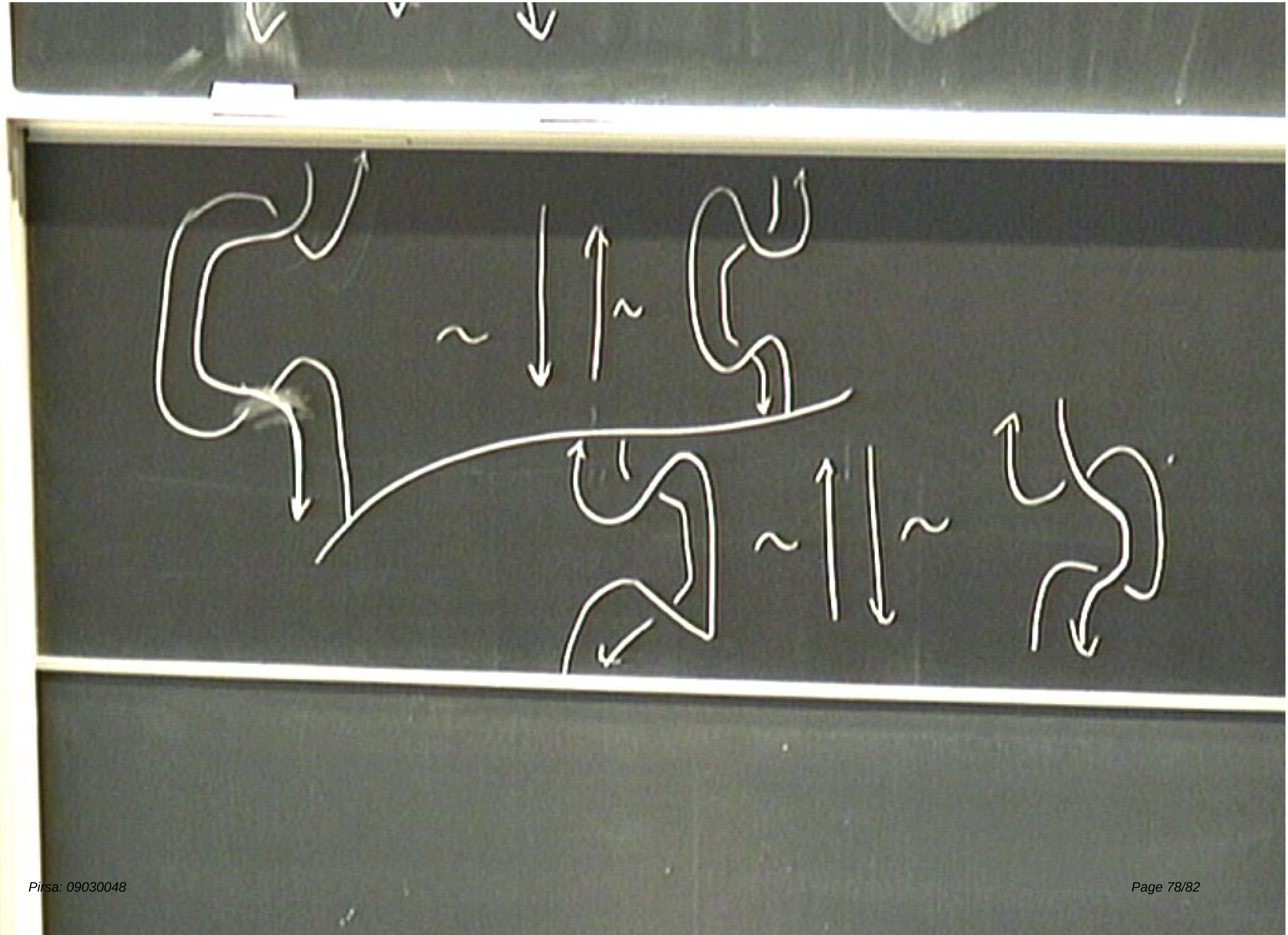






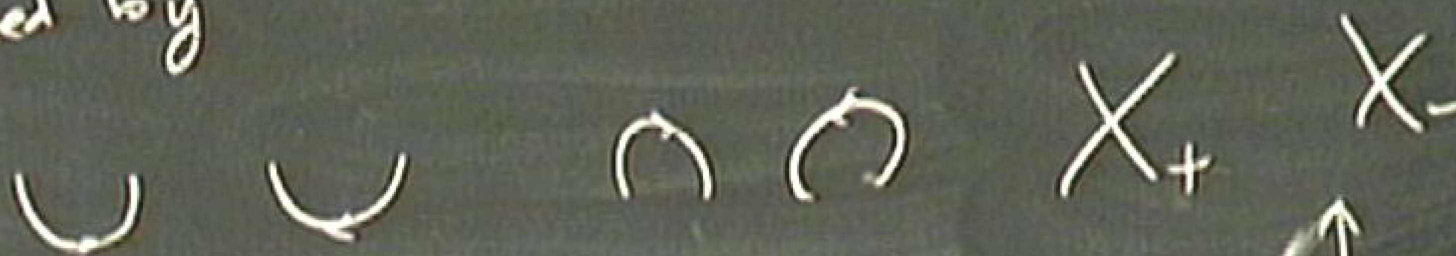




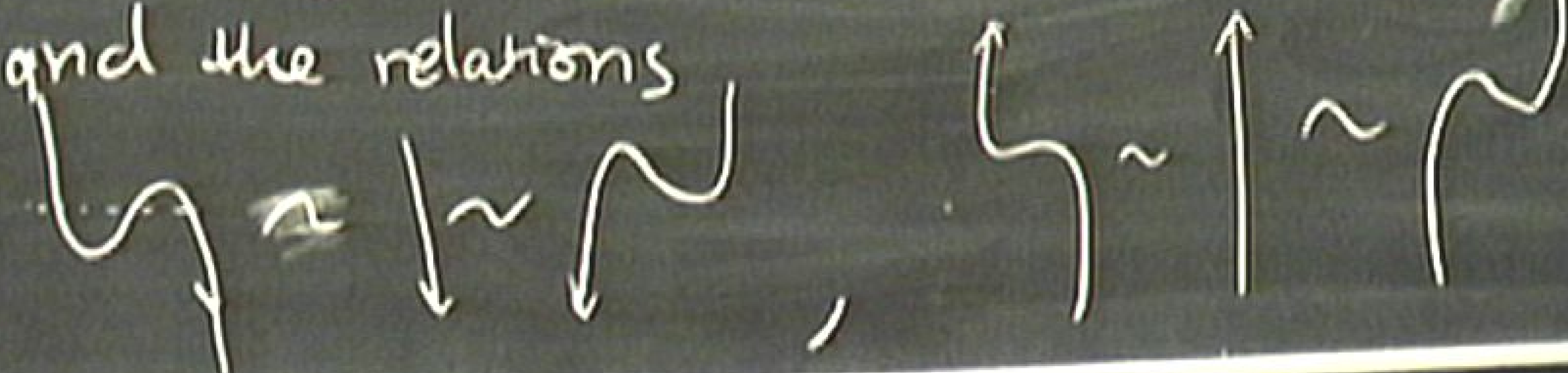


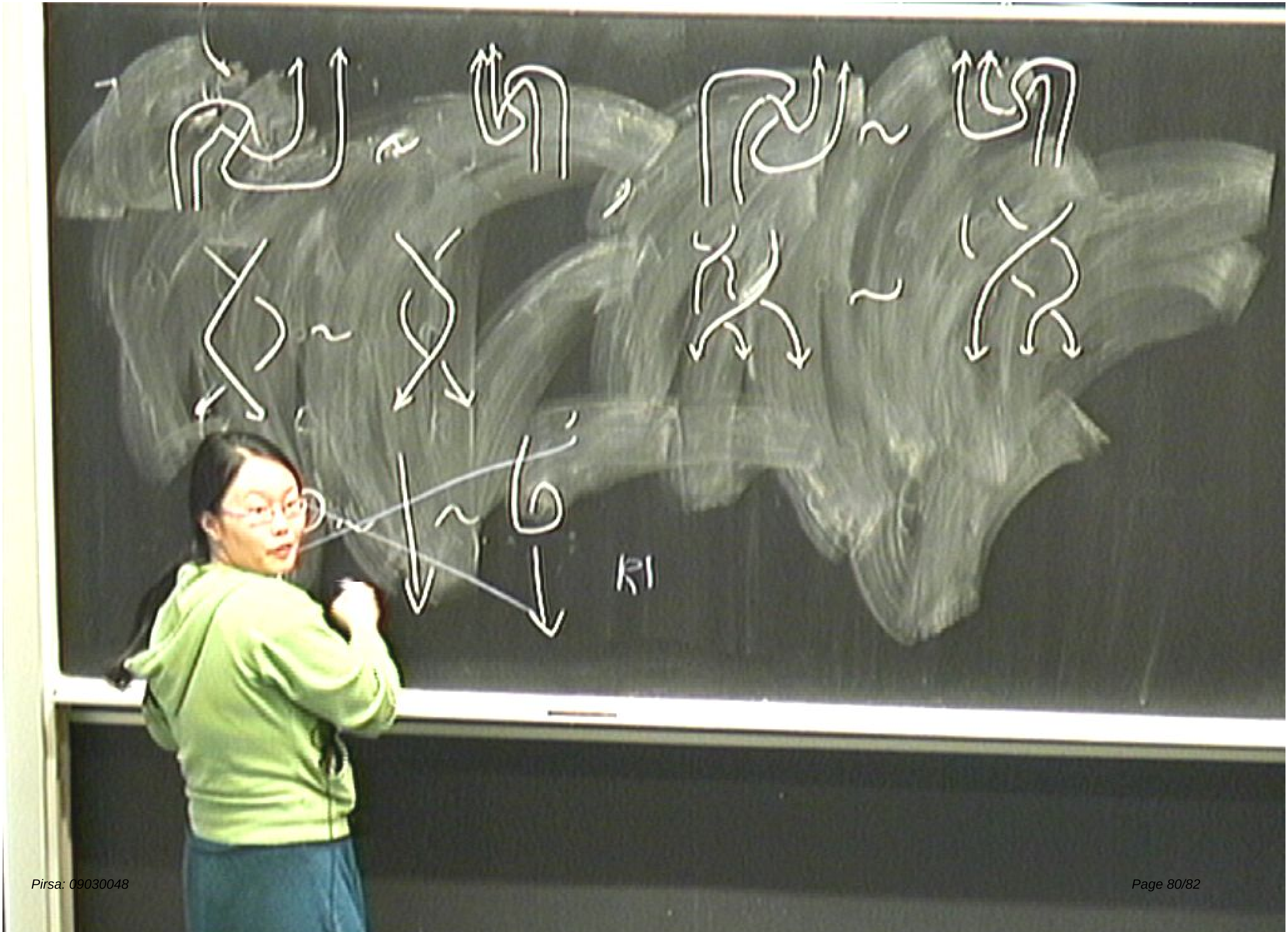
Tangle category is the strict tensor category

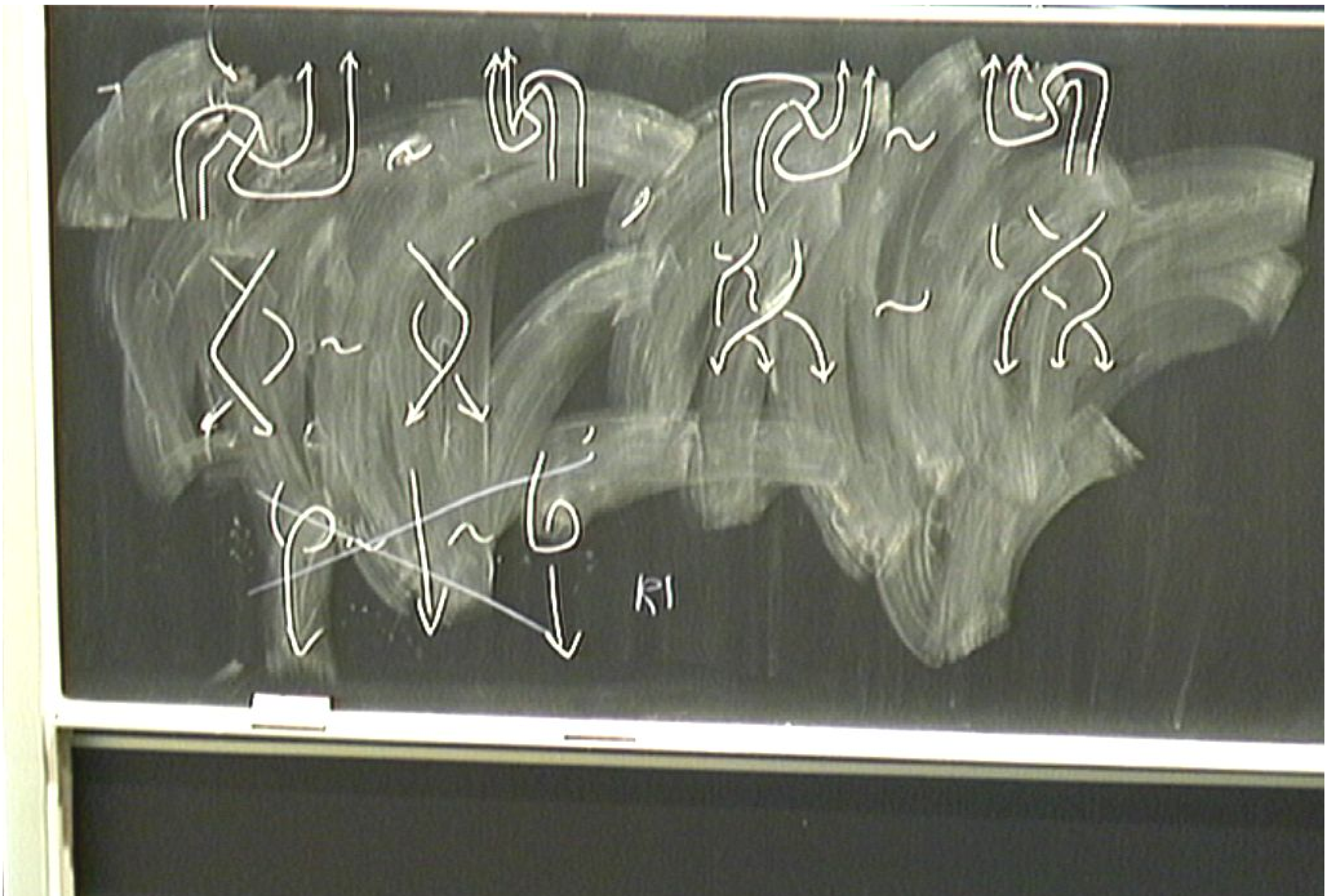
generated by

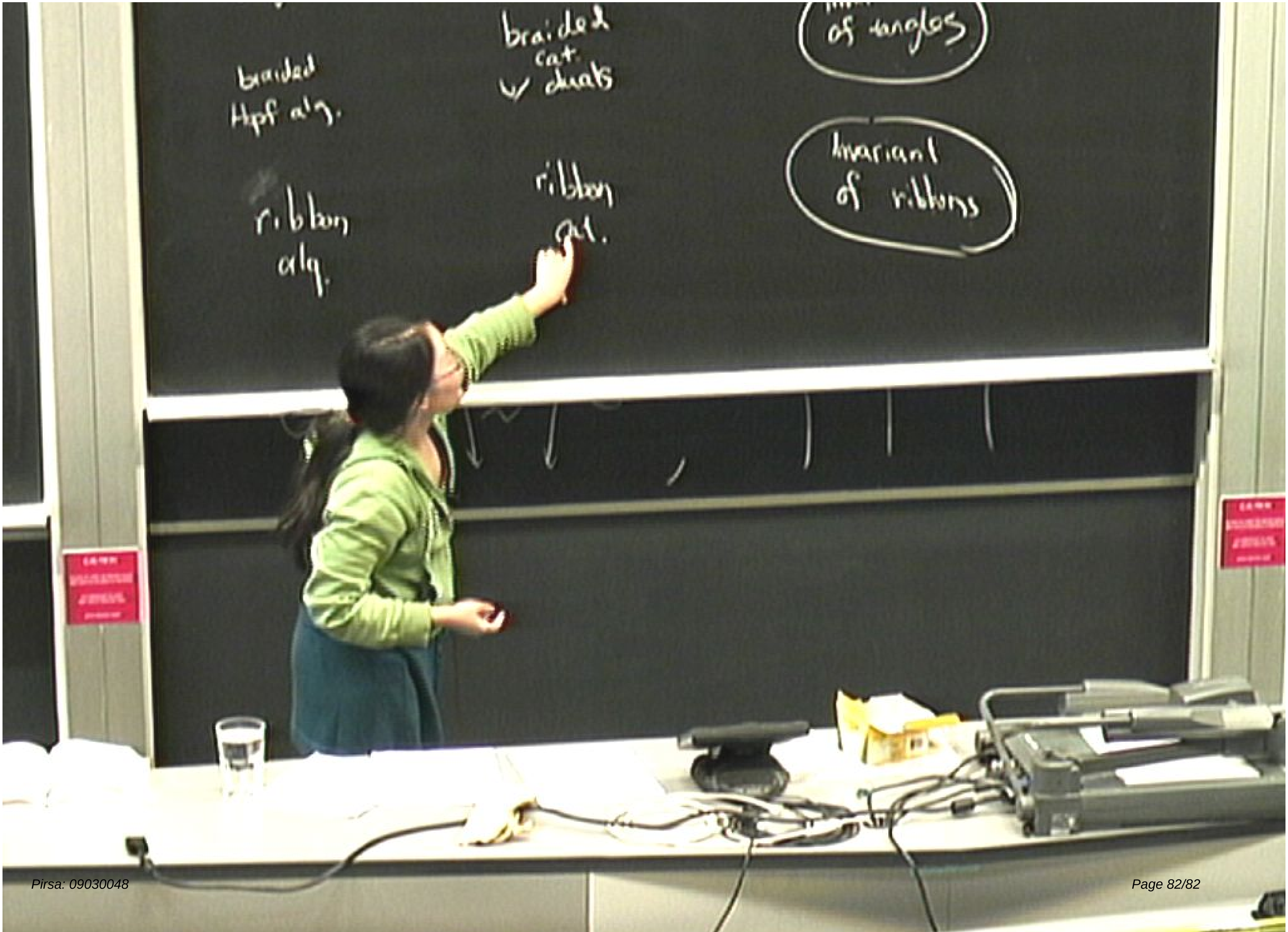


and the relations




R1


R1


braided
Hopf alg.
ribbon
alg.
braided
cat.
w/ duals
ribbon
cat.
of tangles
Invariant
of ribbons