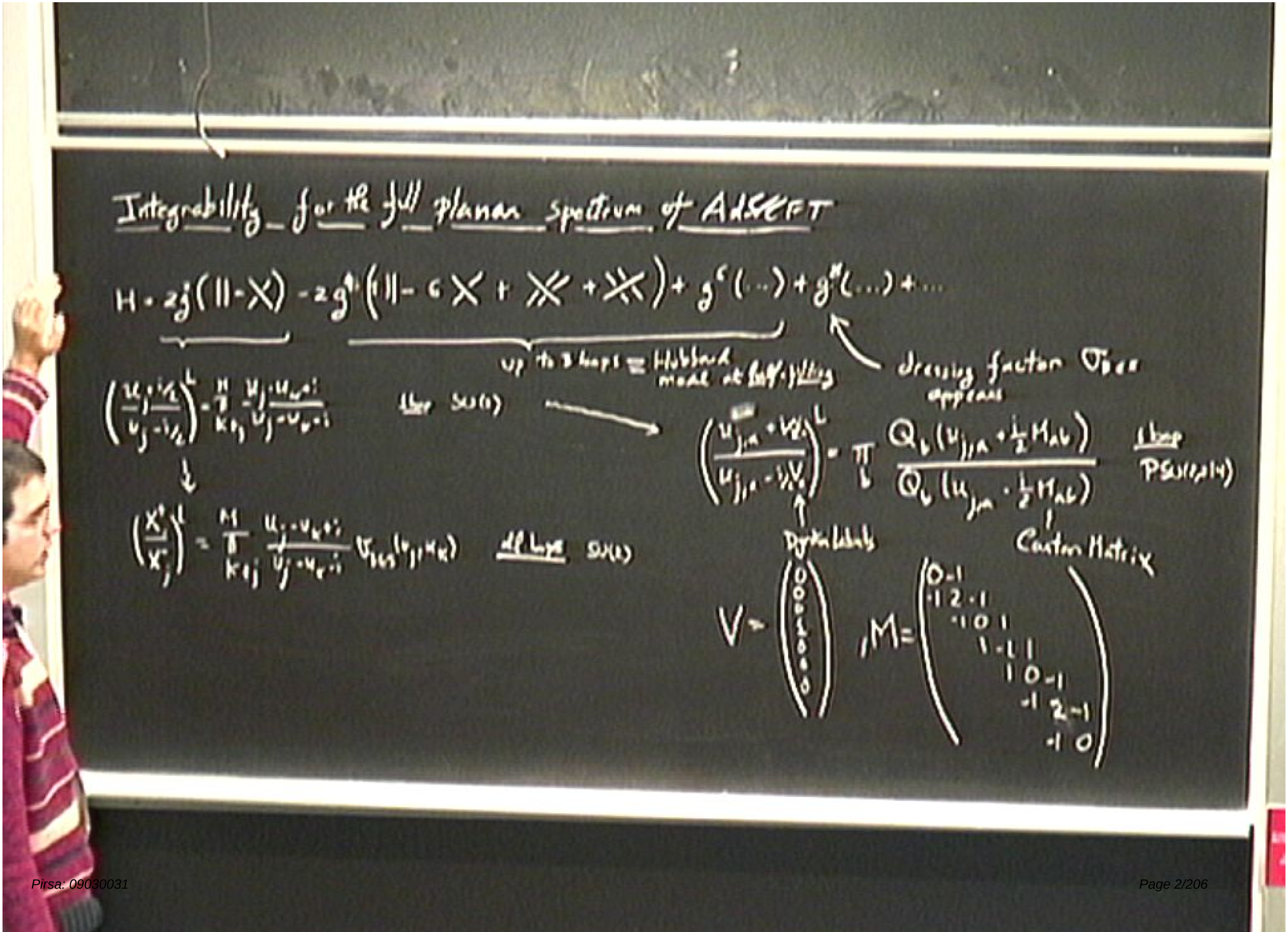
Title: Integrability and planar AdS/CFT - Lecture 3

Date: Mar 03, 2009 11:00 AM

URL: http://pirsa.org/09030031

Abstract: Lecture 3 - The physics of Luscher corrections - Quantum Thermodynamic Bethe Ansatz and Classical Hirota Dynamics - The exact spectrum of planar AdS/CFT



Integrability for the full planar spectrum of AdS/CFT

$$H = \underbrace{2g^2(|| - \times)}_{} \underbrace{- 2g^4(|| - 6\times + \times\!\!| + |\!\!\times) + g^6(\ldots)}_{} + g^8(\ldots) + \ldots$$

up to 3 loops ≡ Hubbard model at half-filling

dressing factor σ_{BES} appears

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} \qquad \text{1 loop } SU(2)$$

$$\downarrow$$

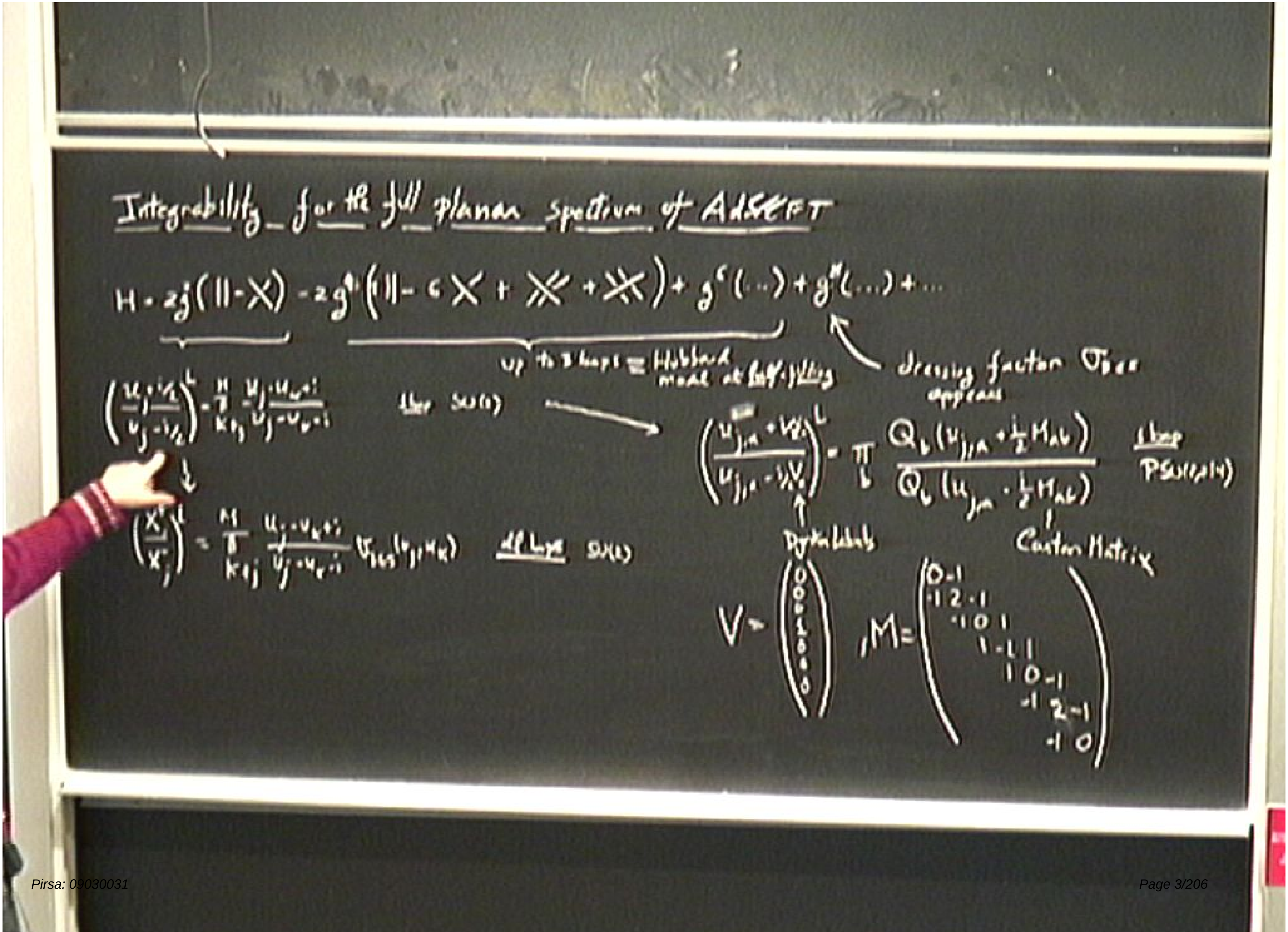
$$\left(\frac{x_j^+}{x_j^-}\right)^L = \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}\,\sigma_{BES}(u_j, u_k) \qquad \underline{\text{All loops}}\ SU(2)$$

$$\longrightarrow \left(\frac{u_{j,a} + \frac{i}{2}V_a}{u_{j,a} - \frac{i}{2}V_a}\right)^L = \prod_b \frac{Q_b(u_{j,a} + \frac{i}{2}M_{ab})}{Q_b(u_{j,a} - \frac{i}{2}M_{ab})} \qquad \underline{\text{1 loop}}\ PSU(2,2|4)$$

Dynkin labels

Cartan Matrix

$$V = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 0 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & 0 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 0 \end{pmatrix}$$



Integrability for the full planar spectrum of AdS/CFT

$$H = \underbrace{2g^2(\mathbb{1} - \times)}\underbrace{- 2g^4(\mathbb{1} - 6\times + \times + \times) + g^6(\ldots)}_{\text{up to 3 loops} \equiv \text{Hubbard model at half-filling}} + g^8(\ldots) + \ldots$$

dressing factor σ_{BES} appears

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}$$ 1 loop SU(2)

$$\Big\downarrow$$

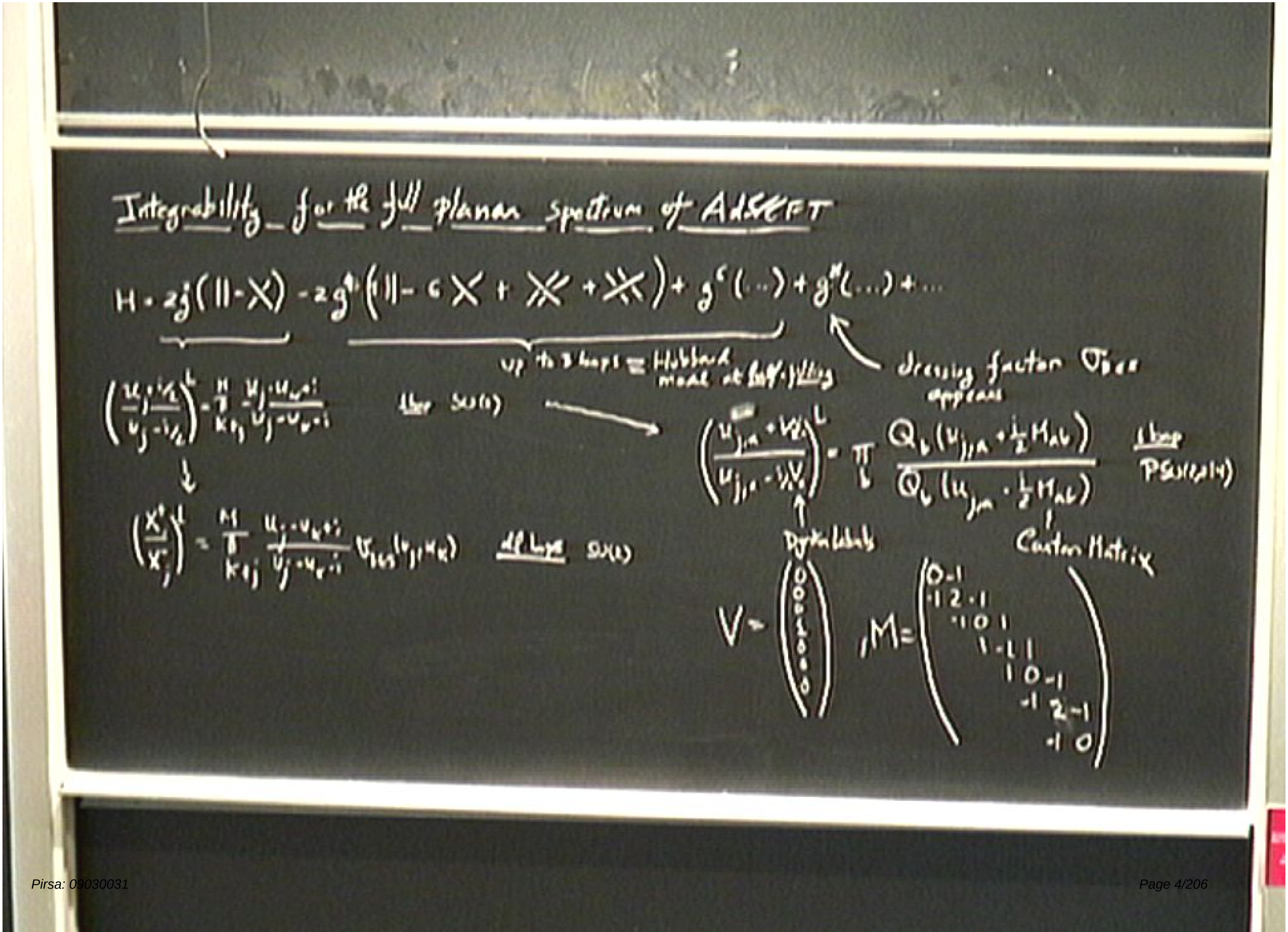
$$\left(\frac{x_j^+}{x_j^-}\right)^L = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}\,\sigma_{BES}^2(u_j, u_k)$$ all loops SU(2)

$$\left(\frac{u_{j,a} + \frac{i}{2}V_a}{u_{j,a} - \frac{i}{2}V_a}\right)^L = \prod_b \frac{Q_b(u_{j,a} + \frac{i}{2}M_{ab})}{Q_b(u_{j,a} - \frac{i}{2}M_{ab})}$$ 1 loop PSU(2,2|4)

Dynkin labels

Cartan Matrix

$$V = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 0 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & 0 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 0 \end{pmatrix}$$



Integrability for the full planar spectrum of AdS/CFT

$$H = \underbrace{2g^2(\mathbb{1} - \times)}\; \underbrace{- 2g^4\left(\mathbb{1} - 6\times + \times\!\times + \times\!\!\times\!\!\times\right) + g^6(\ldots)}_{\text{up to 3 loops} \equiv \text{Hubbard model at half-filling}} + g^8(\ldots) + \ldots$$

dressing factor σ_{BES} appears

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}$$

1 loop SU(2)

$$\downarrow$$

$$\left(\frac{x_j^+}{x_j^-}\right)^L = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}\,\sigma_{BES}(u_j, u_k)$$

all loops SU(2)

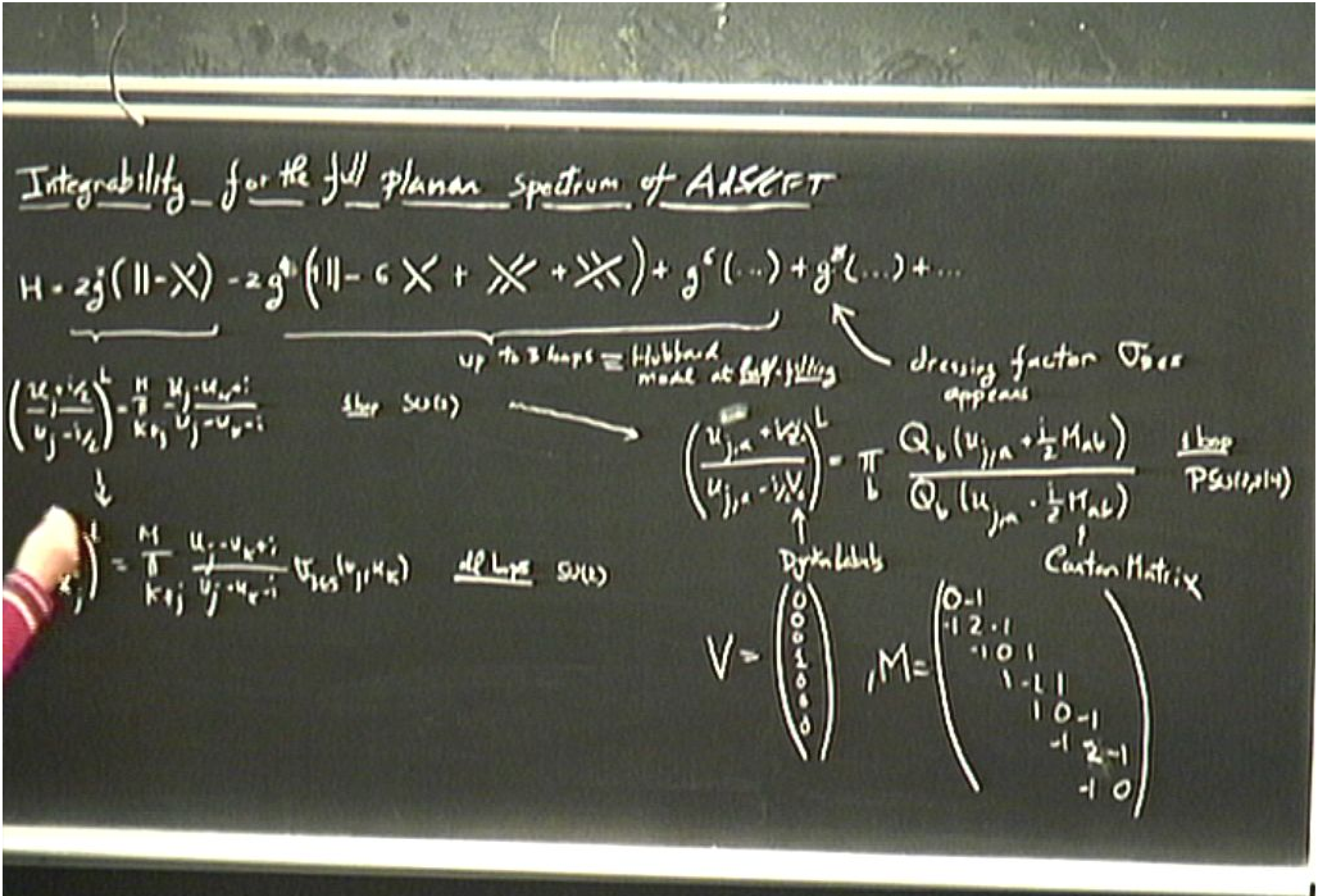
$$\left(\frac{u_{j,a} + \frac{i}{2}V_a}{u_{j,a} - \frac{i}{2}V_a}\right)^L = \prod_b \frac{Q_b(u_{j,a} + \frac{i}{2}M_{ab})}{Q_b(u_{j,a} - \frac{i}{2}M_{ab})}$$

1 loop PSU(2,2|4)

Dynkin labels

Cartan Matrix

$$V = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 0 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & 0 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 0 \end{pmatrix}$$



Integrability for the full planar spectrum of AdS/CFT

$$H = \underbrace{2g^2(\mathbb{1} - X)}\;\underbrace{- 2g^4\left(4\,\mathbb{1} - 6X + \times\!\!\!/\!\!/ + \times\!\!\!\backslash\!\!\backslash\right) + g^6(\ldots)}_{\text{up to 3 loops} \equiv \text{Hubbard model at half-filling}} + g^8(\ldots) + \ldots$$

dressing factor σ_{BES} appears

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} \qquad \underline{\text{1 loop}}\ SU(2)$$

$$\downarrow$$

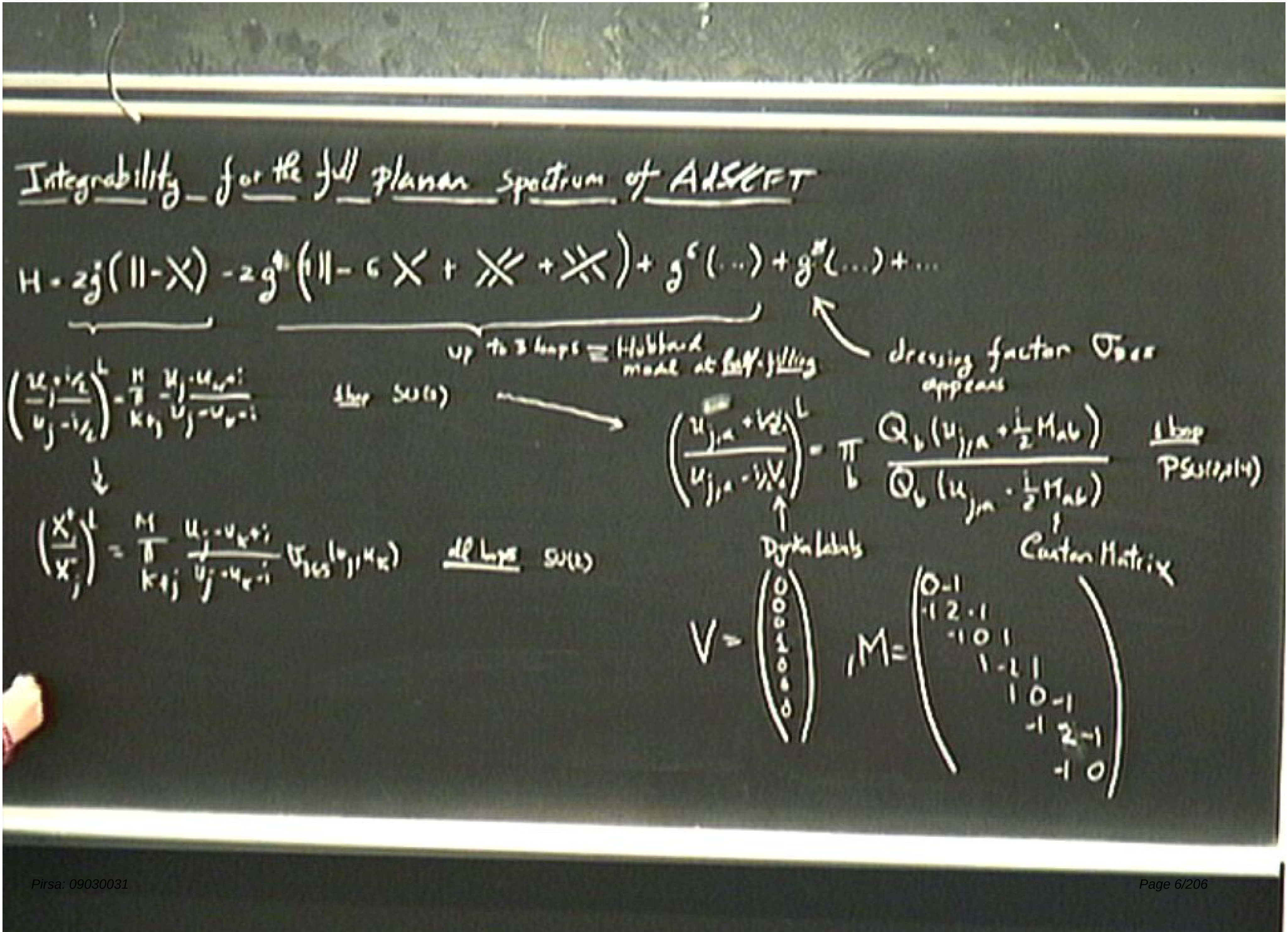
$$\left(\quad x_j\right)^L = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}\,\sigma_{BES}(u_j, u_k) \qquad \underline{\text{all loops}}\ SU(2)$$

$$\left(\frac{u_{j,a} + i/2\, V_a}{u_{j,a} - i/2\, V_a}\right)^L = \prod_b \frac{Q_b\left(u_{j,a} + \frac{i}{2} M_{ab}\right)}{Q_b\left(u_{j,a} - \frac{i}{2} M_{ab}\right)} \qquad \frac{\text{1 loop}}{PSU(2,2|4)}$$

V_a ↑ Dynkin labels

M_{ab} ↑ Cartan Matrix

$$V = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 0 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & 0 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 0 \end{pmatrix}$$



Integrability for the full planar spectrum of AdS/CFT

$$H = \underbrace{2g^2(\,||-X\,)}\;\underbrace{-\,2g^4\left(4\,||-6\,X+X\!\!X+X\!\!X\right)}_{\text{up to 3 loops} \,\equiv\, \text{Hubbard model at half-filling}}+g^6(\ldots)+g^8(\ldots)+\ldots$$

$\nwarrow$ dressing factor σ_{DES} appears

$$\left(\frac{u_j+i/2}{u_j-i/2}\right)^L=\prod_{k\neq j}^{M}\frac{u_j-u_k+i}{u_j-u_k-i} \qquad \underline{\text{1 loop}}\ SU(2) \longrightarrow$$

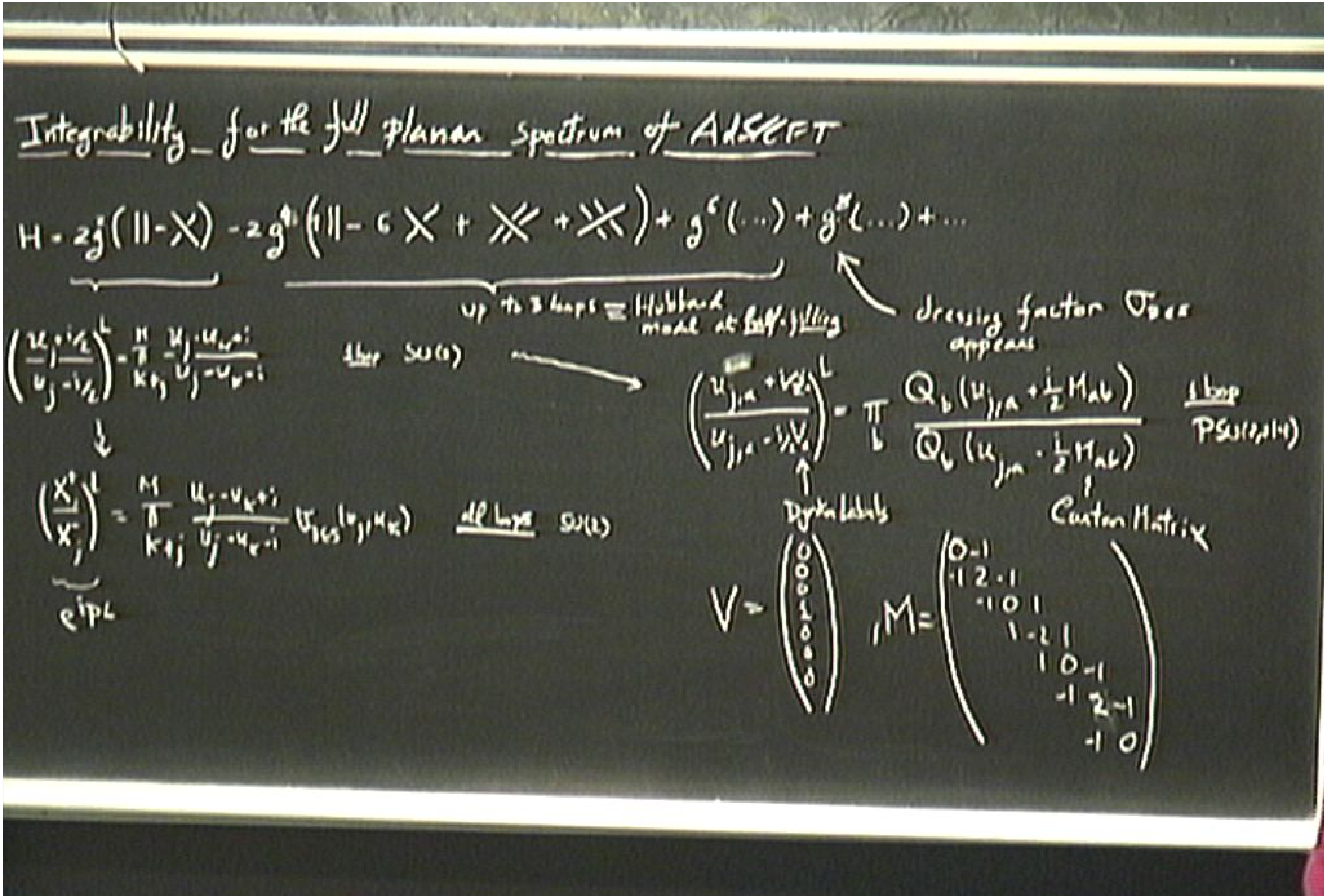
$$\downarrow$$

$$\left(\frac{x_j^+}{x_j^-}\right)^L=\prod_{k\neq j}^{M}\frac{u_j-u_k+i}{u_j-u_k-i}\,\sigma_{DES}(u_j,u_k) \qquad \underline{\text{all loops}}\ SU(2)$$

$$\left(\frac{u_{j,a}+\frac{i}{2}V_a}{u_{j,a}-\frac{i}{2}V_a}\right)^L=\prod_b\frac{Q_b(u_{j,a}+\frac{1}{2}M_{ab})}{Q_b(u_{j,a}-\frac{1}{2}M_{ab})} \qquad \underline{\text{1 loop}}\ PSU(2,2|4)$$

$\uparrow$ Dynkin labels $\qquad\qquad$ $\uparrow$ Cartan Matrix

$$V=\begin{pmatrix}0\\0\\0\\1\\0\\0\\0\end{pmatrix},\quad M=\begin{pmatrix}0&-1&&&&&\\-1&2&-1&&&&\\&-1&0&1&&&\\&&1&-2&1&&\\&&&1&0&-1&\\&&&&-1&2&-1\\&&&&&-1&0\end{pmatrix}$$



Integrability for the full planar spectrum of AdS/CFT

$$H = \underbrace{2g^2(\mathbb{1}-\times)}_{} \underbrace{- 2g^4(4\mathbb{1} - 6\times + \ldots + \ldots) + g^6(\ldots)}_{\text{up to 3 loops} \equiv \text{Hubbard model at half-filling}} + g^8(\ldots) + \ldots$$

dressing factor σ_{BES} appears

$$\left(\frac{u_j+i/2}{u_j-i/2}\right)^L = \prod_{k\neq j}^{M} \frac{u_j-u_k+i}{u_j-u_k-i} \qquad \underline{\text{1 loop}} \quad SU(2)$$

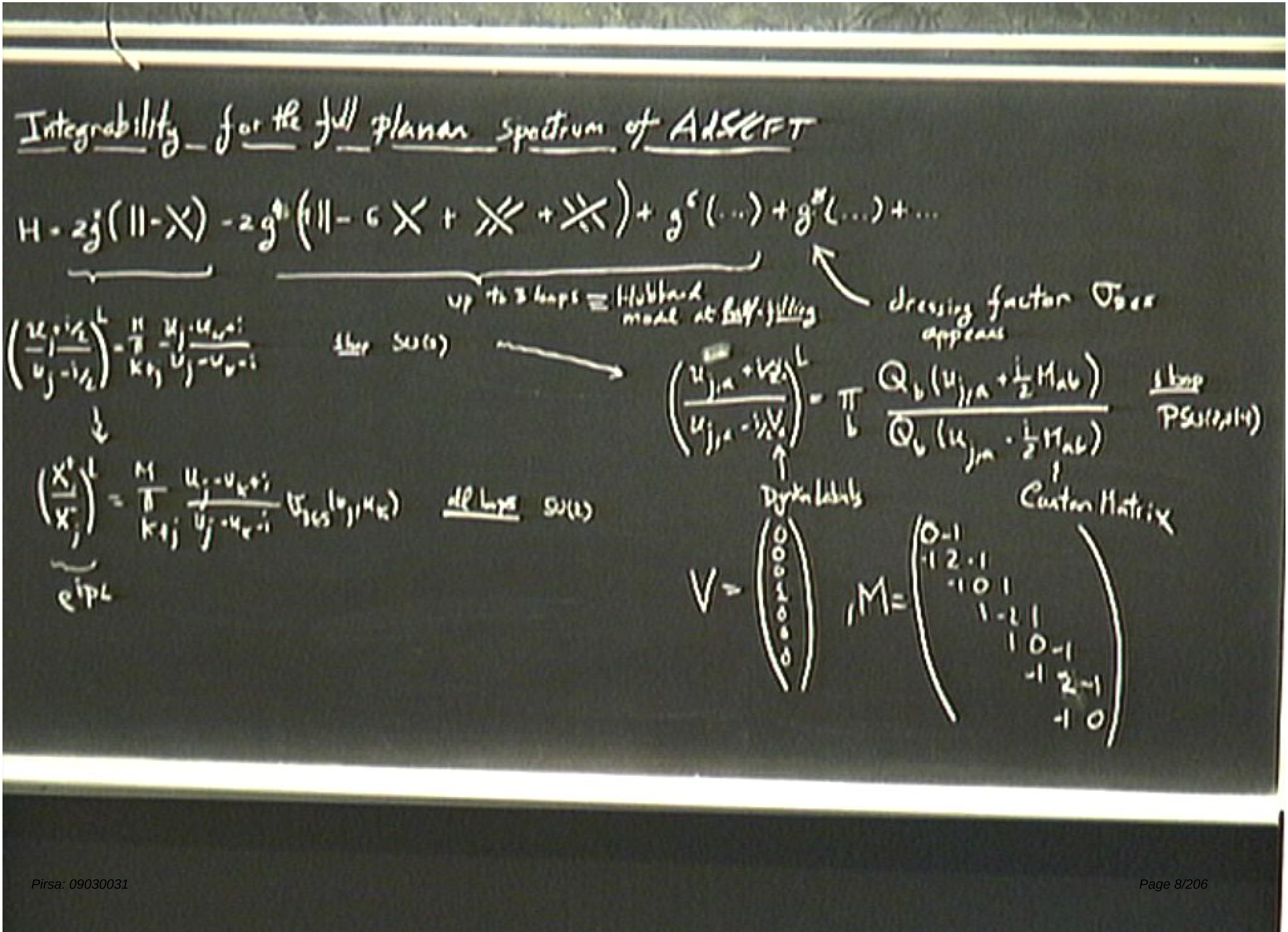
$$\downarrow$$

$$\underbrace{\left(\frac{x_j^+}{x_j^-}\right)^L}_{e^{ipL}} = \prod_{k\neq j}^{M} \frac{u_j-u_k+i}{u_j-u_k-i}\,\sigma_{BES}(u_j,u_k) \qquad \underline{\text{all loops}} \quad SU(2)$$

$$\left(\frac{u_{j,a}+iV_a/2}{u_{j,a}-iV_a/2}\right)^L = \prod_b \frac{Q_b(u_{j,a}+\frac{i}{2}M_{ab})}{Q_b(u_{j,a}-\frac{i}{2}M_{ab})} \qquad \underline{\text{1 loop}} \quad PSU(2,2|4)$$

Dynkin labels $\uparrow$ $\qquad$ $\uparrow$ Cartan Matrix

$$V = \begin{pmatrix} 0\\0\\0\\1\\0\\0\\0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 0 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & 0 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 0 \end{pmatrix}$$



Integrability for the full planar spectrum of AdS/CFT

$$H = \underbrace{2g^2(\| - \times)}_{} \underbrace{- 2g^4(4\| - 6\times + \times\!\!\times + \times\!\!\times\!\!\times) + g^6(\ldots)}_{\text{up to 3 loops} \equiv \text{Hubbard model at half-filling}} + g^8(\ldots) + \ldots$$

dressing factor σ_{BES} appears

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} \qquad \underline{1 \text{ loop}} \quad SU(2)$$

$$\downarrow$$

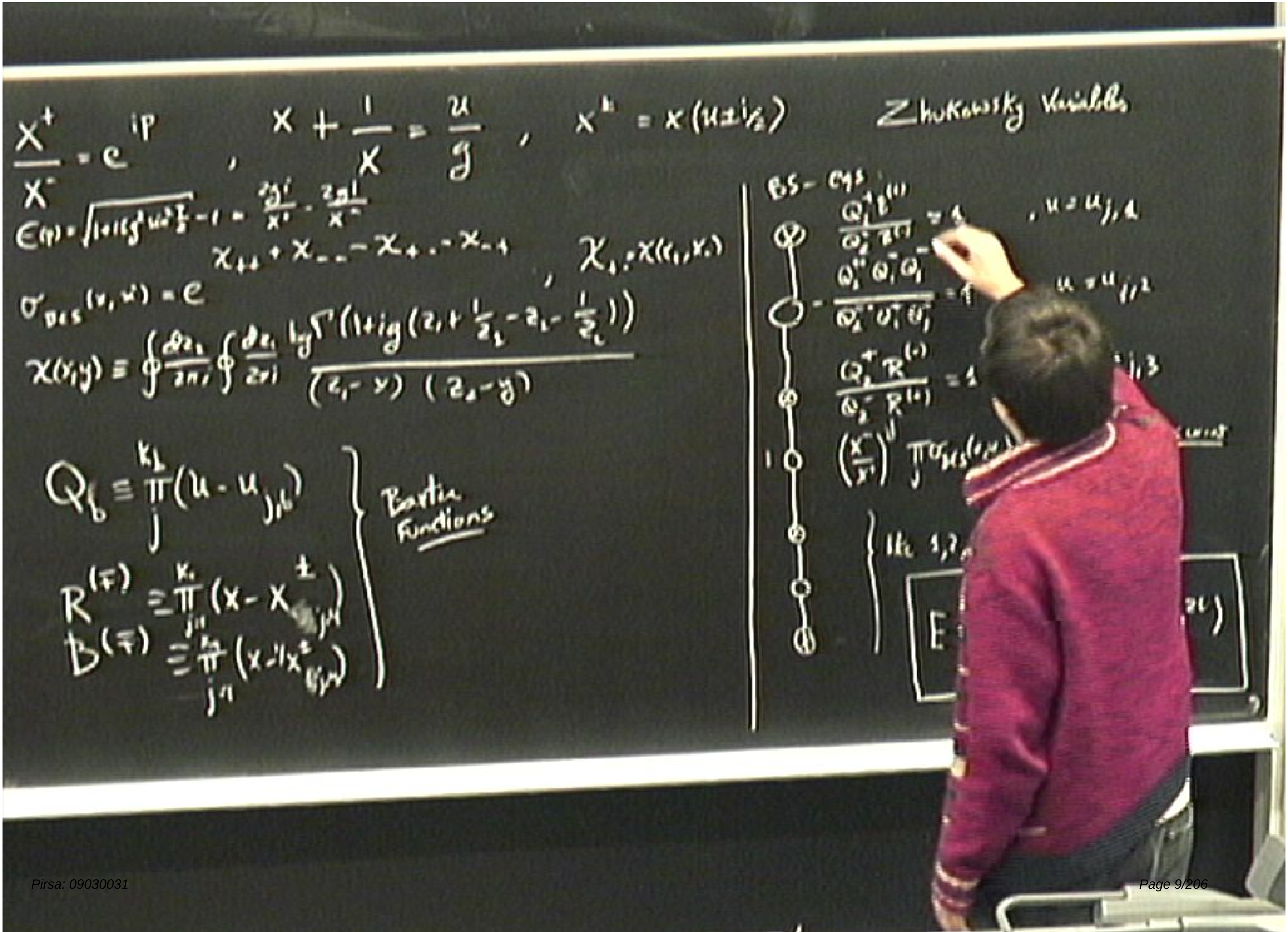
$$\underbrace{\left(\frac{x_j^+}{x_j^-}\right)^L}_{e^{ipL}} = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}\, \sigma_{BES}(u_j, u_k) \qquad \underline{\text{all loops}} \quad SU(2)$$

$$\left(\frac{u_{j,a} + \frac{i}{2}V_a}{u_{j,a} - \frac{i}{2}V_a}\right)^L = \prod_b \frac{Q_b(u_{j,a} + \frac{i}{2}M_{ab})}{Q_b(u_{j,a} - \frac{i}{2}M_{ab})} \qquad \underline{1 \text{ loop}} \quad PSU(2,2|4)$$

Dynkin labels

Cartan Matrix

$$V = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 0 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & 0 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 0 \end{pmatrix}$$



$\frac{x^+}{x^-} = e^{ip}$, $x + \frac{1}{x} = \frac{u}{g}$, $x^{\pm} = x(u \pm i/2)$ Zhukowsky Variables

$\epsilon(p) = \sqrt{1+16 g^2 \sin^2\frac{p}{2}} - 1 = \frac{2gi}{x^+} - \frac{2gi}{x^-}$

$\sigma_{BES}(x, x') = e^{\chi_{++} + \chi_{--} - \chi_{+-} - \chi_{-+}}$, $\chi_{+-} = \chi(x_1^+, x_2^-)$

$$\chi(x,y) = \oint \frac{dz_1}{2\pi i} \oint \frac{dz_2}{2\pi i} \frac{\log\Gamma\left(1 + ig\left(z_1 + \frac{1}{z_1} - z_2 - \frac{1}{z_2}\right)\right)}{(z_1 - x)(z_2 - y)}$$

$Q_b \equiv \prod_j^{K_b} (u - u_{j,b})$

$R^{(\mp)} = \prod_{j=1}^{K_4} (x - x^{\pm}_{j,4})$

$B^{(\mp)} = \prod_{j=1}^{K_4} (x^{-1} - x^{\pm}_{j,4})$

Baxter Functions

BS-eqs

$\frac{Q_2^+ B^{(+)}}{Q_2^- B^{(-)}} = 1$, $u = u_{j,1}$

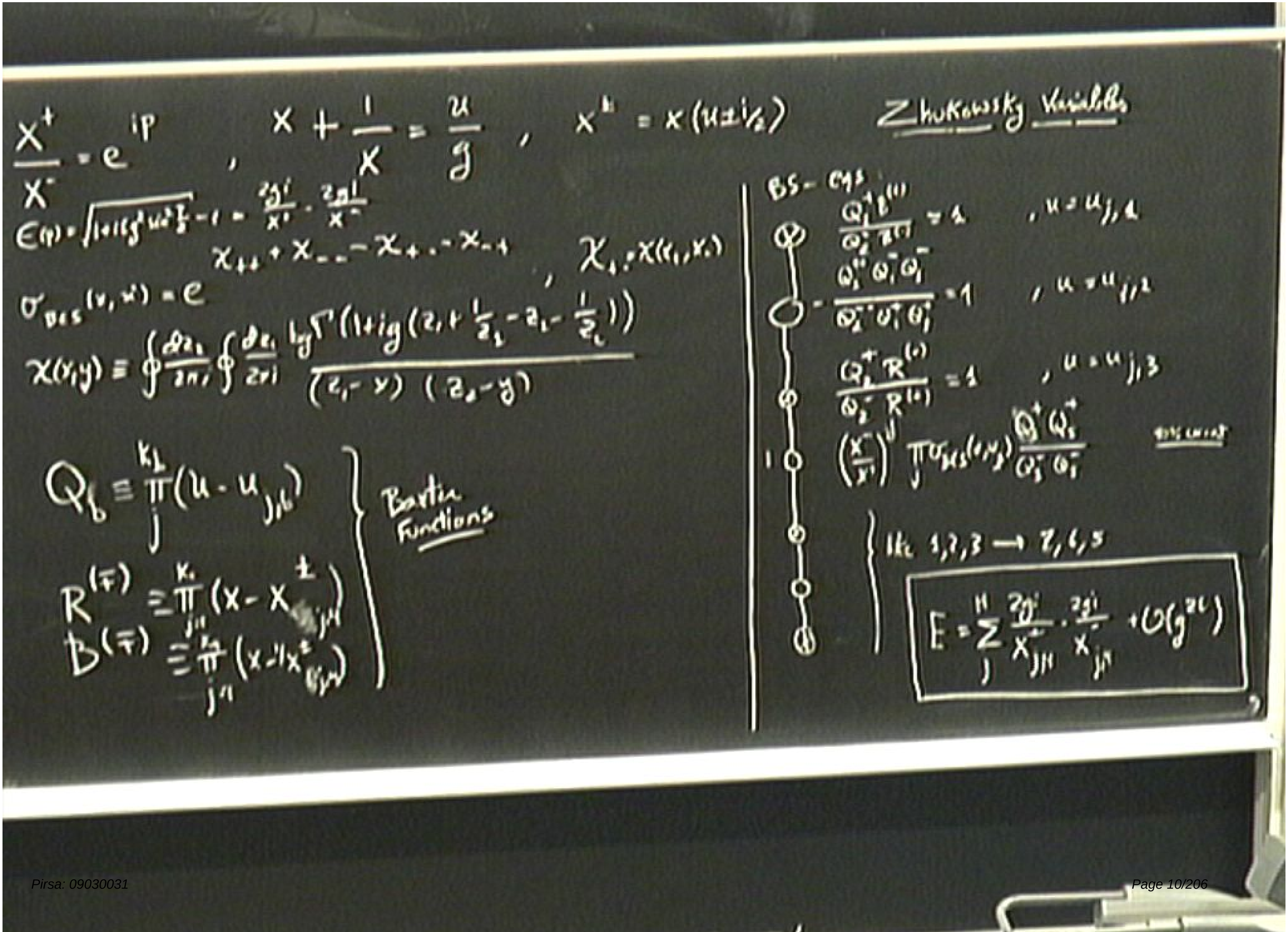
$-\frac{Q_2^{++} Q_1^- Q_3^-}{Q_2^{--} Q_1^+ Q_3^+} = 1$, $u = u_{j,2}$

$\frac{Q_2^+ R^{(-)}}{Q_2^- R^{(+)}} = 1$, $_{j,3}$

$\left(\frac{x^-}{x^+}\right)^J \prod_j \sigma_{BES}(x, x_j)$

like 1,2,

E (2u)



$\frac{x^+}{x^-} = e^{ip}$, $x + \frac{1}{x} = \frac{u}{g}$, $x^{\pm} = x(u \pm i/2)$

Zhukowsky Variables

$\epsilon(p) = \sqrt{1+16g^2\sin^2\frac{p}{2}} - 1 = \frac{2gi}{x^+} - \frac{2gi}{x^-}$

$\sigma_{BES}(x,x') = e^{\chi_{++}+\chi_{--}-\chi_{+-}-\chi_{-+}}$, $\chi_{+-} = \chi(x_1, x_2)$

$$\chi(x,y) = \oint\frac{dz_1}{2\pi i}\oint\frac{dz_2}{2\pi i}\frac{\log\Gamma(1+ig(z_1+\frac{1}{z_1}-z_2-\frac{1}{z_2}))}{(z_1-x)\,(z_2-y)}$$

$$\left.\begin{aligned} Q_l &= \prod_{j}^{K_l}(u-u_{j,l}) \\ R^{(\mp)} &= \prod_{j=1}^{K_4}(x-x^{\mp}_{j,4}) \\ B^{(\mp)} &= \prod_{j=1}^{K_4}(x^{-1}-x^{\mp}_{j,4}) \end{aligned}\right\}$$

Baxter Functions

BS-eqs

$\frac{Q_2^+B^{(+)}}{Q_2^-B^{(-)}} = 1$, $u = u_{j,1}$

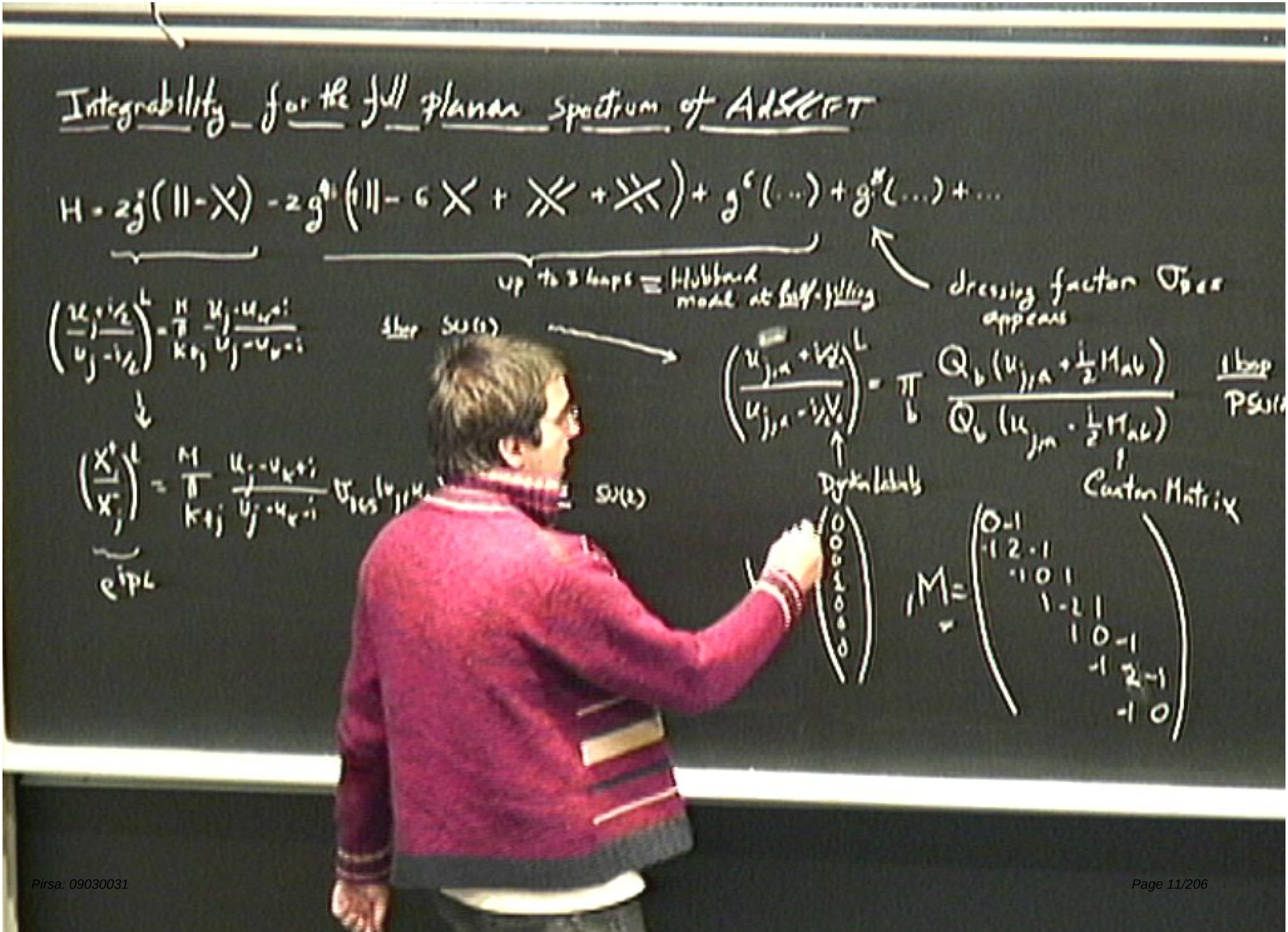
$-\frac{Q_2^{++}Q_1^-Q_3^-}{Q_2^{--}Q_1^+Q_3^+} = 1$, $u = u_{j,2}$

$\frac{Q_2^+R^{(-)}}{Q_2^-R^{(+)}} = 1$, $u = u_{j,3}$

$\left(\frac{x^-}{x^+}\right)^J\prod_j\sigma_{BES}(x,x_j)\frac{Q_3^+Q_5^+}{Q_3^-Q_5^-}$ [illegible]

like 1,2,3 → 7,6,5

$$\boxed{E = \sum_j^{K_4}\frac{2gi}{x^+_{j,4}} - \frac{2gi}{x^-_{j,4}} + O(g^{2L})}$$



Integrability for the full planar spectrum of AdS/CFT

$$H = \underbrace{2g^2(\mathbb{1} - X)}_{} \underbrace{- 2g^4(4\mathbb{1} - 6X + X + X) + g^6(\dots)}_{\text{up to 3 loops} \equiv \text{Hubbard model at half-filling}} + g^8(\dots) + \dots$$

dressing factor σ_{BES} appears

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}$$

1 loop SU(2)

$$\downarrow$$

$$\underbrace{\left(\frac{x_j^+}{x_j^-}\right)^L}_{e^{ipL}} = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} \sigma^2_{BES}(u_j, u_k)$$

SU(2)

$$\left(\frac{u_{j,a} + \frac{i}{2}V_a}{u_{j,a} - \frac{i}{2}V_a}\right)^L = \prod_b \frac{Q_b(u_{j,a} + \frac{i}{2}M_{ab})}{Q_b(u_{j,a} - \frac{i}{2}M_{ab})}$$

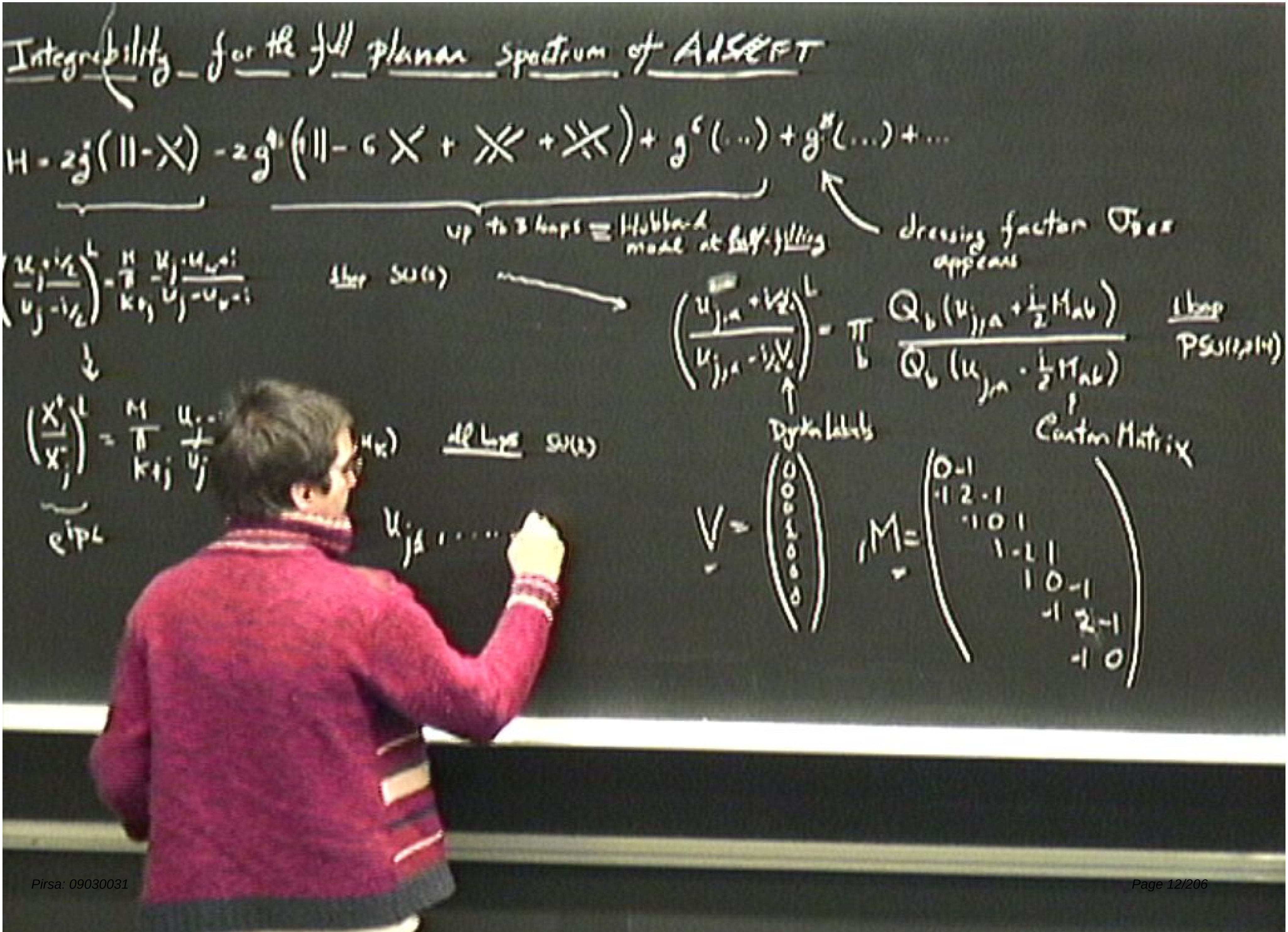
1 loop PSU(

Dynkin labels

Cartan Matrix

$$V = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 0 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & 0 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 0 \end{pmatrix}$$



Integrability for the full planar spectrum of AdS/CFT

$$H = \underbrace{2g^2(\mathbb{I} - \times)}\; \underbrace{- 2g^4(4\mathbb{I} - 6\times + \times\!\!\times + \times\!\!\times) + g^6(\ldots)}_{\text{up to 3 loops} \equiv \text{Hubbard model at half-filling}} + g^8(\ldots) + \ldots$$

dressing factor σ_{BES} appears

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} \quad \underline{\text{1 loop}}\ SU(2) \longrightarrow$$

$$\downarrow$$

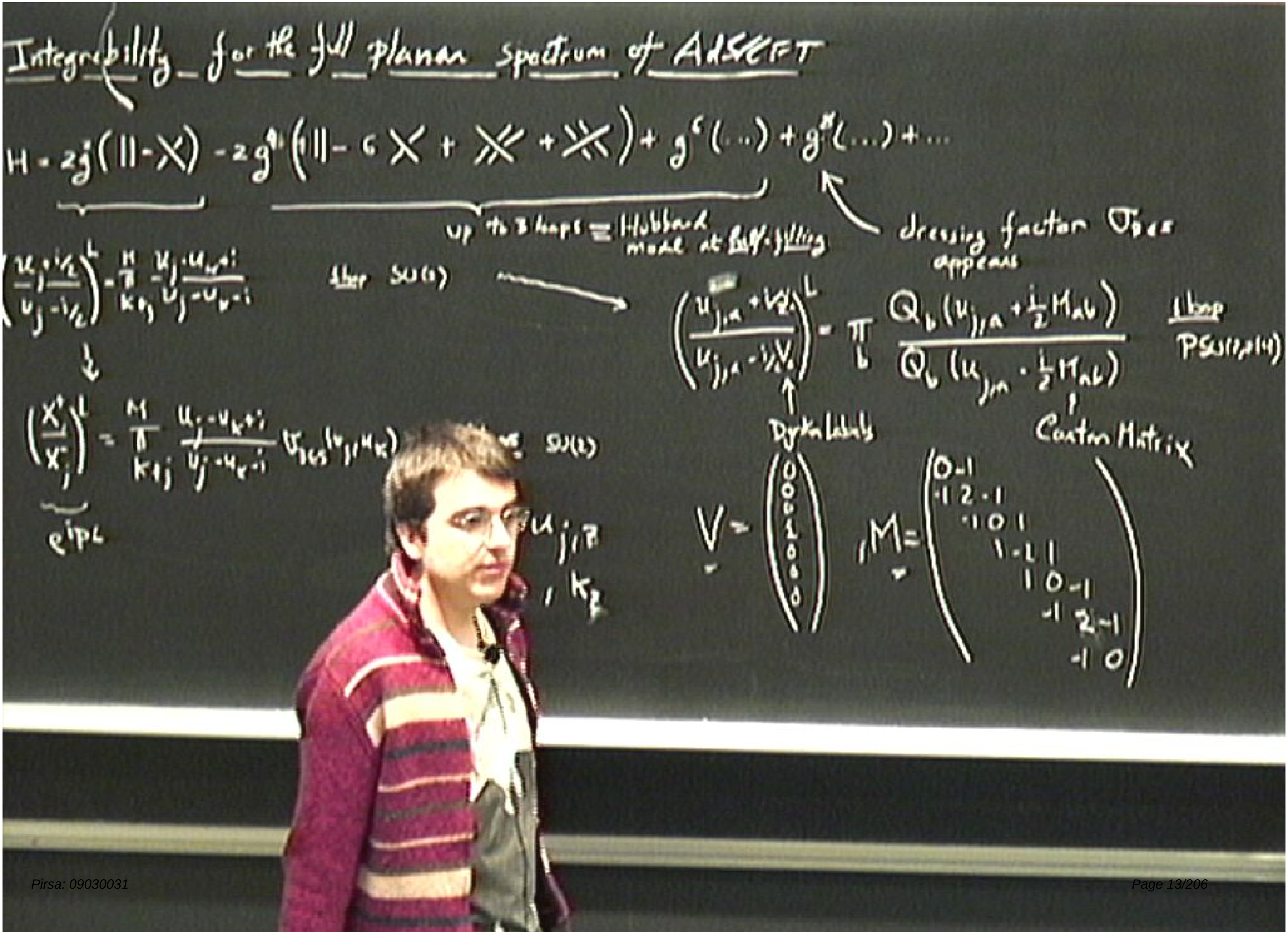
$$\underbrace{\left(\frac{x_j^+}{x_j^-}\right)^L}_{e^{ipL}} = \prod_{k\neq j}^{M} \frac{u_{\cdot}-\cdot}{u_j} \ldots u_k) \quad \underline{\text{all loops}}\ SU(2)$$

$$u_{j,a}, \ldots\ldots$$

$$\left(\frac{u_{j,a} + \frac{i}{2}V_a}{u_{j,a} - \frac{i}{2}V_a}\right)^L = \prod_b \frac{Q_b(u_{j,a} + \frac{i}{2}M_{ab})}{Q_b(u_{j,a} - \frac{i}{2}M_{ab})} \quad \underline{\text{1 loop}}\ PSU(2,2|4)$$

$\uparrow$ Dynkin labels — $\uparrow$ Cartan Matrix

$$\underline{V} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{M} = \begin{pmatrix} 0 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 0 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & 0 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 0 \end{pmatrix}$$



Integrability for the full planar spectrum of AdS/CFT

$$H = \underbrace{2g^2(\| - \times)}\; \underbrace{-\,2g^4(\||| - 6\times + \times\!\!\!\times + \times\!\!\!\times) + g^6(\ldots)} + g^8(\ldots) + \ldots$$

up to 3 loops ≡ Hubbard model at half-filling

dressing factor σ_{BES} appears

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}$$ 1loop SU(2)

$$\downarrow$$

$$\underbrace{\left(\frac{x_j^+}{x_j^-}\right)^L}_{e^{ipL}} = \prod_{k \neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}\,\sigma_{BES}(u_j, u_k)$$ SU(2)

$u_{j,a}$

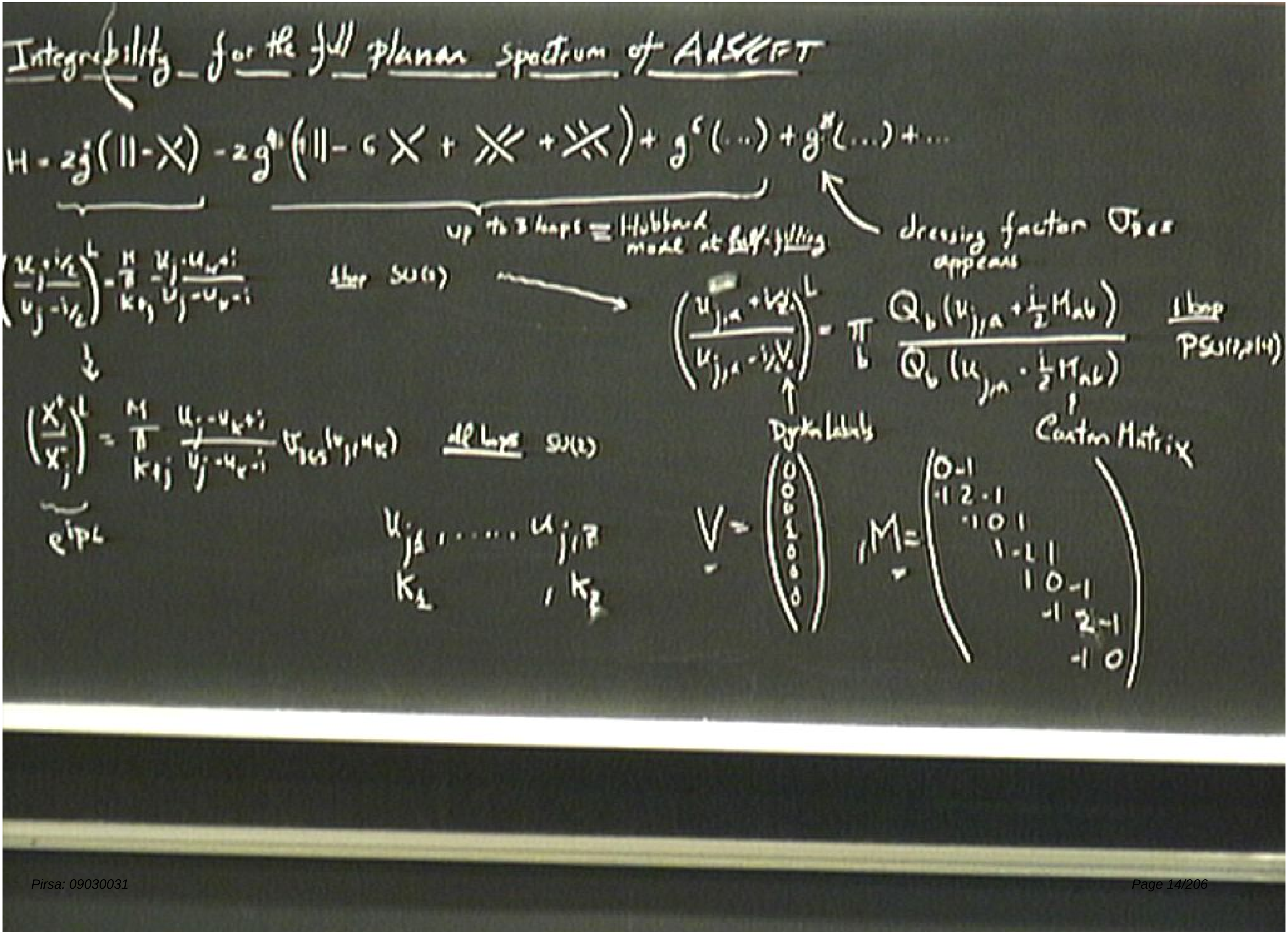
$, K_a$

$$\left(\frac{u_{j,a} + i/2\, V_a}{u_{j,a} - i/2\, V_a}\right)^L = \prod_b \frac{Q_b(u_{j,a} + \frac{i}{2} M_{ab})}{Q_b(u_{j,a} - \frac{i}{2} M_{ab})}$$ 1loop PSU(2,2|4)

Dynkin labels

Cartan Matrix

$$\underline{V} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{M} = \begin{pmatrix} 0 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 0 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & 0 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 0 \end{pmatrix}$$



Integrability for the full planar spectrum of AdS/CFT

$$H = \underbrace{2g^2(\| - \times)}_{} \underbrace{- 2g^4(\| - 6\times + \times\!\!\times + \times\!\!\times) + g^6(\ldots)}_{\text{up to 3 loops} \equiv \text{Hubbard model at half-filling}} + g^8(\ldots) + \ldots$$

dressing factor σ_{BES} appears

$$\left(\frac{u_j+i/2}{u_j-i/2}\right)^L = \prod_{k\neq j}^{M} \frac{u_j-u_k+i}{u_j-u_k-i}$$

1 loop SU(2)

↓

$$\underbrace{\left(\frac{x_j^+}{x_j^-}\right)^L}_{e^{ipL}} = \prod_{k\neq j}^{M} \frac{u_j-u_k+i}{u_j-u_k-i}\,\sigma_{BES}(u_j,u_k)$$

all loops SU(2)

$$\left(\frac{u_{j,a}+\frac{i}{2}V_a}{u_{j,a}-\frac{i}{2}V_a}\right)^L = \prod_b \frac{Q_b(u_{j,a}+\frac{i}{2}M_{ab})}{Q_b(u_{j,a}-\frac{i}{2}M_{ab})}$$

1 loop PSU(2,2|4)

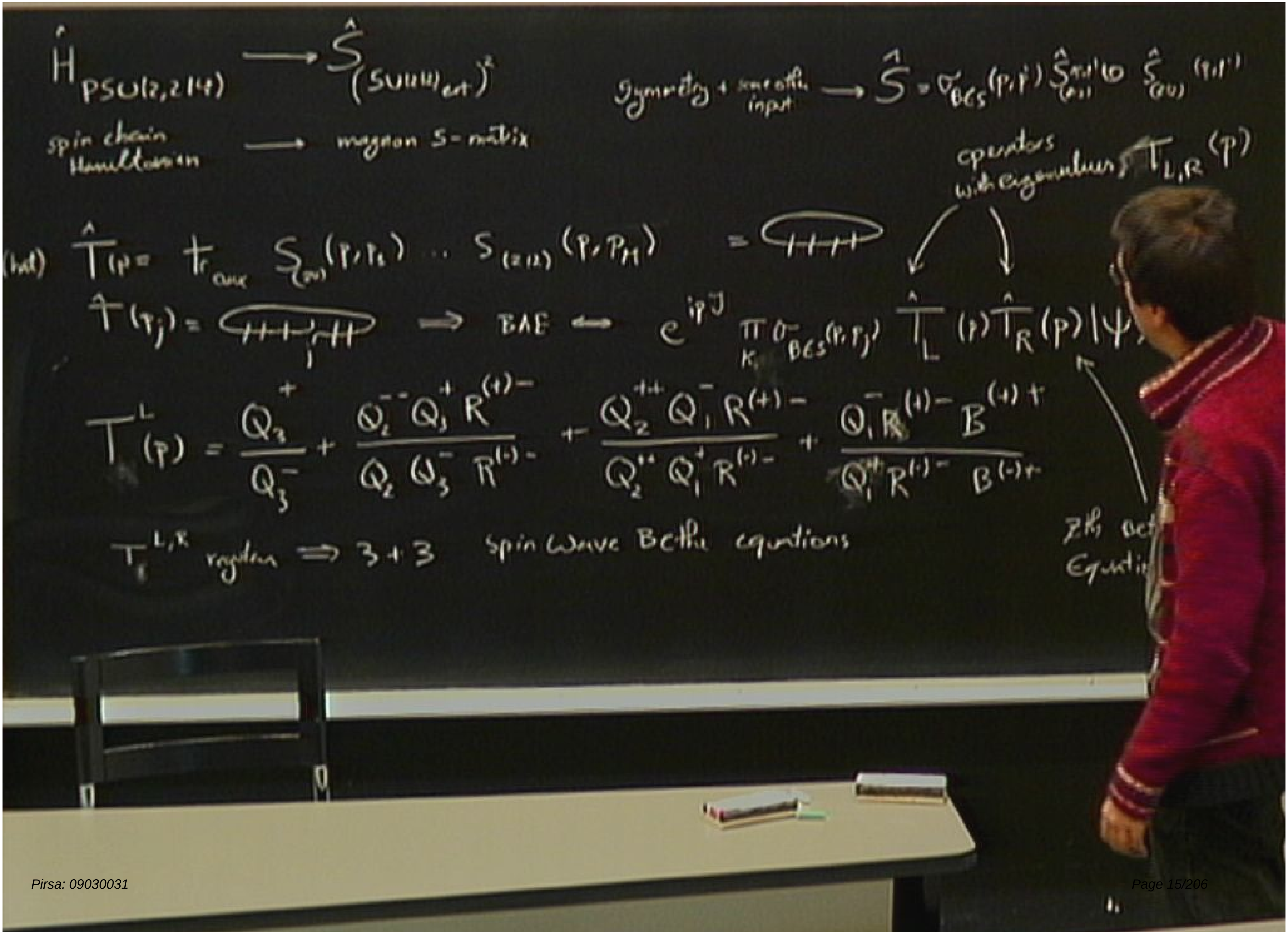
Dynkin labels

Cartan Matrix

$$u_{j,1}, \ldots\ldots, u_{j,7}$$

$$K_1, \quad\quad , K_7$$

$$\underline{V} = \begin{pmatrix} 0\\0\\0\\1\\0\\0\\0 \end{pmatrix}, \quad \underline{M} = \begin{pmatrix} 0 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 0 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & 0 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 0 \end{pmatrix}$$



$\hat{H}_{PSU(2,2|4)} \longrightarrow \hat{S}_{(SU(2|2)_{ext})^2}$

spin chain Hamiltonian ⟶ magnon S-matrix

Symmetry + some other input ⟶ $\hat{S} = \sigma_{BES}(p,p')\,\hat{S}^{(p,p')}_{(2|2)} \otimes \hat{S}_{(2|2)}(p,p')$

operators with eigenvalues $T_{L,R}(p)$

(het) $\hat{T}(p) = \mathrm{tr}_{aux}\, S_{(2|2)}(p,p_1) \cdots S_{(2|2)}(p,p_M) =$

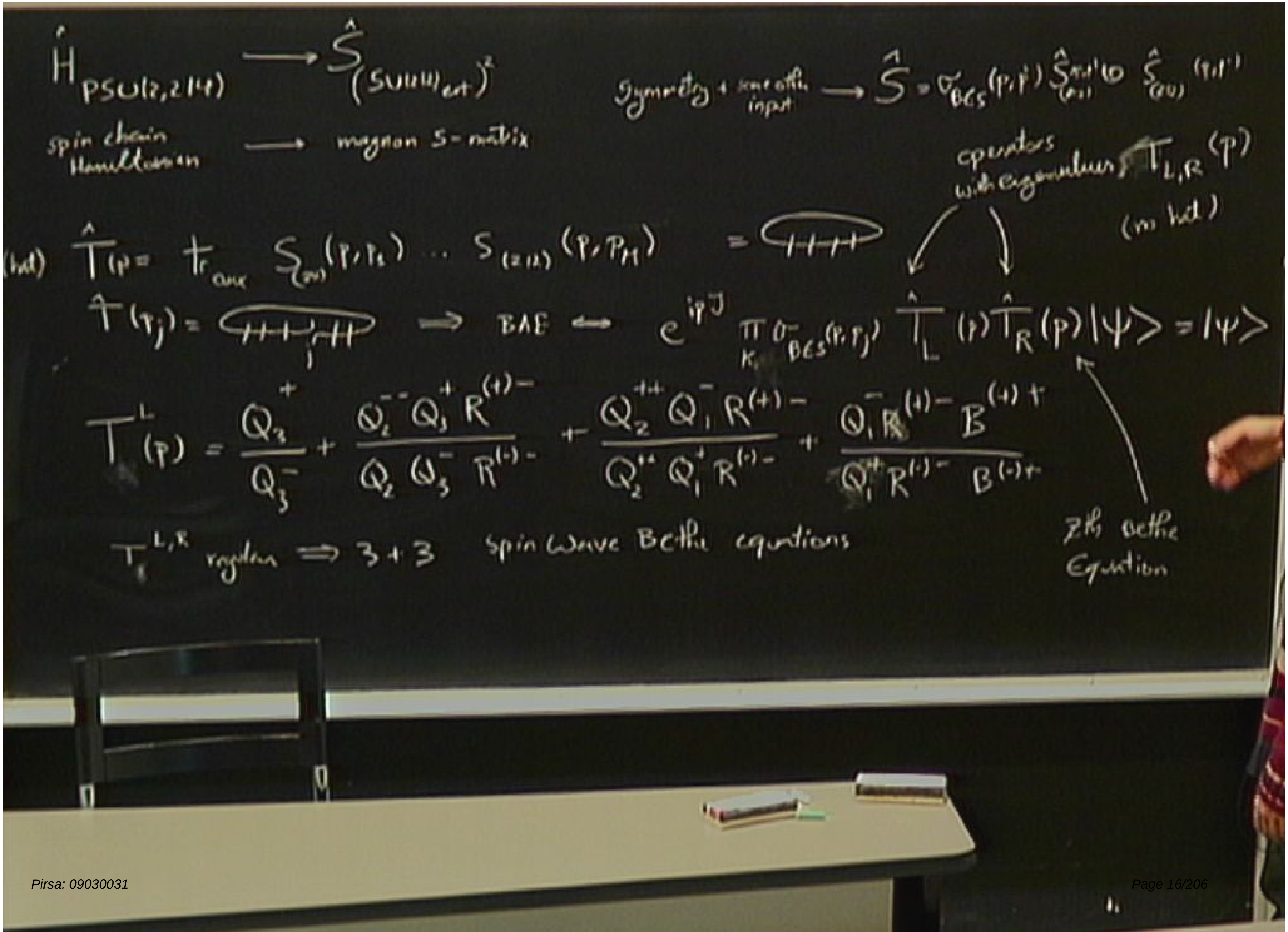
$\hat{T}(p_j) =$ ⟹ BAE ⟺ $e^{ipJ} \prod_{k} \sigma_{BES}(p,p_j)\; \hat{T}_L(p)\hat{T}_R(p)|\psi\rangle$

$$T^{L}(p) = \frac{Q_3^{+}}{Q_3^{-}} + \frac{Q_2^{--}Q_3^{+}R^{(+)-}}{Q_2 Q_3^{-} R^{(-)-}} + \frac{Q_2^{++}Q_1^{-}R^{(+)-}}{Q_2^{++}Q_1^{+}R^{(-)-}} + \frac{Q_1^{-}R^{(+)-}B^{(+)+}}{Q_1^{+}R^{(-)-}B^{(-)+}}$$

$T_1^{L,R}$ regular ⟹ 3 + 3 spin wave Bethe equations

Zh, Bet

Equati



$\hat{H}_{PSU(2,2|4)} \longrightarrow \hat{S}_{(SU(2|2)_{ext})^2}$

spin chain Hamiltonian ⟶ magnon S-matrix

Symmetry + some other input $\longrightarrow \hat{S} = \sigma_{BES}(p,p')\, \hat{S}_{(2|2)}^{(p,p')} \otimes \hat{S}_{(2|2)}^{(p,p')}$

operators with eigenvalues $T_{L,R}(p)$

(no hd)

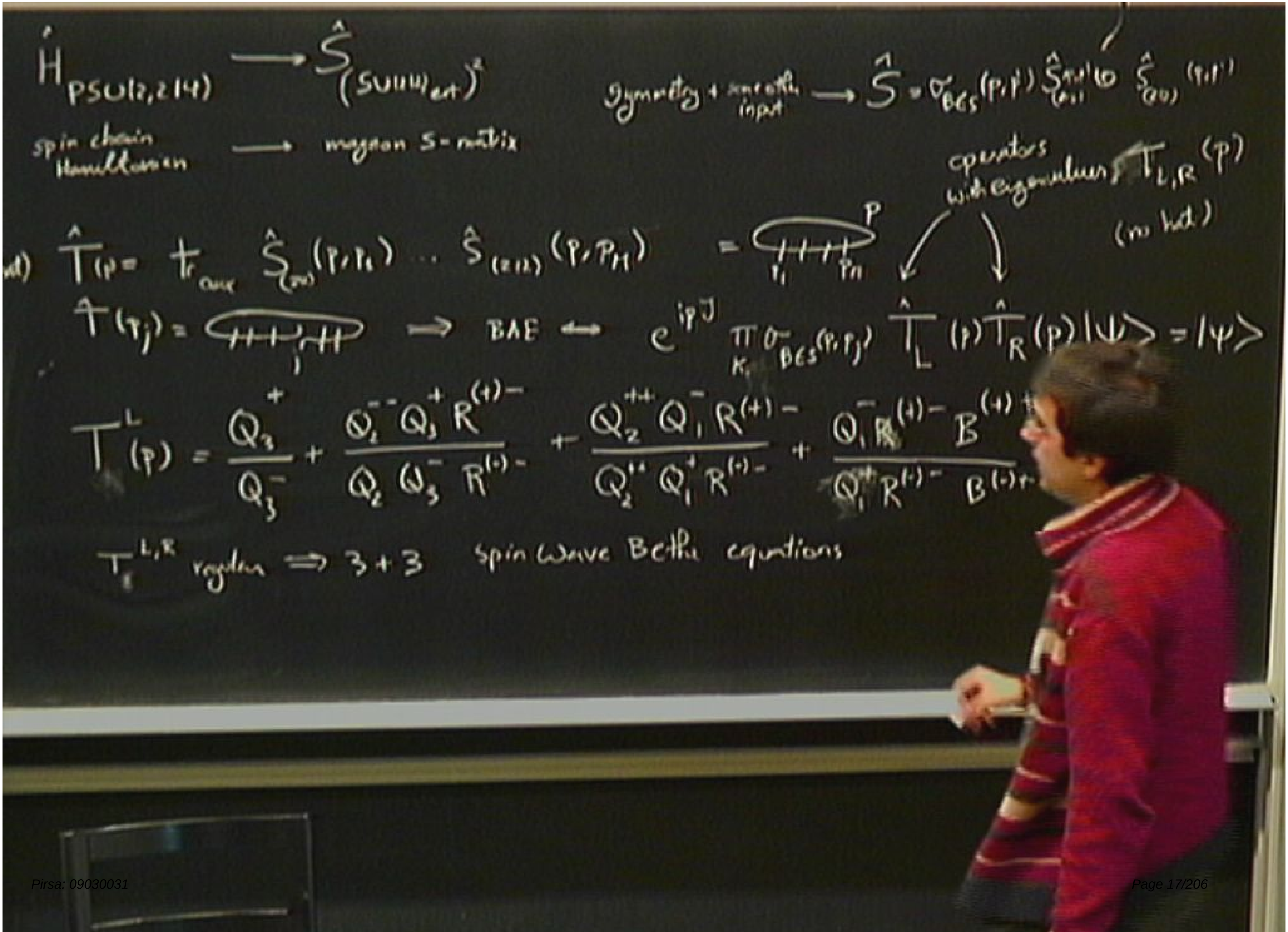
(hd) $\hat{T}(p) = \mathrm{tr}_{aux}\, S_{(2|2)}(p,p_1) \cdots S_{(2|2)}(p,p_M) =$

$\hat{T}(p_j) =$ ⟹ BAE ⟺ $e^{ip_jJ} \prod_{k} \sigma_{BES}(p_j,p_k)\, \hat{T}_L(p)\, \hat{T}_R(p)|\psi\rangle = |\psi\rangle$

$$T^L(p) = \frac{Q_3^+}{Q_3^-} + \frac{Q_2^{--}Q_3^+R^{(+)-}}{Q_2 Q_3^- R^{(-)-}} + \frac{Q_2^{++}Q_1^-R^{(+)-}}{Q_2^{++}Q_1^+R^{(-)-}} + \frac{Q_1^-R^{(+)-}B^{(+)+}}{Q_1^+R^{(-)-}B^{(-)+}}$$

$T_1^{L,R}$ regular ⟹ 3+3 spin wave Bethe equations

Zh's Bethe Equation



$$\hat{H}_{PSU(2,2|4)} \longrightarrow \hat{S}_{(SU(2|2)_{ext})^2}$$

spin chain Hamiltonian ⟶ magnon S-matrix

symmetry + smoothness input ⟶ $\hat{S} = \sigma_{BES}(p,p') \hat{S}^{(p,p')}_{(2|2)} \otimes \hat{S}_{(2|2)}(p,p')$

$$\hat{T}(p) = \mathrm{tr}_{aux}\, \hat{S}_{(2|2)}(p,p_1) \ldots \hat{S}_{(2|2)}(p,p_M) =$$

p, p_1, p_M

operators with eigenvalues $T_{L,R}(p)$

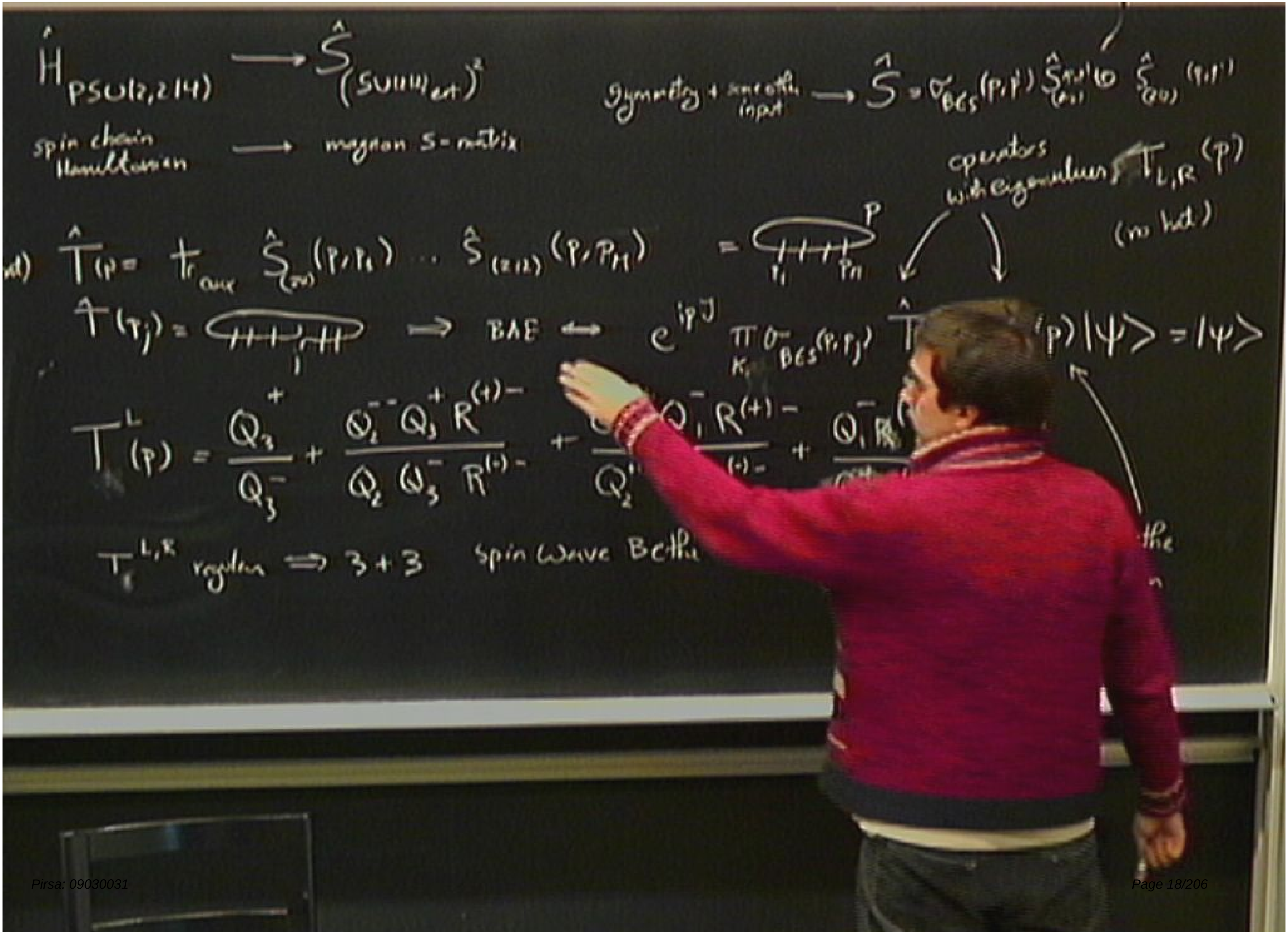
(no hat)

$$\hat{T}(p_j) = \quad \Longrightarrow \text{BAE} \Longleftrightarrow e^{ipJ} \prod_{k} \sigma_{BES}(p,p_j)\, \hat{T}_L(p)\, \hat{T}_R(p) |\psi\rangle = |\psi\rangle$$

j

$$T^{L}(p) = \frac{Q_3^{+}}{Q_3^{-}} + \frac{Q_2^{--} Q_3^{+} R^{(+)-}}{Q_2\, Q_3^{-}\, R^{(-)-}} + \frac{Q_2^{++} Q_1^{-} R^{(+)-}}{Q_2^{+} Q_1^{+} R^{(-)-}} + \frac{Q_1^{-} R^{(+)-}\, B^{(+)+}}{Q_1^{+} R^{(-)-}\, B^{(-)+}}$$

$T^{L,R}$ regular ⟹ 3 + 3 spin wave Bethe equations



$\hat{H}_{PSU(2,2|4)} \longrightarrow \hat{S}_{(SU(1|1)_{ext})^2}$

spin chain Hamiltonian $\longrightarrow$ magnon S-matrix

symmetry + some other input $\longrightarrow \hat{S} = \sigma_{BES}(p,p') \, \hat{S}^{su(1)}_{(2|2)} \otimes \hat{S}_{(2|2)}(p,p')$

operators with eigenvalues $T_{L,R}(p)$ (no hat)

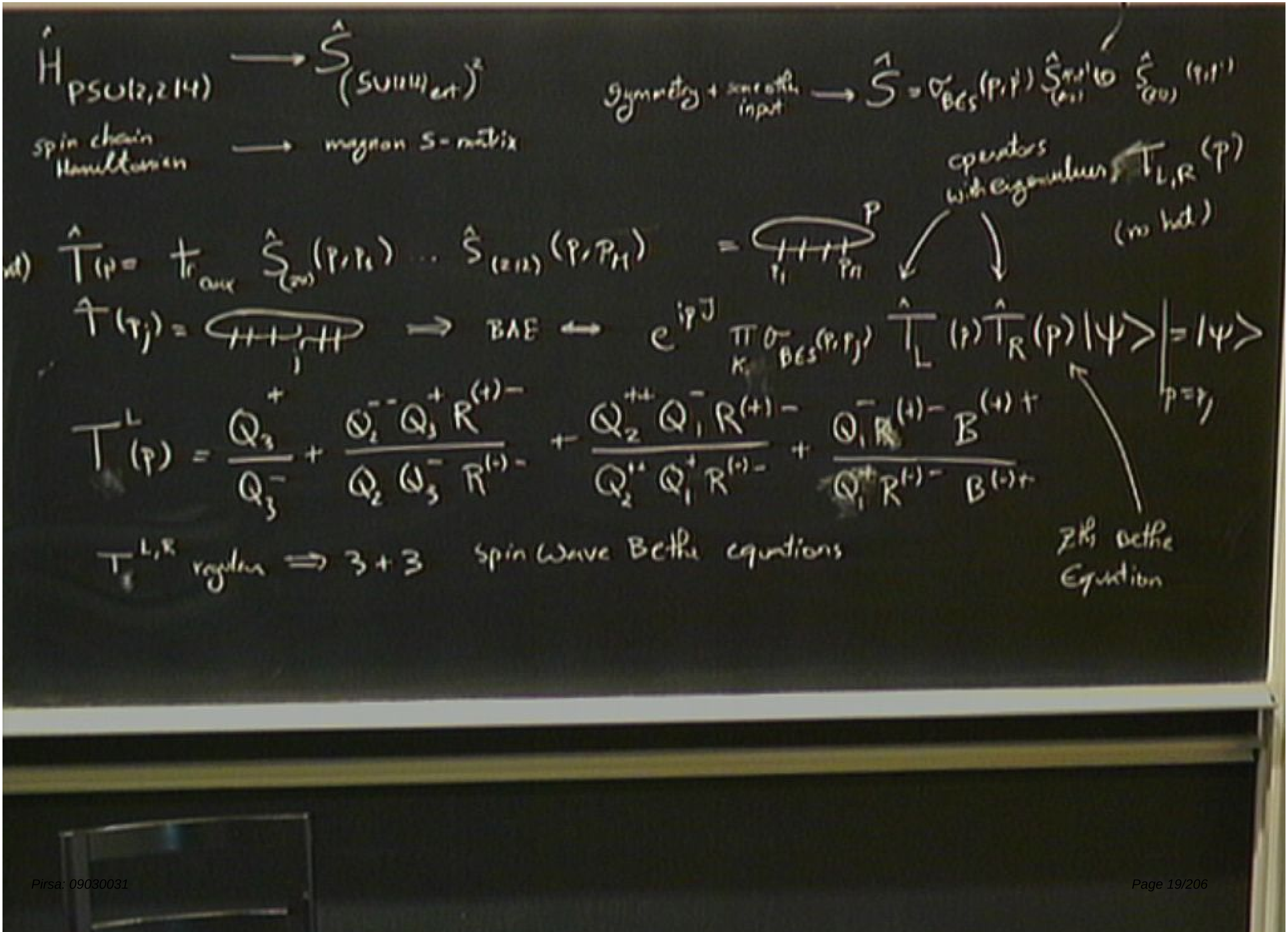
$\hat{T}(p) = \mathrm{tr}_{aux} \, \hat{S}_{(2|2)}(p,p_1) \cdots \hat{S}_{(2|2)}(p,p_M)$ =

$\hat{T}(p_j) =$ $\Longrightarrow$ BAE $\Longleftrightarrow$ $e^{ipJ} \prod_{k} \sigma_{BES}(p,p_j) \, \hat{T}$ $(p)|\psi\rangle = |\psi\rangle$

$$T^{L}(p) = \frac{Q_3^{+}}{Q_3^{-}} + \frac{Q_2^{--} Q_3^{+} R^{(+)-}}{Q_2 \, Q_3^{-} \, R^{(-)-}} + \frac{Q \; Q_1^{-} R^{(+)-}}{Q_2^{+} \quad {}^{(-)-}} + \frac{Q_1^{-} R}{}$$

$T_{\ell}^{L,R}$ regular $\Longrightarrow 3+3$ spin Wave Bethe

the



$$\hat{H}_{PSU(2,2|4)} \longrightarrow \hat{S}_{(SU(2|2)_{ext})^2}$$

spin chain Hamiltonian ⟶ magnon S-matrix

symmetry + some other input $\longrightarrow \hat{S} = \sigma_{BES}(p,p')\, \hat{S}^{(p,p')}_{(2|2)} \otimes \hat{S}_{(2|2)}(p,p')$

operators with eigenvalues $T_{L,R}(p)$

(no hat)

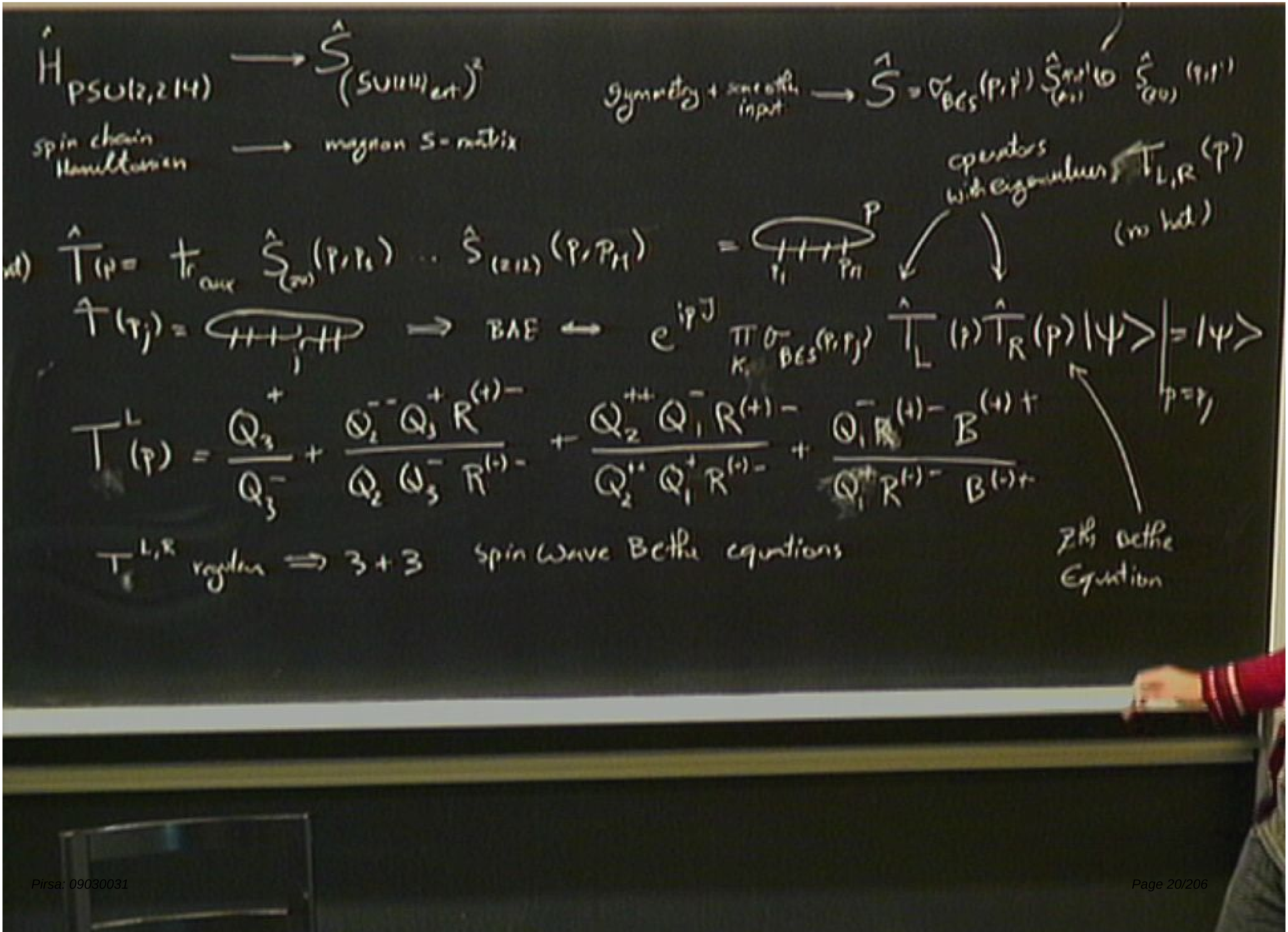
$$\hat{T}(p) = \mathrm{tr}_{aux}\, \hat{S}_{(2|2)}(p,p_1) \ldots \hat{S}_{(2|2)}(p,p_M) = \quad p \quad p_1 \ \ p_M$$

$$\hat{T}(p_j) = \quad j \quad \Longrightarrow \text{BAE} \Longleftrightarrow e^{ipJ} \prod_{k} \sigma_{BES}(p,p_j)\, \hat{T}_L(p)\, \hat{T}_R(p)\, |\psi\rangle \Big|_{p=p_j} = |\psi\rangle$$

$$T^L(p) = \frac{Q_3^+}{Q_3^-} + \frac{Q_2^{--} Q_3^+ R^{(+)-}}{Q_2\, Q_3^-\, R^{(-)-}} + \frac{Q_2^{++} Q_1^- R^{(+)-}}{Q_2^{+} Q_1^+ R^{(-)-}} + \frac{Q_1^- R^{(+)-} B^{(+)+}}{Q_1^+ R^{(-)-} B^{(-)+}}$$

zero Bethe Equation

$T^{L,R}$ regular $\Longrightarrow$ 3 + 3 spin Wave Bethe equations



$\hat{H}_{PSU(2,2|4)} \longrightarrow \hat{S}_{(SU(2|2)_{ext})^2}$

spin chain Hamiltonian ⟶ magnon S-matrix

symmetry + some other input $\longrightarrow \hat{S} = \sigma_{BES}(p,p')\,\hat{S}^{(p,p')}_{(2|2)} \otimes \hat{S}_{(2|2)}(p,p')$

operators with eigenvalues $T_{L,R}(p)$

(no hat)

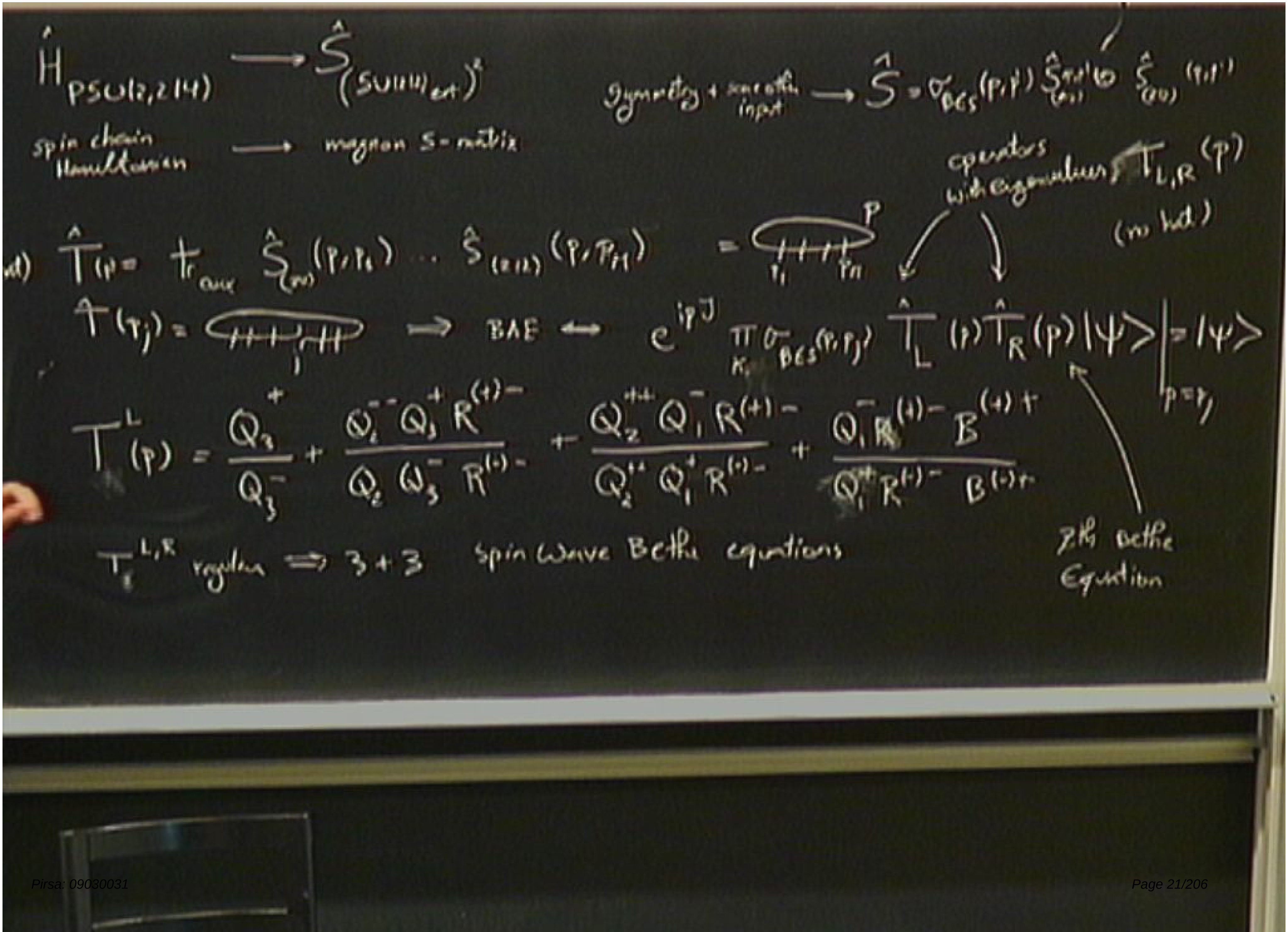
$\hat{T}(p) = \mathrm{tr}_{aux}\, \hat{S}_{(2|2)}(p,p_1) \cdots \hat{S}_{(2|2)}(p,p_M) =$

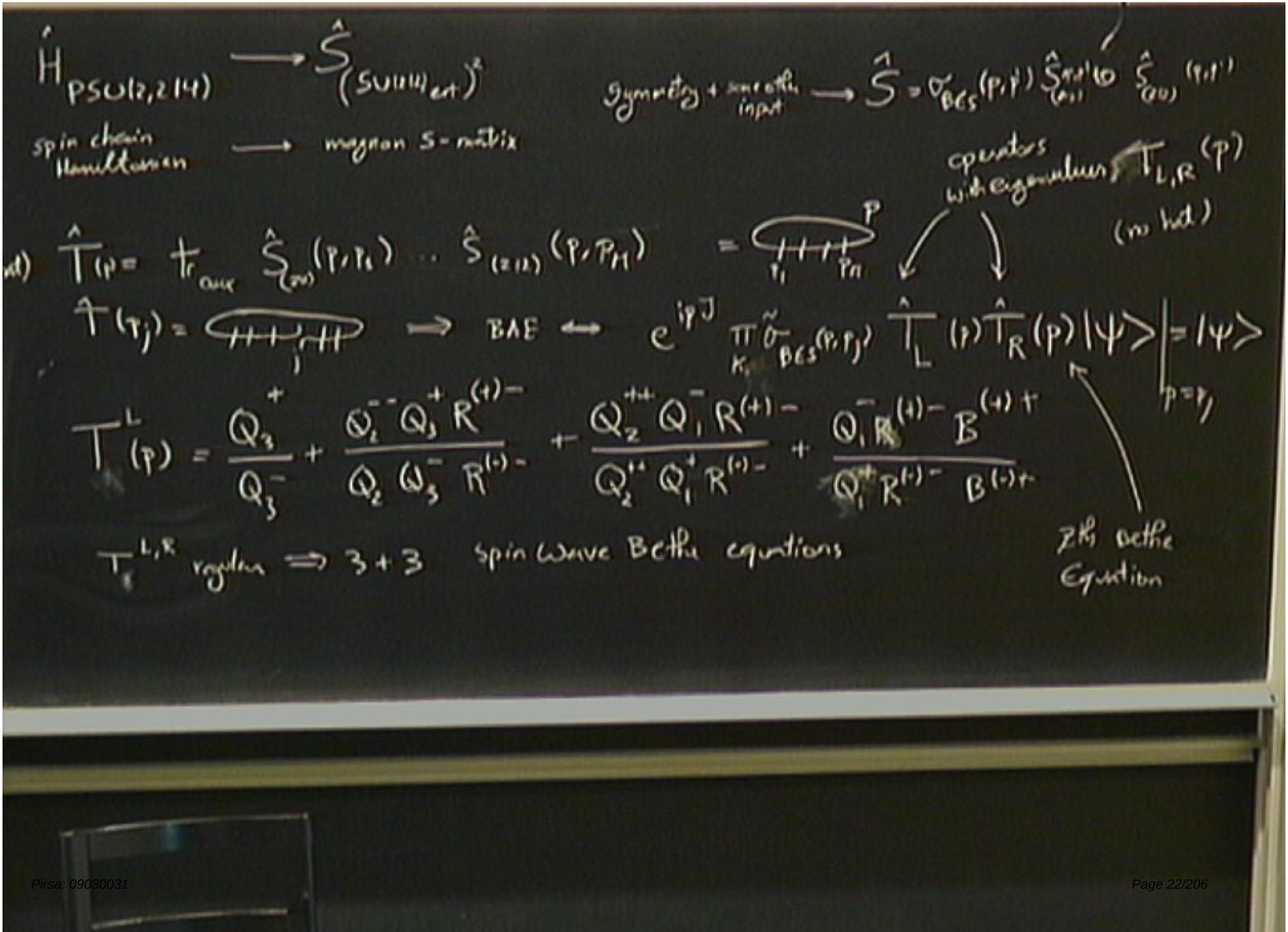
$T(p_j) =$ ⟹ BAE ⟺ $e^{ipJ} \prod_{k} \sigma_{BES}(p,p_j)\, \hat{T}_L(p)\hat{T}_R(p)|\psi\rangle\Big|_{p=p_j} = |\psi\rangle$

$$T^L(p) = \frac{Q_3^{+}}{Q_3^{-}} + \frac{Q_2^{--}Q_3^{+}R^{(+)-}}{Q_2\, Q_3^{-}\, R^{(-)-}} + \frac{Q_2^{++}Q_1^{-}R^{(+)-}}{Q_2^{+}Q_1^{+}R^{(-)-}} + \frac{Q_1^{-}R^{(+)-}\,B^{(+)+}}{Q_1^{+}R^{(-)-}\,B^{(-)+}}$$

$T_1^{L,R}$ regular ⟹ 3 + 3 spin wave Bethe equations

7th Bethe Equation





$$\hat{H}_{PSU(2,2|4)} \longrightarrow \hat{S}_{(SU(2|2)_{ext})^2}$$

spin chain Hamiltonian ⟶ magnon S-matrix

symmetry + some other input $\longrightarrow \hat{S} = \sigma_{BES}(p,p')\,\hat{S}^{(p,p')}_{(2|2)} \otimes \hat{S}_{(2|2)}(p,p')$

operators with eigenvalues $T_{L,R}(p)$ (no hat)

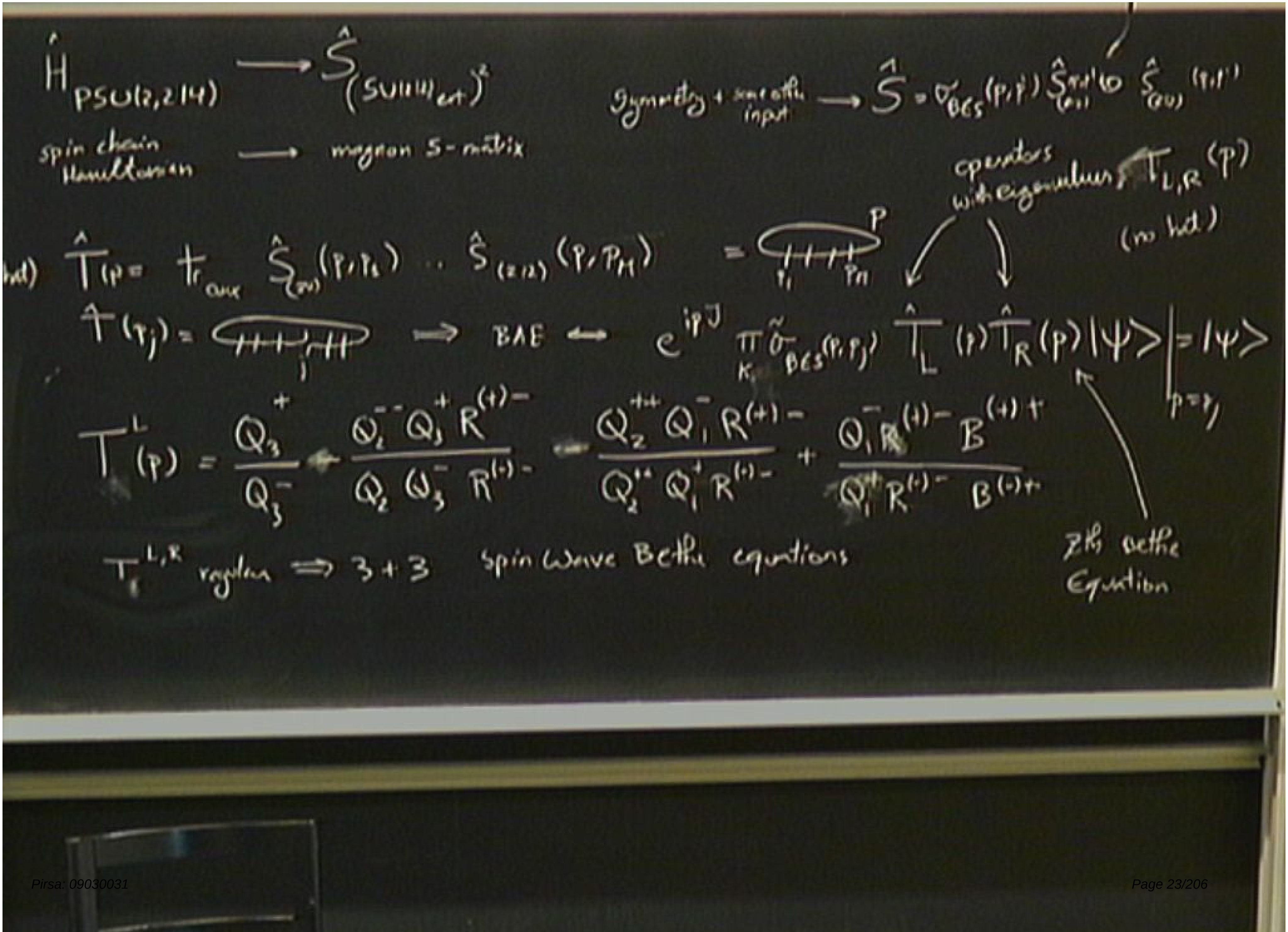
ad) $\hat{T}(p) = \mathrm{tr}_{aux}\, \hat{S}_{(2|2)}(p,p_1) \ldots \hat{S}_{(2|2)}(p,p_M) = \overset{p}{\underset{p_1 \quad p_M}{\bigcirc}}$

$\hat{T}(p_j) = \underset{j}{\bigcirc} \Longrightarrow$ BAE $\Longleftrightarrow$ $e^{ipJ} \prod_{k} \tilde{\sigma}_{BES}(p,p_j)\, \hat{T}_L(p)\hat{T}_R(p)|\psi\rangle\Big|_{p=p_j} = |\psi\rangle$

$$T^L(p) = \frac{Q_3^+}{Q_3^-} + \frac{Q_2^{--}Q_3^+R^{(+)-}}{Q_2 Q_3^- R^{(-)-}} + \frac{Q_2^{++}Q_1^-R^{(+)-}}{Q_2^{+}Q_1^+R^{(-)-}} + \frac{Q_1^-R^{(+)-}B^{(+)+}}{Q_1^+R^{(-)-}B^{(-)+}}$$

$T_1^{L,R}$ regular $\Longrightarrow$ 3 + 3 spin wave Bethe equations

Zhu Bethe Equation



$\hat{H}_{PSU(2,2|4)} \longrightarrow \hat{S}_{(SU(1|1)_{ext})^2}$

spin chain Hamiltonian ⟶ magnon S-matrix

Symmetry + some other input ⟶ $\hat{S} = \sigma_{BES}(p,p')\,\hat{S}^{(p,p')}_{(2|2)} \otimes \hat{S}_{(2|2)}(p,p')$

operators with eigenvalues $T_{L,R}(p)$

(no hd.)

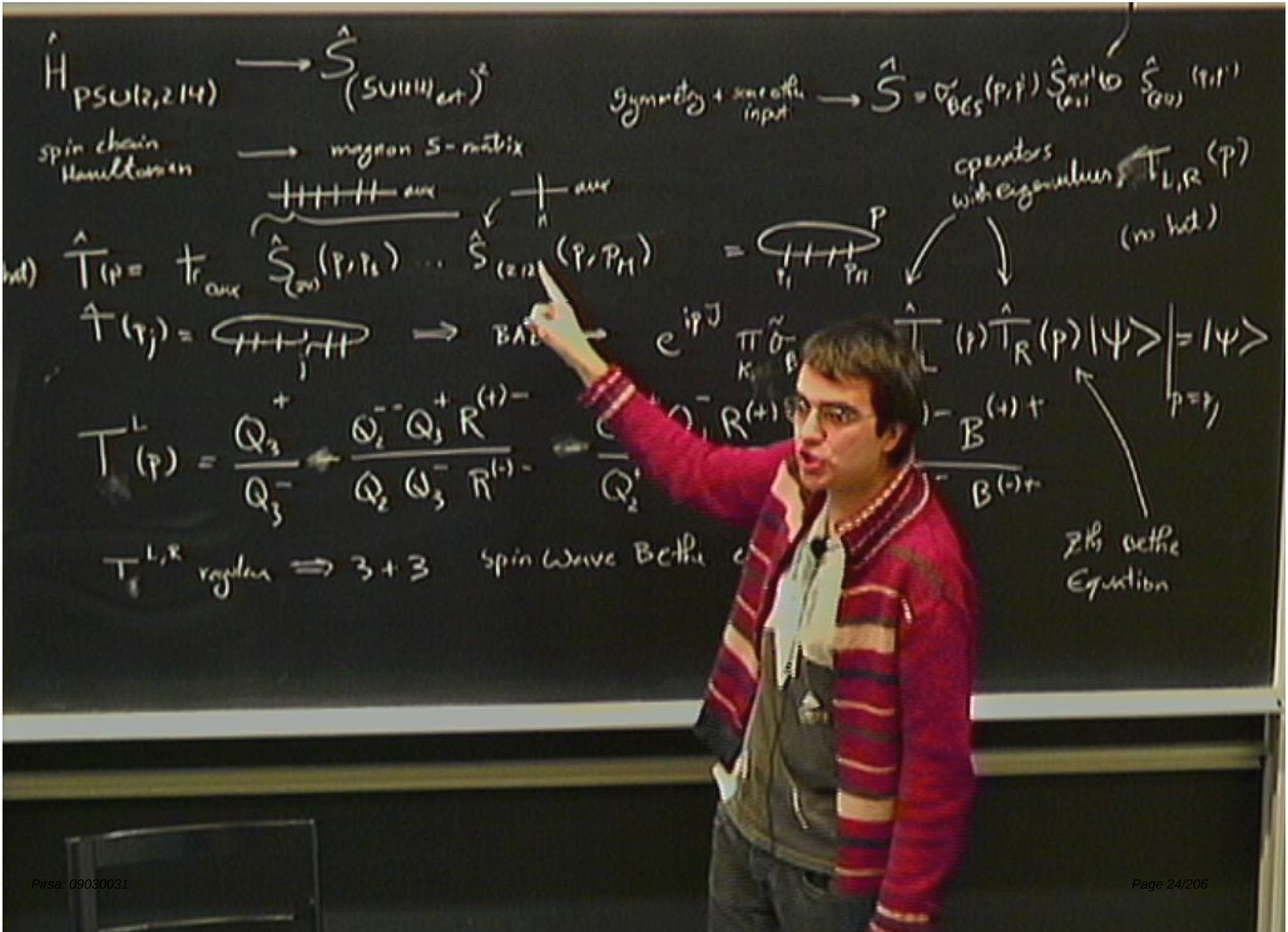
hd) $\hat{T}(p) = \mathrm{tr}_{aux}\, \hat{S}_{(2|2)}(p,p_1) \cdots \hat{S}_{(2|2)}(p,p_M) =$ p p_1 p_M

$\hat{T}(p_j) =$ j $\Longrightarrow$ BAE $\longleftrightarrow$ $e^{ipJ} \prod_{K} \tilde{\sigma}_{BES}(p,p_j)\, \hat{T}_L(p)\hat{T}_R(p)|\psi\rangle\Big|_{p=p_j} = |\psi\rangle$

$$T^L(p) = \frac{Q_3^+}{Q_3^-} - \frac{Q_2^{--}Q_3^+R^{(+)-}}{Q_2\,Q_3^-\,R^{(-)-}} - \frac{Q_2^{++}Q_1^-R^{(+)-}}{Q_2^{+}Q_1^+R^{(-)-}} + \frac{Q_1^-R^{(+)-}B^{(+)+}}{Q_1^+R^{(-)-}B^{(-)+}}$$

$T_1^{L,R}$ regular $\Longrightarrow$ 3+3 spin wave Bethe equations

7th Bethe Equation



$\hat{H}_{PSU(2,2|4)} \longrightarrow \hat{S}_{(SU(1|1)_{ext})^2}$

Symmetry + some other input $\longrightarrow \hat{S} = \sigma_{BES}(p,p')\, \hat{S}_{(2|1)}(p,p') \otimes \hat{S}_{(2|1)}(p,p')$

spin chain Hamiltonian $\longrightarrow$ magnon S-matrix

aux

aux

operators with eigenvalues $T_{L,R}(p)$

(no lvl.)

lvl.) $\hat{T}(p) = \mathrm{tr}_{aux}\, \hat{S}_{(2|1)}(p,p_1) \ldots \hat{S}_{(2|1)}(p,p_M)$ =

p_1 p_M p

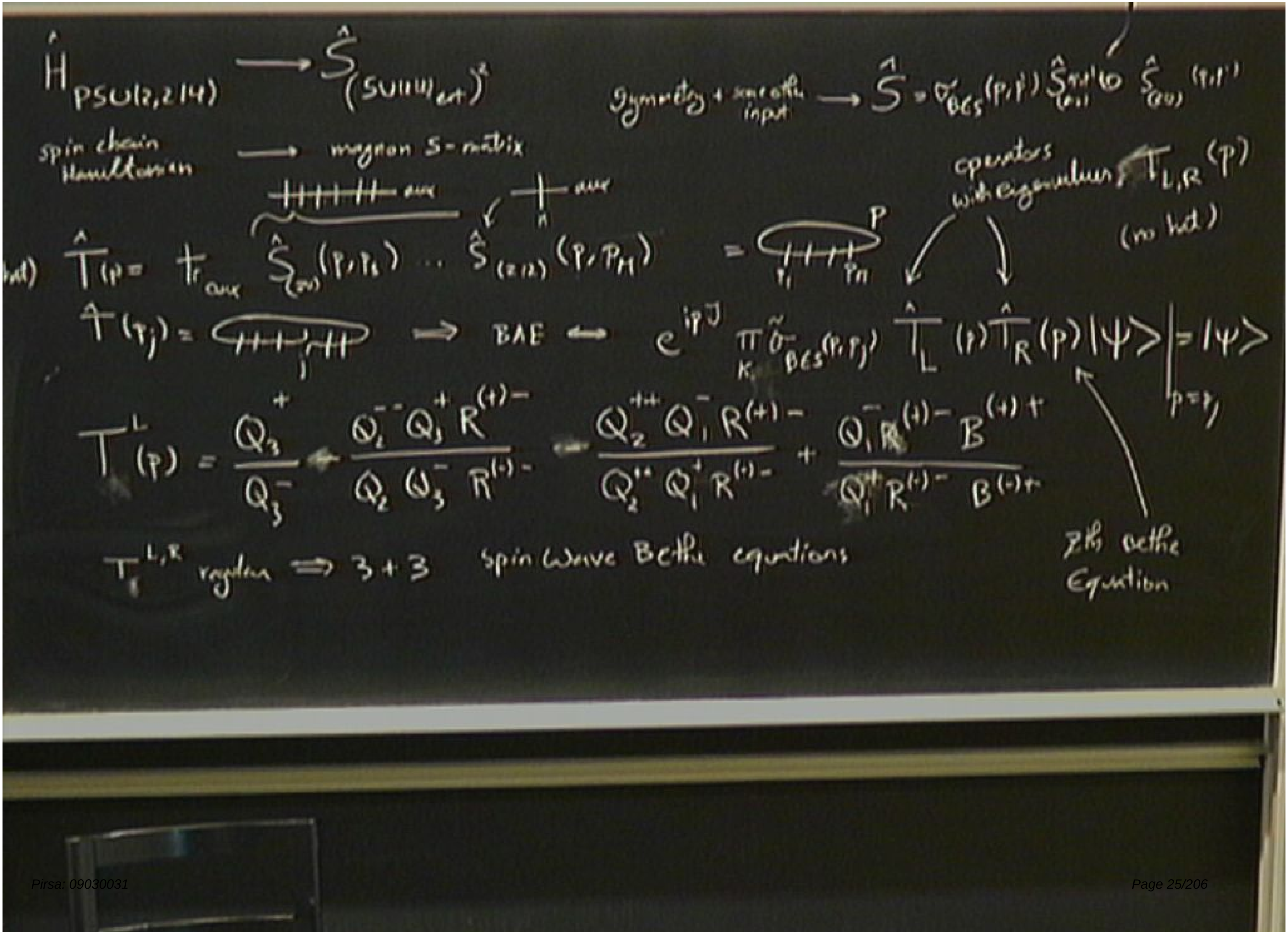
$\hat{T}(p_j) =$ j $\Longrightarrow$ BAE $\Longrightarrow$ $e^{ipJ} \prod_{K_i} \tilde{\sigma}_{B}$

$\hat{T}_L(p)\, \hat{T}_R(p)\, |\psi\rangle \Big|_{p=p_j} = |\psi\rangle$

$T^L(p) = \frac{Q_3^+}{Q_3^-} \leftarrow \frac{Q_2^{--} Q_3^+ R^{(+)-}}{Q_2\, Q_3^-\, R^{(-)-}} \leftarrow \frac{Q}{Q_2^+}\; Q_1^- R^{(+)} \quad)- \frac{B^{(+)+}}{B^{(-)+}}$

$T_1^{L,R}$ regular $\Longrightarrow$ 3 + 3 spin wave Bethe e

2nd bethe Equation



$\hat{H}_{PSU(2,2|4)} \longrightarrow \hat{S}_{(SU(1|1)_{ext})^2}$

spin chain Hamiltonian ⟶ magnon S-matrix

symmetry + some other input $\longrightarrow \hat{S} = \sigma_{BES}(p,p')\,\hat{S}^{(p,p')}_{(2|2)} \otimes \hat{S}_{(2|2)}(p,p')$

operators with eigenvalues $T_{L,R}(p)$

(no hat)

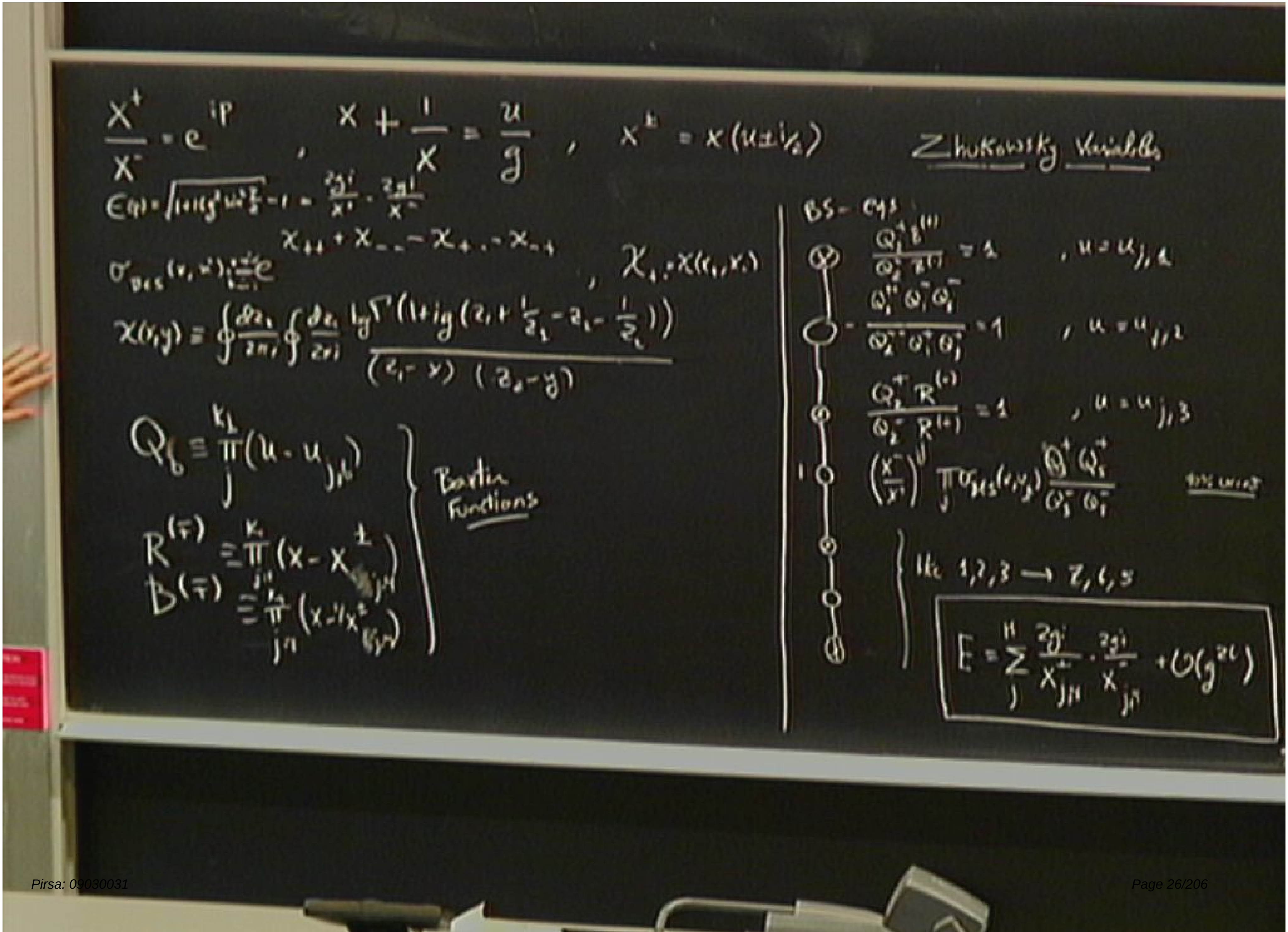
hd) $\hat{T}(p) = \mathrm{tr}_{aux}\, \underbrace{\hat{S}_{(2|2)}(p,p_1) \cdots}\, \hat{S}_{(2|2)}(p,p_M)$ =

$\hat{T}(p_j) = \;\Longrightarrow\;$ BAE $\longleftrightarrow$ $e^{ipJ} \prod_{k} \tilde{\sigma}_{BES}(p,p_j)\; \hat{T}_L(p)\hat{T}_R(p)|\psi\rangle\Big|_{p=p_j} = |\psi\rangle$

$$T^L(p) = \frac{Q_3^+}{Q_3^-} - \frac{Q_2^{--}Q_3^+R^{(+)-}}{Q_2 Q_3^- R^{(-)-}} - \frac{Q_2^{++}Q_1^- R^{(+)-}}{Q_2^{++}Q_1^+R^{(-)-}} + \frac{Q_1^- R^{(+)-} B^{(+)+}}{Q_1^+ R^{(-)-} B^{(-)+}}$$

$T_1^{L,R}$ regular $\Longrightarrow$ 3 + 3 spin wave Bethe equations

7th Bethe Equation



$$\frac{x^+}{x^-} = e^{ip} \;,\quad x + \frac{1}{x} = \frac{u}{g} \;,\quad x^\pm = x(u \pm i/2)$$

Zhukowsky Variables

$$\epsilon(q) = \sqrt{1+16g^2\sin^2\frac{p}{2}} - 1 = \frac{2gi}{x^+} - \frac{2gi}{x^-}$$

$$\sigma_{BES}(x,x') = e^{\chi_{++}+\chi_{--}-\chi_{+-}-\chi_{-+}} \;,\quad \chi_{+-} = \chi(x_1^+, x_2^-)$$

$$\chi(x,y) = \oint\frac{dz_1}{2\pi i}\oint\frac{dz_2}{2\pi i}\frac{\log\Gamma\left(1+ig\left(z_1+\frac{1}{z_1}-z_2-\frac{1}{z_2}\right)\right)}{(z_1-x)(z_2-y)}$$

$$Q_l \equiv \prod_j^{K_l}(u-u_{j,l})$$

$$R^{(\mp)} \equiv \prod_{j}^{K_4}(x-x^\pm_{j,4})$$

$$B^{(\mp)} \equiv \prod_{j}^{K_4}(x^{-1}-x^\pm_{j,4})$$

Baxter Functions

BS-eqs

$$\frac{Q_2^+ B^{(+)}}{Q_2^- B^{(-)}} = 1 \;,\quad u = u_{j,1}$$

$$-\frac{Q_2^{++} Q_1^- Q_3^-}{Q_2^{--} Q_1^+ Q_3^+} = -1 \;,\quad u = u_{j,2}$$

$$\frac{Q_2^+ R^{(-)}}{Q_2^- R^{(+)}} = 1 \;,\quad u = u_{j,3}$$

$$\left(\frac{x^-}{x^+}\right)^L \prod_j \sigma_{BES}(u,u_j)\frac{Q_3^+ Q_5^+}{Q_3^- Q_5^-}$$

this wing

like 1,2,3 → 7,6,5

$$\boxed{E = \sum_j^{K_4}\frac{2gi}{x^+_{j,4}} - \frac{2gi}{x^-_{j,4}} + O(g^{2L})}$$

1+1 d QFT

Suppose m_∞ is known , $m_L = ?$

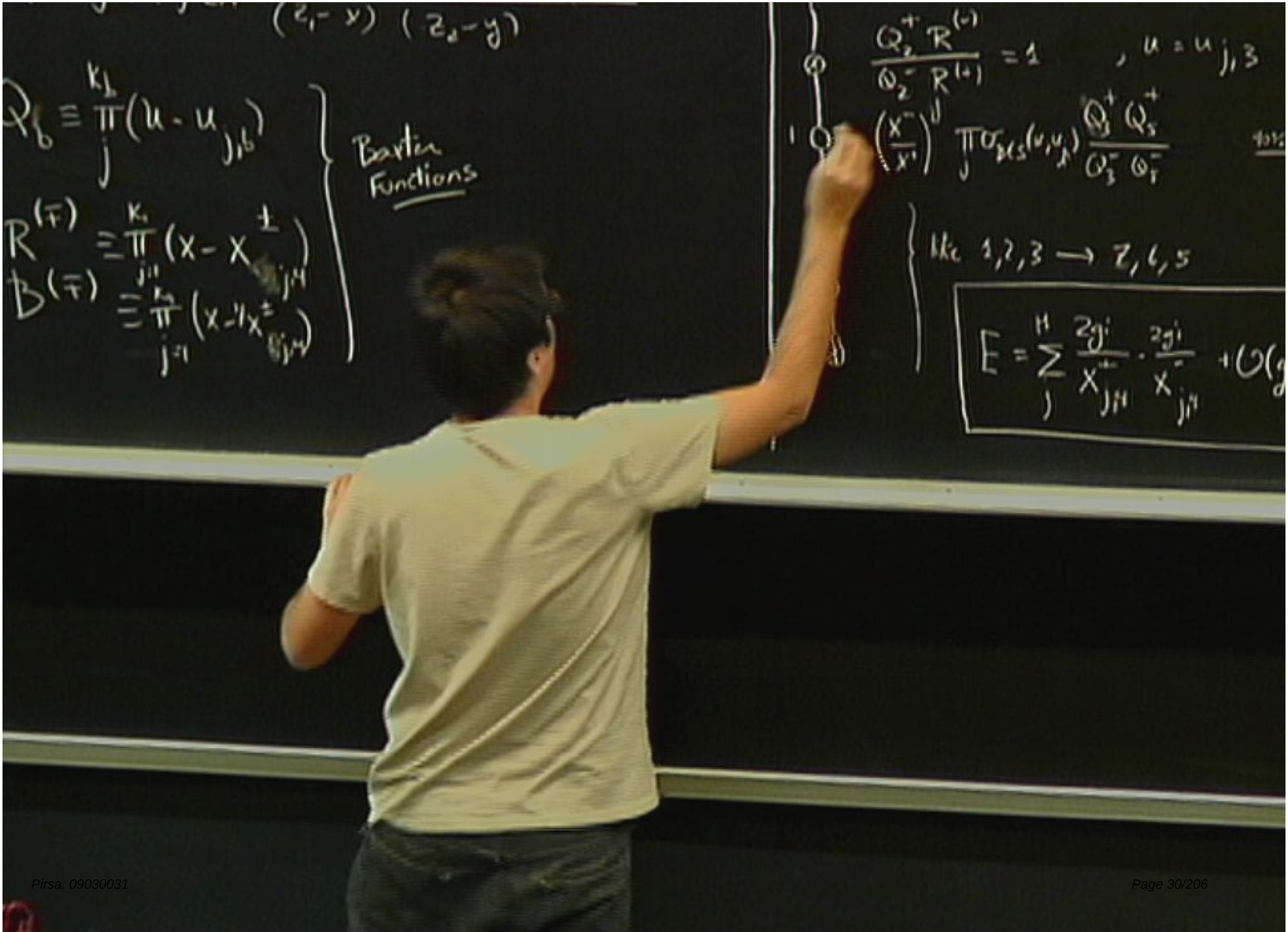
1+1 d QFT

Suppose m_∞ is known , $m_L = ?$

1+1 d QFT

Suppose m_∞ is known, $m_L = ?$

$\Sigma \quad \delta_L m \leftrightarrow \delta_L \Sigma = ?$



$(z_1 - x) \quad (z_2 - y)$

$Q_b \equiv \prod_j^{K_b} (u - u_{j,b})$

$R^{(\mp)} \equiv \prod_{j=1}^{K_4} (x - x^{\pm}_{j,4})$

$B^{(\mp)} \equiv \prod_{j=1}^{K_4} (x^{-1} - x^{\pm}_{j,4})$

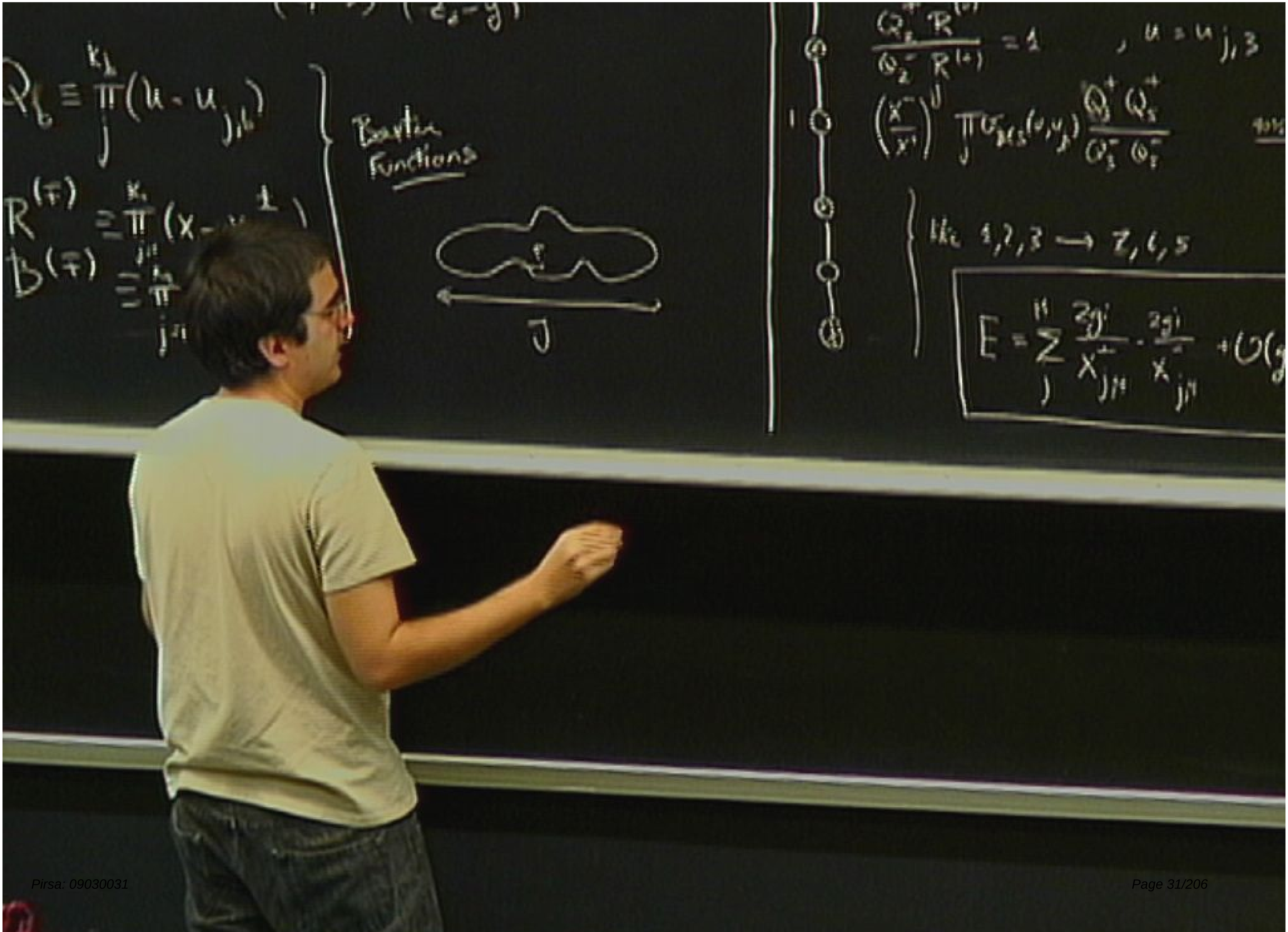
Baxter Functions

$\frac{Q_2^+ R^{(-)}}{Q_2^- R^{(+)}} = 1 \quad , u = u_{j,3}$

$\left(\frac{x^-}{x^+}\right)^J \prod_j \sigma_{BES}(u, u_j) \frac{Q_3^+ Q_5^+}{Q_3^- Q_5^-}$

like 1,2,3 ⟶ 7,6,5

$$\boxed{E = \sum_j^{K_4} \frac{2gi}{x^+_{j,4}} - \frac{2gi}{x^-_{j,4}} + O(g}$$


$Q_\ell \equiv \prod_j^{K_\ell}(u - u_{j,\ell})$
Baxter Functions
$R^{(\mp)} = \prod_{j=1}^{K_4}(x - x \ldots$
$B^{(\mp)} = \prod_{j=1}^{K_4}$
J
$\frac{Q_2^+ R^{(-)}}{Q_2^- R^{(+)}} = 1 \quad , u = u_{j,3}$
$\left(\frac{x^-}{x^+}\right)^J \prod_j \sigma_{BES}(u,u_j) \frac{Q_3^+ Q_5^{++}}{Q_3^- Q_5^-}$
like 1,2,3 → 7,6,5
$E = \sum_j^{K_4} \frac{2gi}{x^+_{j,4}} - \frac{2gi}{x^-_{j,4}} + O(g$

1+1 d QFT

Suppose m_∞ is known, $m_L = ?$

$\delta_L m \leftrightarrow \delta_L \Sigma = ?$

1+1 d QFT

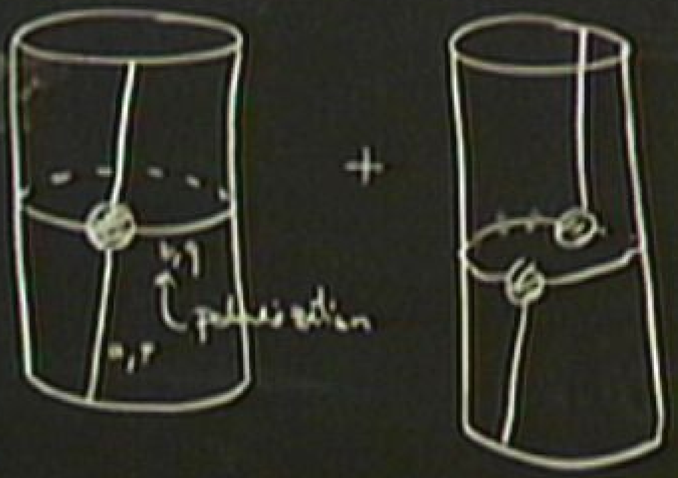
Suppose m_∞ is known, $m_L = ?$

Σ $\quad \delta_L m \longleftrightarrow \delta_L \Sigma =$

1+1 d QFT

Suppose m_∞ is known, $m_L = ?$

Σ $\quad \delta_L m \leftrightarrow \delta_L \Sigma =$

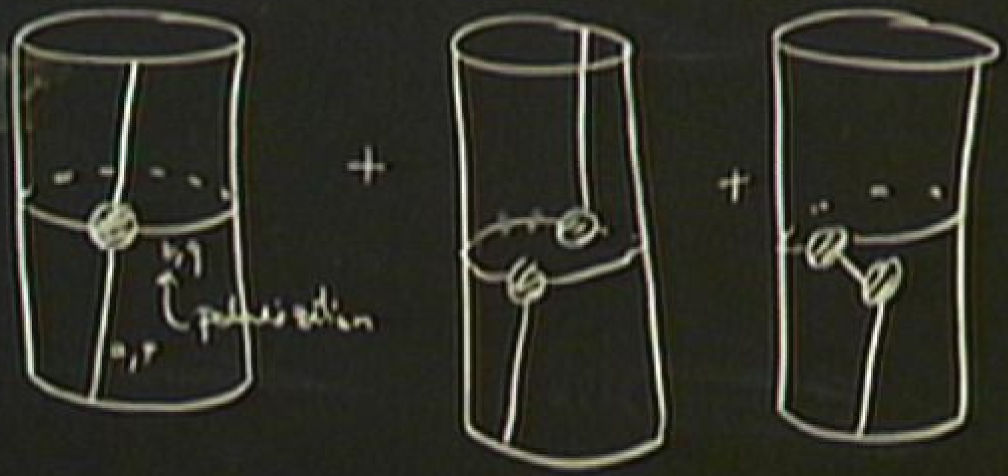


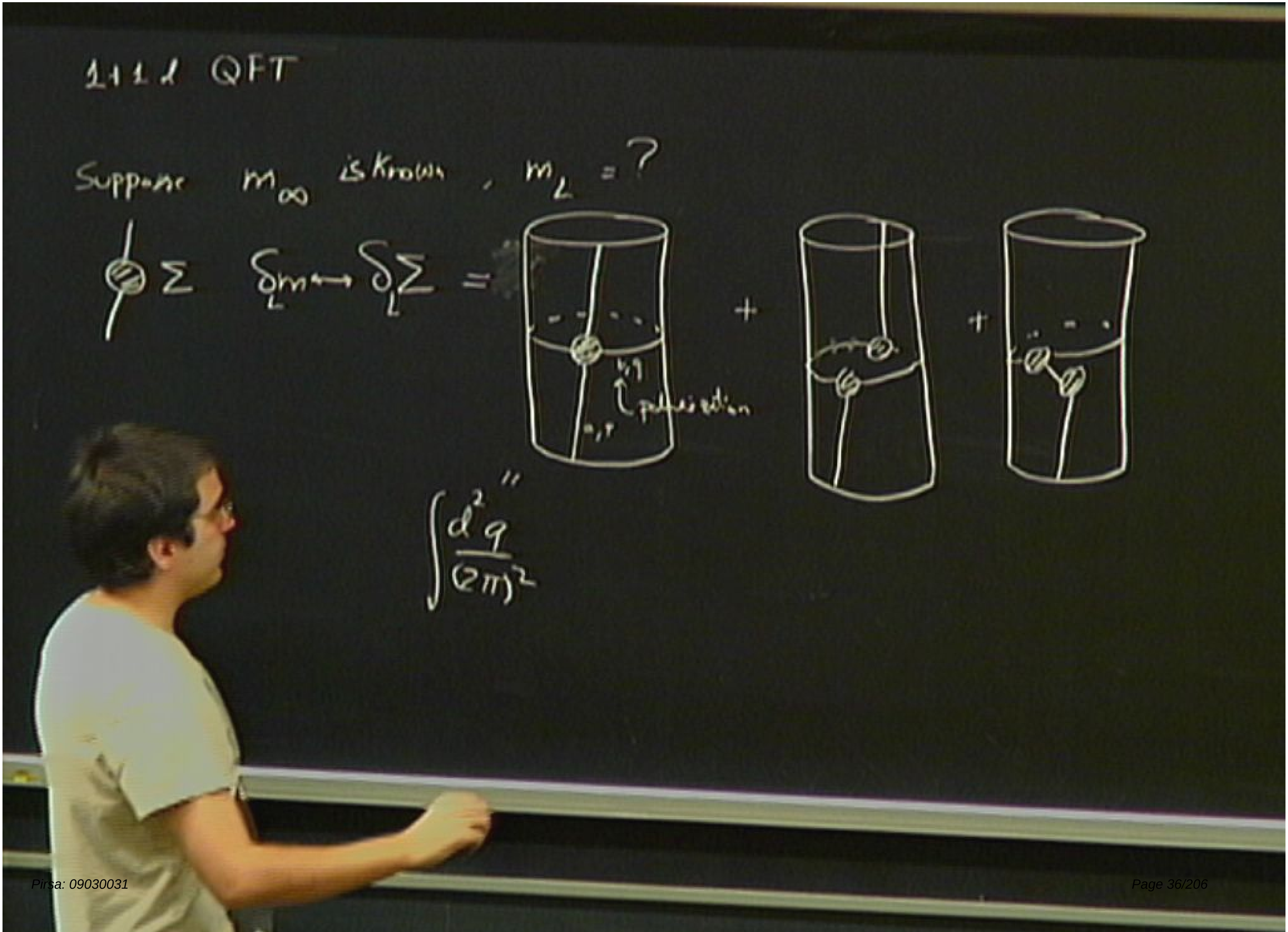
1+1 d QFT

Suppose m_∞ is known, $m_L = ?$

Σ

$\delta_L m \leftrightarrow \delta_L \Sigma =$





1+1 d QFT

Suppose m_∞ is known, $m_L = ?$

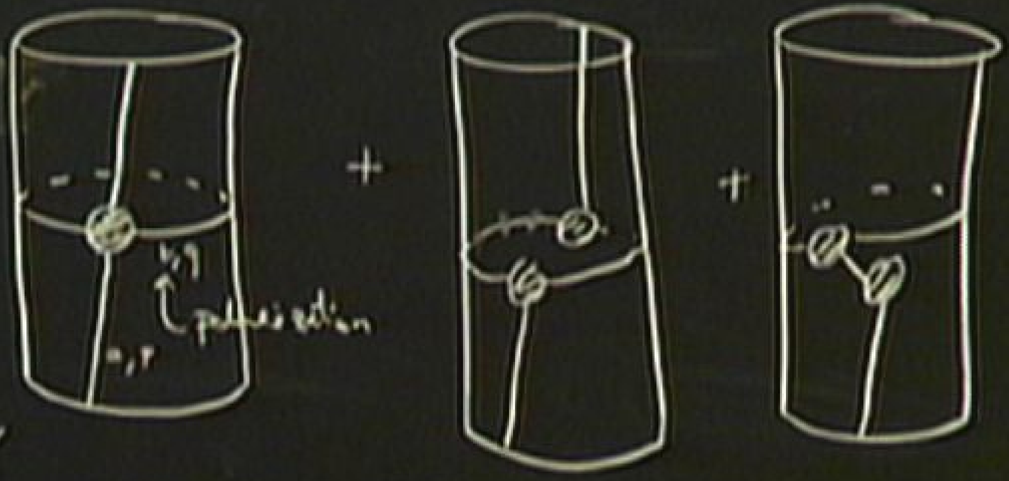
$\Sigma \quad \delta_L m \longleftrightarrow \delta_L \Sigma = \quad + \quad +$

$$\int \frac{d^2 q}{(2\pi)^2} \quad ''$$

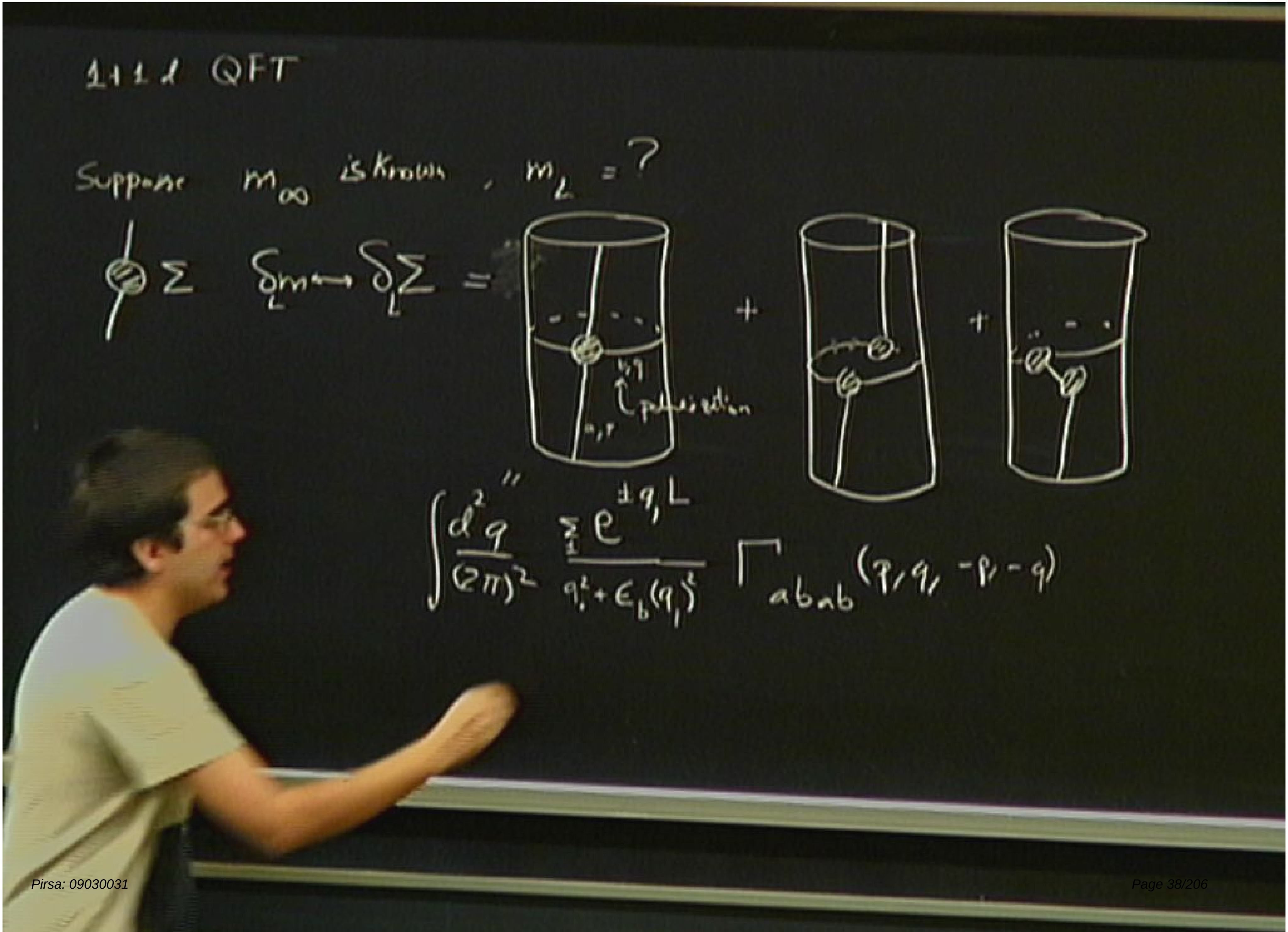
1+1 d QFT

Suppose m_∞ is known, $m_L = ?$

 Σ $\quad \delta_L m \longleftrightarrow \delta_L \Sigma =$



$$\text{"} \int \frac{d^2 q}{(2\pi)^2} \frac{1}{q_0^2 + \epsilon_b(q_1)^2} \Gamma_{abab}(p, q, -p, -q)$$



1+1 d QFT

Suppose m_∞ is known, $m_L = ?$

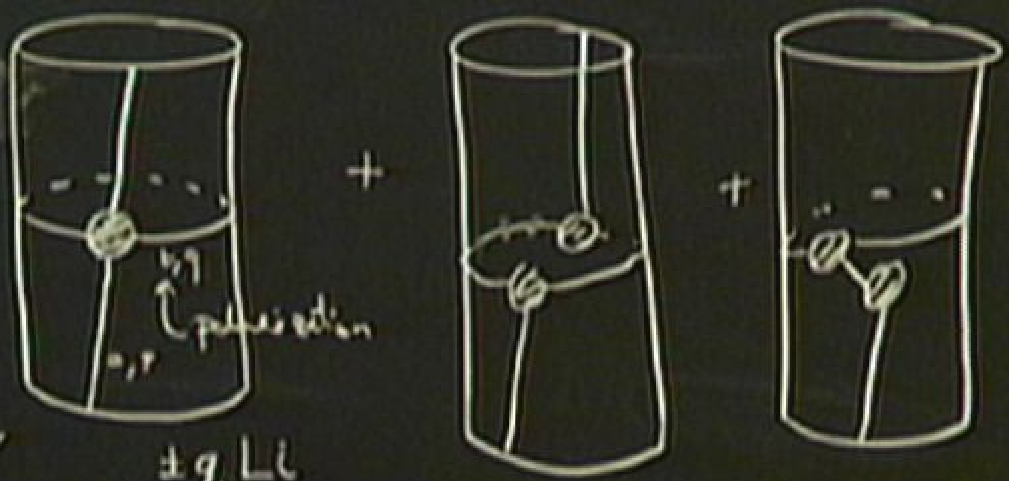
Σ $\quad \delta_L m \longleftrightarrow \delta_L \Sigma =$

$$\int \frac{d^2 q}{(2\pi)^2} {}'' \frac{\sum_{\pm} e^{\pm q_1 L}}{q_0^2+\epsilon_b(q_1)^2} \Gamma_{abab}(p, q, -p, -q)$$

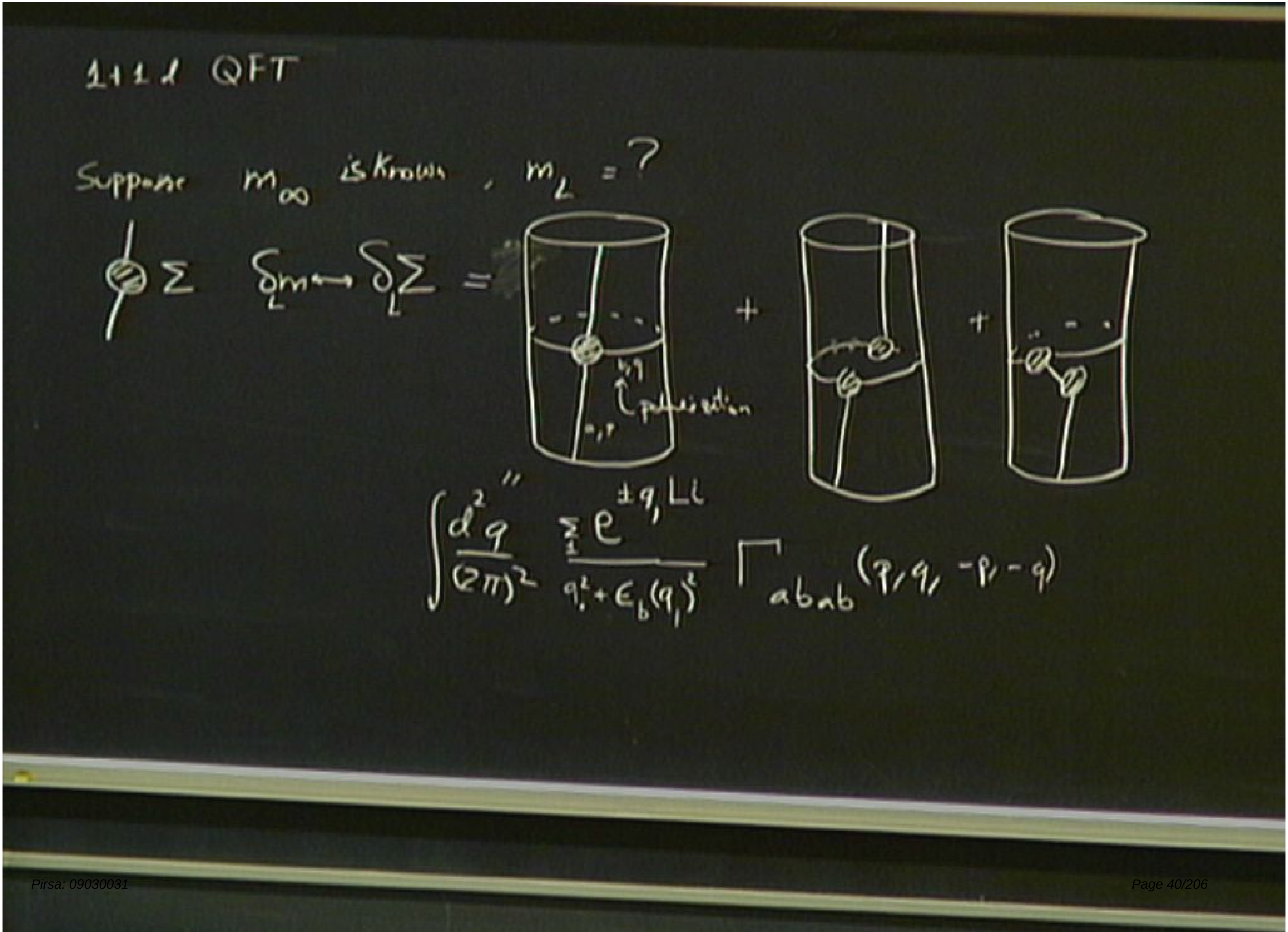
1+1 d QFT

Suppose m_∞ is known, $m_L = ?$

$\Sigma \quad \delta_L m \longleftrightarrow \delta_L \Sigma =$



$$\int \frac{d^2 q}{(2\pi)^2}'' \frac{\sum_{\pm} e^{\pm q_1 L i}}{q_0^2 + E_b(q_1)^2} \Gamma_{abab}(p,q,-p,-q)$$



1+1 d QFT

Suppose m_∞ is known, $m_L = ?$

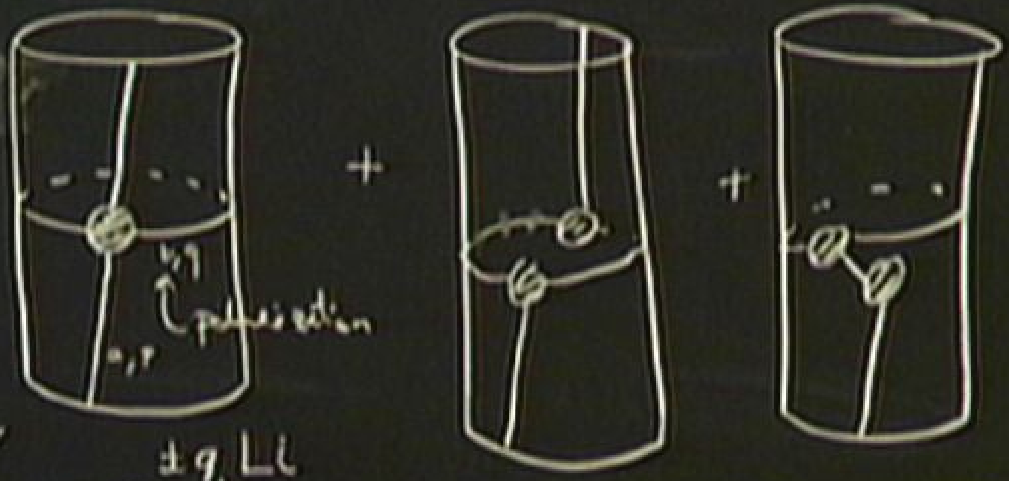
$\Sigma \quad \delta_L m \longleftrightarrow \delta_L \Sigma = \quad + \quad +$

$$\int \frac{d^2q}{(2\pi)^2}{}'' \frac{\sum_{\pm} e^{\pm q_1 L i}}{q_0^2+\epsilon_b(q_1)^2} \Gamma_{abab}(p,q,-p,-q)$$

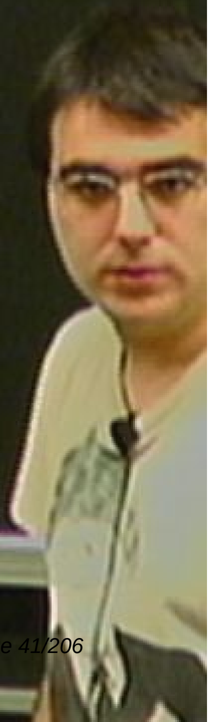
1+1 d QFT

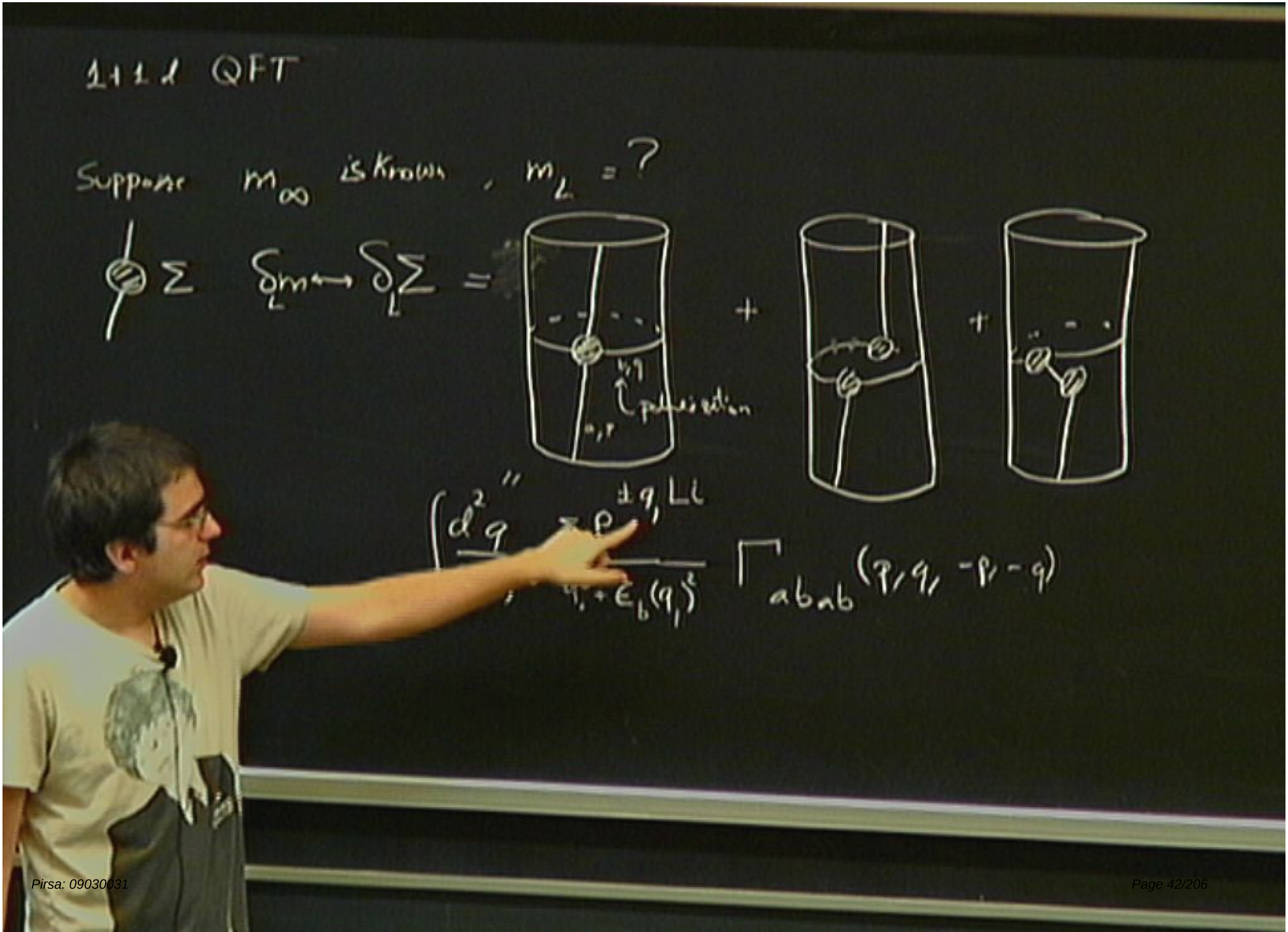
Suppose m_∞ is known , $m_L = ?$

$\Sigma \quad \delta_L m \leftrightarrow \delta_L \Sigma =$



$$\int \frac{d^2 q}{(2\pi)^2}'' \frac{\sum_{\pm} e^{\pm q_1 L i}}{q_1^2 + \epsilon_b(q_1)^2} \Gamma_{abab}(p, q, -p, -q)$$



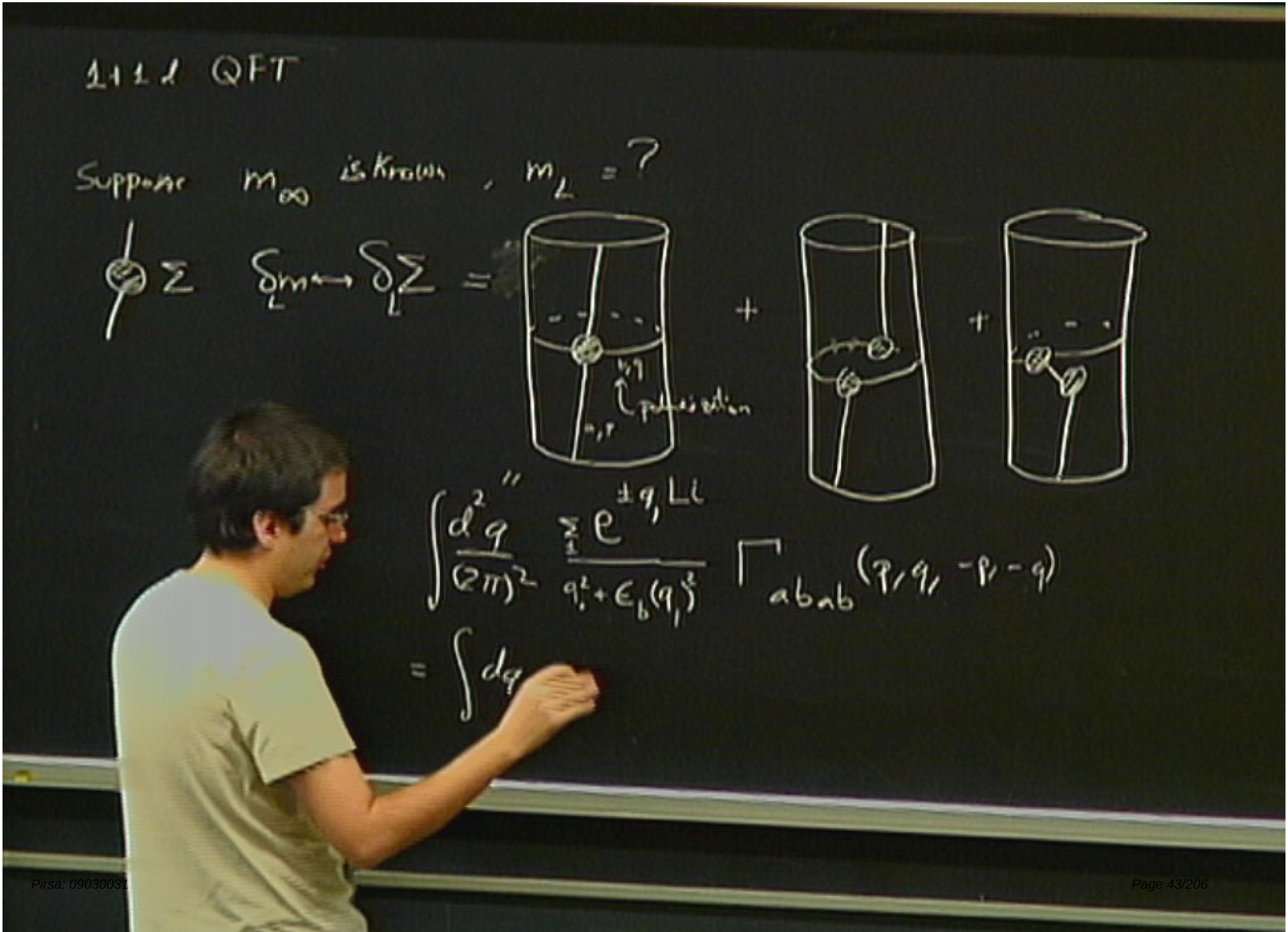


1+1 d QFT

Suppose m_∞ is known, $m_L = ?$

$\Sigma \quad \delta_L m \longleftrightarrow \delta_L \Sigma = \quad + \quad +$

$$\int \frac{d^2q}{\;} \text{''} \; \frac{e^{\pm q_1 L i}}{q_1 + \epsilon_b(q_1)^2} \Gamma_{abab}(p,q,-p-q)$$



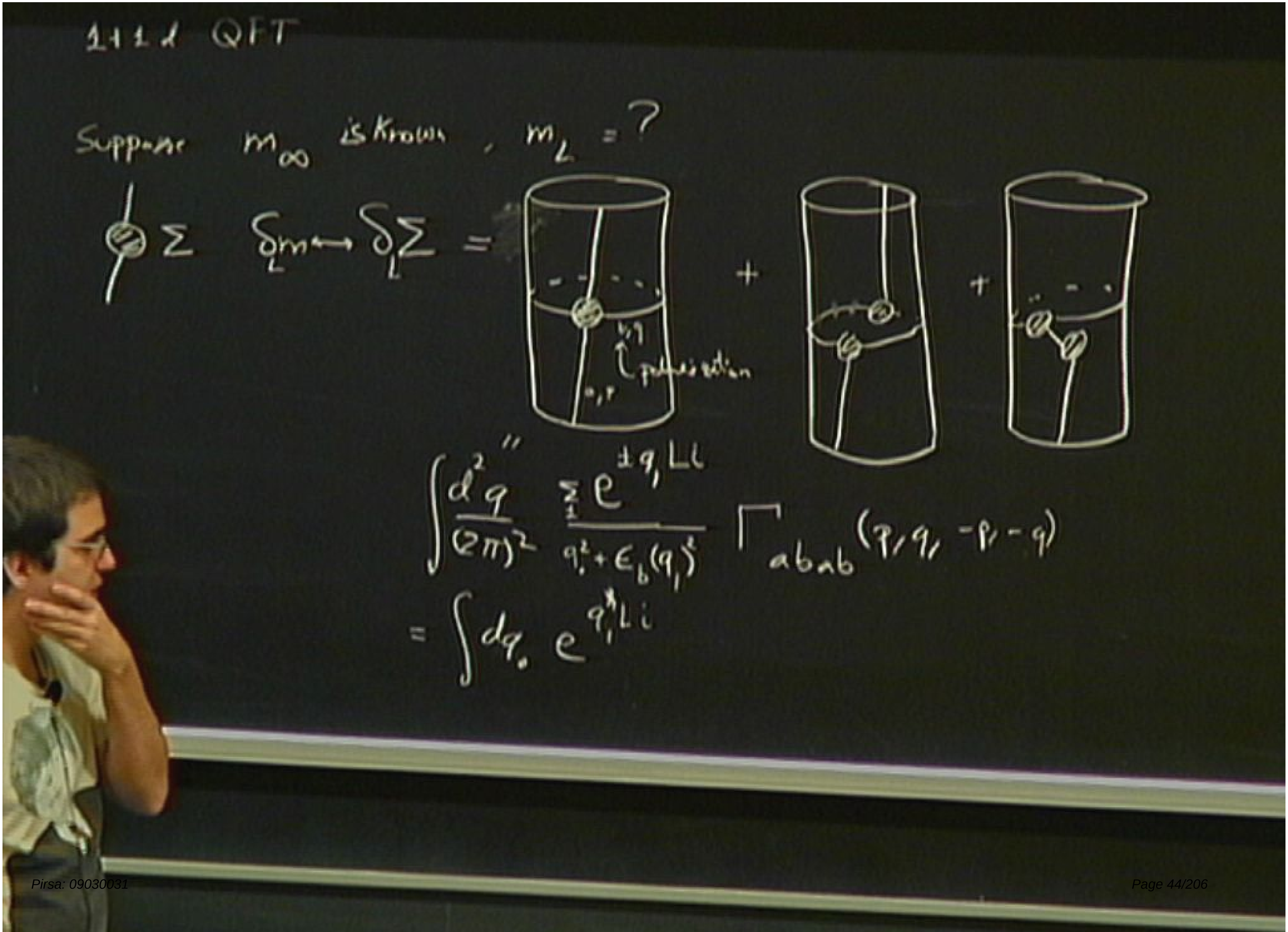
1+1 d QFT

Suppose m_∞ is known , $m_L = ?$

Σ $\quad \delta_L m \longleftrightarrow \delta_L \Sigma =$ + +

$$\int \frac{d^2 q}{(2\pi)^2}'' \frac{\sum_{\pm} e^{\pm q_1 L i}}{q_1^2 + \epsilon_b(q_1)^2} \Gamma_{abab}(p, q, -p, -q)$$

$$= \int dq$$



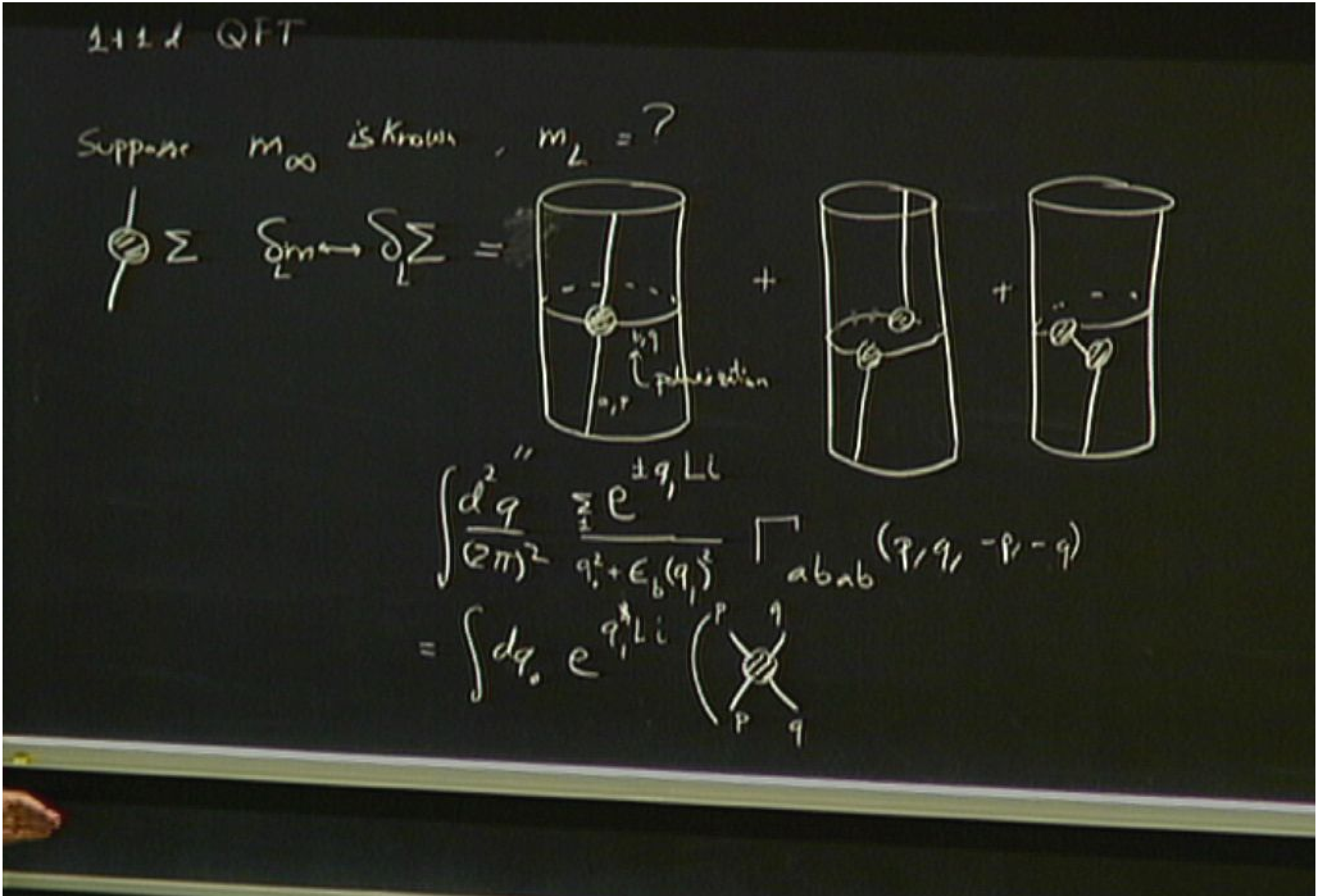
1+1 d QFT

Suppose m_∞ is known , $m_L = ?$

$\Sigma \qquad \delta_L m \longleftrightarrow \delta_L \Sigma =$ + +

$$\text{"}\int \frac{d^2 q}{(2\pi)^2} \frac{\sum_{\pm} e^{\pm q_1 L i}}{q_0^2 + \epsilon_b(q_1)^2} \Gamma_{abab}(p, q, -p, -q)$$

$$= \int dq_0 \, e^{q_1^* L i}$$



1+1 d QFT

Suppose m_∞ is known, $m_L = ?$

$\Sigma \quad \delta_L m \longleftrightarrow \delta_L \Sigma =$ + +

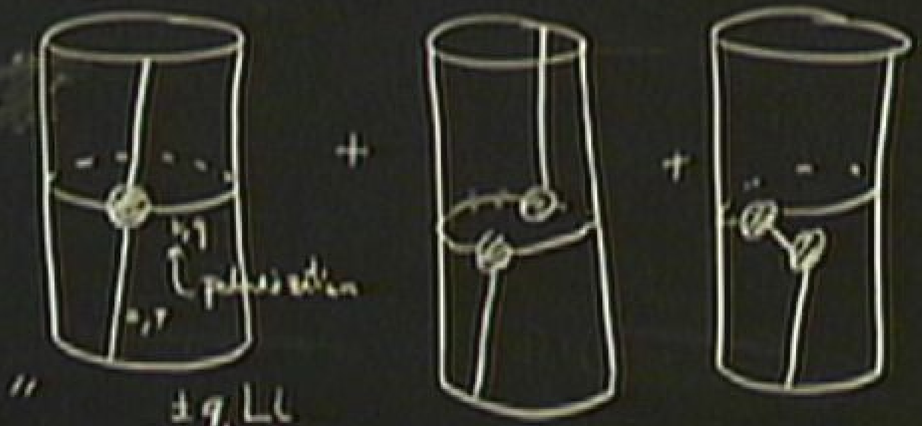
$$\text{''} \int \frac{d^2 q}{(2\pi)^2} \frac{\sum_{\pm} e^{\pm q_1 L i}}{q_0^2 + \epsilon_b(q_1)^2} \Gamma_{abab}(p, q, -p, -q)$$

$$= \int dq_0 \, e^{q_1^* L i} \Big($$

1+1 d QFT

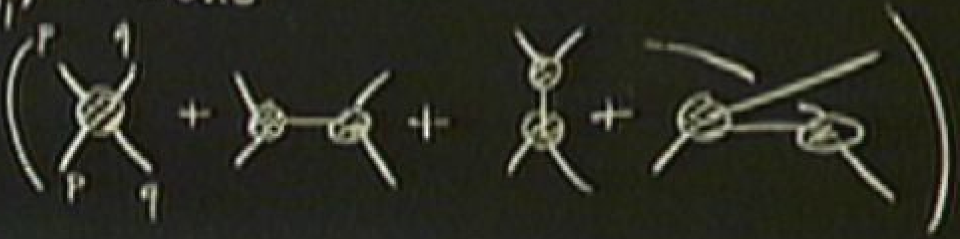
Suppose m_∞ is known, $m_L = ?$

$\Sigma \quad \delta_L m \longleftrightarrow \delta_L \Sigma =$



$$\int \frac{d^2 q}{(2\pi)^2} {}'' \frac{\sum e^{\pm q_1 L i}}{q_0^2 + \epsilon_b(q_1)^2} \Gamma_{abab}(p, q, -p, -q)$$

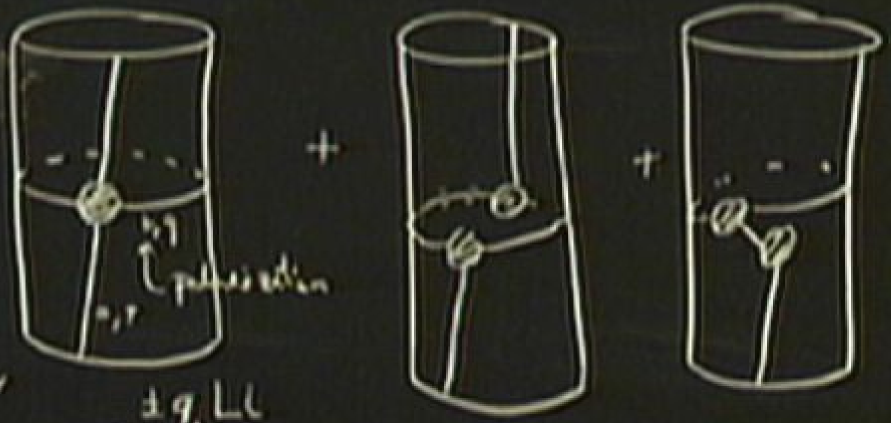
$$= \int dq_0 \, e^{q_1 L i} \Big(\quad \Big)$$



1+1 d QFT

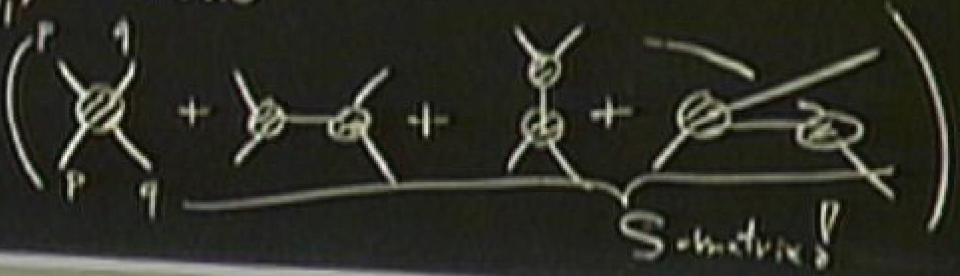
Suppose m_∞ is known, $m_L = ?$

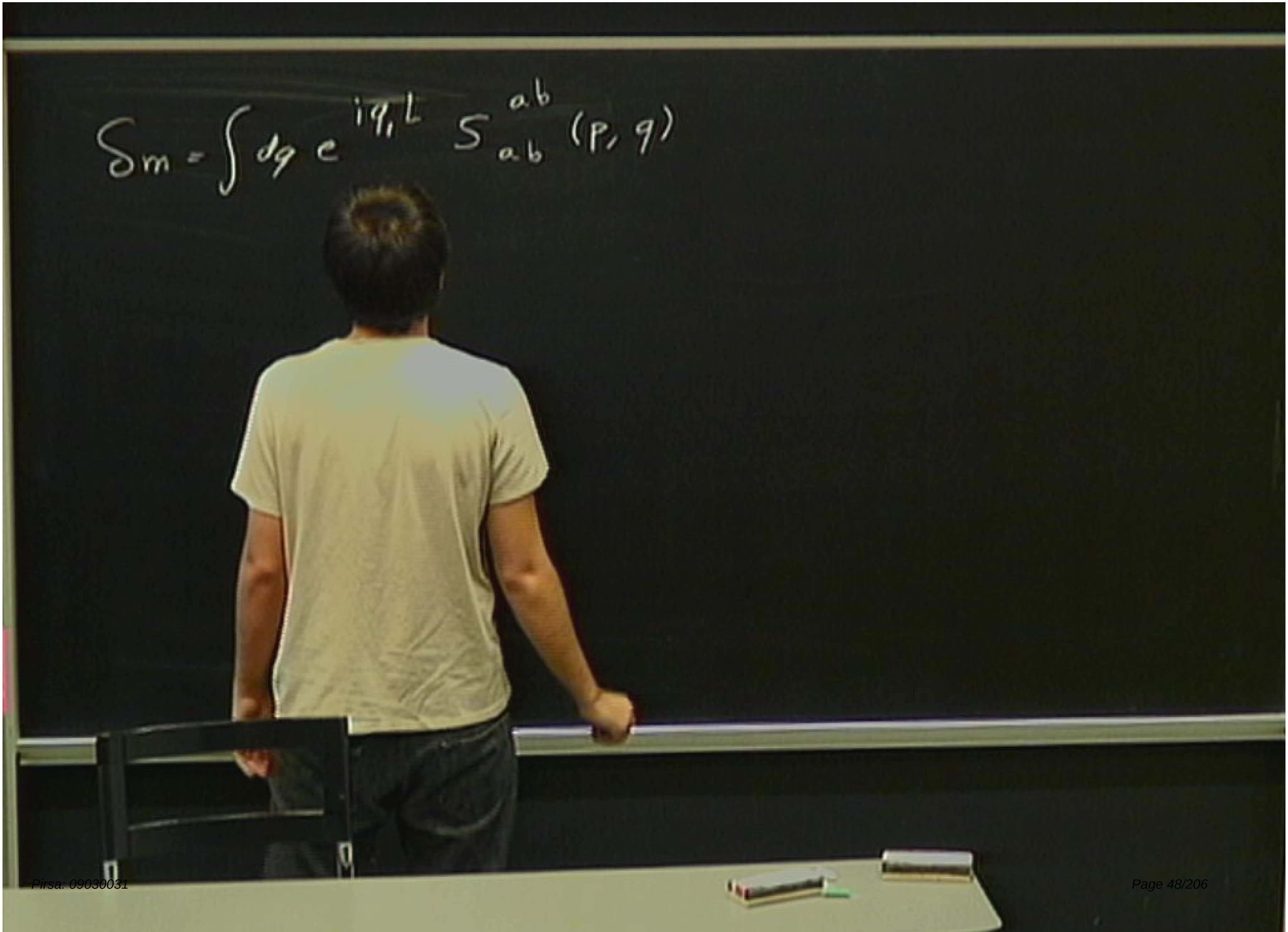
Σ $\quad \delta_L m \longleftrightarrow \delta_L \Sigma =$



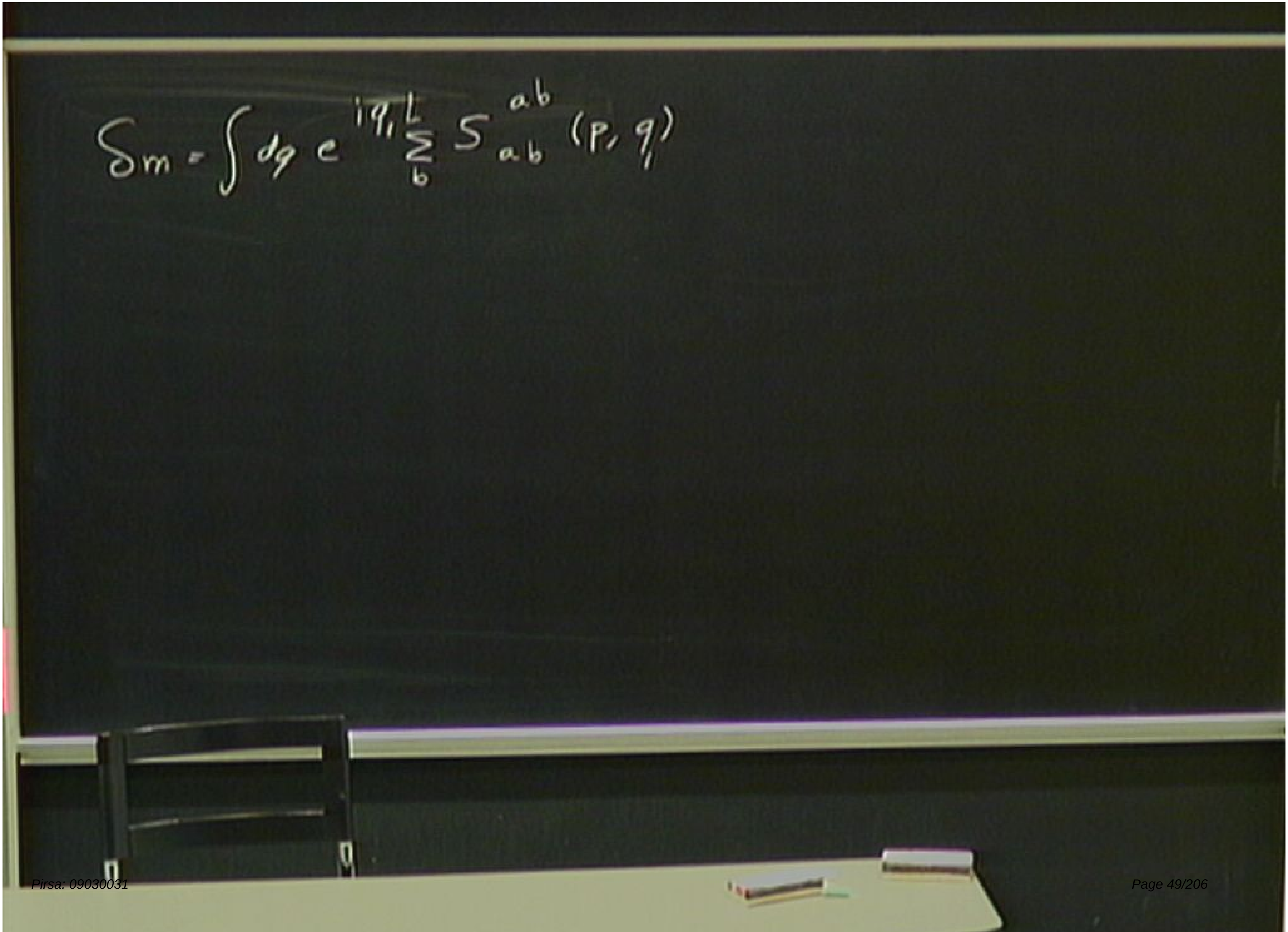
$$\text{''} \int \frac{d^2q}{(2\pi)^2} \frac{\sum_{\pm} e^{\pm q_1 L i}}{q_0^2+\epsilon_b(q_1)^2} \Gamma_{abab}(p,q,-p,-q)$$

$$= \int dq_0 \, e^{q_1^{*} L i} \Bigg(\quad \Bigg)$$

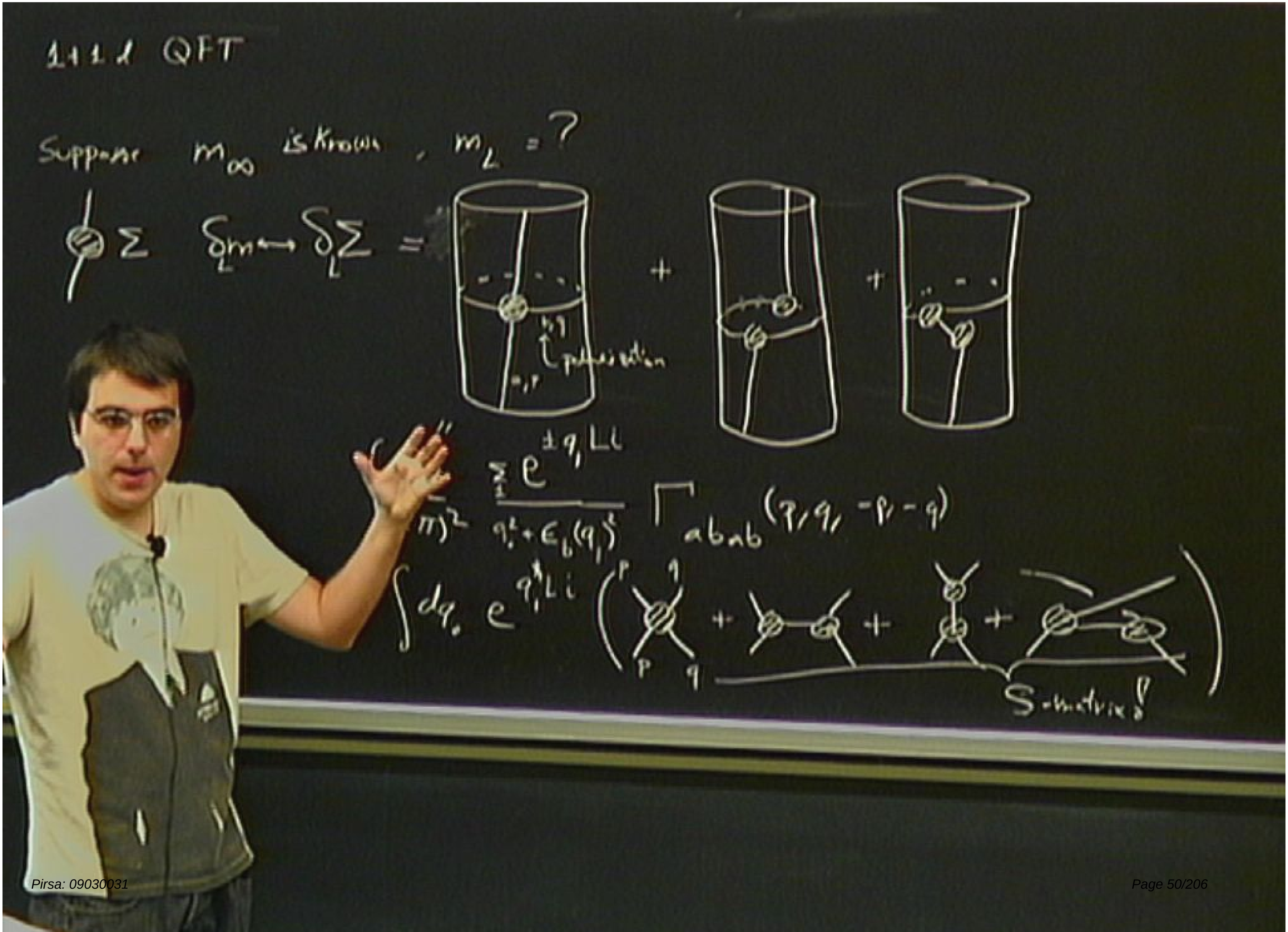




$$\delta m = \int dq \, e^{i q_1 L} \, S^{ab}_{ab}(p, q)$$



$$\delta m = \int dq \, e^{i q_1 L} \sum_b S^{ab}_{ab}(p, q)$$



1+1 d QFT

Suppose m_∞ is known, $m_L = ?$

$\Sigma \quad \delta_L m \leftrightarrow \delta_L \Sigma =$ + +

polarization

$\frac{\sum e^{\pm q_1 L i}}{q_0^2+\epsilon_b(q_1)^2}\,\Gamma_{abab}(p,q,-p-q)$

$m)^2$

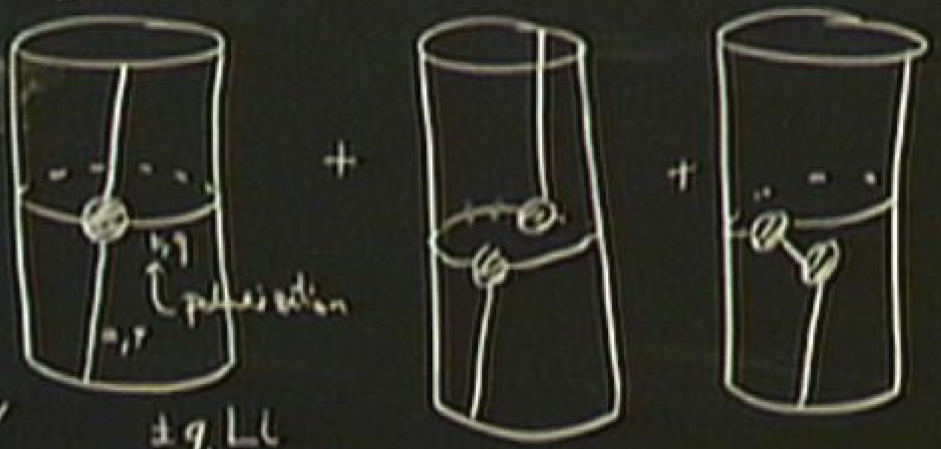
$\int dq_0\, e^{q_1^{\pm} L i}\Big(+ + + \Big)$

S-matrix!

1+1 d QFT

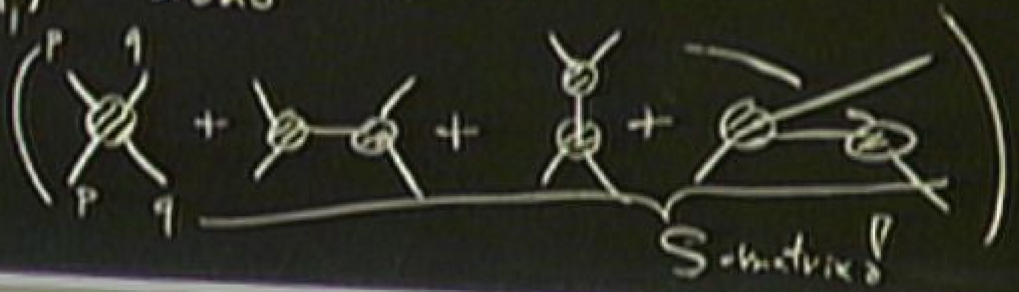
Suppose m_∞ is known, $m_L = ?$

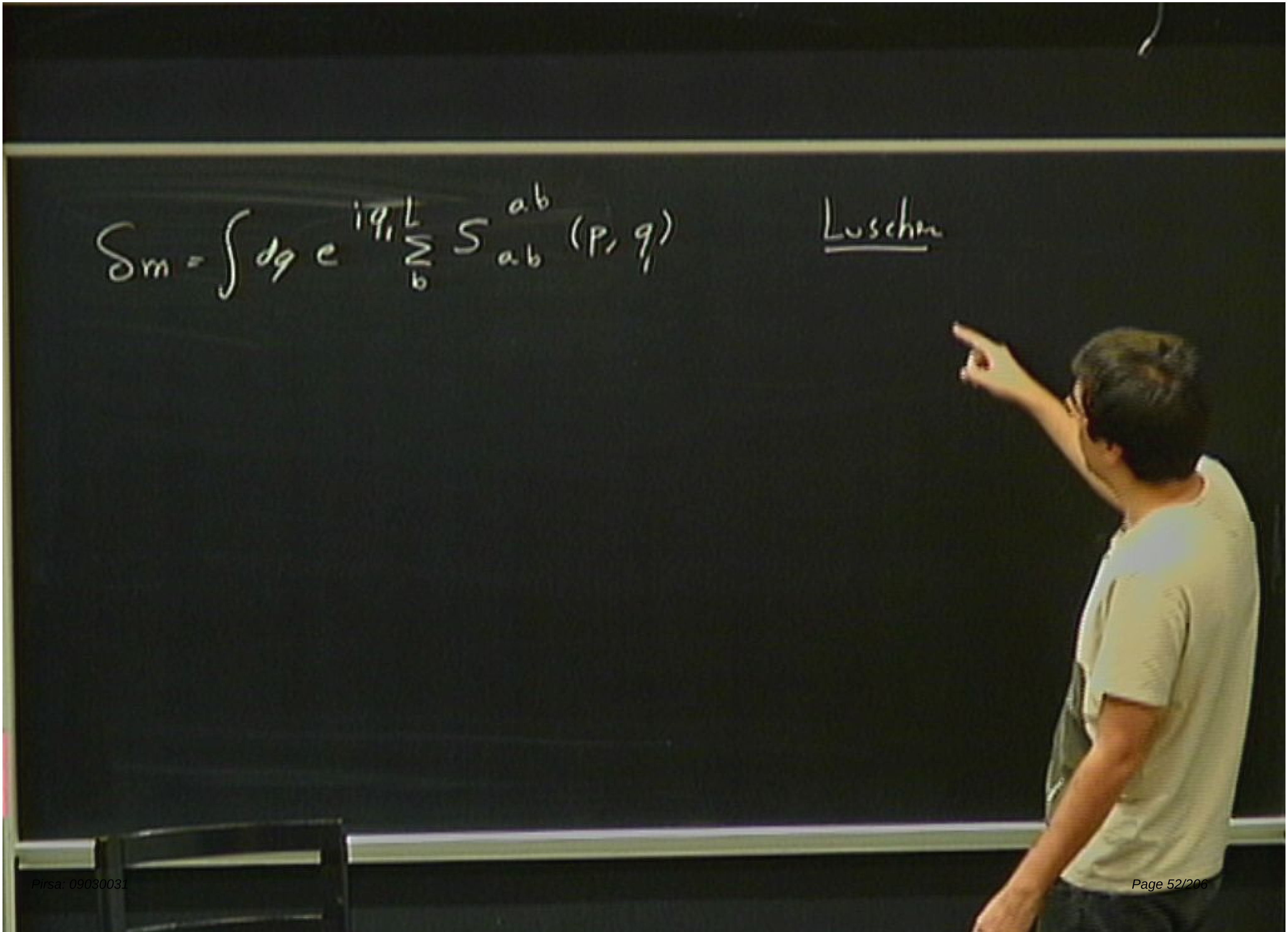
$\delta_L m \leftrightarrow \delta_L \Sigma =$



$$\int \frac{d^2 q}{(2\pi)^2} \, '' \, \frac{\sum_{\pm} e^{\pm q_1 L i}}{q_0^2 + \epsilon_b(q_1)^2} \, \Gamma_{abab}(p, q, -p, -q)$$

$$= \int dq_0 \, e^{q_1^{*} L i}$$





$$\delta m = \int dq\, e^{iq_1 L} \sum_b S_{ab}^{ab}(p, q_1)$$

Luschn

$$\delta m = \int dq\, e^{iq_1 L} \sum_b S_{ab}^{ab}(p, q_1)$$

Luschn

$$\delta m = \int dq\, e^{i q_1 L} \sum_b S_{ab}^{ab}(p, q_1)$$

Luscher

many particles

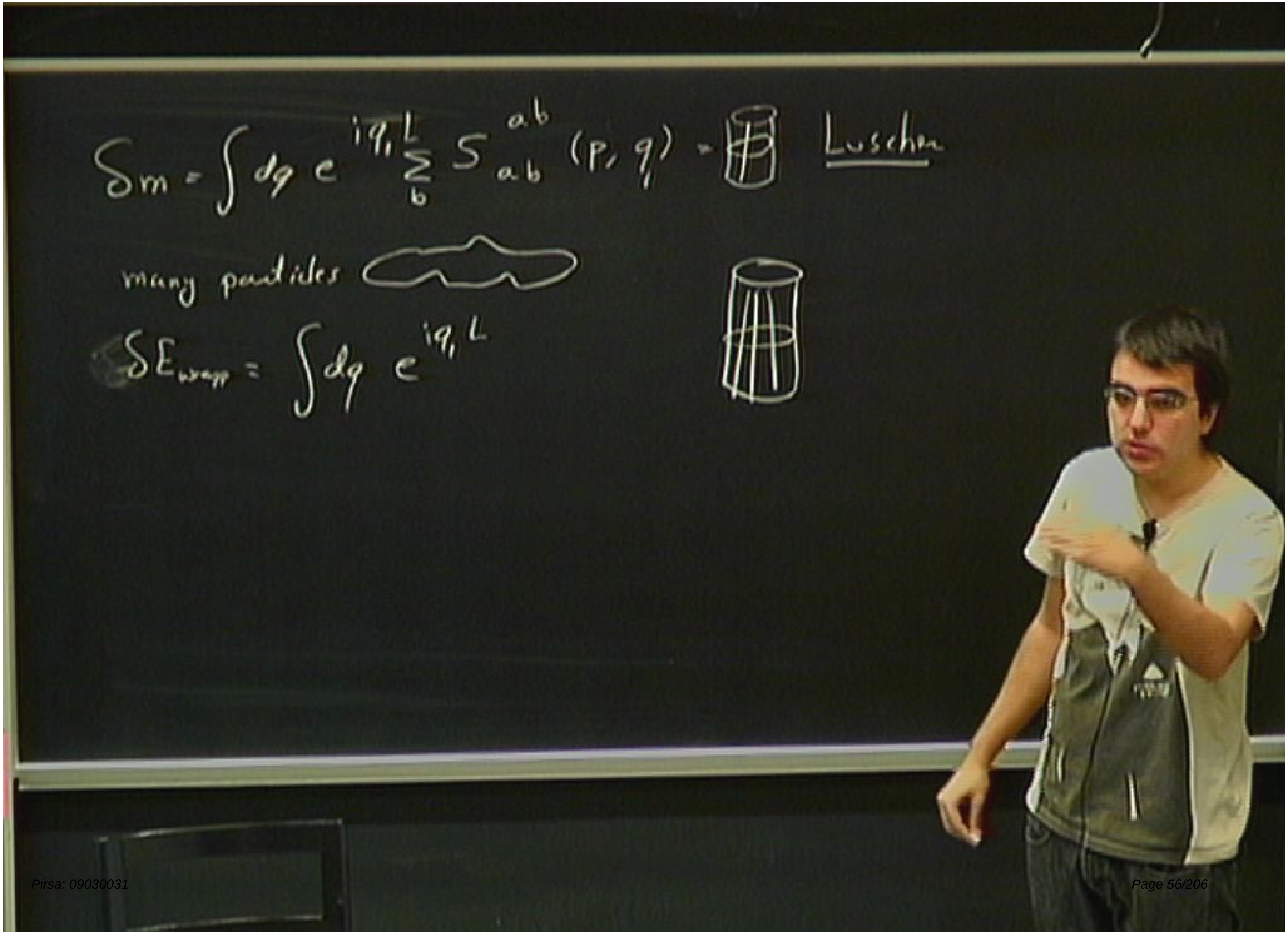
$$\delta E_{\text{wra}} \quad \int dq\, e^{i q_1 L}$$

$$\delta m = \int dq\, e^{iq_1 L} \sum_b S_{ab}^{ab}(p, q_1) =$$

Luscher

many particles

$$\delta E_{wrap} = \int dq\, e^{iq_1 L}$$

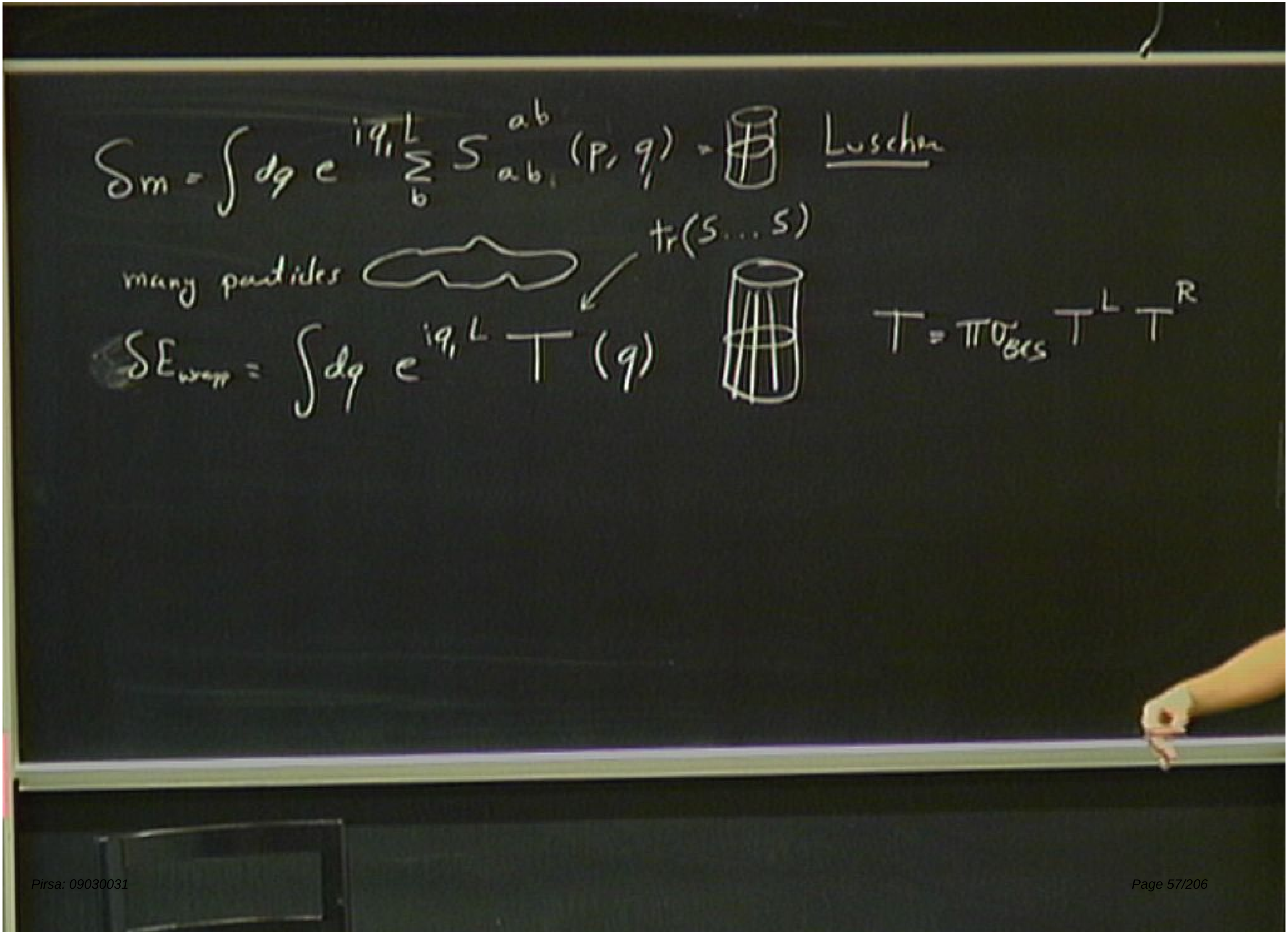


$$\delta m = \int dq\, e^{iq_1 L} \sum_b S_{ab}^{ab}(p, q_1) =$$

Luscher

many particles

$$\delta E_{wrapp} = \int dq\, e^{iq_1 L}$$



$$\delta m = \int dq\, e^{iq_1 L} \sum_b S_{ab_1}^{ab}(p, q_1) =$$

Luscher

$tr(S \ldots S)$

many particles

$$\delta E_{wrap} = \int dq\, e^{iq_1 L}\, T(q)$$

$T = \pi\sigma_{BCS}\, T^L\, T^R$

$$\delta m = \int dq\, e^{iq_1 L} \sum_b S^{ab}_{ab_1}(p, q_1) =$$

Luscher

many particles

$\mathrm{tr}(S \ldots S)$

$$\delta E_{wrap} = \int dq\, e^{iq_1 L}\, T(q)$$

$$T = \Pi\,\sigma'_{BES}\,T^L\,T^R$$

$$\delta m = \int dq \, e^{i q_1 L} \sum_b S^{ab}_{ab}(p, q_1) =$$

<u>Luscher</u>

$\mathrm{tr}(S \ldots S)$

many particles

$$\delta E_{wrap} = \int dq \, e^{i q_1 L} \, T(q)$$

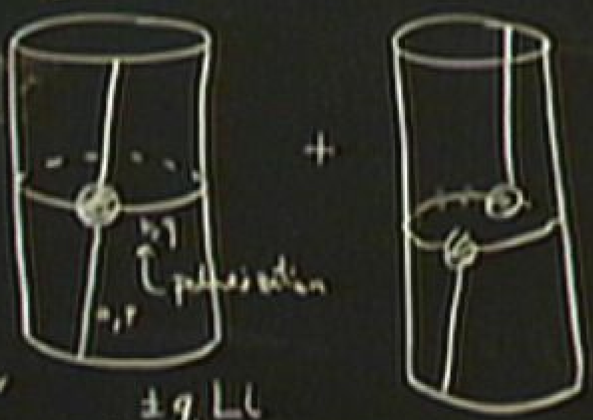
$$T = \Pi \, \sigma_{BES} \, T^L \, T^R$$

$$\delta E_{wrap} =$$

1+1 d QFT

Suppose m_∞ is known, $m_L = ?$

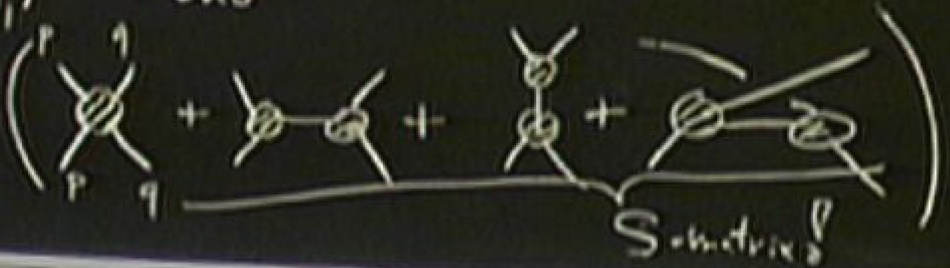
$\Sigma \quad \delta_L m \longleftrightarrow \delta_L \Sigma =$



$+ \mathcal{O}(e^{-2mL})$
$+ \mathcal{O}(g^{2L})$

$$" \int \frac{d^2 q}{(2\pi)^2} \frac{\sum_{\pm} e^{\pm q_1 L i}}{q_0^2 + \epsilon_b(q_1)^2} \Gamma_{abab}(p,q,-p,-q) "$$

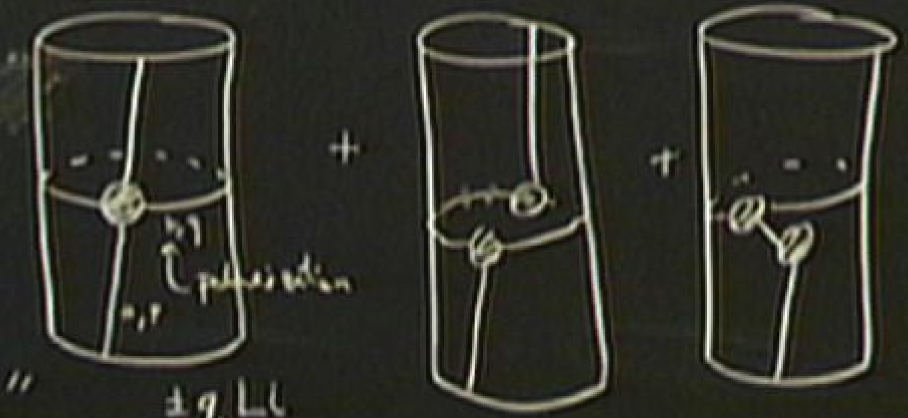
$$= \int dq_0 \, e^{q_1^{\pm} L i}$$



1+1 d QFT

Suppose m_∞ is known, $m_L = ?$

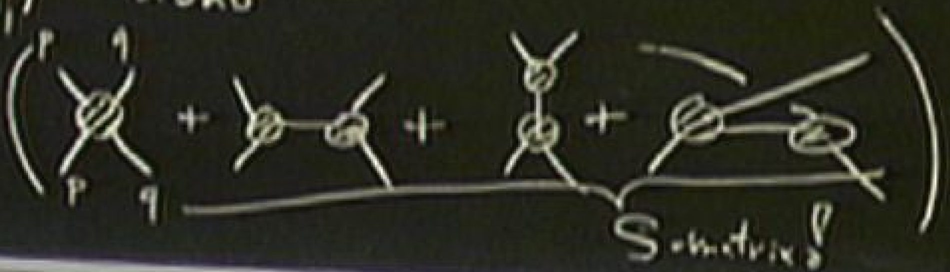
Σ $\quad \delta_L m \longleftrightarrow \delta_L \Sigma =$

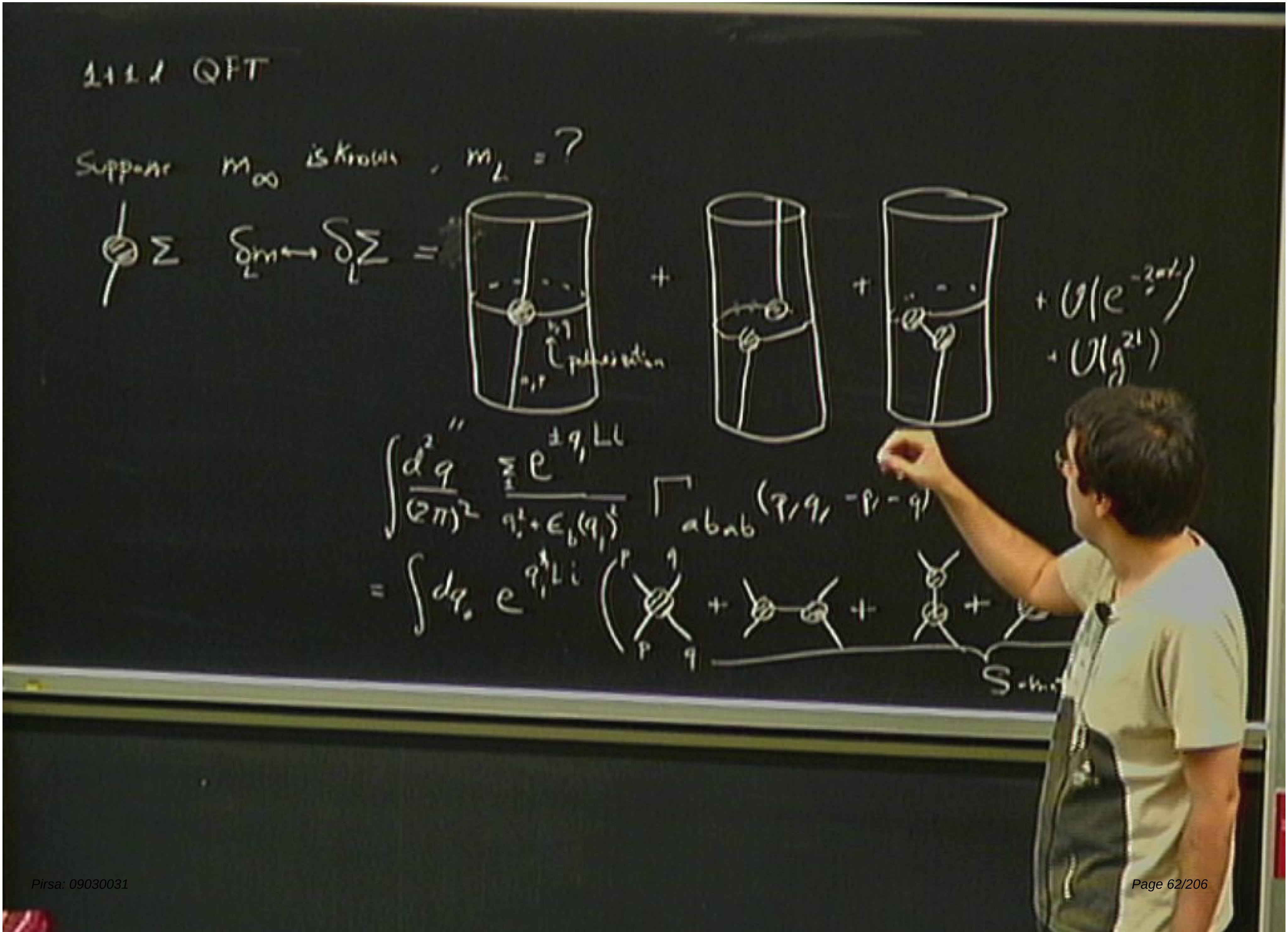


$+\,\mathcal{O}(e^{-\frac{3}{2}mL})$
$+\,\mathcal{O}(g^{2l})$

$$\int \frac{d^2q}{(2\pi)^2}\,'' \frac{\sum_{\pm} e^{\pm q_1 L i}}{q_0^2+\epsilon_b(q_1)^2}\,\Gamma_{abab}(p,q,-p,-q)$$

$$= \int dq_0\, e^{q_1^{*} L i} \Big(\quad \Big)$$





1+1 d QFT

Suppose m_∞ is known, $m_L = ?$

Σ $\quad \delta_L m \longleftrightarrow \delta_L \Sigma =$ $+$ $+$ $+\, \mathcal{O}(e^{-3mL}) + \mathcal{O}(g^{2L})$

$$\text{''}\int \frac{d^2 q}{(2\pi)^2} \frac{\sum_b e^{\pm q_1 L i}}{q_0^2 + \epsilon_b(q_1)^2} \Gamma_{abab}(p, q, -p, -q)$$

$$= \int dq_0\, e^{q_1^* L i} \Big(\quad + \quad + \quad + \quad$$

$$\delta m = \int dq\, e^{iq_1 L} \sum_b S^{ab}_{ab_1}(p, q_1) =$$

Luschm

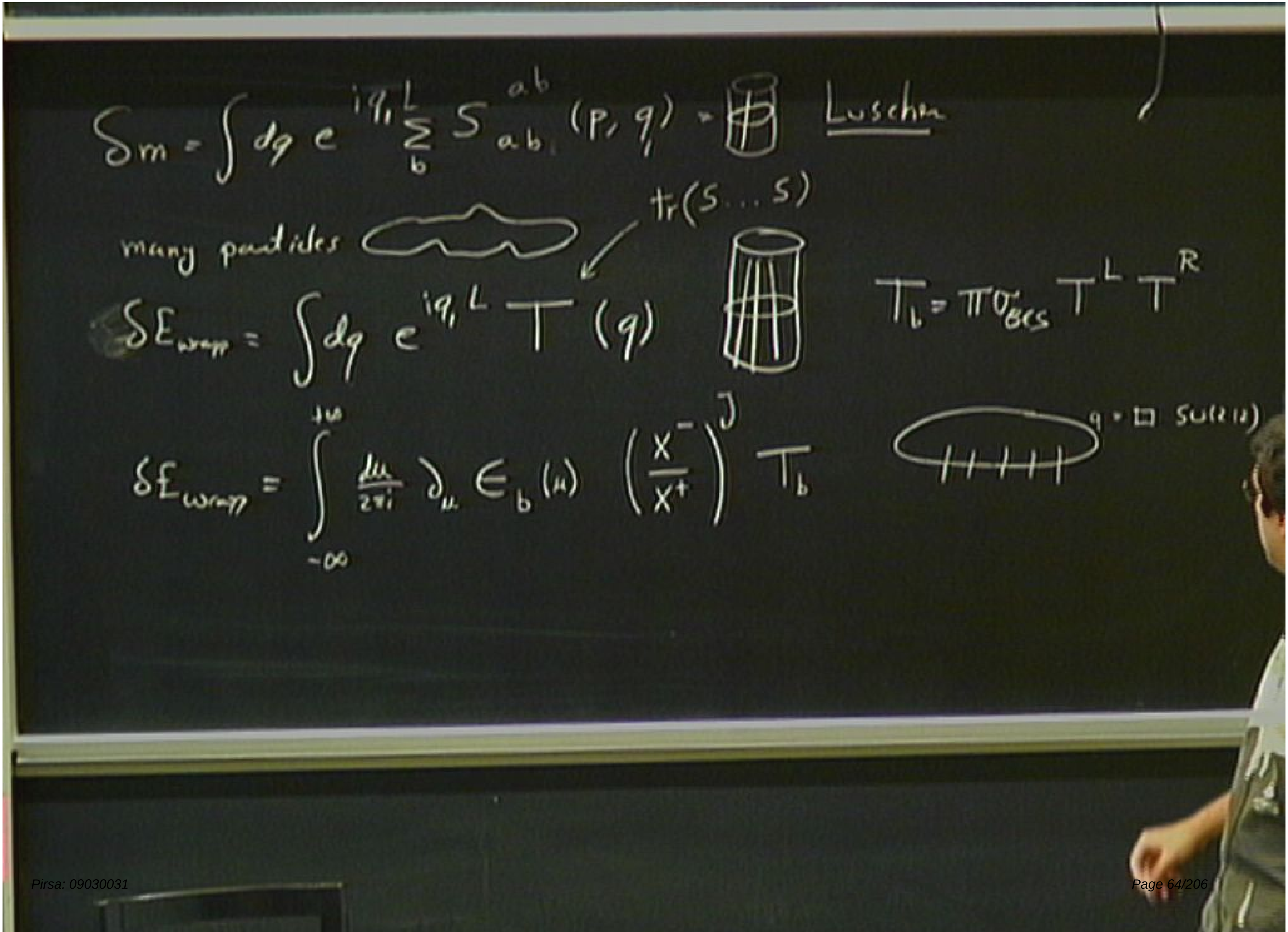
many particles

$\mathrm{tr}(S \ldots S)$

$$\delta E_{wrapp} = \int dq\, e^{iq_1 L}\, T(q)$$

$$T = \pi\sigma_{BES}\, T^L\, T^R$$

$$\delta E_{wrapp} = \int_{-\infty}^{+\infty} \frac{du}{2\pi i}\, \partial_u\, \epsilon_b(u) \left(\frac{X^-}{X^+}\right)^J T$$



$$\delta m = \int dq\, e^{iq_1 L} \sum_b S_{ab_1}^{ab}(p, q_1) =$$

Luschen

tr(S … S)

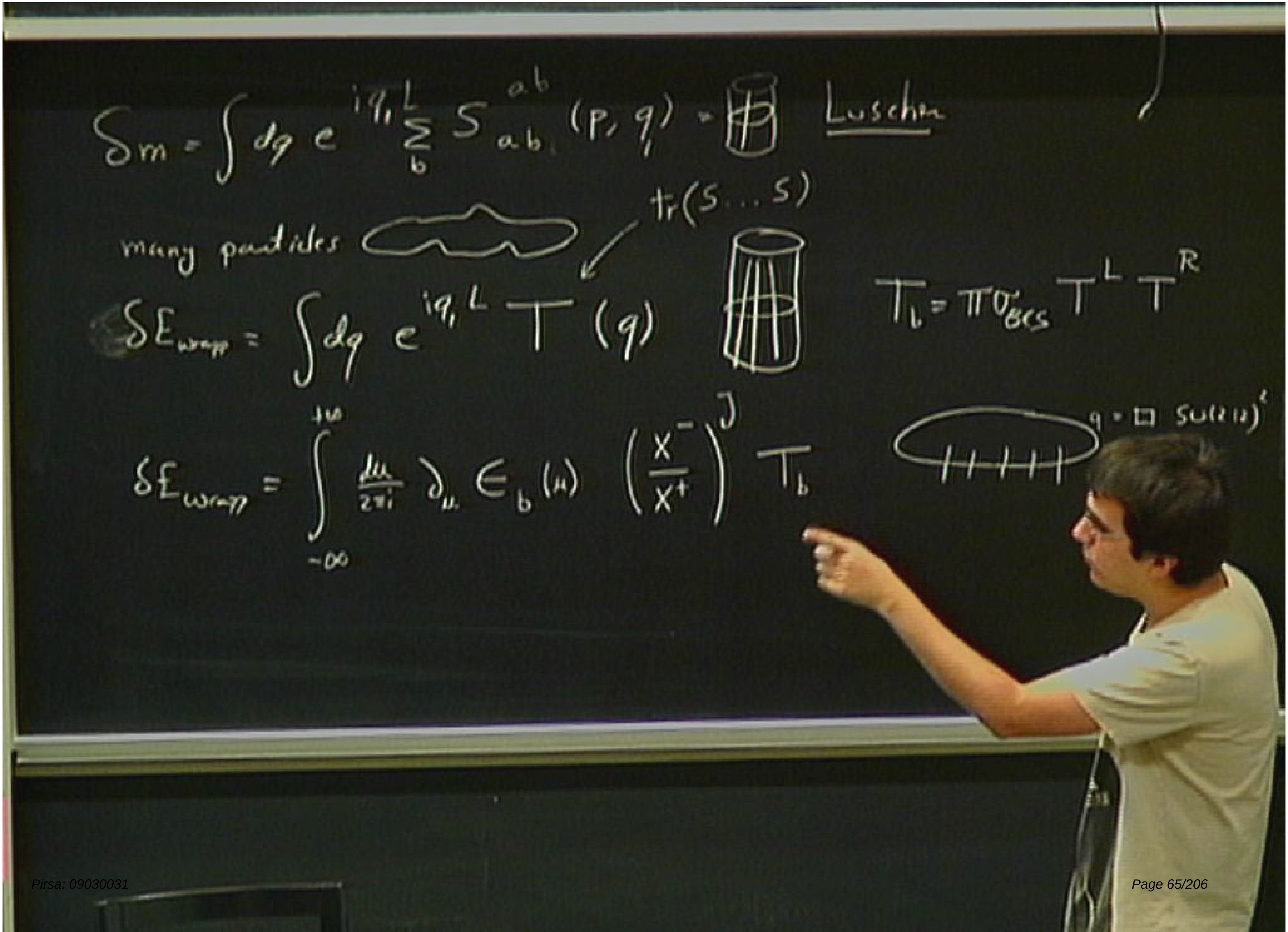
many particles

$$\delta E_{wrap} = \int dq\, e^{iq_1 L}\, T(q)$$

$$T_b = \Pi\, \sigma_{BES}\, T^L\, T^R$$

$$\delta E_{wrap} = \int_{-\infty}^{+\infty} \frac{du}{2\pi i}\, \partial_u \epsilon_b(u) \left(\frac{x^-}{x^+}\right)^J T_b$$

q = □ su(2|2)



$$\delta m = \int dq\, e^{iq_1 L} \sum_b S_{ab_1}^{ab}(p, q_1) =$$

<u>Luschm</u>

$\mathrm{tr}(S \ldots S)$

many particles

$$\delta E_{wrapp} = \int dq\, e^{iq_1 L}\, T(q)$$

$$T_b = \pi \sigma'_{BES}\, T^L\, T^R$$

$$\delta E_{wrapp} = \int_{-\infty}^{+\infty} \frac{du}{2\pi i}\, \partial_u \epsilon_b(u) \left(\frac{x^-}{x^+}\right)^J T_b$$

$q = \square\ SU(2|2)^2$

$$\delta m = \int dq\, e^{iq_1 L} \sum_b S_{ab_1}^{ab}(p, q_1) =$$

Luscher

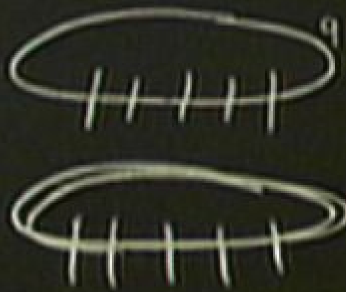
many particles

tr(S . . . S)

$$\delta E_{wrapp} = \int dq\, e^{iq_1 L}\, T(q)$$

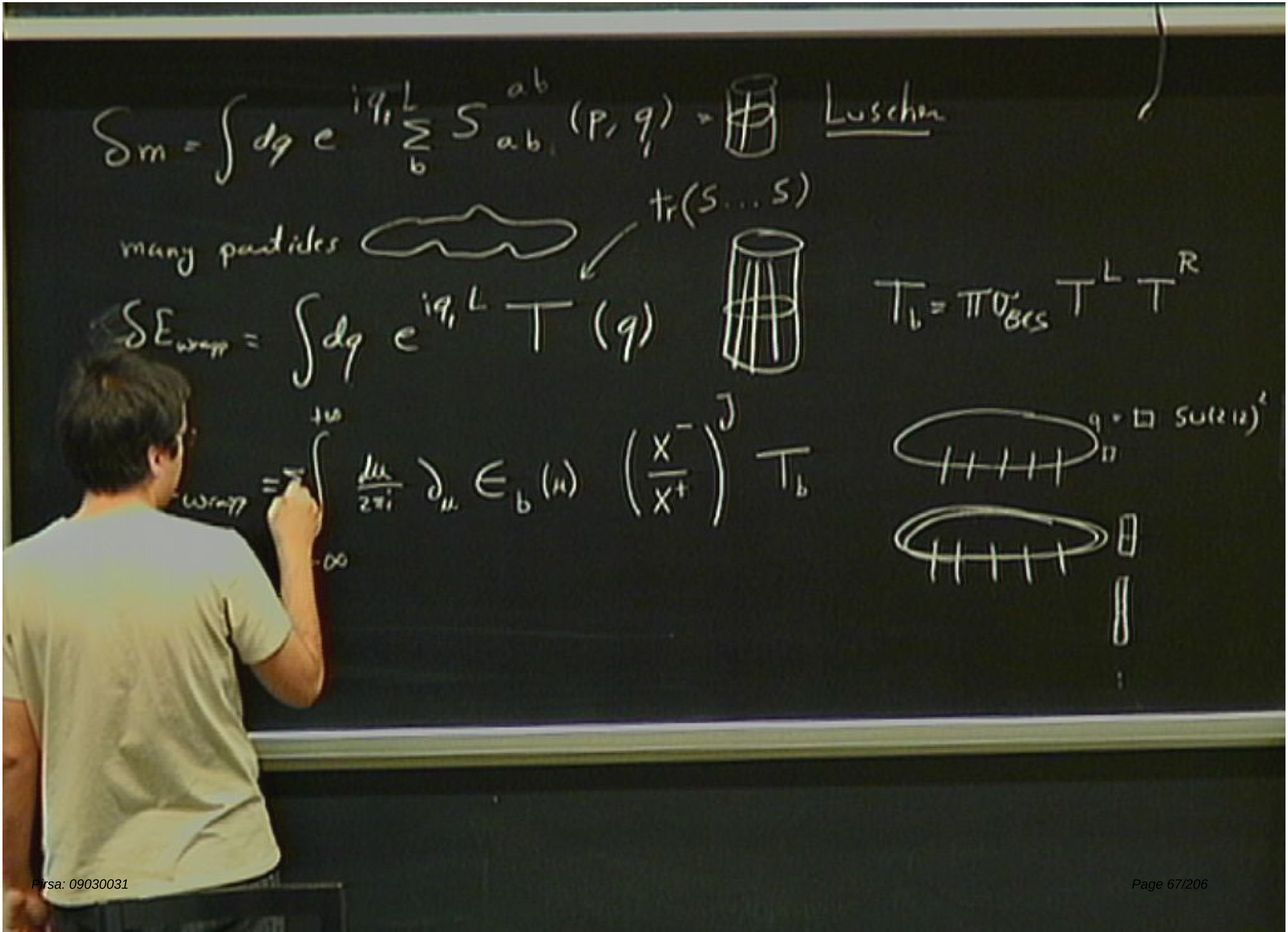
$$T_b = \prod \sigma_{BES}\, T^L\, T^R$$

$$\delta E_{wrapp} = \int_{-\infty}^{+\infty} \frac{du}{2\pi i}\, \partial_u \epsilon_b(u) \left(\frac{x^-}{x^+}\right)^J T_b$$



q = □ $SU(2|2)^2$

⊟



$$\delta m = \int dq\, e^{iq_1 L} \sum_b S_{ab_1}^{ab}(p, q_1) =$$

Luschen

many particles

tr(S ... S)

$$\delta E_{wrapp} = \int dq\, e^{iq_1 L}\, T(q)$$

$$T_b = \pi\, O_{BES}\, T^L\, T^R$$

$$\delta E_{wrapp} = \int_{-\infty}^{+\infty} \frac{du}{2\pi i}\, \partial_u \epsilon_b(u) \left(\frac{X^-}{X^+}\right)^J T_b$$

q = □ $SU(2|2)^2$

□

$$\delta m = \int dq\, e^{iq_1 L} \sum_b S_{ab}^{ab}(p, q_1) =$$

<u>Luschn</u>

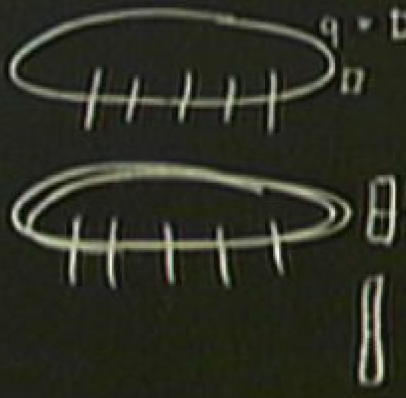
$\mathrm{tr}(S \ldots S)$

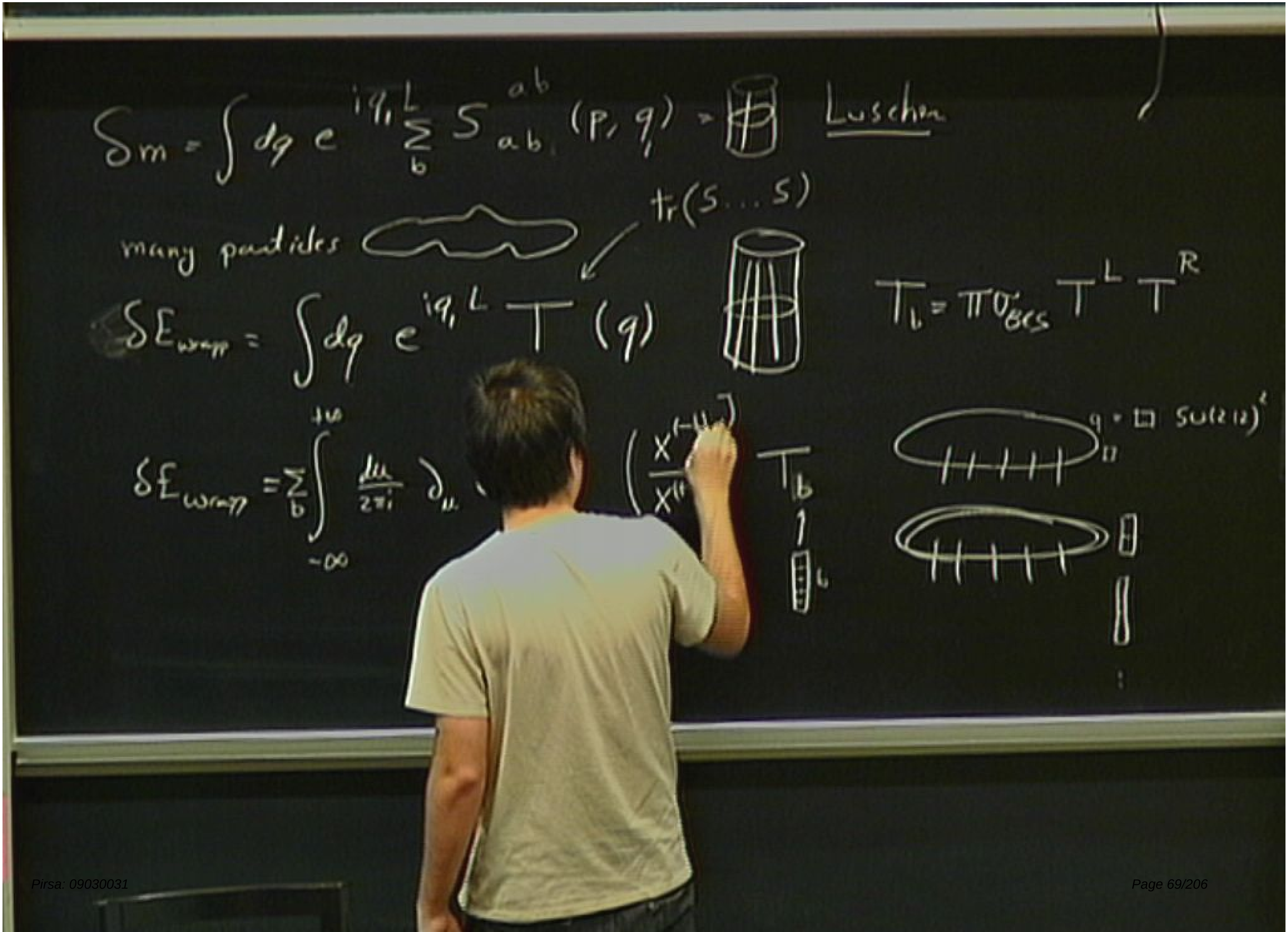
many particles

$$\delta E_{wrap} = \int dq\, e^{iq_1 L}\, T(q)$$

$$T_b = \pi \sigma_{BES}\, T^L\, T^R$$

$$\delta E_{wrap} = \sum_b \int_{-\infty}^{+\infty} \frac{du}{2\pi i}\, \partial_u \epsilon_b(u) \left(\frac{X^-}{X^+}\right)^J T_b$$





$$\delta m = \int dq\, e^{i q_1 L} \sum_b S^{ab}_{ab}(p, q_1) = $$

Luscher

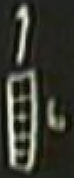
many particles

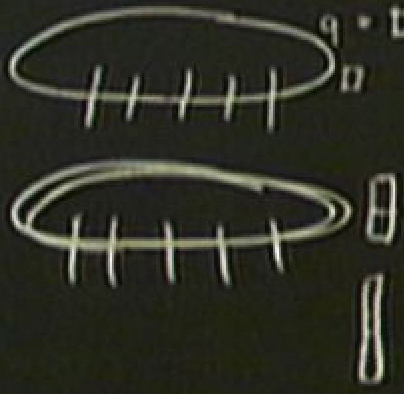
tr(S … S)

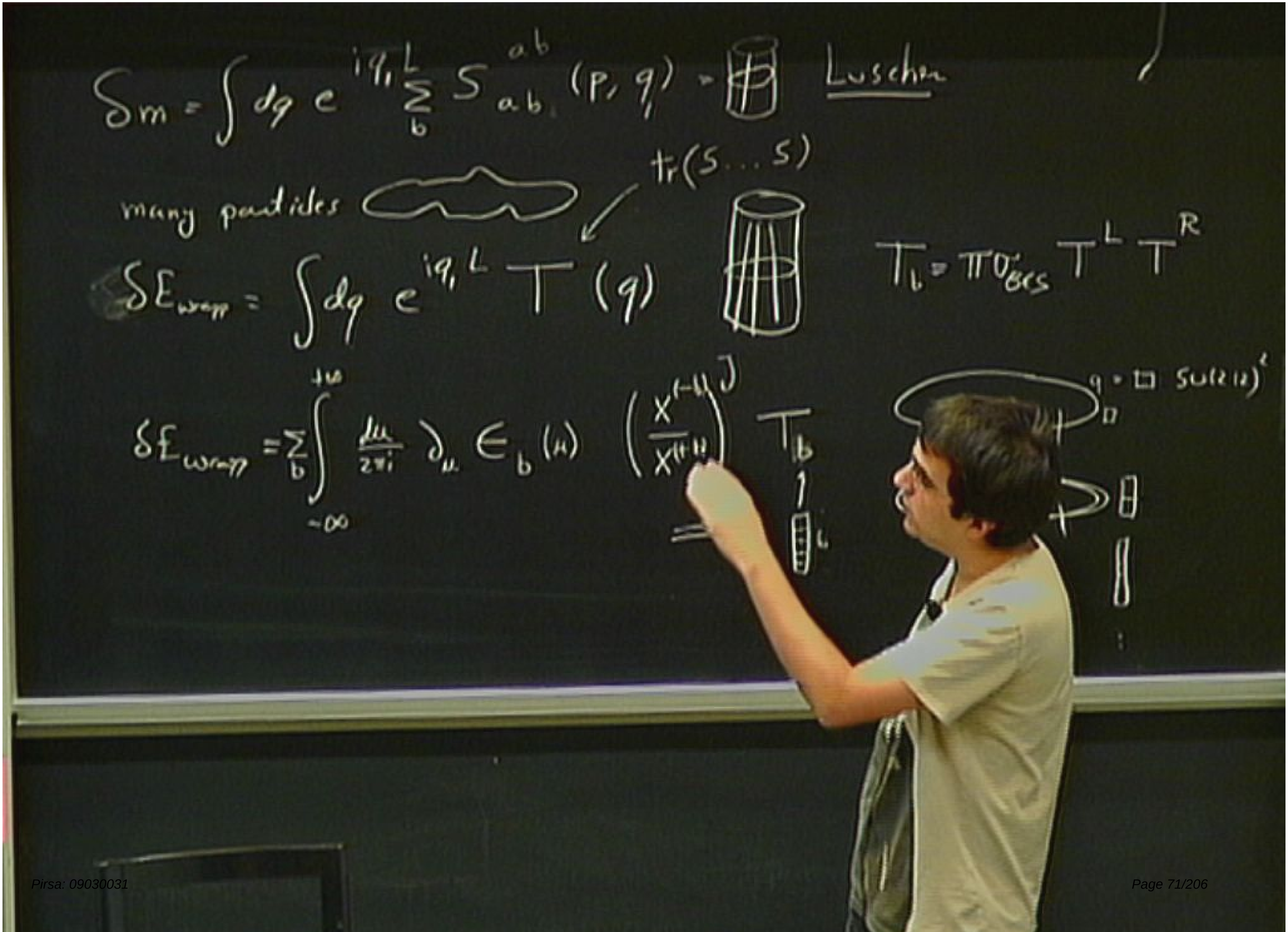
$$\delta E_{wrapp} = \int dq\, e^{i q_1 L}\, T(q)$$

$$T_b = \pi \sigma_{BES}\, T^L\, T^R$$

$$\delta E_{wrapp} = \sum_b \int_{-\infty}^{+\infty} \frac{du}{2\pi i}\, \partial_u \epsilon_b(u) \left(\frac{x^{(-b)}}{x^{(+b)}}\right)^J T_b$$







$$\delta m = \int dq\, e^{iq_1 L} \sum_b S^{ab}_{ab_1}(p,q) =$$

Luscher

$$\mathrm{tr}(S \ldots S)$$

many particles

$$\delta E_{wrap} = \int dq\, e^{iq_1 L}\, T(q)$$

$$T_b = \Pi \sigma'_{BES}\, T^L T^R$$

$$\delta E_{wrap} = \sum_b \int_{-\infty}^{+\infty} \frac{du}{2\pi i}\, \partial_u \epsilon_b(u) \left(\frac{x^{[-b]}}{x^{[+b]}}\right)^J T_b$$

$$q = \square \quad SU(2|2)^2$$

$$\delta m = \int dq\, e^{iq_1 L} \sum_b S_{ab}^{ab}(p,q) =$$

Luschm

many particles

tr(S … S)

$$\delta E_{wrap} = \int dq\, e^{iq_1 L}\, T(q)$$

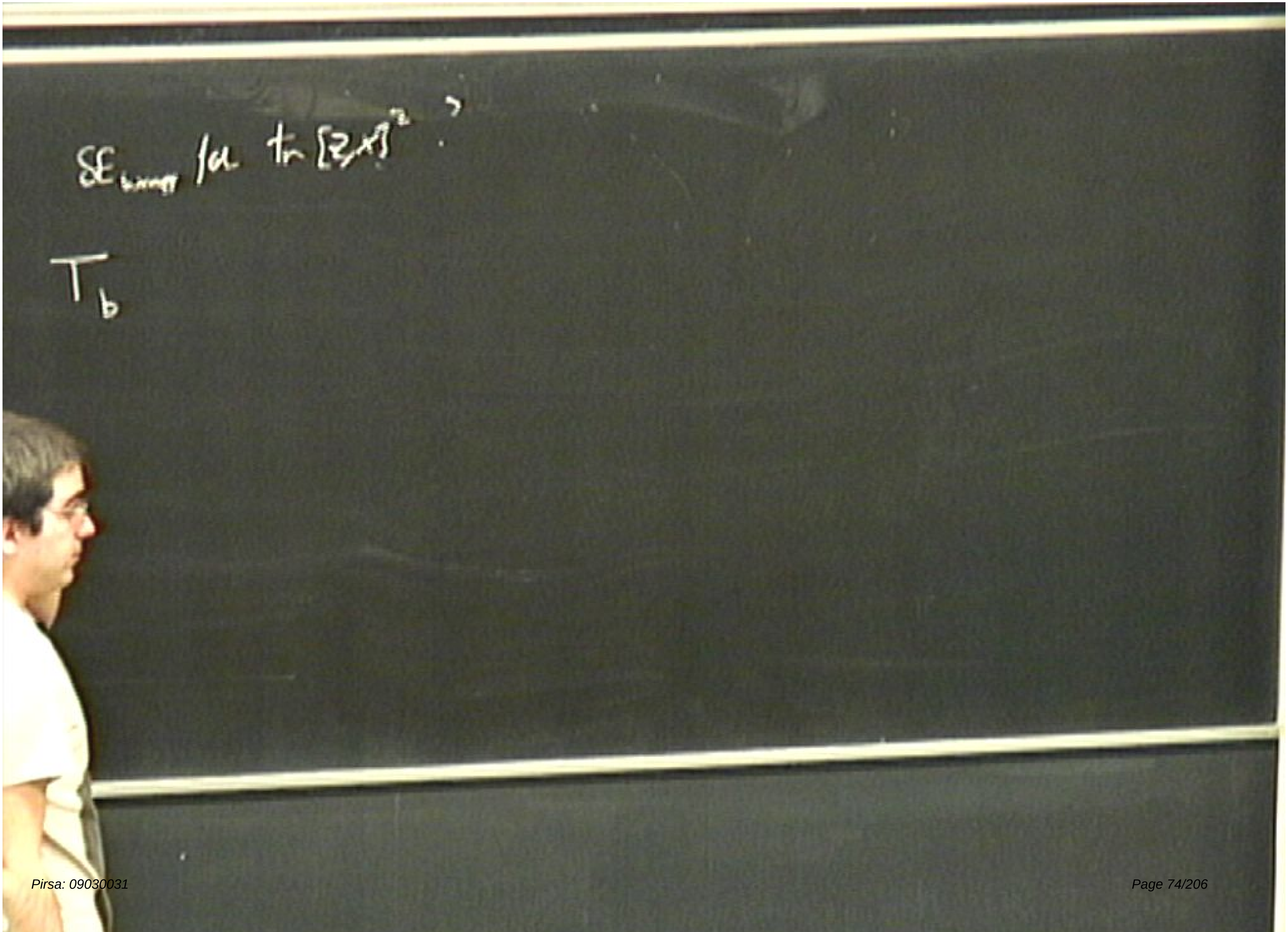
$$T_b = \prod \sigma'_{BES}\, T^L\, T^R$$

$$\delta E_{wrap} = \sum_b \int_{-\infty}^{+\infty} \frac{du}{2\pi i}\, \partial_u \epsilon_b(u) \left(\frac{X^{(-1)}}{X^{(+1)}}\right)^J T_b$$

$$T_b = $$

$q = \square \quad su(2|2)^2$

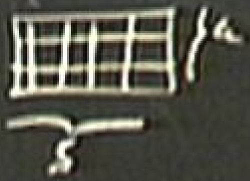
$\delta E_{\text{wrapp}} \int dt \ \mathrm{tr}\,[Z,X]^2$?



$\delta E_{binding} \int dt \, \mathrm{tr}[Z,X]^2$?

T_b

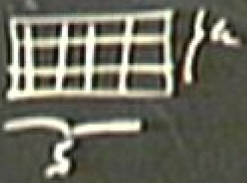
δE_{wrapp} for $\mathrm{tr}[Z,X]^2$?



$(x_{a,s})^2 = X_{a,s+1} X_{a,s-1}$

T_b

δE_{wrapp} for $\mathrm{tr}[Z,X]^2$?



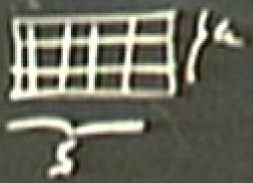
$$(\chi_{a,s})^2 = \chi_{a,s+1}\chi_{a,s-1} + \chi_{a+1,s}\chi_{a-1,s}$$

$T_{a,s}$

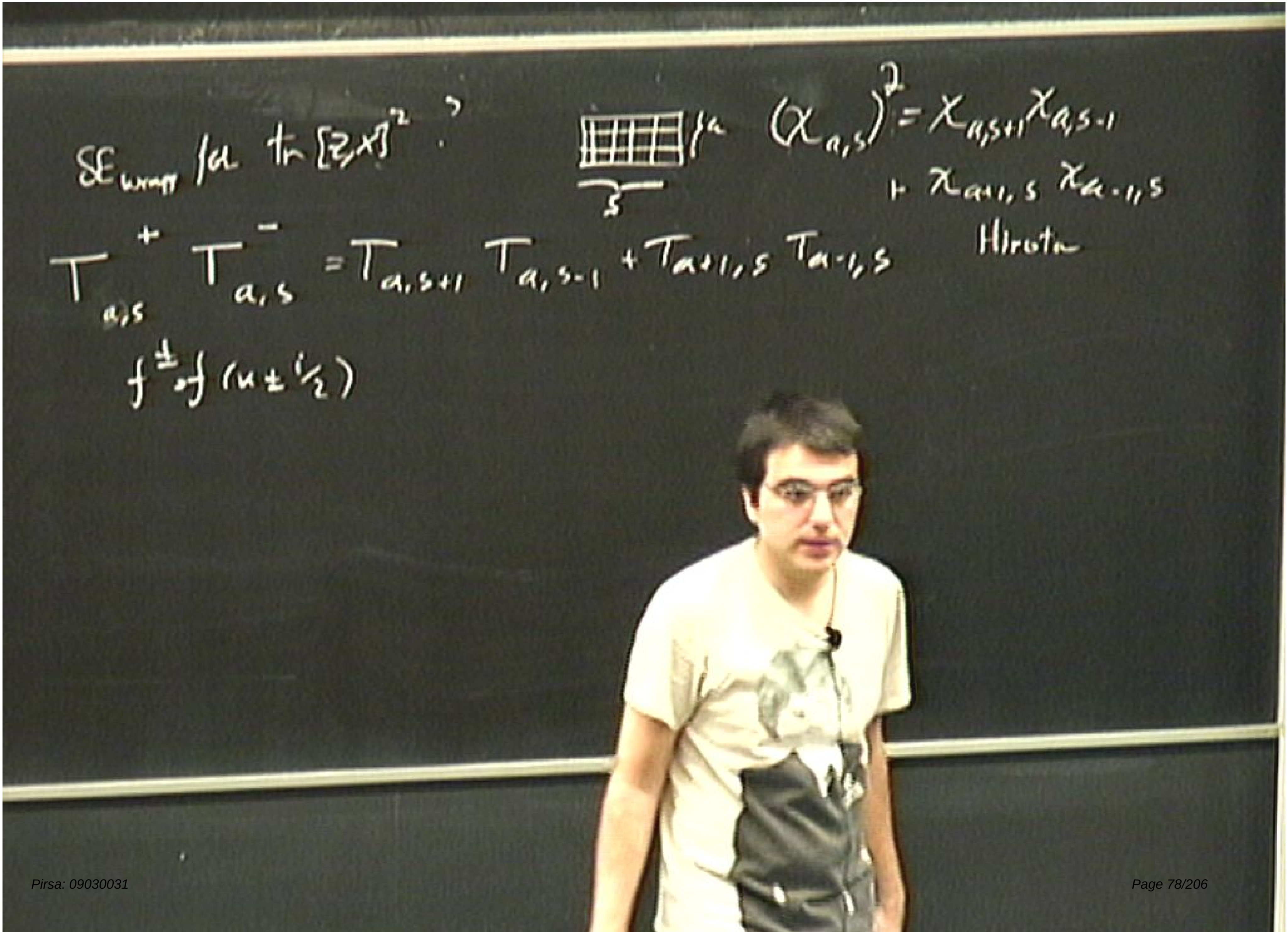
δE_{wrapp} for $\mathrm{tr}\,[Z,X]^2$?

$T^{+}_{a,s}\;T^{-}_{a,s}$

$f^{\pm} = f(u \pm i/2)$



$$(\chi_{a,s})^2 = \chi_{a,s+1}\chi_{a,s-1} + \chi_{a+1,s}\chi_{a-1,s}$$



δE_{wrapp} for $\text{tr}[Z,X]^2$?

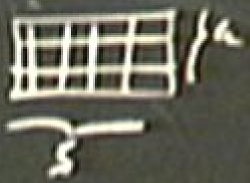
$$(\chi_{a,s})^2 = \chi_{a,s+1}\chi_{a,s-1} + \chi_{a+1,s}\chi_{a-1,s}$$

Hirota

$$T^{+}_{a,s}\,T^{-}_{a,s} = T_{a,s+1}\,T_{a,s-1} + T_{a+1,s}\,T_{a-1,s}$$

$f^{\pm} = f(u \pm i/2)$

SE_{wrapp} for $\mathrm{tr}[Z,X]^2$?



$$(\chi_{a,s})^2 = \chi_{a,s+1}\chi_{a,s-1} + \chi_{a+1,s}\chi_{a-1,s}$$

$$T^{+}_{a,s}\,T^{-}_{a,s} = T_{a,s+1}\,T_{a,s-1} + T_{a+1,s}\,T_{a-1,s}$$ <u>Hirota</u>

$$f^{\pm} = f(u \pm i/2)$$

T

SE_{wrapp} for $\mathrm{tr}[Z,X]^2$?

$$(\chi_{a,s})^2 = \chi_{a,s+1}\chi_{a,s-1} + \chi_{a+1,s}\chi_{a-1,s}$$

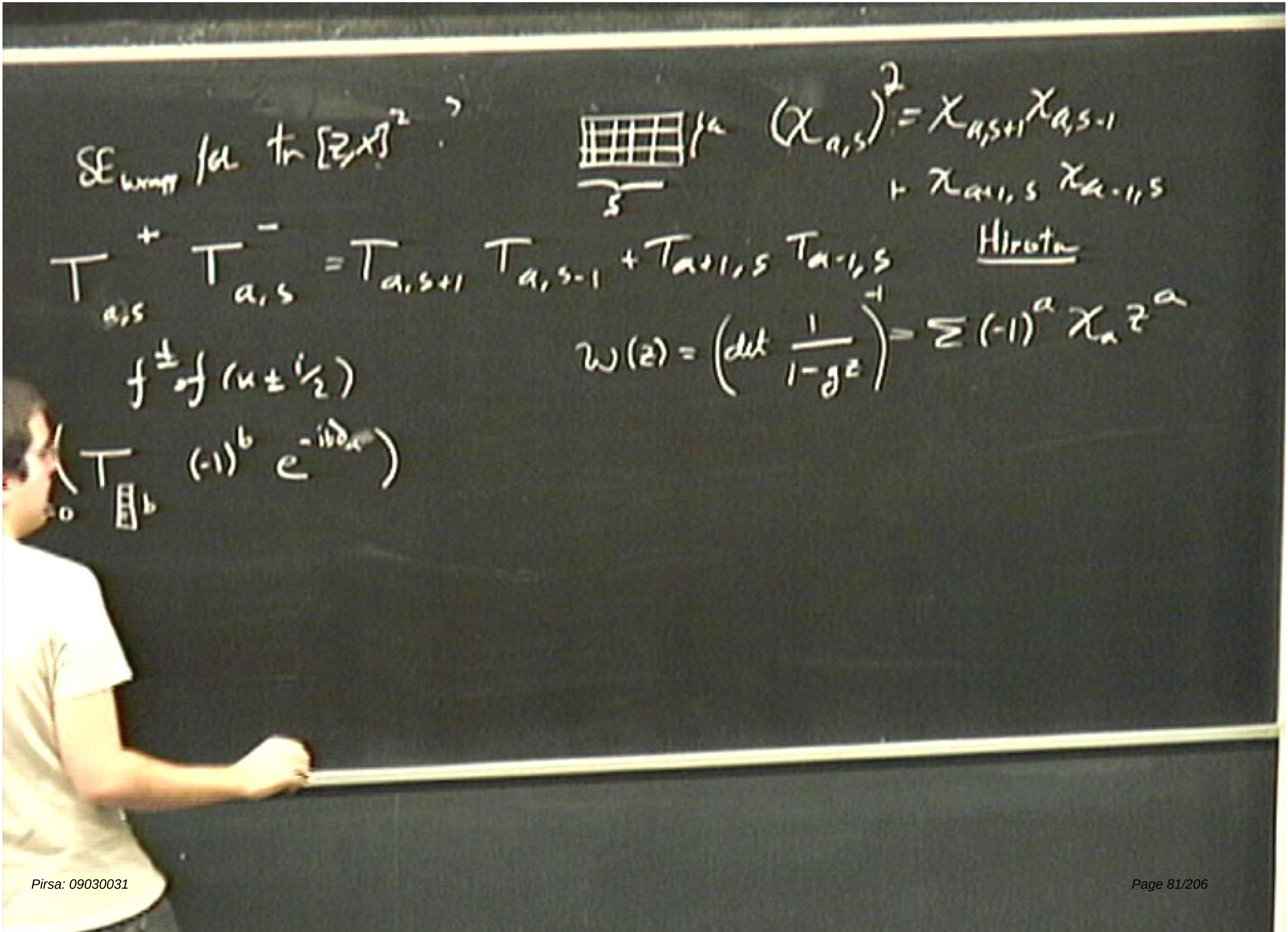
$$T^{+}_{a,s}\, T^{-}_{a,s} = T_{a,s+1}\, T_{a,s-1} + T_{a+1,s}\, T_{a-1,s}$$

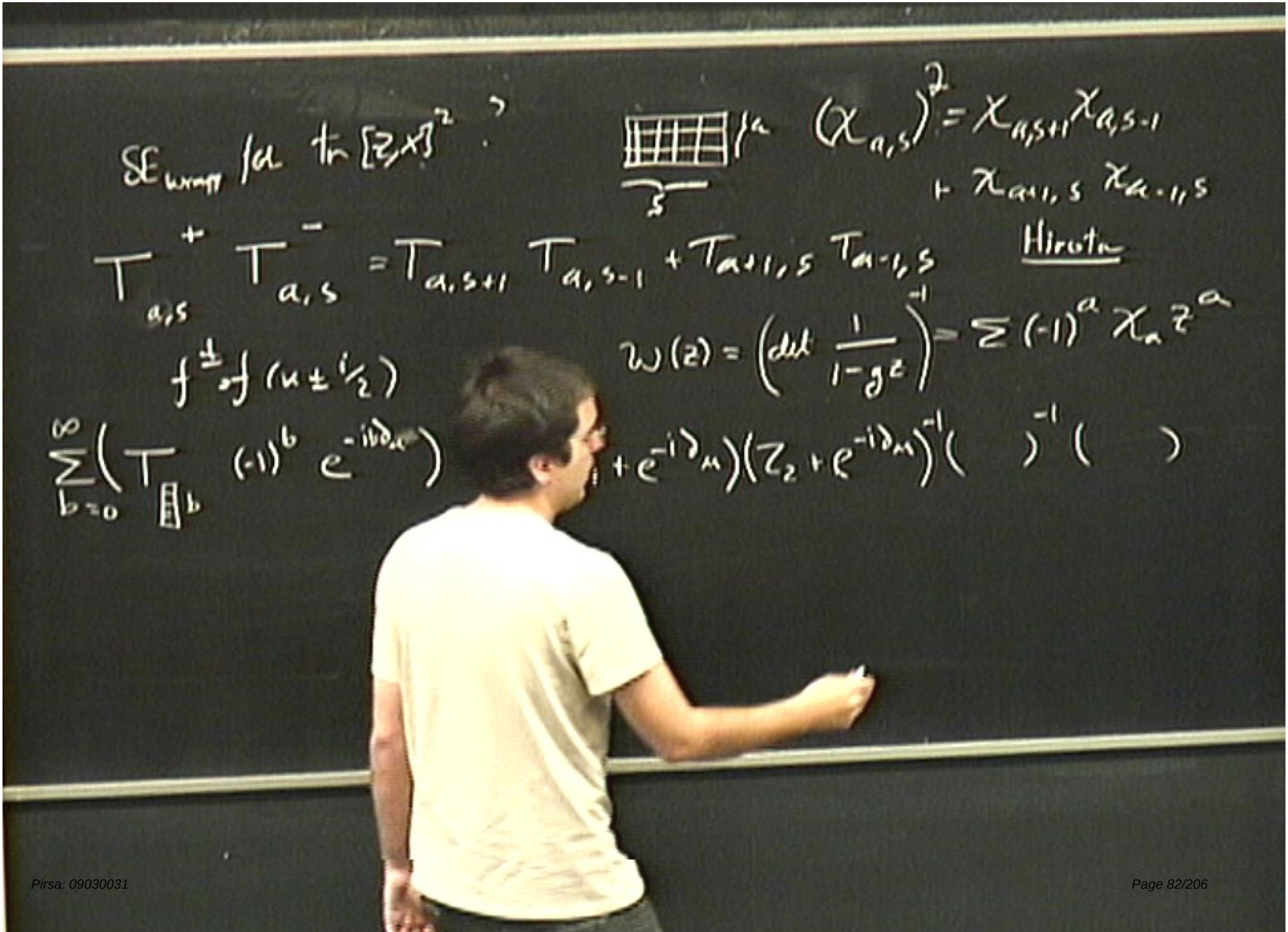
Hirota

$$f^{\pm} = f(u \pm i/2)$$

$$w(z) = \left(\det \frac{1}{1-gz}\right)^{-1} = \sum (-1)^a \chi_a z^a$$

T





$\mathcal{SE}_{\text{wrapp}}$ for $\mathrm{tr}\,[Z,X]^2$?

$\left(\chi_{a,s}\right)^2 = \chi_{a,s+1}\chi_{a,s-1}$

$+ \chi_{a+1,s}\,\chi_{a-1,s}$

$T^{+}_{a,s}\,T^{-}_{a,s} = T_{a,s+1}\,T_{a,s-1} + T_{a+1,s}\,T_{a-1,s}$ **Hirota**

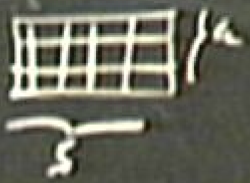
$f^{\pm}$ of $(u \pm i/2)$

$w(z) = \left(\det \frac{1}{1-gz}\right)^{-1} = \sum (-1)^a \chi_a z^a$

$\sum_{b=0}^{\infty}\left(T_{b}\,(-1)^b\,e^{-ib\partial_u}\right)$

$+e^{-i\partial_u})(Z_2+e^{-i\partial_u})^{-1}(\quad)^{-1}(\quad)$

SE_{wrapp} for $\text{tr}\,[Z,X]^2$?



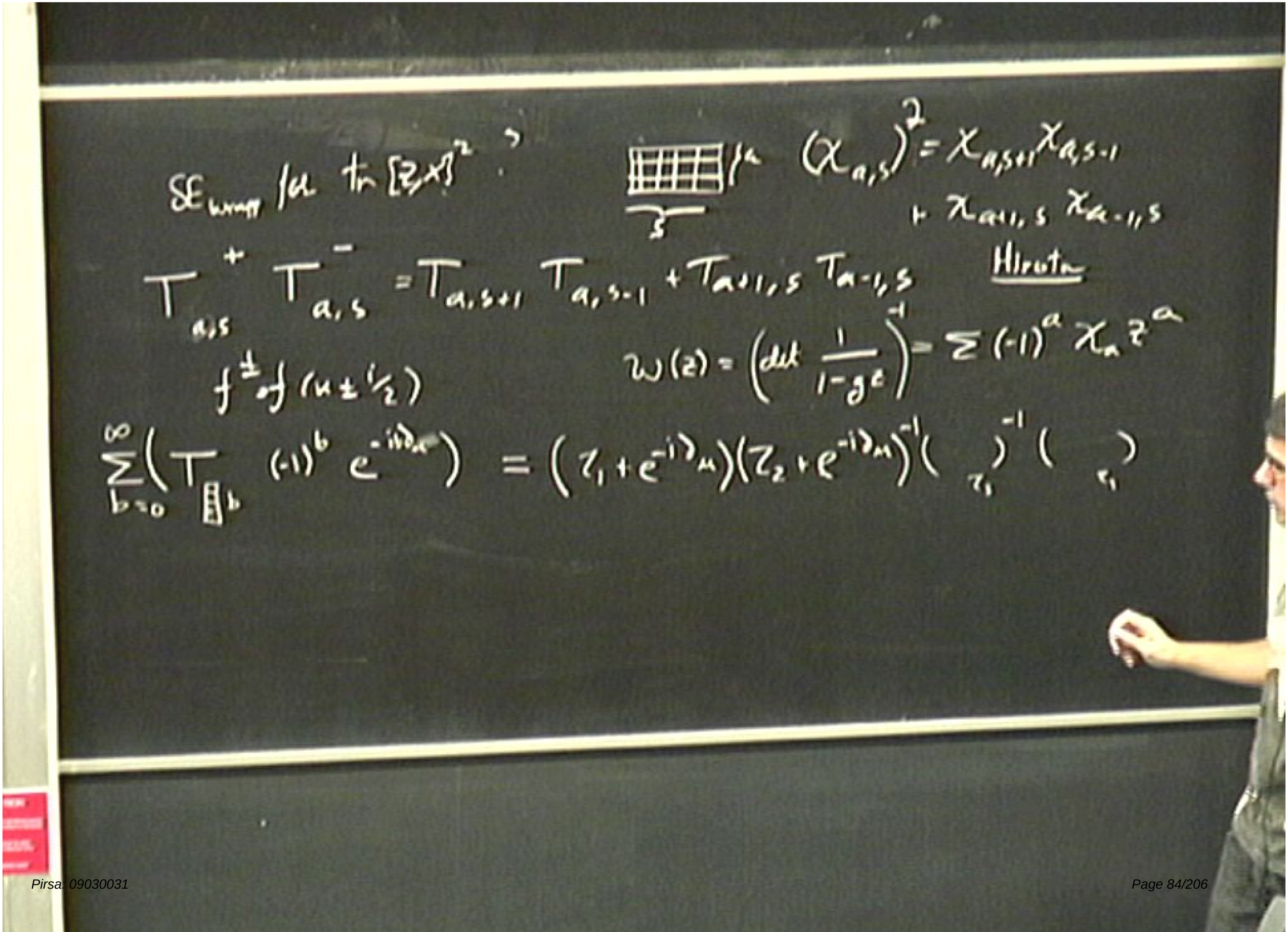
$$(\chi_{a,s})^2 = \chi_{a,s+1}\chi_{a,s-1} + \chi_{a+1,s}\chi_{a-1,s}$$

$$T^{+}_{a,s}\,T^{-}_{a,s} = T_{a,s+1}\,T_{a,s-1} + T_{a+1,s}\,T_{a-1,s}$$ **<u>Hirota</u>**

$f^{\pm} = f(u \pm \tfrac{i}{2})$

$$w(z) = \left(\det \frac{1}{1-gz}\right)^{-1} = \sum (-1)^a \chi_a z^a$$

$$\sum_{b=0}^{\infty}\left(T_{\boxed{\vdots}\,b}\,(-1)^b e^{-ib\partial_u}\right) = \left(z_1 + e^{-i\partial_u}\right)\left(z_2 + e^{-i\partial_u}\right)^{-1}\left(\quad\right)^{-1}_{z_1}\left(\quad\right)_{z_1}$$



SE_{wrapp} for $\mathrm{tr}[Z,X]^2$?

$$(\chi_{a,s})^2 = \chi_{a,s+1}\chi_{a,s-1} + \chi_{a+1,s}\chi_{a-1,s}$$

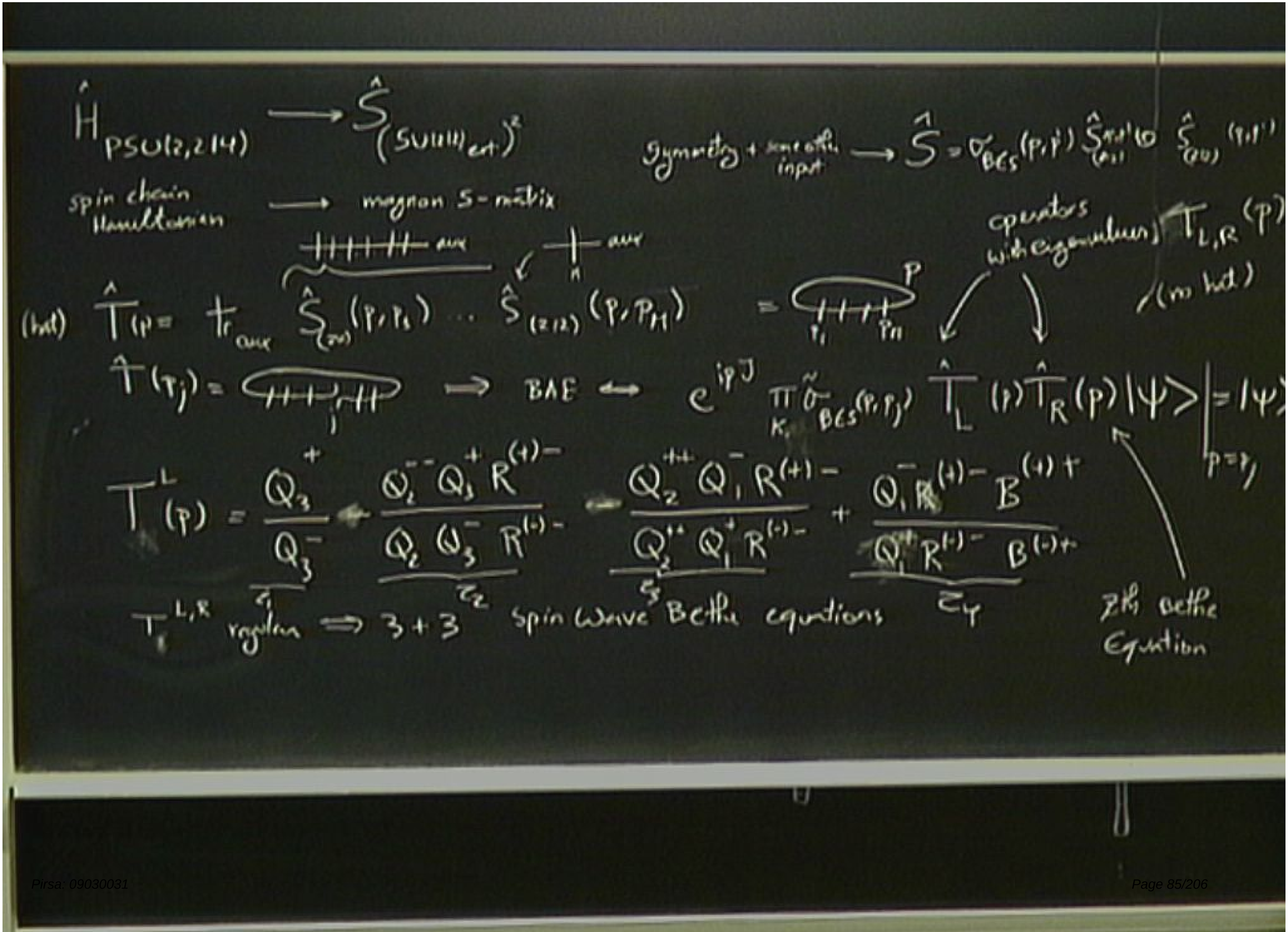
Hirota

$$T^{+}_{a,s}\, T^{-}_{a,s} = T_{a,s+1}\, T_{a,s-1} + T_{a+1,s}\, T_{a-1,s}$$

$f^{\pm}$ of $(u \pm \tfrac{1}{2})$

$$w(z) = \left(\det \frac{1}{1-gz}\right)^{-1} = \Sigma (-1)^a \chi_a z^a$$

$$\sum_{b=0}^{\infty}\left(T_{\square\, b}\, (-1)^b e^{-ib\partial_u}\right) = \left(z_1 + e^{-i\partial_u}\right)\left(z_2 + e^{-i\partial_u}\right)^{-1}\left(\quad\right)^{-1}_{z_1}\left(\quad\right)_{z_1}$$



$$\hat H_{PSU(2,2|4)} \longrightarrow \hat S_{(SU(1|1)_{c.e.})^2}$$

spin chain Hamiltonian → magnon S-matrix

Symmetry + some other input $\longrightarrow \hat S = \sigma_{BES}(p,p')\, \hat S^{(p,p')}_{(2|2)} \otimes \hat S_{(2|2)}(p,p')$

(half) $\hat T(p) = \mathrm{tr}_{aux}\, \hat S_{(2|2)}(p,p_1) \cdots \hat S_{(2|2)}(p,p_M) =$

operators with eigenvalues $T_{L,R}(p)$

(no half)

$\hat T(p_j) =$ $\Longrightarrow$ BAE $\longleftrightarrow$ $e^{ipJ} \prod_{k} \tilde\sigma_{BES}(p,p_j)\, \hat T_L(p) \hat T_R(p) |\psi\rangle \Big|_{p=p_j} = |\psi\rangle$

$$T^L(p) = \underbrace{\frac{Q_3^+}{Q_3^-}}_{z_1} - \underbrace{\frac{Q_2^{--} Q_3^{+} R^{(+)-}}{Q_2 Q_3^{-} R^{(-)-}}}_{z_2} - \underbrace{\frac{Q_2^{++} Q_1^{-} R^{(+)-}}{Q_2^{+} Q_1^{+} R^{(-)-}}}_{z_3} + \underbrace{\frac{Q_1^{-} R^{(+)-} B^{(+)+}}{Q_1^{+} R^{(-)-} B^{(-)+}}}_{z_4}$$

7th Bethe Equation

$T^{L,R}$ regular $\Longrightarrow$ 3 + 3 spin wave Bethe equations

δE_{wrapp} for $\mathrm{tr}[Z,X]^2$?

$$(\chi_{a,s})^2 = \chi_{a,s+1}\chi_{a,s-1} + \chi_{a+1,s}\chi_{a-1,s}$$

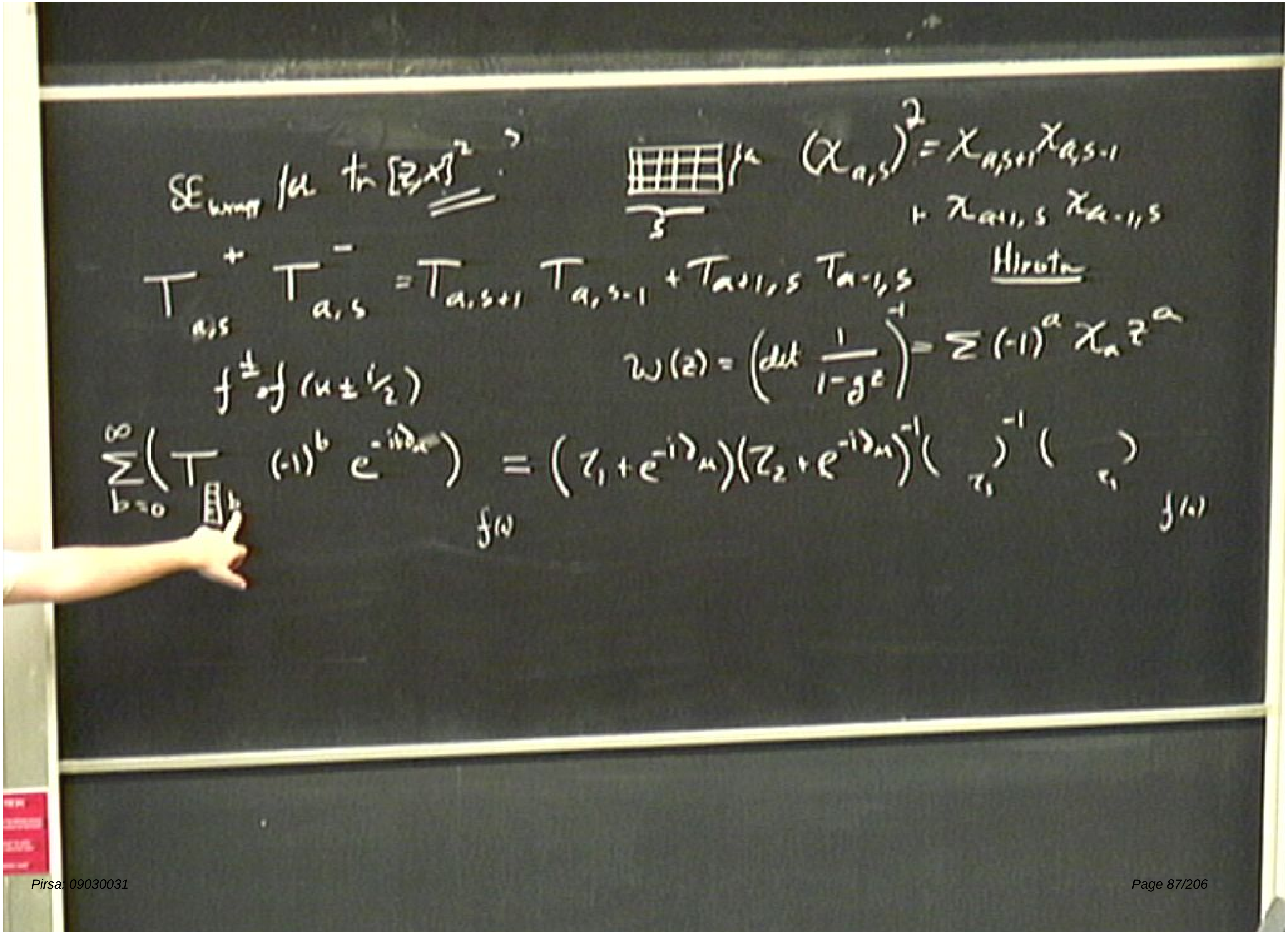
$$T^{+}_{a,s}\, T^{-}_{a,s} = T_{a,s+1}\, T_{a,s-1} + T_{a+1,s}\, T_{a-1,s}$$

Hirota

$$f^{\pm} = f(u \pm \tfrac{1}{2})$$

$$w(z) = \left(\det \frac{1}{1-gz}\right)^{-1} = \sum (-1)^a \chi_a z^a$$

$$\sum_{b=0}^{\infty}\left(T_{\square^{b}}\, (-1)^b\, e^{-ib\partial_u}\right)_{f(u)} = \left(z_1 + e^{-i\partial_u}\right)\left(z_2 + e^{-i\partial_u}\right)^{-1}\left(\quad\right)^{-1}_{z_1}\left(\quad\right)_{z_1}\ f(u)$$



δE_{wrapp} for $\mathrm{tr}\,[Z,X]^2$?

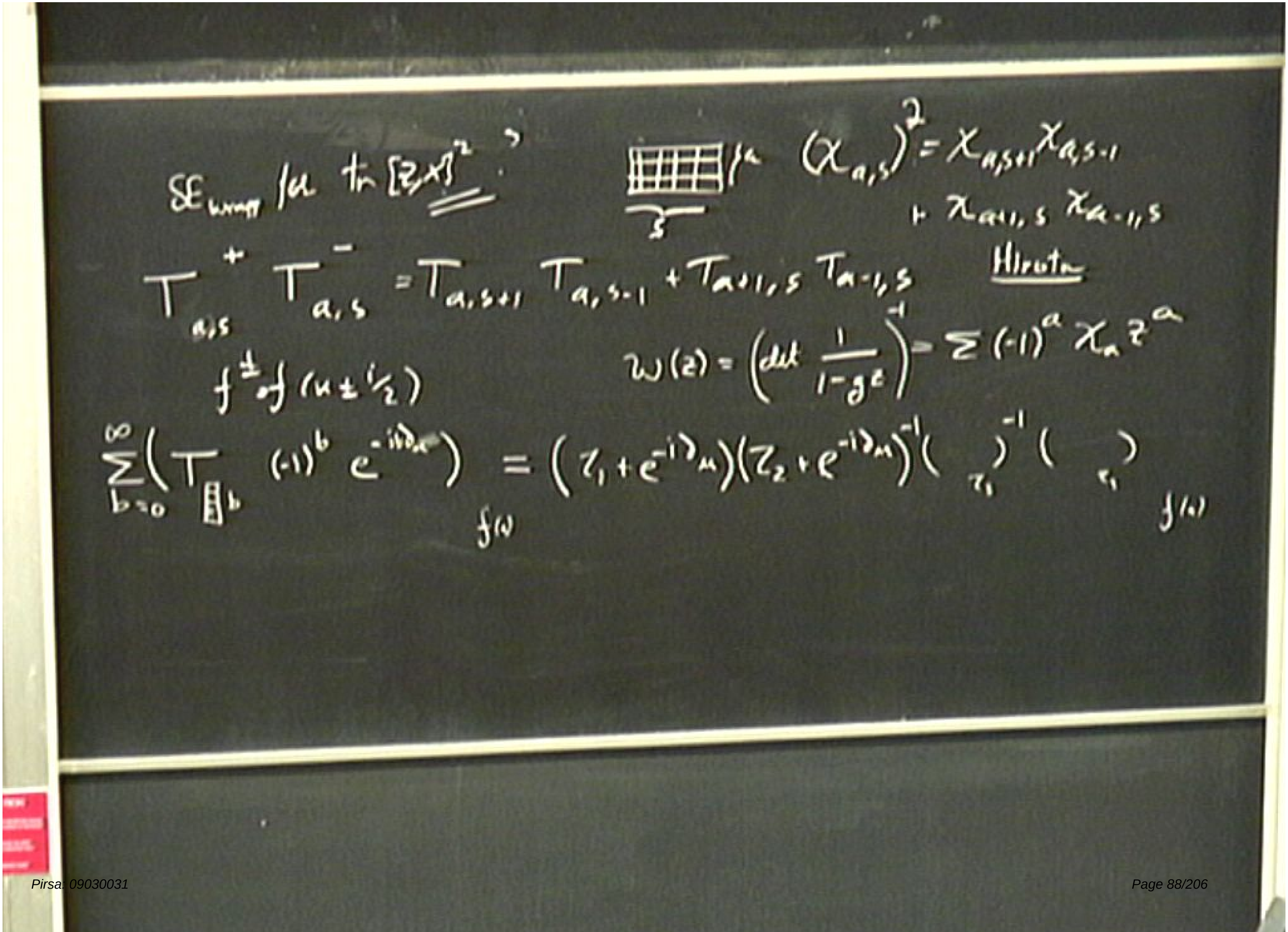
$\underbrace{\boxed{}}_{s}\}a \quad (\chi_{a,s})^2 = \chi_{a,s+1}\chi_{a,s-1} + \chi_{a+1,s}\chi_{a-1,s}$

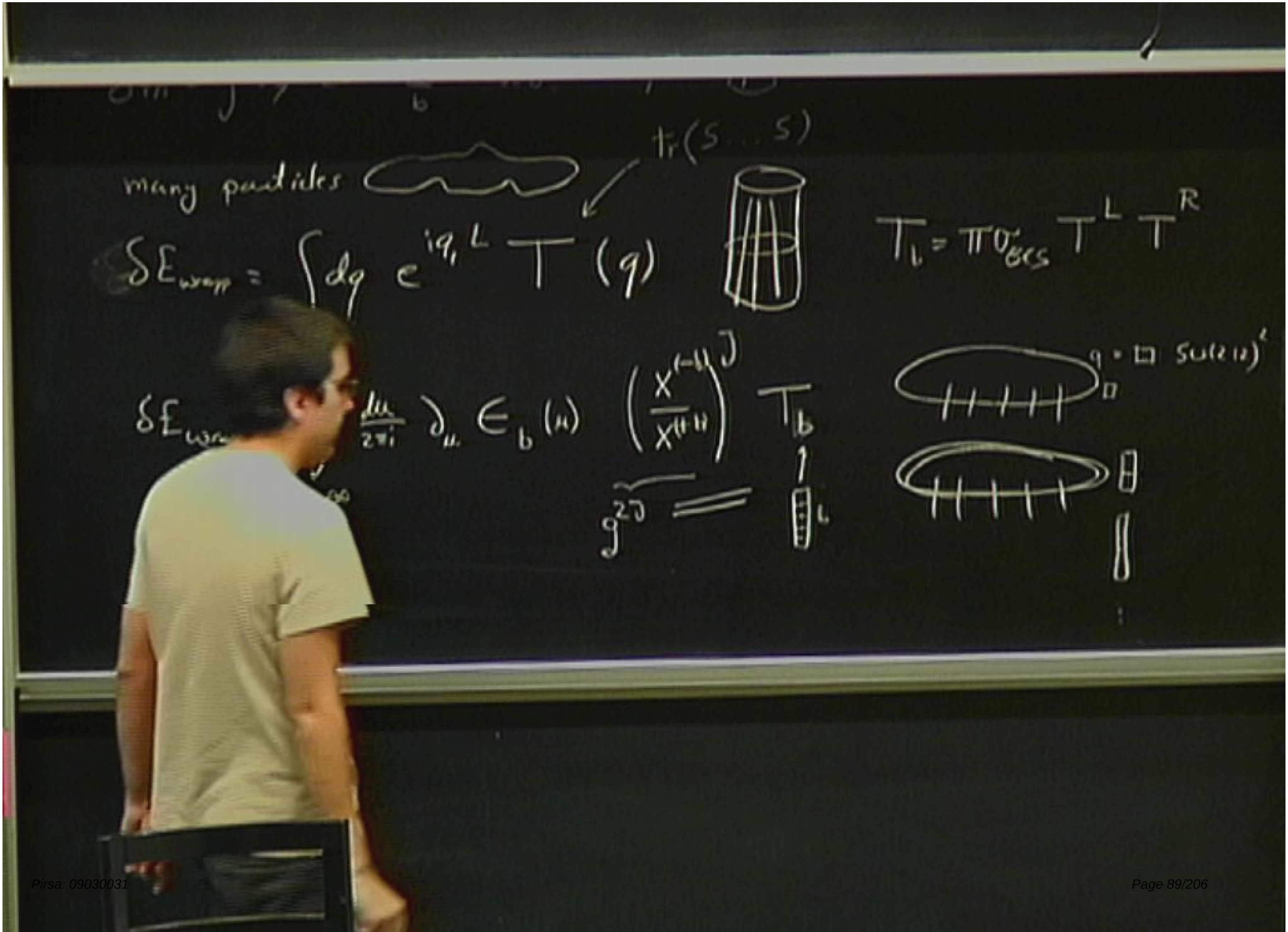
$$T^{+}_{a,s}\,T^{-}_{a,s} = T_{a,s+1}\,T_{a,s-1} + T_{a+1,s}\,T_{a-1,s}$$ Hirota

$f^{\pm} = f(u \pm i/2)$

$$w(z) = \left(\det \frac{1}{1-gz}\right)^{-1} = \sum (-1)^a \chi_a z^a$$

$$\sum_{b=0}^{\infty}\left(T_{\boxed{\,}\,b}\,(-1)^b e^{-ib\partial_u}\right)_{f(u)} = \left(z_1 + e^{-i\partial_u}\right)\left(z_2 + e^{-i\partial_u}\right)^{-1}\left(\quad\right)^{-1}_{z_1}\left(\quad\right)_{z_1} f(u)$$





many particles

$\mathrm{tr}(S \ldots S)$

$$\delta E_{wrap} = \int dq\, e^{iq_1 L}\, T(q)$$

$$T_b = \pi \sigma_{BES}\, T^L T^R$$

$$\delta E_{wr} \ldots \frac{du}{2\pi i}\, \partial_u \epsilon_b(u) \left(\frac{x^{(-1)}}{x^{(+1)}}\right)^J T_b$$

$$g^{2J} = \boxed{\vdots}_L$$

$q = \square \quad SU(2|2)^2$

many particles

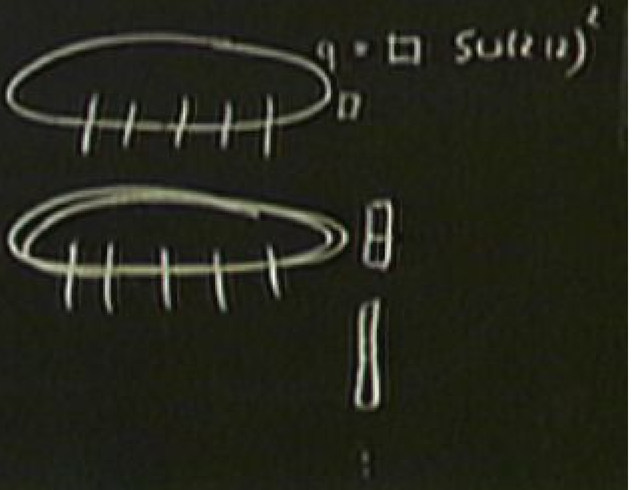
$\mathrm{tr}(S \ldots S)$

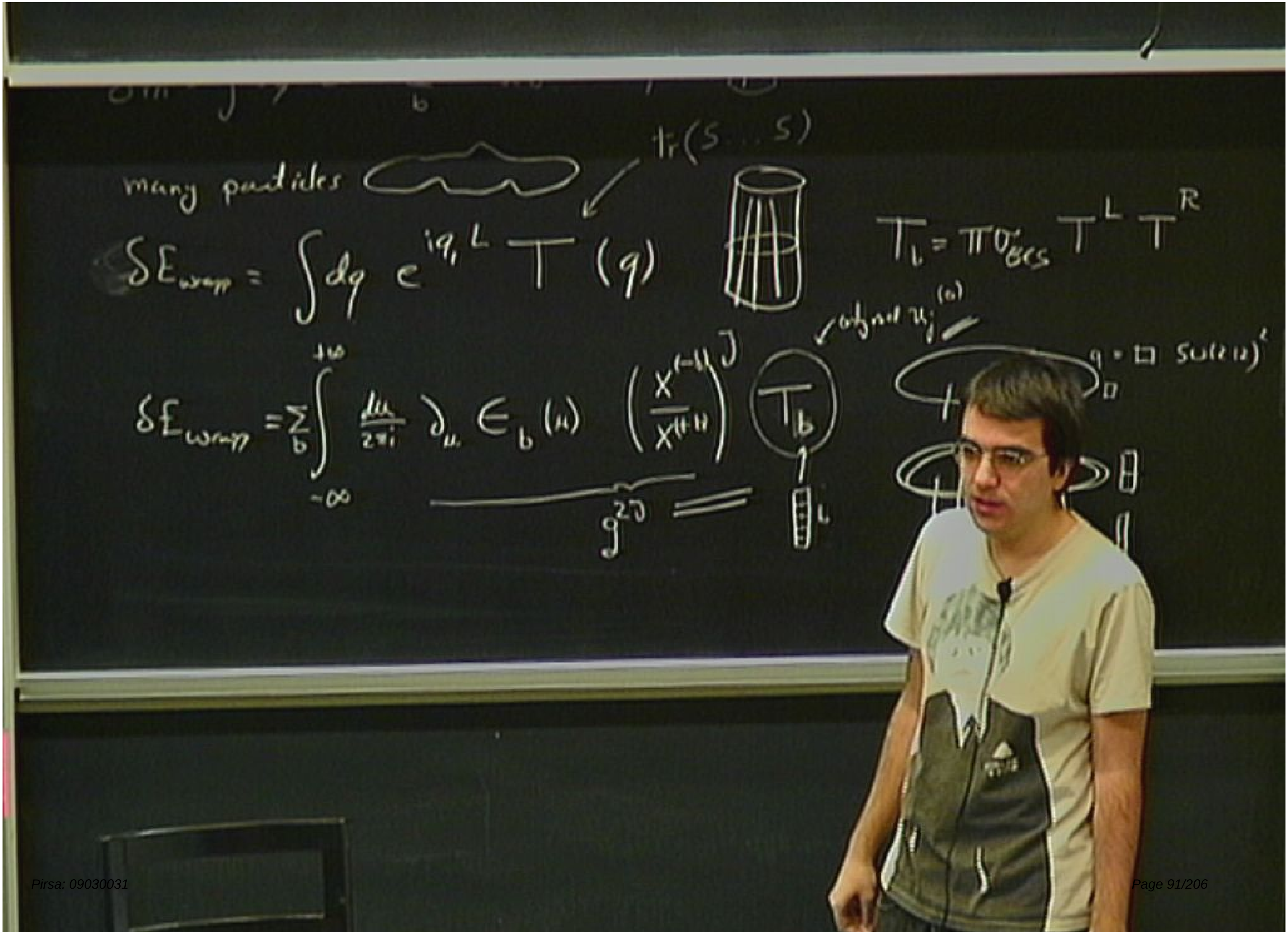
$$\delta E_{\text{wrap}} = \int dq \; e^{iq_1 L} \, T(q)$$

$$T_b = \pi \sigma_{BES}' \, T^L \, T^R$$

$$\delta E_{\text{wrap}} = \sum_b \int_{-\infty}^{+\infty} \frac{du}{2\pi i} \, \partial_u \epsilon_b(u) \left(\frac{X^{(-)}}{X^{(+)}} \right)^J T_b$$

$$\overbrace{g^{2J}}^{} = \quad T_b \; \text{—} \; b$$





many particles

tr(S … S)

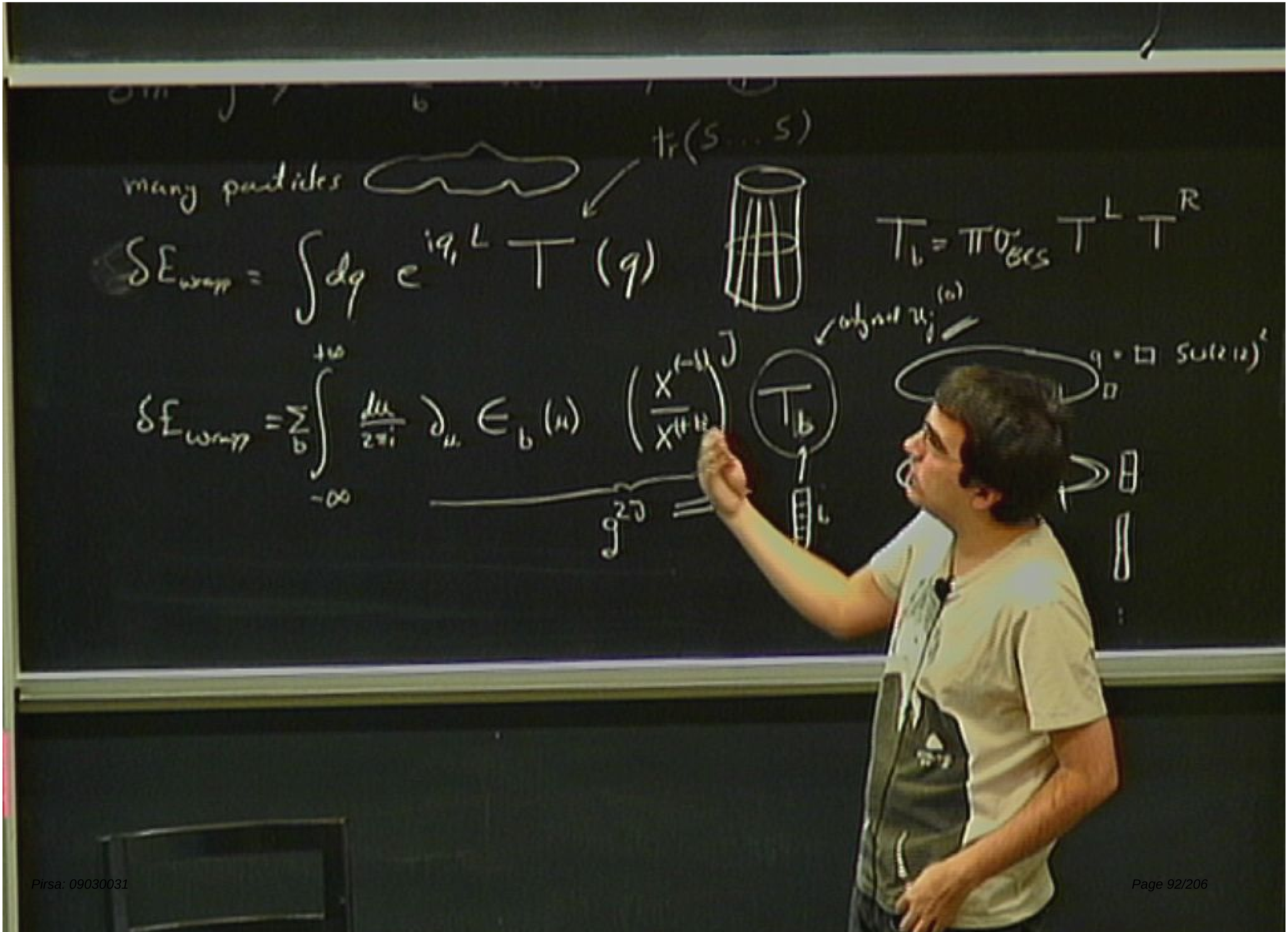
$$\delta E_{wrap} = \int dq\, e^{iq_1 L}\, T(q)$$

$$T_L = \prod \sigma'_{BCS}\, T^L\, T^R$$

obtained $u_j^{(a)}$

$$\delta E_{wrap} = \sum_b \int_{-\infty}^{+\infty} \underbrace{\frac{du}{2\pi i}\, \partial_u \epsilon_b(u) \left(\frac{X^{(-)}}{X^{(+)}}\right)^J}_{g^{2J}} \boxed{T_b}$$

$q = \square$ $SU(2|2)^2$



many particles

$\mathrm{tr}(S \ldots S)$

$$\delta E_{wrap} = \int dq \, e^{i q_1 L} \, T(q)$$

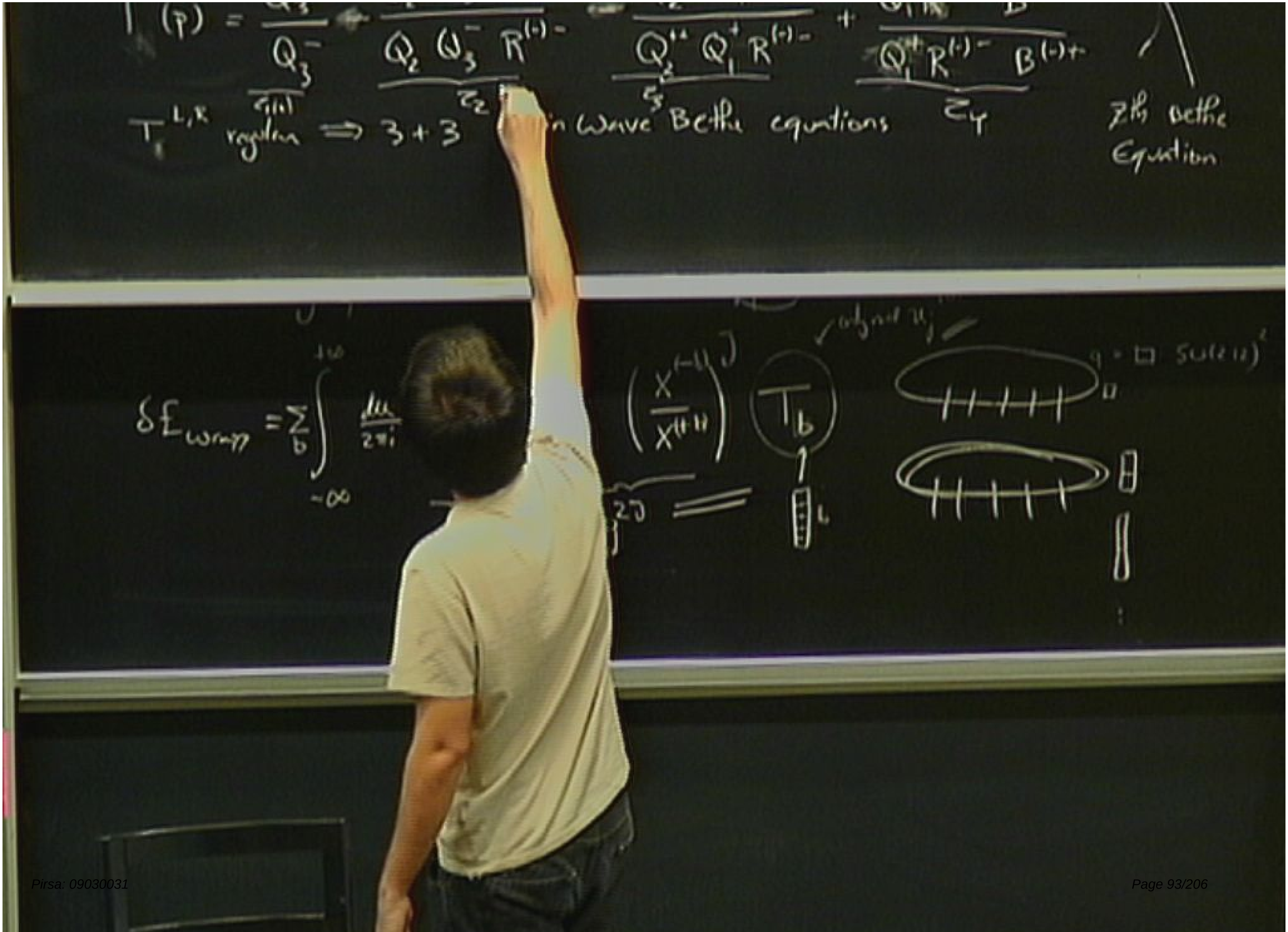
$$T_b = \pi \sigma_{BES}' \, T^L \, T^R$$

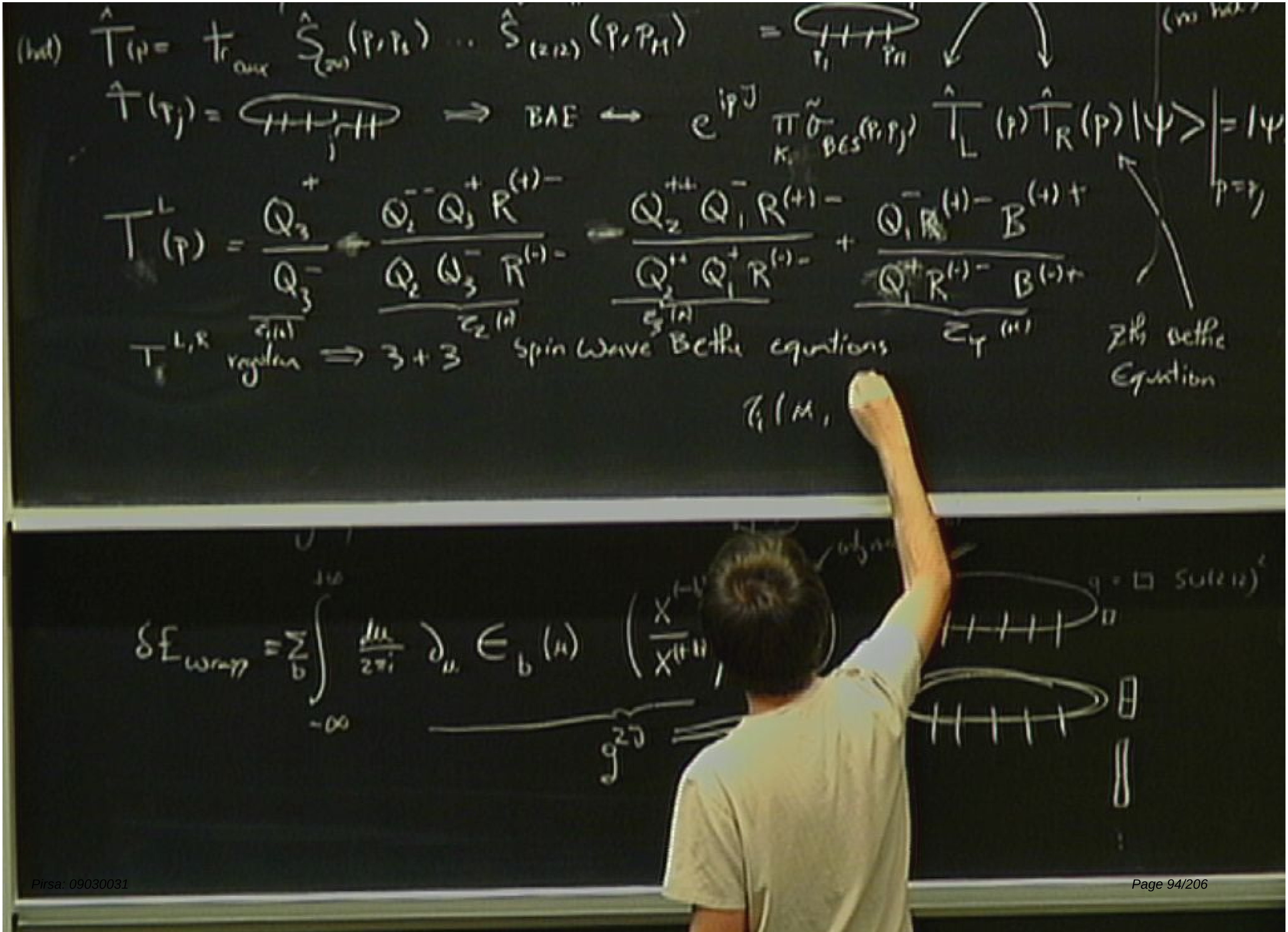
defined $x_j^{(a)}$

$q = \square \quad SU(2|2)^2$

$$\delta E_{wrap} = \sum_b \int_{-\infty}^{+\infty} \frac{du}{2\pi i} \, \partial_u \epsilon_b(u) \, \left(\frac{X^{(-)}}{X^{(+)}} \right)^J \boxed{T_b}$$

g^{2J}


$T(p) = \frac{Q_3}{Q_3^-}$
$Q_2 Q_3^- R^{(-)-}$
z_2
$Q_2^{++} Q_1^+ R^{(-)-}$
z_3
$Q_1^+ R^{(-)-} B^{(-)+}$
z_4
$T_1^{L,R}$ regular ⟹ 3 + 3 in Wave Bethe equations
7th Bethe Equation
$\delta E_{wrapp} = \sum_b \int_{-\infty}^{+\infty} \frac{du}{2\pi i}$
$\left(\frac{X^{[-1]}}{X^{[+1]}}\right)^J$
$2J$
T_b
b
$q = \square$ $su(2|2)^2$



(hol) $\hat{T}(p) = \mathrm{tr}_{aux}\, \hat{S}_{(2|2)}(p,p_1) \dots \hat{S}_{(2|2)}(p,p_M)$ = $p_1 \dots p_M$ (no hol)

$\hat{T}(p_j) =$ j $\Longrightarrow$ BAE $\longleftrightarrow$ $e^{ipJ} \prod_{k} \tilde{\sigma}_{BES}(p,p_j)\, \hat{T}_L(p)\hat{T}_R(p)|\psi\rangle\Big|_{p=p_j} = |\psi$

$$T^L(p) = \underbrace{\frac{Q_3^{+}}{Q_3^{-}}}_{z_1(u)} + \underbrace{\frac{Q_1^{--}Q_3^{+}R^{(+)-}}{Q_2 Q_3^{-} R^{(-)-}}}_{z_2(u)} - \underbrace{\frac{Q_2^{++}Q_1^{-}R^{(+)-}}{Q_2^{+}Q_1^{+}R^{(-)-}}}_{z_3(u)} + \underbrace{\frac{Q_1^{-}R^{(+)-}B^{(+)+}}{Q_1^{+}R^{(-)-}B^{(-)+}}}_{z_4(u)}$$

$T_1^{L,R}$ regular $\Longrightarrow$ 3 + 3 spin wave Bethe equations

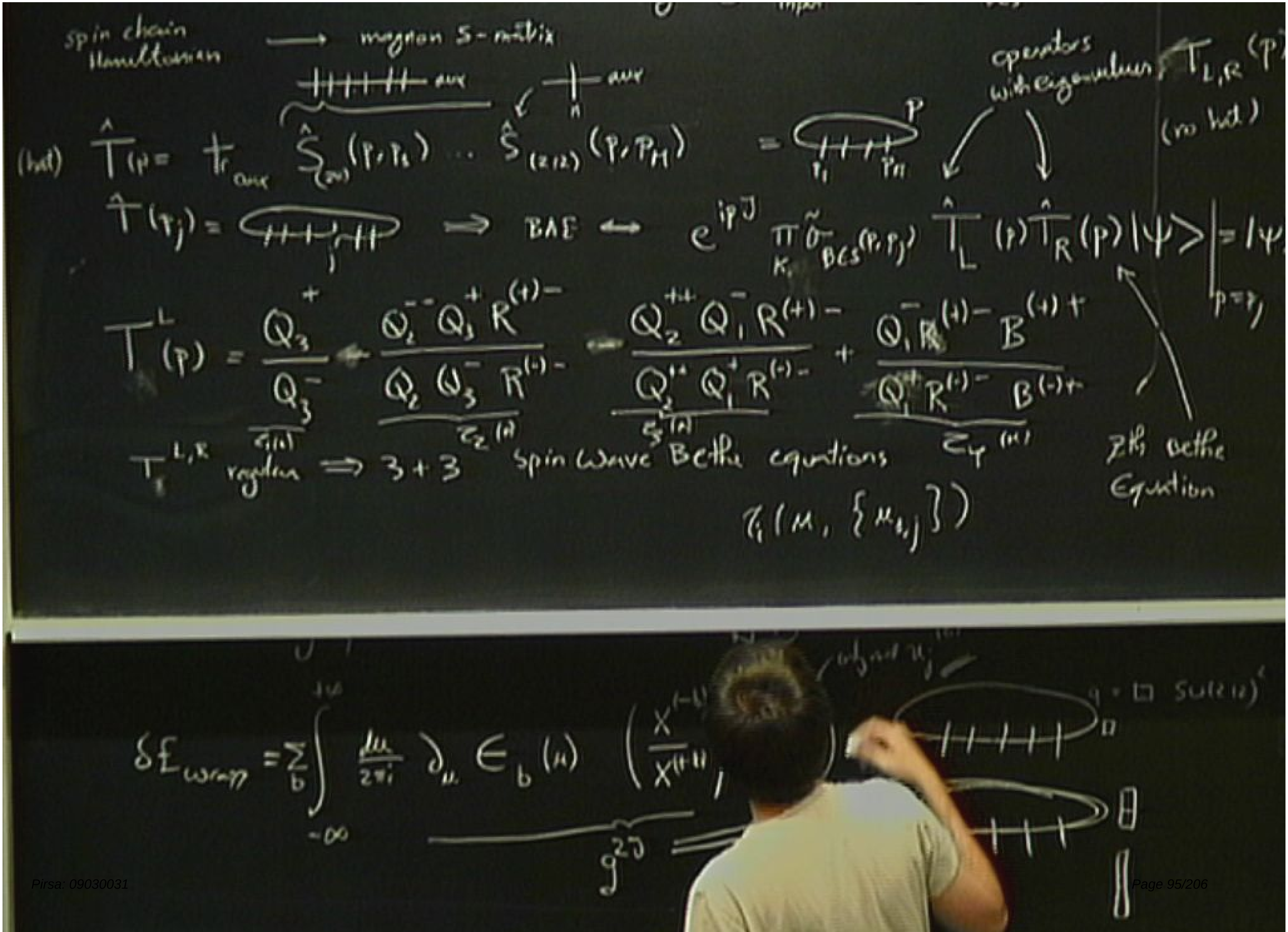
2nd Bethe Equation

$z_1(u,$

$$\delta E_{wrapp} = \sum_b \int_{-\infty}^{+\infty} \frac{du}{2\pi i}\, \partial_u \epsilon_b(u) \left(\frac{X^{[-b]}}{X^{[+b]}}\right.$$

g^{2J}

$q = \square$ $SU(2|2)^2$



spin chain Hamiltonian ⟶ magnon S-matrix

$$\text{(hol)}\quad \hat{T}(p) = \mathrm{tr}_{\text{aux}}\, \underbrace{\hat{S}_{(2|2)}(p,p_1)\ \cdots\ \hat{S}_{(2|2)}(p,p_M)}\ = $$

$$\hat{T}(p_j) = \quad \Longrightarrow \text{BAE} \longleftrightarrow e^{ipJ} \prod_{k} \tilde{\sigma}_{\text{BES}}(p,p_j)\ \hat{T}_L(p)\,\hat{T}_R(p)\,|\psi\rangle\Big|_{p=p_j} = |\psi$$

operators with eigenvalues $T_{L,R}(p)$ (no hol)

$$T^{L}(p) = \underbrace{\frac{Q_3^{+}}{Q_3^{-}}}_{\epsilon_1(u)} - \underbrace{\frac{Q_2^{--}\,Q_3^{+}\,R^{(+)-}}{Q_2\,Q_3^{-}\,R^{(-)-}}}_{\epsilon_2(u)} - \underbrace{\frac{Q_2^{++}\,Q_1^{-}\,R^{(+)-}}{Q_2^{+}\,Q_1^{+}\,R^{(-)-}}}_{\epsilon_3(u)} + \underbrace{\frac{Q_1^{-}\,R^{(+)-}\,B^{(+)+}}{Q_1^{+}\,R^{(-)-}\,B^{(-)+}}}_{\epsilon_4(u)}$$

$T_1^{L,R}$ regular ⟹ 3 + 3 spin wave Bethe equations

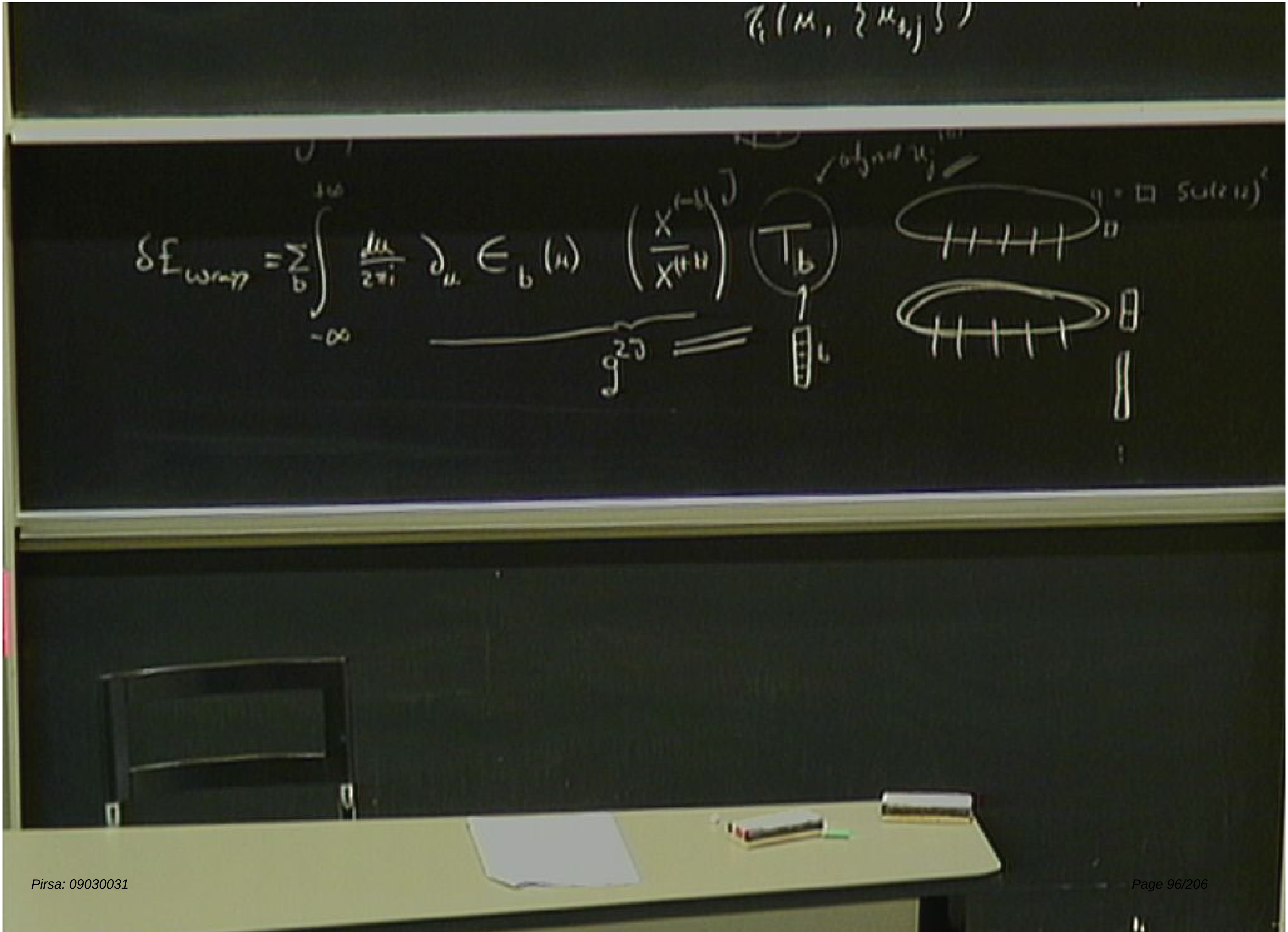
$\epsilon_1(u, \{u_{k,j}\})$

7th Bethe Equation

$$\delta E_{\text{wrap}} = \sum_b \int_{-\infty}^{+\infty} \frac{du}{2\pi i}\, \partial_u\, \epsilon_b(u)\ \underbrace{\left(\frac{X^{(-)}}{X^{(+)}}\right.}_{g^{2J}},$$

adjoint u_j

$a = \square\ SU(2|2)^2$

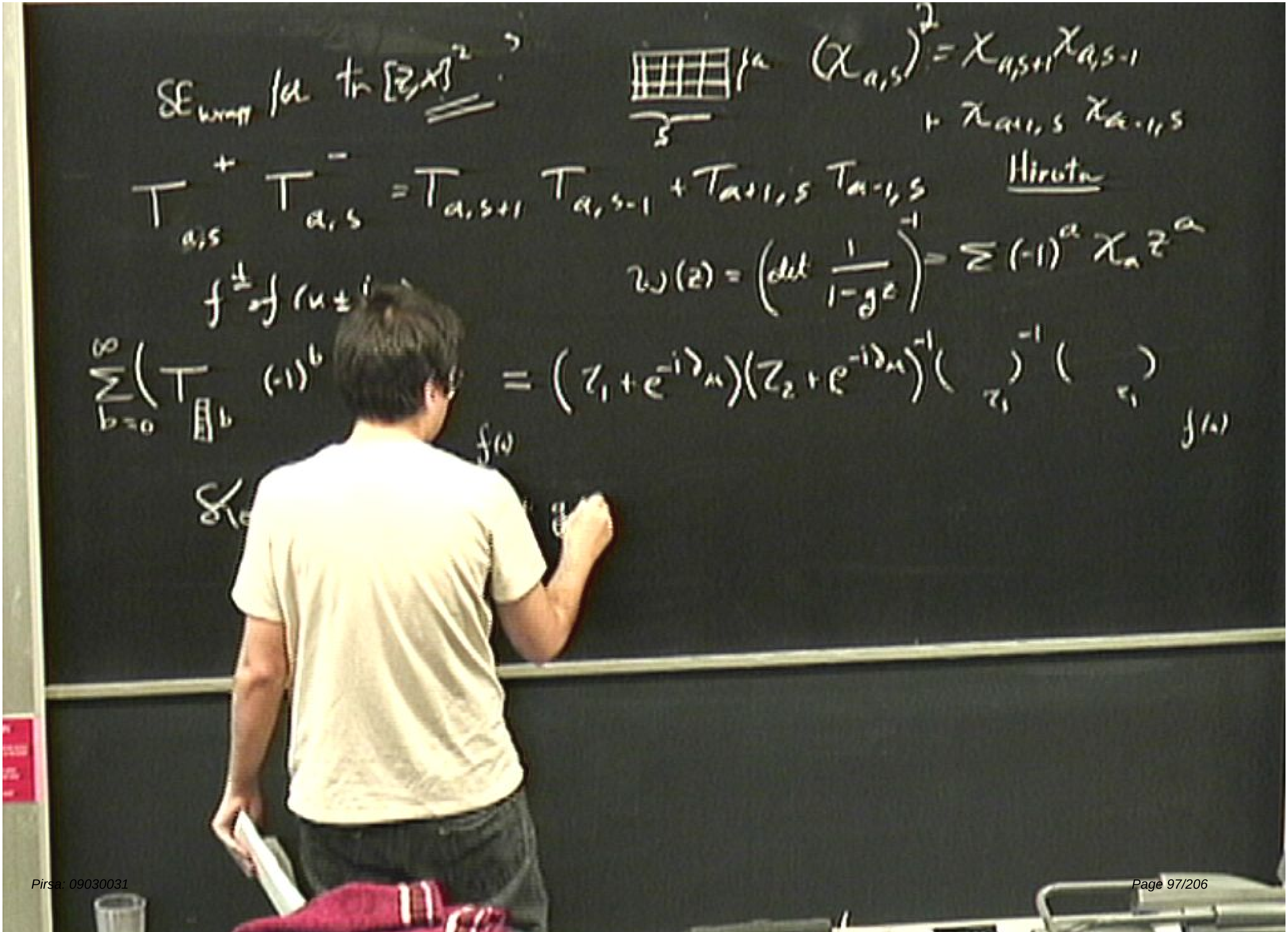


$T_1(u, \{u_{4,j}\})$

$$\delta\mathcal{E}_{wrapp} = \sum_b \int_{-\infty}^{+\infty} \frac{du}{2\pi i}\, \partial_u \epsilon_b(u) \underbrace{\left(\frac{X^{(-)}}{X^{(+)}}\right)^J}_{g^{2J}} T_b$$

adjoint u_j

$\square$ $SU(2|2)^2$



δE_{wrap} for $\mathrm{tr}[Z,X]^2$?

$(\chi_{a,s})^2 = \chi_{a,s+1}\chi_{a,s-1} + \chi_{a+1,s}\chi_{a-1,s}$

$T^{+}_{a,s}\, T^{-}_{a,s} = T_{a,s+1}\, T_{a,s-1} + T_{a+1,s}\, T_{a-1,s}$ **Hirota**

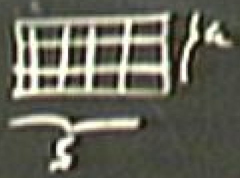
$f^{\pm} = f(u \pm i$

$w(z) = \left(\det \frac{1}{1-gz}\right)^{-1} = \sum (-1)^a \chi_a z^a$

$\sum_{b=0}^{\infty} \Big(T_{\square\, b}\ (-1)^b$ $\quad_{f(u)}$ $= \left(z_1 + e^{-i\partial_u}\right)\left(z_2 + e^{-i\partial_u}\right)^{-1}\left(\quad\right)^{-1}_{z_1}\left(\quad\right)_{z_1}\ f(u)$

$\delta($

δE_{wrapp} for $\underline{\underline{\mathrm{tr}[Z,X]^2}}$;



$(\chi_{a,s})^2 = \chi_{a,s+1}\chi_{a,s-1} + \chi_{a+1,s}\chi_{a-1,s}$

$T^+_{a,s}\, T^-_{a,s} = T_{a,s+1}\, T_{a,s-1} + T_{a+1,s}\, T_{a-1,s}$ <u>Hirota</u>

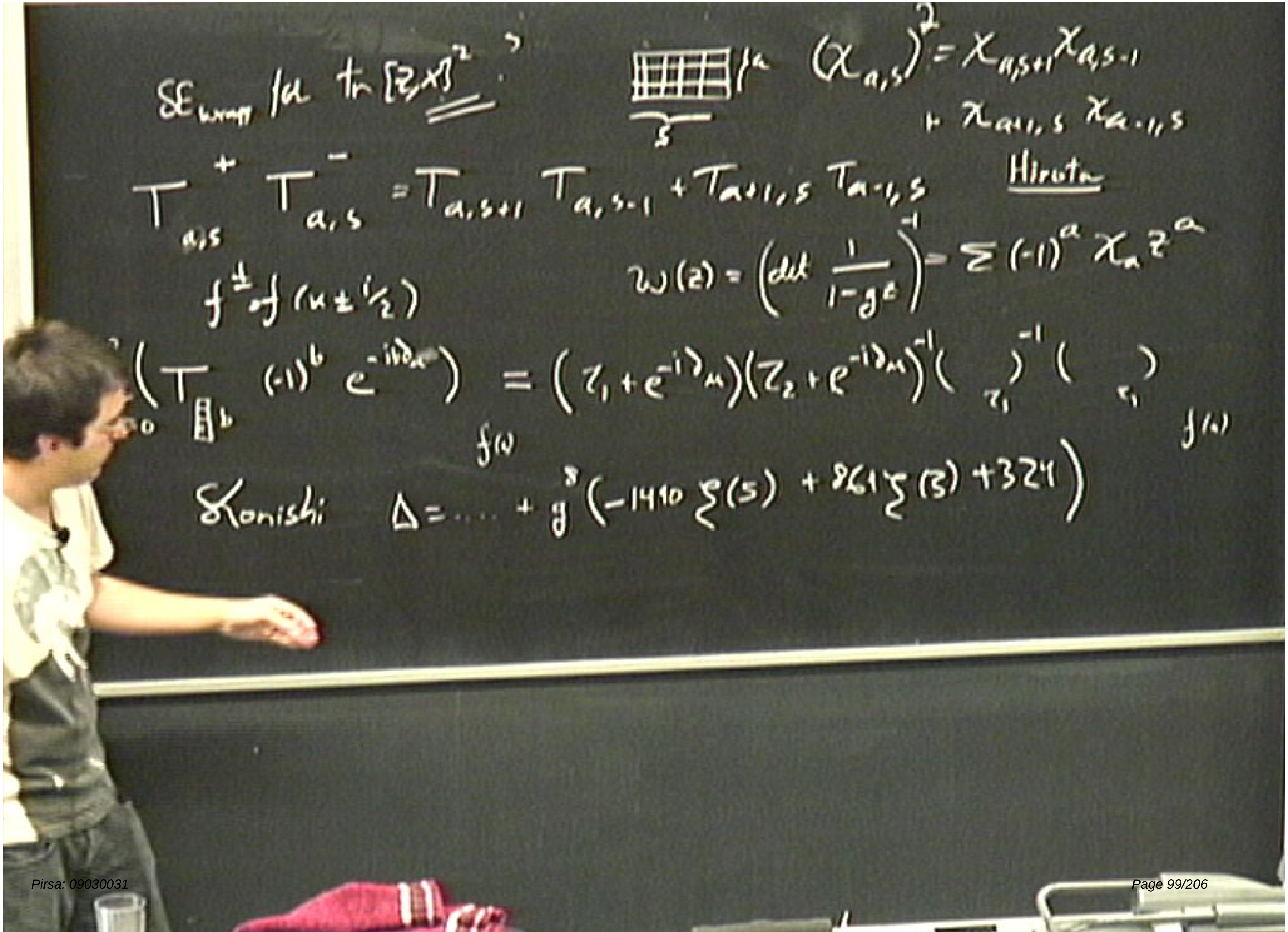
$f^{\pm} = f(u \pm i/2)$

$w(z) = \left(\det \frac{1}{1-gz}\right)^{-1} = \sum (-1)^a \chi_a z^a$

$$\sum_{b=0}^{\infty}\left(T_{b}\,(-1)^b\, e^{-ib\partial_u}\right) = \left(z_1 + e^{-i\partial_u}\right)\left(z_2 + e^{-i\partial_u}\right)^{-1}\left(\quad\right)^{-1}_{z_1}\left(\quad\right)_{z_1}$$

$f(u)$ $\qquad f(u)$

δKonishi $\Delta = \ldots + g^8\left(-1440\,\zeta(5) + 864\,\zeta(3) + 324\right)$



δE_{wrapp} for $\text{tr}[Z,X]^2$?

$$\left(\chi_{a,s}\right)^2 = \chi_{a,s+1}\chi_{a,s-1} + \chi_{a+1,s}\chi_{a-1,s}$$

$$T^+_{a,s}\,T^-_{a,s} = T_{a,s+1}\,T_{a,s-1} + T_{a+1,s}\,T_{a-1,s}$$

Hirota

$$w(z) = \left(\det \frac{1}{1-gz}\right)^{-1} = \sum (-1)^a \chi_a z^a$$

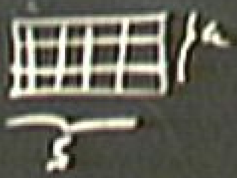
$f^{\pm}$ of $f(u \pm i/2)$

$$\sum_{b=0}\left(T_{b}\,(-1)^b\,e^{-ib\partial_u}\right) = \left(z_1 + e^{-i\partial_u}\right)\left(z_2 + e^{-i\partial_u}\right)^{-1}\left(\;\right)^{-1}_{z_1}\left(\;\right)_{z_1}$$

$f(u)$ $\qquad f(u)$

δKonishi $\quad \Delta = \dots + g^8\left(-1410\,\zeta(5) + 861\,\zeta(3) + 324\right)$

δE_{wrap} for $\mathrm{tr}[Z,X]^2$?



$$(\chi_{a,s})^2 = \chi_{a,s+1}\chi_{a,s-1} + \chi_{a+1,s}\chi_{a-1,s}$$

$$T^{+}_{a,s}\, T^{-}_{a,s} = T_{a,s+1}\, T_{a,s-1} + T_{a+1,s}\, T_{a-1,s}$$

Hirota

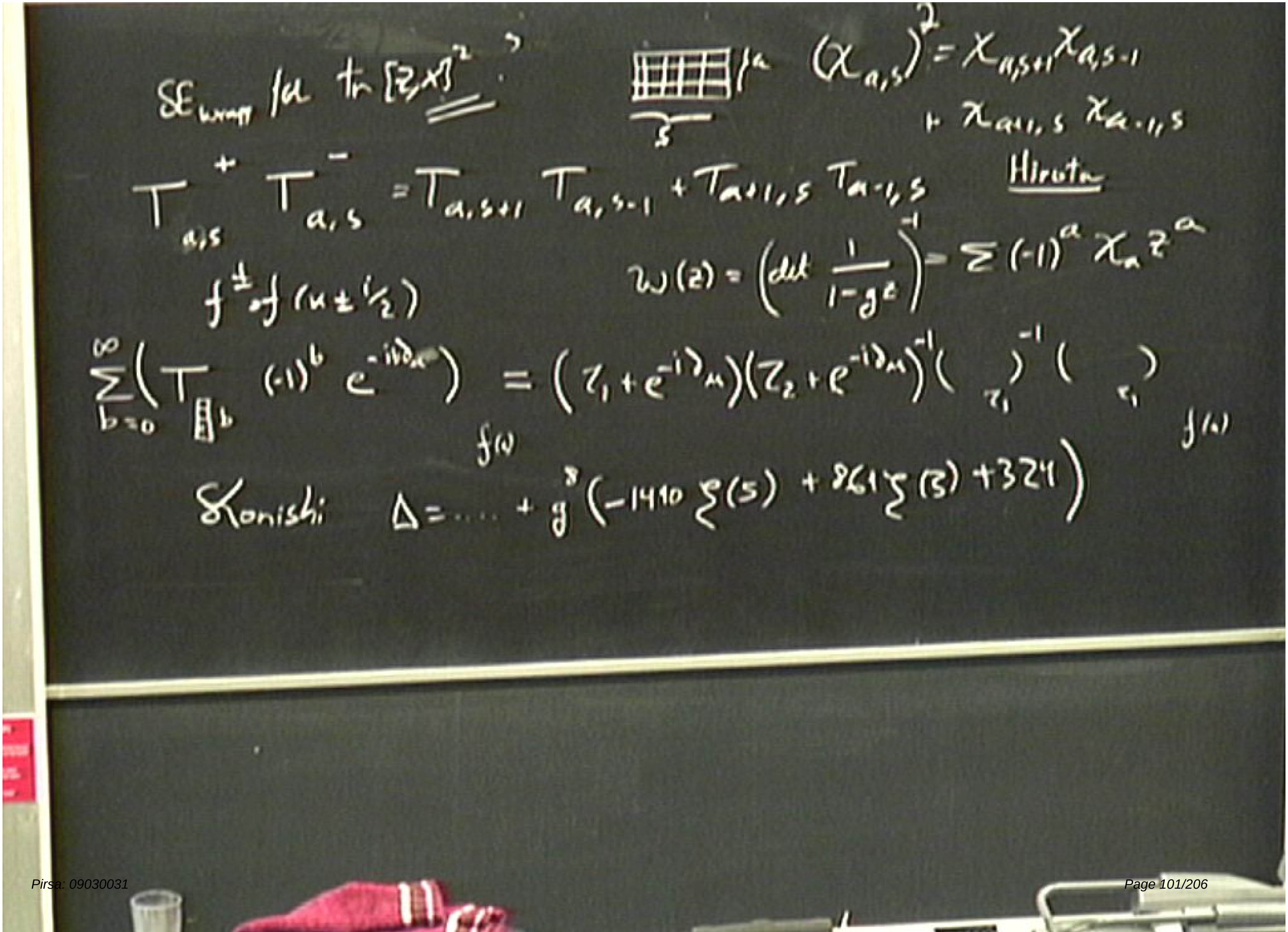
$$w(z) = \left(\det \frac{1}{1-gz}\right)^{-1} = \sum (-1)^a \chi_a z^a$$

$f^{\pm}$ of $(u \pm i/2)$

$$\sum_{b=0}^{\infty}\left(T_{b}\,(-1)^b e^{-i b \partial_u}\right) = \left(z_1 + e^{-i\partial_u}\right)\left(z_2 + e^{-i\partial_u}\right)^{-1}\left(\quad_{z_1}\right)^{-1}\left(\quad_{z_1}\right)$$

$f(u)$ $f(u)$

$$\text{Konishi} \quad \Delta = \ldots + g^8\left(-1410\,\zeta(5) + 861\,\zeta(3) + 324\right)$$



δE_{wrap} for $\text{tr}[Z,X]^2$?

$\underbrace{\boxed{}}_{s}\}a \quad (\chi_{a,s})^2 = \chi_{a,s+1}\chi_{a,s-1} + \chi_{a+1,s}\chi_{a-1,s}$

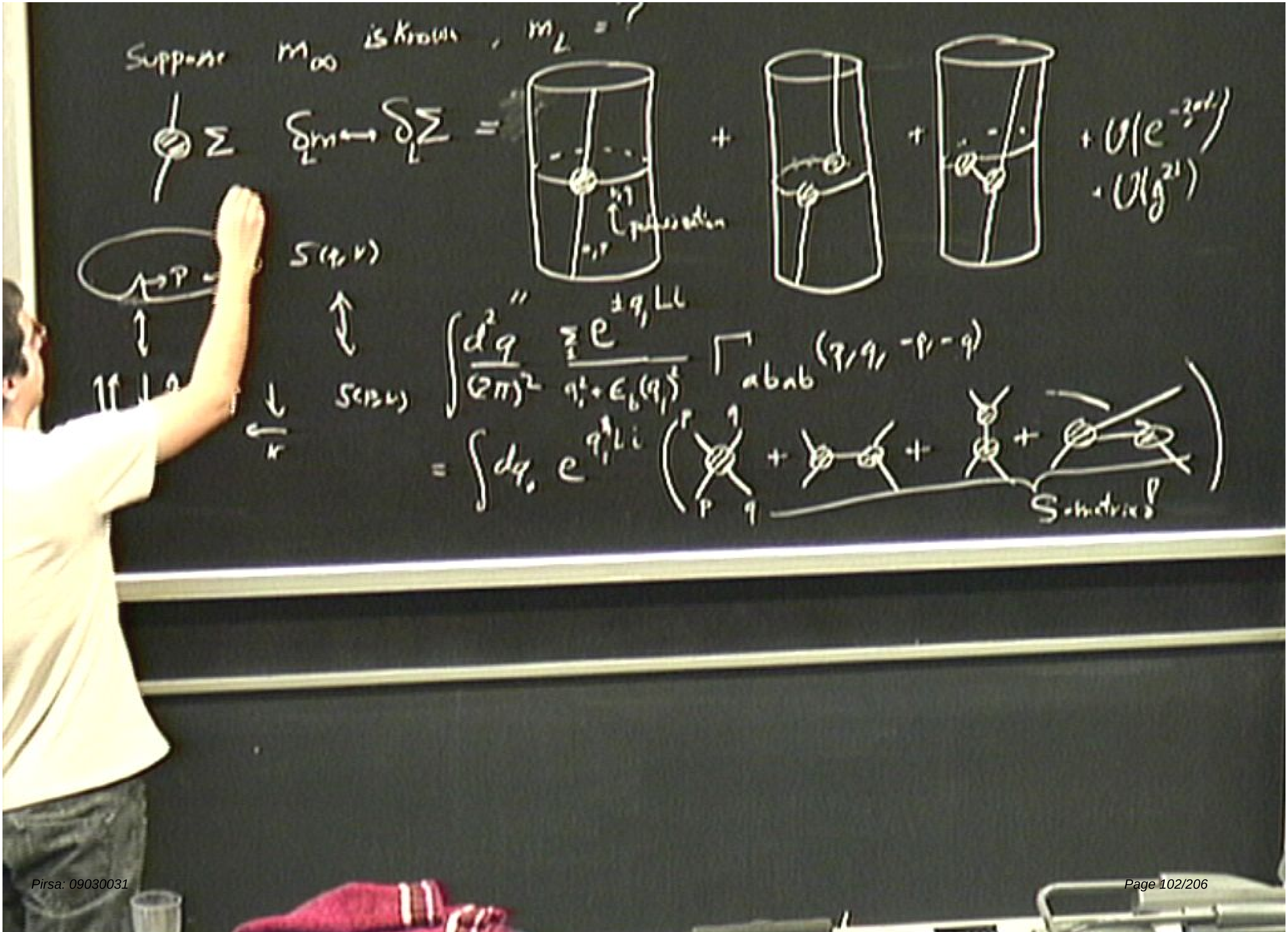
$T^{+}_{a,s}\, T^{-}_{a,s} = T_{a,s+1}\, T_{a,s-1} + T_{a+1,s}\, T_{a-1,s}$ **Hirota**

$f^{\pm} = f(u \pm i/2)$

$w(z) = \left(\det \frac{1}{1-gz}\right)^{-1} = \sum (-1)^a \chi_a z^a$

$$\sum_{b=0}^{\infty}\left(T_{b}\,(-1)^b\, e^{-ib\partial_u}\right) f(u) = \left(z_1 + e^{-i\partial_u}\right)\left(z_2 + e^{-i\partial_u}\right)^{-1}\left(\quad {}_{z_1}\right)^{-1}\left(\quad {}_{z_1}\right) f(u)$$

Konishi $\Delta = \dots + g^8\left(-1410\,\zeta(5) + 861\,\zeta(3) + 324\right)$



Suppose m_∞ is known, $m_L = ?$

$\Sigma \quad \delta_L m \longleftrightarrow \delta_L \Sigma = \quad + \quad + \quad + \mathcal{O}(e^{-3m L}) + \mathcal{O}(g^{21})$

$S(q,v)$

$\Updownarrow$

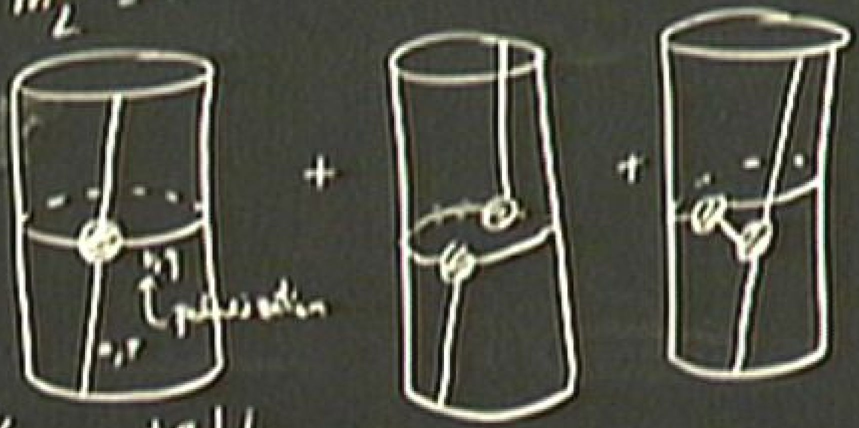
$S(B,v) \quad \int \frac{d^2 q}{(2\pi)^2} \frac{\sum_{\pm} e^{\pm q_1 L i}}{q_1^2 + \epsilon_b(q_1)^2} \Gamma_{abab}(q, q, -p, -q)$

$= \int dq_0 \, e^{q_1 L i} \left(\quad + \quad + \quad + \quad \right)$

Suppose m_∞ is known, $m_L = ?$

$\Sigma \quad \delta_L m \longleftrightarrow \delta_L \Sigma =$

1+1 dFT



$+ O(e^{-3\mu L})$
$+ O(g^{2})$

$S(q,\nu)$

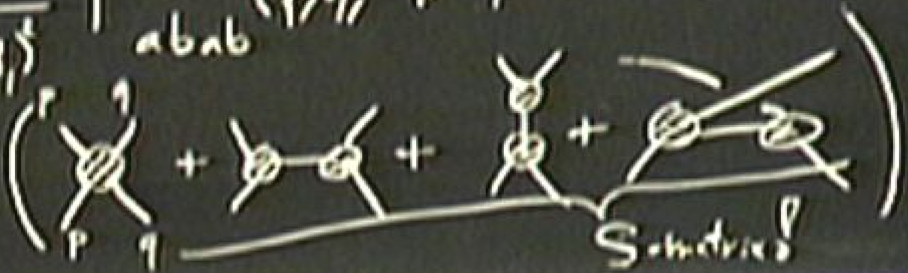
$\updownarrow \qquad \Updownarrow$

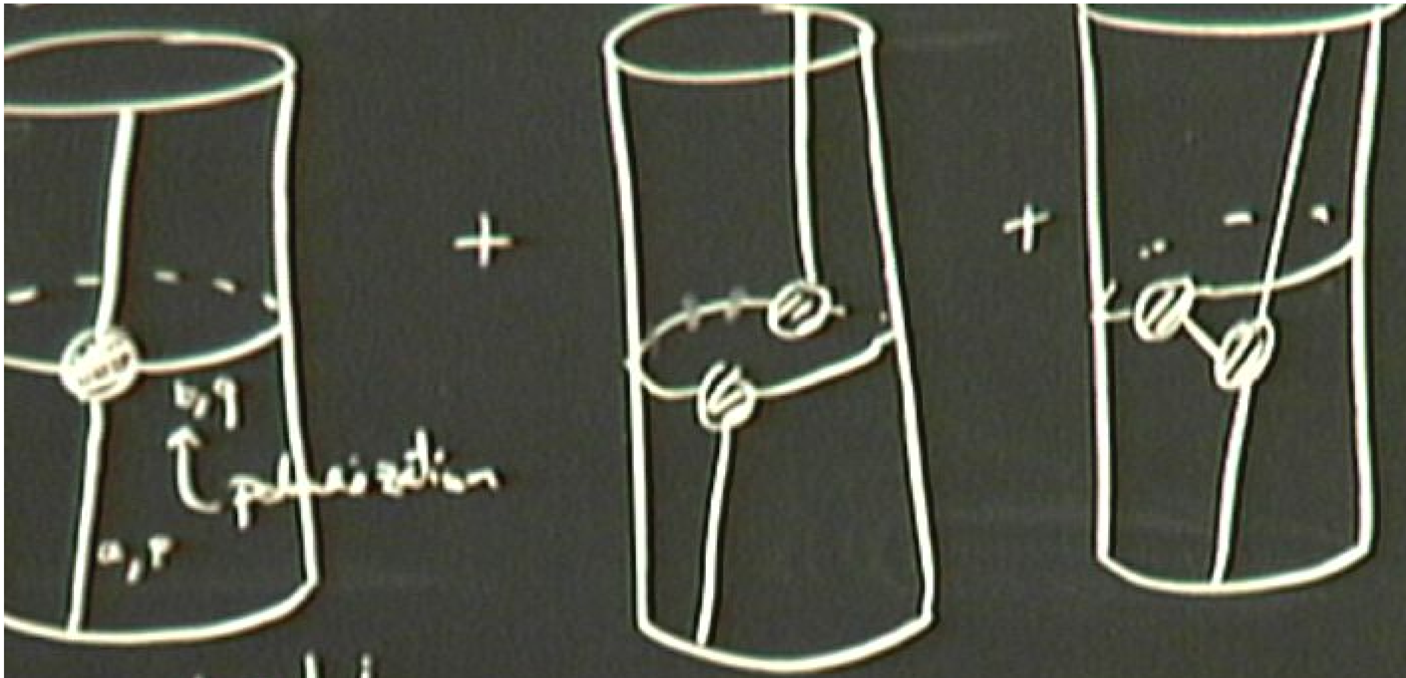
↑↑ ↓ ↑ ↑ ↑ ↓ $\quad S(p,\nu)$

$\overrightarrow{q} \qquad \overleftarrow{k}$

$$\int \frac{d^2q}{(2\pi)^2}\ \frac{\sum e^{\pm q_1 L i}}{q_1^2+\epsilon_b(q_1)^2}\ \Gamma_{abab}(q, q, -p, -q)$$

$$= \int dq_0\ e^{q_1 L i} \Big(\Big)$$

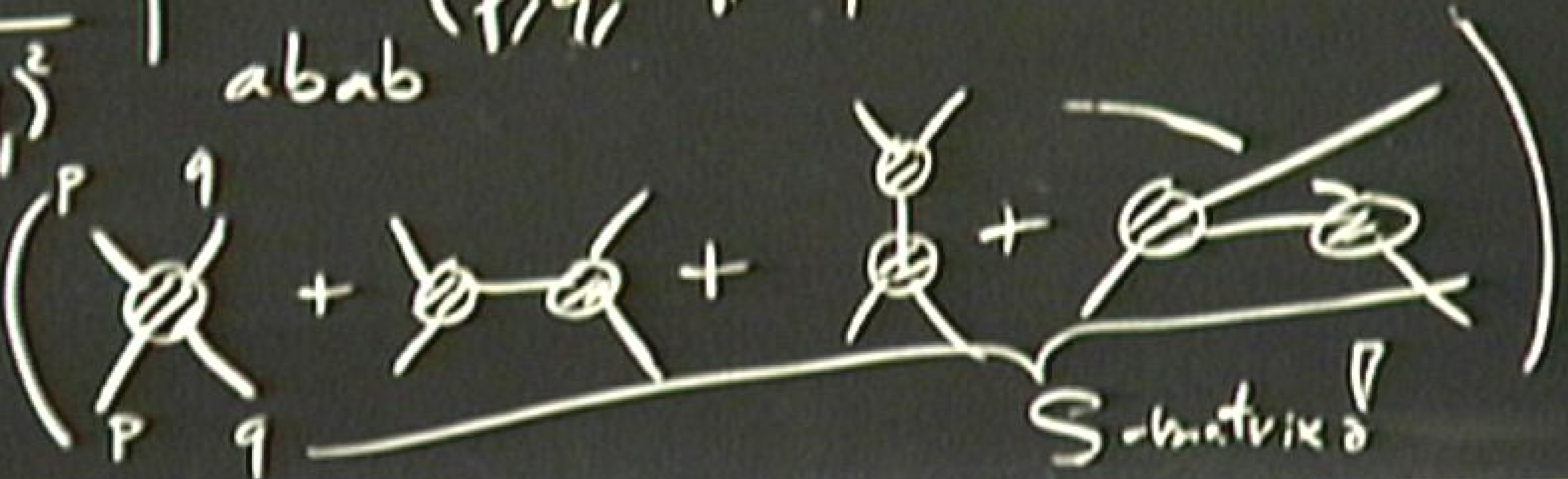




$+\mathcal{O}(e^{-2mL})$
$+\mathcal{O}(g^{2L})$

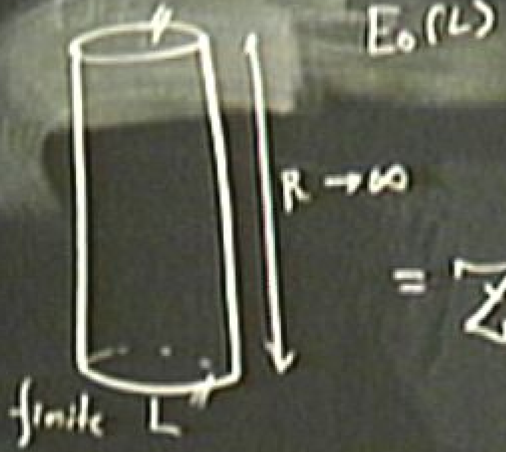
$$\frac{\sum_{\pm} e^{\pm q_1 L i}}{q_0^2+\epsilon_b(q_1)^2}\,\Gamma_{abab}(p,q,-p,-q)$$

$e^{q_1^{*} L i}$



Today

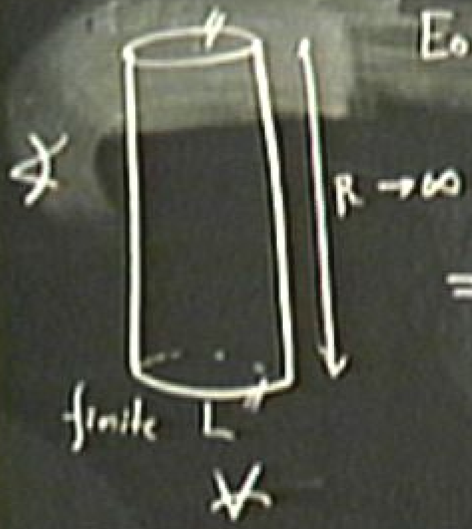
[with N. Gromov, V. Kazakov]



$$= Z = e^{-R E_0(L)}$$

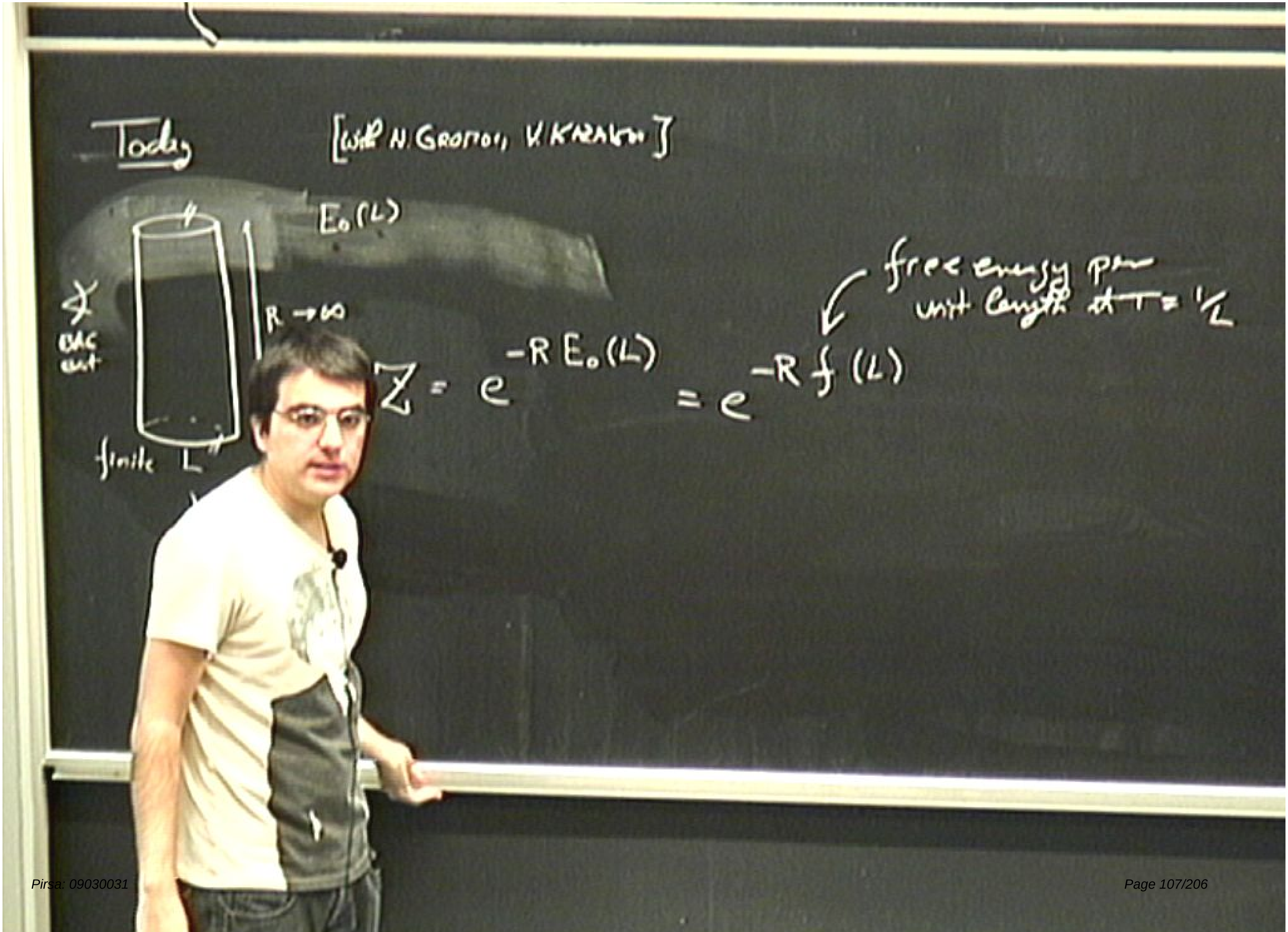
Today

[with N. GROMOV, V. KAZAKOV]



$$= Z = e^{-R E_0(L)} = e^{-R f(L)}$$

free energy per unit length at T = 1/L

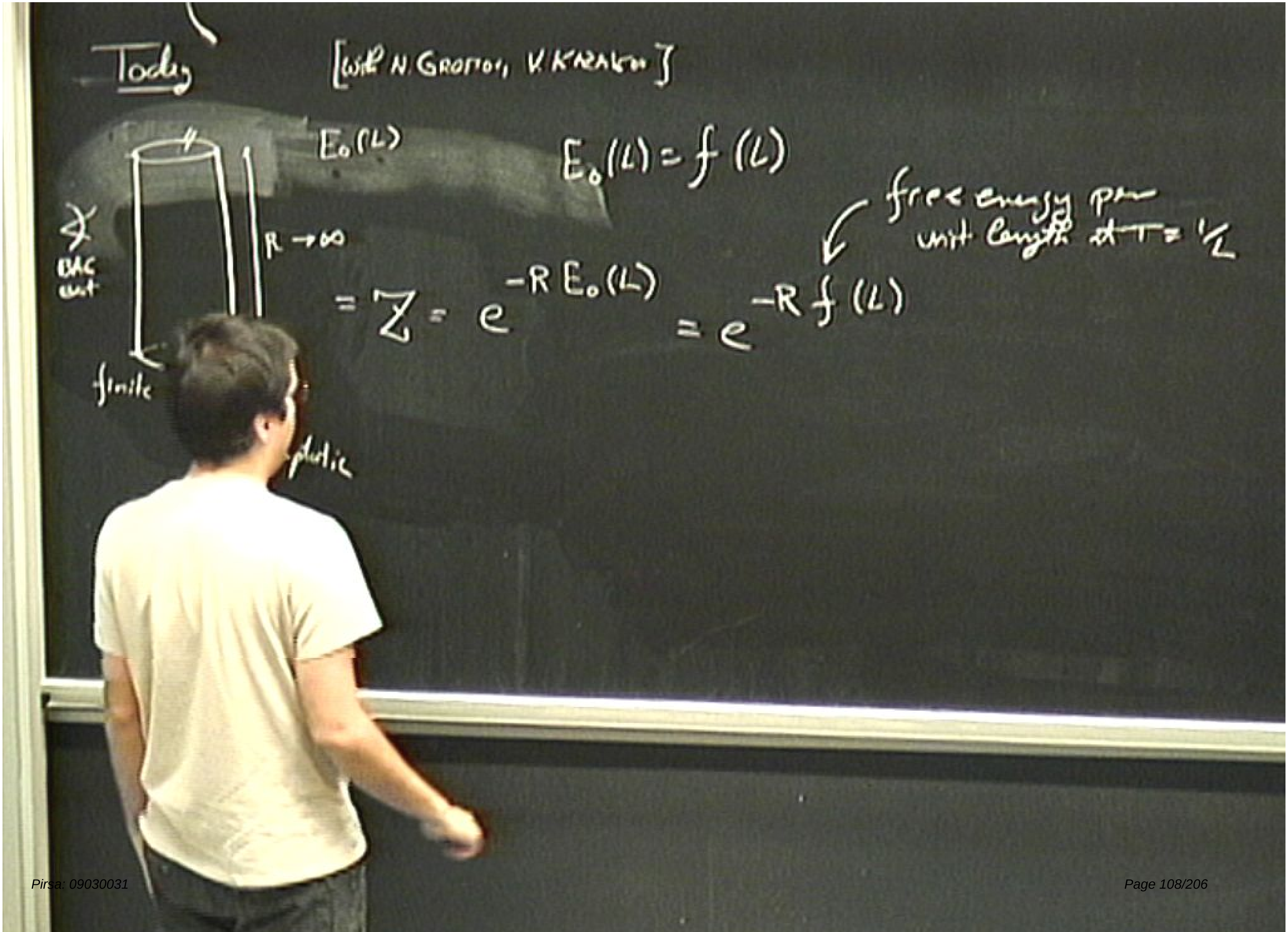


Today

[with N. Gromov, V. Kazakov]

$$Z = e^{-R E_0(L)} = e^{-R f(L)}$$

free energy per unit length at $T = 1/L$



Today

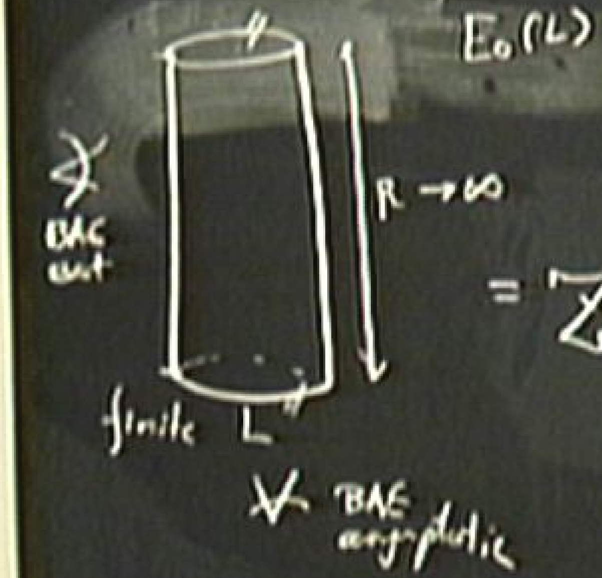
[with N. Gromov, V. Kazakov]

$$E_0(L) = f(L)$$

free energy per unit length at $T = 1/L$

$$= Z = e^{-R E_0(L)} = e^{-R f(L)}$$

Today [with N. Gromov, V. Kazakov]

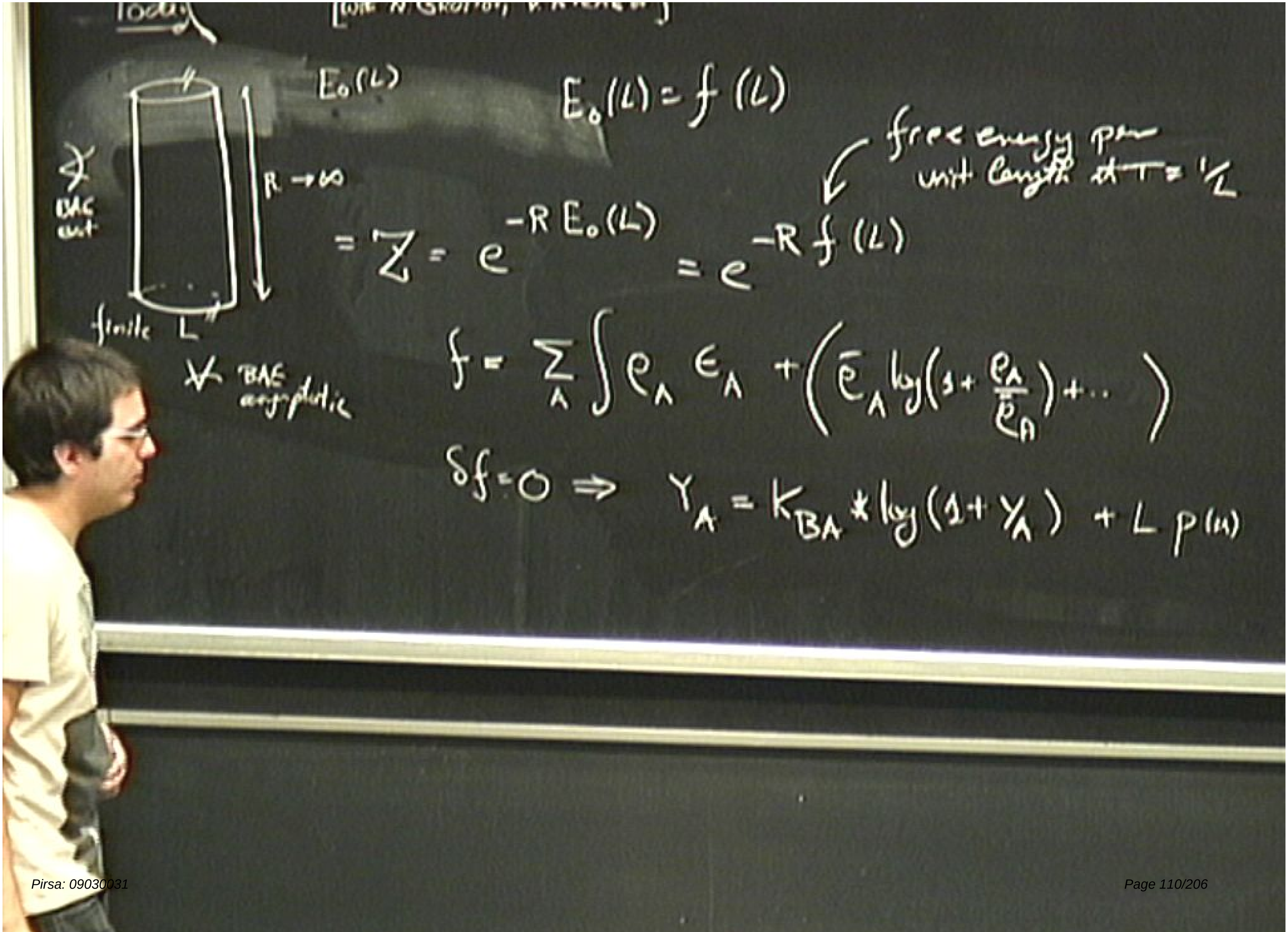


$$E_0(L) = f(L)$$

free energy per unit length at T = 1/L

$$= Z = e^{-R E_0(L)} = e^{-R f(L)}$$

$$f = \sum_{\Lambda} \int \rho_\Lambda \epsilon_\Lambda + \left(\bar{\rho}_\Lambda \log\left(1 + \frac{\rho_\Lambda}{\bar{\rho}_\Lambda}\right) + \dots \right)$$



$E_0(L)$

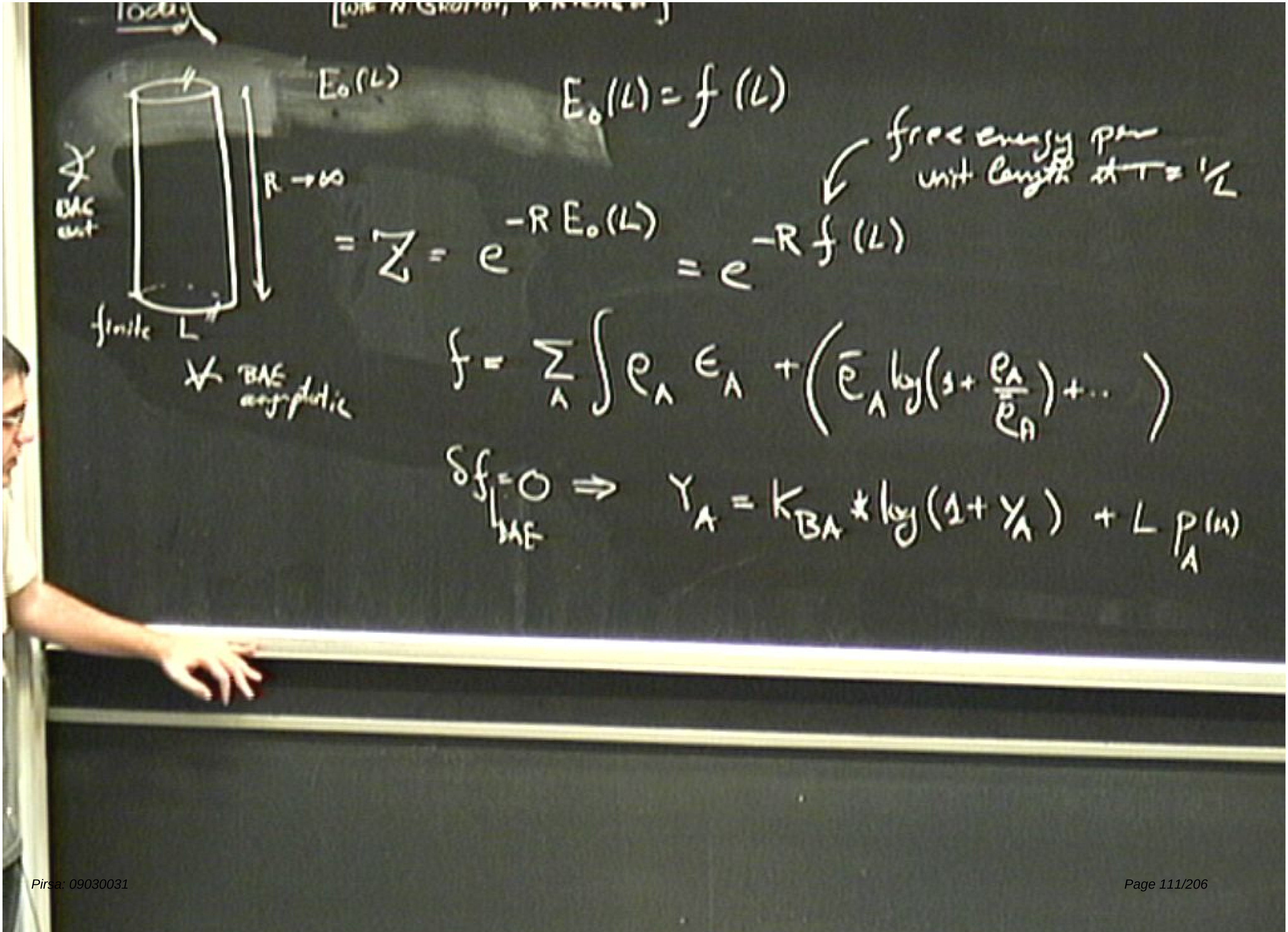
$$E_0(L) = f(L)$$

free energy per unit length at T = 1/L

$$= Z = e^{-R E_0(L)} = e^{-R f(L)}$$

$$f = \sum_A \int \rho_A \epsilon_A + \left(\bar{\rho}_A \log\left(1 + \frac{\rho_A}{\bar{\rho}_A}\right) + \dots \right)$$

$$\delta f = 0 \Rightarrow Y_A = K_{BA} * \log(1 + Y_A) + L\, p(u)$$



[with N. Gromov, [illegible]]

$$E_0(L) = f(L)$$

free energy per unit length at T = 1/L

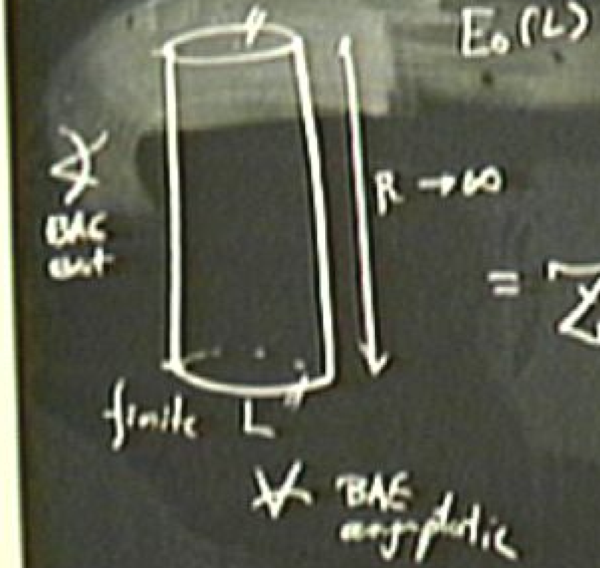
$$= Z = e^{-R E_0(L)} = e^{-R f(L)}$$

$$f = \sum_A \int \rho_A \, \epsilon_A + \left(\bar{\rho}_A \log\left(1 + \frac{\rho_A}{\bar{\rho}_A}\right) + \dots \right)$$

$$\delta f\big|_{BAE} = 0 \Rightarrow Y_A = K_{BA} * \log(1 + Y_A) + L\, p_A^{(u)}$$

Today

[with N. Gromov, V. Kazakov]



$$E_0(L) = f(L)$$

free energy per unit length at T = 1/L

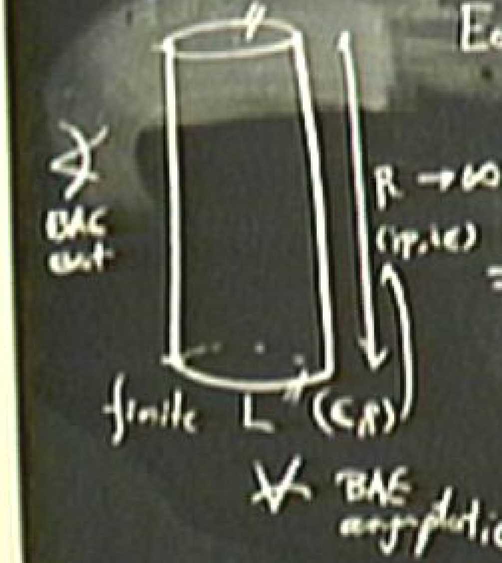
$$= Z = e^{-R E_0(L)} = e^{-R f(L)}$$

$$f = \sum_A \int \rho_A \epsilon_A + \left(\bar\rho_A \log\left(1 + \frac{\rho_A}{\bar\rho_A}\right) + \dots \right)$$

$$\delta f\big|_{BAE} = 0 \Rightarrow Y_A = K_{BA} * \log(1 + Y_A) + L\, p_A^{(u)}$$

Today

[with N. Gromov, V. Kazakov]



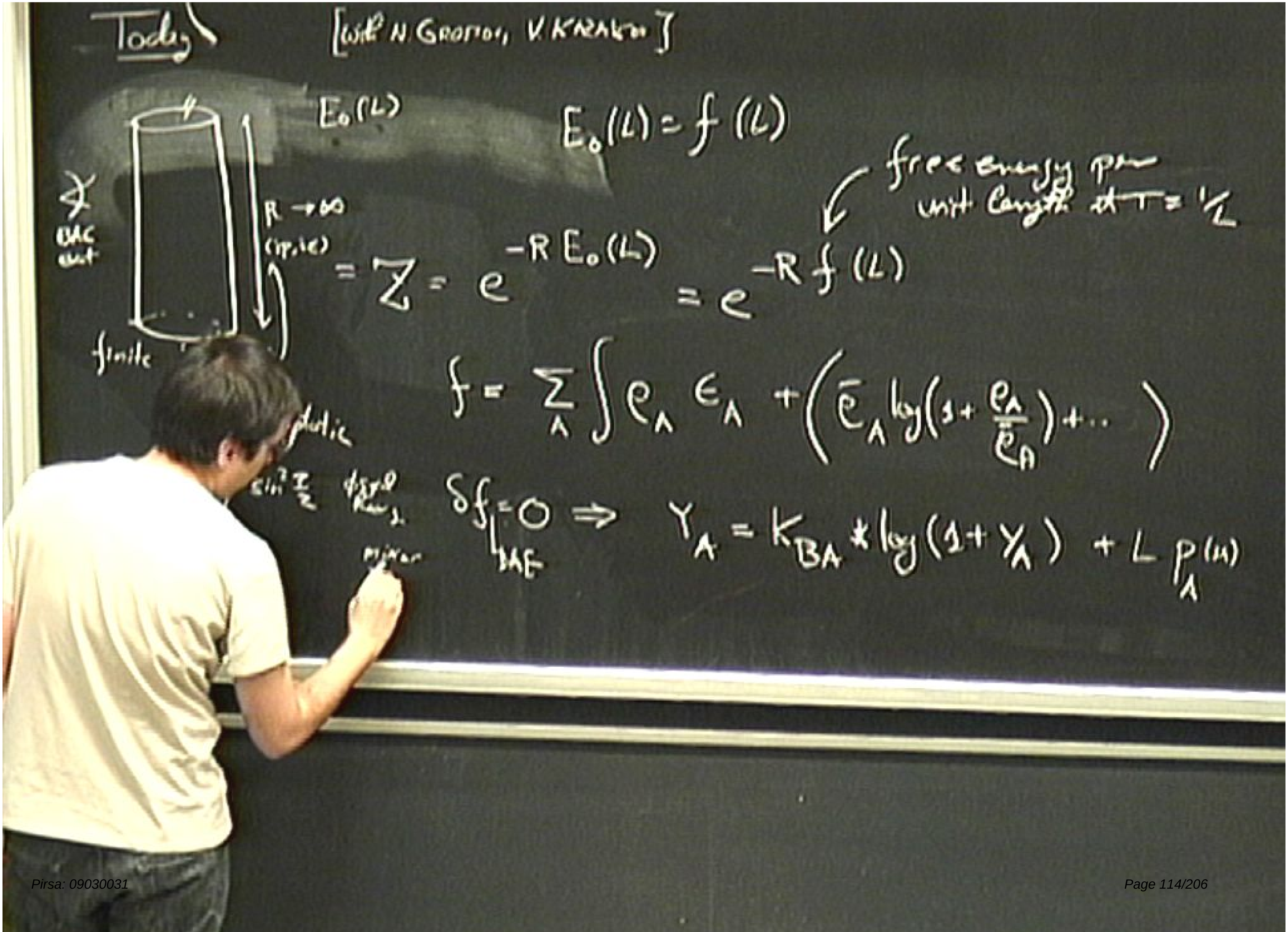
$$E_0(L) = f(L)$$

free energy per unit length at T = 1/L

$$= Z = e^{-R E_0(L)} = e^{-R f(L)}$$

$$f = \sum_A \int \rho_A \epsilon_A + \left(\bar{\rho}_A \log\left(1 + \frac{\rho_A}{\bar{\rho}_A}\right) + \dots \right)$$

$$\delta f\Big|_{BAE} = 0 \Rightarrow Y_A = K_{BA} * \log(1 + Y_A) + L\, p_A^{(u)}$$



Today [with N. Gromov, V. Kazakov]

$E_0(L)$

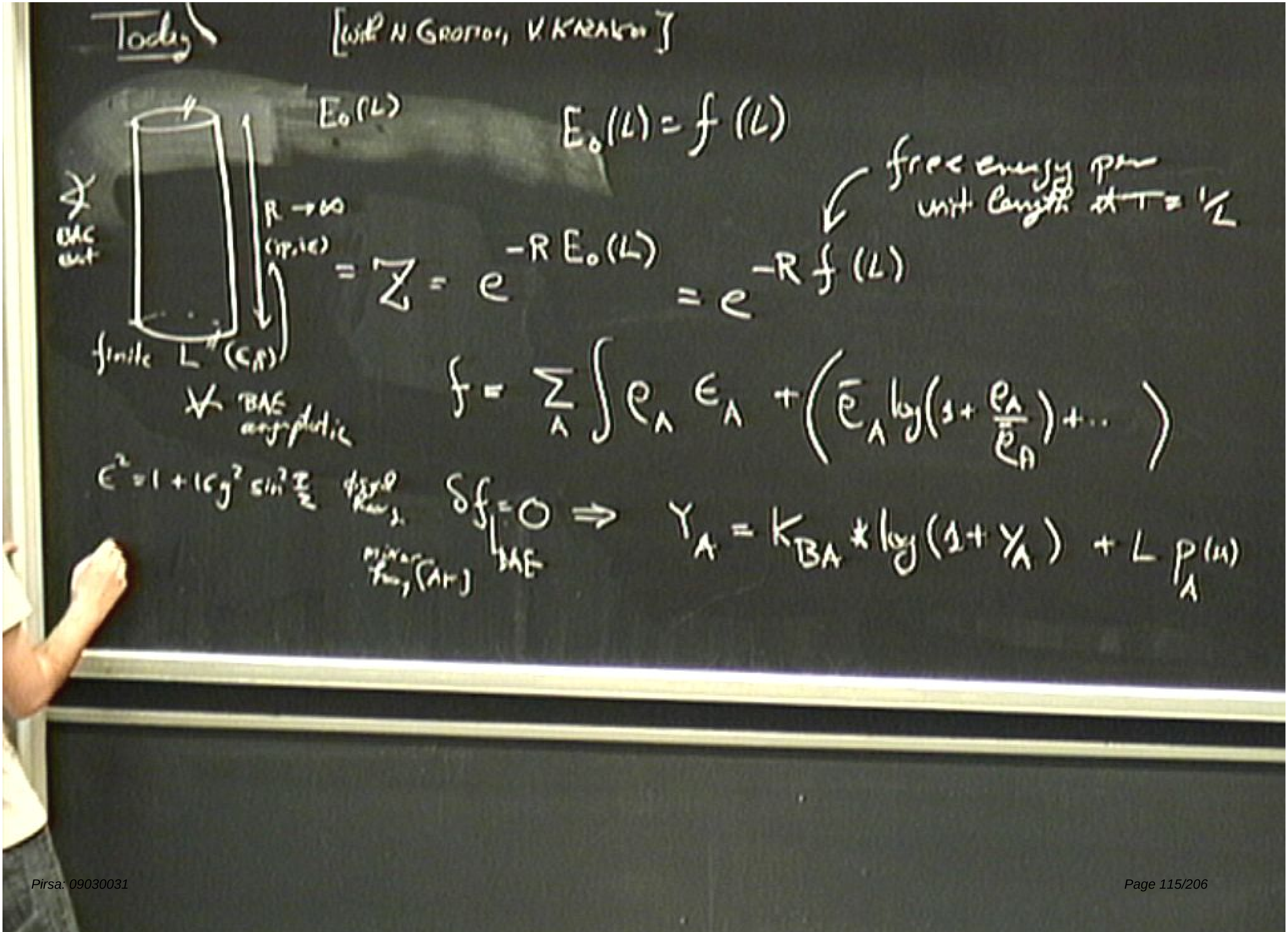
$$E_0(L) = f(L)$$

free energy per unit length at $T = 1/L$

$$= Z = e^{-R E_0(L)} = e^{-R f(L)}$$

$$f = \sum_A \int \rho_A \epsilon_A + \left(\bar{\rho}_A \log\left(1 + \frac{\rho_A}{\bar{\rho}_A}\right) + \dots \right)$$

$$\delta f \big|_{BAE} = 0 \Rightarrow Y_A = K_{BA} * \log(1 + Y_A) + L\, p_A^{(n)}$$



Today

[with N. Gromov, V. Kazakov]

$E_0(L)$

$E_0(L) = f(L)$

free energy per unit length at $T = 1/L$

$R \to \infty$ (p, E)

BAE exact

$= Z = e^{-R E_0(L)} = e^{-R f(L)}$

finite L $(\tilde{E}, \tilde{p})$

TBA asymptotic BAE

$$f = \sum_A \int \rho_A \epsilon_A + \left(\bar{\rho}_A \log\left(1 + \frac{\rho_A}{\bar{\rho}_A}\right) + \dots \right)$$

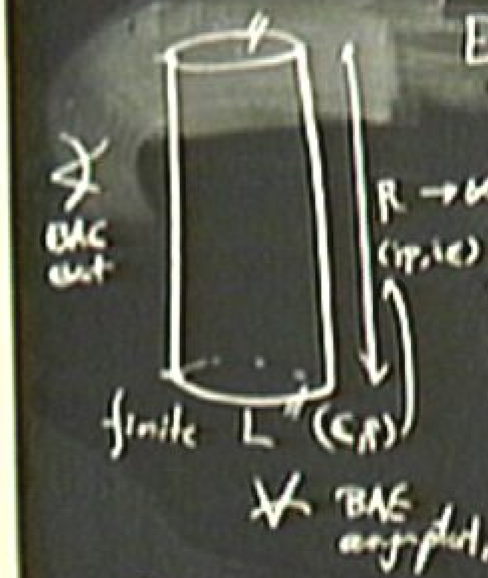
$\epsilon^2 = 1 + 16 g^2 \sin^2 \frac{p}{2}$ physical theory

mirror theory (AF)

$$\delta f \Big|_{BAE} = 0 \Rightarrow Y_A = K_{BA} * \log(1 + Y_A) + L \, p_A^{(m)}$$

Today

[with N. Gromov, V. Kazakov]



$$E_0(L)$$

$$E_0(L) = f(L)$$

free energy per unit length at T = 1/L

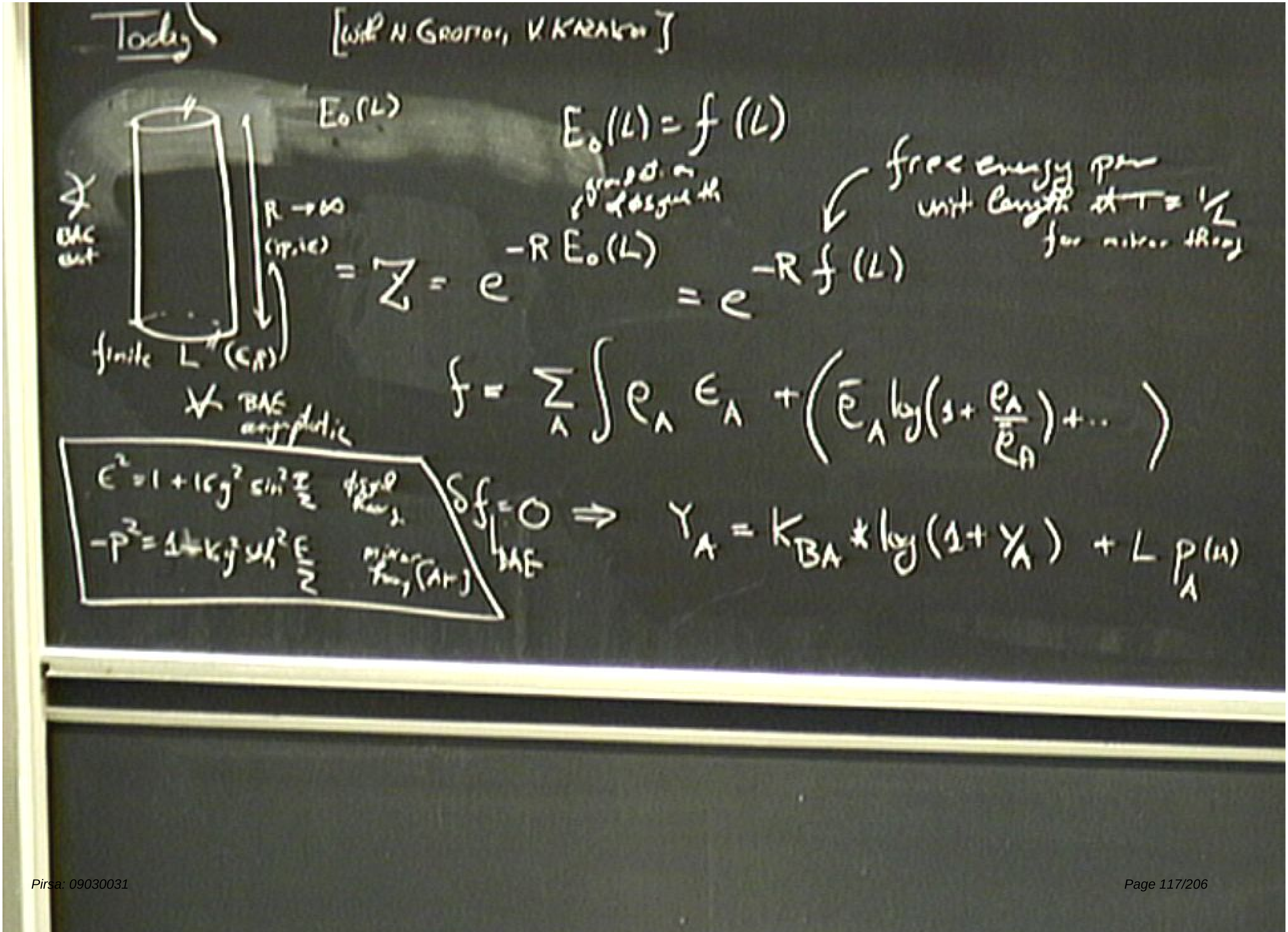
$$= Z = e^{-R E_0(L)} = e^{-R f(L)}$$

$$f = \sum_A \int \rho_A \, \epsilon_A + \left(\bar{\rho}_A \log\left(1 + \frac{\rho_A}{\bar{\rho}_A}\right) + \dots \right)$$

$\epsilon^2 = 1 + 16 g^2 \sin^2 \frac{p}{2}$ physical theory

$-p^2 = 1 + 16 g^2 \sinh^2 \frac{\epsilon}{2}$ mirror theory (AF)

$$\delta f \Big|_{BAE} = 0 \Rightarrow Y_A = K_{BA} * \log(1 + Y_A) + L \, p_A^{(m)}$$



Today

[with N. Gromov, V. Kazakov]

$E_0(L)$

BAE exact

$R \to \infty$

(p, ϵ)

finite L (ϵ, p)

$$E_0(L) = f(L)$$

ground st. en of size L

free energy per unit length at $T = 1/L$ for mirror theory

$$= Z = e^{-R E_0(L)} = e^{-R f(L)}$$

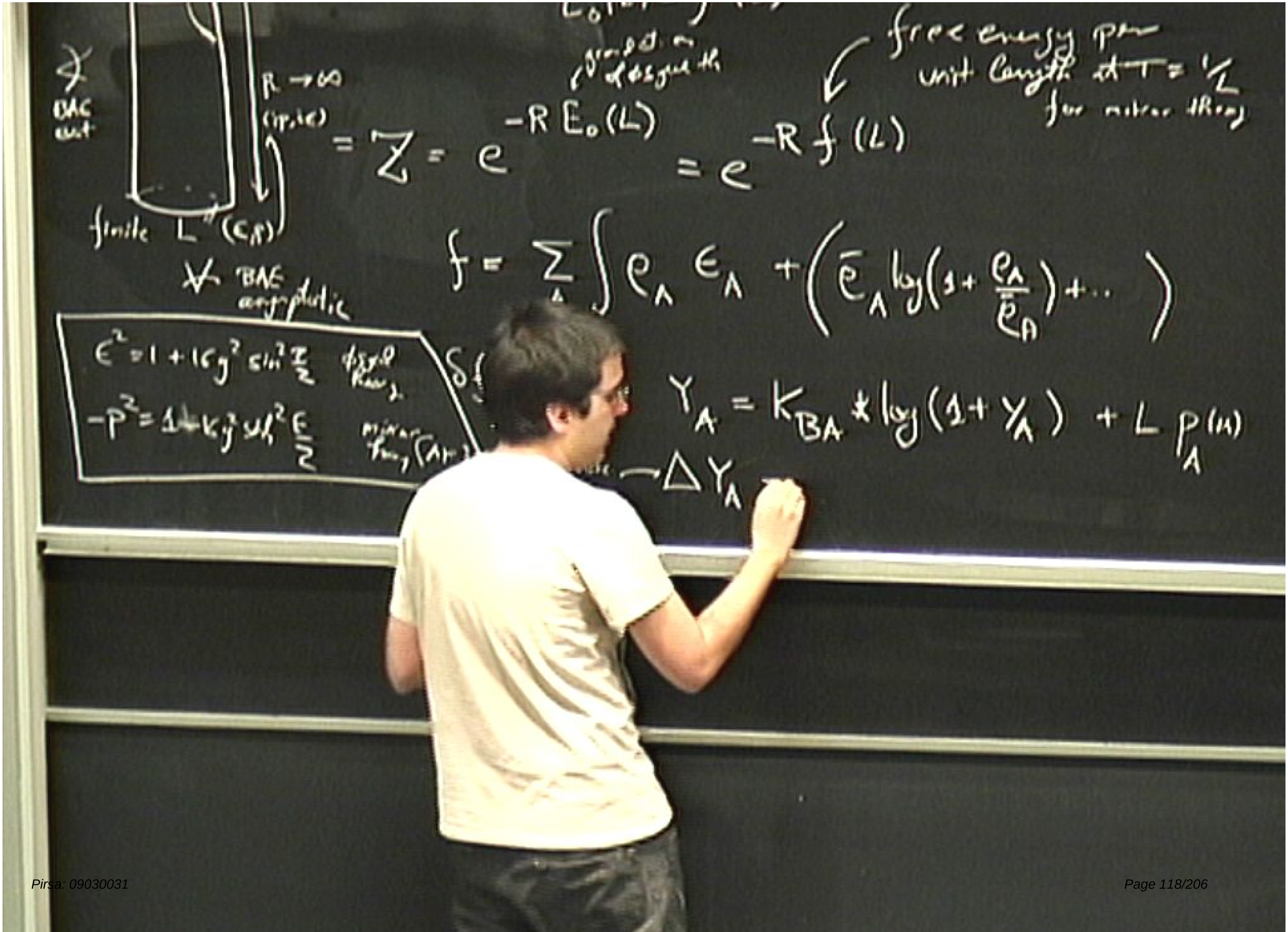
∀ BAE asymptotic

$$\epsilon^2 = 1 + 16 g^2 \sin^2 \frac{p}{2}$$ physical theory

$$-p^2 = 1 + 16 g^2 \sinh^2 \frac{\epsilon}{2}$$ mirror theory (AF)

$$f = \sum_A \int \rho_A \, \epsilon_A + \left(\bar{\rho}_A \log\left(1 + \frac{\rho_A}{\bar{\rho}_A}\right) + \dots \right)$$

$$\delta f \big|_{BAE} = 0 \Rightarrow Y_A = K_{BA} * \log(1 + Y_A) + L \, p_A^{(m)}$$



BAE cut

R → ∞ (p,ε)

finite L (ε,p)

$= Z = e^{-R E_0(L)} = e^{-R f(L)}$

free energy per unit length at T = 1/L for mirror theory

∀ BAE asymptotic

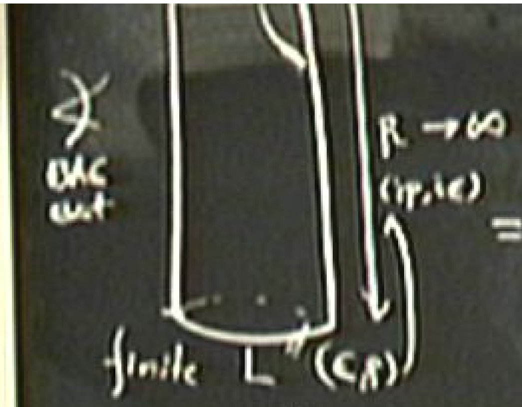
$$f = \sum_A \int \rho_A \epsilon_A + \left(\bar{\rho}_A \log\left(1 + \frac{\rho_A}{\bar{\rho}_A}\right) + \ldots \right)$$

$$\boxed{\begin{array}{ll} \epsilon^2 = 1 + 16 g^2 \sin^2 \frac{p}{2} & \text{physical theory} \\ -p^2 = 1 + 16 g^2 \sinh^2 \frac{\epsilon}{2} & \text{mirror theory } (A \ldots) \end{array}}$$

δf

$$Y_A = K_{BA} * \log(1 + Y_A) + L\, p_A^{(m)}$$

$\to \Delta Y_A =$



ground st. en. of ϕ-string th.

free energy per unit length at T = 1/L for mirror theory

$$= Z = e^{-R E_0(L)} = e^{-R f(L)}$$

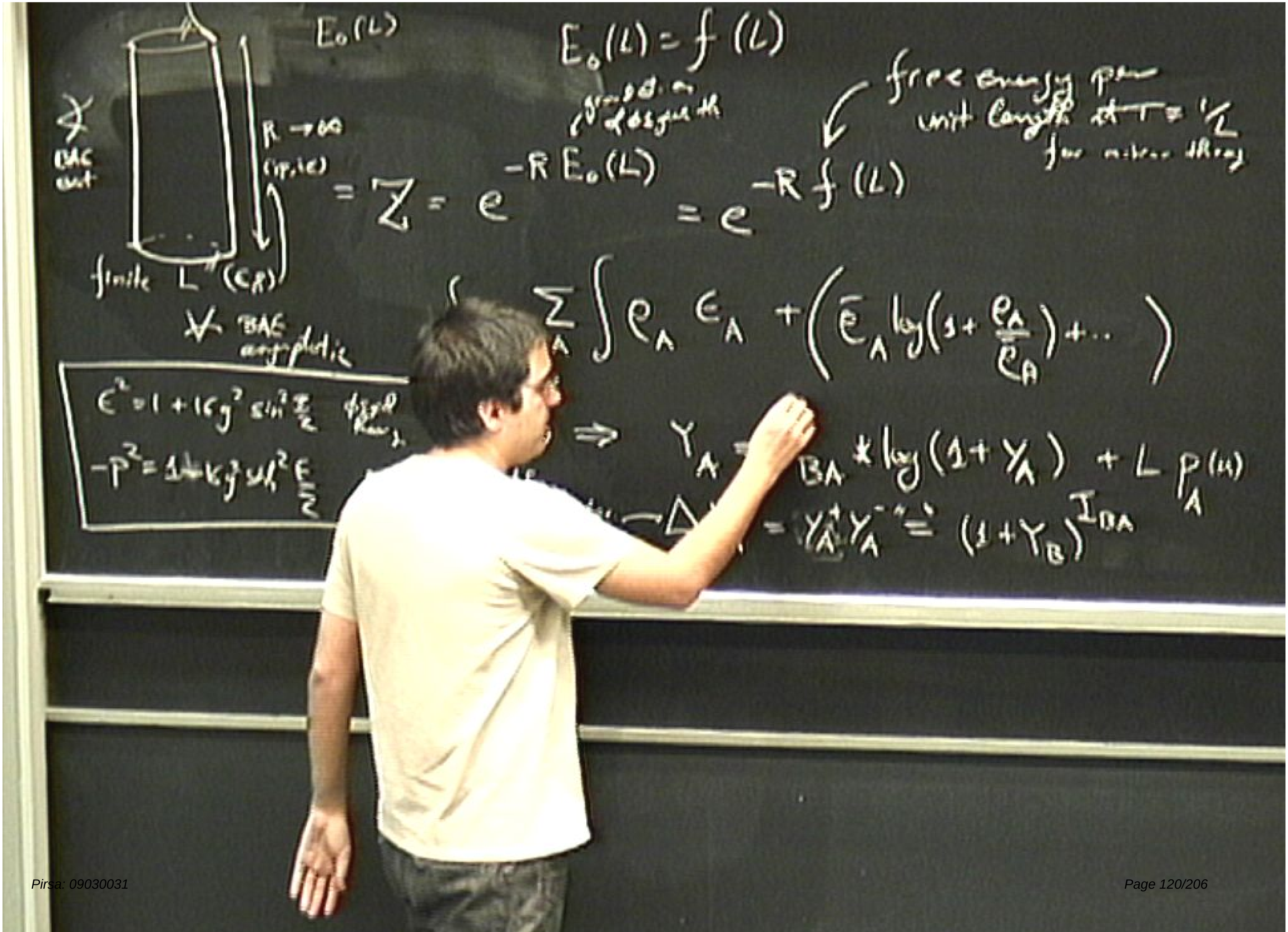
∀ BAE asymptotic

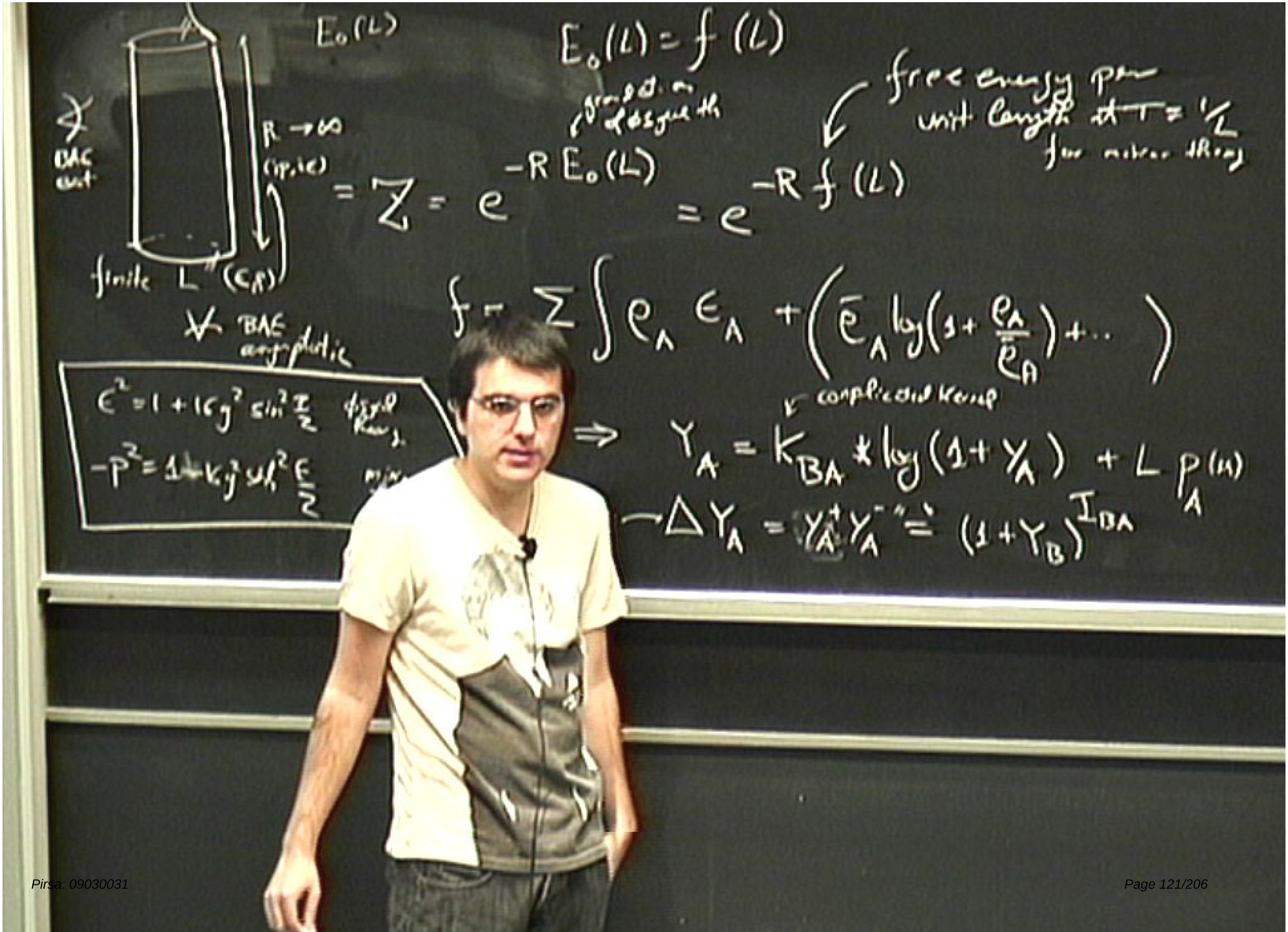
$$f = \sum_A \int \rho_A \epsilon_A + \left(\bar{\rho}_A \log\left(1 + \frac{\rho_A}{\bar{\rho}_A}\right) + \dots \right)$$

$$\boxed{\begin{array}{ll} \epsilon^2 = 1 + 16 g^2 \sin^2 \frac{p}{2} & \text{ϕ-string theory} \\ -\tilde{p}^2 = 1 + 16 g^2 \sinh^2 \frac{\tilde{E}}{2} & \text{mirror theory (AF)} \end{array}}$$

$$\delta f\big|_{BAE} = 0 \Rightarrow Y_A = K_{BA} * \log(1 + Y_A) + L\, p_A^{(n)}$$

disc. inf. $\to \Delta Y_A = Y_A^{+} Y_A^{-} \text{"="} (1 + Y_B)^{I_{BA}}$


E₀(L)
E₀(L) = f(L)
free energy per unit length at T = 1/L for mirror theory
BAE exact
R → ∞
finite L
= Z = e^{-R E₀(L)} = e^{-R f(L)}
Σ_A ∫ ρ_A ε_A + (ε̄_A log(1 + ρ_A/ρ̄_A) + …)
BAE asymptotic
ε² = 1 + 16g² sin² p/2
-p² = 1 + 16g² sin² E/2
⇒ Y_A = … log(1 + Y_A) + L p̃_A(u)
Δ … = Y_A⁺ Y_A⁻ = (1 + Y_B)^{I_BA}



$E_0(L)$

$E_0(L) = f(L)$

[illegible]

free energy per unit length at $T = 1/L$ for mirror theory

BAE exact

$R \to \infty$

(p, ϵ)

finite L

$(\tilde{\epsilon}, \tilde{p})$

✗ BAE asymptotic

$$= Z = e^{-R E_0(L)} = e^{-R f(L)}$$

$$f = \sum \int \rho_A \epsilon_A + \left(\bar{\rho}_A \log\left(1 + \frac{\rho_A}{\bar{\rho}_A}\right) + \dots \right)$$

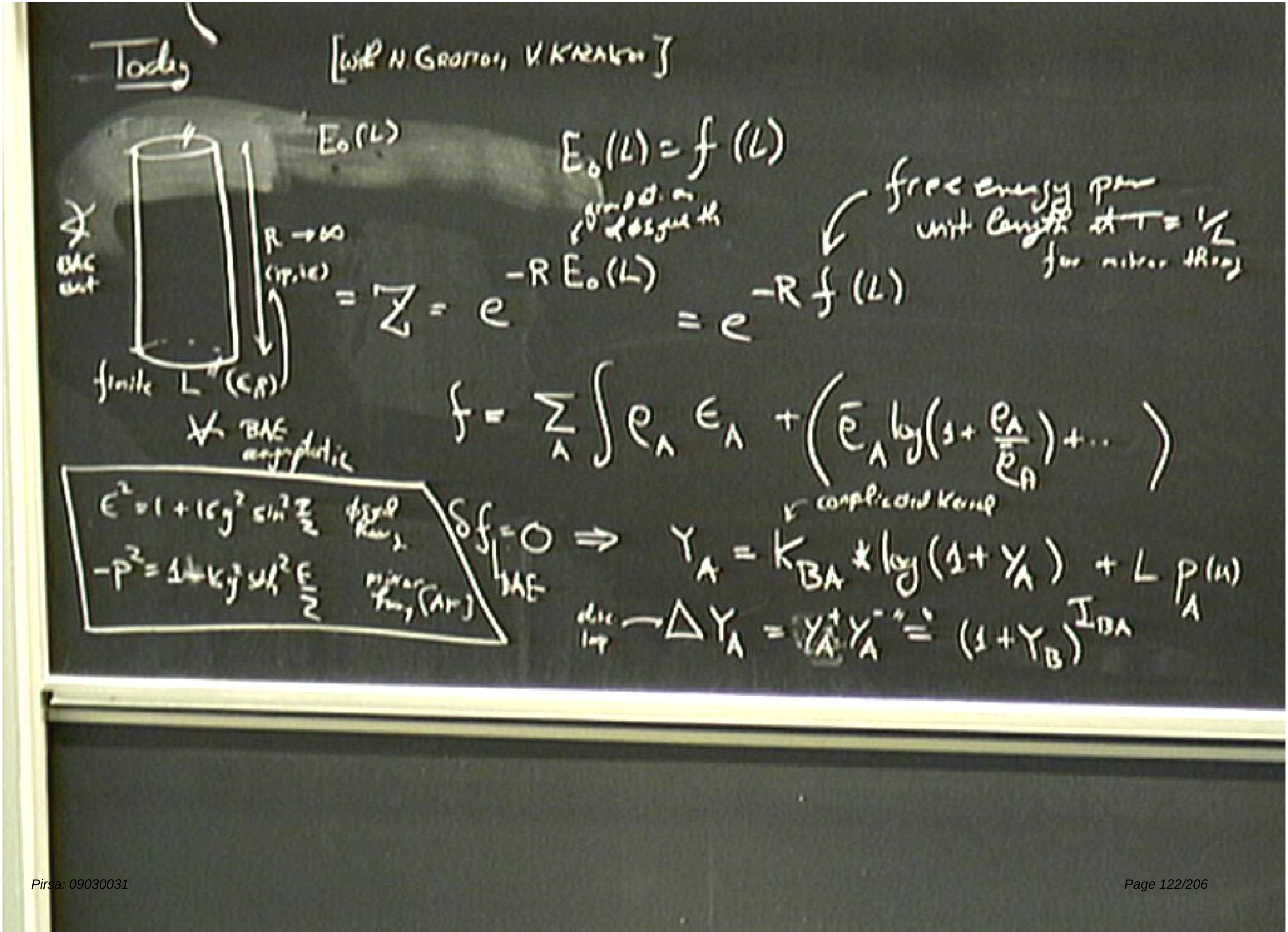
$$\epsilon^2 = 1 + 16 g^2 \sin^2 \frac{p}{2}$$ physical theory

$$-\tilde{p}^2 = 1 + 16 g^2 \sinh^2 \frac{\tilde{\epsilon}}{2}$$ mirror

complicated kernel

$$\Rightarrow Y_A = K_{BA} * \log(1 + Y_A) + L\, p_A^{(m)}$$

$$\to \Delta Y_A = \text{"} Y_A^+ Y_A^- \text{"} \doteq (1 + Y_B)^{I_{BA}}$$



Today [with N. Gromov, V. Kazakov]

$E_0(L)$

$R \to \infty$ (p, ϵ)

finite L (ϵ, p)

∃ BAE exact

∀ BAE asymptotic

$$= Z = e^{-R E_0(L)} = e^{-R f(L)}$$

$$E_0(L) = f(L)$$

ground st. en. of AdS gauge th

free energy per unit length at $T = 1/L$ for mirror theory

$$f = \sum_A \int \rho_A \epsilon_A + \left(\bar{\rho}_A \log\left(1 + \frac{\rho_A}{\bar{\rho}_A}\right) + \dots \right)$$

$$\boxed{\begin{array}{ll} \epsilon^2 = 1 + 16 g^2 \sin^2 \frac{p}{2} & \text{physical theory} \\ -p^2 = 1 + 16 g^2 \sinh^2 \frac{\epsilon}{2} & \text{mirror theory (AF)} \end{array}}$$

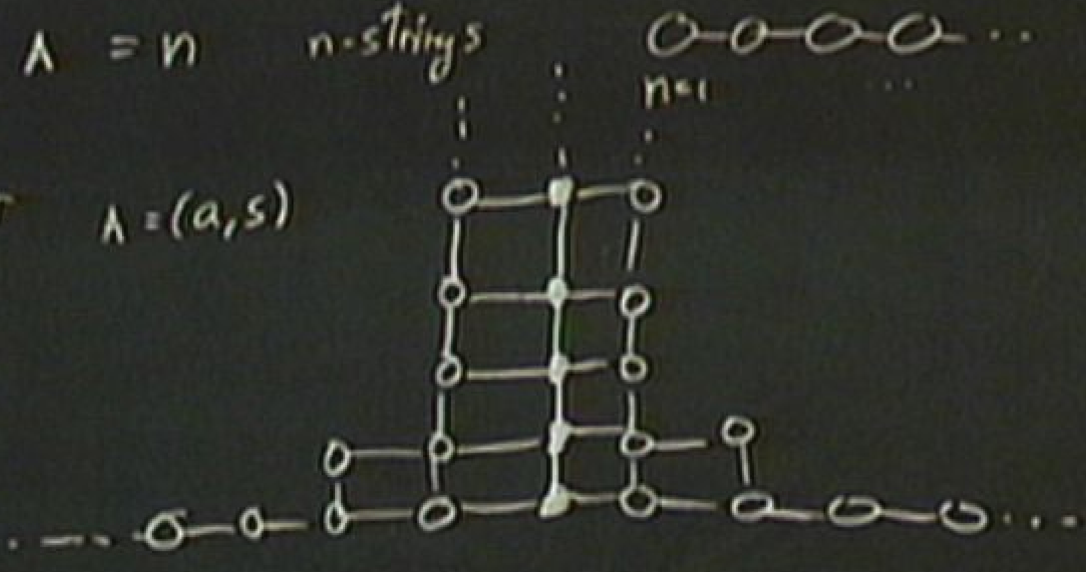
complicated kernel

$$\delta f \big|_{BAE} = 0 \Rightarrow Y_A = K_{BA} * \log(1 + Y_A) + L\, p_A^{(m)}$$

$$\text{disc log} \to \Delta Y_A = Y_A^+ Y_A^- \doteq (1 + Y_B)^{I_{BA}}$$

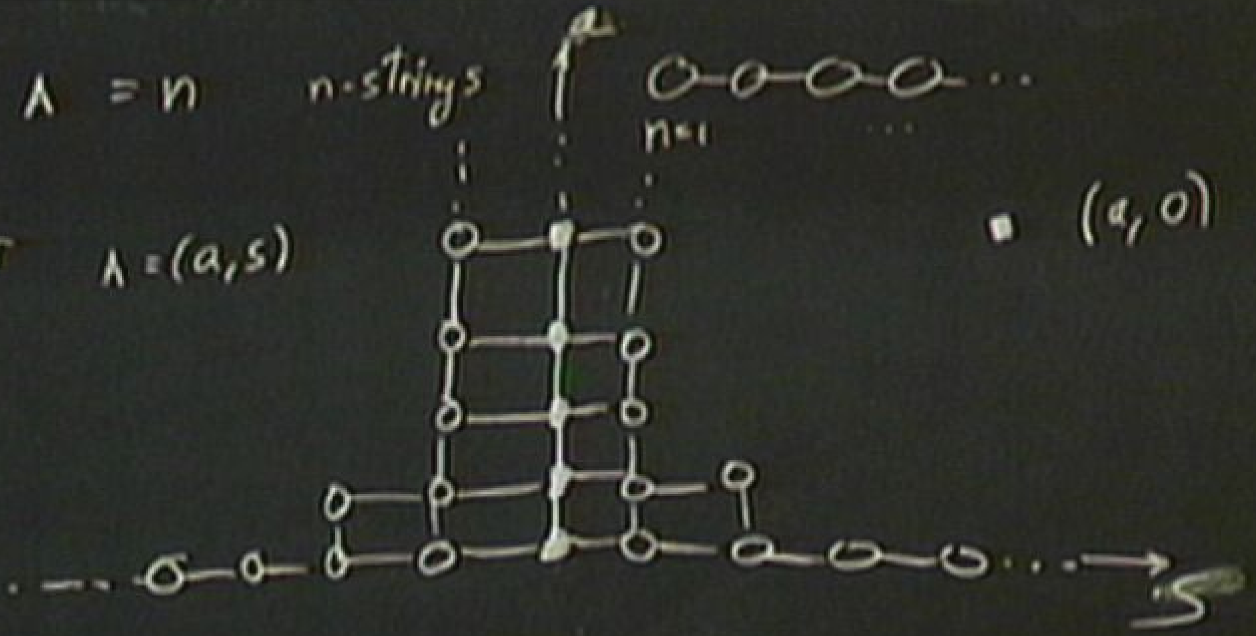
SU(n) $\Lambda = n$ n-strings

AdS/CFT $\Lambda = (a,s)$



$SU(n)$ $\Lambda = n$ n-strings

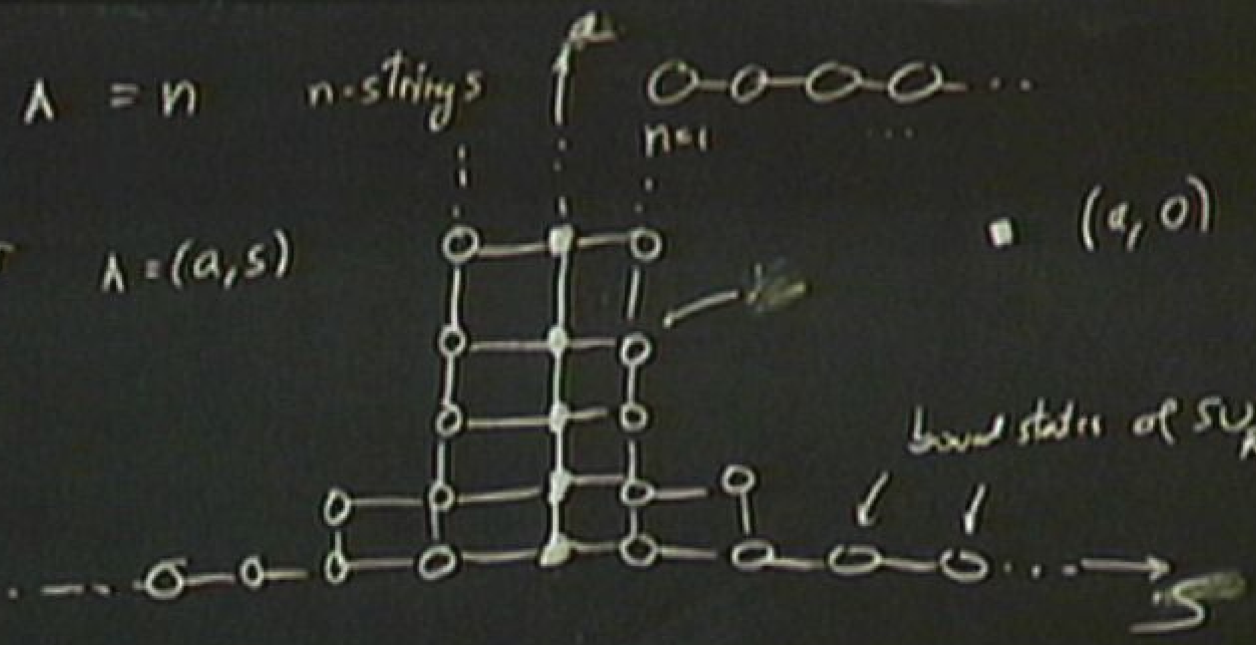
AdS/CFT $\Lambda = (a,s)$



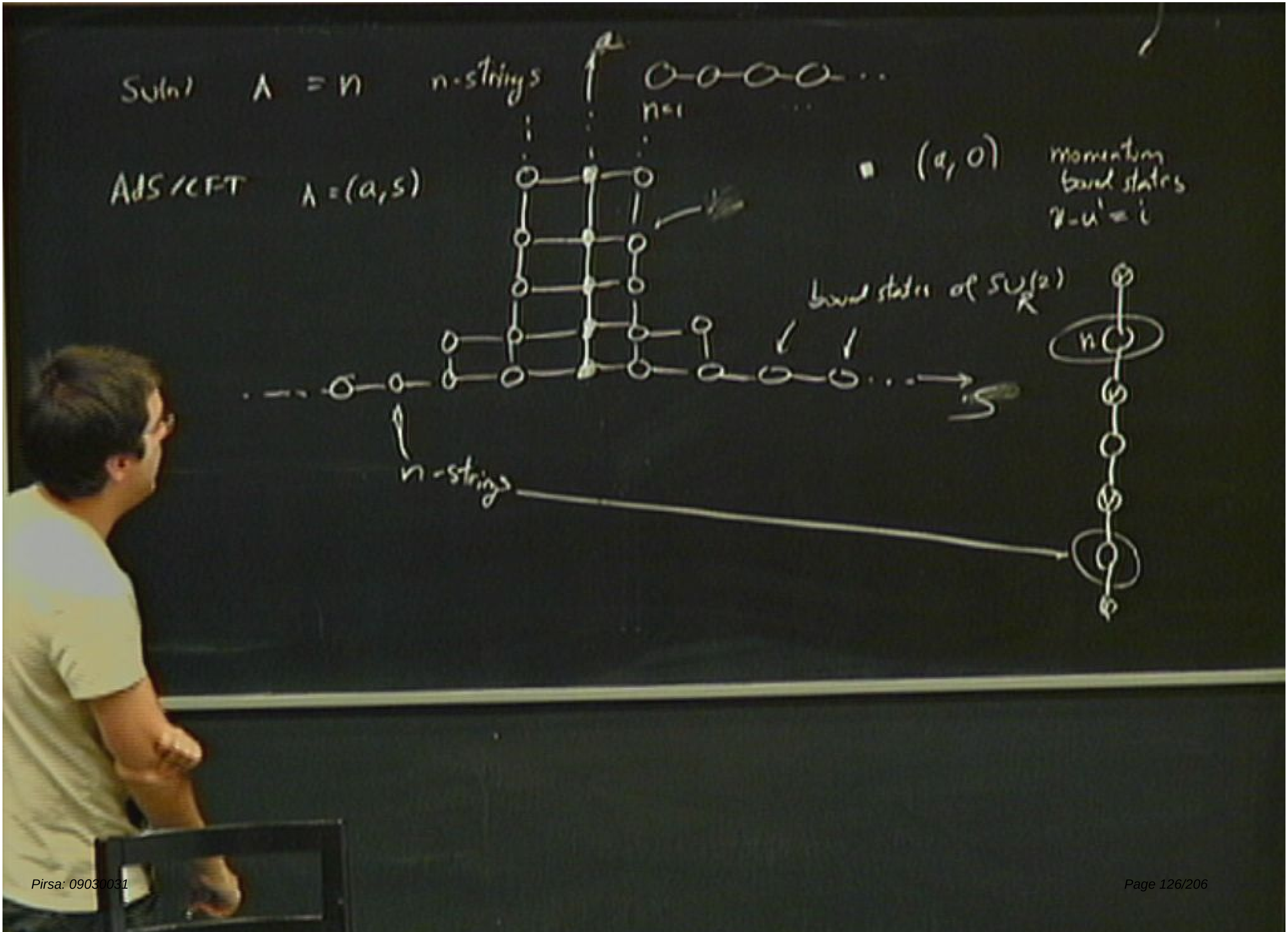
■ $(a,0)$ momentum bound states $u-u' = i$

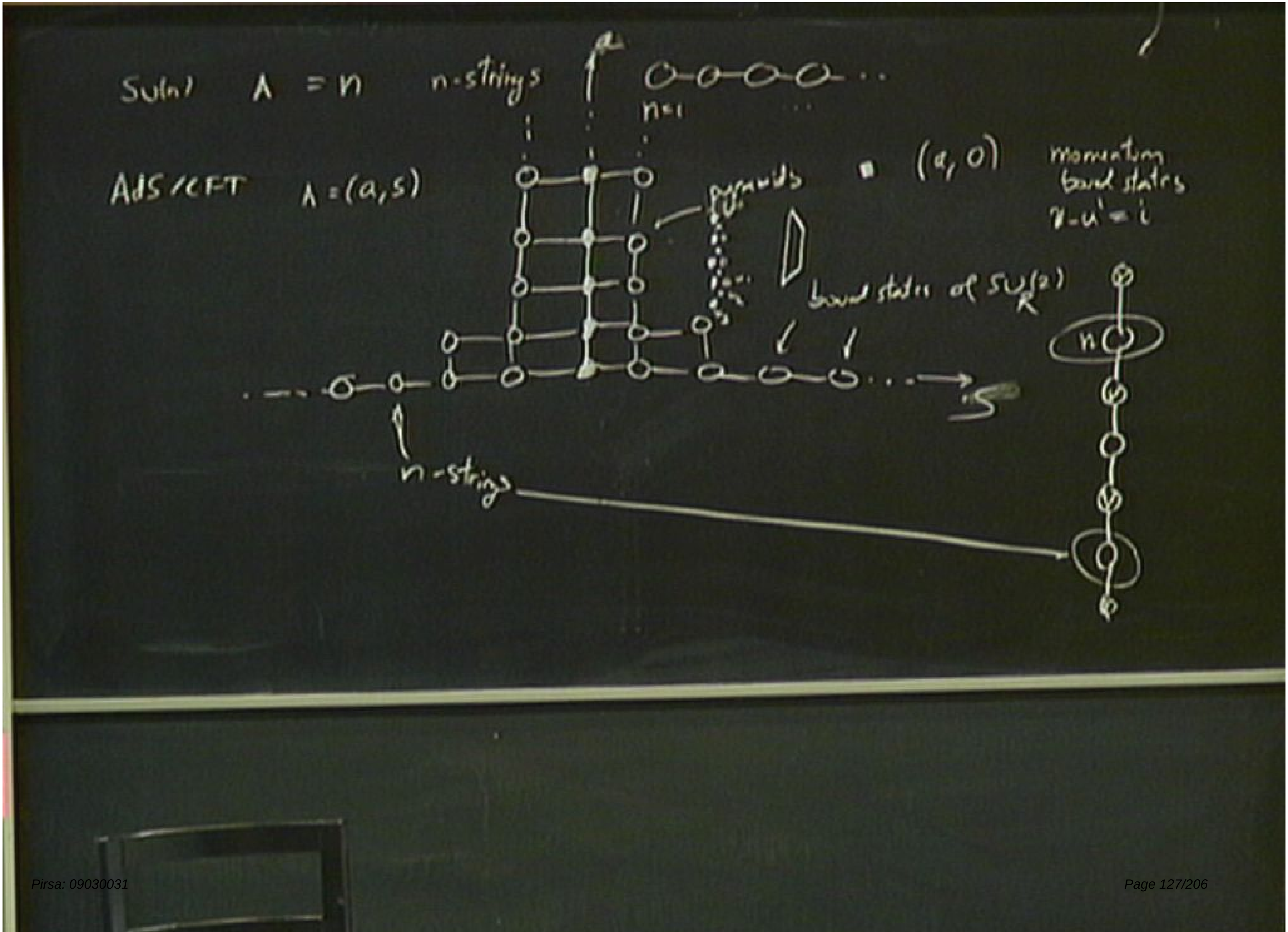
SU(n) $\Lambda = n$

AdS/CFT $\Lambda = (a,s)$



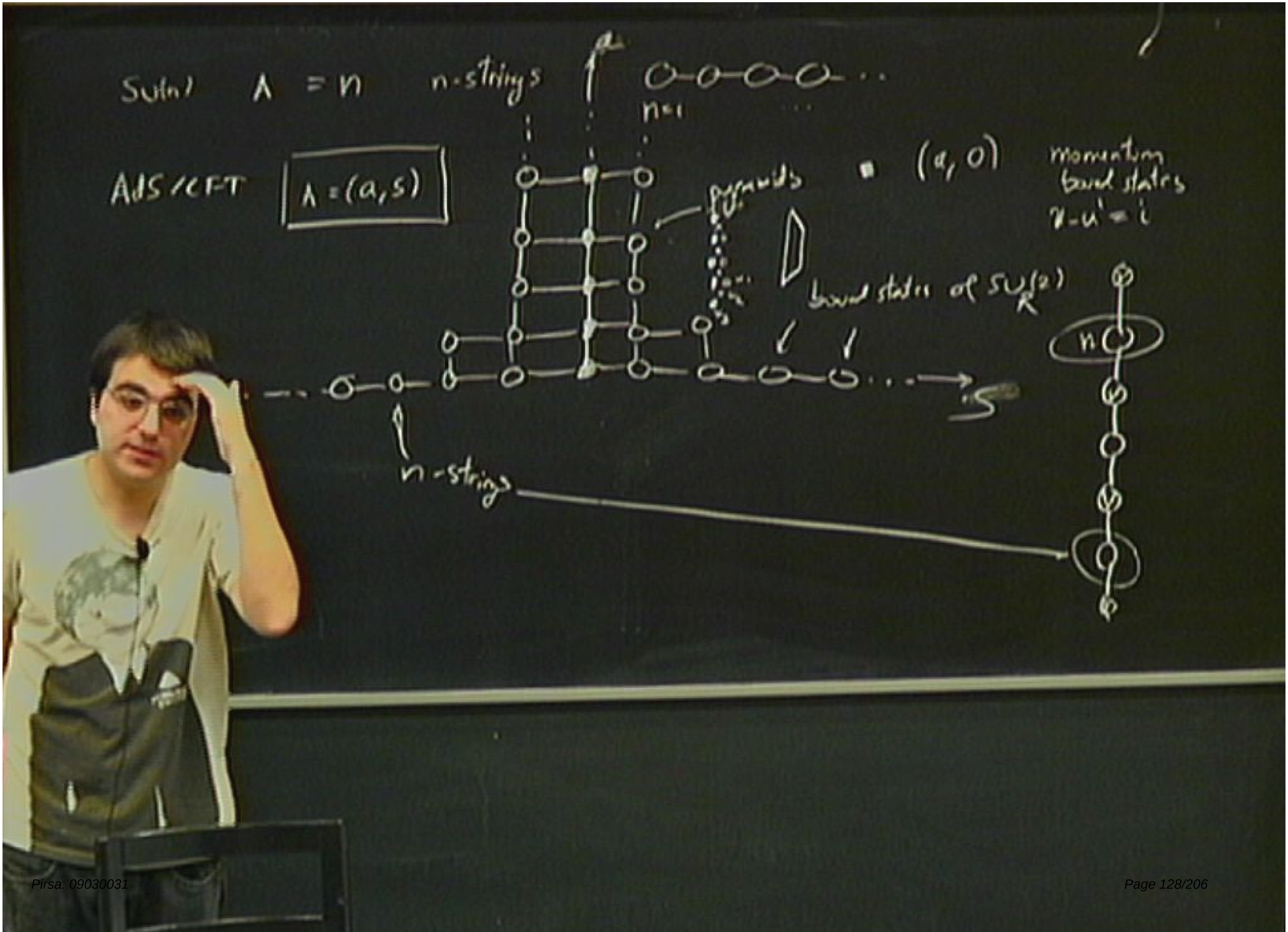
- $(a,0)$ momentum bound states $x - u' = i$


SU(n) Λ = n
n-strings
a
n=1
AdS/CFT Λ = (a,s)
(a,0)
momentum
bound states
x-u' = i
bound states of SU(2)_R
n
s
n-strings



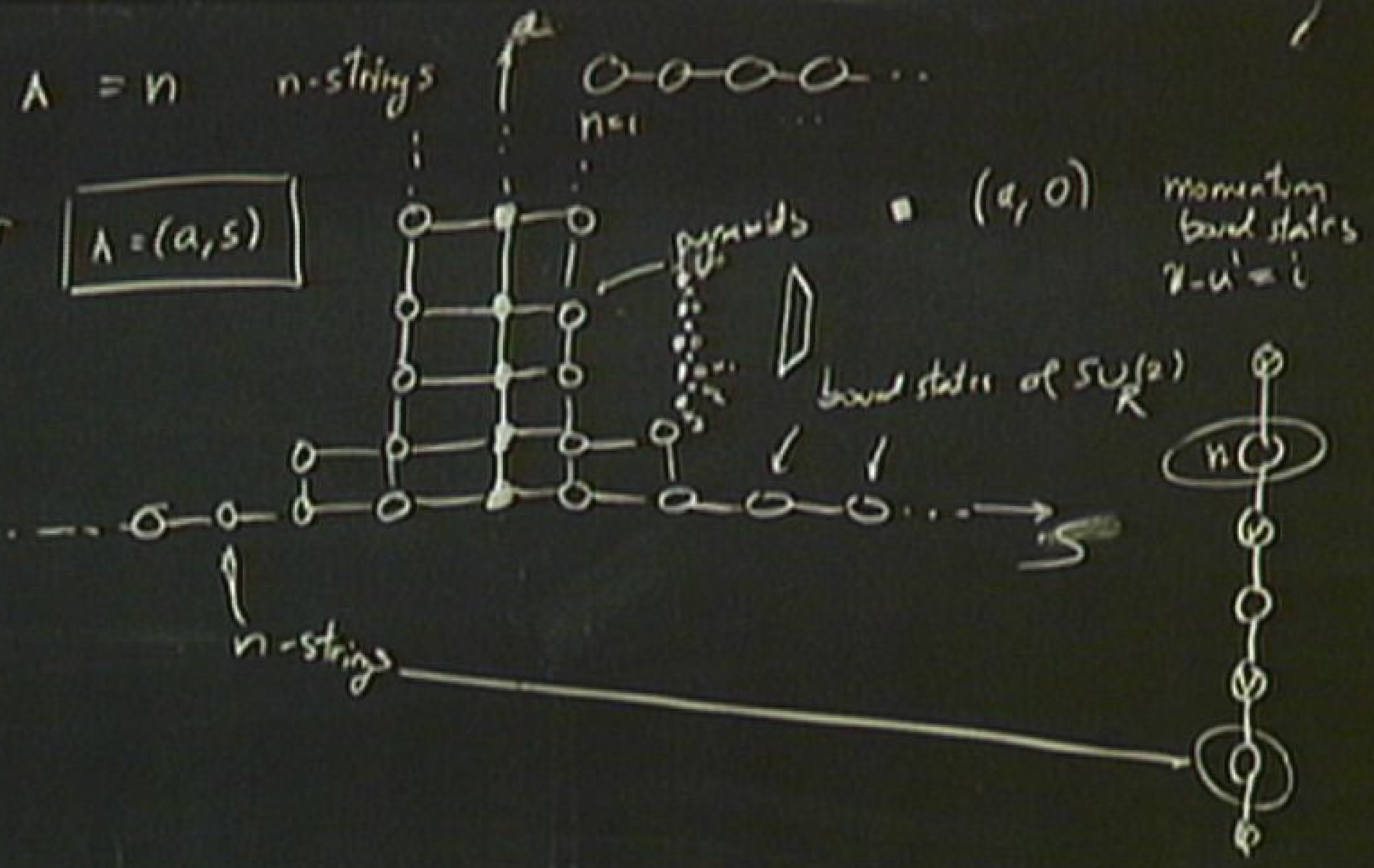
SU(n) $\Lambda = n$

AdS/CFT $\Lambda = (a,s)$


SU(n) Λ = n n-strings
n=1
AdS/CFT
Λ=(a,s)
pyramids
(a,0) momentum bound states
u-u' = i
bound states of SU(2)_R
n
n-strings
s

SU(n) $\Lambda = n$

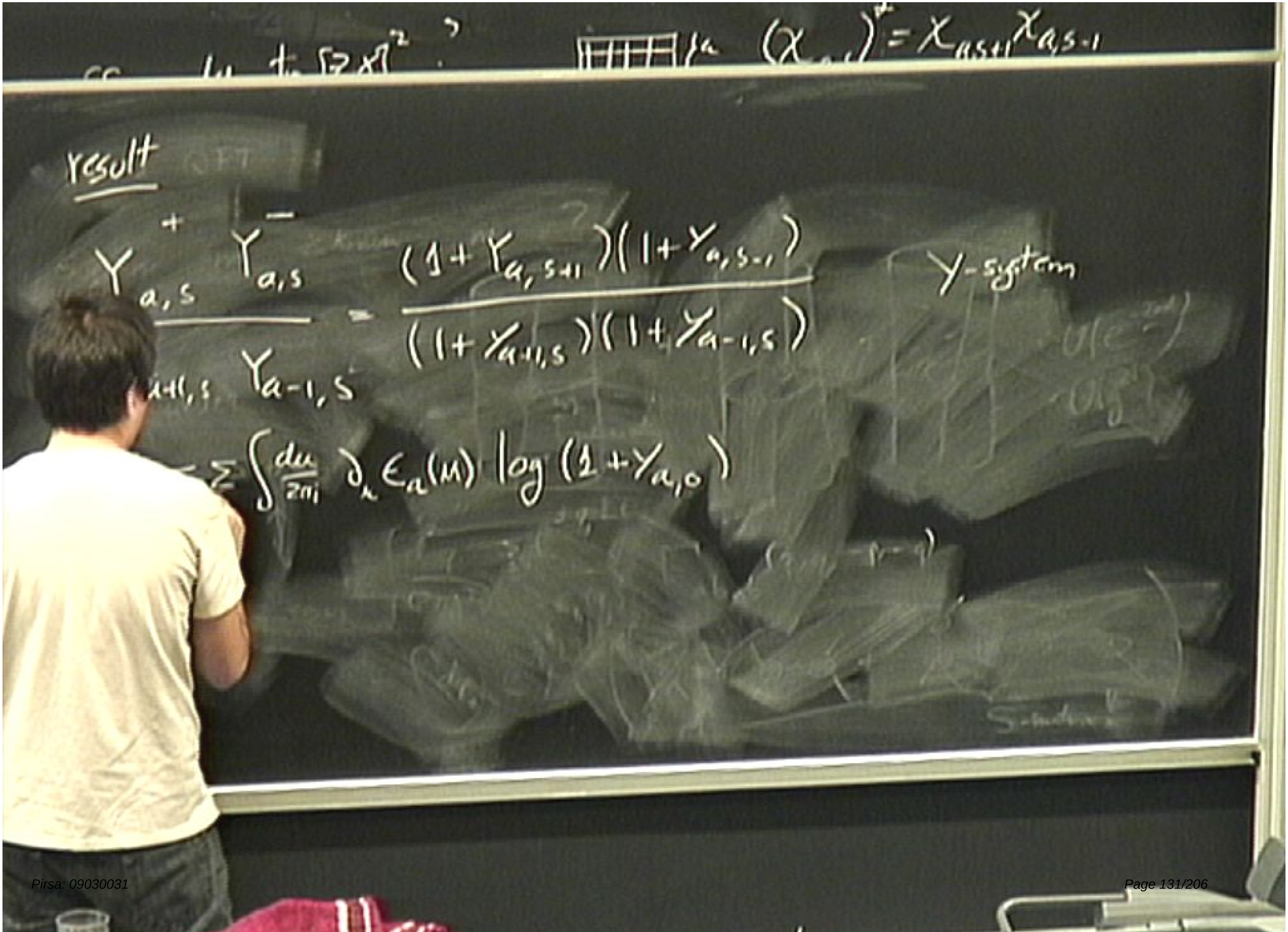
AdS/CFT $\boxed{\Lambda = (a,s)}$



$\mathrm{tr}\,[T X]^2$ $\quad (\chi \quad)^{\pm} = \chi_{a,s+1}\chi_{a,s-1}$

result

$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$



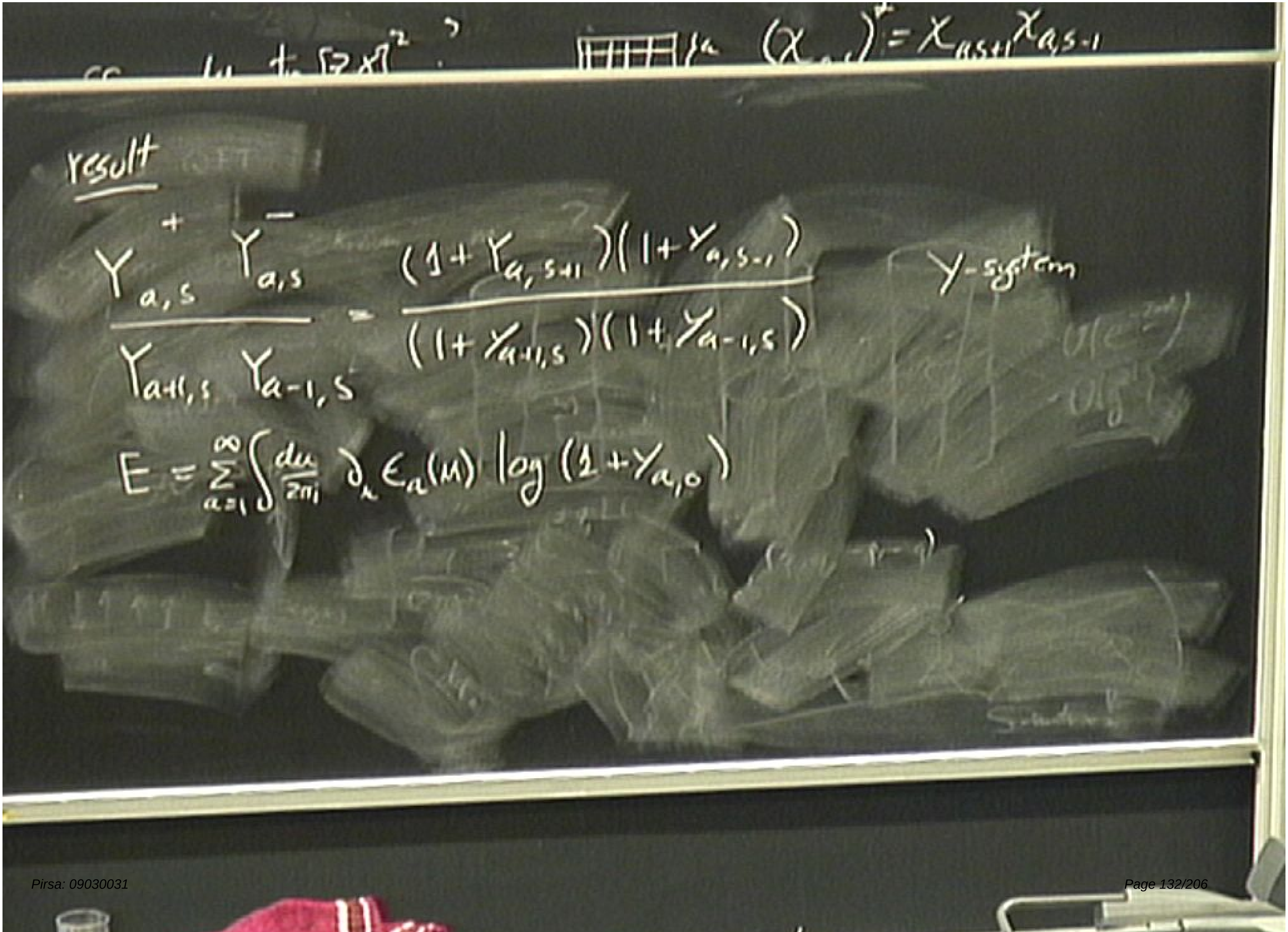
$= \chi_{a,s+1}\chi_{a,s-1}$

result

$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

Y-system

$$\Sigma \int \frac{du}{2\pi i}\, \partial_u \epsilon_a(u) \log(1+Y_{a,0})$$



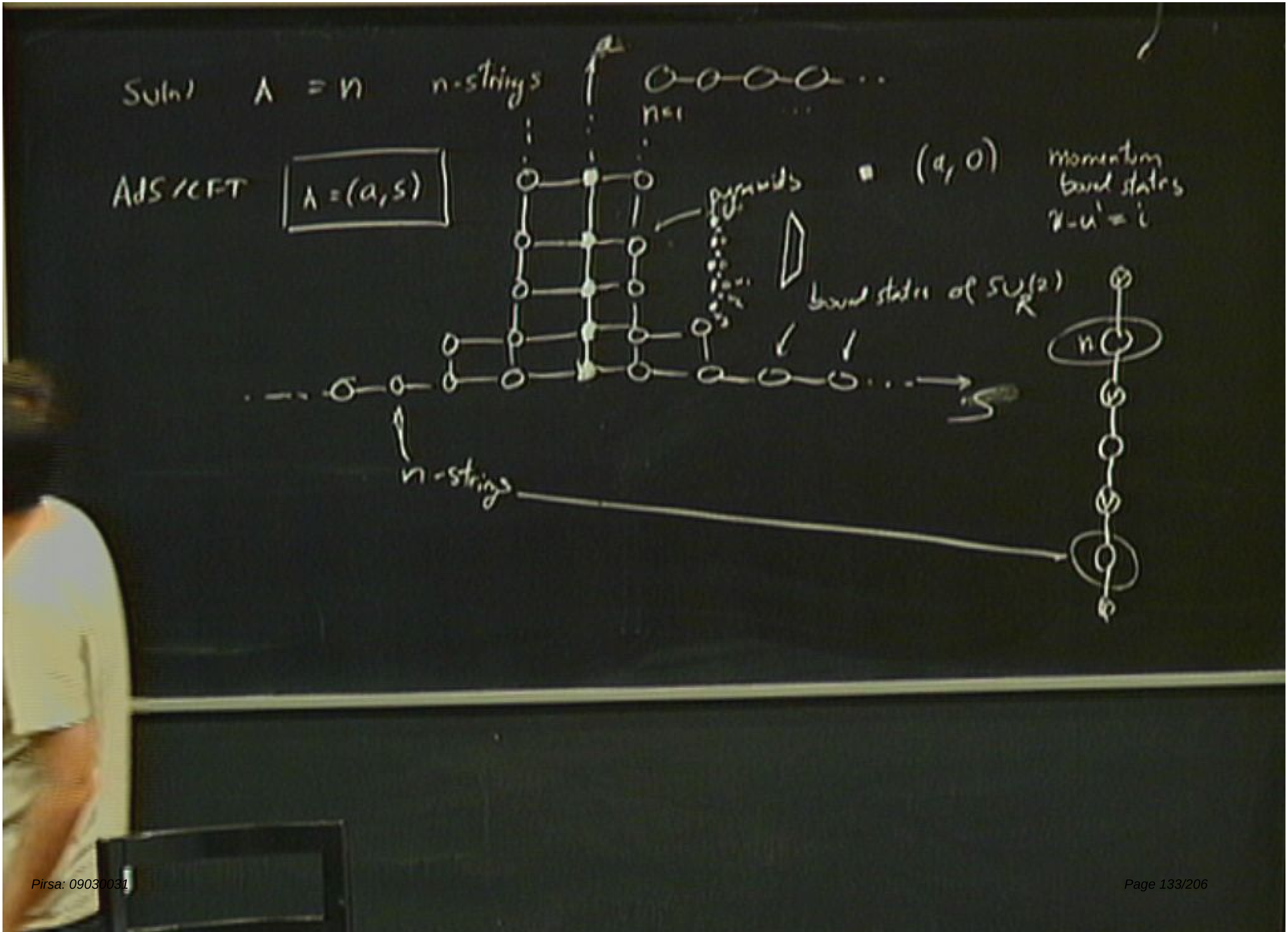
$$(\chi_{a,s})^2 = \chi_{a,s+1}\chi_{a,s-1}$$

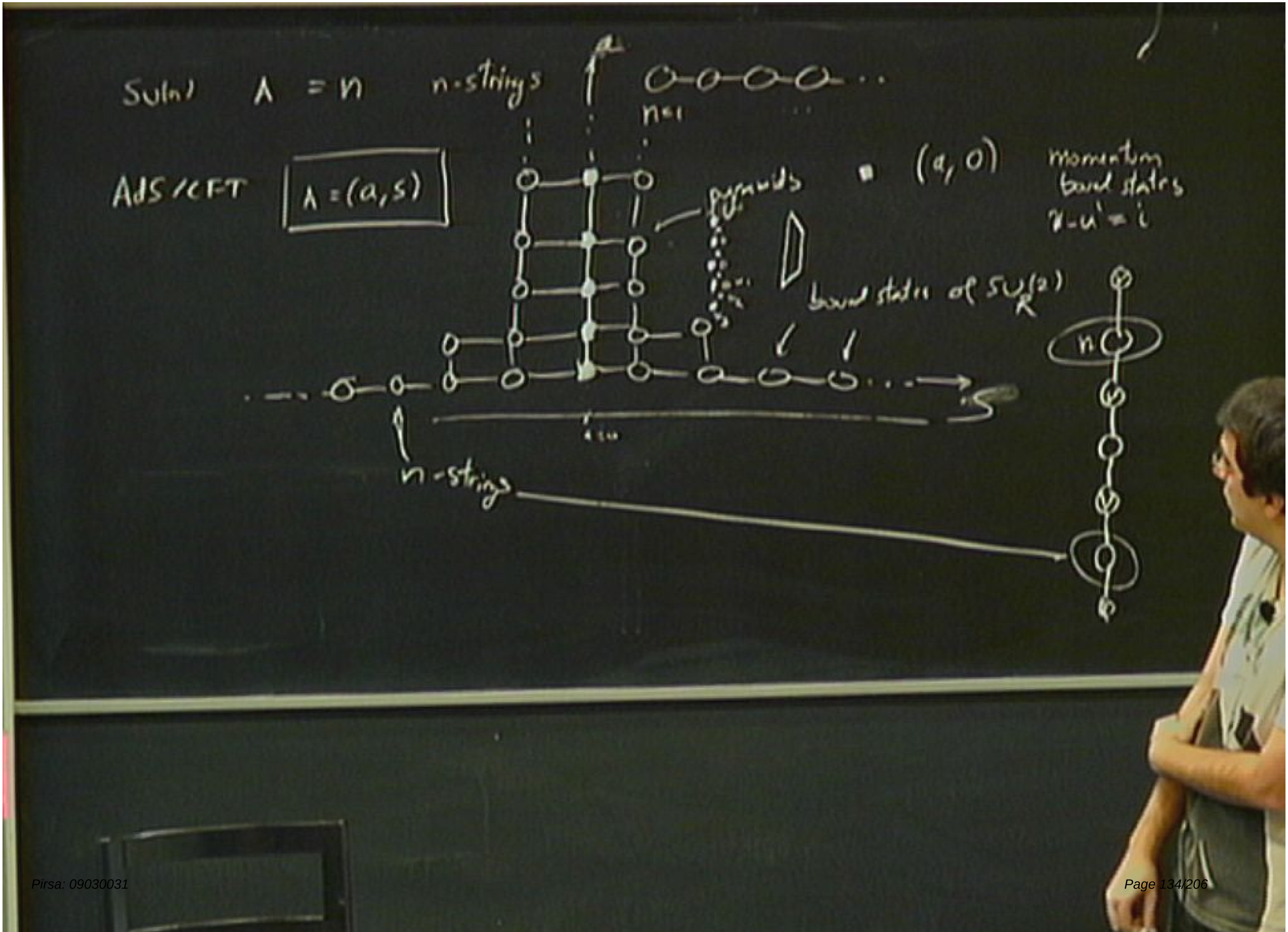
result

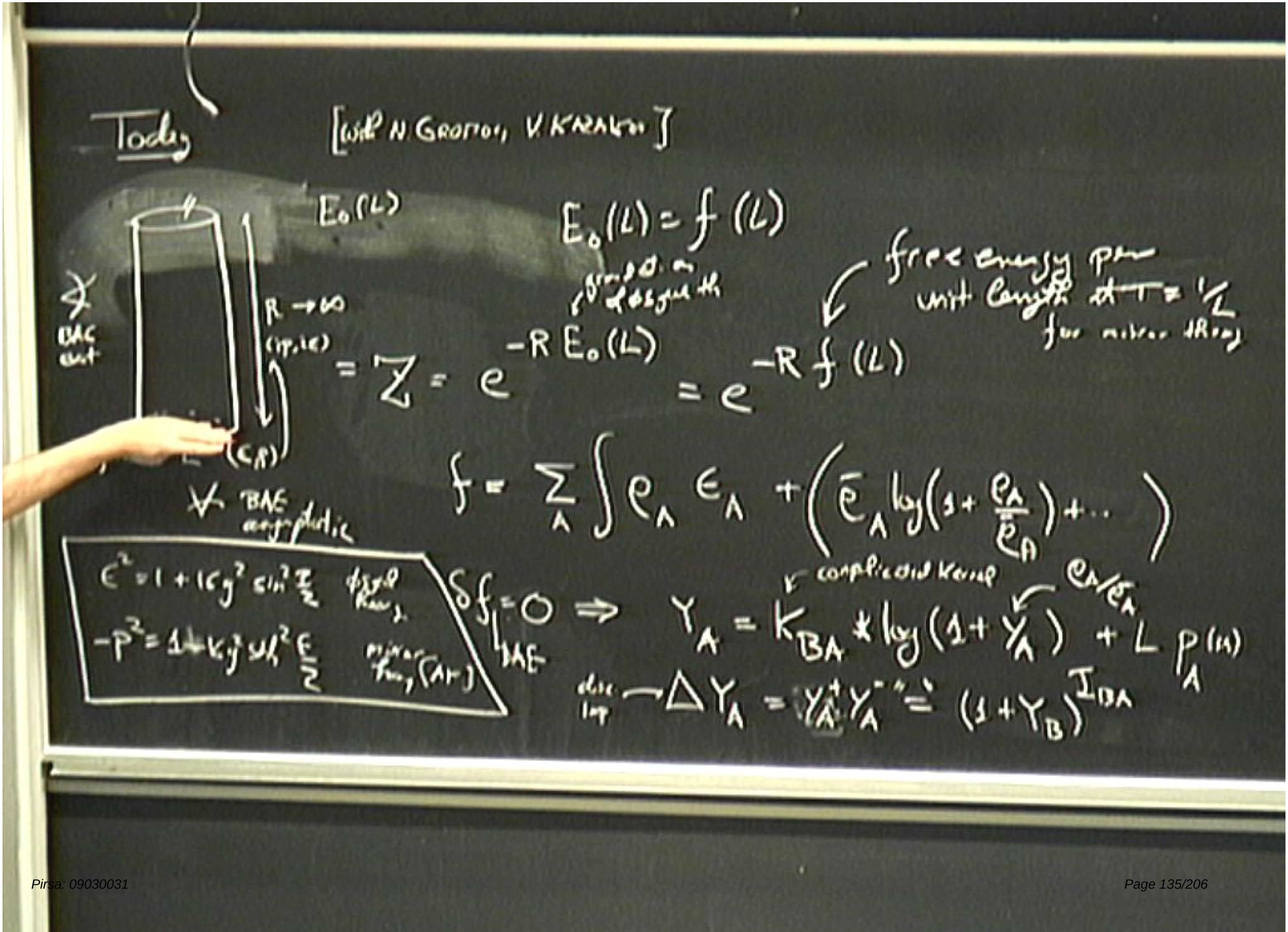
$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

Y-system

$$E = \sum_{a=1}^{\infty}\int\frac{du}{2\pi i}\,\partial_u\epsilon_a(u)\,\log\left(1+Y_{a,0}\right)$$


SU(n)
Λ = n
n-strings
n=1
AdS/CFT
Λ = (a,s)
pyramids
(a, 0)
momentum bound states
u - u' = i
bound states of SU_R(2)
n
s
n-strings


SU(n) Λ = n
n-strings
n=1
AdS/CFT
Λ = (a,s)
pyramids
(a,0)
momentum bound states
x - u' = i
bound states of SU_R(2)
s
n
n-strings



Today [with N. Gromov, V. Kazakov]

E_0(L)

R → ∞

(sp. dir)

(L,R)

BAE exact

BAE asymptotic

$$E_0(L) = f(L)$$

symmetry in space & length

free energy per unit length at T = 1/L for mirror theory

$$= Z = e^{-R\,E_0(L)} = e^{-R\,f(L)}$$

$$f = \sum_A \int \rho_A \epsilon_A + \left(\bar{\rho}_A \log\left(1 + \frac{\rho_A}{\bar{\rho}_A}\right) + \dots \right)$$

$$\boxed{\begin{array}{ll} \epsilon^2 = 1 + 16 g^2 \sin^2 \frac{p}{2} & \text{physical theory} \\ -p^2 = 1 + 16 g^2 \sinh^2 \frac{\epsilon}{2} & \text{mirror theory (AF)} \end{array}}$$

complicated kernel

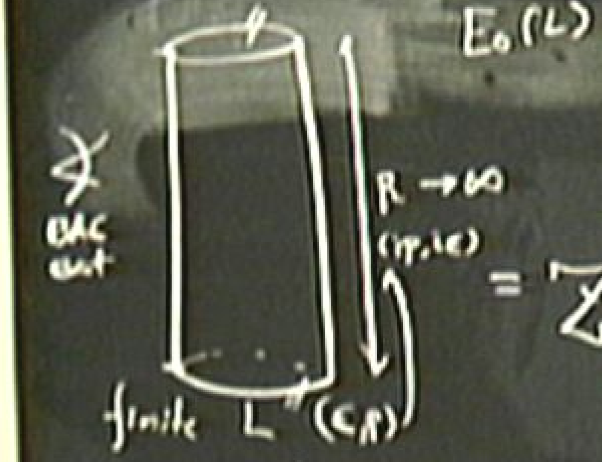
$\rho_A / \bar{\rho}_A$

$$\delta f \Big|_{BAE} = 0 \Rightarrow Y_A = K_{BA} * \log(1 + Y_A) + L\, p_A^{(m)}$$

disc. lim → $$\Delta Y_A = Y_A^+ Y_A^- \text{"="} (1 + Y_B)^{I_{BA}}$$

Today

[with N. Gromov, V. Kazakov]



∀ BAE asymptotic

$$E_0(L) = f(L)$$

ground st. en. of AdS₅ string th.

free energy per unit length at T = 1/L for mirror theory

$$= Z = e^{-R E_0(L)} = e^{-R f(L)}$$

$$f = \sum_A \int \rho_A \epsilon_A + \left(\bar{\rho}_A \log\left(1 + \frac{\rho_A}{\bar{\rho}_A}\right) + \dots \right)$$

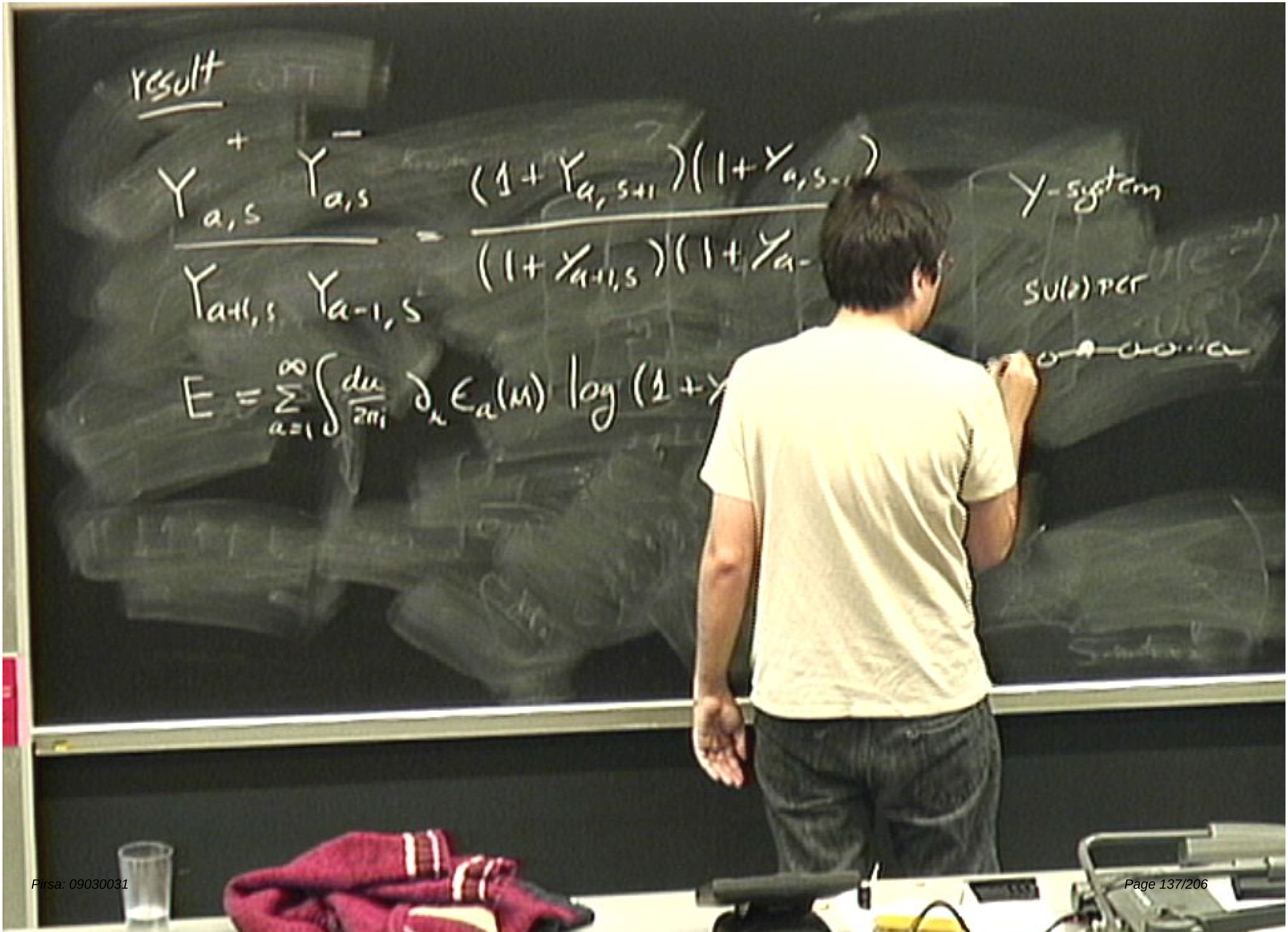
complicated kernel

$\rho_A / \bar{\rho}_A$

$$\boxed{\begin{array}{ll} \epsilon^2 = 1 + 16 g^2 \sin^2 \frac{p}{2} & \text{physical theory} \\ -p^2 = 1 + 16 g^2 \sinh^2 \frac{\epsilon}{2} & \text{mirror theory (AF)} \end{array}}$$

$$\delta f \Big|_{BAE} = 0 \Rightarrow Y_A = K_{BA} * \log(1 + Y_A) + L\, p_A^{(m)}$$

disc. lim $\longrightarrow \Delta Y_A = Y_A^+ Y_A^- \text{ "}=\text{" } (1 + Y_B)^{I_{BA}}$



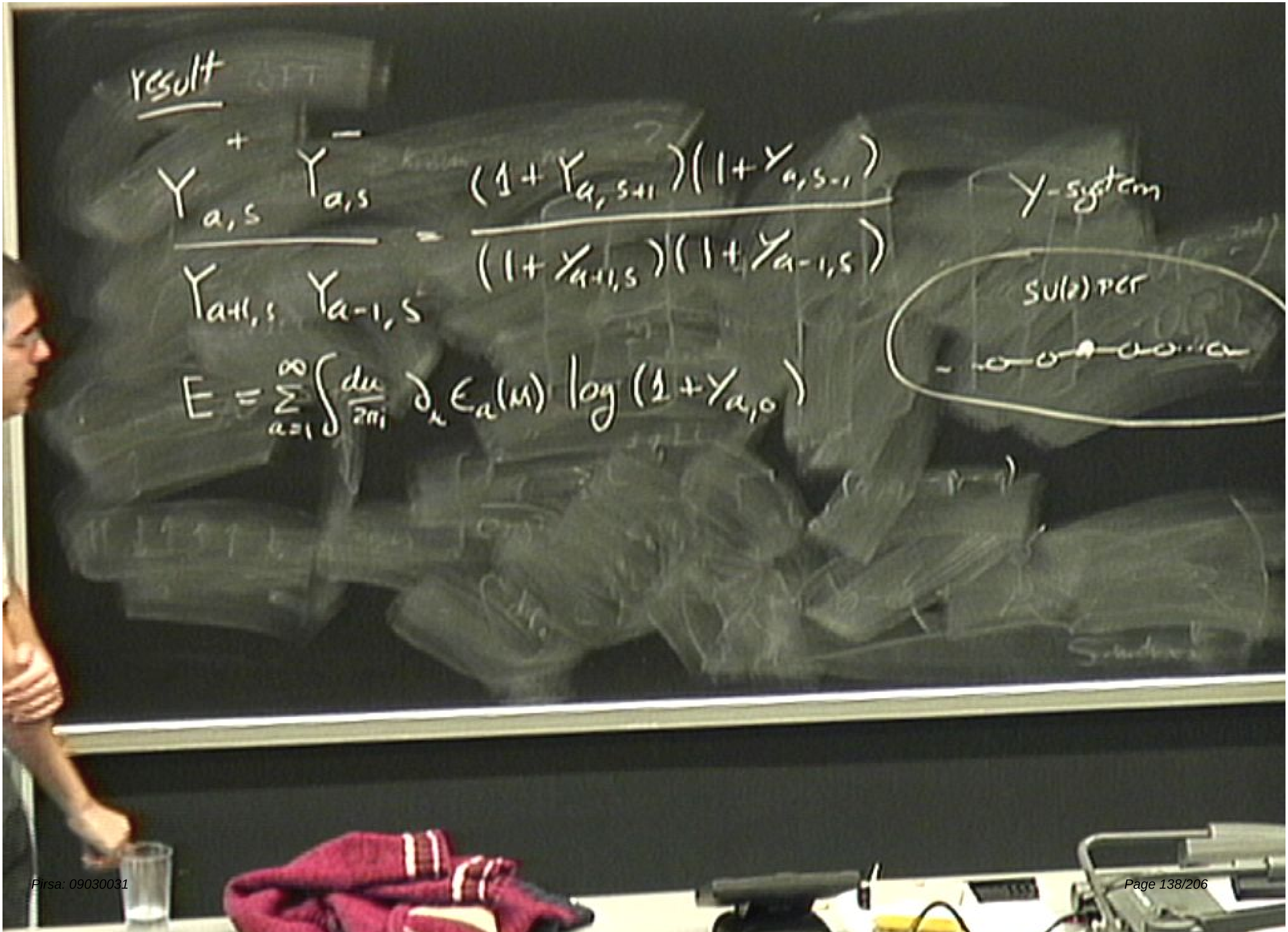
result

$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-}}$$

Y-system

SU(2) PCF

$$E = \sum_{a=1}^{\infty}\int\frac{du}{2\pi i}\,\partial_u\epsilon_a(u)\,\log(1+Y$$

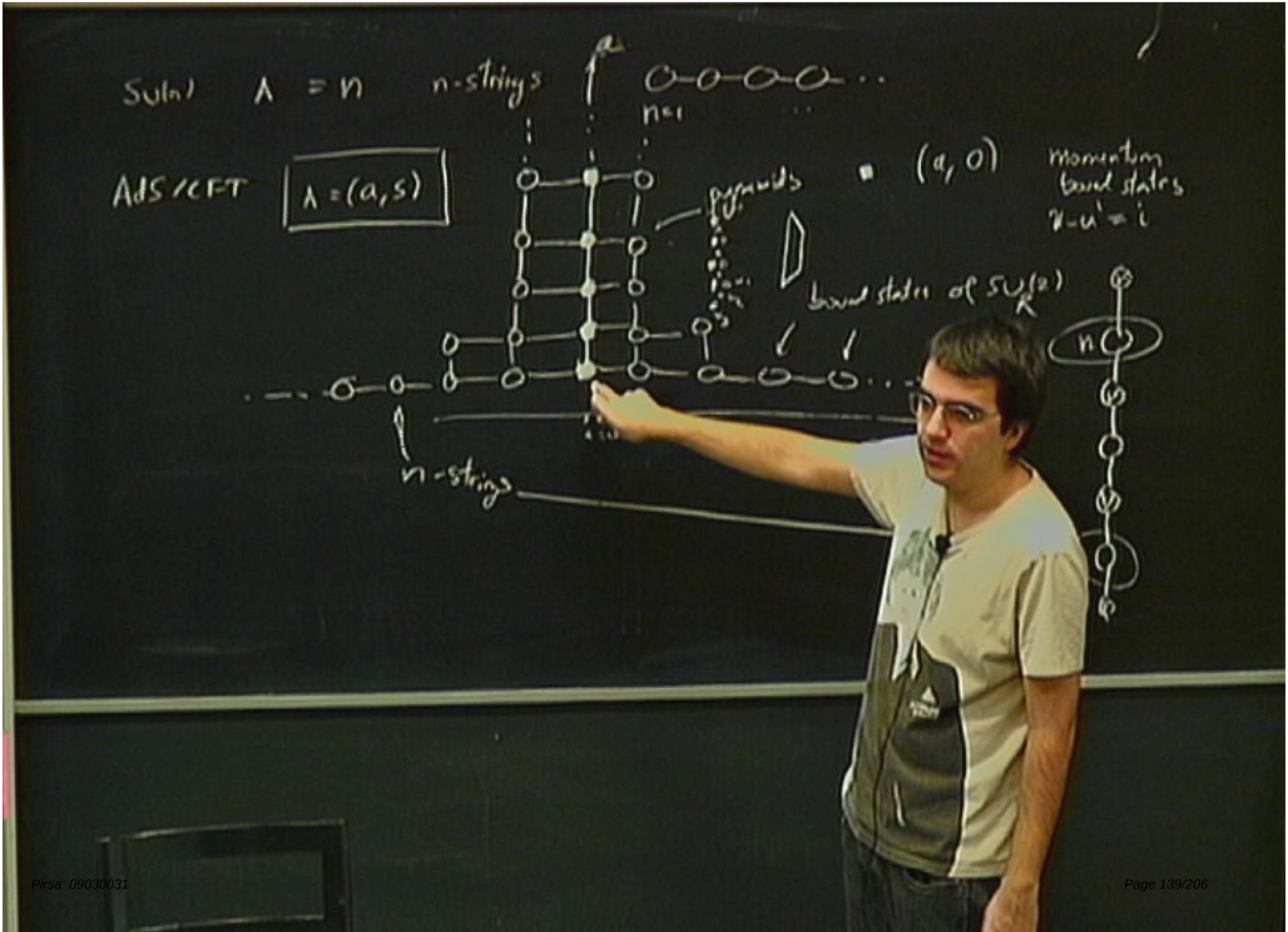


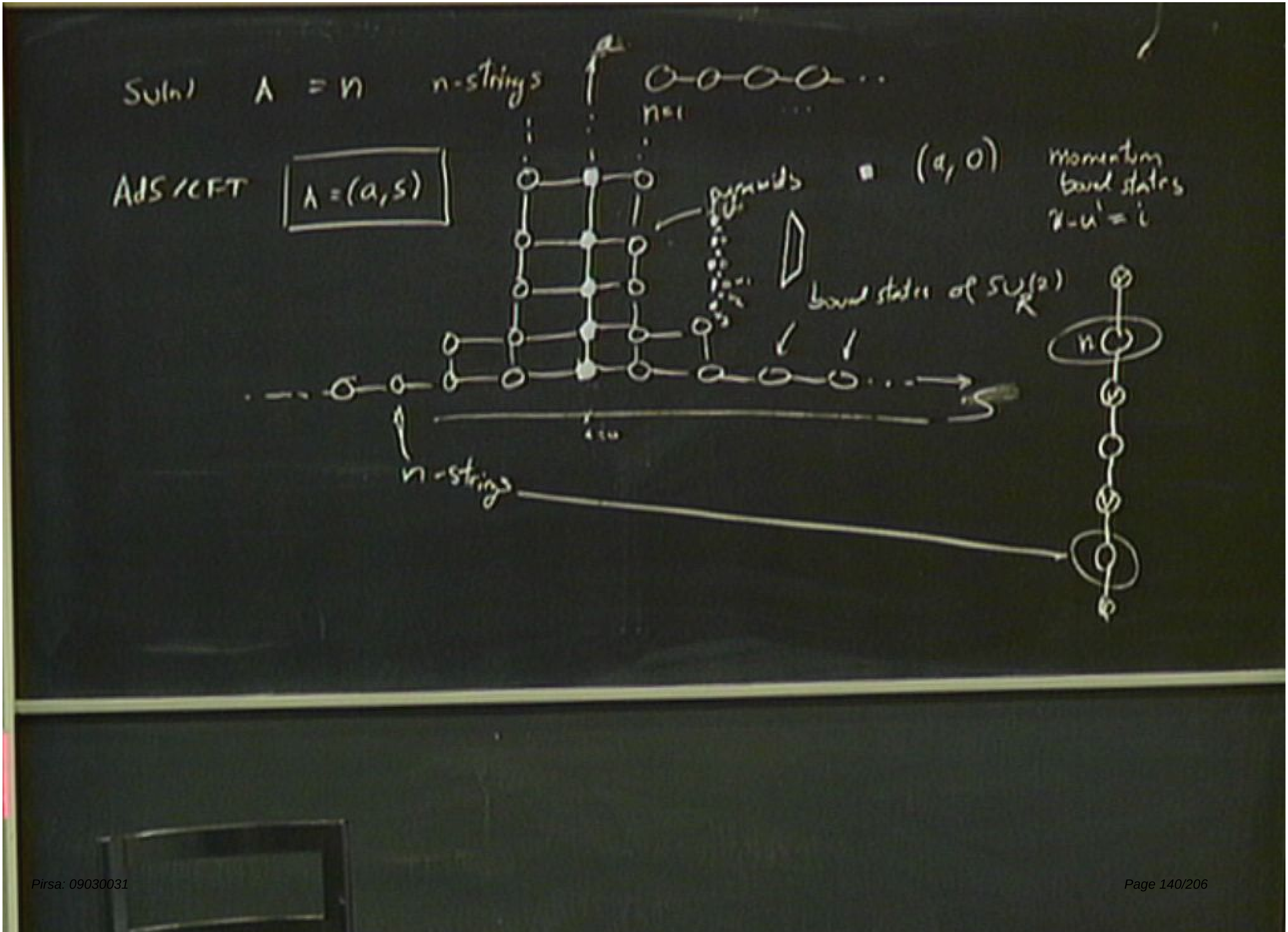
result

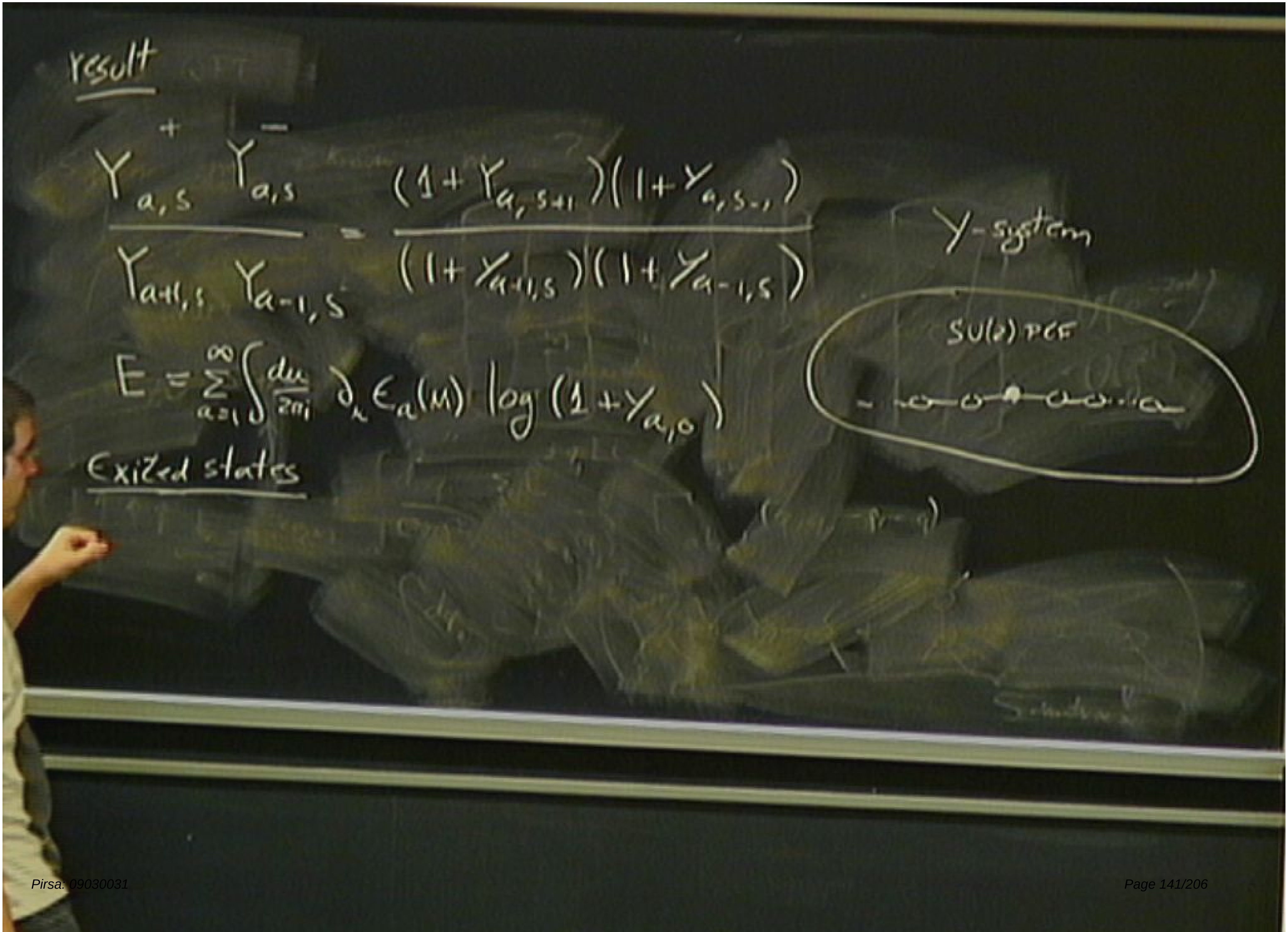
$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

Y-system

$$E = \sum_{a=1}^{\infty} \int \frac{du}{2\pi i}\, \partial_u \epsilon_a(u)\, \log(1+Y_{a,0})$$


SU(n) λ = n n-strings
n=1
AdS/CFT
λ = (a, s)
pyramids
(a, 0)
momentum
bound states
x - u' = i
bound states of SU(2)_R
n
n-strings


SU(n)
Λ = n
n-strings
n=1
AdS/CFT
Λ = (a,s)
pyramids
(a, 0)
momentum
bound states
x - u' = i
bound states of SU_R(2)
n
n-strings



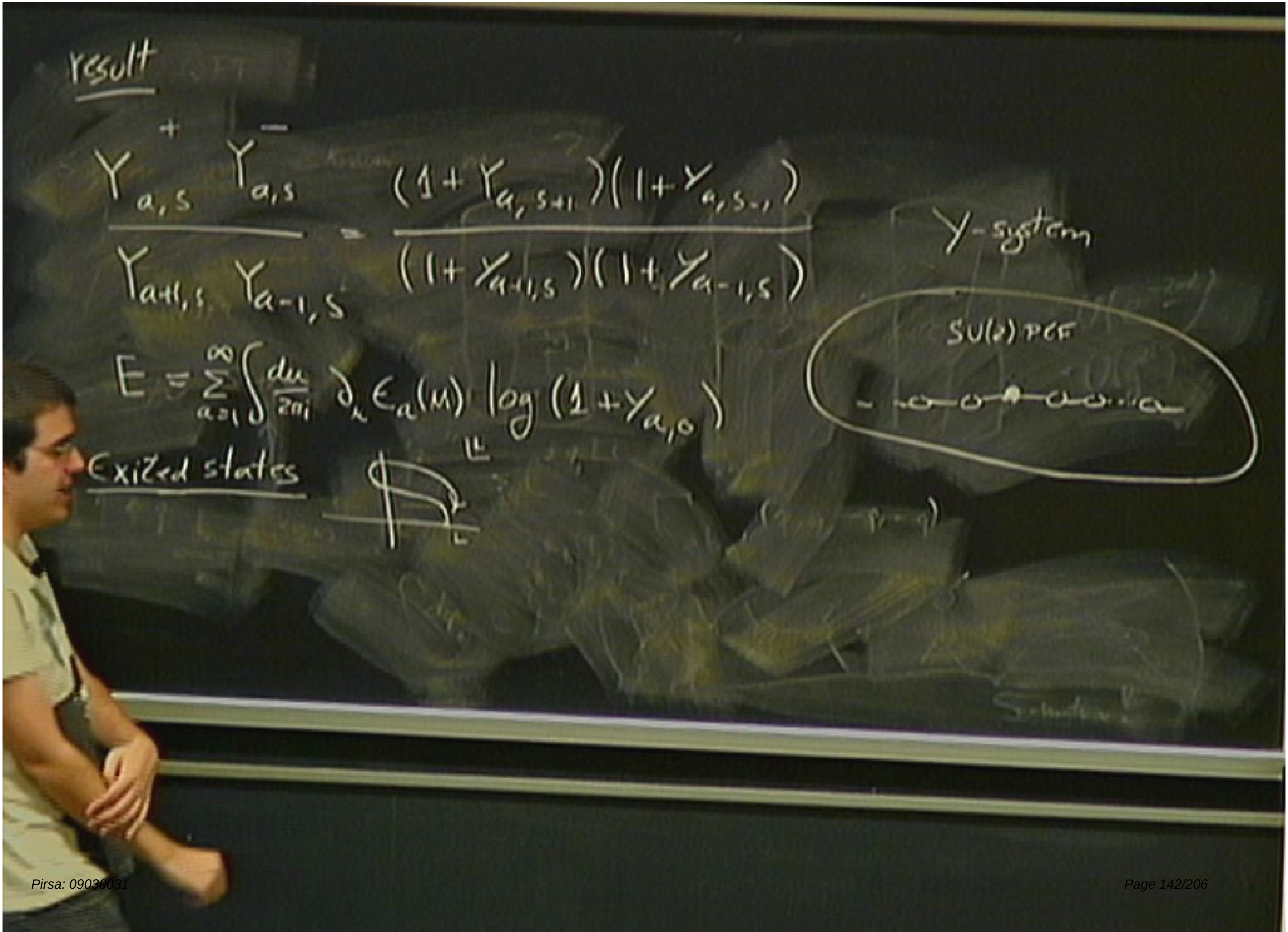
result

$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

Y-system

$$E = \sum_{a=1}^{\infty}\int \frac{du}{2\pi i}\,\partial_u \epsilon_a(u)\,\log\left(1+Y_{a,0}\right)$$

Exized states



result

$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

Y-system

SU(2) PCF

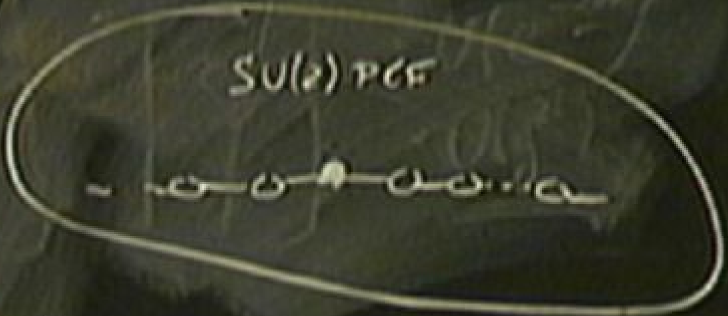
$$E = \sum_{a=1}^{\infty}\int\frac{du}{2\pi i}\,\partial_u\epsilon_a(u)\,\log(1+Y_{a,0})$$

Exited states

result

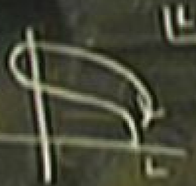
$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

Y-system

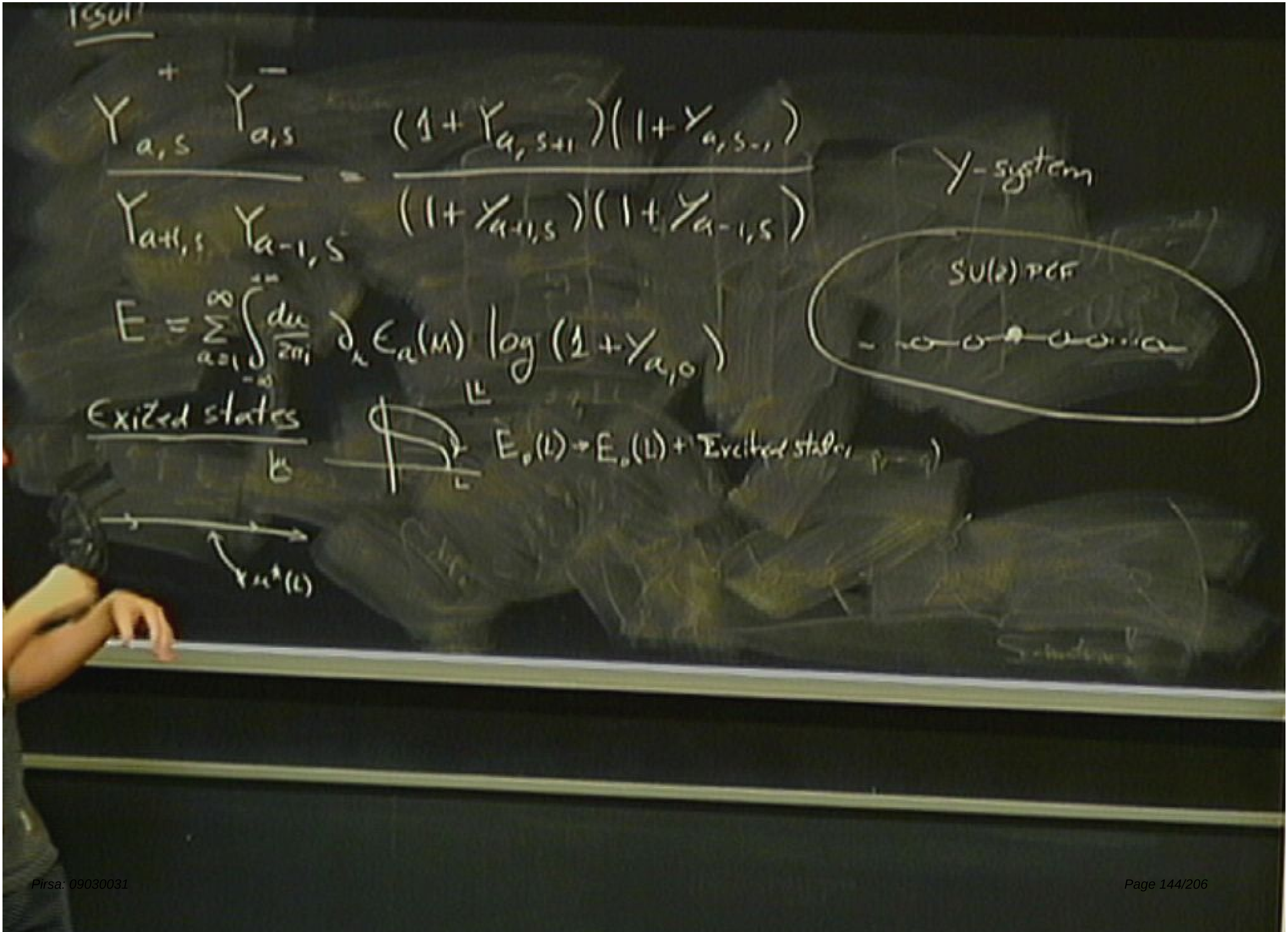


$$E = \sum_{a=1}^{\infty}\int\frac{du}{2\pi i}\,\partial_u \epsilon_a(u)\,\log(1+Y_{a,0})$$

Exited states



$E_0(L) \to E_0(L) +$ Excited states, ( )



Ysol

$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

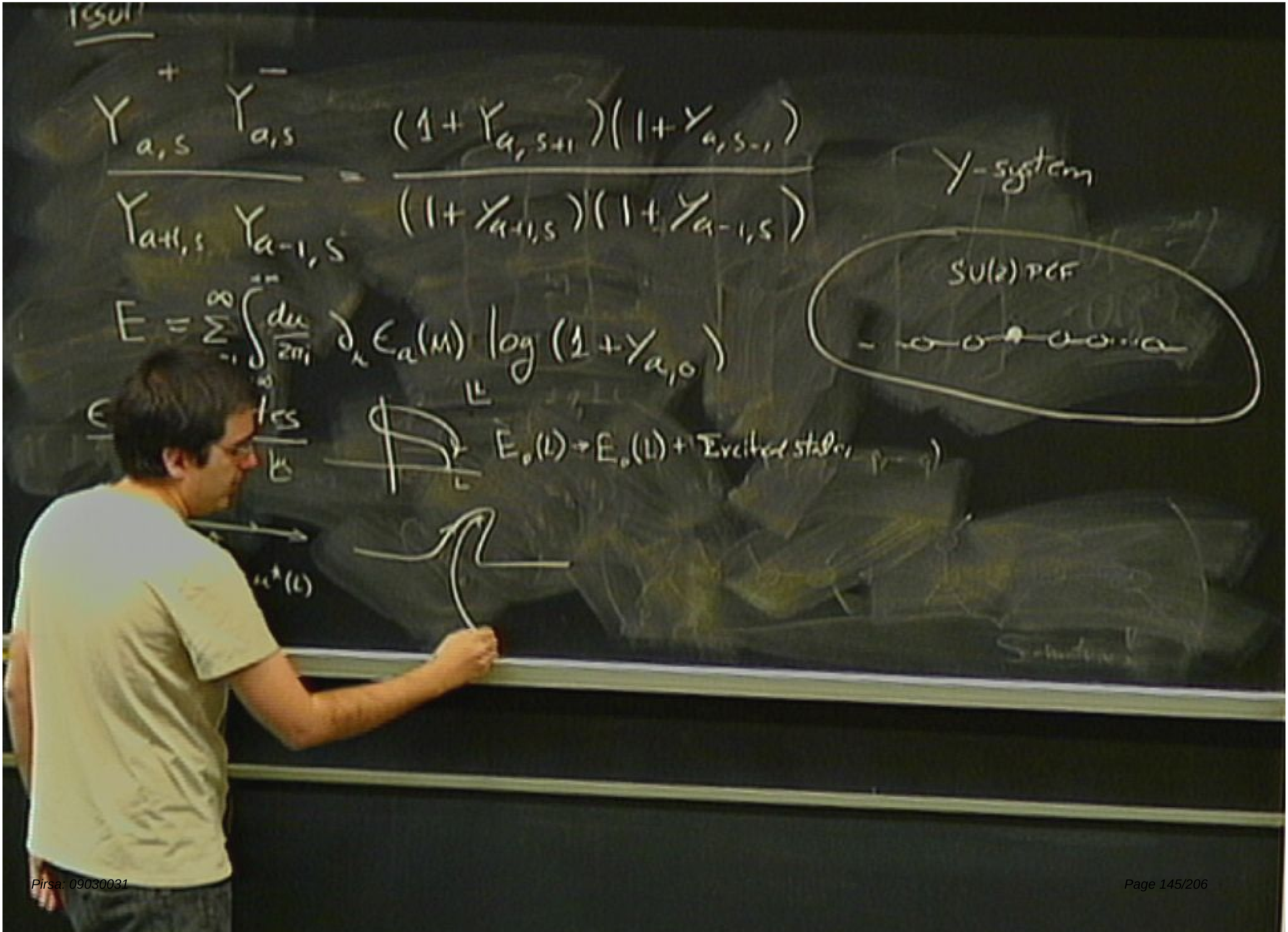
Y-system

$$E = \sum_{a=1}^{\infty}\int_{-\infty}^{+\infty}\frac{du}{2\pi i}\,\partial_u \epsilon_a(u)\,\log(1+Y_{a,0})$$

SU(2) PCF

Exited states

$E_0(L) \to E_0(L) +$ Excited states $(\ldots)$



result

$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

Y-system

$$E = \sum_{a=1}^{\infty}\int_{-\infty}^{\infty}\frac{du}{2\pi i}\,\partial_u \epsilon_a(u)\,\log\left(1+Y_{a,0}\right)$$

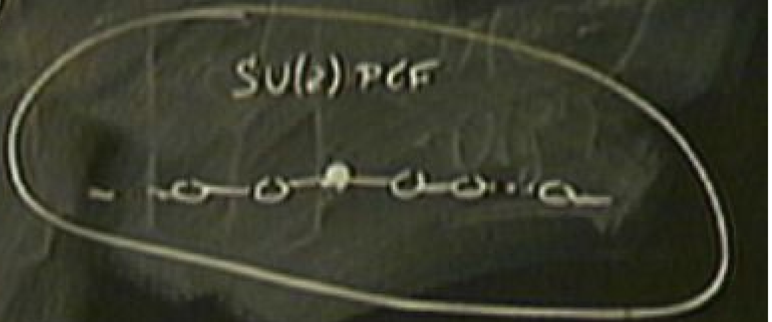
SU(2) PCF

$E_0(L) \to E_0(L) +$ Excited states

result

$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

Y-system



$$E = \sum_{a=1}^{\infty}\int_{-\infty}^{+\infty}\frac{du}{2\pi i}\,\partial_u \epsilon_a(u)\,\log(1+Y_{a,0})$$

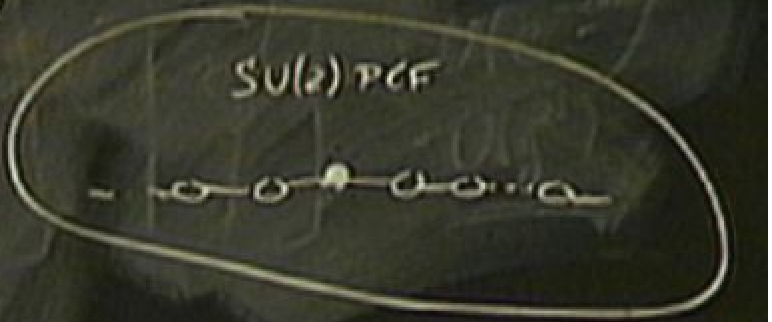
Exited states

$E_0(L) \to E_0(L) +$ Excited states $(\ldots)$

result

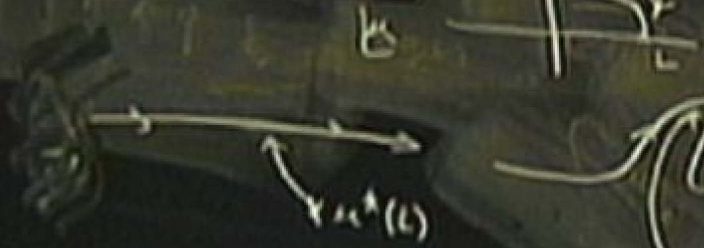
$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

Y-system

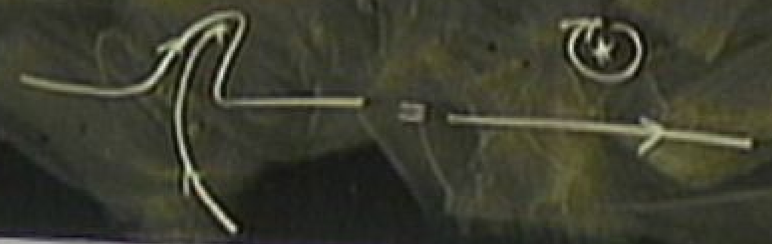


$$E = \sum_{a=1}^{\infty}\int_{-\infty}^{\infty}\frac{du}{2\pi i}\,\partial_u \epsilon_a(u)\,\log(1+Y_{a,0})$$

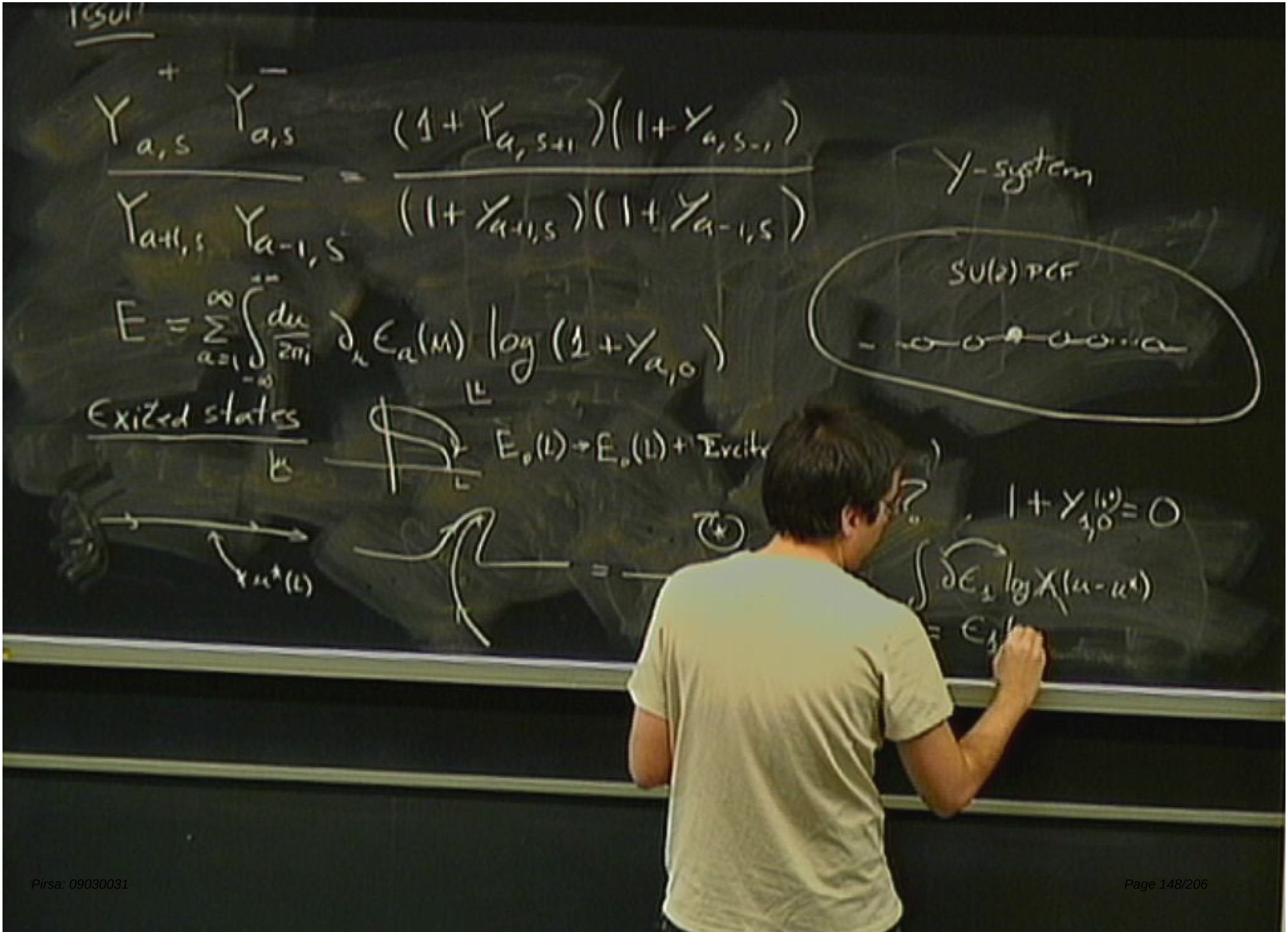
Exited states



$E_0(L) \to E_0(L) +$ Excited states



$* = ?$, $1+Y_{1,0}=0$



result

$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

Y-system

SU(2) PCF

$$E = \sum_{a=1}^{\infty}\int_{-\infty}^{\infty}\frac{du}{2\pi i}\,\partial_u \epsilon_a(u)\,\log\left(1+Y_{a,0}\right)$$

Excited states

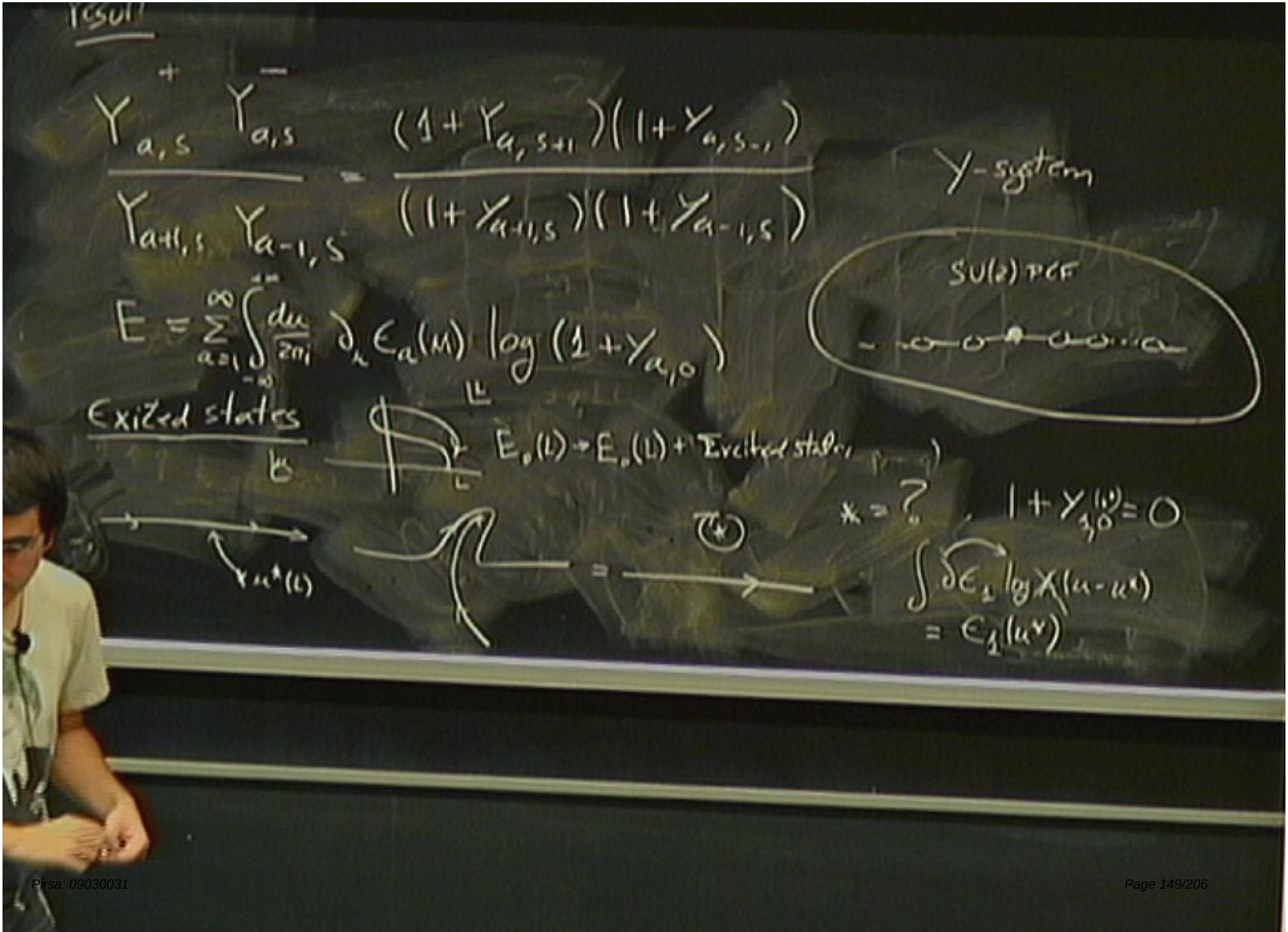
$E_0(L) \to E_0(L) + \text{Excit}$

$y\,u^*(L)$

$1+Y^{(0)}_{1,0}=0$

$\int \partial\epsilon_1 \log X(u-u^*)$

$= \epsilon_1($



result

$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

Y-system

SU(2) PCF

$$E = \sum_{a=1}^{\infty}\int_{-\infty}^{+\infty}\frac{du}{2\pi i}\,\partial_u \epsilon_a(u)\,\log(1+Y_{a,0})$$

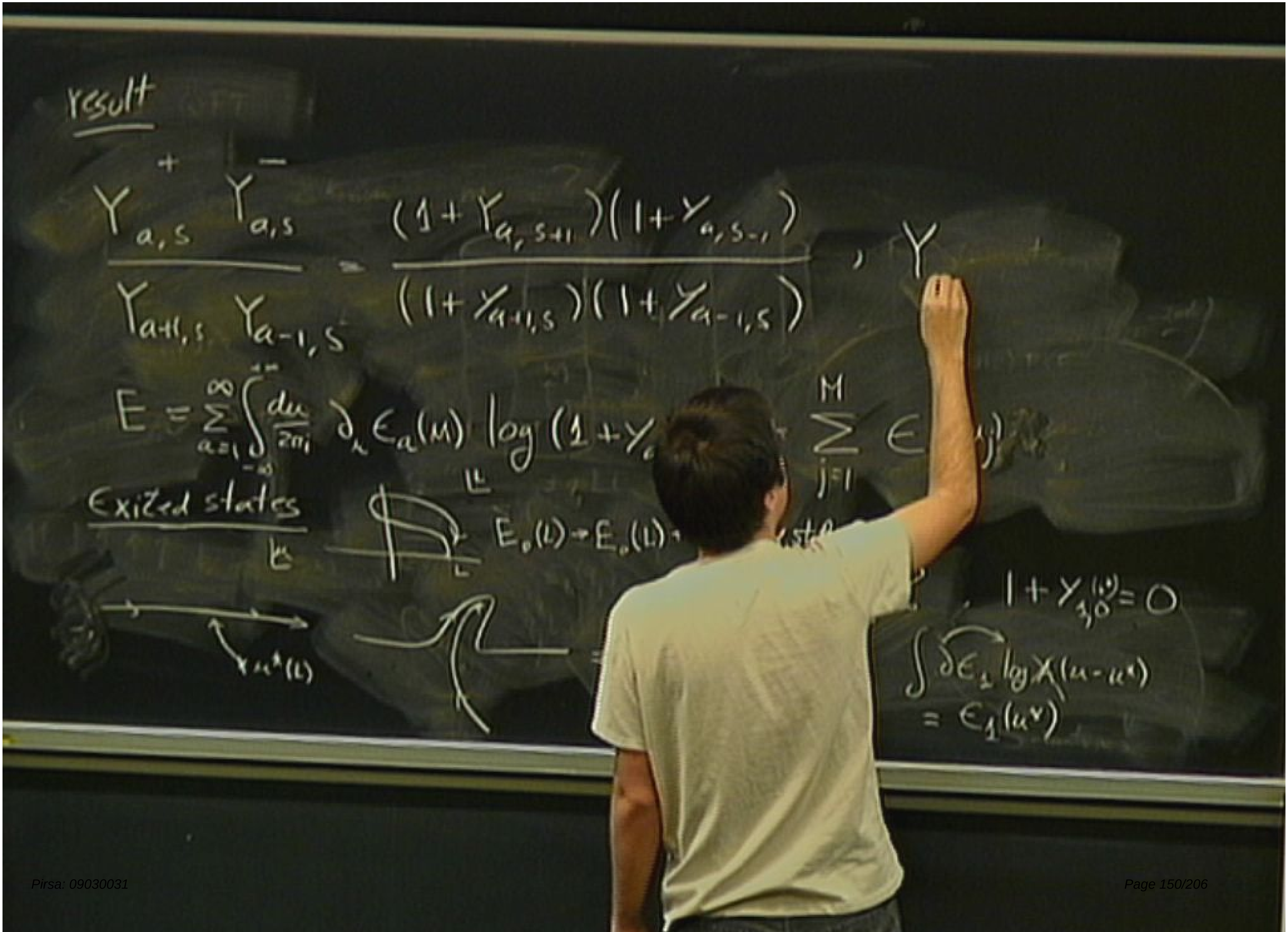
Excited states

$E_0(L) + E_0(L) + $ Excited states

$x = ?$, $1+Y^{(u)}_{1,0}=0$

$u^*(L)$

$$\int \partial\epsilon_1 \log X(u-u^*) = \epsilon_1(u^*)$$



result

$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}, \quad Y$$

$$E = \sum_{a=1}^{\infty}\int_{-\infty}^{+\infty}\frac{du}{2\pi i}\,\partial_u \epsilon_a(u)\,\log(1+Y_a \quad \sum_{j=1}^{M} \epsilon \quad (u_j)$$

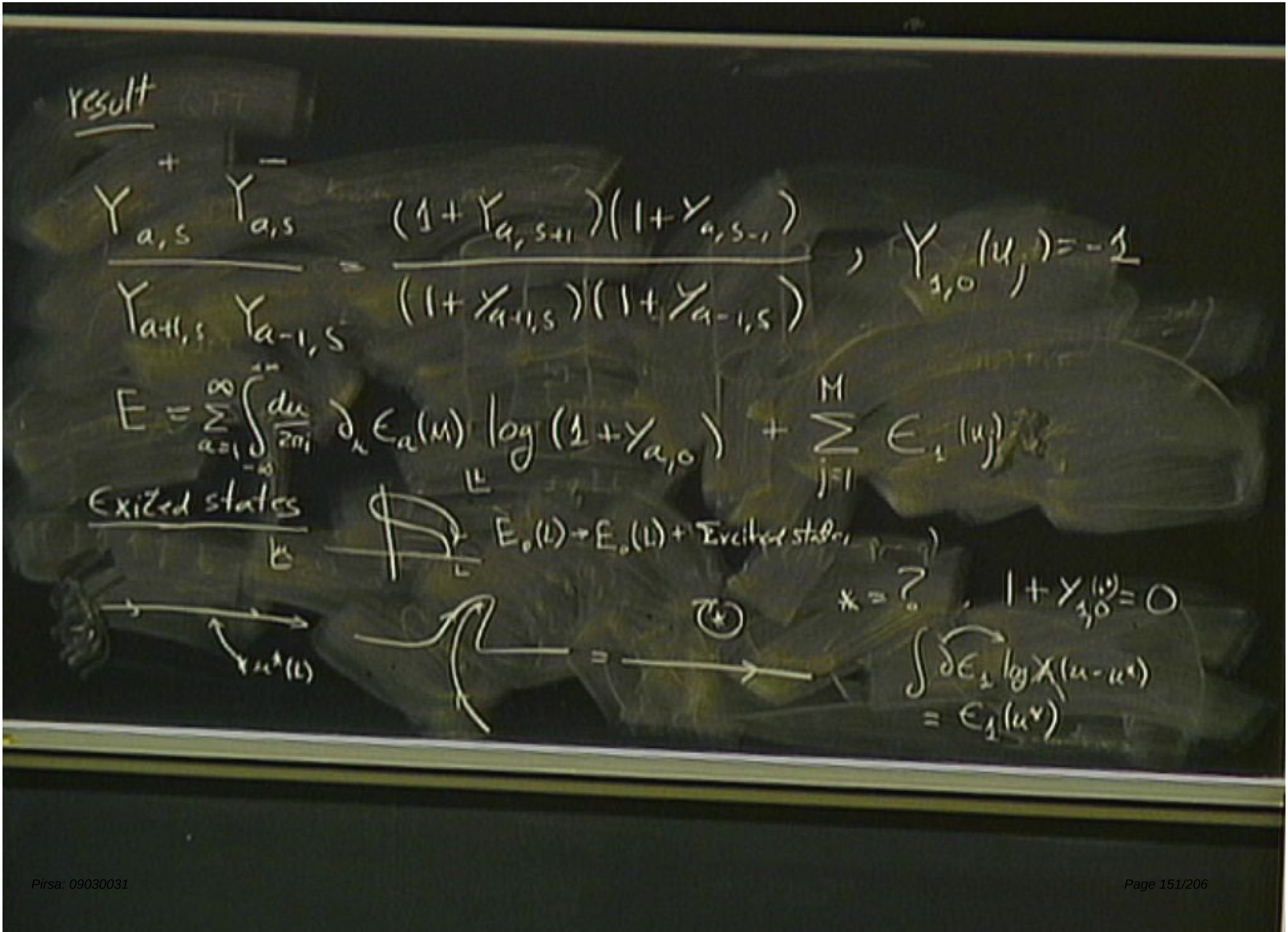
Exited states

$E_0(L) \to E_n(L)$

$u^*(L)$

$1+Y^{(j)}_{3,0}=0$

$$\int \partial\epsilon_1 \log X(u-u^*) = \epsilon_1(u^*)$$



result

$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}, \quad Y_{1,0}(u_j) = -1$$

$$E = \sum_{a=1}^{\infty} \int_{-\infty}^{+\infty} \frac{du}{2\pi i}\, \partial_u \epsilon_a(u) \log(1+Y_{a,0}) + \sum_{j=1}^{M} \epsilon_1(u_j)$$

Exited states

$E_0(L) \to E_0(L) +$ Excited states

$* = ?$, $1 + Y^{(*)}_{1,0} = 0$

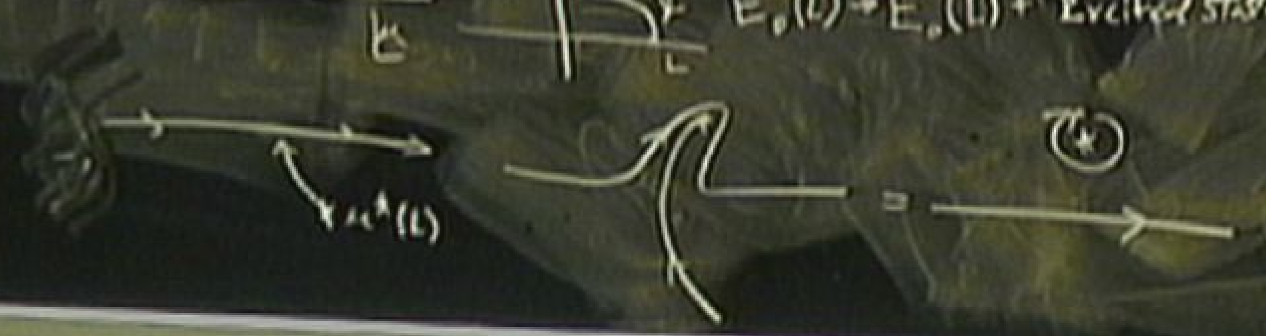
$$\int \partial\epsilon_1 \log X(u-u^*) = \epsilon_1(u^*)$$

result

$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})} , \quad Y_{1,0}(u_j) = -1$$

$$E = \sum_{a=1}^{\infty}\int_{-\infty}^{+\infty}\frac{du}{2\pi i}\,\partial_u \epsilon_a(u)\,\log(1+Y_{a,0}) + \sum_{j=1}^{M}\epsilon_1(u_j)$$

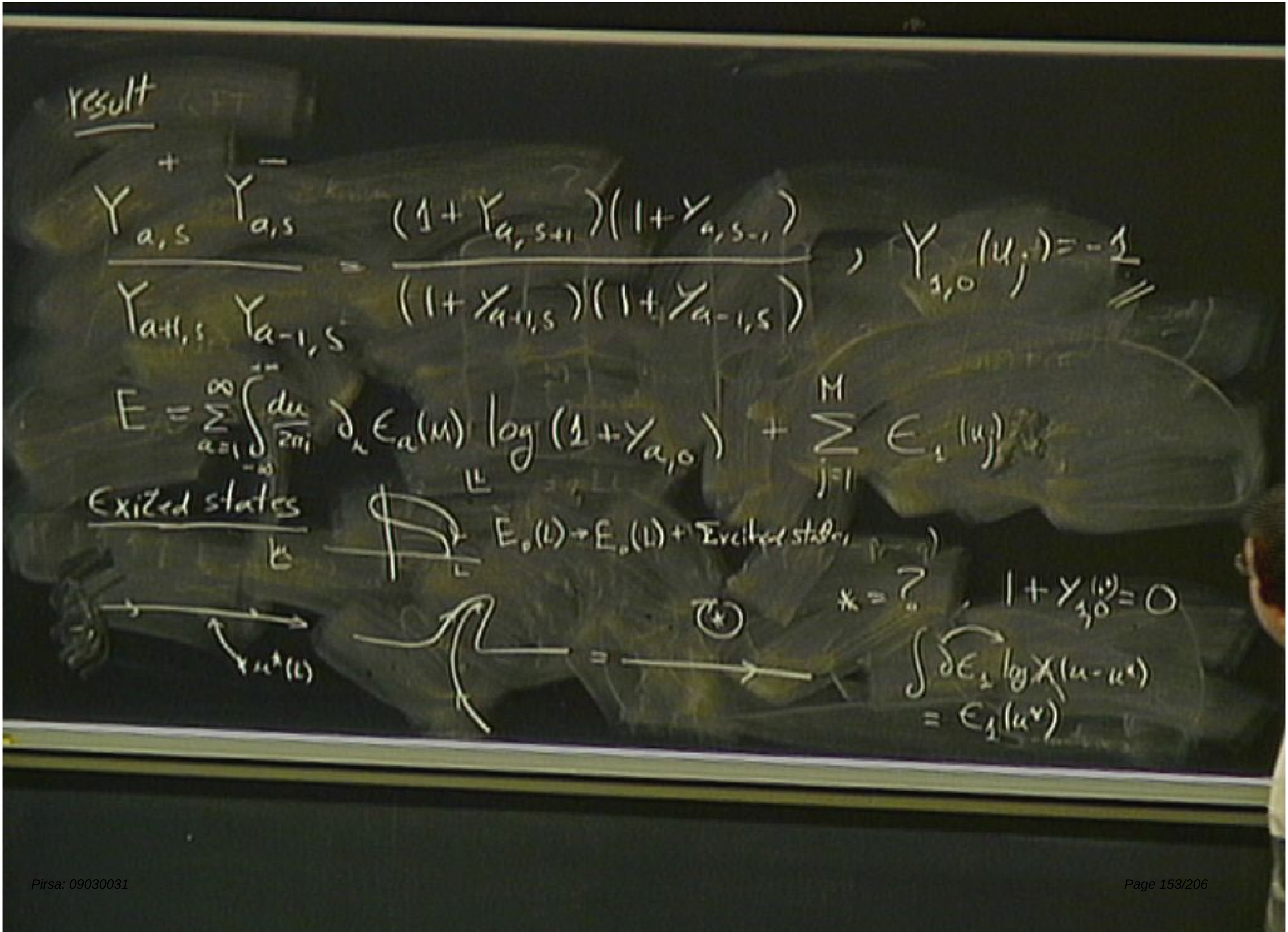
Exized states



$E_0(L) \to E_0(L) + \text{Excited states}$

$* = ?$, $1 + Y^{(*)}_{1,0} = 0$

$$\int \partial\epsilon_1 \log X(u-u^*) = \epsilon_1(u^*)$$



result

$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}, \quad Y_{1,0}(u_j) = -1$$

$$E = \sum_{a=1}^{\infty}\int_{-\infty}^{+\infty}\frac{du}{2\pi i}\,\partial_u \epsilon_a(u)\,\log(1+Y_{a,0}) + \sum_{j=1}^{M}\epsilon_1(u_j)$$

Exited states

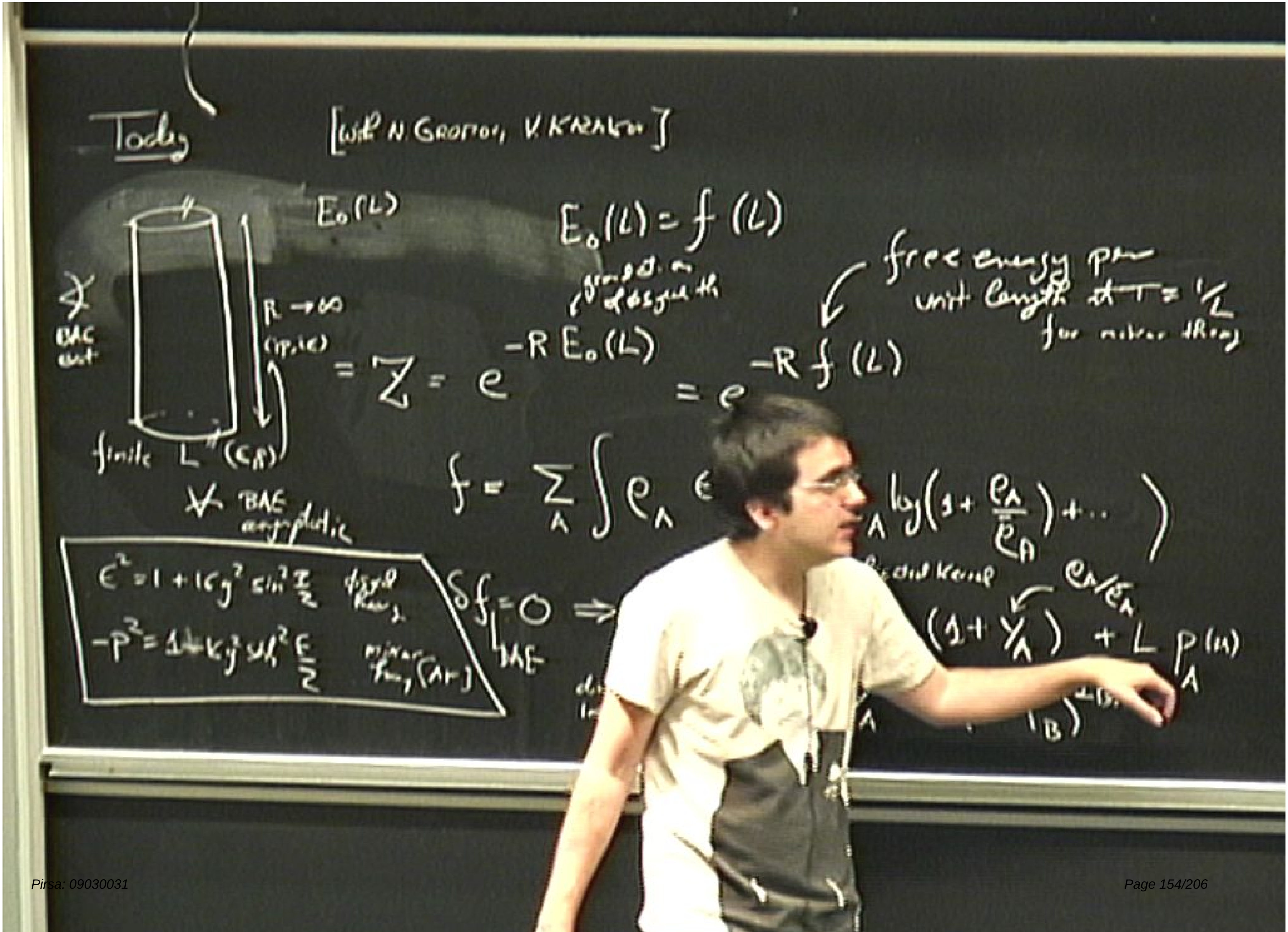
L

$E_0(L) \to E_0(L) + \text{Excited states}$

$u^*(L)$

$* = ?\,, \quad 1+Y^{(0)}_{1,0} = 0$

$$\int \partial\epsilon_1 \log X(u-u^*) = \epsilon_1(u^*)$$



Today

[with N. Gromov, V. Kazakov]

$E_0(L)$

$R \to \infty$

(p, E)

BAE exact

finite L (ϵ, p)

$E_0(L) = f(L)$

$$= Z = e^{-R\,E_0(L)} = e^{-R\,f(L)}$$

free energy per unit length at $T = 1/L$ for mirror theory

$$f = \sum_A \int \rho_A\, \epsilon \quad \dots \quad \log\left(1 + \frac{\rho_A}{\bar{\rho}_A}\right) + \dots \quad)$$

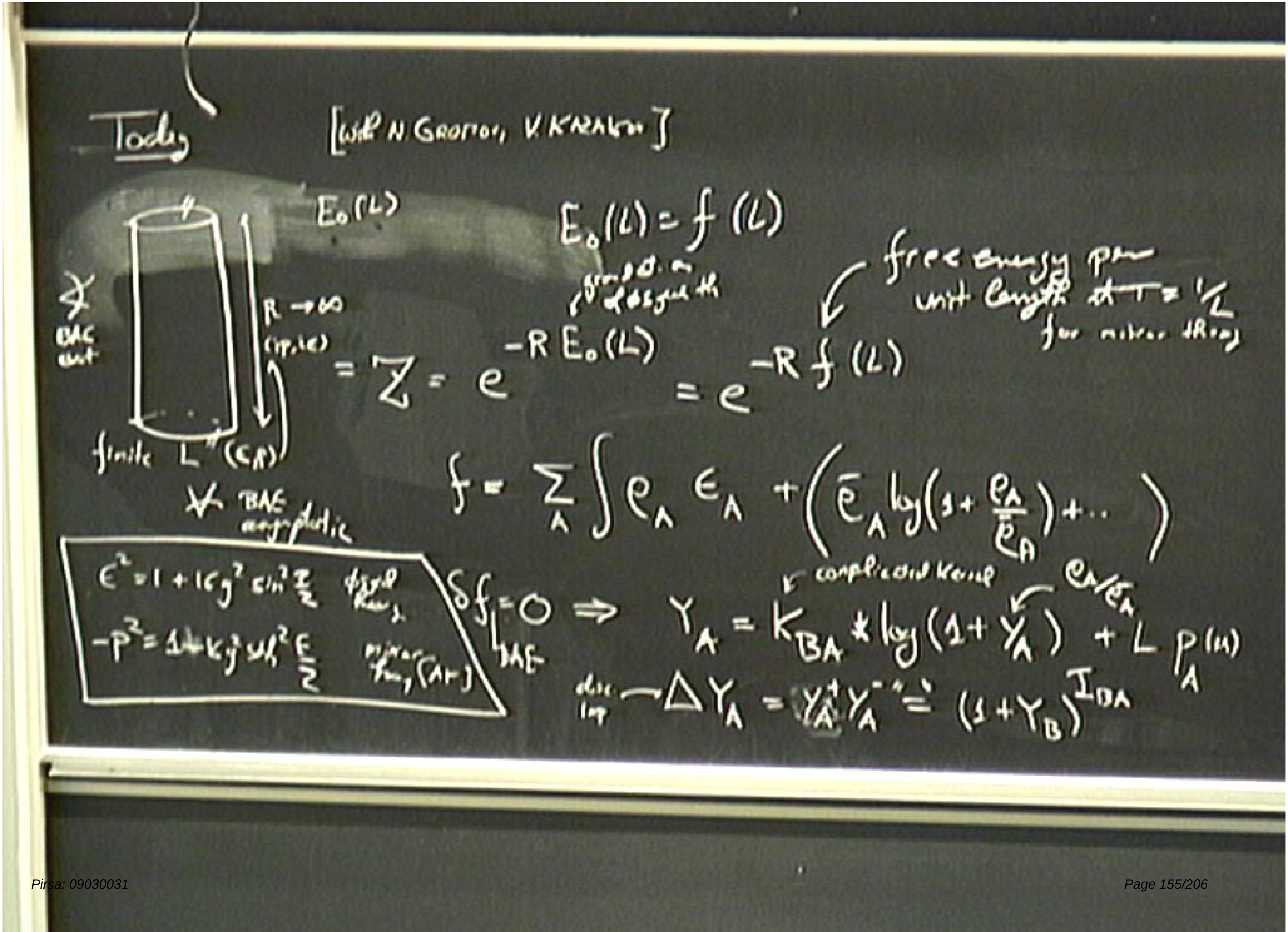
BAE asymptotic

$$\boxed{\begin{array}{ll} \epsilon^2 = 1 + 16 g^2 \sin^2 \frac{p}{2} & \text{[illegible]} \\ -p^2 = 1 + 16 g^2 \sinh^2 \frac{\epsilon}{2} & \text{mirror theory (AFS)} \end{array}}$$

$$\delta f \Big|_{BAE} = 0 \Rightarrow$$

$\rho_A / \bar{\rho}_A$

$$(1 + Y_A) + L\, p_A(u)$$



Today

[with N. Gromov, V. Kazakov]

$E_0(L)$

$R \to \infty$

(p,ϵ)

BAE exact

finite L (ϵ,p)

$$= Z = e^{-R E_0(L)} = e^{-R f(L)}$$

$$E_0(L) = f(L)$$

ground st. en of [illegible]

free energy per unit length at $T = 1/L$ for mirror theory

$$f = \sum_A \int \rho_A \epsilon_A + \left(\bar{\rho}_A \log\left(1 + \frac{\rho_A}{\bar{\rho}_A}\right) + \dots \right)$$

X BAE asymptotic

$\epsilon^2 = 1 + 16 g^2 \sin^2 \frac{p}{2}$ physical theory

$-p^2 = 1 + 16 g^2 \sinh^2 \frac{\epsilon}{2}$ mirror theory (Ar)

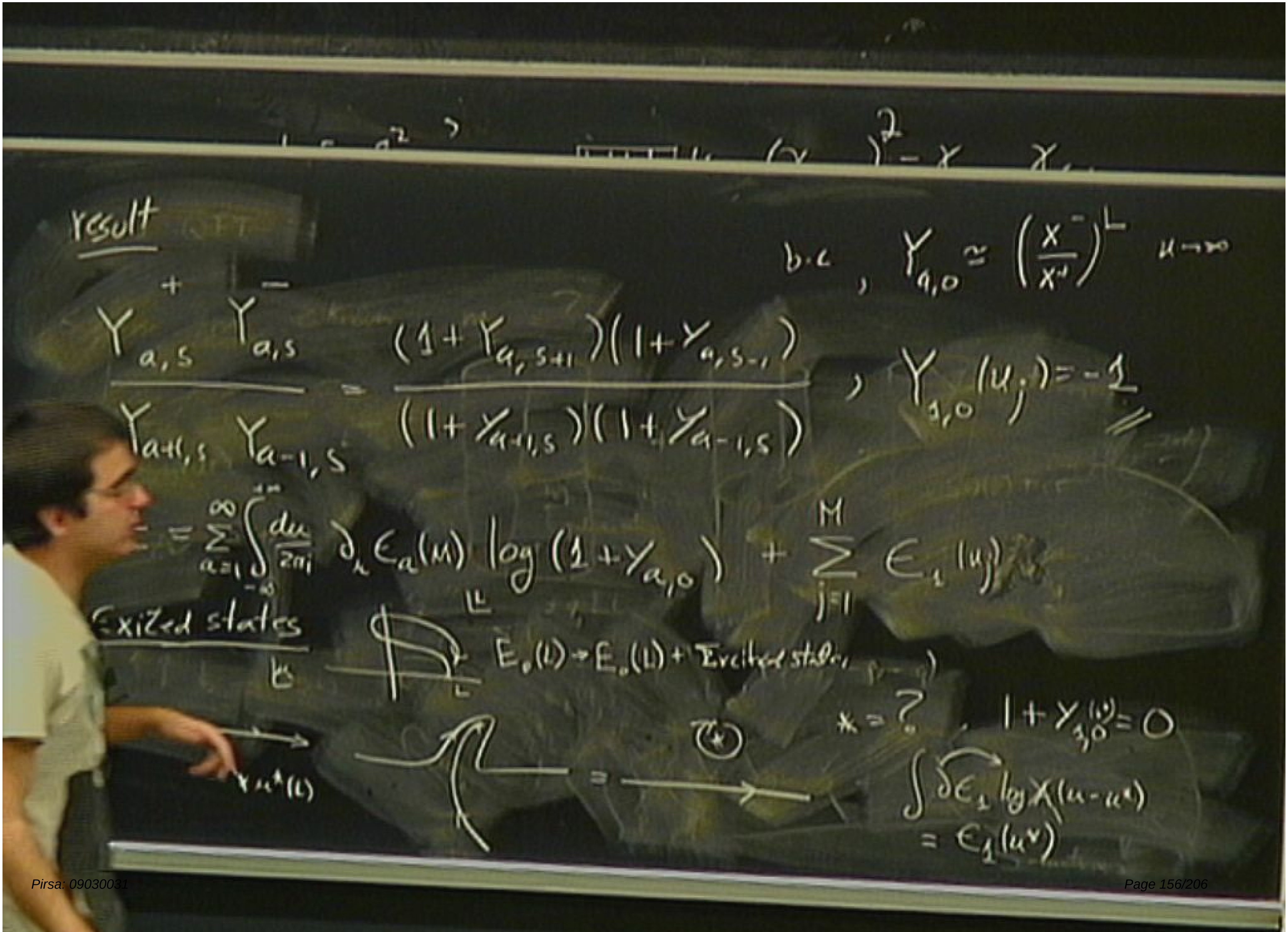
complicated kernel

$\rho_A / \bar{\rho}_A$

$$\delta f\big|_{BAE} = 0 \Rightarrow Y_A = K_{BA} * \log(1 + Y_A) + L\, p_A^{(m)}$$

disc. eq $\to$

$$\Delta Y_A = Y_A^+ Y_A^- \text{“}=\text{”} (1 + Y_B)^{I_{BA}}$$



result

b.c , $Y_{1,0} \simeq \left(\frac{x^-}{x^+}\right)^L$ $u \to \infty$

$$\frac{Y^+_{a,s}\, Y^-_{a,s}}{Y_{a+1,s}\, Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})} , \quad Y_{1,0}(u_j) = -1$$

$$= \sum_{a=1}^{\infty} \int_{-\infty}^{+\infty} \frac{du}{2\pi i}\, \partial_u \epsilon_a(u) \log\left(1+Y_{a,0}\right) + \sum_{j=1}^{M} \epsilon_1(u_j)$$

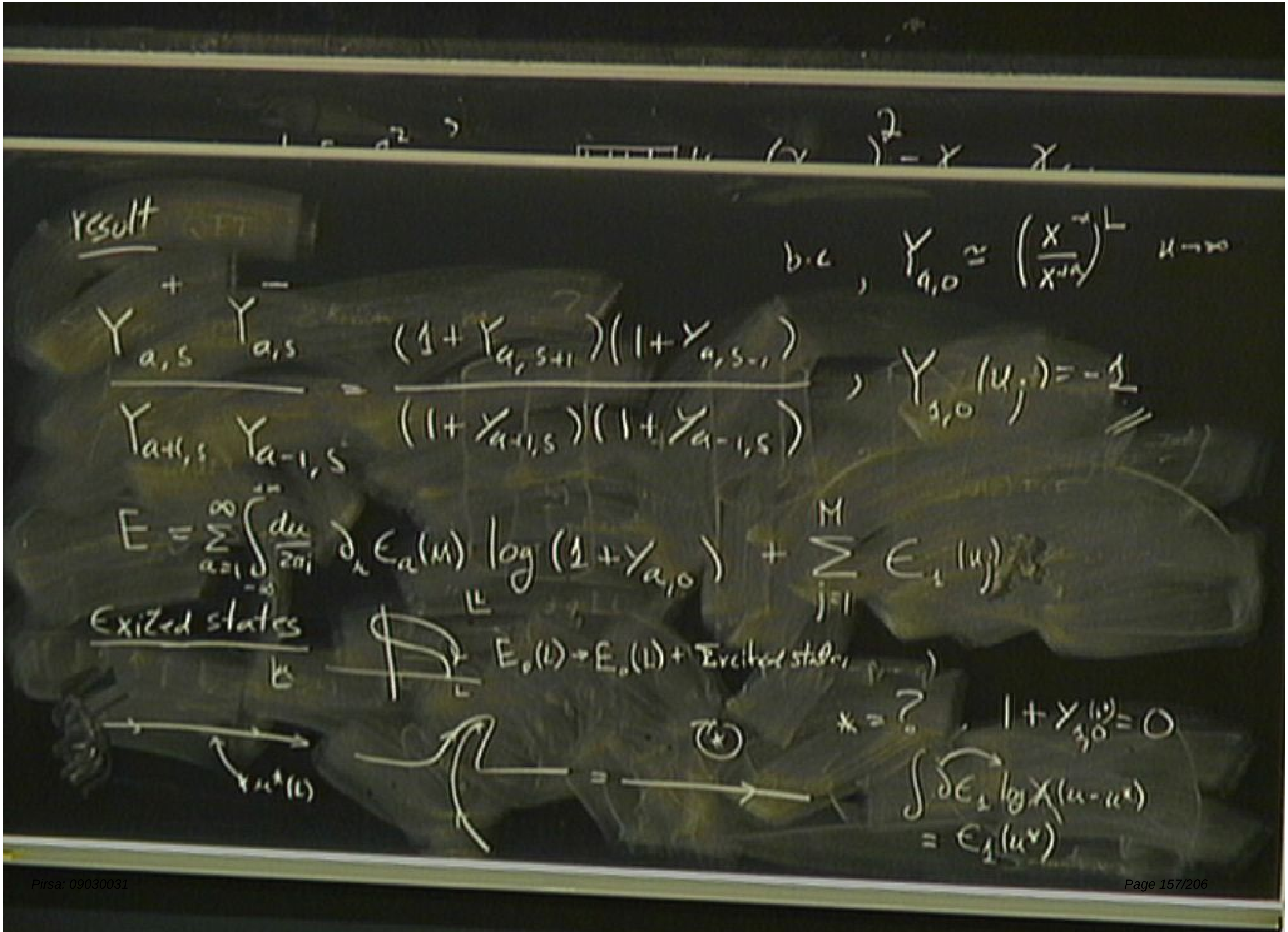
Exized states

$E_0(L) \to E_0(L) +$ Excited states

$* = ?$, $1+Y^{(*)}_{1,0} = 0$

$u^*(L)$

$$\int \partial \epsilon_1 \log X(u-u^*) = \epsilon_1(u^*)$$



result

b.c , $Y_{1,0} \simeq \left(\frac{x^-}{x^+}\right)^L$ $u \to \infty$

$$\frac{Y^+_{a,s}\,Y^-_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}$$

, $Y_{1,0}(u_j) = -1$

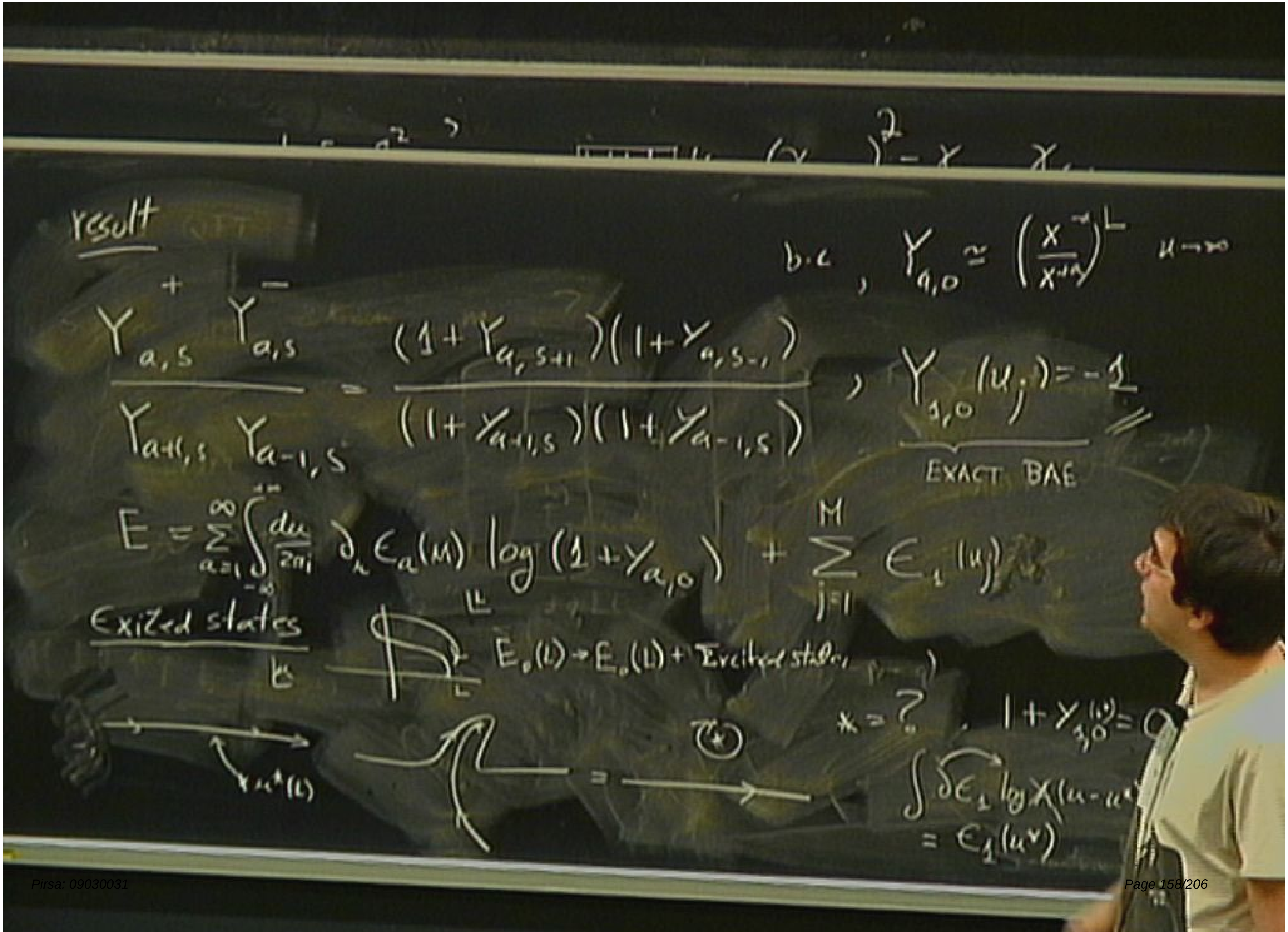
$$E = \sum_{a=1}^{\infty}\int_{-\infty}^{+\infty}\frac{du}{2\pi i}\,\partial_u \epsilon_a(u)\,\log(1+Y_{a,0}) + \sum_{j=1}^{M}\epsilon_1(u_j)$$

Exized states

$E_0(L) \to E_0(L) +$ Excited states

$* = ?$, $1+Y^{(u)}_{1,0}=0$

$$\int \partial\epsilon_1 \log X(u-u^*) = \epsilon_1(u^*)$$



result

b.c , $Y_{a,0} \simeq \left(\frac{x^-}{x^+}\right)^L$ $u \to \infty$

$$\frac{Y^+_{a,s}\,Y^-_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})} \quad , \quad \underbrace{Y_{1,0}(u_j) = -1}_{\text{EXACT BAE}}$$

$$E = \sum_{a=1}^{\infty} \int_{-\infty}^{+\infty} \frac{du}{2\pi i}\, \partial_u \epsilon_a(u) \log(1+Y_{a,0}) + \sum_{j=1}^{M} \epsilon_1(u_j)$$

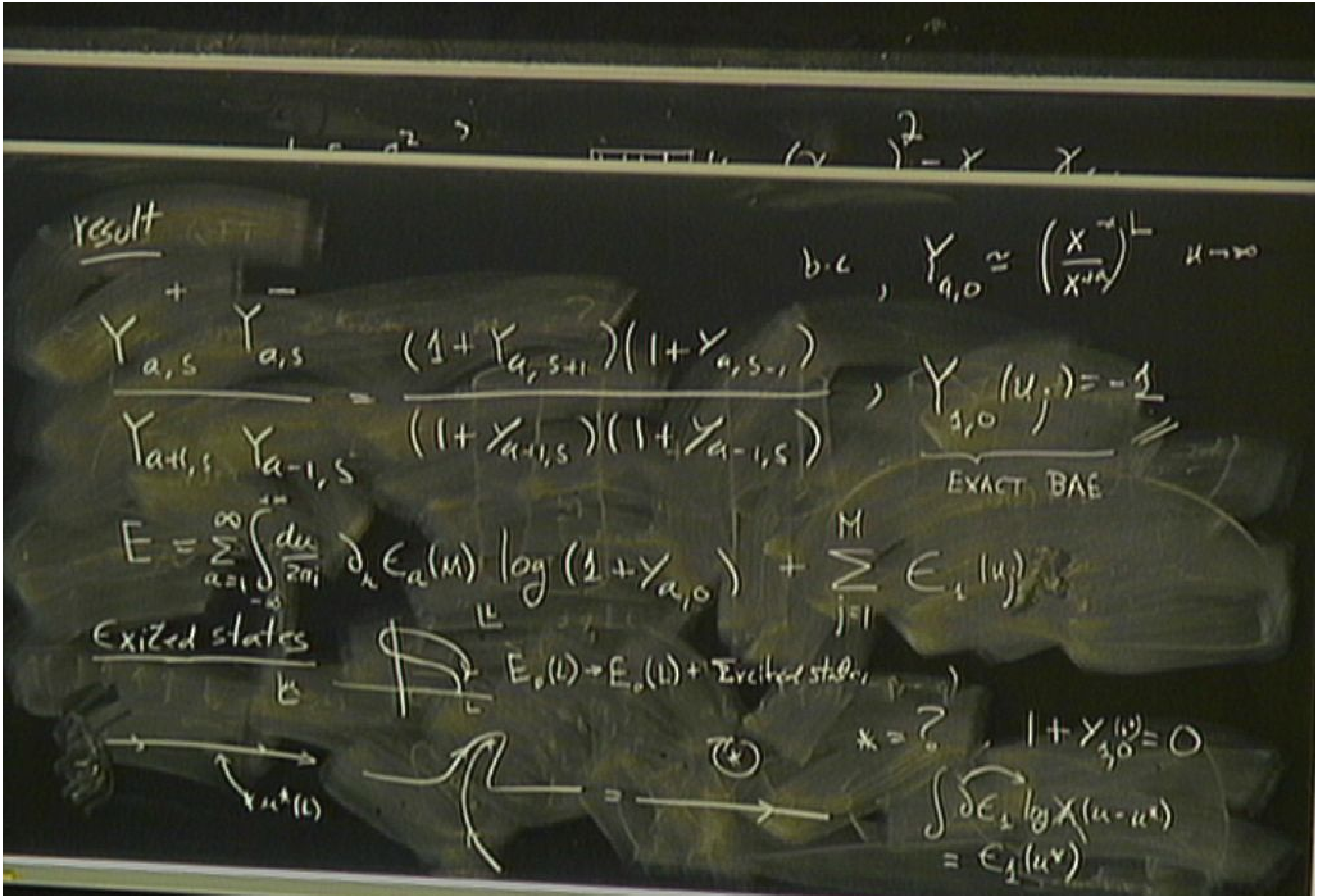
Exized states

$E_0(L) \to E_0(L) + $ Excited states

$* = ?$, $1+Y^{(*)}_{1,0} = 0$

$\int \partial\epsilon_1 \log X(u-u^*)$

$= \epsilon_1(u^*)$



result

$$\text{b.c.}\,,\quad Y_{1,0} \simeq \left(\frac{x^{-}}{x^{+}}\right)^{L} \quad u \to \infty$$

$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}\,, \quad \underbrace{Y_{1,0}(u_j) = -1}_{\text{EXACT BAE}}$$

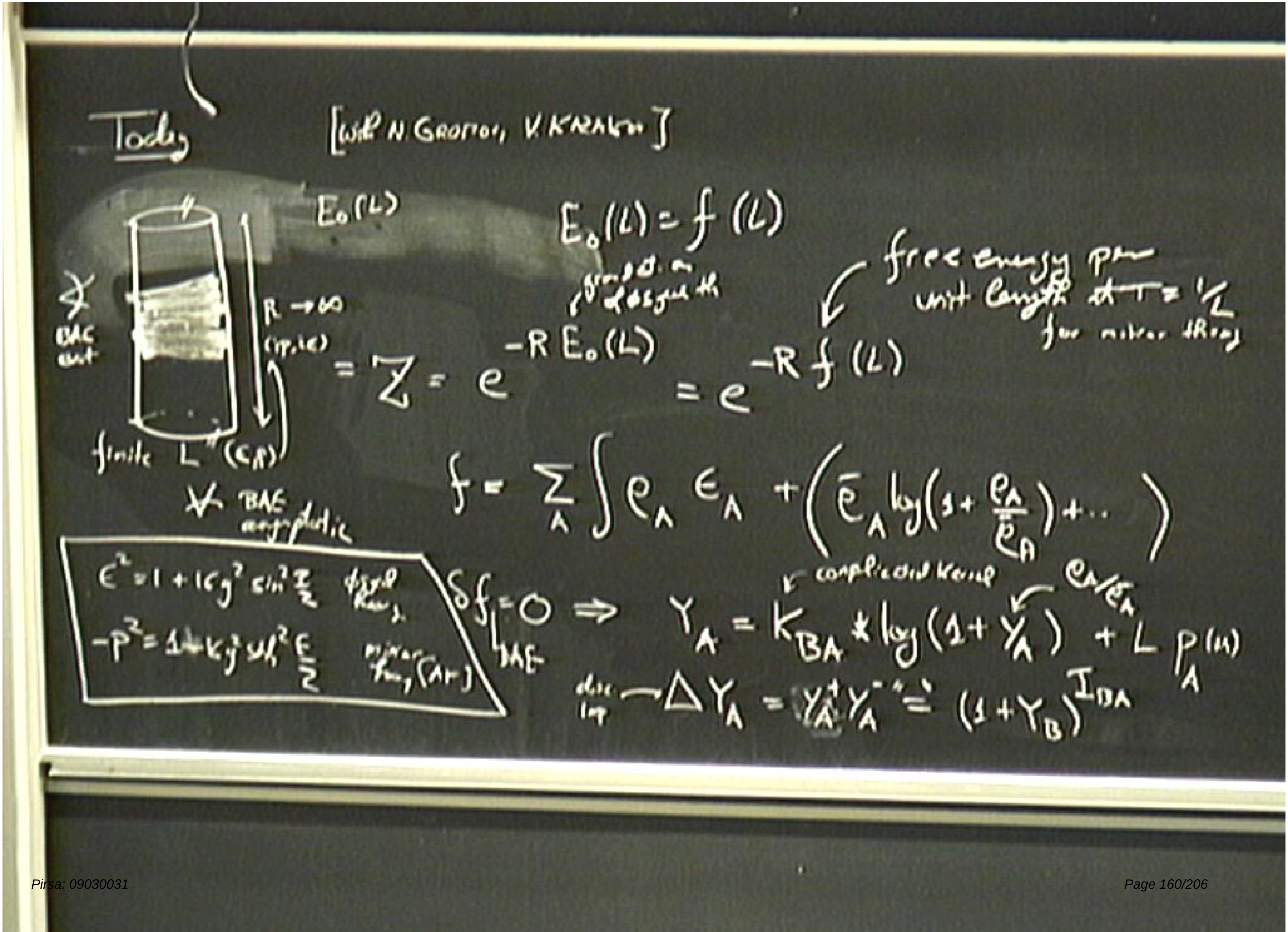
$$E = \sum_{a=1}^{\infty} \int_{-\infty}^{+\infty} \frac{du}{2\pi i}\, \partial_u \epsilon_a(u) \log(1+Y_{a,0}) + \sum_{j=1}^{M} \epsilon_1(u_j)$$

Exited states

$E_0(L) \to E_0(L)$ + Excited states

$\ast = ?\,, \quad 1 + Y^{(0)}_{1,0} = 0$

$$\int \partial \epsilon_1 \log X(u-u^\ast) = \epsilon_1(u^\ast)$$



Today [with N. Gromov, V. Kazakov]

∀ BAE asymptotic

$$E_0(L) = f(L)$$

ground st. en. of string th.

free energy per unit length at T = 1/L for mirror theory

$$= Z = e^{-R E_0(L)} = e^{-R f(L)}$$

$$f = \sum_A \int \rho_A \epsilon_A + \left(\bar\rho_A \log\left(1 + \frac{\rho_A}{\bar\rho_A}\right) + \dots \right)$$

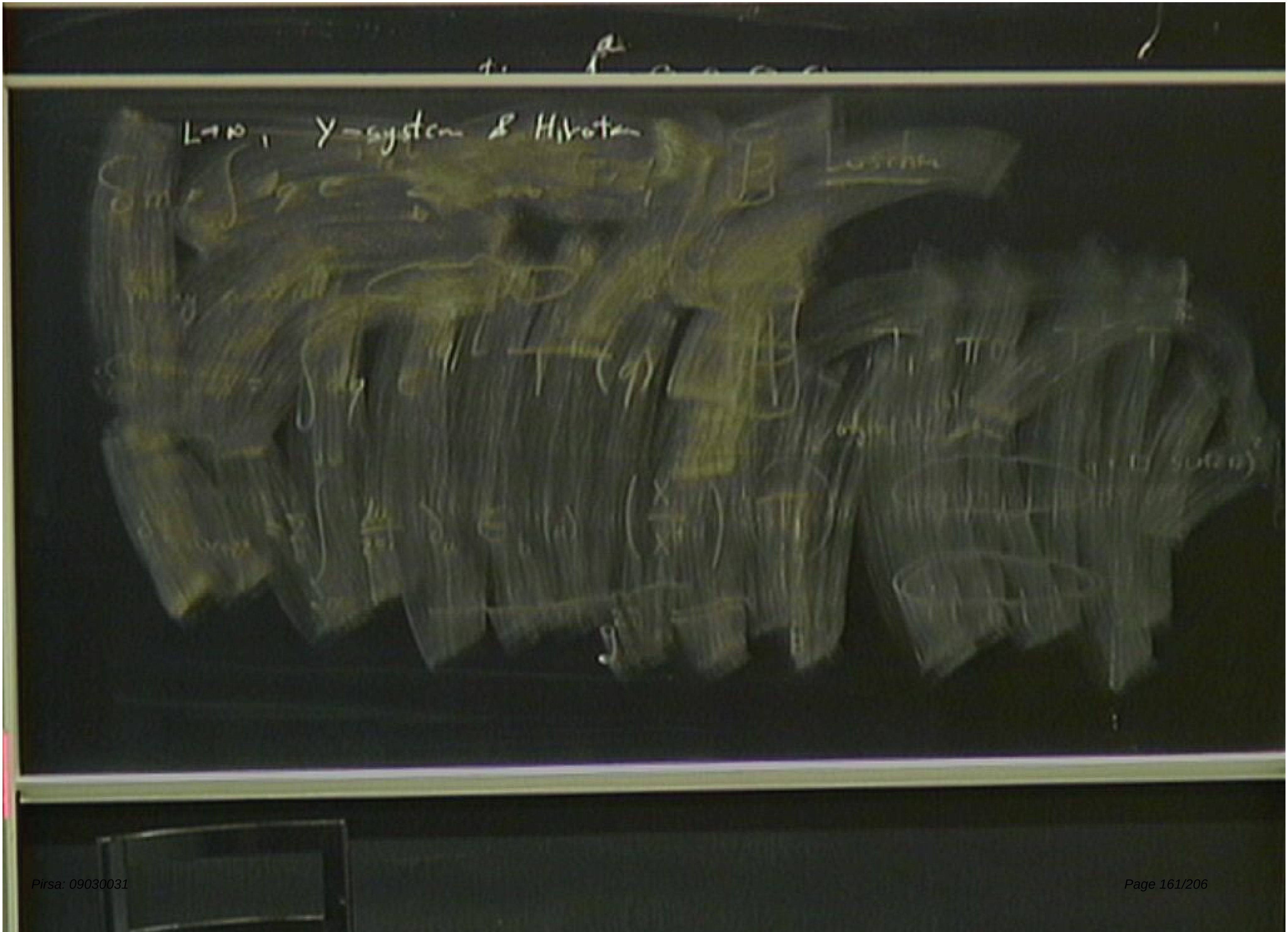
$$\boxed{\begin{array}{ll} \epsilon^2 = 1 + 16 g^2 \sin^2 \frac{p}{2} & \text{physical theory} \\ -\tilde p^2 = 1 + 16 g^2 \sinh^2 \frac{\tilde\epsilon}{2} & \text{mirror theory (AF)} \end{array}}$$

complicated kernel

$\rho_A/\bar\rho_A$

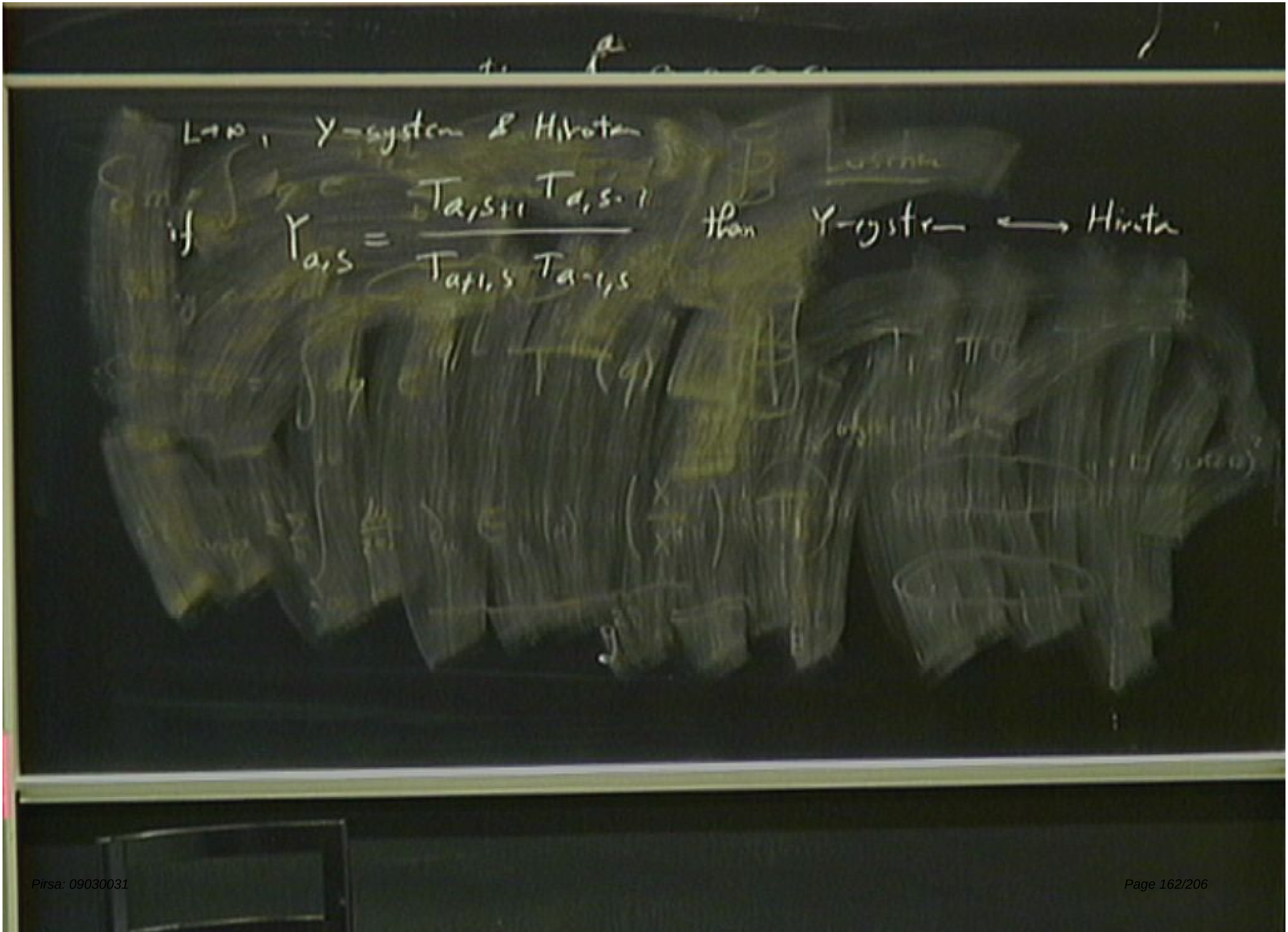
$$\delta f \Big|_{BAE} = 0 \Rightarrow Y_A = K_{BA} * \log(1 + Y_A) + L\, p_A^{(m)}$$

$$\text{disc. inf.} \longrightarrow \Delta Y_A = Y_A^+ Y_A^- \text{"="} (1 + Y_B)^{I_{BA}}$$



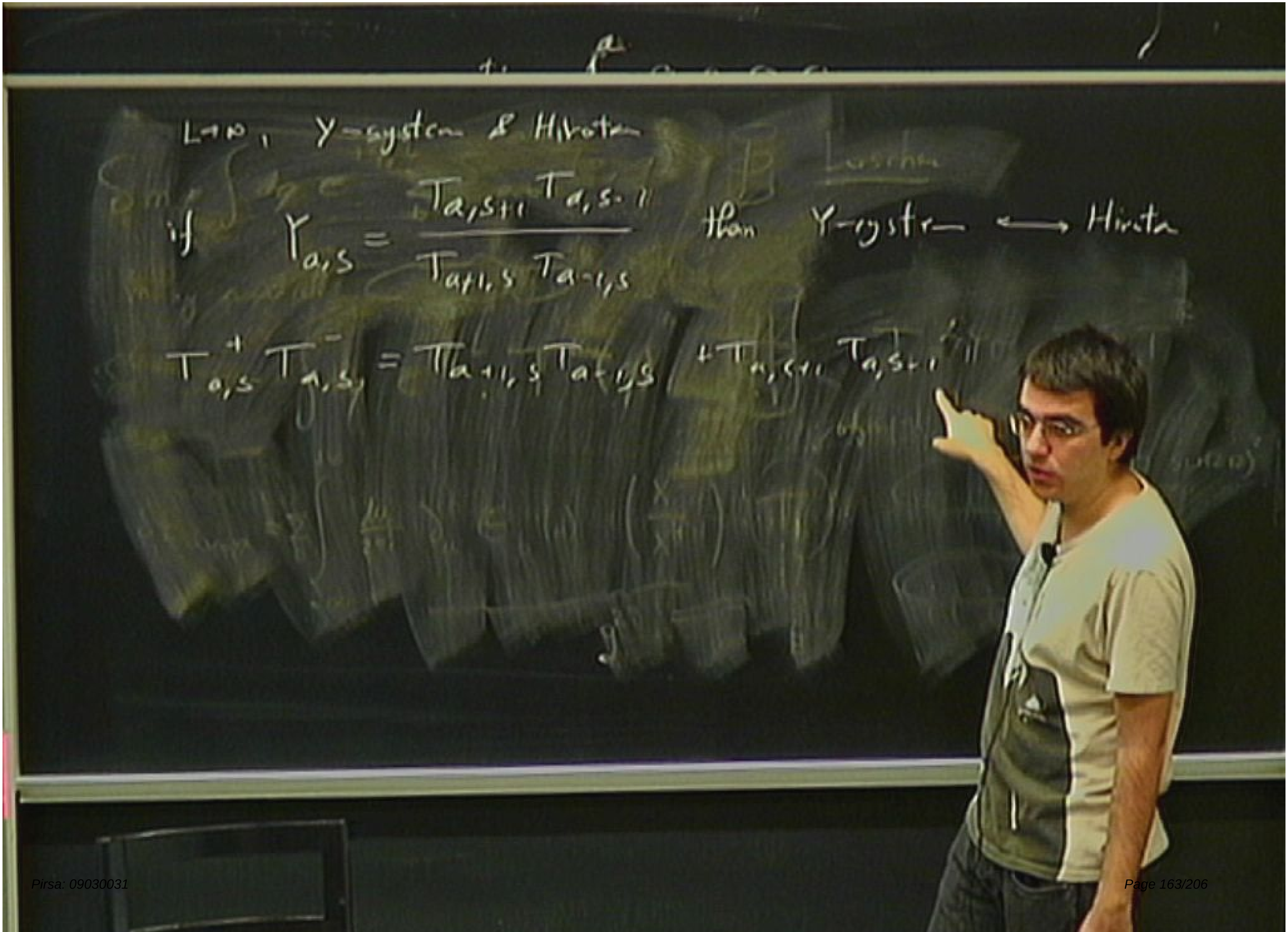
LNP, Y-system & Hirota

Luscher



LAW, Y-system & Hirota

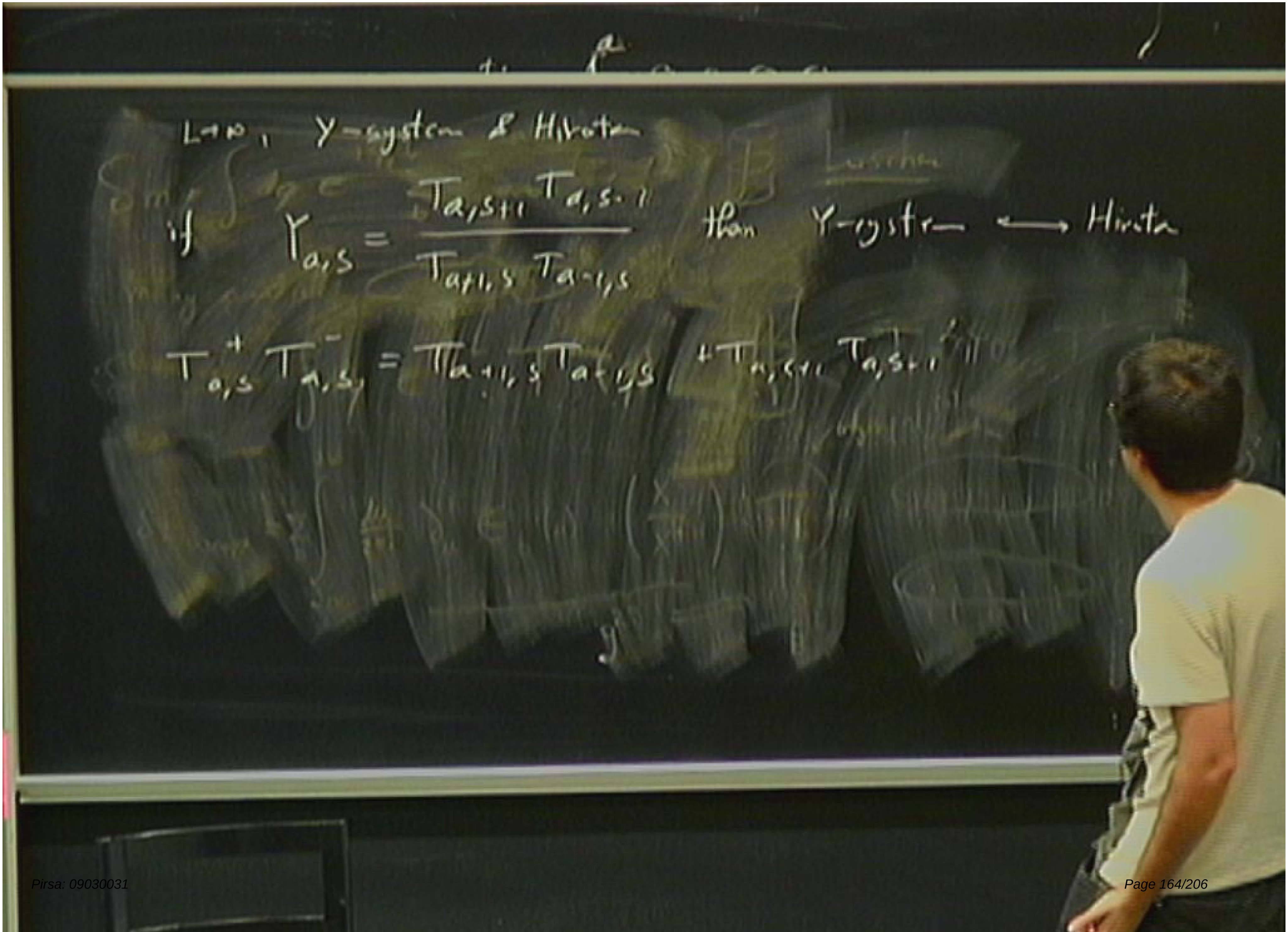
if $Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}}$ then Y-system $\longleftrightarrow$ Hirota



Lax, Y-system & Hirota

if $Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}$ than Y-system ⟷ Hirota

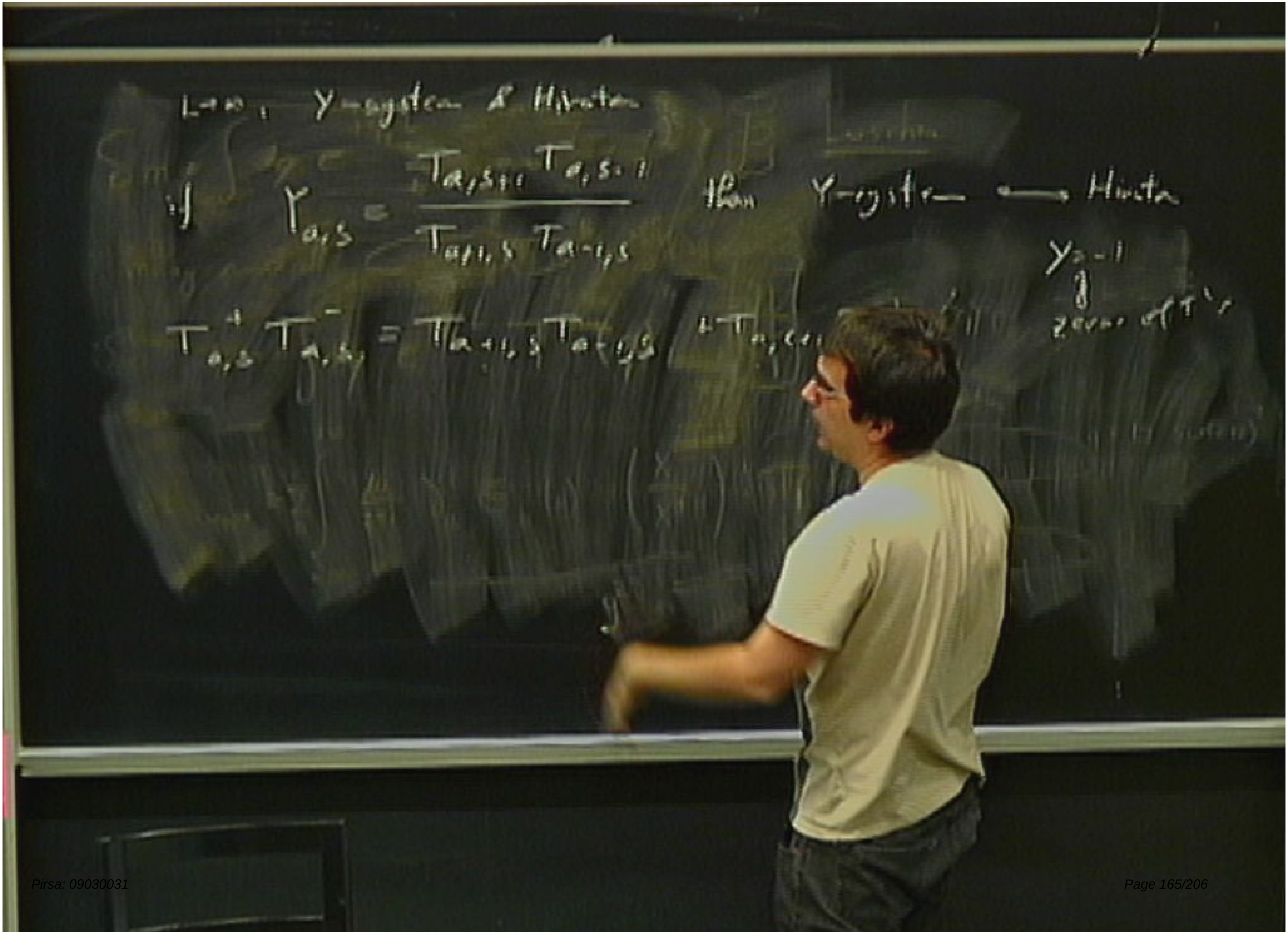
$T^{+}_{a,s}T^{-}_{a,s} = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s+1}$



Lam, Y-system & Hirota

if $Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}}$ than Y-system ⟷ Hirota

$$T^{+}_{a,s} T^{-}_{a,s} = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1}$$



Lem: Y-system & Hirota

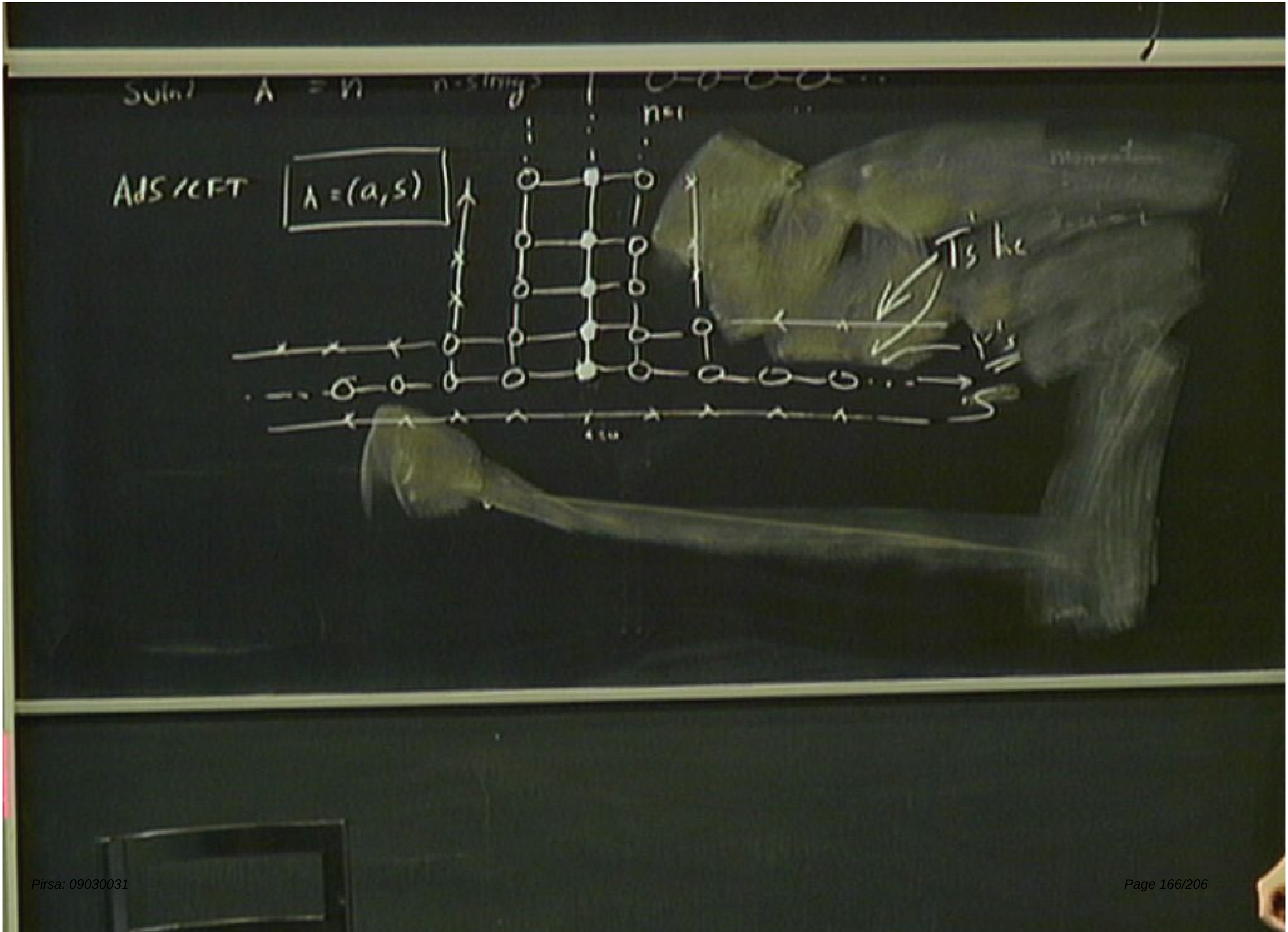
if $Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}$ then Y-system $\longleftrightarrow$ Hirota

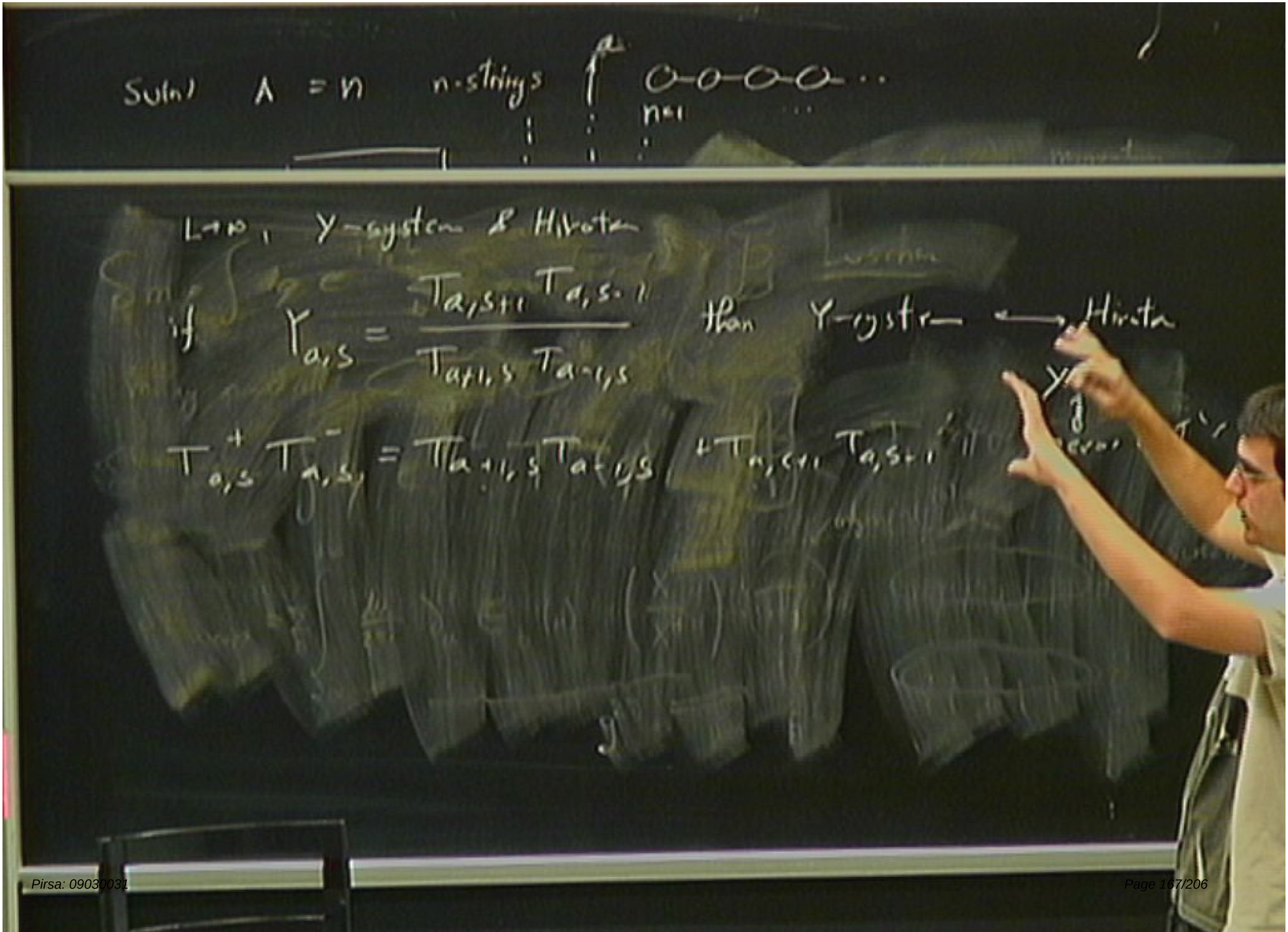
$Y_{a,s} = -1$

$\Updownarrow$

zeros of T's

$T^{+}_{a,s}T^{-}_{a,s} = T_{a+1,s}T_{a-1,s} + T_{a,s+1}$


SU(n) λ = n n-strings
n=1
AdS/CFT
λ = (a,s)
T_s
Y_s
s

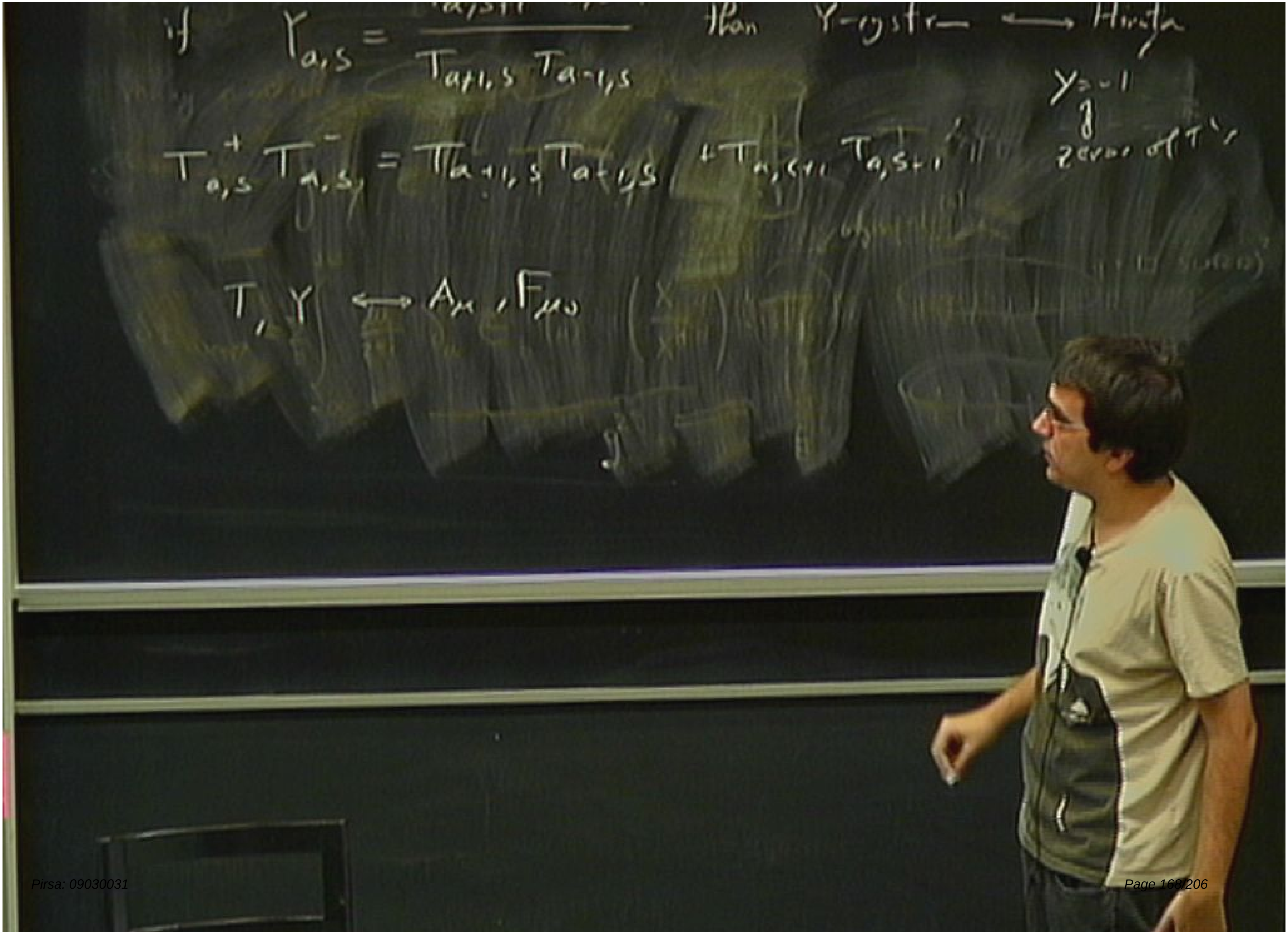


SU(n) $\Lambda = n$ n-strings

LAN, Y-system & Hirota

if $Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}}$ then Y-system ⟷ Hirota

$$T^{+}_{a,s} T^{-}_{a,s} = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1}$$



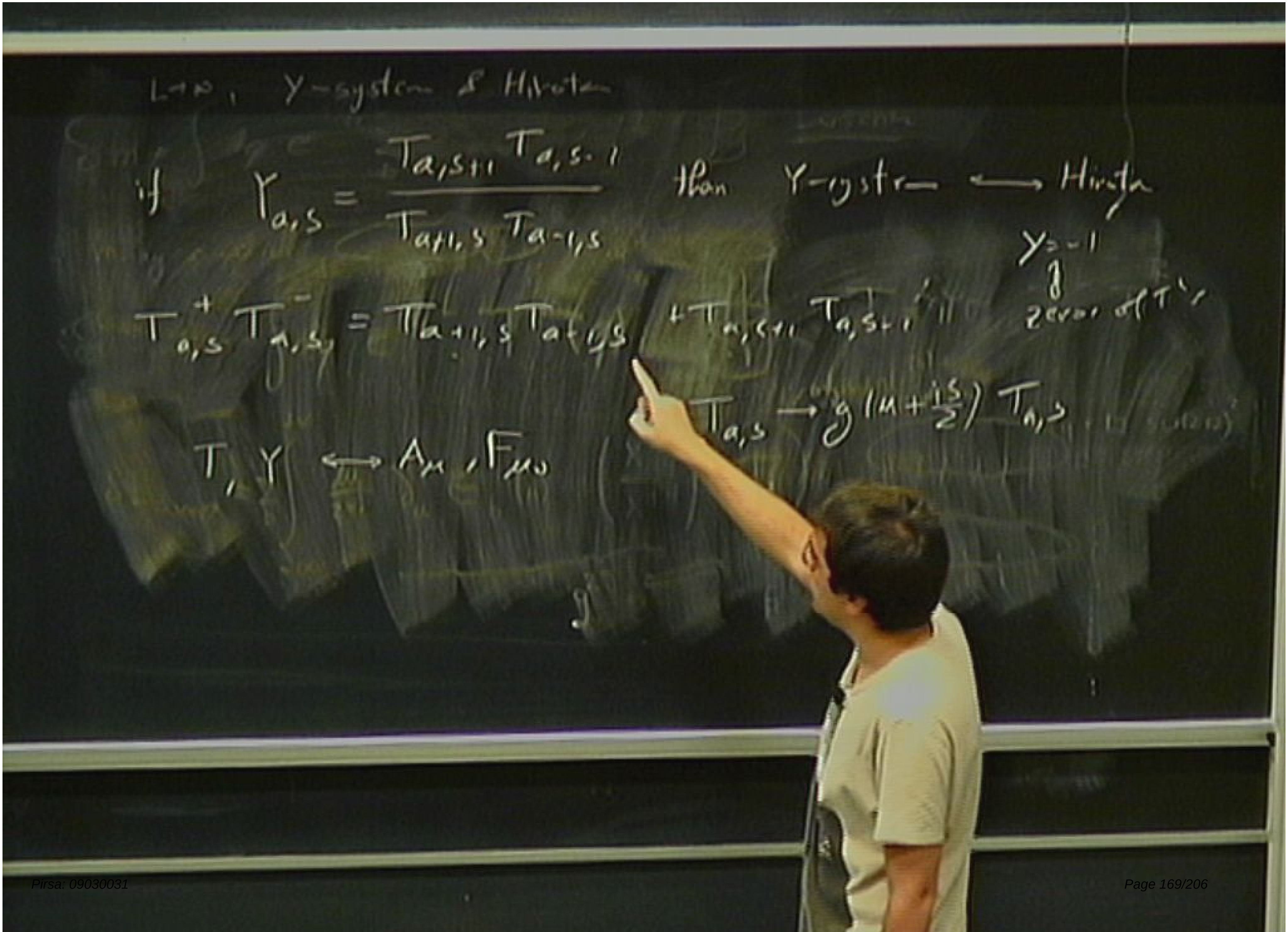
if $Y_{a,s} = \frac{\;_{a,s+1}}{T_{a+1,s}\, T_{a-1,s}}$ than Y-system $\longleftrightarrow$ Hirota

$Y = -1$

zeros of T's

$T^{+}_{a,s}\, T^{-}_{a,s} = T_{a+1,s}\, T_{a-1,s} + T_{a,s+1}\, T_{a,s+1}$

$T, Y \longleftrightarrow A_{\mu}, F_{\mu\nu}$



[illegible], Y-system & Hirota

if $Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}}$ then Y-system ⟷ Hirota

$Y = -1$

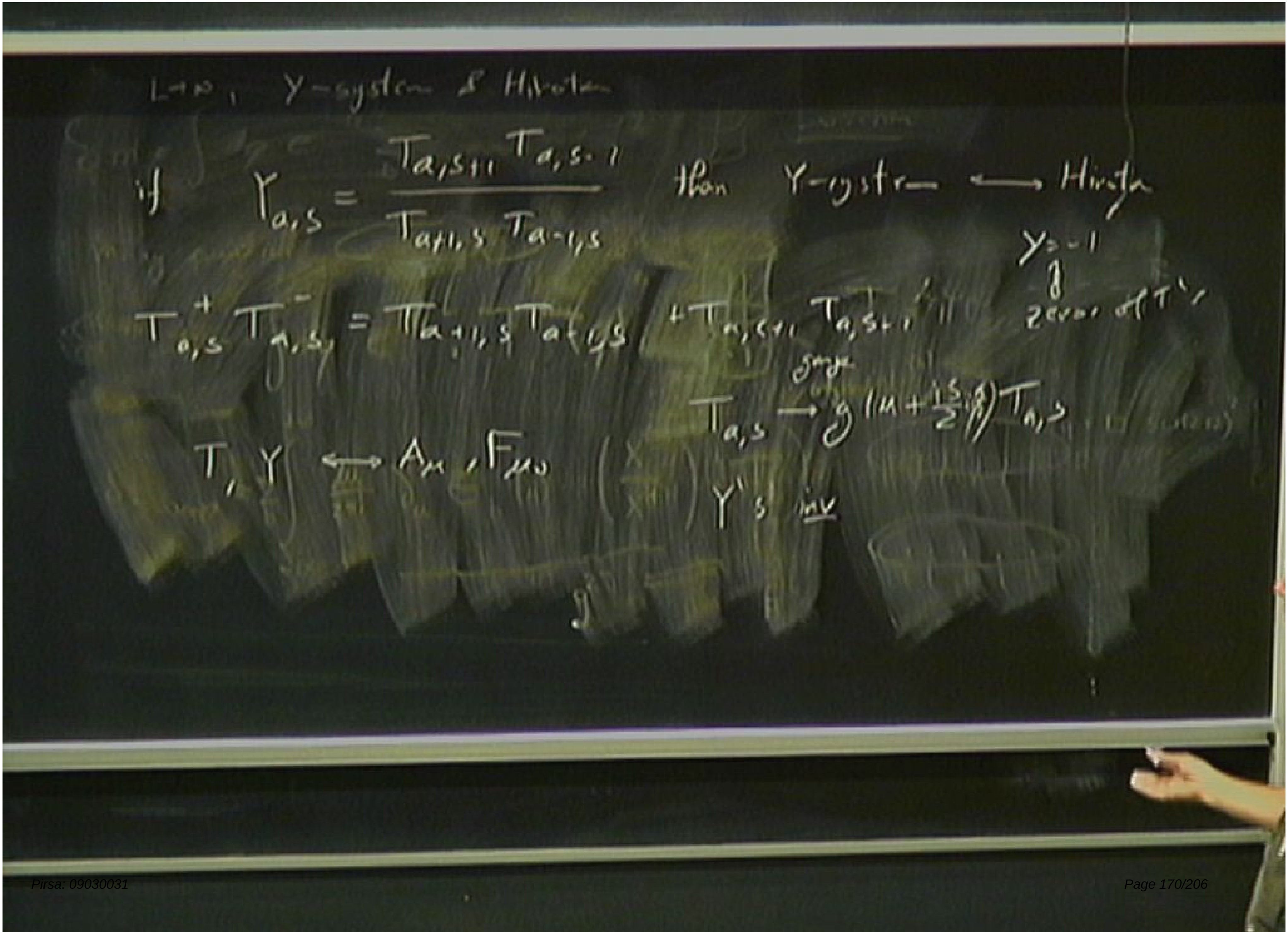
$\updownarrow$

zeros of T's

$T^+_{a,s} T^-_{a,s} = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1}$

$T_{a,s} \to g(u + \frac{is}{2}) T_{a,s}$

$T, Y \longleftrightarrow A_\mu, F_{\mu\nu}$



L+N, Y-system & Hirota

if $Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}}$ then Y-system ⟷ Hirota

$Y_s = -1$

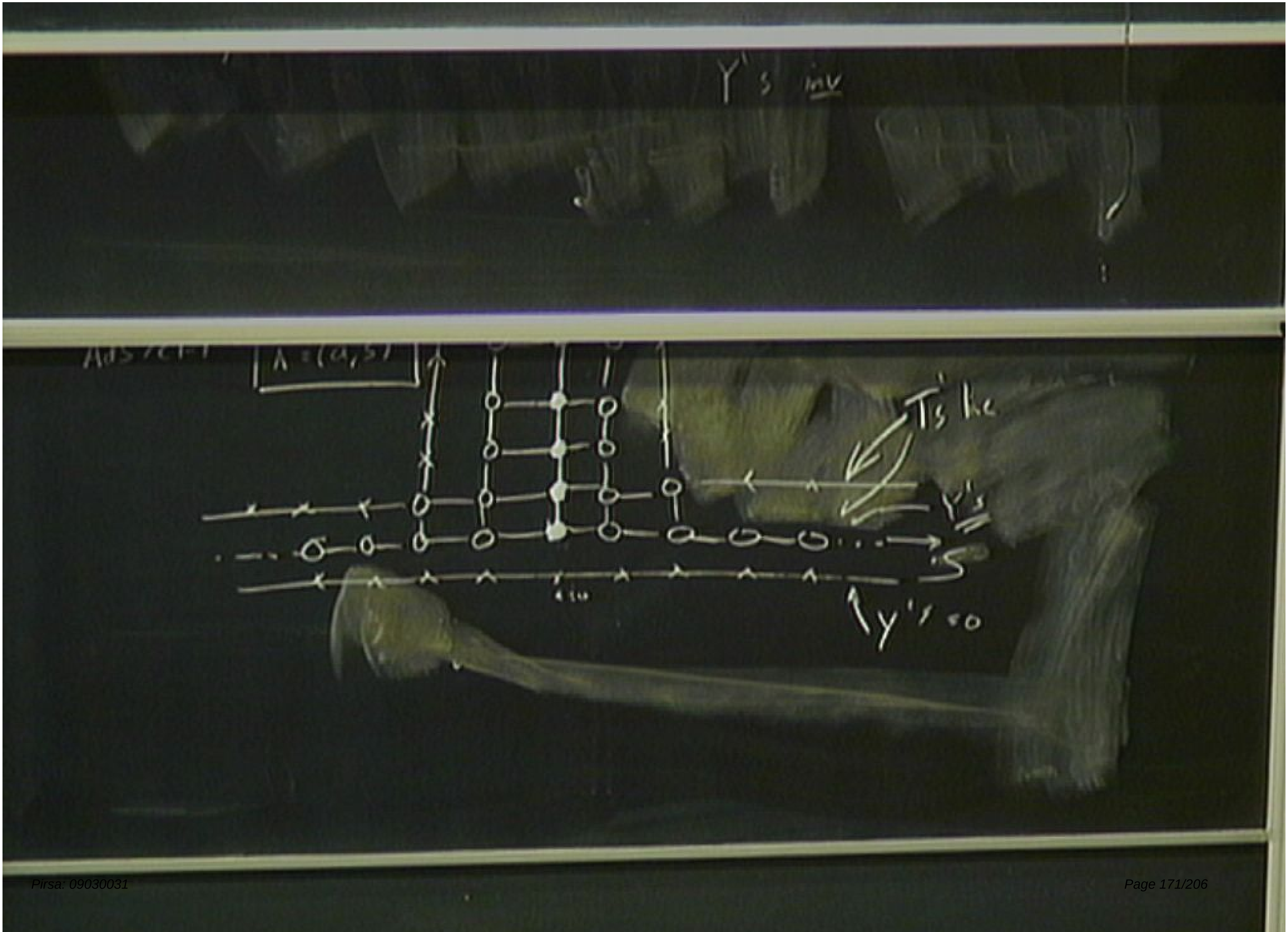
zeros of T's

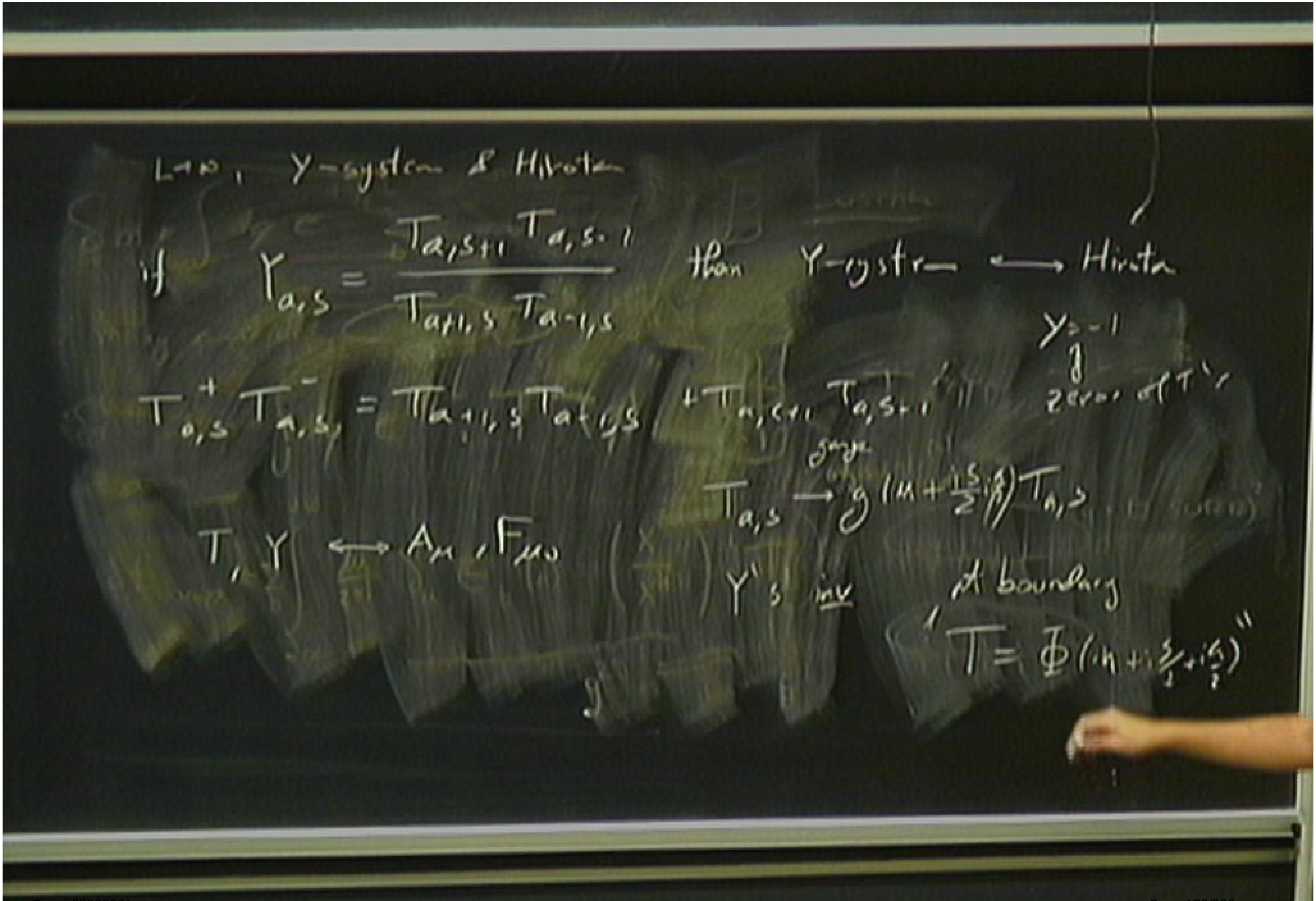
$T^+_{a,s} T^-_{a,s} = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1}$

$T_{a,s} \xrightarrow{\text{gauge}} g\left(u+\frac{is}{2}\right) T_{a,s}$

$T, Y \longleftrightarrow A_\mu, F_{\mu\nu}$

Y's inv


Y's inv
Λ=(a,s)
T's
Y's
S
y's=0



LAW, Y-system & Hirota

if $Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}}$ then Y-system $\longleftrightarrow$ Hirota

$T^{+}_{a,s} T^{-}_{a,s} = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s+1}$

$Y_s - 1$

$\updownarrow$

zeros of T's

$T, Y \longleftrightarrow A_\mu, F_{\mu\nu}$

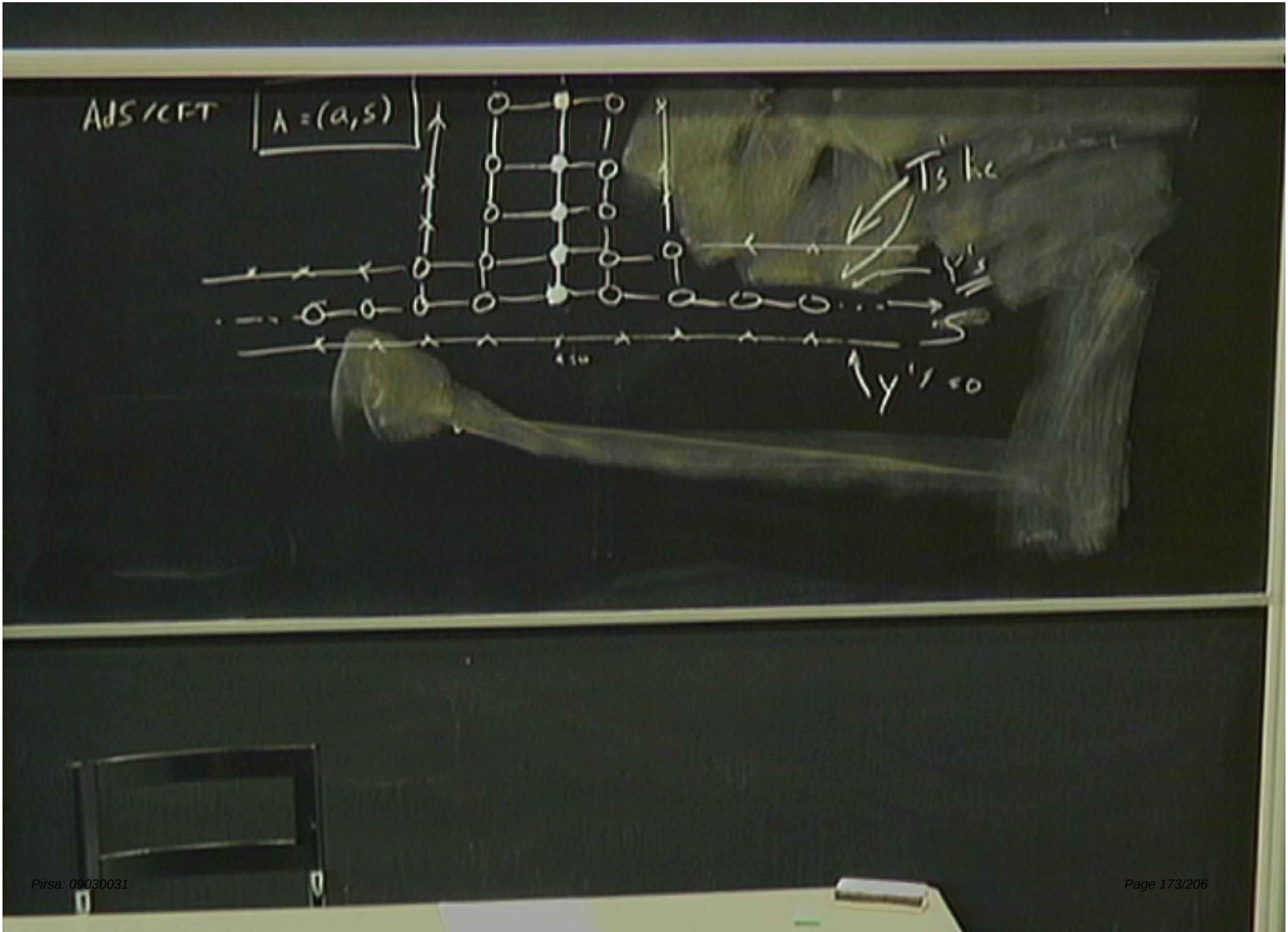
gauge

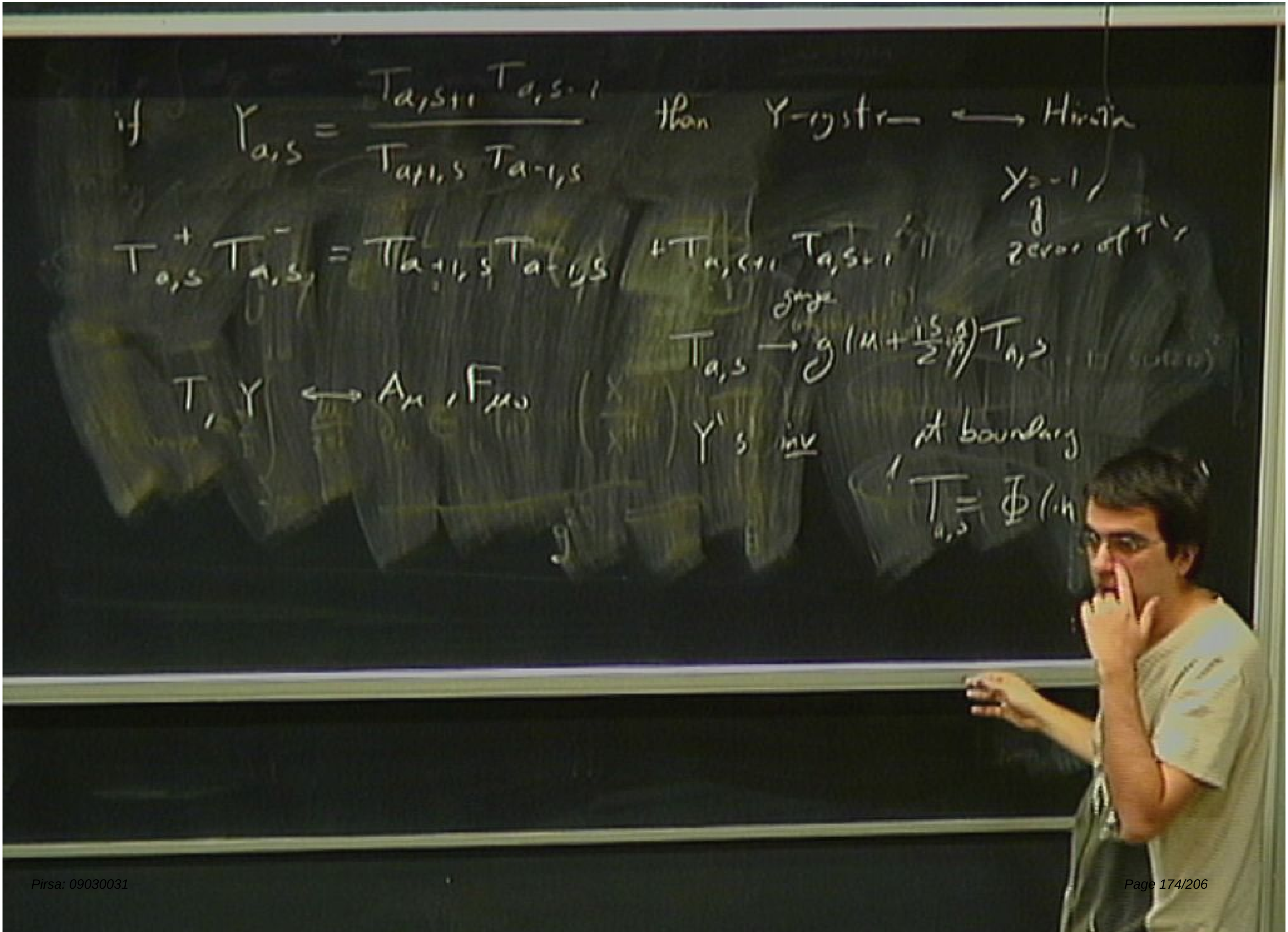
$T_{a,s} \longrightarrow g\left(u + i\frac{s}{2} + i\frac{a}{2}\right) T_{a,s}$

Y's inv

at boundary

"$T = \Phi\left(u + i\frac{s}{2} + i\frac{a}{2}\right)$"


AdS/CFT
Λ = (a,s)
T's live
Y's
S
Y's = 0



if $Y_{a,s} = \frac{T_{a,s+1}\,T_{a,s-1}}{T_{a+1,s}\,T_{a-1,s}}$ then Y-system ⟷ Hirota

$Y_{?} = -1$

zeros of T's

$T^+_{a,s}\,T^-_{a,s} = T_{a+1,s}\,T_{a-1,s} + T_{a,s+1}\,T_{a,s+1}$

$T_{a,s} \xrightarrow{\text{gauge}} g\left(u+\frac{is}{2}\,\frac{ia}{2}\right)T_{a,s}$

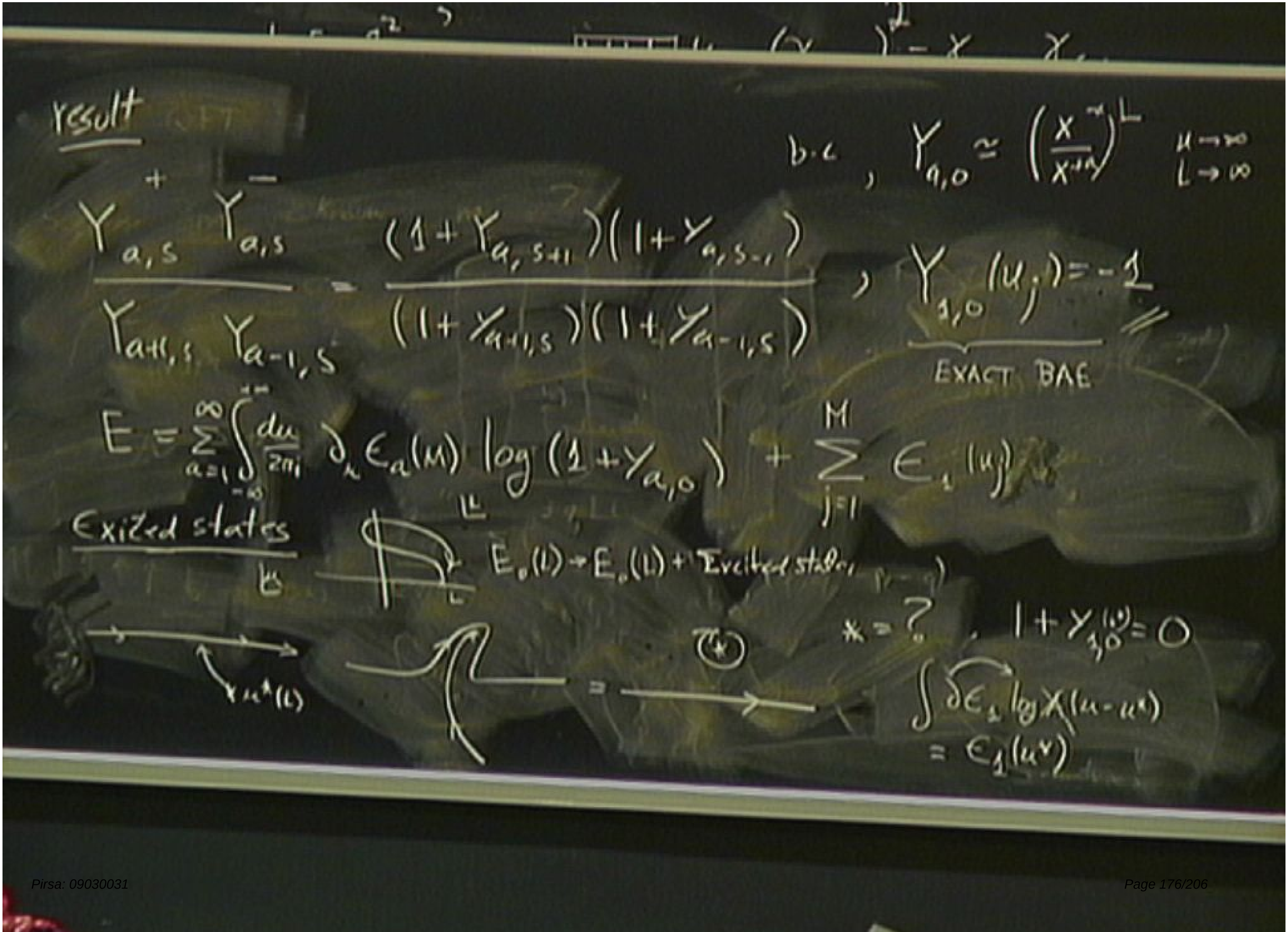
$T, Y \longleftrightarrow A_\mu, F_{\mu\nu}$

Y's inv

at boundary

$T_{a,s} = \Phi(u$

$L \to \infty$



result

$$\frac{Y^{+}_{a,s}\,Y^{-}_{a,s}}{Y_{a+1,s}\,Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}\,,\quad \underbrace{Y_{1,0}(u_j) = -1}_{\text{EXACT BAE}}$$

b.c , $Y_{a,0} \simeq \left(\frac{x^{-}}{x^{+}}\right)^{L}$ $u \to \infty$ $L \to \infty$

$$E = \sum_{a=1}^{\infty} \int_{-\infty}^{+\infty} \frac{du}{2\pi i}\, \partial_u \epsilon_a(u) \log(1+Y_{a,0}) + \sum_{j=1}^{M} \epsilon_1(u_j)$$

Exized states

$E_0(L) \to E_0(L) +$ Excited states

$\times u^{*}(L)$

$\circledast$ $\ast = ?$, $1+Y^{(0)}_{1,0} = 0$

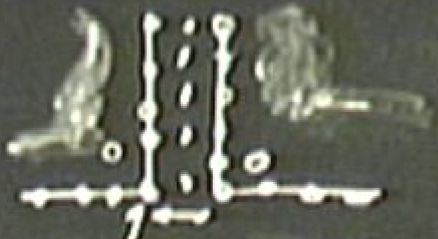
$$\int \partial \epsilon_1 \log X(u-u^{*}) = \epsilon_1(u^{*})$$

$L \to \infty$

$Y_{a,0} \ll 1$

$L \to \infty$

$Y_{a,0} \ll 1$



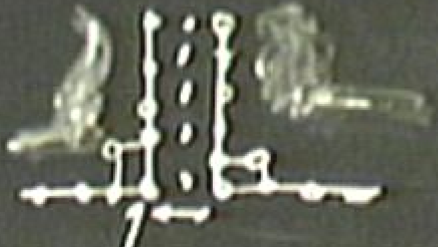
$1 + \xcancel{Y_{a,0}}$

$L \to \infty$

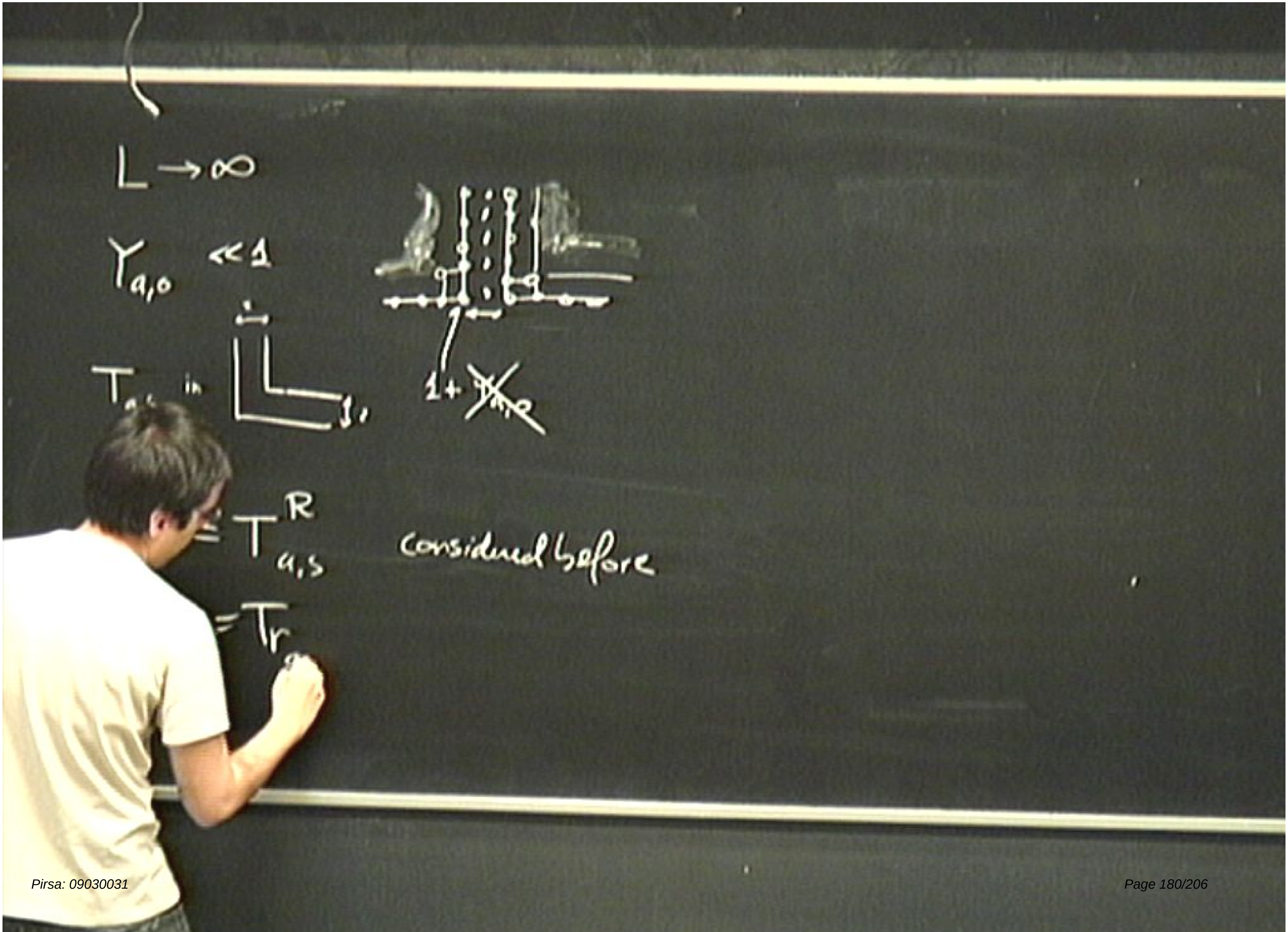
$Y_{a,0} \ll 1$

$\doteq$

$T_{a,s}$ in



$1+\cancel{Y_{1,2}}$



$L \to \infty$

$Y_{a,0} \ll 1$

$T_{a,}$ in $\overset{\dot{-}}{\llcorner}_{3,1}$

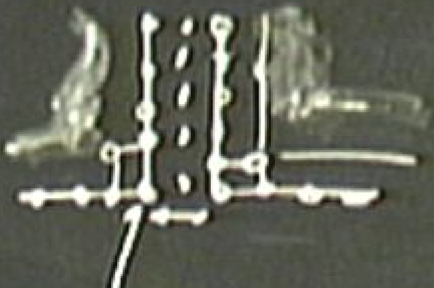
$1 + \cancel{Y_{1,2}}$

$= T^{R}_{a,s}$ considered before

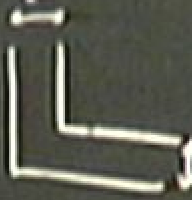
$= \mathrm{Tr}$

$L \to \infty$

$Y_{a,0} \ll 1$



$1 + \cancel{Y_{a,0}}$

$T_{a,s}$ in  3,

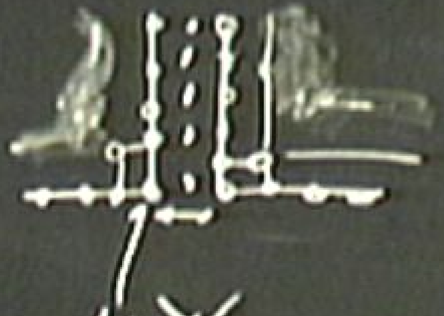
$\Downarrow$

$T_{a,s} = T^{R}_{a,s}$ considered before

$= \mathrm{Tr}_{a,s}\, S \ldots\, S$

$L \to \infty$

$Y_{a,0} \ll 1$



$\dot{-}$

$T_{a,s}$ in  3,

$1 + \cancel{Y_{1,2}}$

$\Downarrow$

$T_{a,s} \simeq T^{R}_{a,s}$ considered before

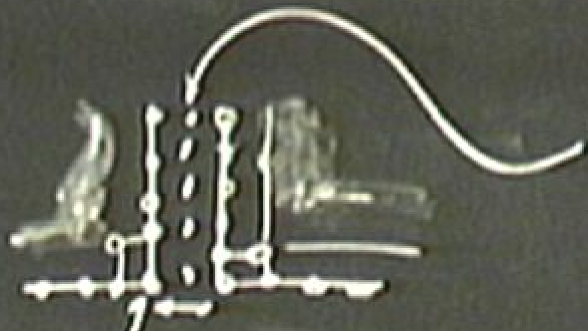
$= \mathrm{Tr}_{a,s}\, S \ldots S$

$L \to \infty$

$Y_{a,0} \ll 1$

$\doteq$

$T_{a,s}$ in



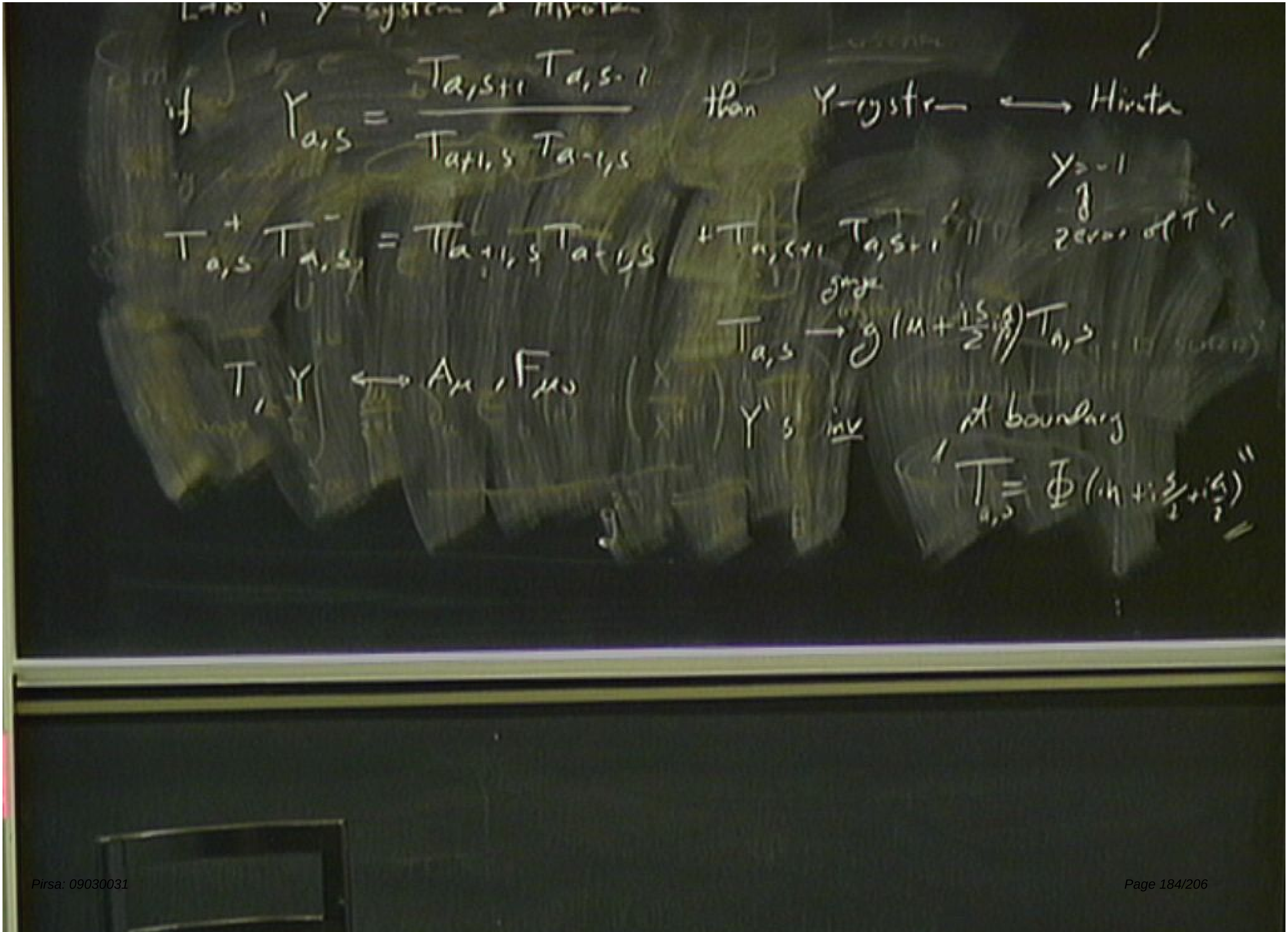
$1+\cancel{Y_{a,2}}$

$$\frac{Y^{+}_{a,0}\,Y^{-}_{a,0}}{Y_{a+1,0}\,Y_{a-1,0}} \cdot \frac{(1+Y_{a,1})(1+Y_{a,-1})}{(1+\cancel{Y_{a,0}})(1+\cancel{Y_{a,0}})}$$

$\Downarrow$

$T_{a,s} \simeq T^{R}_{a,s}$ considered before

$= \mathrm{Tr}_{a,s}\, S \ldots\ldots S$



Y-system & Hirota

if $Y_{a,s} = \frac{T_{a,s+1}\,T_{a,s-1}}{T_{a+1,s}\,T_{a-1,s}}$ then Y-system ⟷ Hirota

$Y_s - 1$

≀

zeros of T's

$$T^+_{a,s}\,T^-_{a,s} = T_{a+1,s}\,T_{a-1,s} + T_{a,s+1}\,T_{a,s+1}$$

gauge

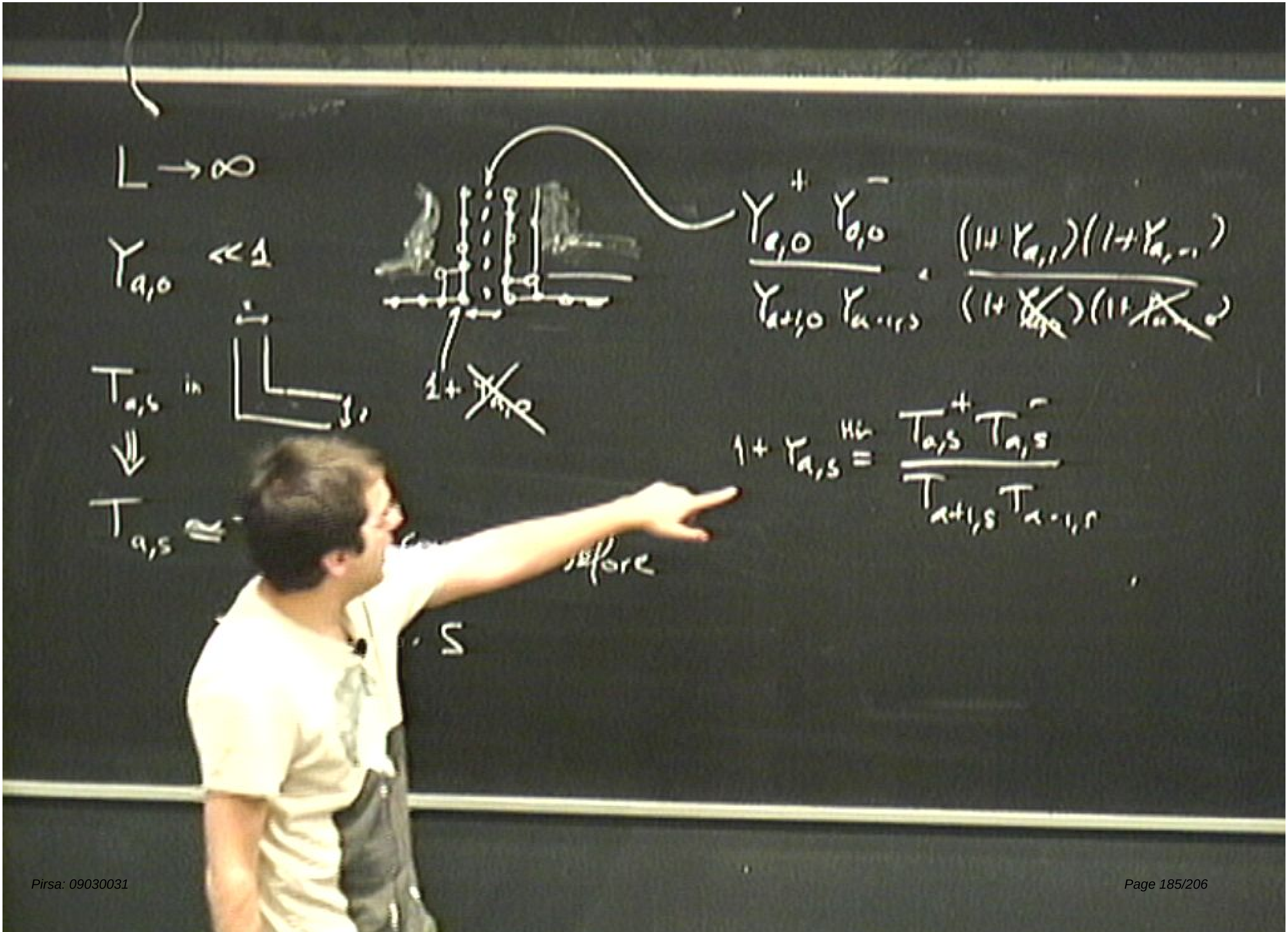
$$T_{a,s} \to g\left(u+\frac{is}{2}+\frac{ia}{2}\right)T_{a,s}$$

$$T, Y \longleftrightarrow A_\mu, F_{\mu\nu}$$

Y's inv

at boundary

"$T_{a,s} = \Phi\left(u+i\frac{s}{2}+i\frac{a}{2}\right)$"



$L \to \infty$

$Y_{a,0} \ll 1$

$\doteq$

$T_{a,s}$ in 3,

$\Downarrow$

$T_{a,s} \simeq$

before

S

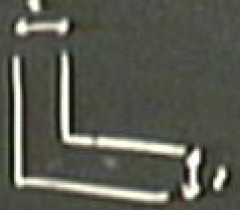
$$\frac{Y^{+}_{a,0}\,Y^{-}_{a,0}}{Y_{a+1,0}\,Y_{a-1,0}} \cdot \frac{(1+Y_{a,1})(1+Y_{a,-1})}{(1+\cancel{Y_{a+1,0}})(1+\cancel{Y_{a-1,0}})}$$

$1+\cancel{Y_{a,0}}$

$$1+Y_{a,s} \overset{\text{Hb}}{=} \frac{T^{+}_{a,s}\,T^{-}_{a,s}}{T_{a+1,s}\,T_{a-1,s}}$$

$L \to \infty$

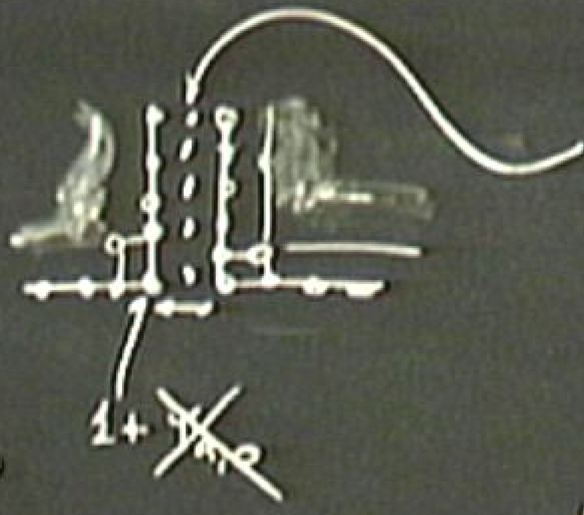
$Y_{a,0} \ll 1$

$T_{a,s}$ in 

$\Downarrow$

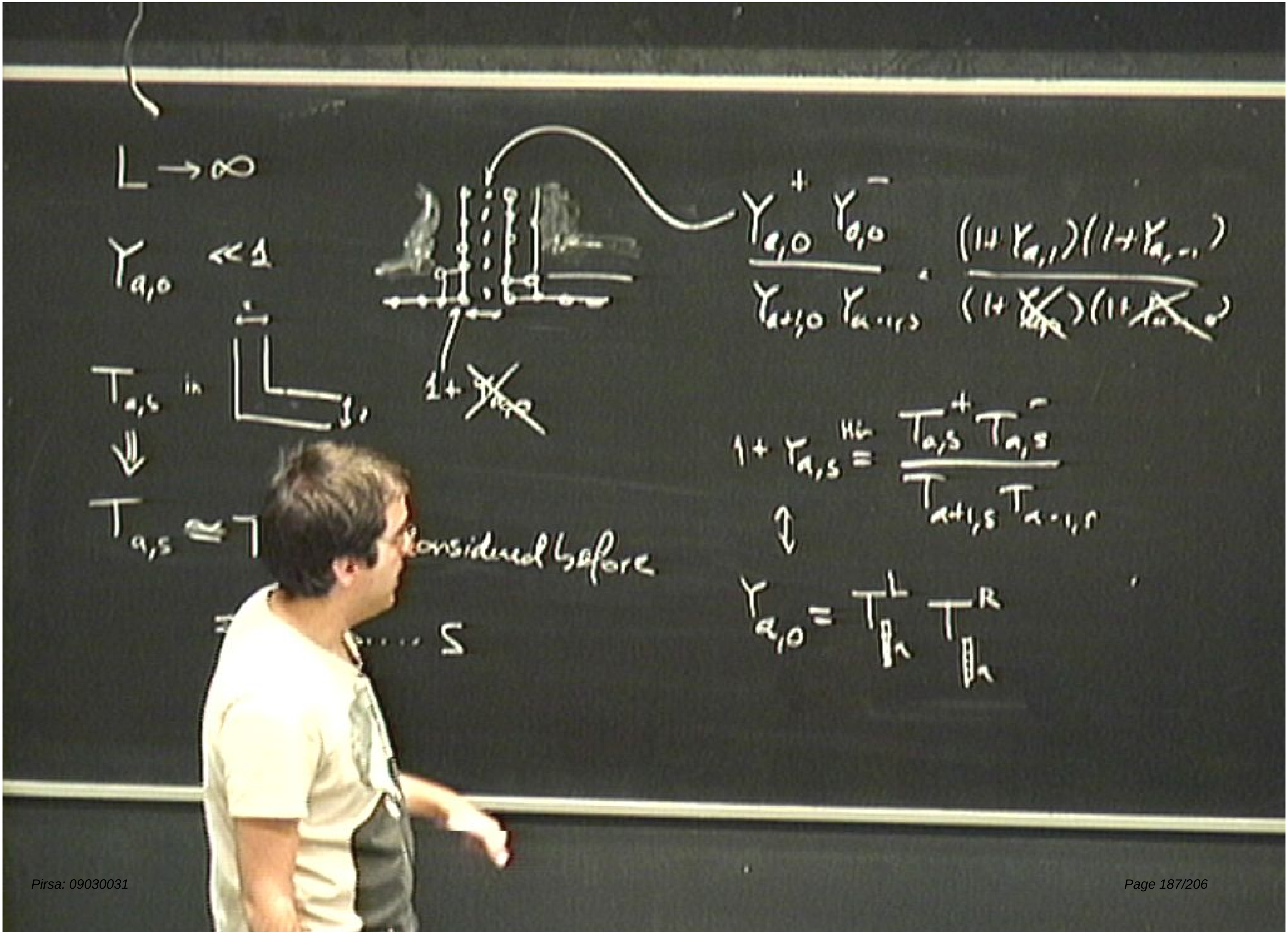
$T_{a,s} \simeq T^{R}_{a,s}$ considered before

$= \mathrm{Tr}_{a,s}\, S \dots S$



$$\frac{Y^{+}_{a,0}\, Y^{-}_{a,0}}{Y_{a+1,0}\, Y_{a-1,s}} \cdot \frac{(1+Y_{a,1})(1+Y_{a,-1})}{(1+\cancel{Y_{a+1,0}})(1+\cancel{Y_{a-1,0}})}$$

$$1+Y_{a,s} \overset{\text{Hir}}{=} \frac{T^{+}_{a,s}\, T^{-}_{a,s}}{T_{a+1,s}\, T_{a-1,s}}$$



$L \to \infty$

$Y_{a,0} \ll 1$

$\doteq$

$T_{a,s}$ in $s,$

$\Downarrow$

$T_{a,s} \simeq T$ considered before

$= \dots S$

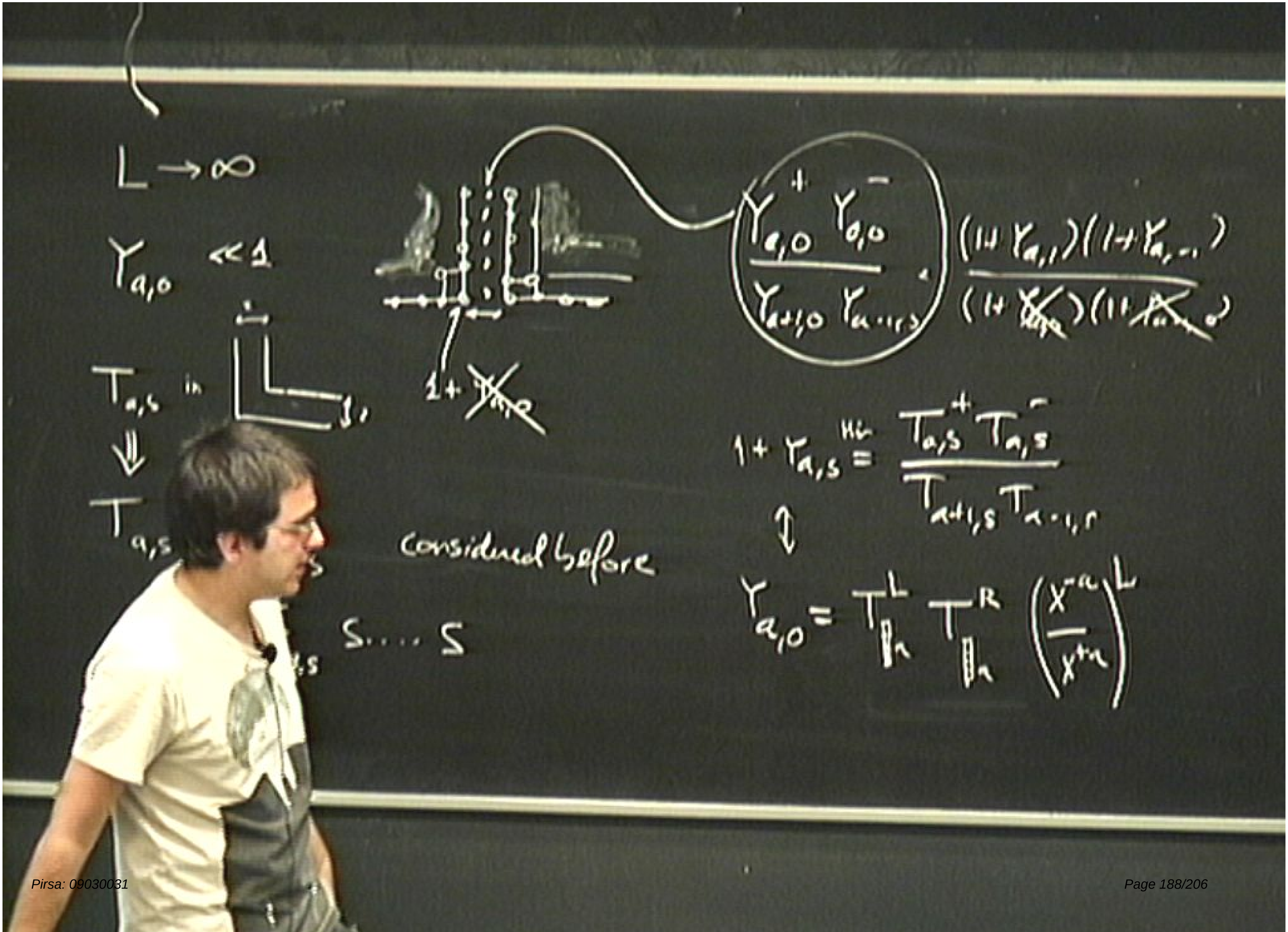
$1 + \cancel{Y_{a,0}}$

$$\frac{Y^{+}_{a,0}\, Y^{-}_{a,0}}{Y_{a+1,0}\, Y_{a-1,0}} \cdot \frac{(1+Y_{a,1})(1+Y_{a,-1})}{(1+\cancel{Y_{a,0}})(1+\cancel{Y_{a,0}})}$$

$$1 + Y_{a,s}^{Hh} = \frac{T^{+}_{a,s}\, T^{-}_{a,s}}{T_{a+1,s}\, T_{a-1,s}}$$

$\Updownarrow$

$$Y_{a,0} = T^{L}_{\Box_n}\, T^{R}_{\Box_n}$$



$L \to \infty$

$Y_{a,0} \ll 1$

$T_{a,s}$ in

$\Downarrow$

$T_{a,s}$

considered before

$S \ldots S$

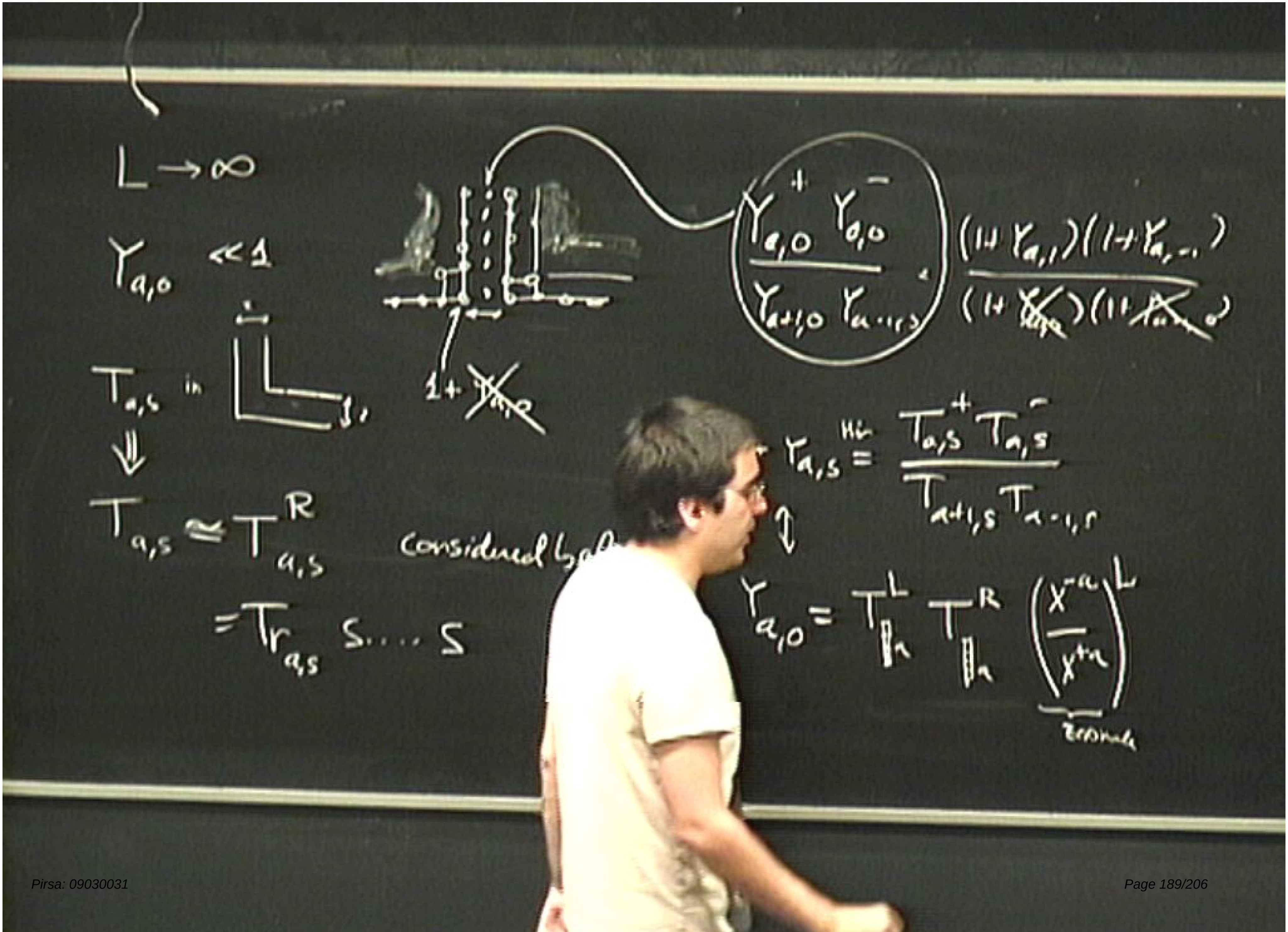
$1 + \cancel{Y_{1,2}}$

$$\frac{Y^{+}_{a,0}\,Y^{-}_{a,0}}{Y_{a+1,0}\,Y_{a-1,s}} = \frac{(1+Y_{a,1})(1+Y_{a,-1})}{(1+\cancel{Y_{a+1,0}})(1+\cancel{Y_{a-1,0}})}$$

$$1 + Y_{a,s} \overset{\text{Hir}}{=} \frac{T^{+}_{a,s}\,T^{-}_{a,s}}{T_{a+1,s}\,T_{a-1,s}}$$

$\Updownarrow$

$$Y_{a,0} = T^{L}_{\|a}\,T^{R}_{\|a}\left(\frac{x^{-a}}{x^{+a}}\right)^{L}$$



$L \to \infty$

$Y_{a,0} \ll 1$

$T_{a,s}$ in $\dot{=}$ L-shaped domain

$\Downarrow$

$T_{a,s} \simeq T^{R}_{a,s}$ considered bef

$= \mathrm{Tr}_{a,s}\, S \ldots S$

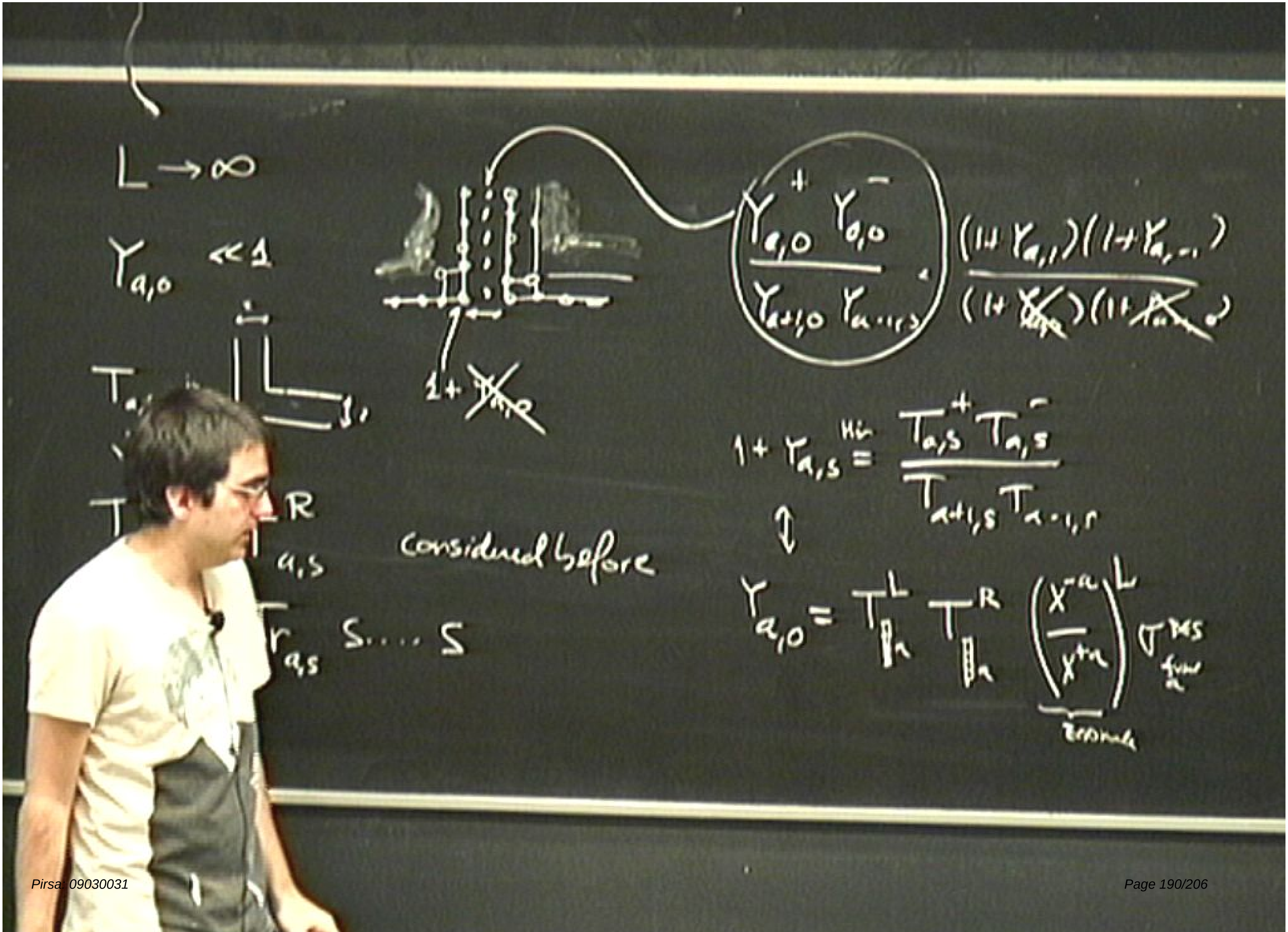
$1+\cancel{Y_{1,2}}$

$$\frac{Y^{+}_{a,0}\,Y^{-}_{a,0}}{Y_{a+1,0}\,Y_{a-1,0}} = \frac{(1+Y_{a,1})(1+Y_{a,-1})}{(1+\cancel{Y_{a,0}})(1+\cancel{Y_{a,0}})}$$

$$Y^{\mathrm{Hir}}_{a,s} = \frac{T^{+}_{a,s}\,T^{-}_{a,s}}{T_{a+1,s}\,T_{a-1,s}}$$

$\Updownarrow$

$$Y_{a,0} = T^{L}_{\|n}\,T^{R}_{\|n}\,\underbrace{\left(\frac{x^{-n}}{x^{+n}}\right)^{L}}_{\text{Zeeman}}$$



$L \to \infty$

$Y_{a,0} \ll 1$

$\doteq$

$T_{a,}$ … $3,$

$1 + \cancel{Y_{\dots}}$

T … R

$T_{a,s}$ considered before

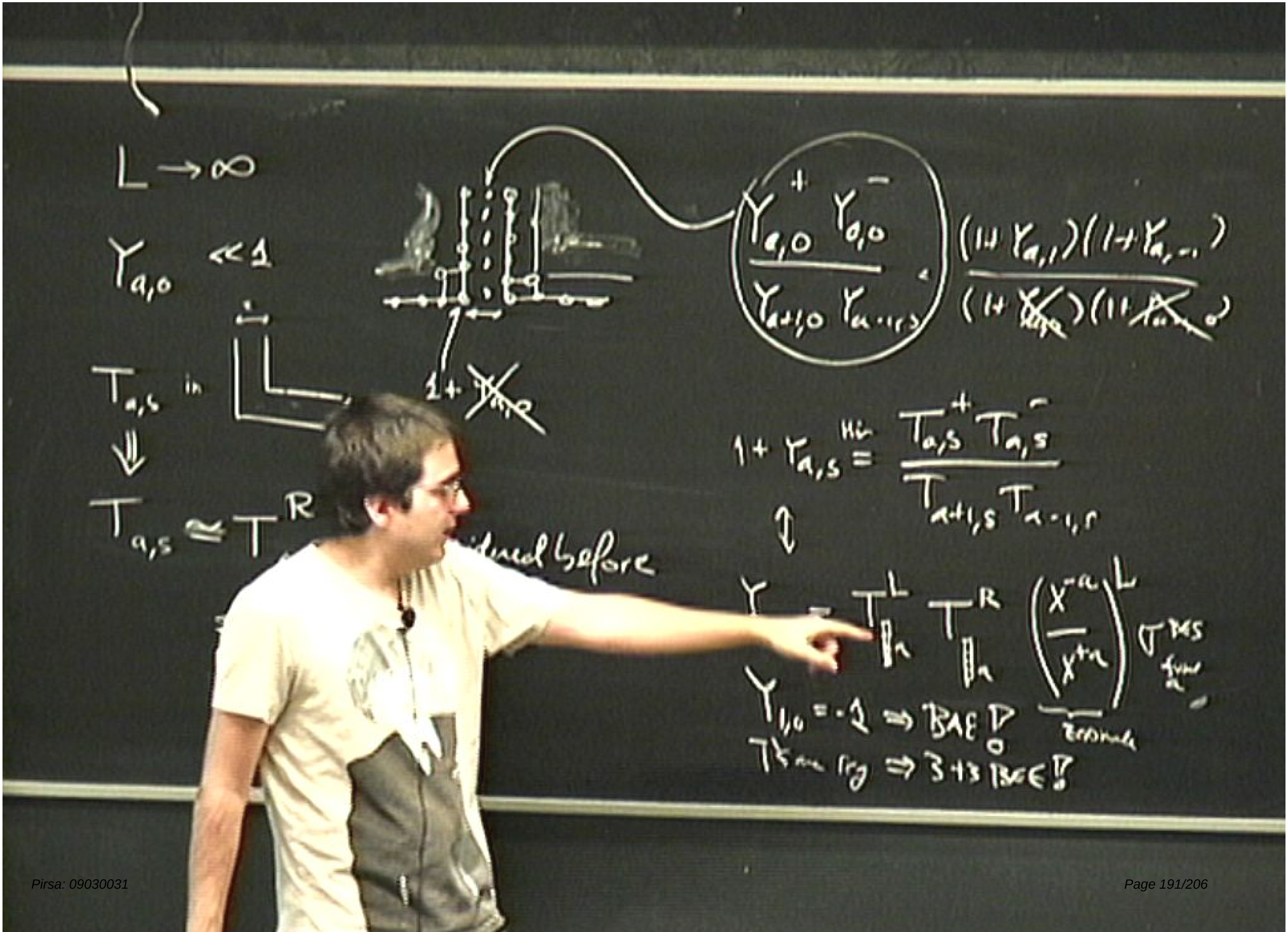
$T_{a,s}$ S…. S

$$\frac{Y^{+}_{a,0}\, Y^{-}_{a,0}}{Y_{a+1,0}\, Y_{a-1,s}} \cdot \frac{(1+Y_{a,1})(1+Y_{a,-1})}{(1+\cancel{Y_{\dots}})(1+\cancel{Y_{\dots}})}$$

$$1 + Y_{a,s} \overset{Hir}{=} \frac{T^{+}_{a,s}\, T^{-}_{a,s}}{T_{a+1,s}\, T_{a-1,s}}$$

$\Updownarrow$

$$Y_{a,0} = T^{L}_{\square a}\, T^{R}_{\square a} \underbrace{\left(\frac{x^{-a}}{x^{+a}}\right)^{L}}_{\text{[illegible]}} \sigma^{MS}_{fund\,a}$$



$L \to \infty$

$Y_{a,0} \ll 1$

$\doteq$

$T_{a,s}$ in

$\Downarrow$

$T_{a,s} \simeq T^R$ … studied before

$$\frac{Y^+_{a,0}\, Y^-_{a,0}}{Y_{a+1,0}\, Y_{a-1,0}} = \frac{(1+Y_{a,1})(1+Y_{a,-1})}{(1+\cancel{Y_{a+1,0}})(1+\cancel{Y_{a-1,0}})}$$

$1+\cancel{Y_{1,0}}$

$$1+Y_{a,s} \overset{\text{Hir}}{=} \frac{T^+_{a,s}\, T^-_{a,s}}{T_{a+1,s}\, T_{a-1,s}}$$

$\Updownarrow$

$$Y = T^L_{\|_n}\, T^R_{\|_n} \underbrace{\left(\frac{x^{-a}}{x^{+a}}\right)^L}_{\text{dressing}} \sigma^{BES}_{\text{fun}}$$

$Y_{1,0} = -1 \Rightarrow$ BAE !

T's reg $\Rightarrow$ 3+3 BAE !!

$L \to \infty$

$Y_{a,0} \ll 1$

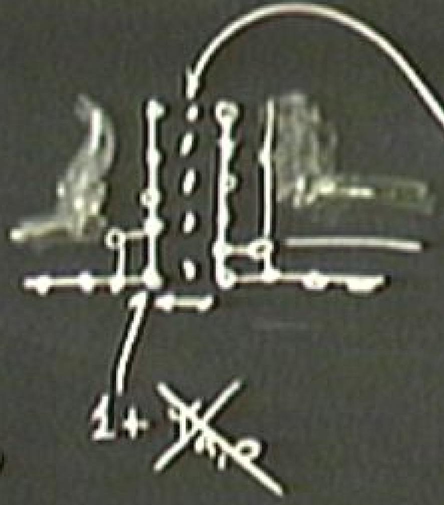
$\doteq$

$T_{a,s}$ in 

$\Downarrow$

$T_{a,s} \simeq T^{R}_{a,s}$ considered before

$= \mathrm{Tr}_{a,s}\, S \dots S$



$$\boxed{\frac{Y^{+}_{a,0}\,Y^{-}_{a,0}}{Y_{a+1,0}\,Y_{a-1,0}}} = \frac{(1+Y_{a,1})(1+Y_{a,-1})}{(1+\cancel{Y_{a+1,0}})(1+\cancel{Y_{a-1,0}})}$$

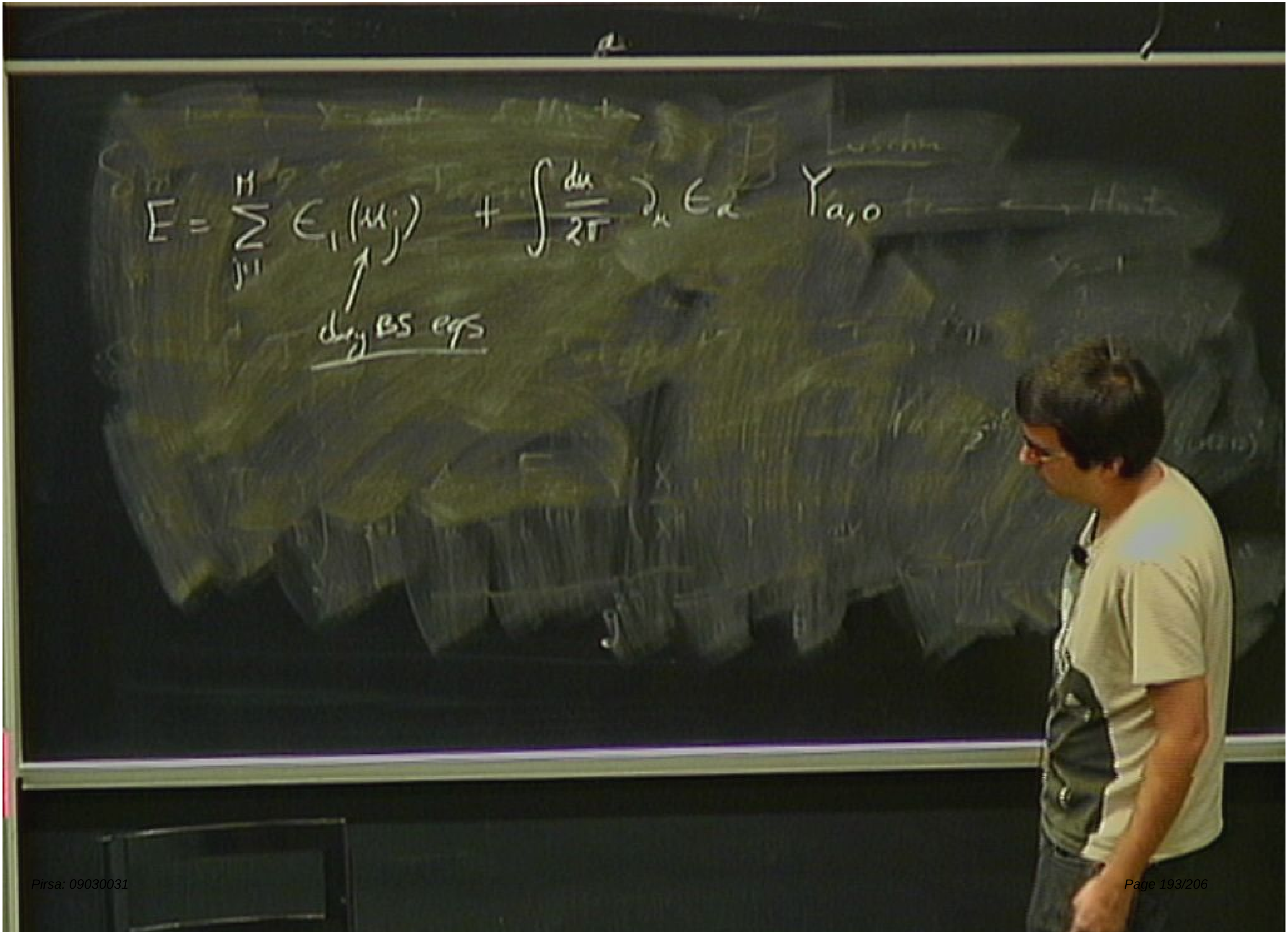
$$1+Y_{a,s} \overset{\text{Hir}}{=} \frac{T^{+}_{a,s}\,T^{-}_{a,s}}{T_{a+1,s}\,T_{a-1,s}}$$

$\Updownarrow$

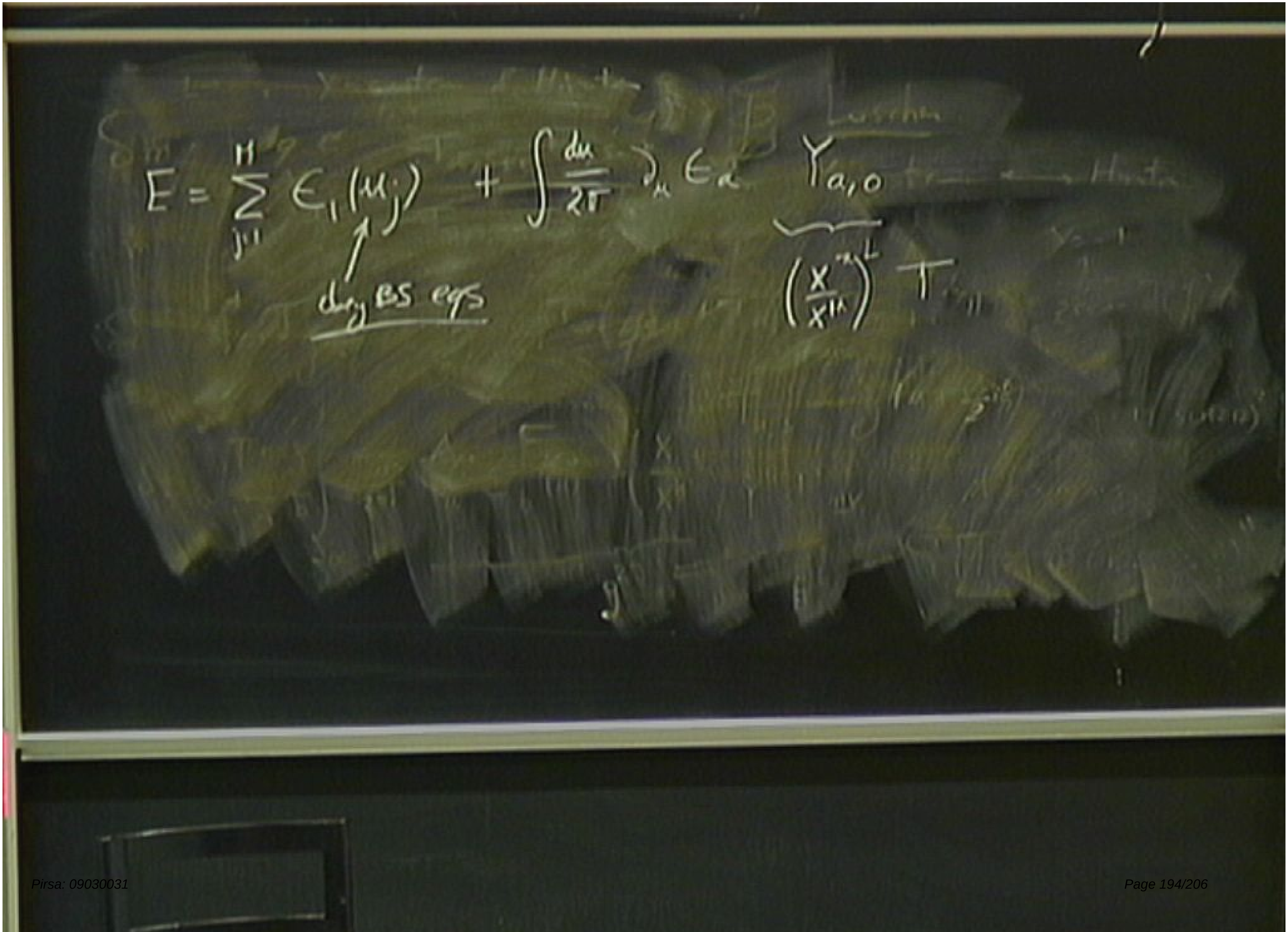
$$Y_{a,0} = T^{L}_{\square_a}\, T^{R}_{\square_a} \underbrace{\left(\frac{x^{-a}}{x^{+a}}\right)^{L}}_{\text{[illegible]}} \sigma^{BES}_{\text{fuse }a}$$

$Y_{1,0} = -1 \Rightarrow \text{BAE}\,\overset{!}{\circ}$

T's in Y $\Rightarrow$ 3+3 BAE !!



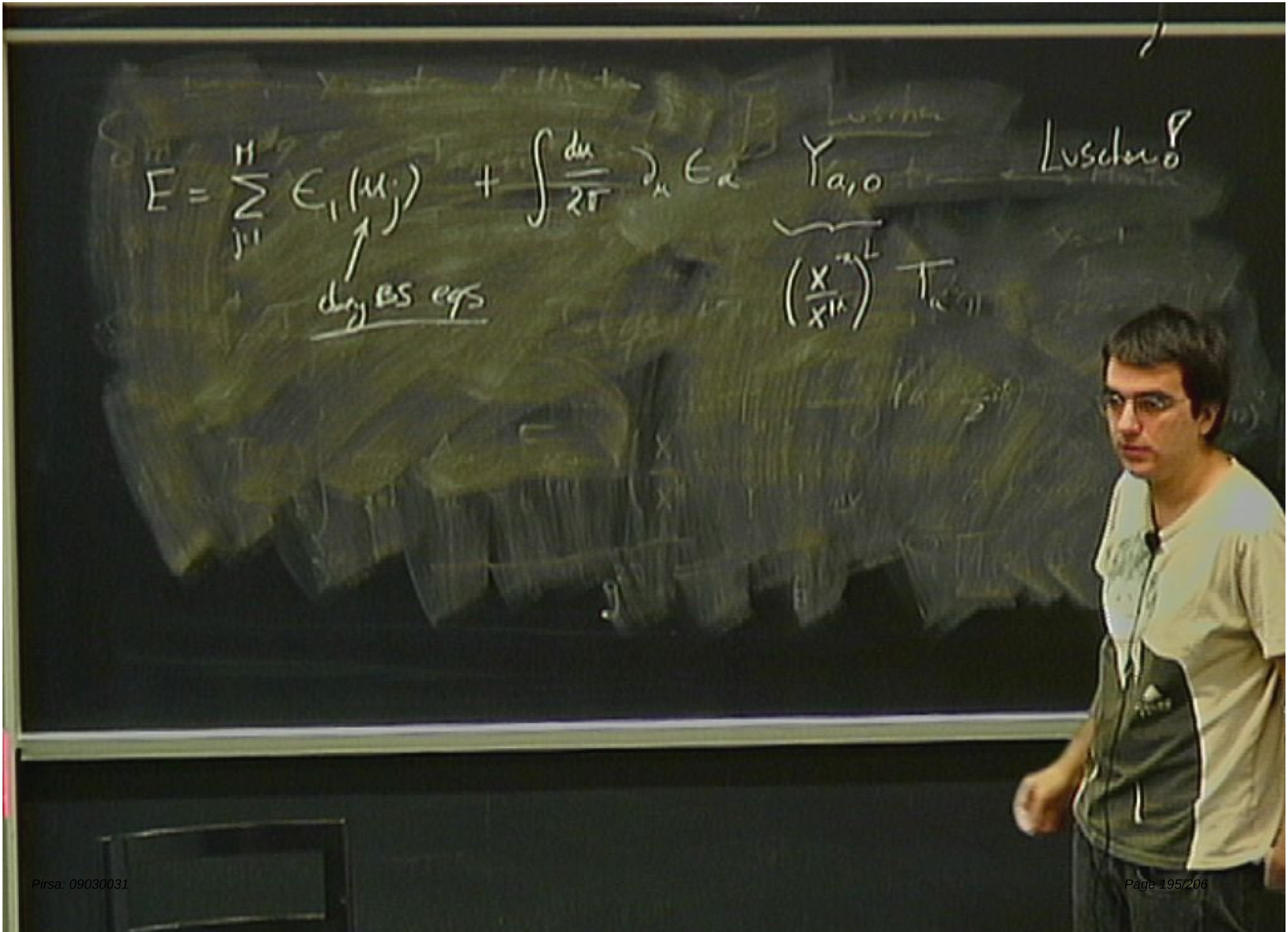
$$E = \sum_{j=1}^{M} \epsilon_1(\underset{\substack{\uparrow \\ \underline{\text{diag BS eqs}}}}{u_j}) + \int \frac{du}{2\pi} \partial_u \epsilon_a \quad Y_{a,0}$$



$$E = \sum_{j=1}^{M} \epsilon_1(u_j) + \int \frac{du}{2\pi}\, \partial_u \epsilon_a\, \underbrace{Y_{a,0}}_{\left(\frac{x^{-}}{x^{+}}\right)^{L} T_{a,1}}$$

$\uparrow$

diag BS eqs



$$E = \sum_{j=1}^{M} \epsilon_1(u_j) + \int \frac{du}{2\pi} \partial_u \epsilon_a \underbrace{Y_{a,0}}_{\left(\frac{x^-}{x^+}\right)^L T_{a,1}}$$

diag BS eqs

Luscher!

Luscher!

$$E = \sum_{j=1}^{M} \underset{\substack{\uparrow \\ \text{diag BS eqs}}}{\epsilon_1(u_j)} + \int \frac{du}{2\pi} \partial_u \epsilon_a \underbrace{Y_{a,0}}_{\left(\frac{x^{[-a]}}{x^{[+a]}}\right)^L T_{a,1}}$$

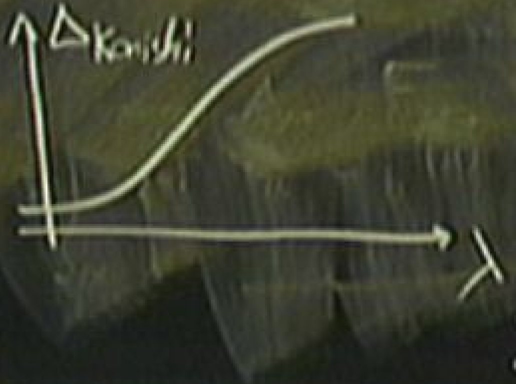
$$E = \sum_{j=1}^{M} \epsilon_1(u_j) + \int \frac{du}{2\pi}\, \partial_u \epsilon_a \underbrace{Y_{a,0}}_{\left(\frac{x^-}{x^+}\right)^L T_{a,1}}$$

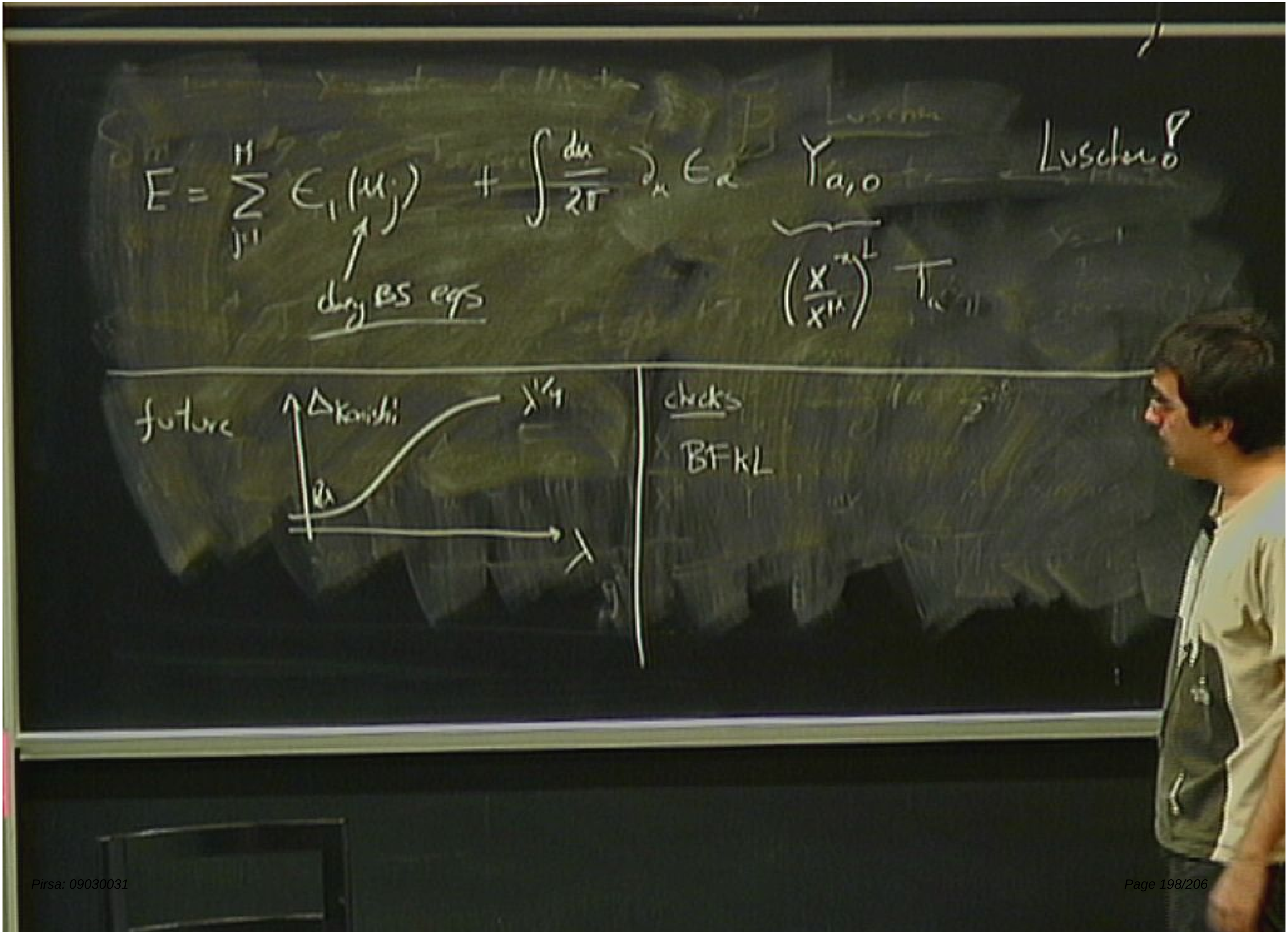
Luscher!

diag BS eqs

---

future





$$E = \sum_{j=1}^{M} \epsilon_1(u_j) + \int \frac{du}{2\pi}\, \partial_u \epsilon_a \underbrace{Y_{a,0}}_{\left(\frac{x^-}{x^+}\right)^L T_{a,1}}$$

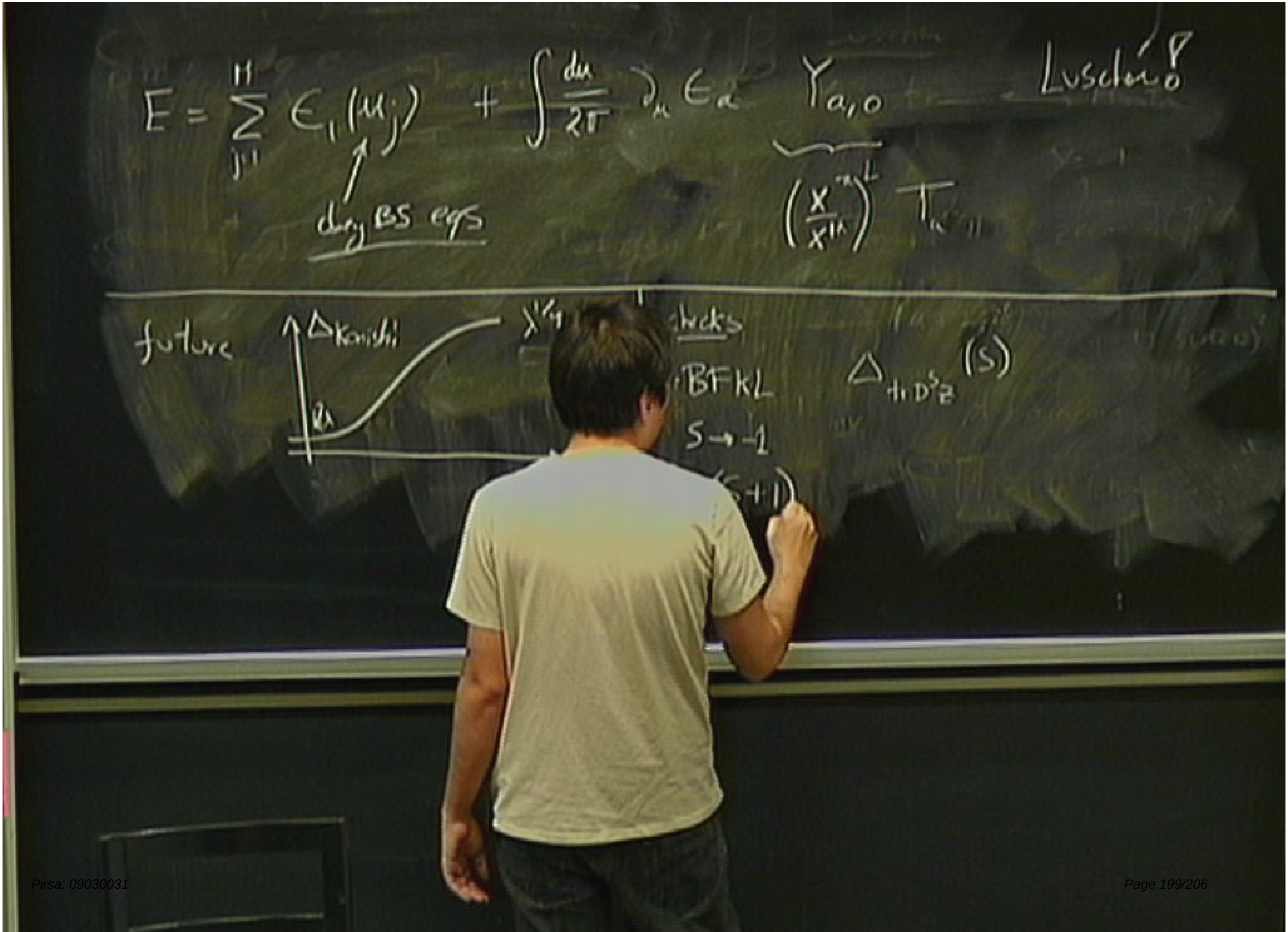
Luscher!

asy BS eqs

future

checks

BFKL



$$E = \sum_{j=1}^{M} \epsilon_1(u_j) + \int \frac{du}{2\pi} \partial_u \epsilon_a \underbrace{Y_{a,0}}_{\left(\frac{x^{-[a]}}{x^{+[a]}}\right)^L T_{a,1}}$$

Luscher!

dry BS eqs

future

checks

BFKL

$S \to -1$

$(S+1)$

$\Delta_{\text{tr} D^S Z}(S)$

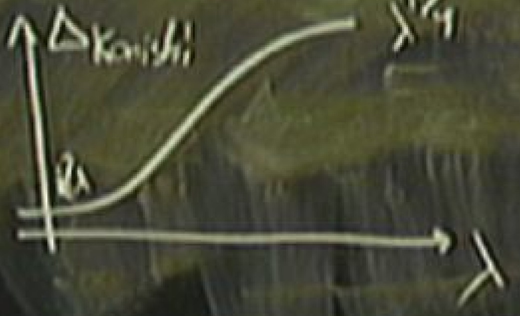
$$E = \sum_{j=1}^{M} \epsilon_1(u_j) + \int \frac{du}{2\pi} \partial_u \epsilon_a \underbrace{Y_{a,0}}_{\left(\frac{x^{-}}{x^{+}}\right)^{L} T_{a,1}}$$

Luscher!

deg BS eqs

---

future

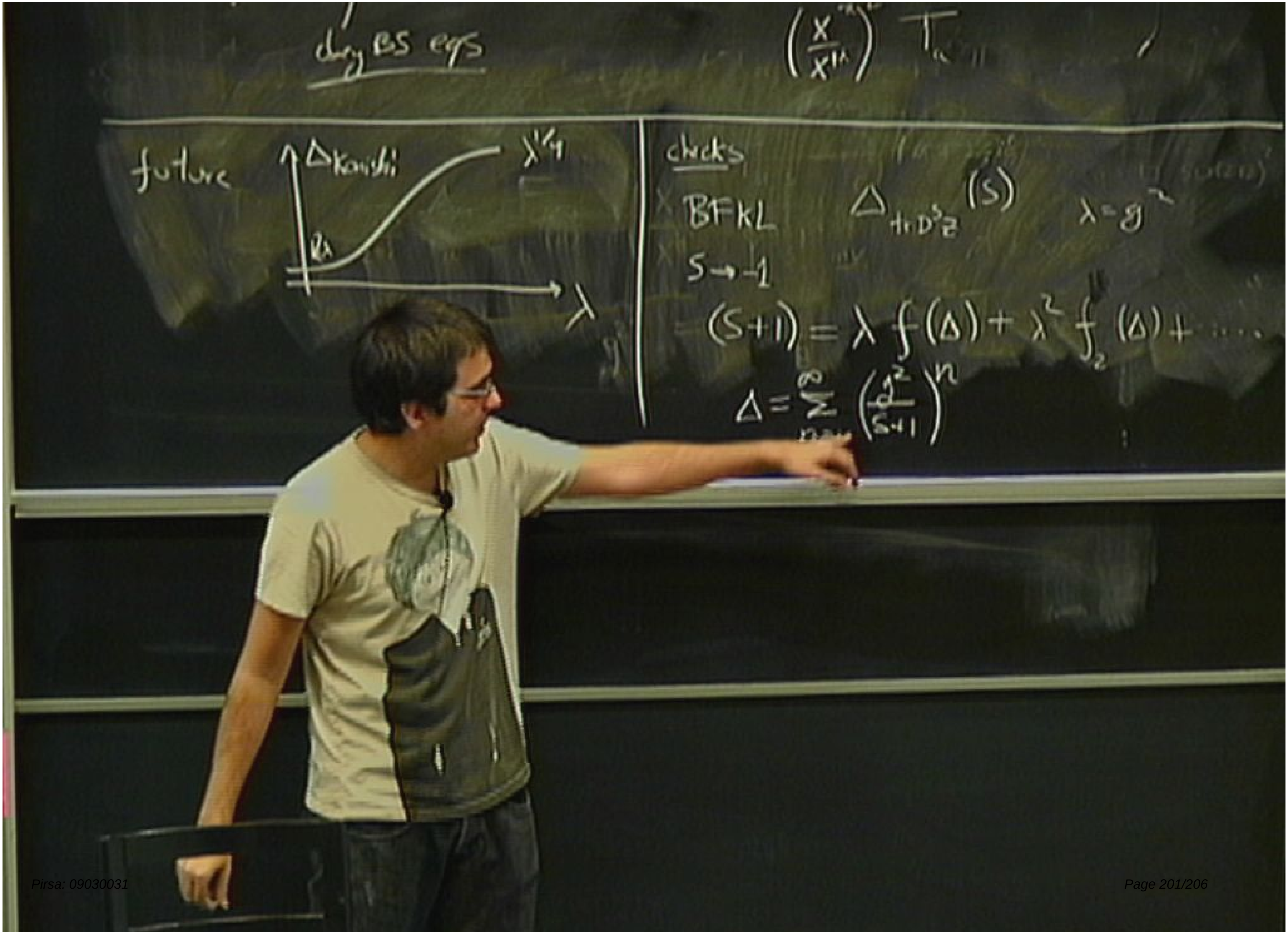


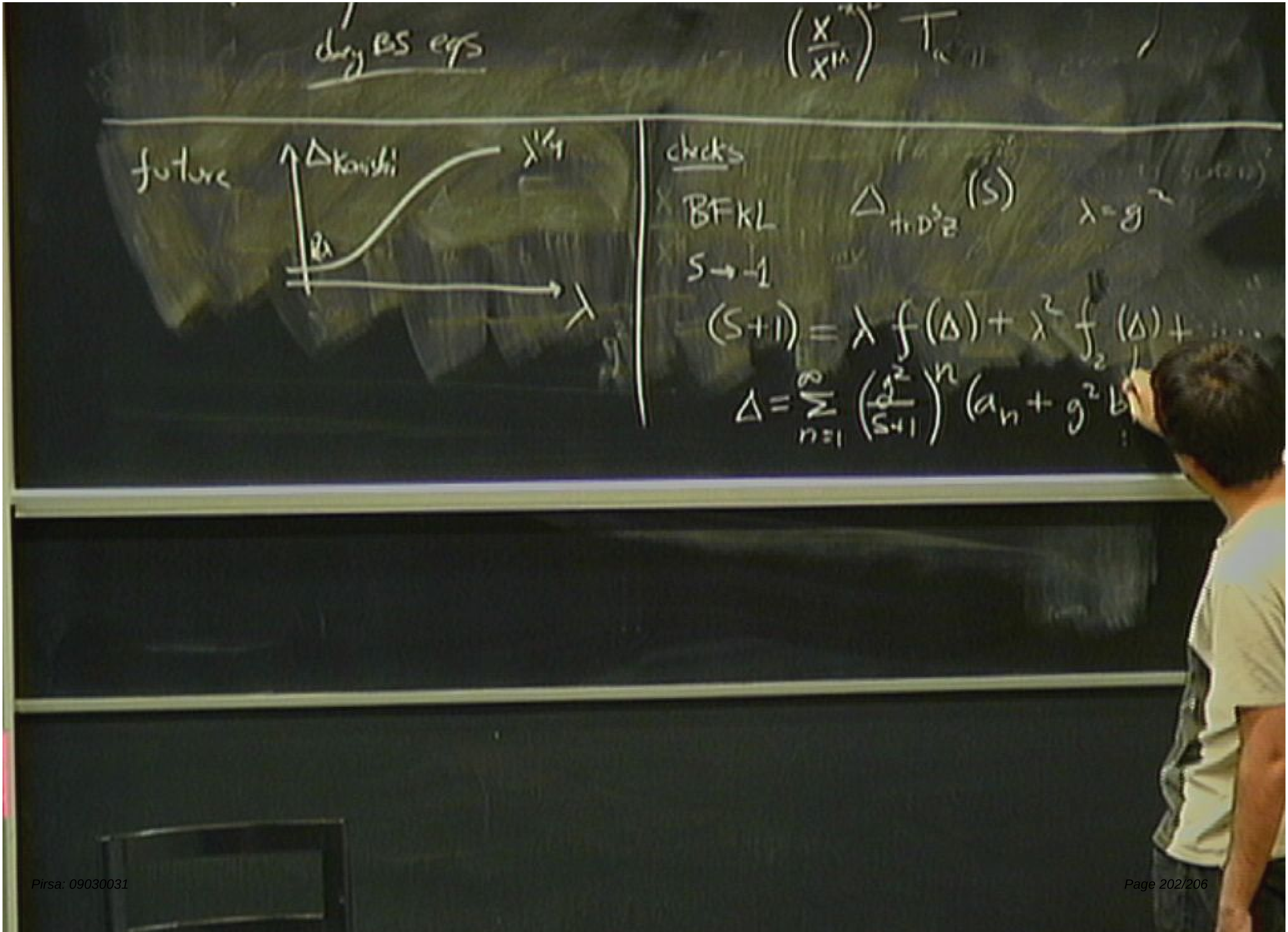
checks

BFKL $\quad \Delta_{\mathrm{tr} D^S Z}(S)$

$S \to -1$

$$(S+1) = \lambda f(\Delta) + \lambda^2 f_2(\Delta) + \cdots$$


diag BS eqs
$\left(\frac{x}{x^{1n}}\right)$ T
future
Δ_{Konishi}
$\lambda^{1/4}$
λ
checks
BFKL
$\Delta_{\text{tr} D^3 Z}(S)$
$\lambda = g^2$
$S \to -1$
$(S+1) = \lambda f(\Delta) + \lambda^2 f_2(\Delta) + \dots$
$\Delta = \sum_{n=1}^{\infty} \left(\frac{g^2}{S+1}\right)^n$



diag BS eqs

$\left(\frac{X}{X^{1n}}\right) \quad T_n$

future

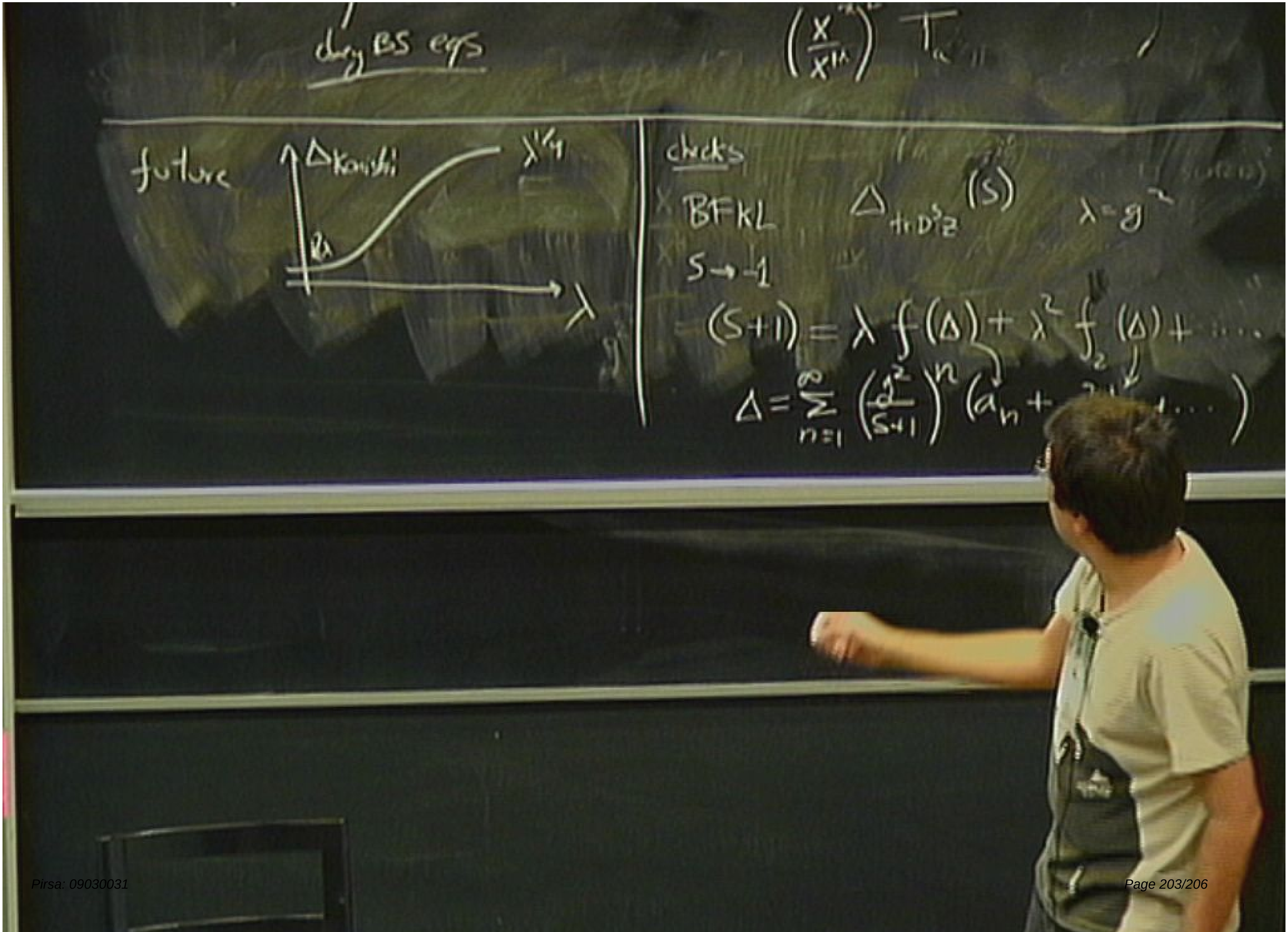
checks

BFKL $\qquad \Delta_{\text{tr} D^3 Z}(S) \qquad \lambda = g^2$

$S \to -1$

$$(S+1) = \lambda f(\Delta) + \lambda^2 f_2(\Delta) + \ldots$$

$$\Delta = \sum_{n=1}^{\infty} \left(\frac{g^2}{S+1}\right)^n (a_n + g^2 b$$



checks

BFKL $\quad \Delta_{\mathrm{tr}\,D^s Z}(S) \quad \lambda = g^2$

$S \to -1$

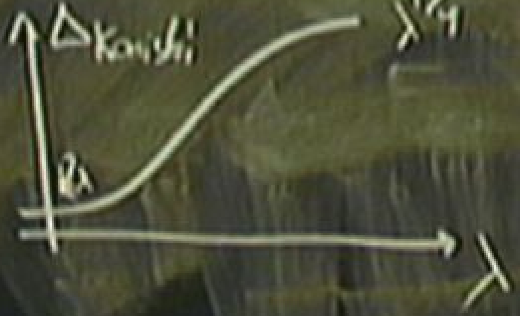
$$(S+1) = \lambda f(\Delta) + \lambda^2 f_2(\Delta) + \dots$$

$$\Delta = \sum_{n=1}^{\infty} \left(\frac{g^2}{S+1}\right)^n (a_n + \dots)$$

diag BS eqs

$\left(\frac{x}{x^{\prime n}}\right) T_n$

future



checks

BFKL $\quad \Delta_{\text{tr}\,D^S Z^2}(S) \quad \lambda = g^2$

$S \to -1$

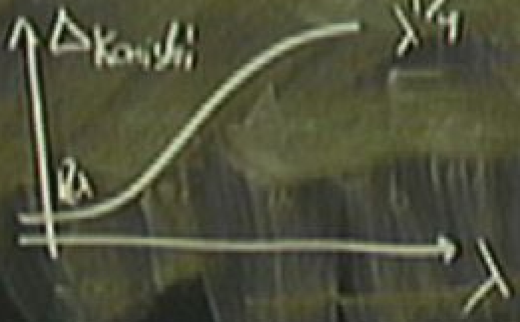
$$(S+1) = \lambda f(\Delta) + \lambda^2 f_2(\Delta) + \dots$$

$$\Delta = \sum_{n=1}^{\infty} \left(\frac{g^2}{S+1}\right)^n \left(a_n + g^2 b_n + \dots\right)$$

diag BS eqs

$\left(\frac{x}{x^{1n}}\right) T_{n}$

future



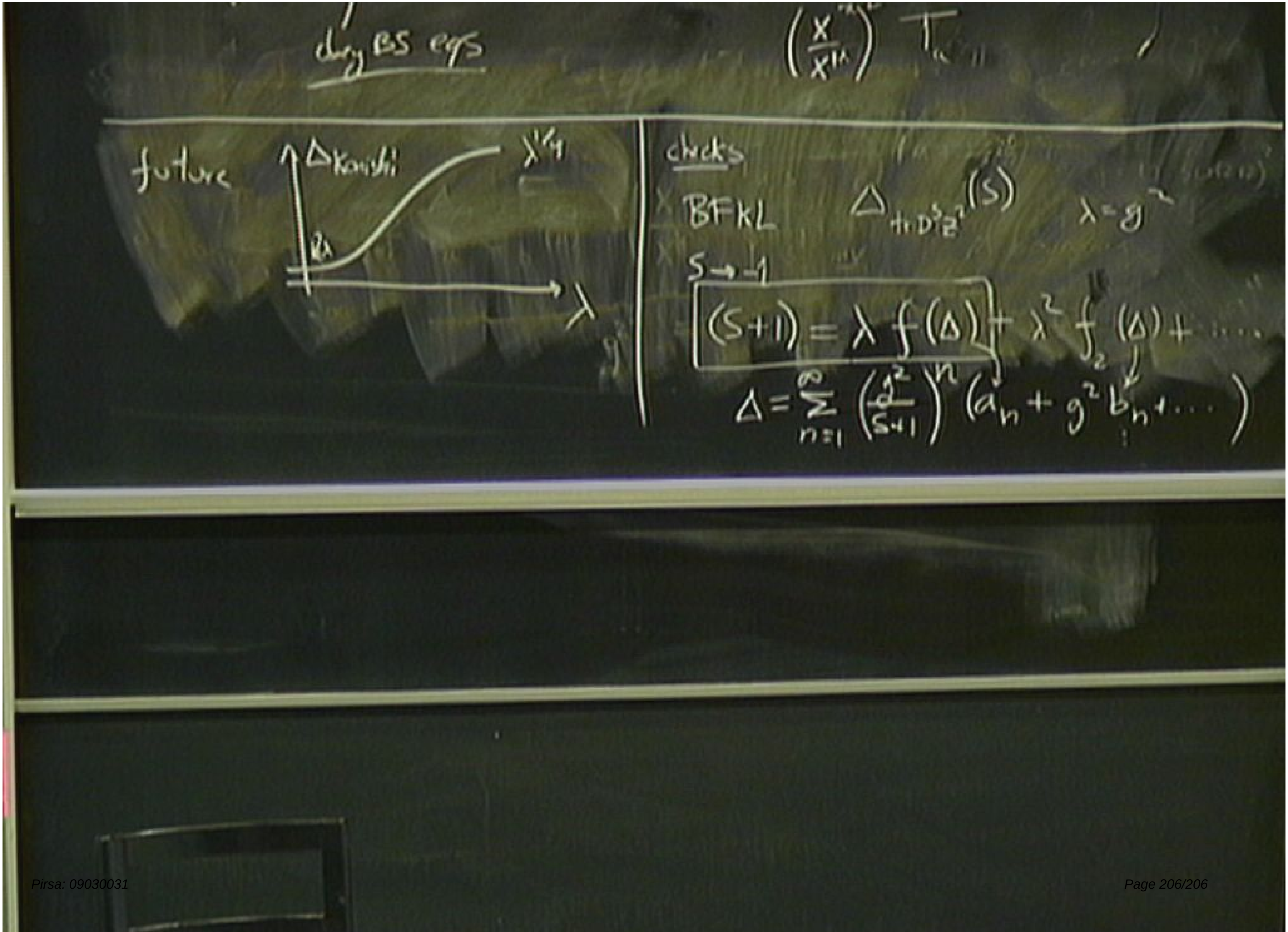
checks

BFKL $\quad \Delta_{\text{tr}\, D^3 Z^2}(S) \quad \lambda = g^2$

$S \to -1$

$$(S+1) = \lambda f(\Delta) + \lambda^2 f_2(\Delta) + \ldots$$

$$\Delta = \sum_{n=1}^{\infty} \left(\frac{g^2}{S+1}\right)^n (a_n + g^2 b_n + \ldots)$$



dry BS eqs

$\left(\frac{X}{X^{1n}}\right) T_{n,1}$

future

checks

BFKL $\quad \Delta_{\mathrm{tr} D^S Z^2}(S) \quad \lambda = g^2$

$S \to -1$

$$\boxed{(S+1) = \lambda f(\Delta)} + \lambda^2 f_2(\Delta) + \ldots$$

$$\Delta = \sum_{n=1}^{\infty} \left(\frac{g^2}{S+1}\right)^n (a_n + g^2 b_n + \ldots)$$