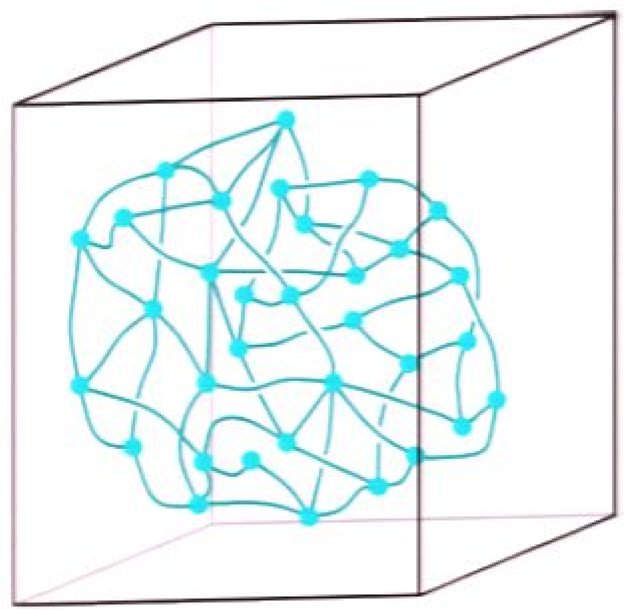
Title: Permutational quantum computation and spin networks

Date: Mar 04, 2009 04:00 PM

URL: http://pirsa.org/09030004

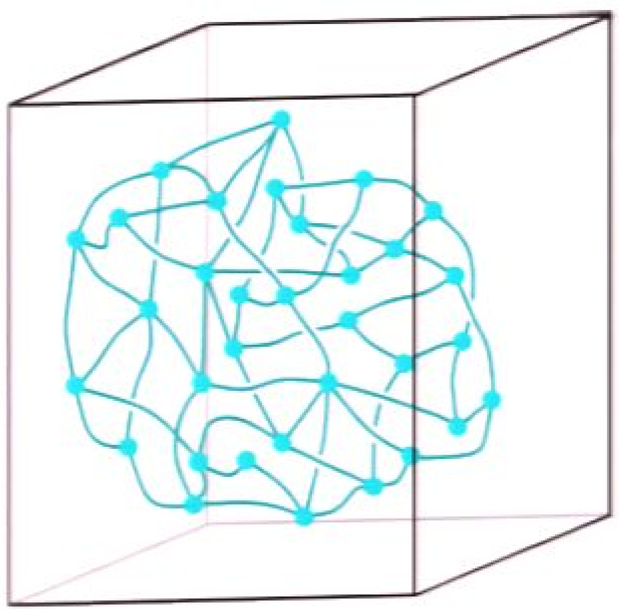
Abstract: In topological quantum computation the geometric details of a particle trajectory become irrelevant; only the topology matters. This is one reason for the inherent fault tolerance of topological quantum computation. I will speak about a model in which this idea is taken one step further. Even the topology is irrelevant. The computation is determined solely by the permutation of the particles. Unlike topological quantum computation, which requires anyons confined to two dimensions, permutational quantum computations can in principle be performed by permuting a set of ordinary spin-1/2 particles with definite total angular momentum in three dimensions. The resulting model of computation appears to be intermediate in power between classical computation (P) and standard quantum computation (BQP). The model may be equivalently defined in terms of spin networks, which are an important concept in loop quantum gravity.

Permutational Quantum Computation & Spin-Networks



Stephen Jordan

Permutational Quantum Computation & Spin-Networks



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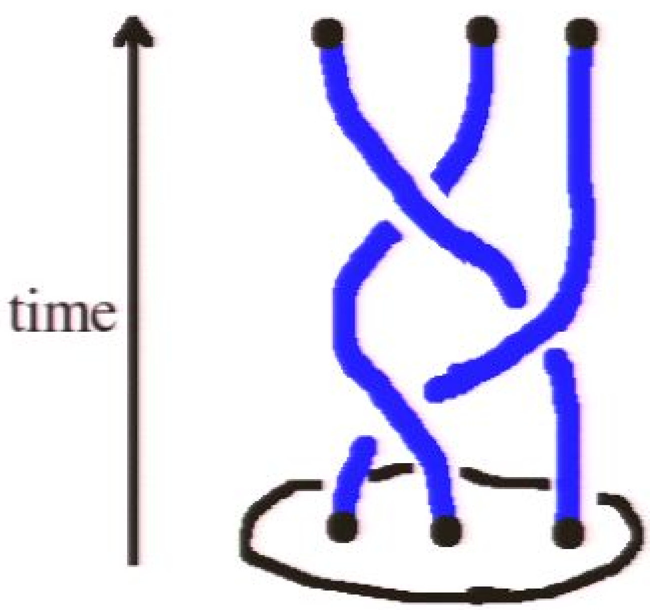
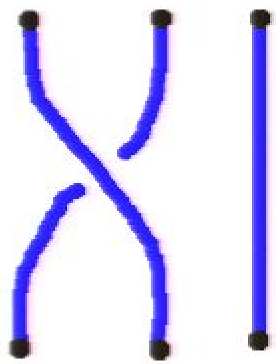
- Degenerate ground space
- Particle-like excitations (anyons)
- Adiabatically drag them around (braid)



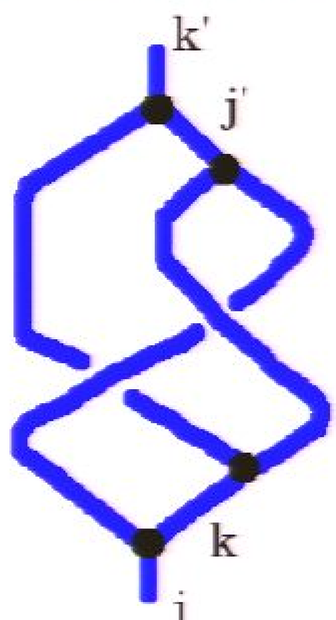
- system ends up in different part of ground space

- The transformation of the ground space is a unitary representation of the braid group.



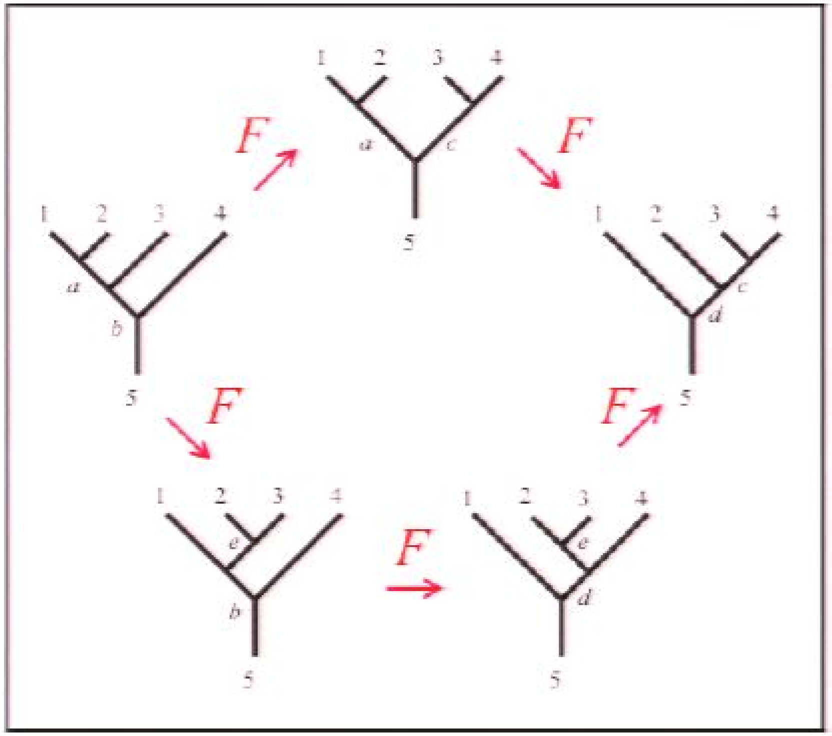
$$\xrightarrow{\rho_{\mathrm{Fib}}} \begin{bmatrix} e^{i3\pi/5} & 0 \\ 0 & e^{-i4\pi/5} \end{bmatrix}$$

- We can also fuse and split particles.



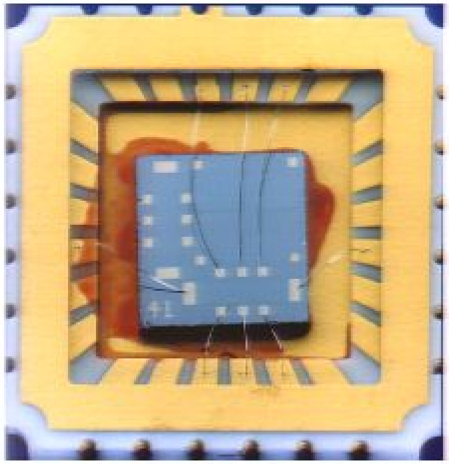
$$\longrightarrow \langle \mathrm{j}',\mathrm{k}' | \rho_{(\mathrm{b})} | \mathrm{j},\mathrm{k} \rangle$$

- The map from braiding and fusing to linear transformations obeys certain consistency rules, for example:



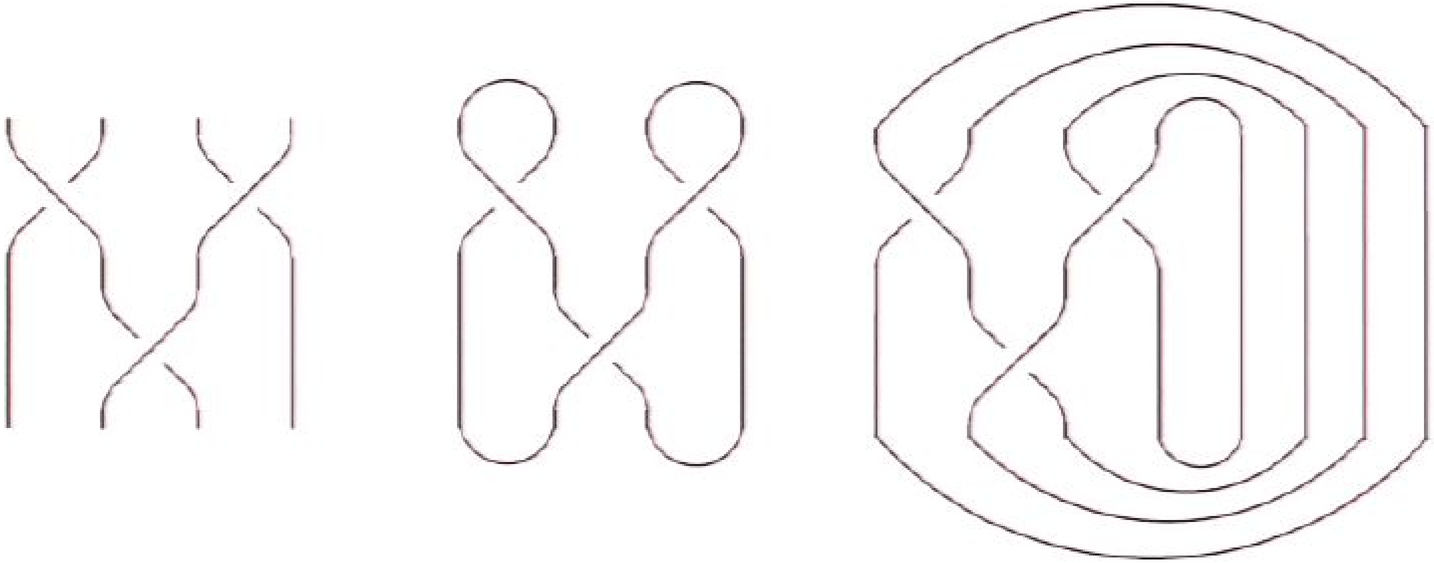
- (This is a modular tensor category.)

- If we find the right kind of anyons we can do universal quantum computation (BQP) by braiding.



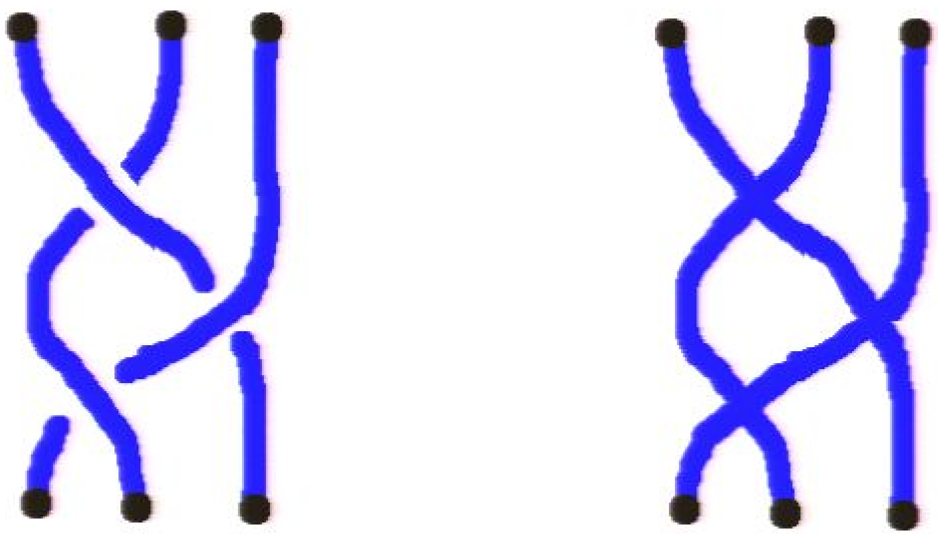
- Thinking about topological QC also led to discovery of new quantum algorithms such as for the Jones polynomial.

- By closing a braid we can make a link:



- A certain matrix element of ρ_{fib} is the Jones polynomial of the plat closure at $e^{i2\pi/5}$ (BQP-complete)
- The trace of ρ_{fib} is the Jones polynomial of the trace closure (DQC1-complete)

- In topological QC local geometry doesn't matter. Only global topology does.
- This helps fault tolerance. (Also information encoded in nonlocal degrees of freedom.)
- What if we ignore even topology? All that's left is a permutation.

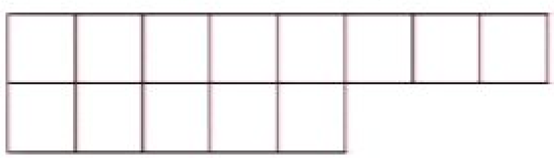


Topological	Permutational
Anyons	Spin-1/2
Braid (B_n)	Permute (S_n)
Fuse	Measure Angular Momentum
Braided Tensor Category	Racah-Wigner Tensor Category

Angular momentum of n spins

$$\vec{S} = \sum_{j=1}^{n} \vec{S}_j \qquad S^2 = \vec{S} \cdot \vec{S} \qquad S^2|j\rangle = j(j+1)|j\rangle$$

- S^2 commutes with any permutation
- the eigenspaces of S^2 transform as irreducible representations of S_n
- The Young diagrams have two rows:



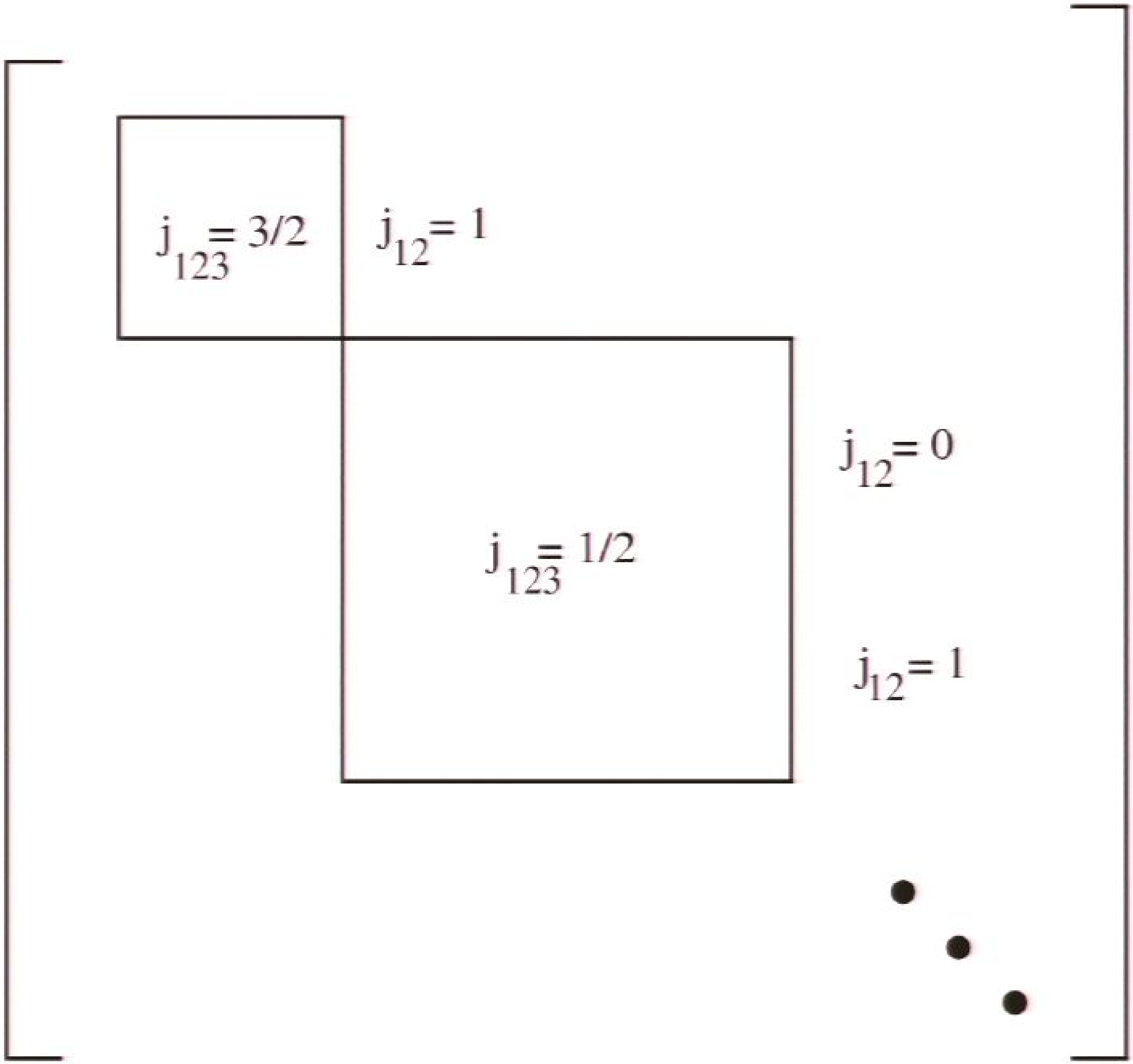
- The overhang is *2j*

- Reminiscent of anyon braiding, but:
  - what about fusion?
  - what about a basis for the representations?
- Example: 3 particles

$$\left.\begin{array}{l} (\vec{S}_1+\vec{S}_2+\vec{S}_3)^2 \\ (\vec{S}_1+\vec{S}_2)^2 \\ Z_1+Z_2+Z_3 \end{array}\right\} \text{complete set of commuting observables}$$

- This gives us a basis.

- How do the representations of S_3 look in this basis?
- $(\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2$ tells us which irrep
- $(\vec{S}_1 + \vec{S}_2)^2$ labels the basis states within an irrep
- $(Z_1 + Z_2 + Z_3)$ is an irrelevant degree of freedom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$j_{12} = 1$
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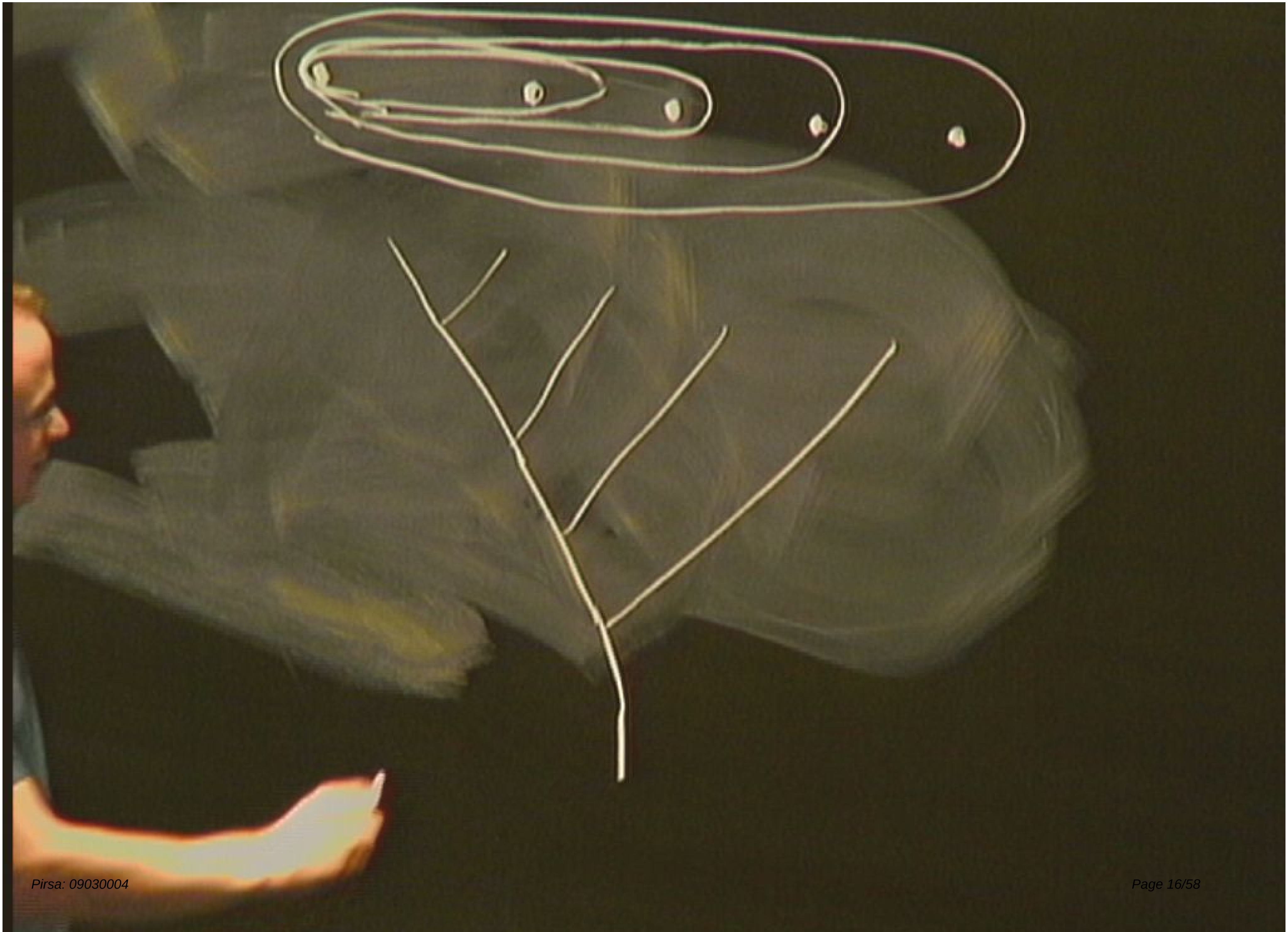
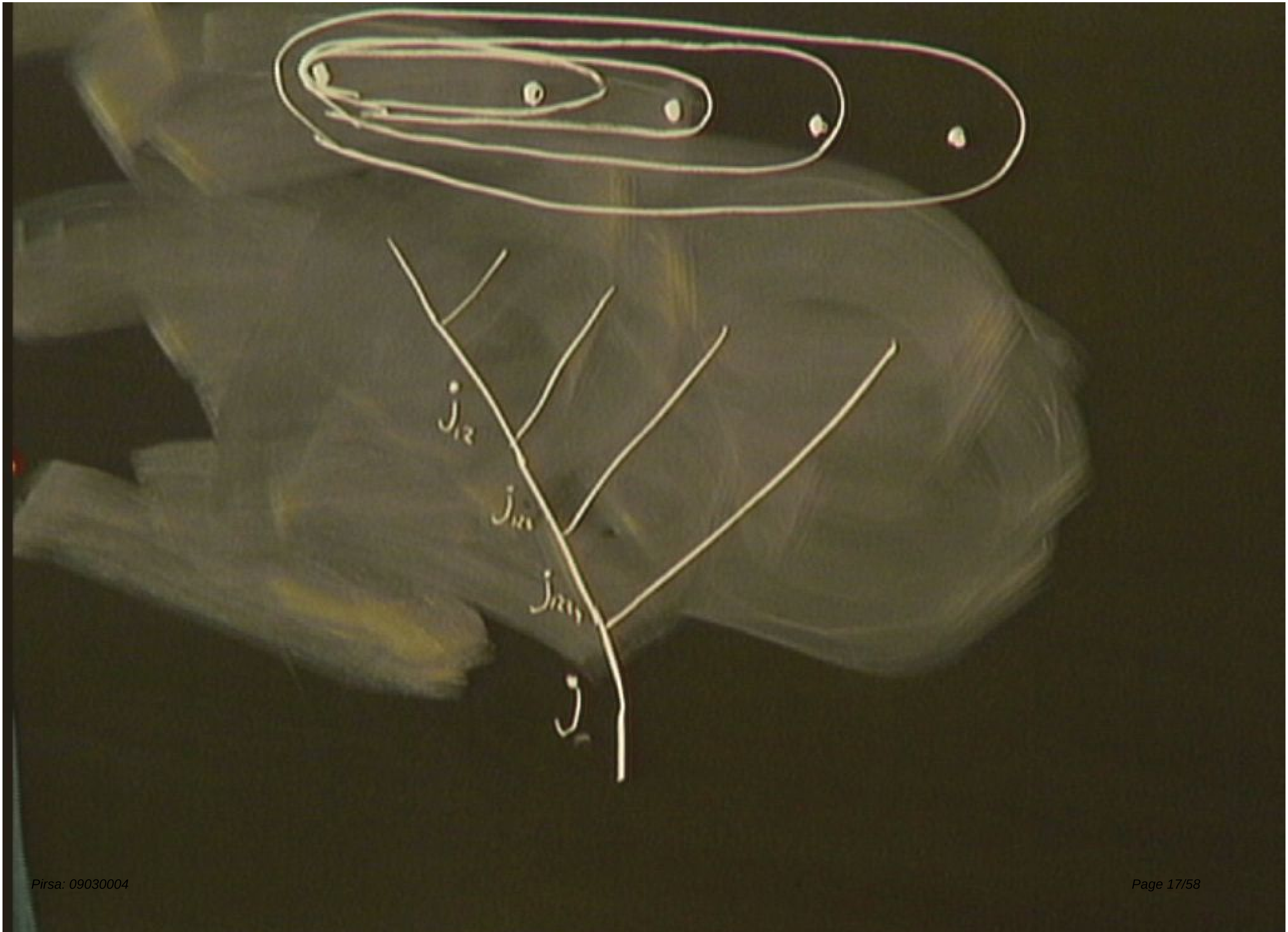
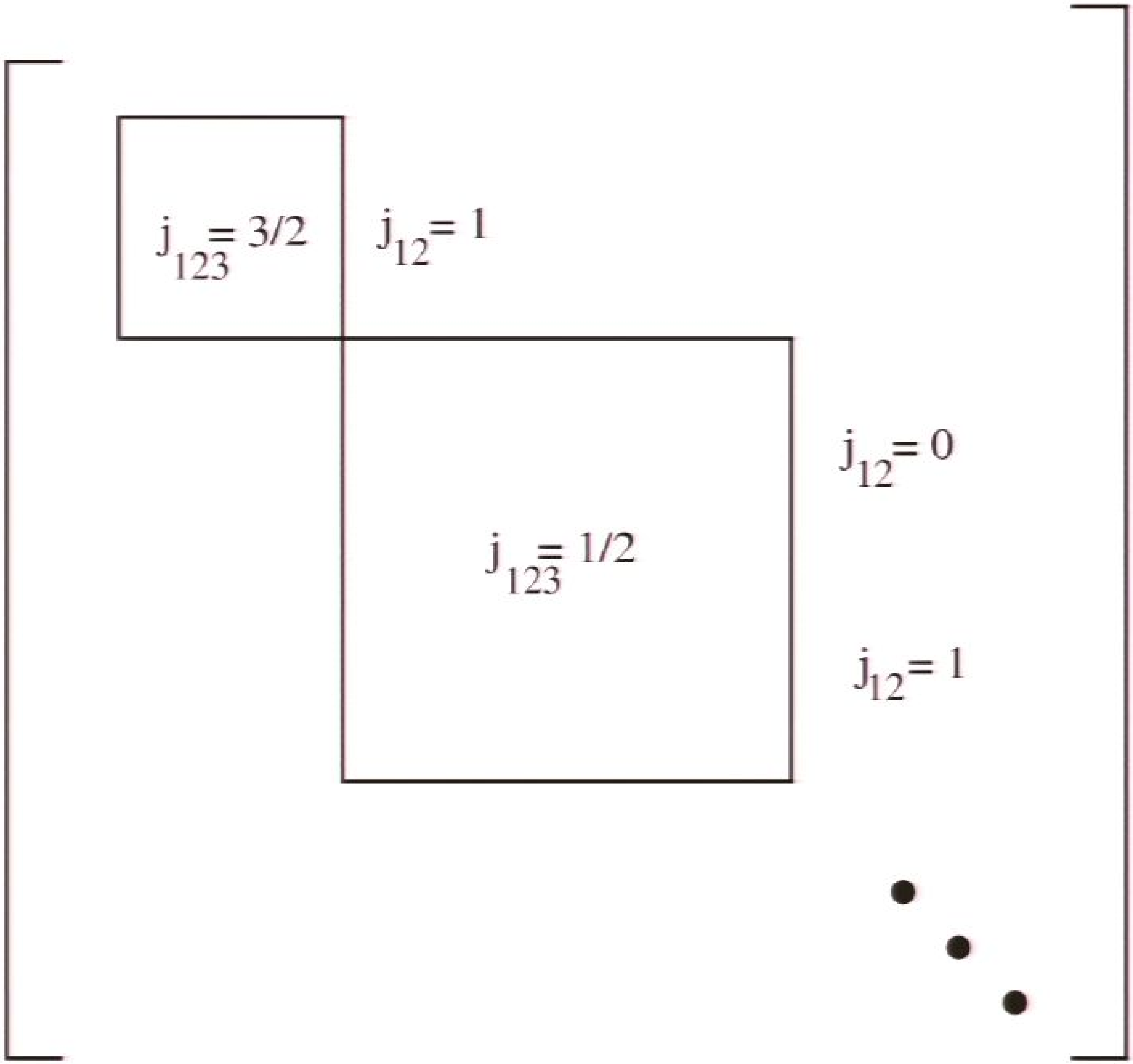



j_{12}
j_{123}
j_{1234}
j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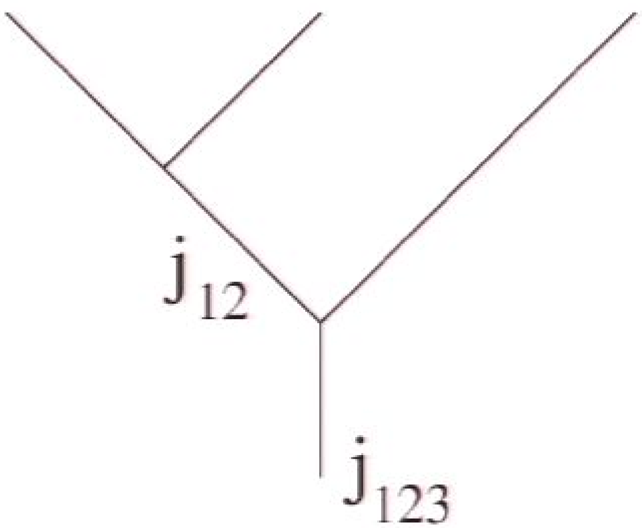
- We have a choice of basis:

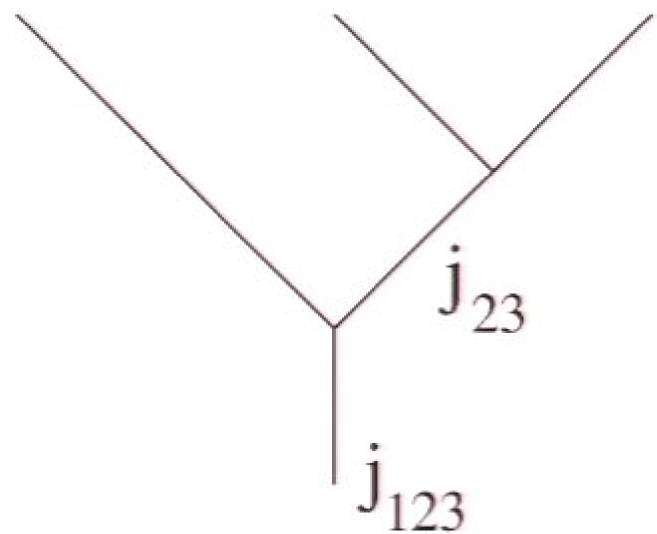
$$(\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2$$

$$(\vec{S}_1 + \vec{S}_2)^2$$

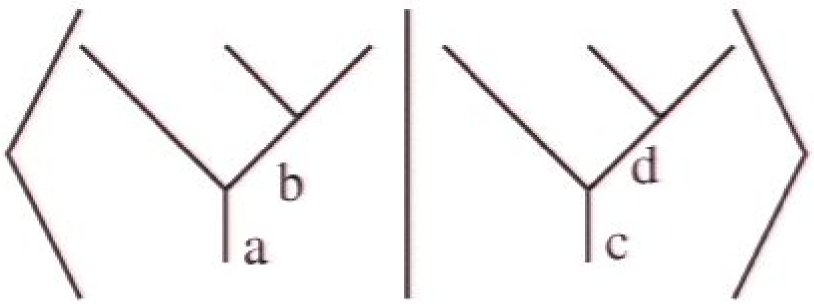


$$(\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2$$

$$(\vec{S}_2 + \vec{S}_3)^2$$

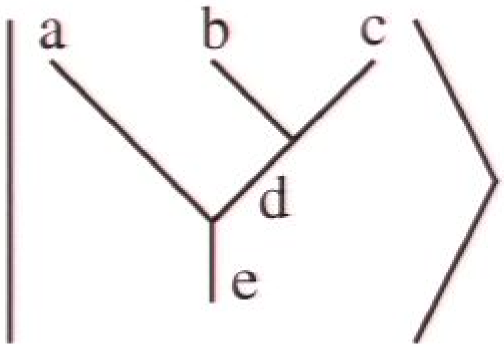


- For a given tree, different labellings correspond to orthogonal states

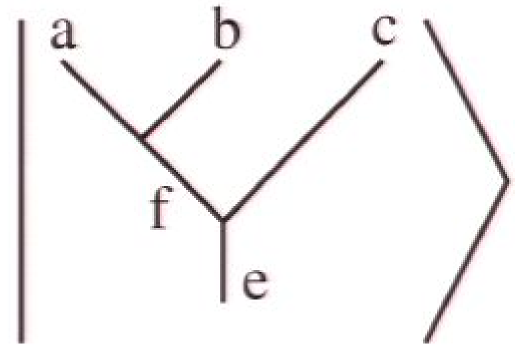


$$= \delta_{ac}\ \delta_{bd}$$

- Different trees are related by recoupling coefficients



$$= \sum_{f} \begin{bmatrix} a & b & f \\ c & e & d \end{bmatrix}$$



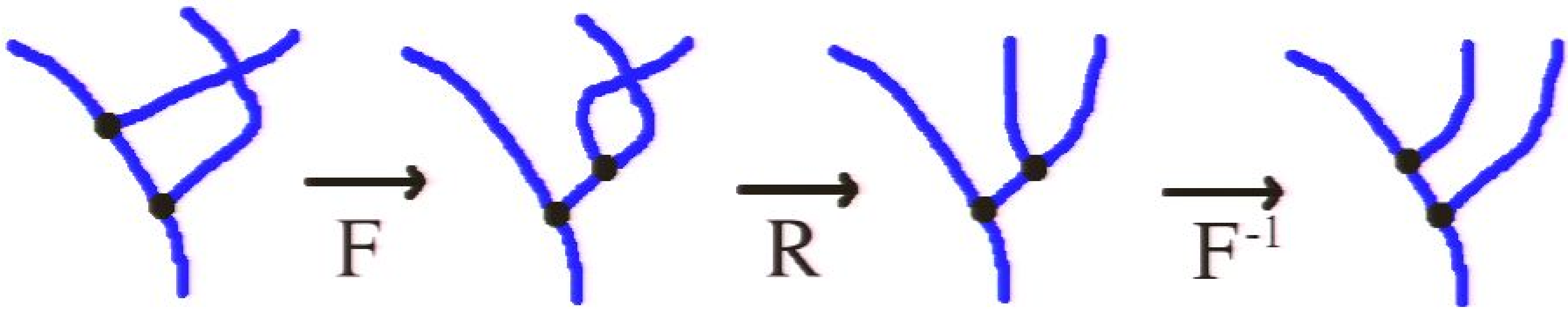
- The recoupling coefficients are:

$$\begin{bmatrix} a & b & f \\ c & e & d \end{bmatrix} = (-1)^{a+b+c+f} \sqrt{(2d+1)(2f+1)} \begin{Bmatrix} a & b & f \\ c & e & d \end{Bmatrix}$$

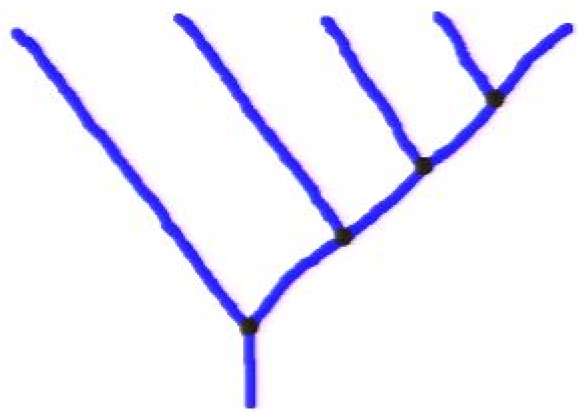
- The *6j* symbols $\begin{Bmatrix} a & b & f \\ c & e & d \end{Bmatrix}$ can be computed in $\mathrm{poly}(a+b+c+d+e+f)$ time using the Racah formula.

- The matrix elements of the S_n irrep are determined by the recoupling coefficients plus the two-particle exchange rule:

$$= (-1)^{b+c-a}$$



- If we couple like this:



Then we get Young's orthogonal form. [Kotani]

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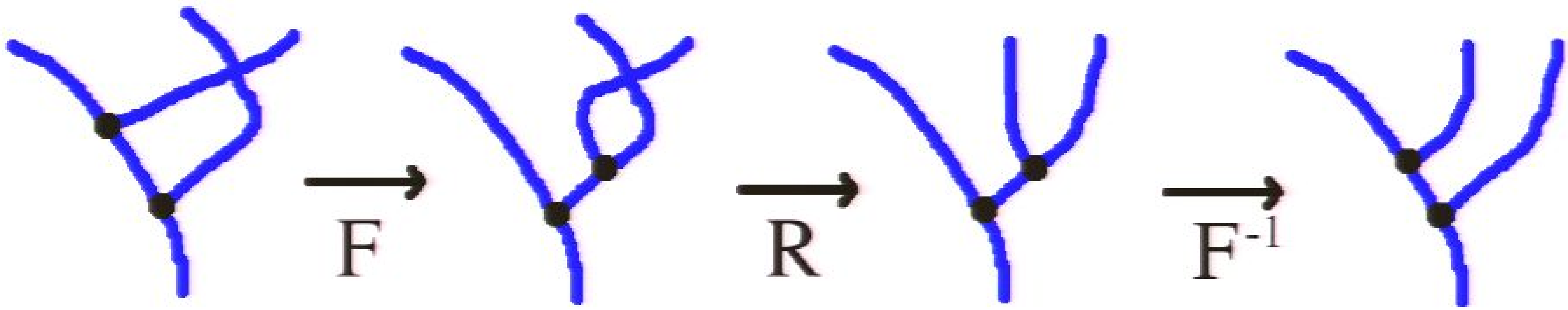

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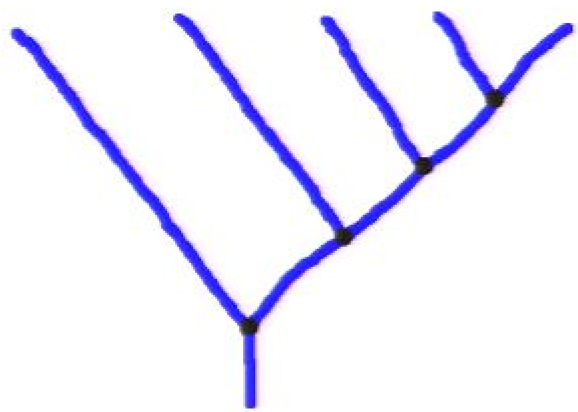

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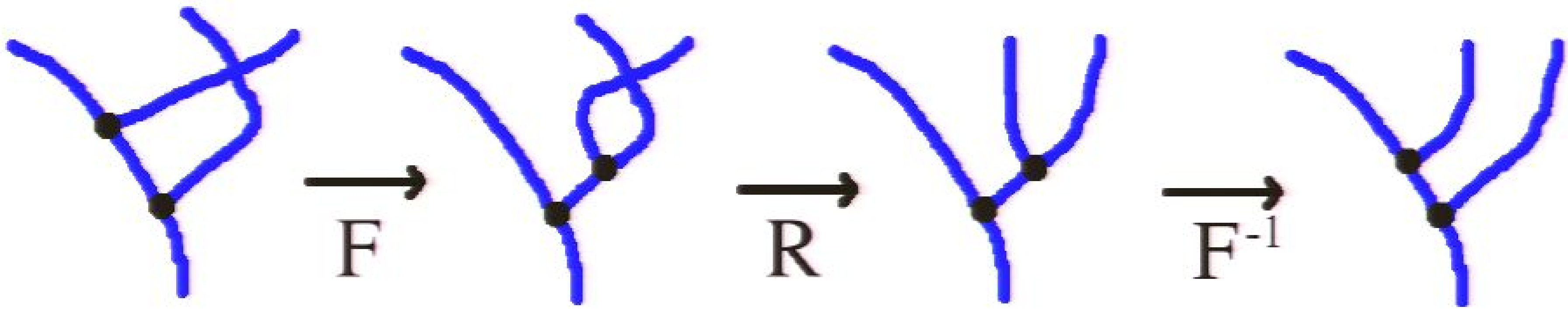

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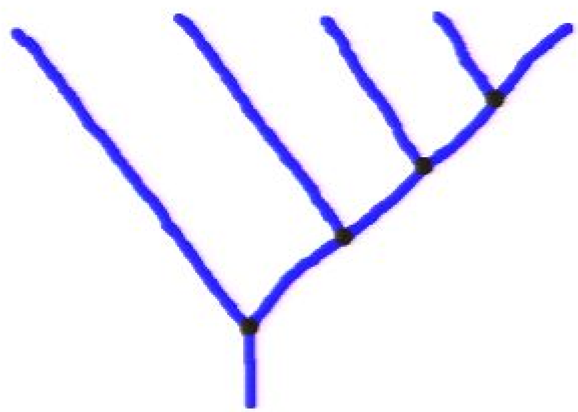
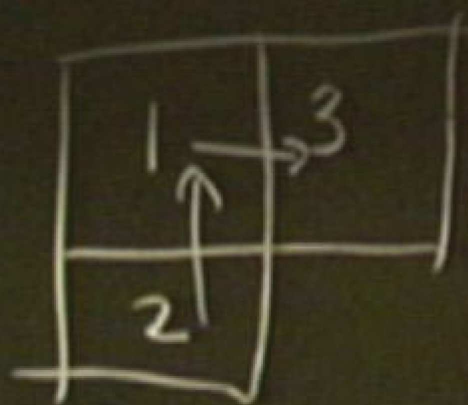


Then we get Young's orthogonal form. [Kotani]

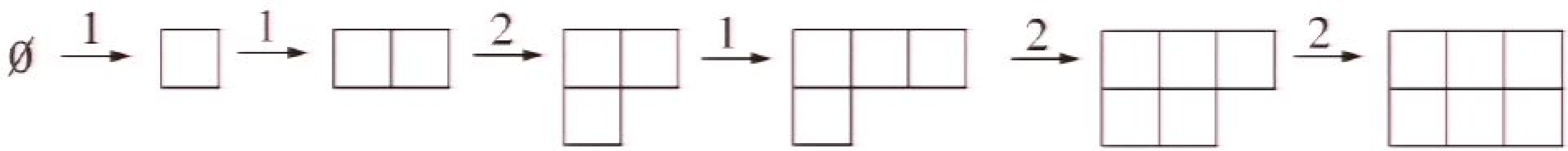
Young's Orthogonal Form

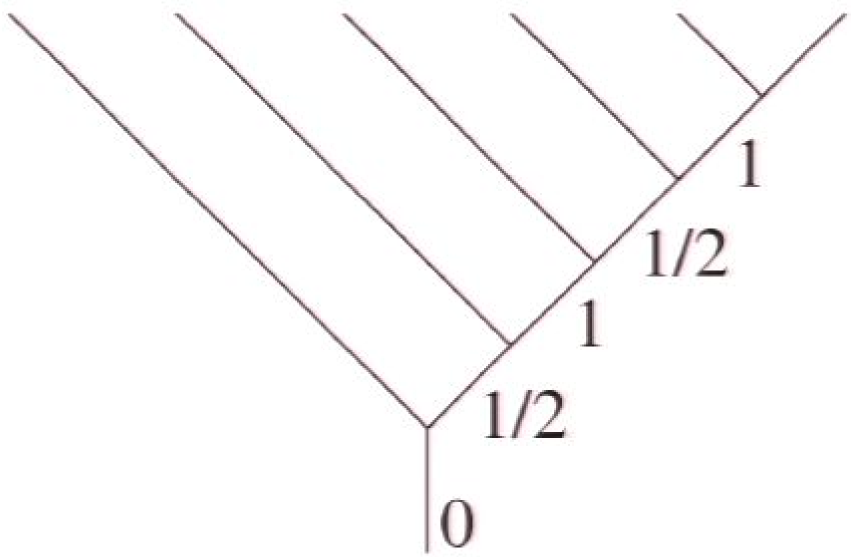
$$\rho_\lambda(\sigma_i)\Lambda = \frac{1}{\tau_i^\Lambda}\Lambda + \sqrt{1 - \frac{1}{(\tau_i^\Lambda)^2}}\Lambda'$$

$$\rho_{\begin{array}{|c|c|}\hline \; & \; \\ \hline \; \\ \cline{1-1}\end{array}}(\sigma_2)\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} = -\frac{1}{2}\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} + \frac{\sqrt{3}}{2}\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$


1
3
2

1	2	4
3	5	6





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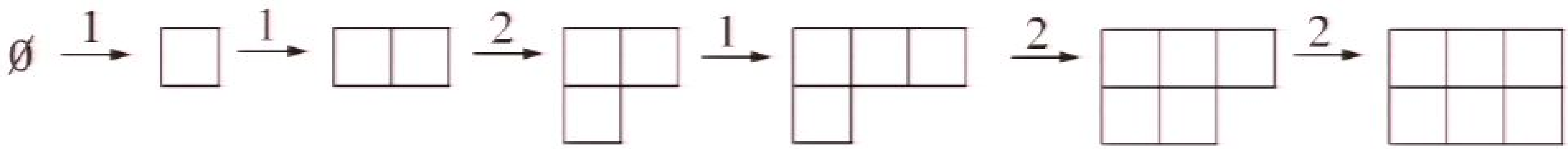
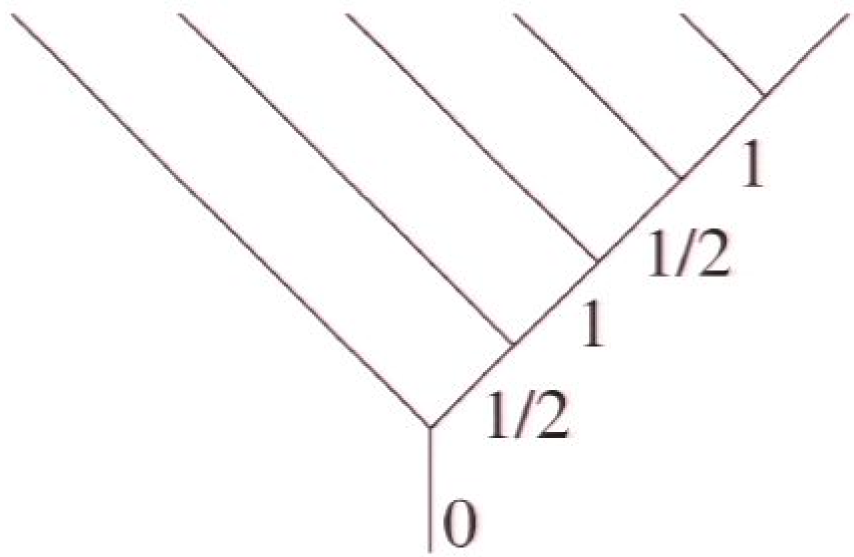
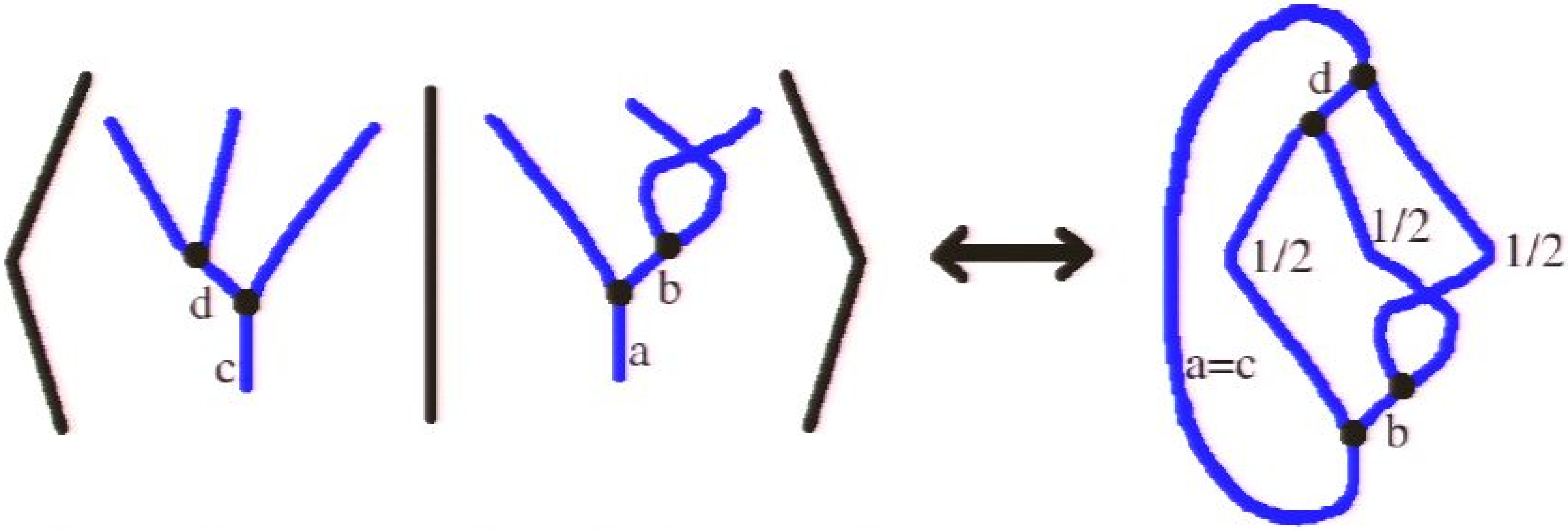
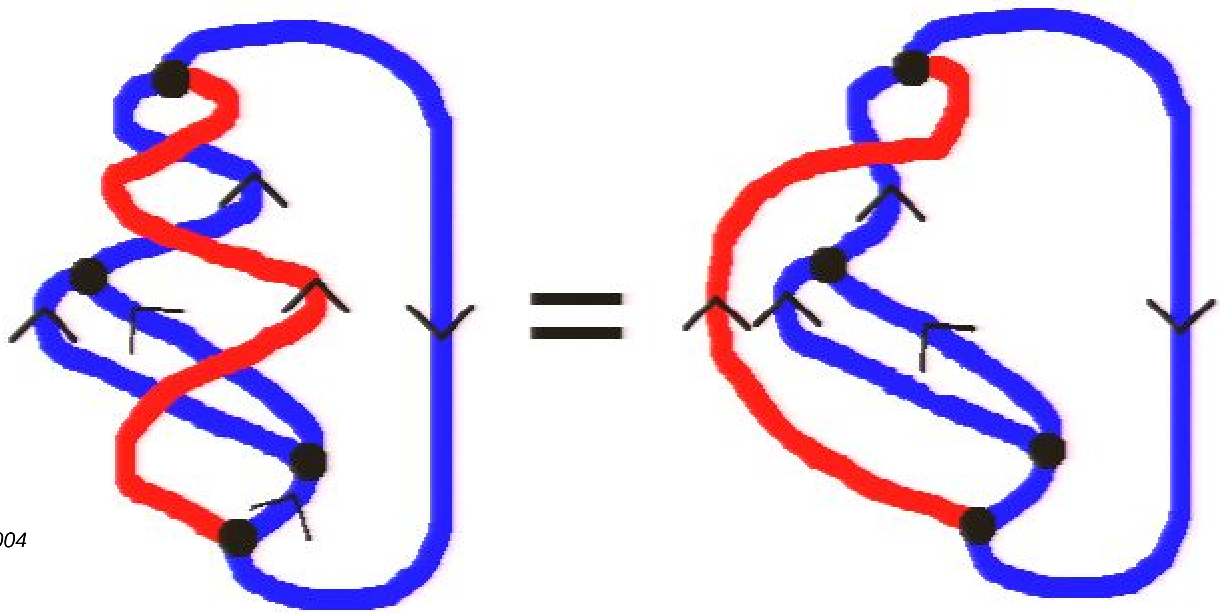




- Most general process:



- Invariant under deformation:



“spin network”

- We now have a model of computation:

  1) Prepare a basis state from some complete set of commuting angular momentum operators.

  2) Permute the qubits.

  3) Measure some other complete set of commuting angular momentum operators

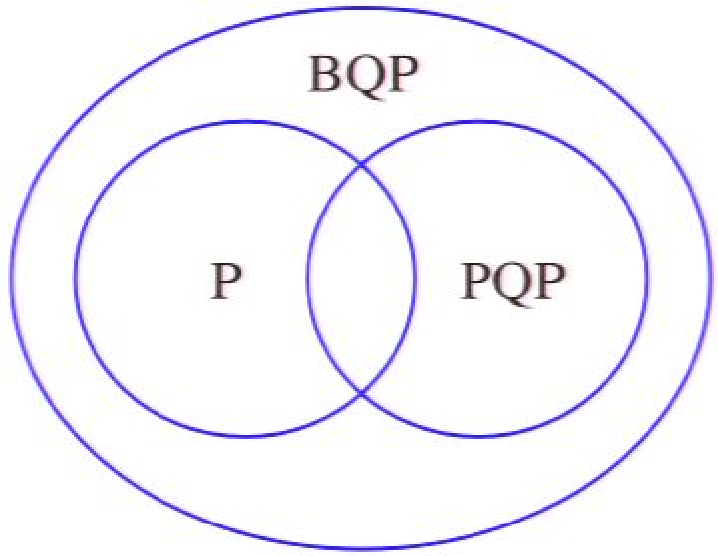
- We also have a problem it can solve:

  Approximate a matrix element from Young's orthogonal form

[See Also Marzuoli et al.]

How Powerful Is It?

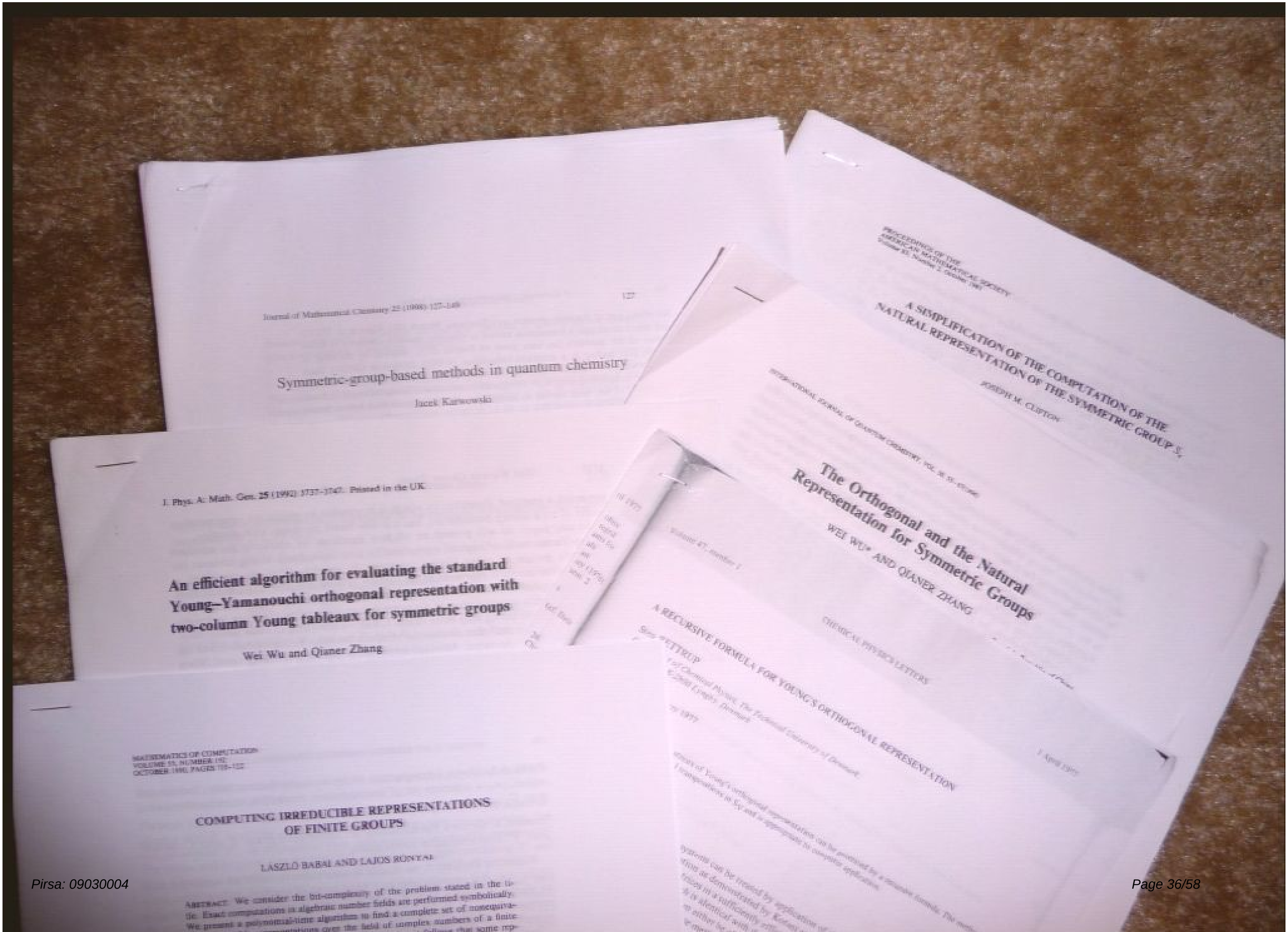
- What I think:



- What I know:

$$\mathrm{PQP} \subset \mathrm{BQP}$$

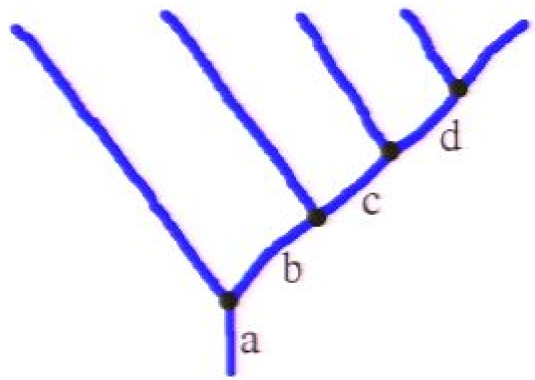
Best classical algorithms for Young's orthogonal form are exponential time


Symmetric-group-based methods in quantum chemistry
Jacek Karwowski
A SIMPLIFICATION OF THE COMPUTATION OF THE NATURAL REPRESENTATION OF THE SYMMETRIC GROUP S_n
JOSEPH M. CLIFTON
J. Phys. A: Math. Gen. 25 (1992) 3737–3747. Printed in the UK
An efficient algorithm for evaluating the standard Young–Yamanouchi orthogonal representation with two-column Young tableaux for symmetric groups
Wei Wu and Qianer Zhang
The Orthogonal and the Natural Representation for Symmetric Groups
WEI WU* AND QIANER ZHANG
CHEMICAL PHYSICS LETTERS
A RECURSIVE FORMULA FOR YOUNG'S ORTHOGONAL REPRESENTATION
COMPUTING IRREDUCIBLE REPRESENTATIONS OF FINITE GROUPS
LASZLO BABAI AND LAJOS RONYAI

$\mathrm{PQP} \subset \mathrm{BQP}$

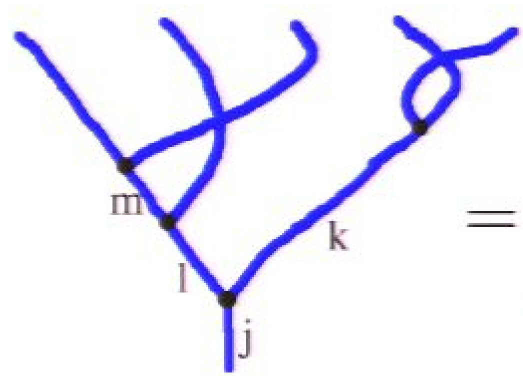
Proof Sketch:

work in this basis:



$$= |abcd\rangle$$

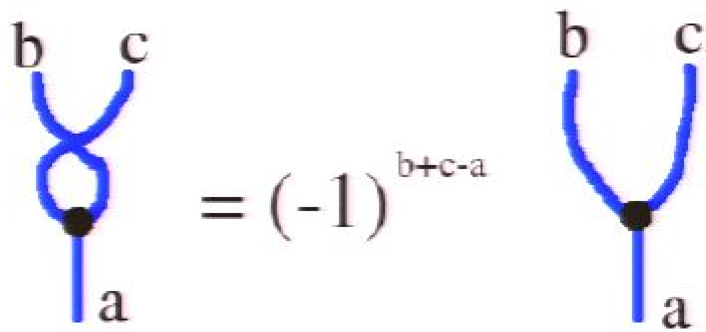
make any PQP state
by polynomially many
F and R moves



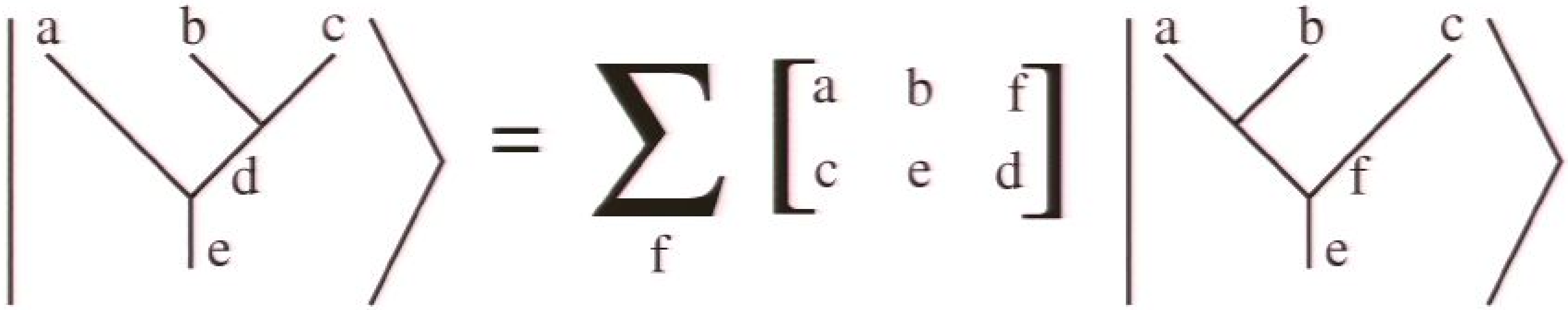
$$= \sum_{abcd} \psi(abcd)|abcd\rangle$$

use Hadamard test

- R is easy to implement: just a phase



- How about F?



  - it is sparse
  - we can efficiently compute the nonzero entries using the Racah formula

- We know how to implement any sparse row-computable Hamiltonian.
- From this we can implement any sparse row- and column-computable unitary.

$$H = \begin{bmatrix} 0 & U \\ U^\dagger & 0 \end{bmatrix}$$

$$e^{iH\pi/2} = i \begin{bmatrix} 0 & U \\ U^\dagger & 0 \end{bmatrix}$$

End of Proof Sketch.

[Joint work with Pawel Wocjan]

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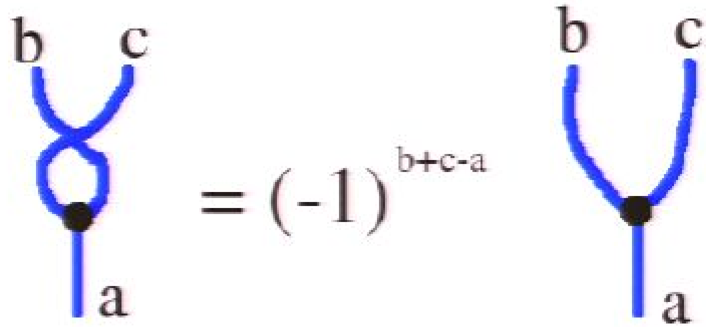

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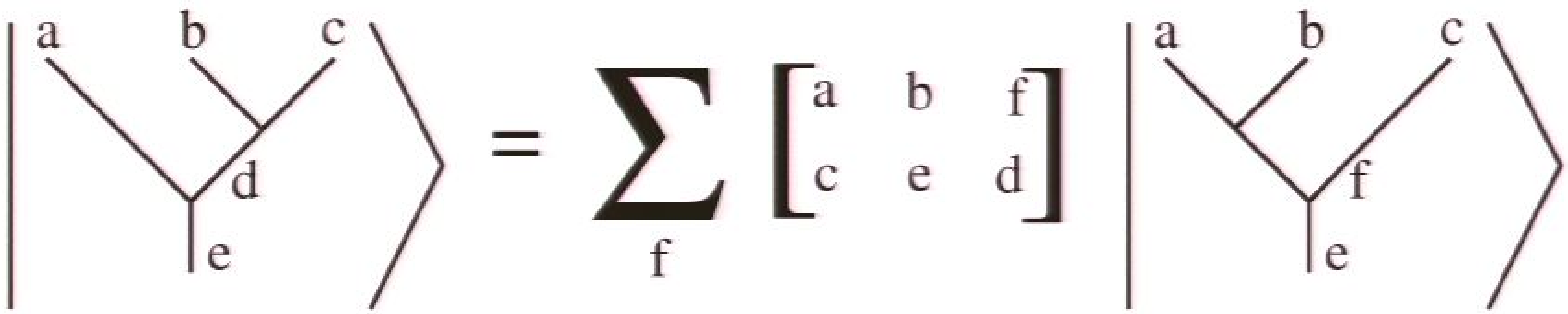

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  - $\mathrm{PQP} \subset \mathrm{BQP}$
  - probably $\mathrm{PQP} \not\subset \mathrm{P}$
- $\mathrm{PQP} = \mathrm{BQP}$?
- I doubt it because:
  - S_n is finite.
  - $\therefore$ no representation S_n can be dense in any unitary group.
  - $\therefore$ cannot use Solovay-Kitaev

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PQP $\subset$ BQP

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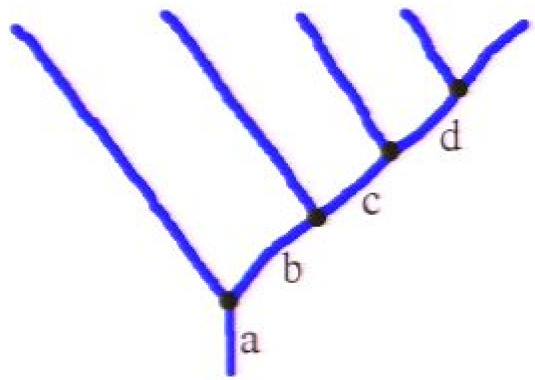

$$= |abcd\rangle$$

make any PQP state
by polynomially many
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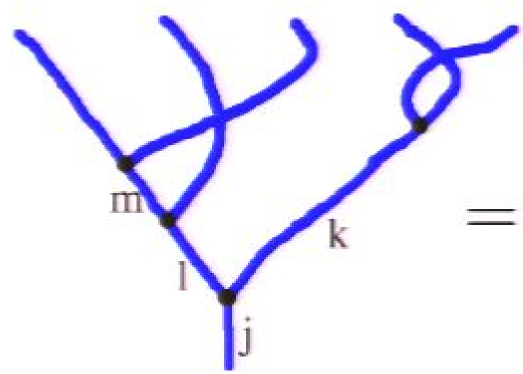

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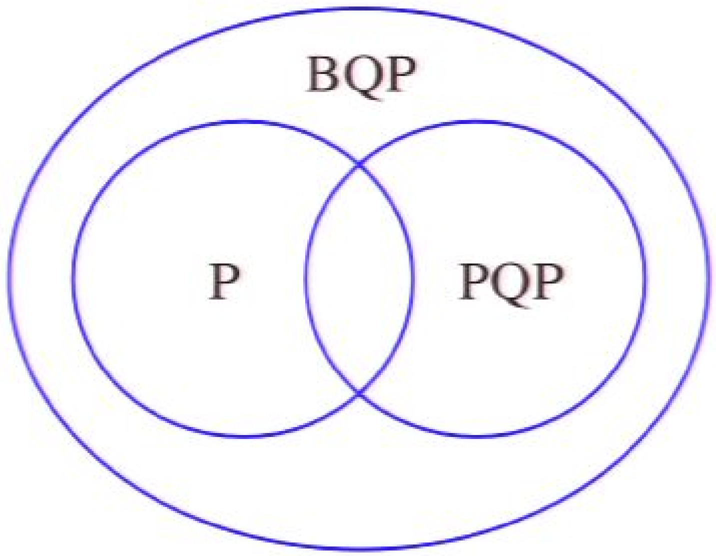

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	B_n	S_n
Matrix Elements	BQP-complete	$\subset \mathrm{BQP}$
Characters	DQC1-complete	$\subset \mathrm{BPP}$

- all irreps of S_n are implementable in BQP
- exact characters of S_n are #P-complete [Hepler]
- normalized characters of S_n are approximable to polynomial precision in BPP
- further evidence $\mathrm{PQP} \neq \mathrm{BQP}$?

Normalized Characters of S_n in BPP Proof:

Theorem 1 (Roichman) *For any partitions* $\mu = (\mu_1, \ldots, \mu_l)$ *and* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *of* n*, the corresponding irreducible character of* S_n *is given by*

$$\chi_\mu^\lambda = \sum_\Lambda W_\mu(\Lambda)$$

where the sum is over all standard Young tableaux Λ *of shape* λ *and*

$$W_\mu(\Lambda) = \prod_{\substack{1 \le i \le k \\ i \notin B(\mu)}} f_\mu(i, \Lambda)$$

where $B(\mu) = \{\mu_1 + \ldots + \mu_r | 1 \le r \le l\}$ *and*

$$f_\mu(i,\Lambda) = \begin{cases} -1 & \text{box } i+1 \text{ of } \Lambda \text{ is in the southwest of box } i \\ 0 & i+1 \text{ is northeast of } i,\ i+2 \text{ is southwest of } i+1, \text{ and } i+1 \notin B(\mu) \\ 1 & \text{otherwise} \end{cases}$$

Theorem 2 (Greene, Nijenhuis, and Wilf) *With polynomial resources, one can sample uniformly from the standard Young Tableaux corresponding to a given shape (n-box Young diagram) using the Hook walk algorithm.*

End of Proof.

Summary

- We formulate a model like topological QC except we permute spin-1/2 particles instead of braiding anyons
- The resulting complexity class PQP is in BQP
- We can compute Young's orthogonal form in PQP thus probably $\mathrm{PQP} \not\subset \mathrm{P}$
- The corresponding computational model based on characters of S_n is in BPP

Usefulness/Open Questions

- Algorithms
  - ✔ Young's Orthogonal Form (& 3nj symbols).
  - ➔ Physics? Geometry? Ponzano-Regge?
- Fault Tolerance/Implementation
  - ✔ angular momentum implementation
  - ➔ parastistical quasiparticles?
- Complexity Theory
  - New complexity class. maybe
  - ➔ Oracle separation between PQP and BQP?

- For an exponentially large unitary matrix the average magnitude of the matrix elements is exponentially small.
- We approximate to polynomial precision?
- Is this trivial?
  - For random instances: yes.
  - In worst case: probably not.

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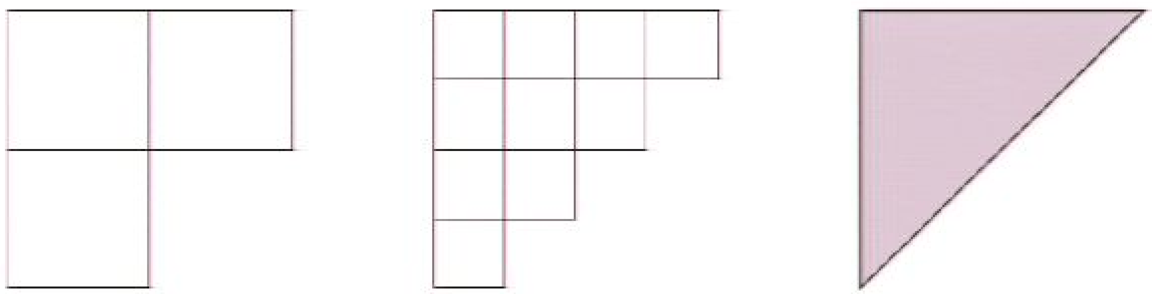
- The normalized character tells us the average diagonal element.
- In certain cases this is large.



3 10 ∞

$$\frac{\chi_{\lambda_n}(\pi)}{d_{\lambda_n}} = C_\pi(\omega) n^{-|\pi|/2} + O(n^{-|\pi|/2-1})$$

[Biane]

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