

Title: deBroglie-Bohm and the Timeless Mechanics of Jacobi

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Abstract: A standard canonical quantization of general relativity yields a time-independent Schroedinger equation whose solutions are static wavefunctions on configuration space. Naively this is in contradiction with the real world where things do change. Broadly speaking, the problem how to reconcile a theory which contains no concept of time with a changing world is called 'the problem of time'. In this seminar we shall study this problem using a reformulation of Newtonian mechanics due to Jacobi (Jacobi's timeless mechanics) which allows one to study the problem of time without all the technical difficulties present in quantized general relativity. We show explicitly that Jacobi's timeless mechanics is a straightforward counterexample to the claim that all first class constraints generate gauge transformations, i.e. physically indistinguishable states. The implications of this is unclear. By making use of deBroglie-Bohm trajectories we derive a necessary and sufficient condition for a time-dependent Schroedinger equation to emerge for subsystems. The importance of 'strong' entanglement between subsystem and environment for the emergence of time is stressed.

de Broglie-Bohm & Jauch's Timeless Mechanics

de Broglie-Bohm & Jauch's Timeless Mechanics

Newtonian Mechanics

de Broglie-Bohm & Jauch's Timeless Mechanics

Newtonian Mechanics

$$L = \sum_n \frac{1}{2} m_n \frac{d\mathbf{x}_n}{dt} - V(\mathbf{x}) \quad \mathbf{x} = (x_1, \dots, x_n)$$

de Broglie-Bohm & Jauch's Timeless Mechanics

Newtonian Mechanics

$$L = \sum_n \frac{1}{2} m_n \frac{dx_n}{dt} - V(\mathbf{x}) \quad \mathbf{x} = (x_1, \dots, x_n)$$



Elimination of time

Step ① Formulate NM on extended configuration space.



de Broglie-Bohm & Jauch's Timeless Mechanics

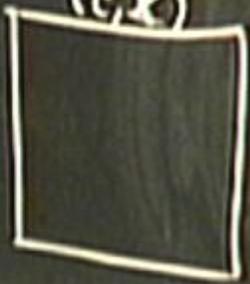
Newtonian Mechanics

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \sum_n \frac{1}{2} m_n \frac{d\mathbf{x}_n}{dt}^2 - V(\mathbf{x}) \quad \mathbf{x} = (x_1, \dots, x_N)$$



Elimination of time

Step ① Formulate NM in extended configuration space.



$$L(\mathbf{x}, \dot{\mathbf{x}}, t) = L(\mathbf{x}, \frac{d\mathbf{x}}{dt}, t) = \sum_n \frac{1}{2} m_n \frac{d\mathbf{x}_n}{dt}^2 - V(\mathbf{x}, t)$$

de Broglie-Bohm & Jauch's Timeless Mechanics

Newtonian Mechanics

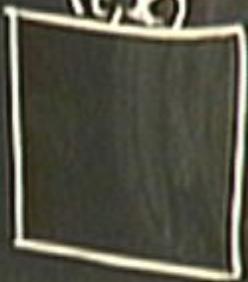
$$L(\mathbf{x}, \dot{\mathbf{x}}) = \sum_n \frac{1}{2} m_n \frac{d\mathbf{x}_n}{dt}^2 - V(\mathbf{x}) \quad \mathbf{x} = (x_1, \dots, x_n)$$

(time)



Elimination of time

Step ① Formulate NM on extended configuration space.



$$\tilde{L}(\mathbf{x}, \dot{\mathbf{x}}, t) = L(\mathbf{x}, \frac{d\mathbf{x}}{dt}, t) = \sum_n \frac{1}{2} m_n \frac{dx_n^2}{dt^2} - V t'$$

Since $V(\mathbf{x})$ is indep of t

$$\frac{\partial V}{\partial t} - \frac{d}{dt} \left(\frac{\partial L}{\partial t'} \right) = 0$$

$$\Rightarrow \frac{\partial L}{\partial t'} = \text{const} = -E$$

$$\frac{\partial L}{\partial t'} = - \left(\sum \frac{1}{2} m_n \frac{x_n'^2}{t'^2} + V \right) = -E$$

Step ②

Since $V(\mathbf{x})$ is indep of t

$$\frac{\partial \bar{L}}{\partial t} - \frac{d}{dt} \left(\frac{\partial \bar{L}}{\partial t'} \right) = 0$$

$$\Rightarrow \frac{\partial \bar{L}}{\partial t'} = \text{const} = -E$$

$$\frac{\partial \bar{L}}{\partial t'} = - \left(\sum \frac{1}{2} m_n \frac{x_n'^2}{t'^2} + V \right) = -E \quad *$$

Step 2

Solve for t' in eq. *

$$\Rightarrow t' = \dots$$

Since $V(\mathbf{x})$ is indep of t

$$\frac{\partial \bar{V}}{\partial t} - \frac{d}{d\lambda} \left(\frac{\partial \bar{L}}{\partial t'} \right) = 0$$

$$\Rightarrow \frac{\partial \bar{L}}{\partial t'} = \text{const} = -E$$

$$\frac{\partial \bar{L}}{\partial t'} = - \left(\sum \frac{1}{2} m_n \frac{x_n'^2}{t'^2} + V \right) = -E \quad *$$

Step ②

Solve for t' in eq. *

$$\Rightarrow t' = \sqrt{\frac{T}{E - V}} \quad T = \sum_n \frac{1}{2} m_n x_n'^2$$

Since $V(\mathbf{x})$ is indep of t

$$\frac{\partial V}{\partial t} - \frac{d}{dt} \left(\frac{\partial \bar{L}}{\partial t'} \right) = 0$$

$$\Rightarrow \frac{\partial \bar{L}}{\partial t'} = \text{const} = -E$$

$$\frac{\partial \bar{L}}{\partial t'} = - \left(\sum_n \frac{1}{2} m_n \frac{x_n'^2}{t'^2} + V \right) = -E \quad *$$

Step ②

Solve for t' in eq. *

$$t' = \sqrt{\frac{T}{E - V}} \quad T = \sum_n \frac{1}{2} m_n x_n'^2$$

Step ③

Define $\bar{L} = \bar{L} - (E)t'$
& insert eq. \square

$$\Rightarrow L_3(\mathbf{x}, \mathbf{x}') = 2 \sqrt{T(E - V)}$$

Since $V(\mathbf{x})$ is indep of t

$$\frac{\partial V}{\partial t} - \frac{d}{dt} \left(\frac{\partial L}{\partial t'} \right) = 0$$

$$\Rightarrow \frac{\partial L}{\partial t'} = \text{const} = -E \quad \text{---} \frac{d}{dt}$$

$$\frac{\partial L}{\partial t'} = - \left(\sum_n \frac{1}{2} m_n \dot{x}_n^2 + V \right) = -E \quad *$$

Step 2

Solve for t' in eq. *

$$\Rightarrow t' = \sqrt{\frac{T}{E - V}} \quad T = \sum_n \frac{1}{2} m_n \dot{x}_n^2$$

Step 3

Define $\bar{L} = L - (E)t'$
& insert eq. \square

$$\Rightarrow L_3(\mathbf{x}, \mathbf{x}') = 2 \sqrt{T(E - V)}$$

Since $V(x)$ is indep of t

$$\frac{\partial V}{\partial t} - \frac{d}{dt} \left(\frac{\partial L}{\partial t'} \right) = 0$$

$$\Rightarrow \frac{\partial L}{\partial t'} = \text{const} = -E \quad \text{---} \frac{d}{dt}$$

$$\frac{\partial L}{\partial t'} = - \left(\sum_n \frac{1}{2} m_n \dot{x}_n^2 + V \right) = -E \quad *$$

Step 2

Solve for t' in eq. *

$$\Rightarrow t' = \sqrt{\frac{T}{E - V}} \quad T = \sum_n \frac{1}{2} m_n \dot{x}_n^2$$

Step 3

Define $\bar{L} = L - (E)t'$
& insert eq. \square

$$\Rightarrow L_3(x, x') = 2 \sqrt{T(E - V)}$$

Since $V(x)$ is indep of t

$$\frac{\partial \bar{L}}{\partial t} - \frac{d}{dt} \left(\frac{\partial \bar{L}}{\partial \dot{t}'} \right) = 0$$

$$\Rightarrow \frac{\partial \bar{L}}{\partial \dot{t}'} = \text{const} = -E \quad \text{---} \frac{d}{dt}$$

$$\frac{\partial \bar{L}}{\partial \dot{t}'} = - \left(\sum \frac{1}{2} m_n \frac{x_n^2}{t'^2} + V \right) = -E \quad *$$

Step ②

Solve for t' in eq. *

$$\Rightarrow t' = \sqrt{\frac{T}{E-V}} \quad T = \sum_n \frac{1}{2} m_n x_n^2$$

Step ③

Define $\bar{L} = \bar{L} - (E)t'$

& insert eq. □

$$\Rightarrow L_3(x, x') = 2 \sqrt{T(E-V)}$$

Reparametrization invariance

L_3 should be homogeneous to 1st deg

$$L_3(x, x') = 1 L(x, x')$$

Since $V(\mathbf{x})$ is indep of t

$$\frac{\partial \bar{L}}{\partial t} - \frac{d}{dt} \left(\frac{\partial \bar{L}}{\partial \dot{t}'} \right) = 0$$

$$\Rightarrow \frac{\partial \bar{L}}{\partial \dot{t}'} = \text{const} = -E \quad \text{---} \frac{d}{dt}$$

$$\frac{\partial \bar{L}}{\partial t'} = - \left(\sum_{i=1}^n \frac{1}{2} m_i \frac{\dot{x}_i^2}{t'^2} + V \right) = -E \quad *$$

Step 2

Solve for t' in eq. *

$$\Rightarrow t' = \sqrt{\frac{T}{E-V}} \quad T = \sum_{i=1}^n \frac{1}{2} m_i \dot{x}_i^2$$

Step 3

Define $\bar{L} = \bar{L} - (E)t'$

& insert eq. \square

$$\Rightarrow L_3(\mathbf{x}, \dot{\mathbf{x}}) = 2 \sqrt{T(E-V)}$$

Reparametrization invariance

L_3 should be homogeneous of 1st order

$$L_3(\mathbf{x}, c\dot{\mathbf{x}}) = c L_3(\mathbf{x}, \dot{\mathbf{x}}) \quad c > 0$$

Eas d met.

$$\sqrt{\frac{E-V}{T}} \frac{d}{d\lambda} \left(m_u \sqrt{\frac{E-V}{T}} \frac{dX_u}{d\lambda} \right) = -\nabla_u V$$

\mathcal{S} -solution space.

$$X(\lambda) \in \mathcal{S} \Rightarrow X(f(\lambda)) \in \mathcal{S}$$

de Broglie-Bohm & Jauch's Timeless Mechanics

Newtonian Mechanics

$$L(\mathbf{X}, \dot{\mathbf{X}}) = \sum_n \frac{1}{2} m_n \frac{d\mathbf{x}_n}{dt} - V(\mathbf{X}) \quad \mathbf{X} = (x_1, \dots, x_N)$$

(time)



$$P_t + P^2 - V = 0$$

$$P^2 + V - E = 0$$

Elimination of time

Step 1 Formulate NM in extended configuration space.



$$L(\mathbf{X}, \mathbf{X}', t) = L(\mathbf{X}, \frac{\mathbf{X}'}{t}) t = \sum_n \frac{1}{2} m_n \frac{\mathbf{x}_n'^2}{t^2} - V t'$$

(\mathbf{X}, t)

Eqs of mot.

$$\sqrt{\frac{E-V}{T}} \frac{d}{d\lambda} \left(m_{\mu} \sqrt{\frac{E-V}{T}} \frac{dx_{\mu}}{d\lambda} \right) = -\nabla_{\mu} V$$

\mathcal{S} - solution space.

$$\underline{\mathcal{X}(\lambda) \in \mathcal{S} \Rightarrow \mathcal{X}(f(\lambda)) \in \mathcal{S}}$$

Important point:

λ cannot be interpreted operationally.

Eqs of mot.

$$\sqrt{\frac{E-V}{T}} \frac{d}{d\lambda} \left(m_u \sqrt{\frac{E-V}{T}} \frac{d\psi_u}{d\lambda} \right) = -\nabla_u V$$

\mathcal{S} - solution space.

$$\underline{\mathcal{X}(\omega) \in \mathcal{S} \Rightarrow \mathcal{X}(f(\omega)) \in \mathcal{S}}$$

Important point:

λ cannot be interpreted operationally.

Recovery of NM

$$dt = \sqrt{\frac{T}{E-V}} d\lambda$$

$$\lambda = t \Rightarrow \frac{T}{E-V} = 1$$

$$\textcircled{T+V=E}$$

$$m_u \frac{d^2 \psi_u}{dt^2} = -\nabla_u V$$

Eqs of mot.

$$\sqrt{\frac{E-V}{T}} \frac{d}{d\lambda} \left(m_n \sqrt{\frac{E-V}{T}} \frac{dx_n}{d\lambda} \right) = -\nabla_n V$$

\mathcal{S} - solution space.

$$\underline{\mathcal{X}(\omega) \in \mathcal{S} \Rightarrow \mathcal{X}(f(\lambda)) \in \mathcal{S}}$$

Important point:

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Recovery of NM

$$dt = \sqrt{\frac{T}{E-V}} d\lambda$$

$$\lambda = t \Rightarrow \frac{T}{E-V} = 1$$

$$\boxed{T+V=E}$$

$$m_n \frac{d^2 x_n}{dt^2} = -\nabla_n V \Big|_{t=\int d\lambda}$$

Hamiltonian Formulation

EAS direct.

$$\sqrt{\frac{E-V}{T}} \frac{d}{d\lambda} \left(m_n \sqrt{\frac{E-V}{T}} \frac{dx_n}{d\lambda} \right) = -\nabla_n V$$

\mathcal{S} -solution space.

$$\underline{\mathcal{X}(\omega) \in \mathcal{S} \Rightarrow \mathcal{X}(f(\lambda)) \in \mathcal{S}}$$

Important point:

λ cannot be interpreted operationally.

Recovery of NM

$$dt = \sqrt{\frac{T}{E-V}} d\lambda$$

$$\lambda = t \Rightarrow \frac{T}{E-V} = 1$$

$$\boxed{T+V=E}$$

$$m_n \frac{d^2 x_n}{dt^2} = -\nabla_n V \sqrt{\frac{T}{E-V}}$$

Hamiltonian Formulation

Eqs of mot.

$$\sqrt{\frac{E-V}{T}} \frac{d}{d\lambda} \left(m_{cl} \sqrt{\frac{E-V}{T}} \frac{dx_{cl}}{d\lambda} \right) = -\nabla_{cl} V$$

\mathcal{S} - solution space.

$$\underline{\mathcal{X}(\omega) \in \mathcal{S} \Rightarrow \mathcal{X}(f(\lambda)) \in \mathcal{S}}$$

Important point:

λ cannot be interpreted operationally.

Recovery of NM

$$dt = \sqrt{\frac{T}{E-V}} d\lambda \quad \lambda = t \rightarrow \frac{T}{E-V} = 1$$

$$\textcircled{T+V=E}$$

$$m_{cl} \frac{d^2 x_{cl}}{dt^2} = -\nabla_{cl} V$$

Hamiltonian Formulation



Eqs of mot.

$$\sqrt{\frac{E-V}{T}} \frac{d}{d\lambda} \left(m_{cl} \sqrt{\frac{E-V}{T}} \frac{dx_n}{d\lambda} \right) = -\nabla_n V$$

\mathcal{S} - solution space.

$$\underline{\mathcal{X}(\lambda) \in \mathcal{S} \Rightarrow \mathcal{X}(f(\lambda)) \in \mathcal{S}}$$

Important point:

λ cannot be interpreted operationally.

Recovery of NM

$$dt = \sqrt{\frac{T}{E-V}} d\lambda \quad \lambda = t \Rightarrow \frac{T}{E-V} = 1$$

$$\textcircled{T+V=E}$$

$$m_{cl} \frac{d^2 x_n}{dt^2} = -\nabla_n V \quad \left(\frac{dx_n}{d\lambda} \right) \left(\frac{d\lambda}{dt} \right) \sqrt{\frac{T}{E-V}}$$

Hamiltonian Formulation



de Broglie-Bohm & Jacobi's Timeless Mechanics

Newtonian Mechanics

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \sum_n \frac{1}{2} m_n \frac{d\mathbf{x}_n}{dt} - V(\mathbf{x}) \quad \mathbf{x} = (x_1, \dots, x_n)$$

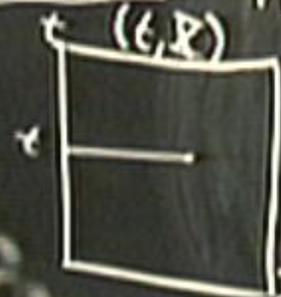
(time)



$$\begin{aligned}
 & \downarrow \\
 & p_t + p^2 - V = 0 \\
 & \downarrow \\
 & \frac{1}{2} m \dot{x}^2 - V = 0 \\
 & \downarrow \\
 & p^2 + V - E = 0
 \end{aligned}$$

Elimination of time

Step 1 Formulate NM on extended config space.



$$L(\mathbf{x}, \mathbf{x}', t) = L(\mathbf{x}, \frac{d\mathbf{x}}{dt}) \Big|_{t'} = \sum_n \frac{1}{2} m_n \frac{d\mathbf{x}_n}{dt'} - V \underline{t'}$$

Eqs of mot.

$$\sqrt{\frac{E-V}{T}} \frac{d}{d\lambda} \left(m_n \sqrt{\frac{E-V}{T}} \frac{dx_n}{d\lambda} \right) = -\nabla_n V$$

\mathcal{S} - solution space.

$$\underline{\mathbf{x}(\lambda)} \in \mathcal{S} \Rightarrow \underline{\mathbf{x}(f(\lambda))} \in \mathcal{S}$$

Important point:

λ cannot be interpreted operationally.

Recovery of NM

$$dt = \sqrt{\frac{T}{E-V}} d\lambda \quad \lambda = t \rightarrow \frac{T}{E-V} = 1$$

$$T + V = E$$

$$m_n \frac{d^2 x_n}{dt^2} = -\nabla_n V$$

Hamiltonian Formulation

Canonical momenta:

$$P_n \equiv \frac{\partial L}{\partial \dot{x}_n} = \frac{m_n \dot{x}_n}{\sqrt{\frac{T}{E-V}}}$$

$$P = (P_n)$$

$$P(\mathbf{x})$$



Eqs of mot.

$$\sqrt{\frac{E-V}{T}} \frac{d}{d\lambda} \left(m_{ii} \sqrt{\frac{E-V}{T}} \frac{dx_{ii}}{d\lambda} \right) = -\nabla_{ii} V$$

S - solution space.

$$\underline{\underline{\mathbf{x}(\lambda) \in \mathcal{S} \Rightarrow \mathbf{x}(f(\lambda)) \in \mathcal{S}}}$$

Important point:

λ cannot be interpreted operationally.

Recovery of NM

$$dt = \sqrt{\frac{I}{E-V}} d\lambda \quad \lambda = t \rightarrow \frac{I}{E-V} = 1$$

$$\textcircled{T+V=E}$$

$$m_{ii} \frac{d^2 x_{ii}}{dt^2} = -\nabla_{ii} V$$

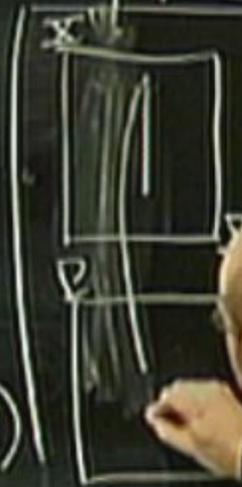
Hamiltonian Formulation

Canonical momenta:

$$p_{ii} \equiv \frac{\partial L_{ii}}{\partial \dot{x}_{ii}} = m_{ii} \dot{x}_{ii} \sqrt{\frac{I}{E-V}}$$

$$\mathbf{P} = (p_1, \dots, p_N)$$

$$P(\mathbf{x}, \dot{\mathbf{x}}) = P(\mathbf{x}, \mathbf{P})$$



Eqs of mot.

$$\sqrt{\frac{E-V}{T}} \frac{d}{d\lambda} \left(m_{ii} \sqrt{\frac{E-V}{T}} \frac{dx_{ii}}{d\lambda} \right) = -\nabla_{ii} V$$

S - Solution space.

$$\underline{\underline{\mathbf{x}(\lambda) \in \mathcal{S} \Rightarrow \mathbf{x}(f(\lambda)) \in \mathcal{S}}}$$

Important point:

λ cannot be interpreted operationally.

Reversal of NM

$$dt = \sqrt{\frac{T}{E-V}} d\lambda \quad \lambda = t \Rightarrow \frac{T}{E-V} = 1$$

$$\textcircled{T+V=E}$$

$$m_{ii} \frac{d^2 x_{ii}}{dt^2} = -\nabla_{ii} V$$

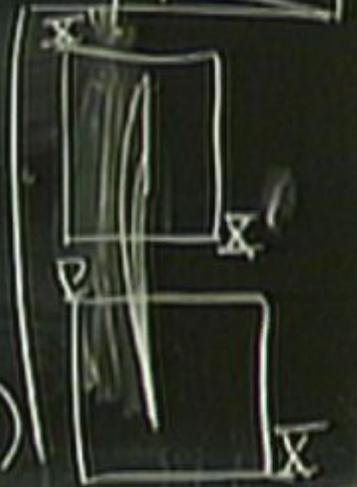
Hamiltonian Formulation

Canonical momenta:

$$p_{ii} \equiv \frac{\partial L_{ii}}{\partial \dot{x}_{ii}} = m_{ii} \dot{x}_{ii} \sqrt{\frac{T}{E-V}}$$

$$\mathbf{P} = (p_1, \dots, p_N)$$

$$P(\mathbf{x}, \dot{\mathbf{x}}) = P(\mathbf{x}, \mathbf{P})$$



Eqs of mot.

$$\sqrt{\frac{E-V}{T}} \frac{d}{d\lambda} \left(m_n \sqrt{\frac{E-V}{T}} \frac{dx_n}{d\lambda} \right) = -\nabla_n V$$

\mathcal{S} - solution space.

$$\underline{x(\lambda)} \in \mathcal{S} \Rightarrow \underline{x}(t(\lambda)) \in \mathcal{S}$$

Important point:

λ cannot be interpreted operationally.

Recovery of NM

$$dt = \sqrt{\frac{T}{E-V}} d\lambda \quad \lambda = t \rightarrow \frac{T}{E-V} = 1$$

$$T + V = E$$

$$m_n \frac{d^2 x_n}{dt^2} = -\nabla_n V$$

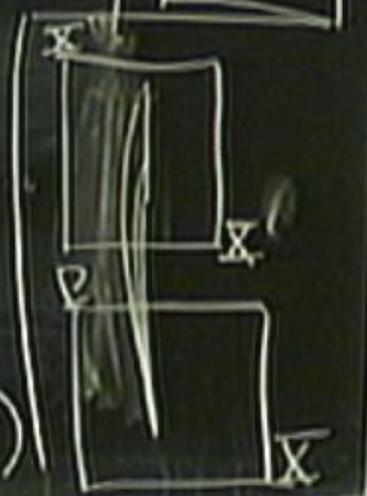
Hamiltonian Formulation

Canonical momenta:

$$p_n \equiv \frac{\partial L}{\partial \dot{x}_n} = \frac{m_n \dot{x}_n}{\sqrt{\frac{T}{E-V}}}$$

$$P = (p_1, \dots, p_N)$$

$$P(\underline{x}, \dot{\underline{x}}) = P(\underline{x}, \underline{P})$$



Canonical Hamiltonian

$$H_c(\mathbf{x}, \mathbf{p}) \equiv \sum_n \mathbf{p}_n \dot{\mathbf{x}}_n - L \equiv 0$$

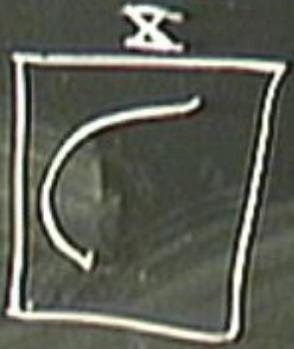
$$X = \sum_n \frac{\mathbf{p}_n^2}{2m_n} + V - E \approx 0$$

Total Hamiltonian

$$H_{\text{tot}} = H_c + N(\mathbf{x}, \mathbf{p})$$

$$L(x) = \sum_n \frac{1}{2} m_n \frac{dx_n}{dt} - V(x)$$

$$x = (x_1, \dots, x_n)$$

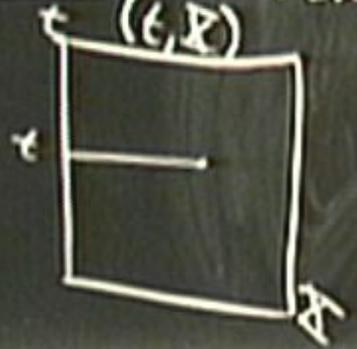


$$p^2 + V - E = 0$$

$$p^2 + V - E = 0$$

Elimination of time

Step 1 Formulate NM in extended configuration space.



$$L(x, \dot{x}, t) = L(x, \frac{x}{t}, t) = \sum_n \frac{1}{2} m_n \dot{x}_n^2 - V(x)$$

$$x = \sum_n \frac{p_n^2}{2m_n} + V - E = 0$$

Total Hamiltonian

$$H_{tot} = H_c + M$$

Canonical Hamiltonian

$$H_c(\mathbf{x}, \mathbf{P}) \equiv \sum_n \mathbf{P}_n \cdot \dot{\mathbf{x}}_n - L \equiv 0$$

$$X = \sum_n \frac{\mathbf{P}_n^2}{2m_n} + V - E \approx 0$$

Total Hamiltonian

$$H_{tot} = \underbrace{H_c}_0 + N(\mathbf{x}, \mathbf{P})X$$

$$H_{tot} = N \left(\sum_n \frac{\mathbf{P}_n^2}{2m_n} + V - E \right)$$

Eqs of motion

$$\dot{\mathbf{x}}_n = \frac{\partial H_{tot}}{\partial \mathbf{P}_n} = N \frac{\mathbf{P}_n}{m_n}$$

$$\dot{\mathbf{P}}_n = - \frac{\partial H_{tot}}{\partial \mathbf{x}_n} = -N \nabla_n V$$

Canonical Hamiltonian

$$H_c(x, p) \equiv \sum_n p_n \dot{x}_n - L \equiv 0$$

$$X = \sum_n \frac{p_n^2}{2m_n} + V - E \approx 0$$

Total Hamiltonian

$$H_{tot} = \underset{0}{H_c} + N(x, p) X$$

$$H_{tot} = N \left(\sum_n \frac{p_n^2}{2m_n} + V - E \right)$$

Eqs of motion

$$\dot{x}_n = \frac{\partial H_{tot}}{\partial p_n} = N \frac{p_n}{m_n}$$

$$p_n' = - \frac{\partial H_{tot}}{\partial x_n} = - N \nabla_n V$$

Canonical Hamiltonian

$$H_c(\mathbf{x}, \mathbf{P}) \equiv \sum_n \mathbf{P}_n \cdot \mathbf{x}'_n - L \equiv 0$$

$$X = \sum_n \frac{\mathbf{P}_n^2}{2m_n} + V - E \approx 0$$

Total Hamiltonian

$$H_{tot} = \underset{0}{H_c} + N(\mathbf{x}, \mathbf{P})X$$

$$H_{tot} = N \left(\sum_n \frac{\mathbf{P}_n^2}{2m_n} + V - E \right)$$

Eqs of motion

$$\mathbf{x}'_n = \frac{\partial H_{tot}}{\partial \mathbf{P}_n} = N \frac{\mathbf{P}_n}{m_n}$$

$$\mathbf{P}'_n = - \frac{\partial H_{tot}}{\partial \mathbf{x}_n} = -N \nabla_n V$$

Canonical Hamiltonian

$$H_c(\mathbf{x}, \mathbf{p}) \equiv \sum_n \mathbf{p}_n \cdot \dot{\mathbf{x}}_n - L \equiv 0$$

$$\chi = \sum_n \frac{\mathbf{p}_n^2}{2m_n} + V - E \approx 0$$

Total Hamiltonian

$$H_{tot} = H_c + N(\mathbf{x}, \mathbf{p}) \chi$$

$$H_{tot} = N \left(\sum_n \frac{\mathbf{p}_n^2}{2m_n} + V - E \right)$$

Eqs of motion

$$\dot{\mathbf{x}}_n = \frac{\partial H_{tot}}{\partial \mathbf{p}_n} = N \frac{\mathbf{p}_n}{m_n}$$

$$\dot{\mathbf{p}}_n = -\frac{\partial H_{tot}}{\partial \mathbf{x}_n} = -N \nabla_n V$$

Should also impose $\chi = 0$

Canonical Hamiltonian

$$H_c(\mathbf{x}, \mathbf{p}) \equiv \sum_n \mathbf{p}_n \cdot \dot{\mathbf{x}}_n - L \equiv 0$$

$$\chi = \sum_n \frac{\mathbf{p}_n^2}{2m_n} + V - E \approx 0$$

Total Hamiltonian

$$H_{tot} = H_c + N(\mathbf{x}, \mathbf{p}) \chi$$

$$H_{tot} = N \left(\sum_n \frac{\mathbf{p}_n^2}{2m_n} + V - E \right)$$

Eqs of motion

$$\dot{\mathbf{x}}_n = \frac{\partial H_{tot}}{\partial \mathbf{p}_n} = N \frac{\mathbf{p}_n}{m_n}$$

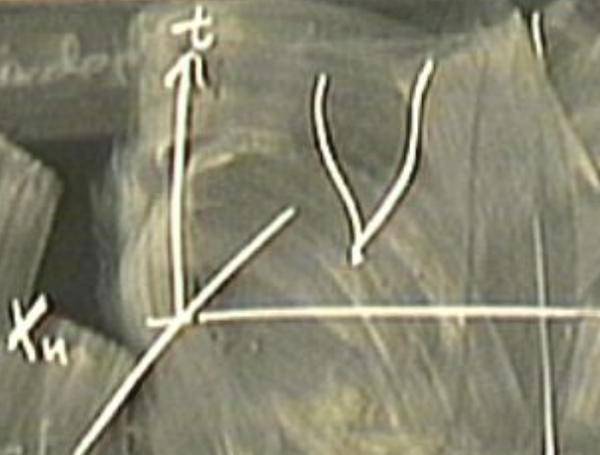
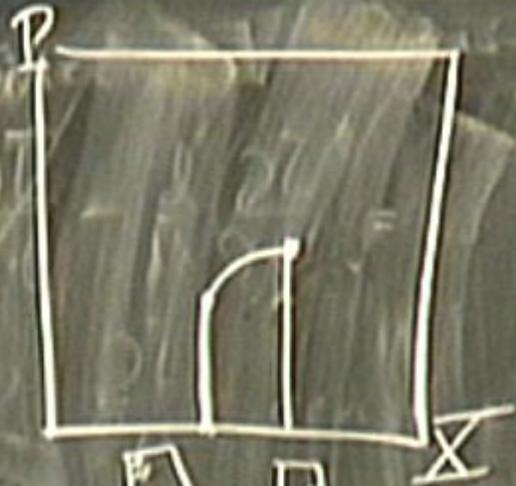
$$\dot{\mathbf{p}}_n = -\frac{\partial H_{tot}}{\partial \mathbf{x}_n} = -N \nabla_n V$$

Should also impose $\chi = 0$



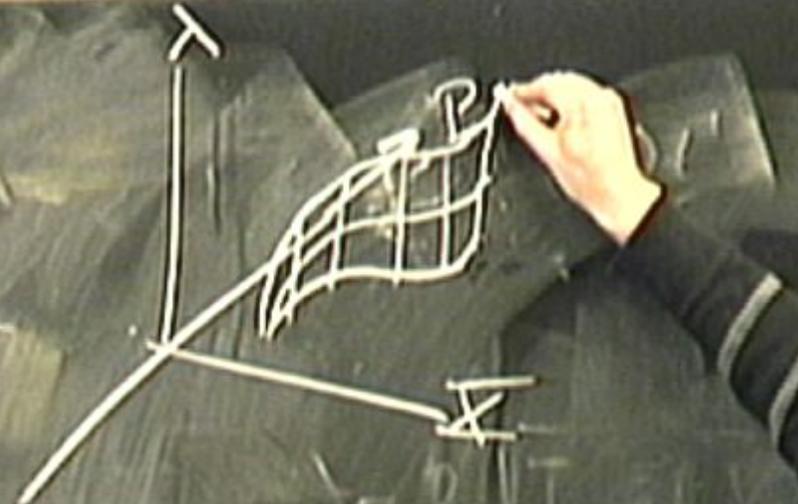
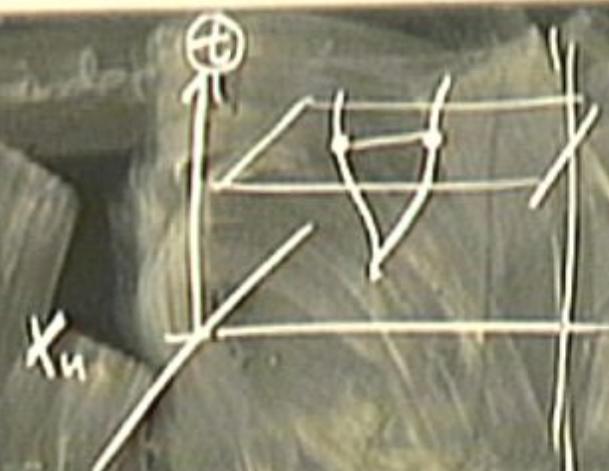
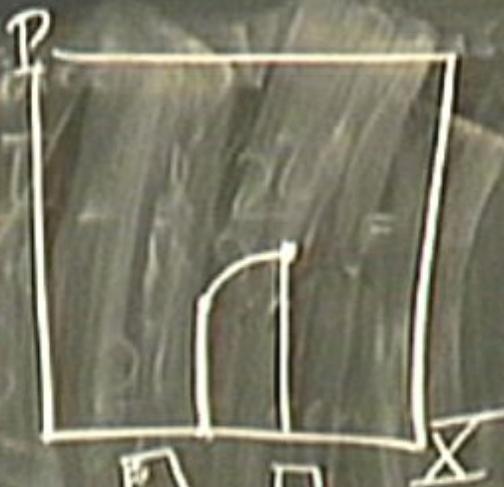
"Gauge" trans.

$$X_{ii} \xrightarrow{\text{'GT'}} X_{ii} + \epsilon \delta X_{ii} = \psi_{ii}$$
$$= X_{ii} + \epsilon \{ X_{ii}, \bar{N} X \} = X_{ii}$$



"Gauge" trans.

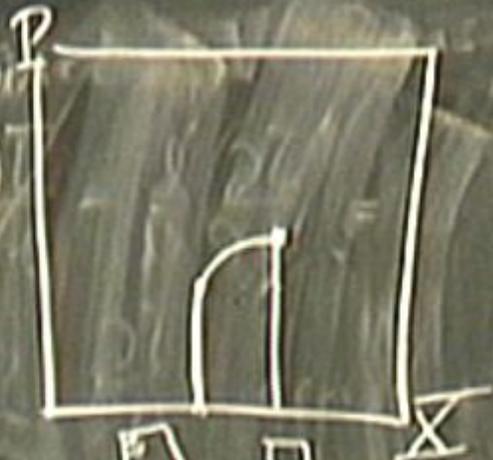
$$\begin{aligned}
 X_n &\xrightarrow{\text{'GT'}} X_n + \epsilon \delta X_n = \tilde{X}_n \\
 &= X_n + \epsilon \{X_n, \bar{N} \chi\} = X_n + \epsilon \bar{N} \frac{K_n^i}{m_n}
 \end{aligned}$$



"Gauge" trans.

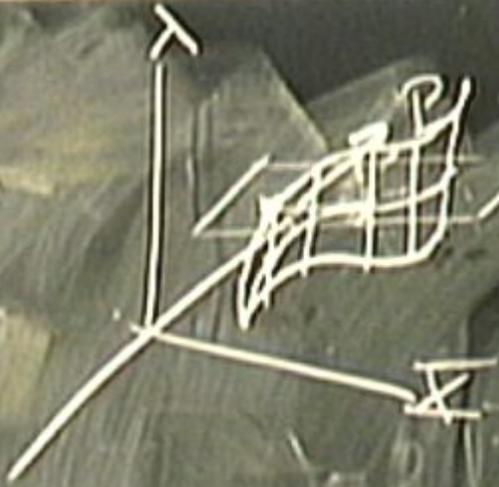
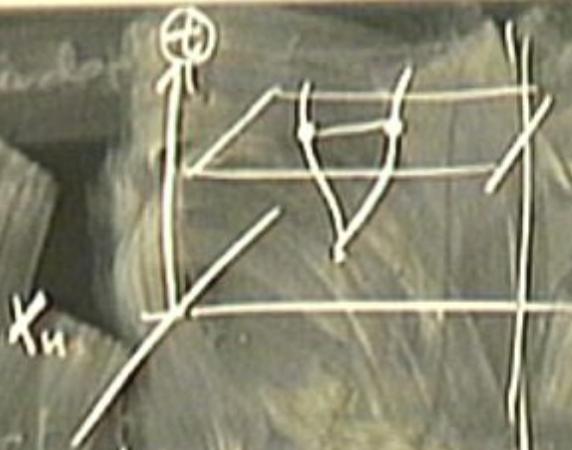
$$X_u \xrightarrow{GT} X_u + \epsilon \delta X_u = \tilde{X}_u$$

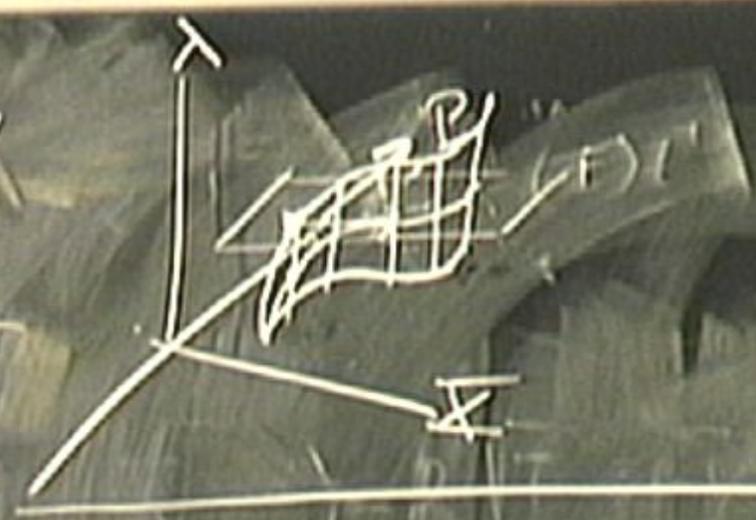
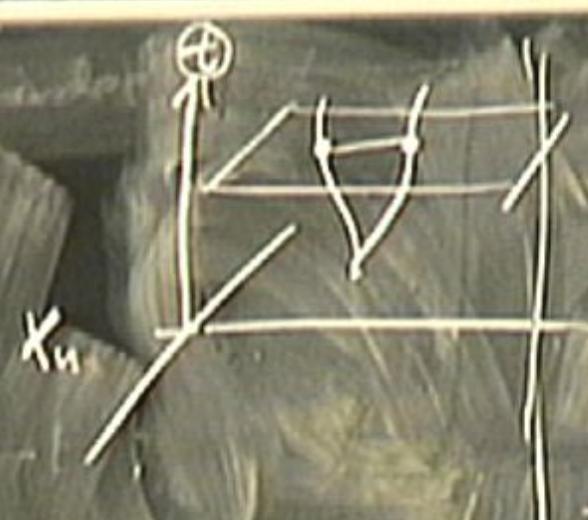
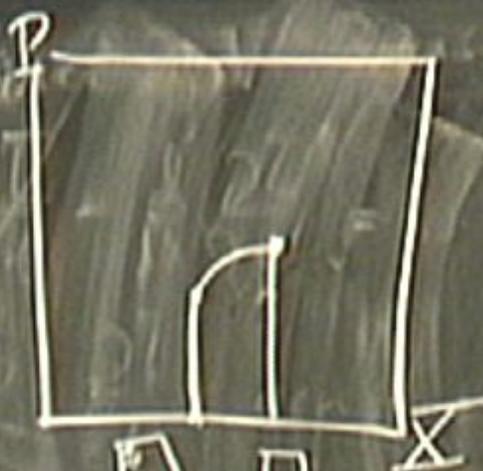
$$= X_u + \epsilon \left\{ X_u, \bar{N} X \right\} = X_u + \epsilon \bar{N} \frac{X_u}{m_u}$$



"Gauge" trans.

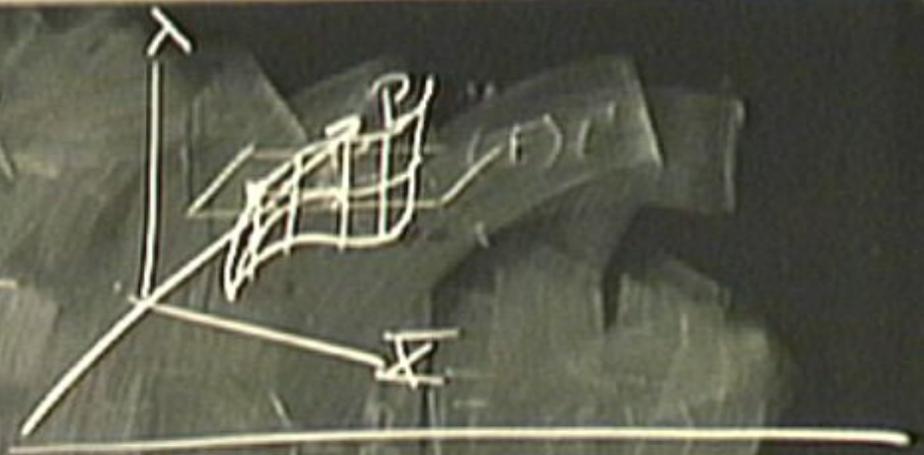
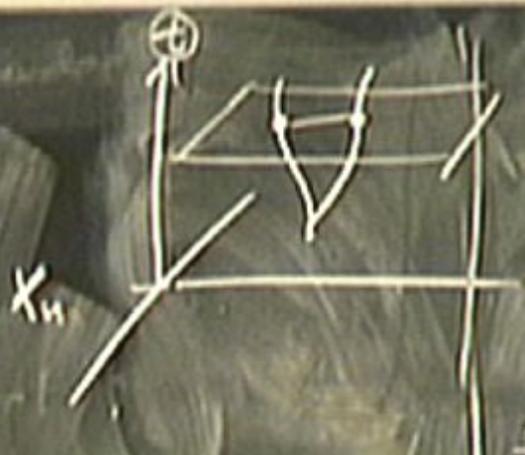
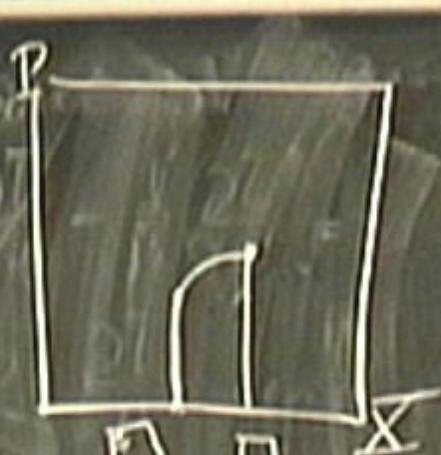
$$\begin{aligned}
 X_{\mu} &\xrightarrow{G^T} X_{\mu} + \epsilon \delta X_{\mu} = \tilde{X}_{\mu} \\
 &= X_{\mu} + \epsilon \{X_{\mu}, \bar{N} \chi\} = X_{\mu} + \epsilon \bar{N} \frac{\chi_{\mu}}{m_{\mu}}
 \end{aligned}$$





'Gauge' trans.

$$\begin{aligned}
 X_u &\xrightarrow{G} X_u + \epsilon \delta X_u = \psi_u \\
 &= X_u + \epsilon \{X_u, \bar{N} X\} = X_u + \epsilon \bar{N} \frac{X_u}{m_u}
 \end{aligned}$$



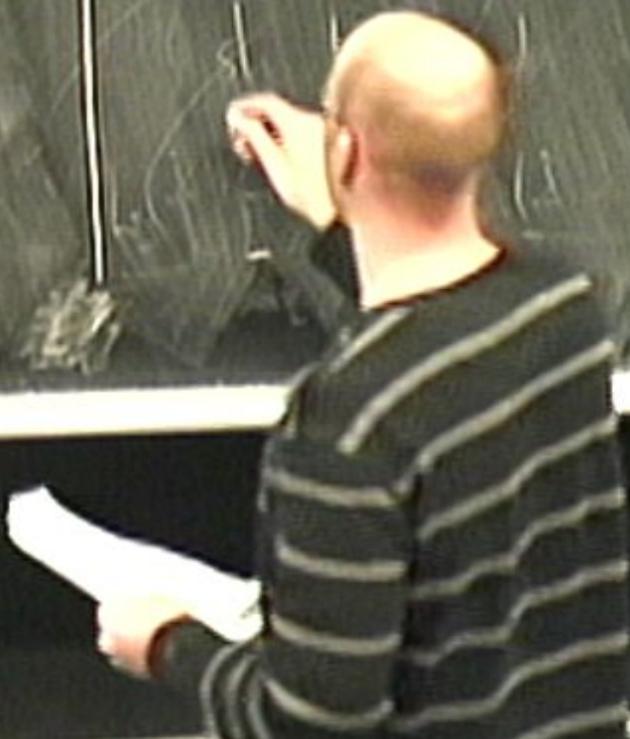
'Gauge' trans.

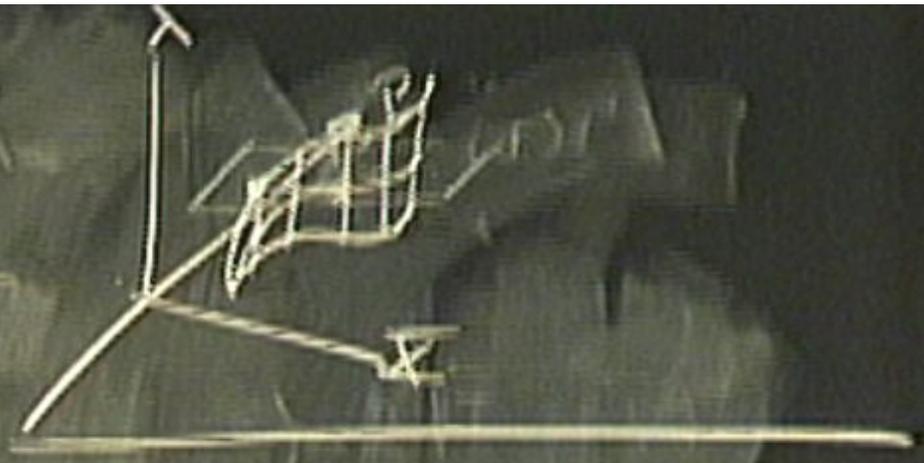
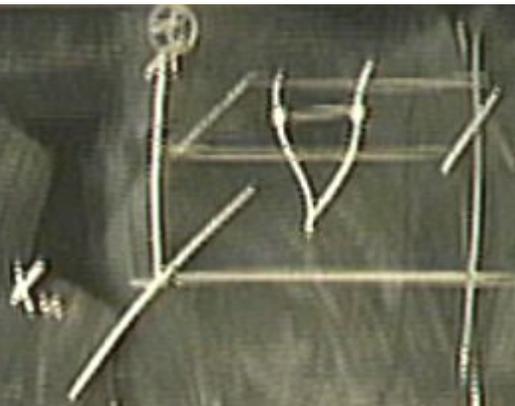
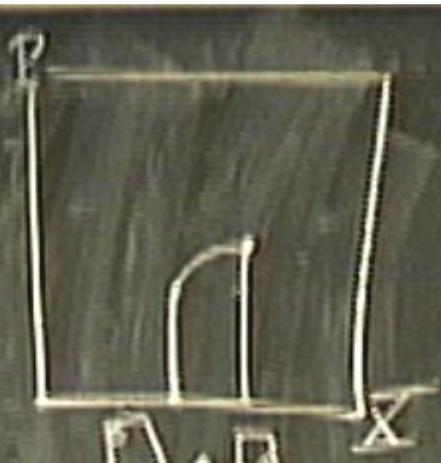
$$X_n \xrightarrow{'ST'} X_n + \epsilon \delta X_n = \tilde{X}_n$$

$$= X_n + \epsilon \{X_n, \bar{N} X\} = X_n + \epsilon \bar{N} \frac{\delta X_n}{\delta X_n}$$

Quantization

$$X_n \rightarrow \hat{X}_n = X_n \quad P_n \rightarrow \hat{P}_n = -iP_n$$





'Gauge' trans.

$$X_n \xrightarrow{\delta T} X_n + \epsilon \delta X_n = \tilde{X}_n$$

$$= X_n + \epsilon \left\{ X_n, \int \bar{N} X \right\} = X_n + \epsilon \bar{N} \frac{\partial X}{\partial X_n}$$

Quantization

$$X_n \rightarrow \hat{X}_n = X_n \quad P_n \rightarrow \hat{P}_n = -i \hbar \frac{\partial}{\partial X_n}$$

$$\{X_n, P_n\} = i$$

$$X \rightarrow \hat{X} = \int \frac{P^2}{2m} + V - E$$

$$P(x, \psi(x)) = P(x, x) \quad | \quad |x$$

$$\hat{\chi}\Psi = 0 \Rightarrow \left(\sum_n \hat{p}_n^2 + V \right) \Psi = E\Psi$$

$$\zeta_{ij} \pi \cdot \pi$$

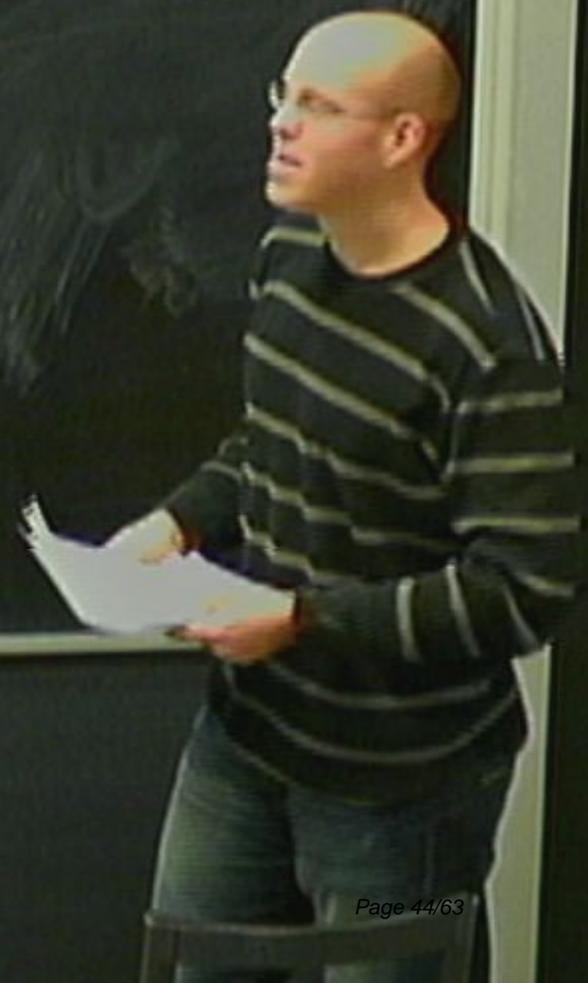


$$P(x, \lambda) = P(x, x) \quad | \quad |x$$

$$\hat{\lambda} \Psi = 0 \Rightarrow \left(\sum_n \hat{L}_n^2 + V \right) \Psi = \hat{E} \Psi$$

$$\left(\frac{1}{2} \sum_{i,j} \hat{\pi}_i \hat{\pi}_j - \hat{K} \right) \Psi = \hat{E} \Psi$$

$$\Psi \in L^2(\mathbb{C}, \mathbb{R}^{3N})$$



EXIT

$$\hat{\chi}\Psi = 0 \Rightarrow \left(\sum_n \frac{\hat{p}_n^2}{2m_n} + V \right) \Psi = E\Psi$$

$$\left(\frac{1}{2} \sum_{i,j} \hat{\pi}_i \hat{\pi}_j - \frac{\hbar^2}{2m} \right) \Psi = E\Psi$$

$$\Psi \in L^2(\mathbb{C}, \mathbb{R}^{3N})$$

$$V = -\frac{1}{r}$$



$$\hat{X}\Psi = 0 \Rightarrow \left(\sum_n \frac{\hat{p}_n^2}{2m_n} + V \right) \Psi = E\Psi$$

$$\left(\frac{1}{2} \sum_{i,j} g^{ij} \hat{\pi}_i \hat{\pi}_j - \hat{R} \right) \Psi = E\Psi$$

$$\Psi \in L^2(\mathbb{C}, \mathbb{R}^{3N})$$

$$V = -\frac{1}{r}$$



$$\hat{\chi}\Psi = 0 \Rightarrow \left(\sum_n \frac{\hat{p}_n^2}{2m_n} + V \right) \Psi = E\Psi$$

$$\left(\frac{1}{2} \sum_{ij} \epsilon^{ijne} \hat{\pi}_i \hat{\pi}_j - \mathcal{R} \right) \Psi = \mathcal{Q} \Psi$$

$$\Psi \in L^2(\mathbb{C}, \mathbb{R}^{3N})$$

$$V = -\frac{1}{r}$$



LBIB theory

$$\Psi = R e^{iS} \rightarrow \Delta$$

$$\begin{cases} \sum_n \frac{(\nabla_n S)^2}{2m_n} + V + Q = E \\ \sum_n \nabla_n \cdot \left(R^2 \frac{\nabla_n S}{m_n} \right) = 0 \end{cases}$$

$$Q = -\sum_n \frac{1}{2m_n} \frac{\nabla_n^2 R}{R}$$

$$P_n = \nabla_n S(\mathbf{x})$$

$$\mathbf{x}_n' = f(\mathbf{x}_n)$$

$$P_n = \nabla_n S(\mathbf{x})$$

$$\mathbf{x}_n' = f(\mathbf{x}_n)$$

$$P_n = \frac{\partial L}{\partial \mathbf{x}_n'}$$

$$P_n = \nabla_n S(\mathbf{X})$$

$$x_n' = f(x_n)$$

$$P_n = \frac{\partial L}{\partial x_n'}$$

$$x_n' = \frac{\partial H_{\text{Tot}}}{\partial P_n} = N \frac{P_n}{m_n}$$

$$P_n = \nabla_n S(\mathbf{X})$$

$$x_n' = f(x_n)$$

$$P_n = \frac{\partial L}{\partial x_n'} \Rightarrow \text{sad face}$$

$$x_n' = \frac{\partial H_{\text{Tot}}}{\partial P_n} = N \frac{P_n}{m_n}$$

$$x_n' = \frac{N}{m_n} \nabla_n S$$

$$P_n = \nabla_n S(\mathbf{x})$$

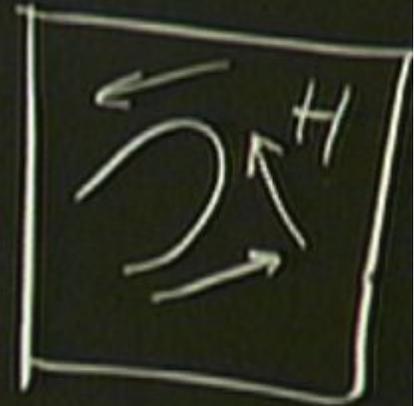
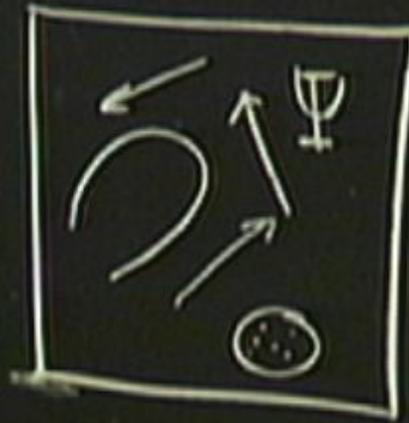
$$x_n' = f(x_n)$$

$$P_n = \frac{\partial L}{\partial x_n'} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{sad face}$$

$$x_n' = \frac{\partial H_{\text{Tot}}}{\partial P_n} =$$

$$x_n' = \frac{N}{m_n}$$

Interpretation of Ψ



$$P_n = \nabla_n S(\mathbf{x})$$

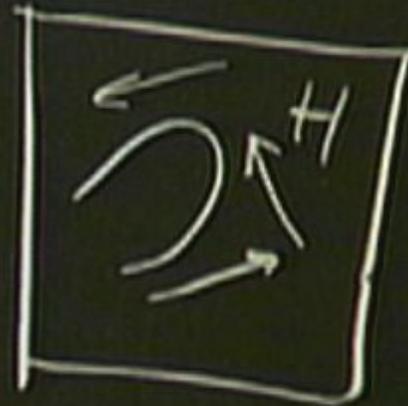
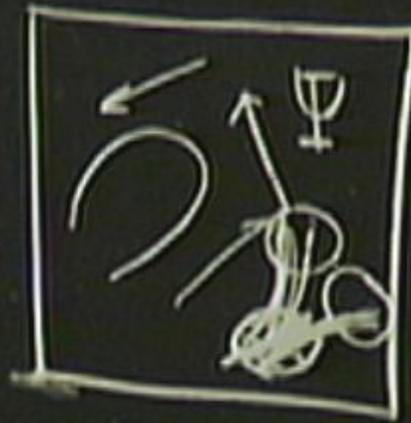
$$x_n' = f(x_n)$$

$$P_n = \frac{\partial L}{\partial x_n'} \} \Rightarrow \text{sad face}$$

$$x_n' = \frac{\partial H_{\text{Tot}}}{\partial P_n} = N \frac{P_n}{m_n}$$

$$x_n' = \frac{N}{m_n} \nabla_n S$$

Interpretation of Ψ



$$|\Psi|^2$$

$$P_n = \nabla_n S(\mathbf{x})$$

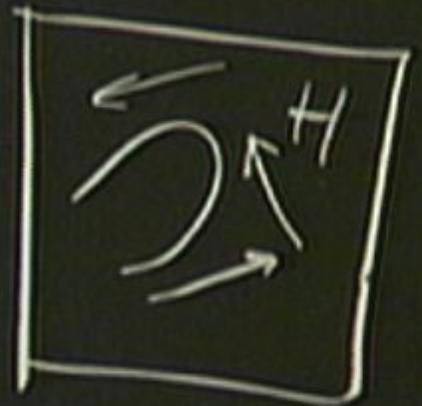
$$x_n' = f(x_n)$$

$$P_n = \frac{\partial L}{\partial x_n'} \Rightarrow \text{sad face}$$

$$x_n' = \frac{\partial H_{\text{Tot}}}{\partial P_n} = N \frac{P_n}{m_n}$$

$$x_n' = \frac{N}{m_n} \nabla_n S$$

Interpretation of Ψ



$$|\Psi|^2$$

Energy spectra

Non-degenerate case

$$\hat{H}\Psi_1 = E\Psi_1 \text{ \& \ } \hat{H}\Psi_2 = E\Psi_2$$

$$\Rightarrow \Psi_1 = \Psi_2$$

$$\Rightarrow \Psi = e^{i\phi} f(\mathbf{x}) \quad f: \mathbb{R}^3 \rightarrow \mathbb{R}$$

Degenerate case

LBSB theory

$$\Psi = R e^{iS} \rightarrow \Delta$$

$$\begin{cases} \sum_n \frac{(\nabla_n S)^2}{2m_n} + V + Q = E \\ \sum_n \nabla_n \cdot \left(R^2 \frac{\nabla_n S}{m_n} \right) = 0 \end{cases}$$

$$Q = -\sum_n \frac{1}{2m_n} \frac{\nabla_n^2 R}{R}$$

Energy spectra

Non-degenerate case

$$\hat{H}\Psi_1 = E\Psi_1 \text{ \& \; } \hat{H}\Psi_2 = E\Psi_2$$

$$\Rightarrow \Psi_1 = \Psi_2$$

$$\Rightarrow \Psi = e^{i\phi} f(\mathbf{x}) \quad (: \mathbb{R}^3 \rightarrow \mathbb{R}$$

Degenerate case

SBS theory

$$\Psi = R e^{iS} \rightarrow \Delta$$

$$\left\{ \begin{array}{l} \sum_n \frac{(\nabla_n S)^2}{2m_n} + V + Q = E \\ \sum_n \nabla_n \cdot \left(R^2 \frac{\nabla_n S}{m_n} \right) = 0 \end{array} \right.$$

$$Q = - \sum_n \frac{1}{2m_n} \frac{\nabla_n^2 R}{R}$$

Energy spectra

Non-degenerate case

$$\hat{H}\Psi_1 = E\Psi_1 \text{ \& \ } \hat{H}\Psi_2 = E\Psi_2$$

$$\Rightarrow \Psi_1 = \Psi_2$$

$$\Rightarrow \Psi = e^{i\omega t} f(\mathbf{x}) \quad f: \mathbb{R}^N \rightarrow \mathbb{R}$$

Degenerate case

Conditional wavefunction

Ψ

Energy spectra

Non-degenerate case

$$\hat{H}\Psi_1 = E\Psi_1 \text{ \& \ } \hat{H}\Psi_2 = E\Psi_2$$

$$\Rightarrow \Psi_1 = \Psi_2$$

$$\Rightarrow \Psi = e^{i\omega t} f(\mathbf{x}) \quad f: \mathbb{R}^{3N} \rightarrow \mathbb{R}$$

Degenerate case

Conditional wavefunction

$$\Psi(Q) \quad Q = (\mathbf{x}, \mathbf{z})$$

$$\Psi(\mathbf{x}, t) = \frac{\Psi(\mathbf{x}, \mathbf{z}(t))}{\sqrt{\int_{\mathbb{R}^N} |\Psi(\mathbf{x}, \mathbf{z})|^2 e^{i\omega(\mathbf{z}, t)}}} \quad N=1$$

Energy spectra

Non-degenerate case

$$\hat{H}\Psi_1 = E\Psi_1 \text{ \& \ } \hat{H}\Psi_2 = E\Psi_2$$

$$\Rightarrow \Psi_1 = \Psi_2$$

$$\Rightarrow \Psi = e^{i\omega t} f(\mathbf{x}) \quad f: \mathbb{R}^N \rightarrow \mathbb{R}$$

Degenerate case

Conditional wavefunction

$$\Psi(Q) \quad Q = (\mathbf{x}, \mathbf{z})$$

$$\Psi(\mathbf{x}, t) = \frac{\Psi(\mathbf{x}, \mathbf{z}(t))}{\left(\int |\Psi(\mathbf{x}, \mathbf{z})|^2 \right)^{1/2}} e^{i\omega(\mathbf{x}, t)} \quad N=1$$

$$i\frac{\partial \Psi}{\partial t} = \left(\frac{\hat{p}^2}{2m} + V + i\Gamma \right) \Psi$$

Energy spectra

Non-degenerate case

$$\hat{H}\Psi_1 = E\Psi_1 \text{ \& \ } \hat{H}\Psi_2 = E\Psi_2$$

$$\Rightarrow \Psi_1 = \Psi_2$$

$$\Rightarrow \Psi = e^{i\omega t} f(\mathbf{x}) \quad f: \mathbb{R}^N \rightarrow \mathbb{R}$$

Degenerate case

Conditional wavefunction

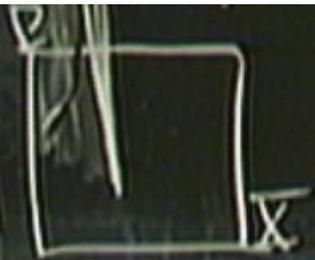
$$\Psi(Q) \quad Q = (x, \mathbf{X})$$

$$\Psi(x, t) = \frac{\Psi(x, \mathbf{X}(t))}{\int \int |\Psi(x, \mathbf{X})|^2 dx^N} e^{i\omega(x, t)} \quad N=1$$

$$i\frac{\partial \Psi}{\partial t} = \left(\frac{\hat{p}^2}{2m} + V + i\Gamma \right) \Psi$$

operationally,

$$L = \frac{1}{2} \sum_{i,j} g^{ij} \left(\frac{\partial}{\partial x^i} \right) \left(\frac{\partial}{\partial x^j} \right) + \sum_i p_i \left(\frac{\partial}{\partial x^i} \right) - P_N$$

$$P(\mathbf{x}, \mathbf{p}) = P(\mathbf{x}, \bar{\mathbf{x}})$$


$$\hat{H}\Psi_1 = E\Psi_1, \quad \hat{H}\Psi_2 = E\Psi_2$$

$$\Rightarrow \Psi_1 = \Psi_2 \quad L = \sum_{i,j} g^{ij} \left(\frac{\partial}{\partial x^i} \right) \left(\frac{\partial}{\partial x^j} \right) + \dots$$

$$\Rightarrow \Psi = e^{i\phi} f(\mathbf{x}) \quad f: \mathbb{R}^N \rightarrow \mathbb{R}$$

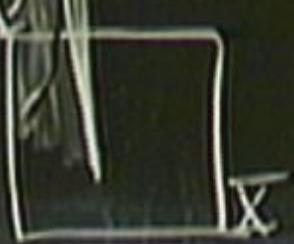
Degenerate case

$$\Psi(\mathbf{x}, t) = \frac{\Psi(\mathbf{x}, \bar{\mathbf{x}}(t))}{\sqrt{\sum_{i,j} g^{ij} \left(\frac{\partial \Psi}{\partial x^i} \right) \left(\frac{\partial \Psi}{\partial x^j} \right) e^{i\phi(\mathbf{x}, t)}}} \quad N=1$$

$$+ V + i\epsilon I \quad \Psi$$

operationally,

$$L = \frac{1}{2} \sum_{i=1}^N \left(\frac{dx_i}{dt} \right)^2 - V(x) = (p_1 \dots p_N)$$

$$P(x, \dot{x}) = P(x, \bar{x})$$


$$\hat{H}\Psi_1 = E\Psi_1, \quad \hat{H}\Psi_2 = E\Psi_2$$

$$\Rightarrow \Psi_1 = \Psi_2$$

$$L = \int dx \left(\frac{1}{2} \dot{x}^2 - V(x) \right)$$

$$\Psi(x, \bar{x}) = \frac{\Psi(x, \bar{x})}{\int dx \Psi(x, \bar{x})}$$

$$\Rightarrow \Psi = e^{i\phi} f(x) \quad f: \mathbb{R}^N \rightarrow \mathbb{R}$$

$$\frac{\partial \Psi}{\partial t} = \left(\frac{\hat{p}^2}{2m} + V + i\epsilon I \right) \Psi$$

Degenerate case

operationally,

$$L = \frac{1}{2} \sqrt{\sum_{i=1}^N \sigma_i^2} \left(\frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \right) \right) \Psi = (p_1 \dots p_N)$$

$$P(x, x') = P(x, \bar{x})$$


$$\hat{H}\Psi_1 = E\Psi_1, \quad \hat{H}\Psi_2 = E\Psi_2$$

$$\Rightarrow \Psi_1 = \Psi_2 \quad L = \int \sum_{i=1}^N \sigma_i^2 (x) + V(x)$$

$$\Rightarrow \Psi = e^{i\phi} f(x) \quad f: \mathbb{R}^N \rightarrow \mathbb{R}$$

Degenerate case

$$\Psi(x, t) = \frac{\Psi(x, \bar{x}(t))}{\sqrt{\sum_{i=1}^N \sigma_i^2(x) e^{i\phi(x,t)}}} \quad N=1$$

$$\frac{\partial \Psi}{\partial t} = \left(\frac{\hbar^2}{2m} \nabla^2 + V + i\epsilon I \right) \Psi$$