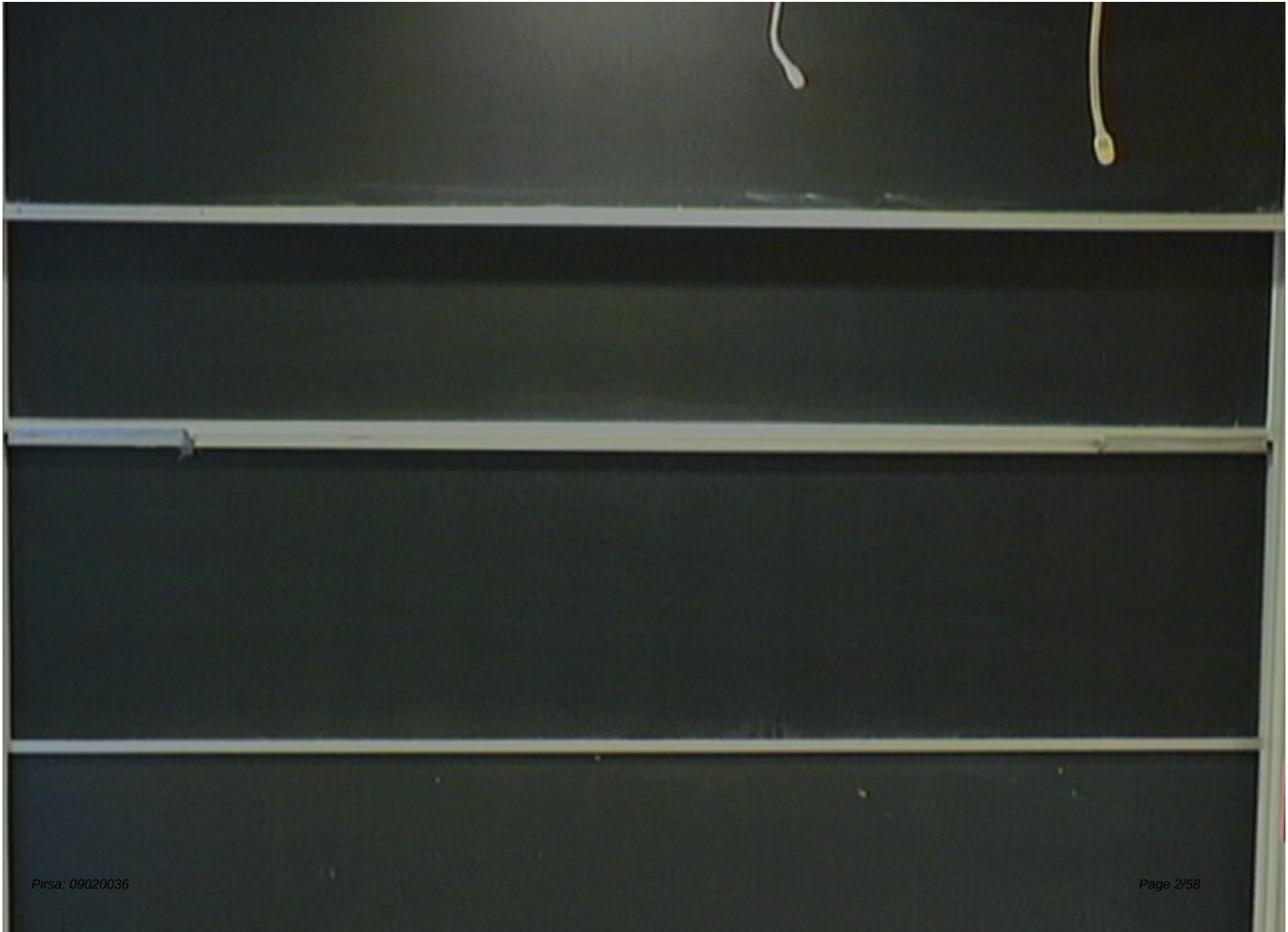
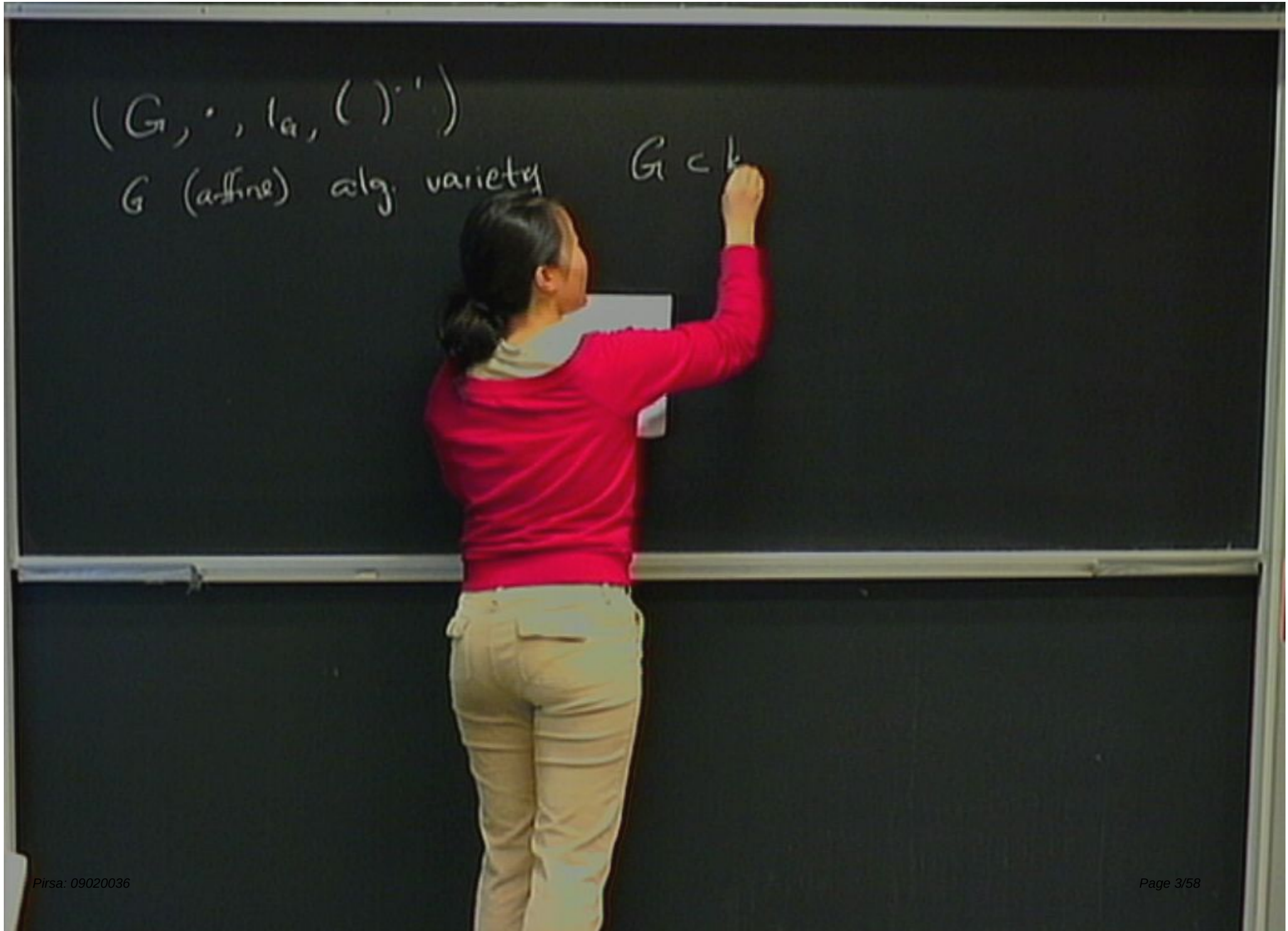
Title: Grad Talks 3

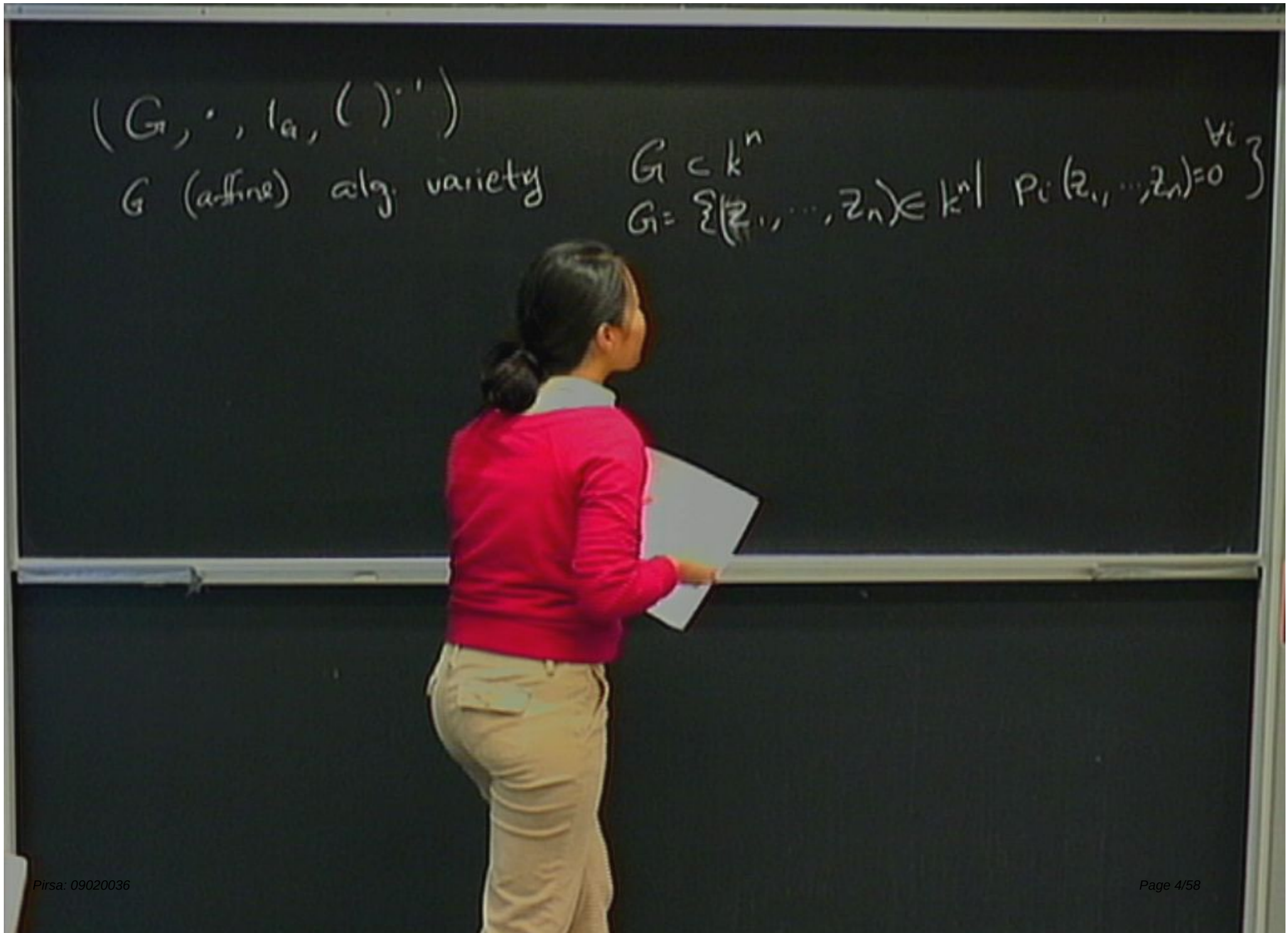
Date: Feb 23, 2009  11:00 AM

URL: http://pirsa.org/09020036

Abstract: Lecture on Quantum Groups by Lucy Zhang



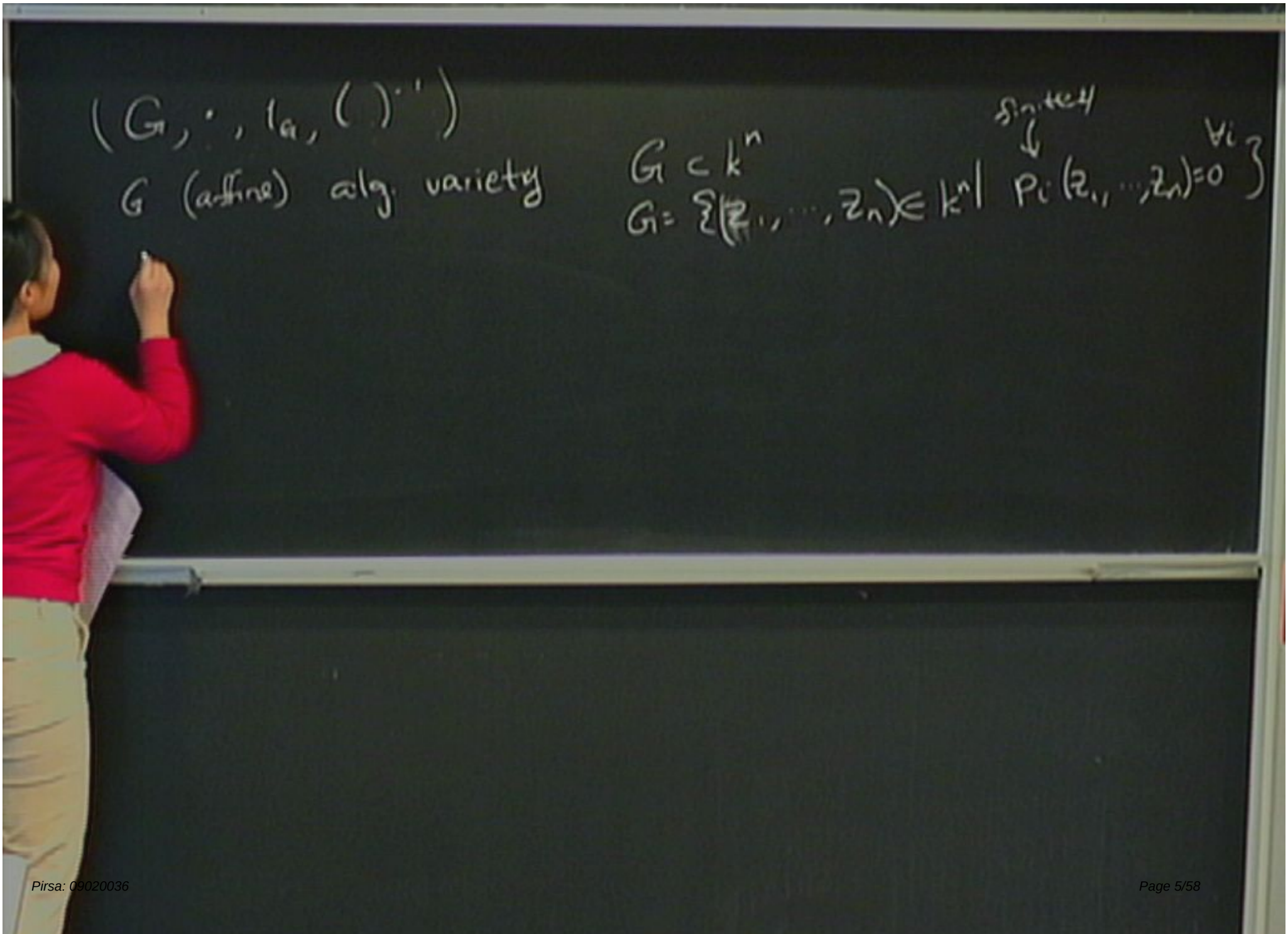

$(G, \cdot, 1_G, (\,)^{-1})$
$G$ (affine) alg. variety
$G \subset k$


$(G, \cdot, 1_G, (\,)^{-1})$
$G$ (affine) alg. variety
$G \subset k^n$
$G = \{(z_1, \dots, z_n) \in k^n \mid p_i(z_1, \dots, z_n) = 0 \ \forall i\}$

$(G, \cdot, 1_G, (\,)^{-1})$

$G$ (affine) alg. variety $\quad G \subset k^n$

$G = \{(z_1, \cdots, z_n) \in k^n \mid p_i(z_1, \cdots, z_n) = 0 \ \forall i\}$ (finitely)



$(G, \cdot, 1_G, (\,)^{-1})$

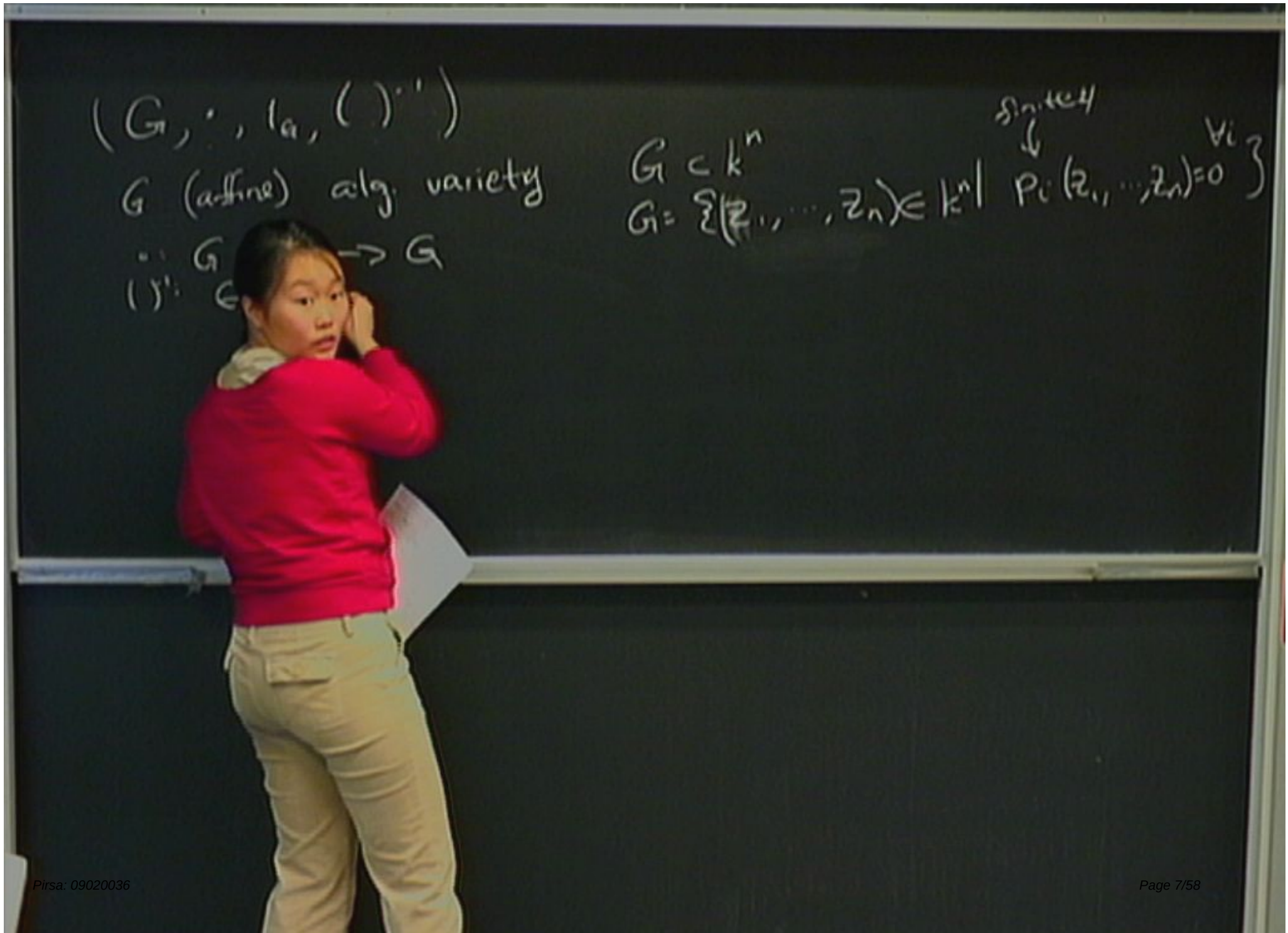
$G$ (affine) alg. variety

$\cdot : G \times G \longrightarrow G$

$(\,)^{-1} : G \longrightarrow G$

$G \subset k^n$

$G = \{(z_1, \dots, z_n) \in k^n \mid p_i(z_1, \dots, z_n) = 0 \ \forall i\}$ (finitely many $i$)


(G, ·, 1_G, ( )^{-1})
G (affine) alg. variety
G ⊂ k^n
G = {(z_1, ..., z_n) ∈ k^n | p_i(z_1, ..., z_n) = 0 ∀i}
finite #
· : G → G
( )^{-1} : G

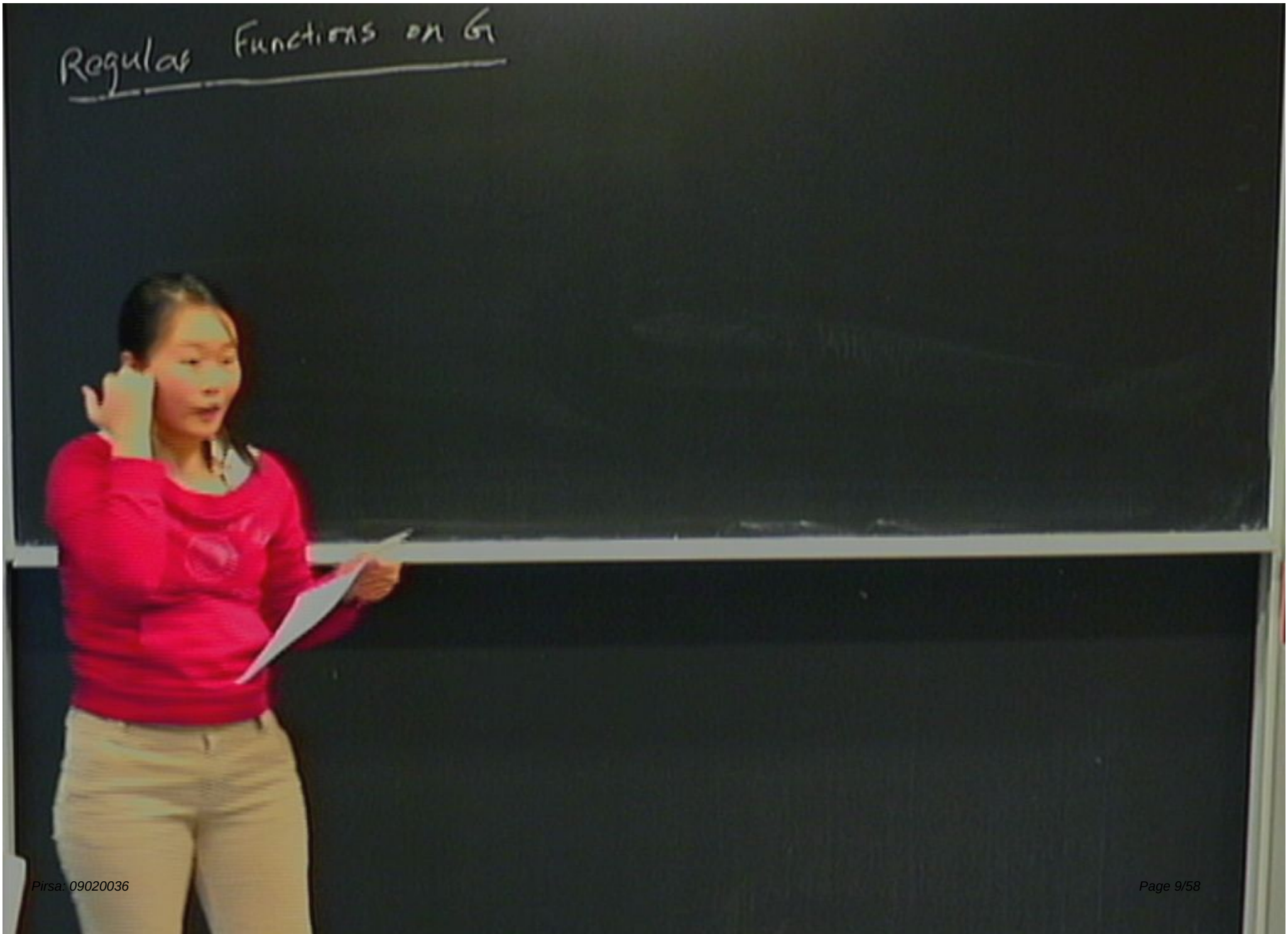
$(G, \cdot, 1_G, (\,)^{-1})$

$G$ (affine) alg. variety $\quad G \subset k^n$

$G = \{(z_1, \cdots, z_n) \in k^n \mid p_i(z_1, \cdots, z_n) = 0 \ \forall i\}$ (finitely many $p_i$)

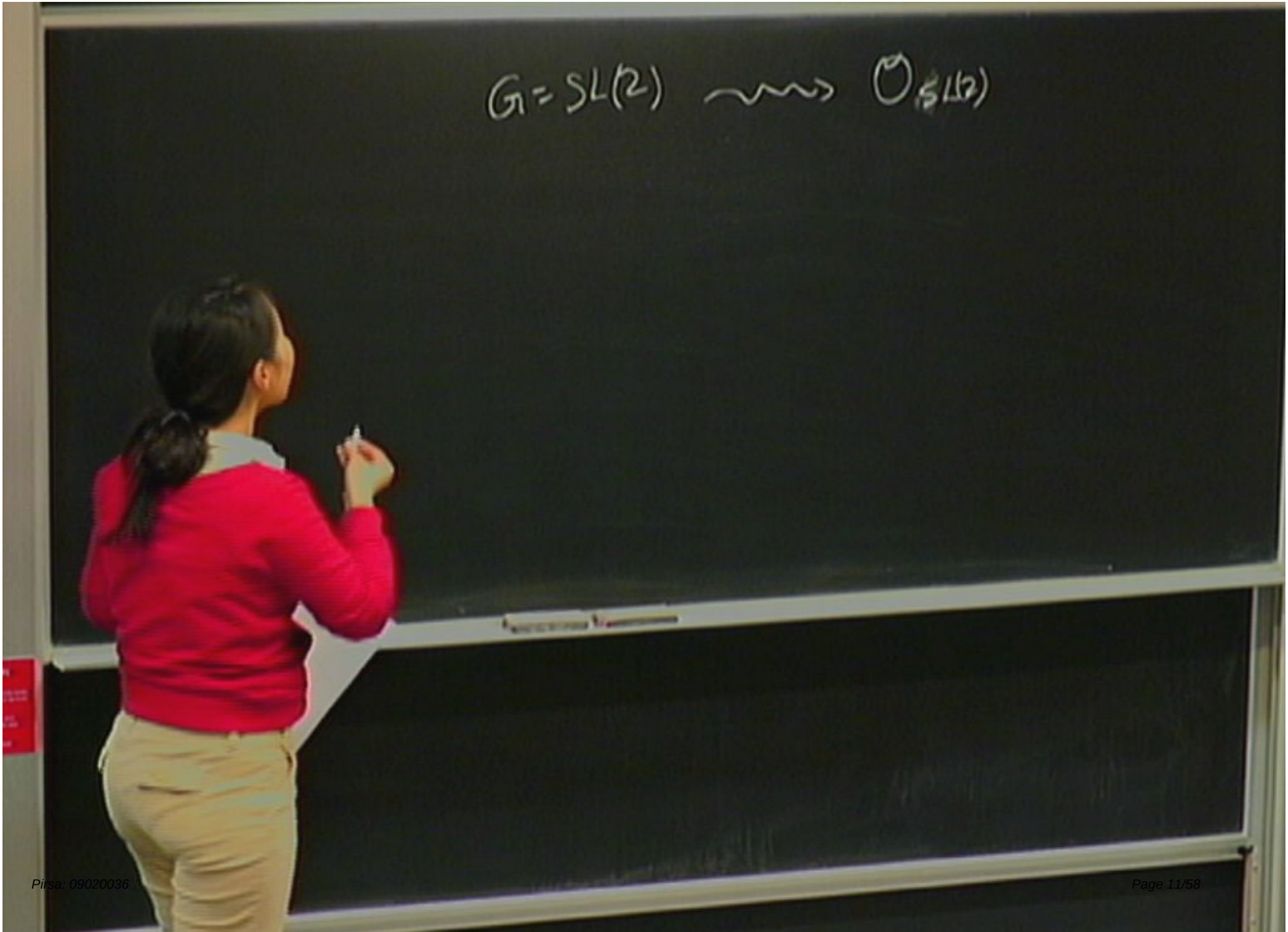
$\cdot : G \times G \to G$

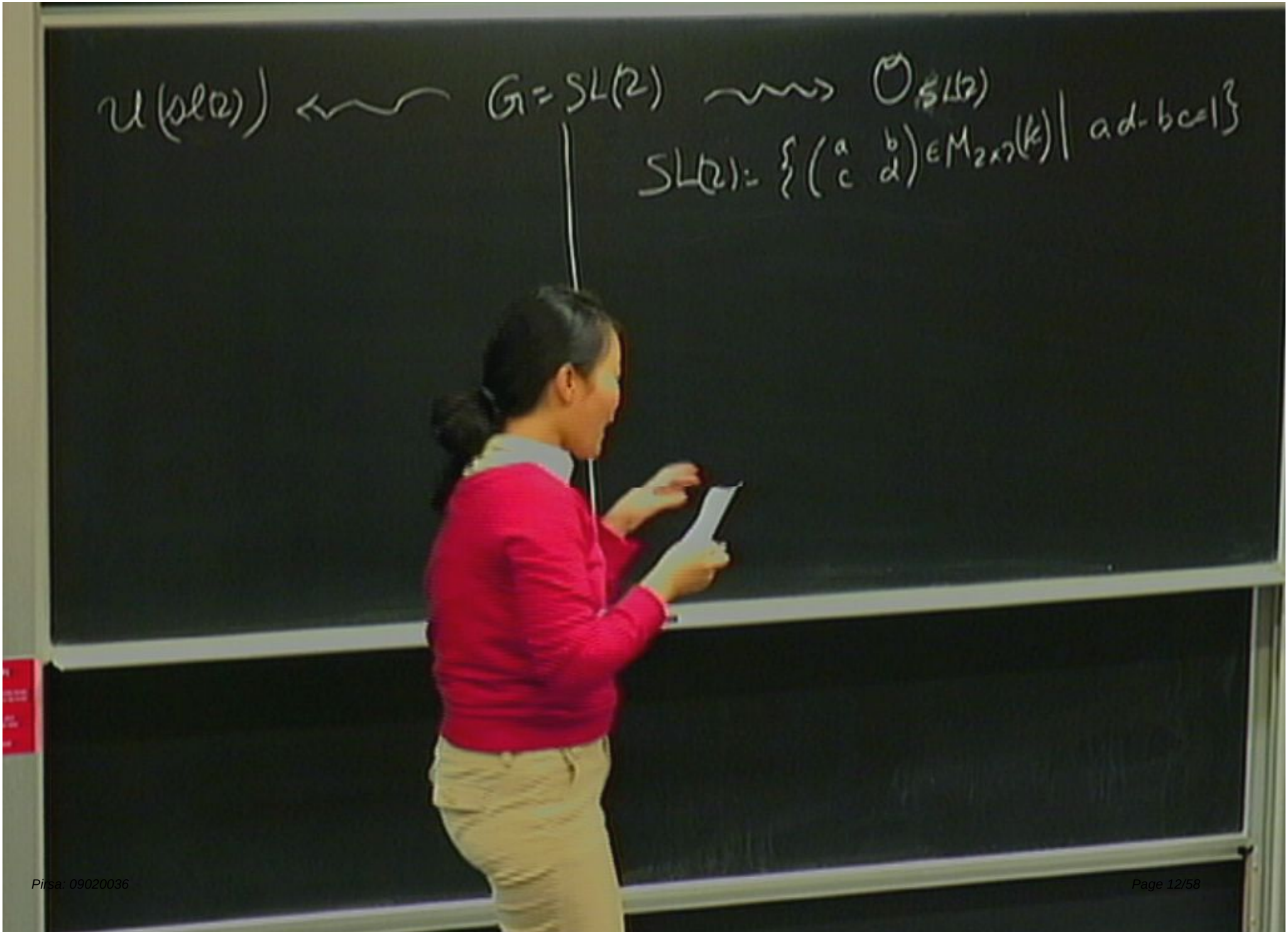
$(\,)^{-1} : G \to G$ } regular maps.

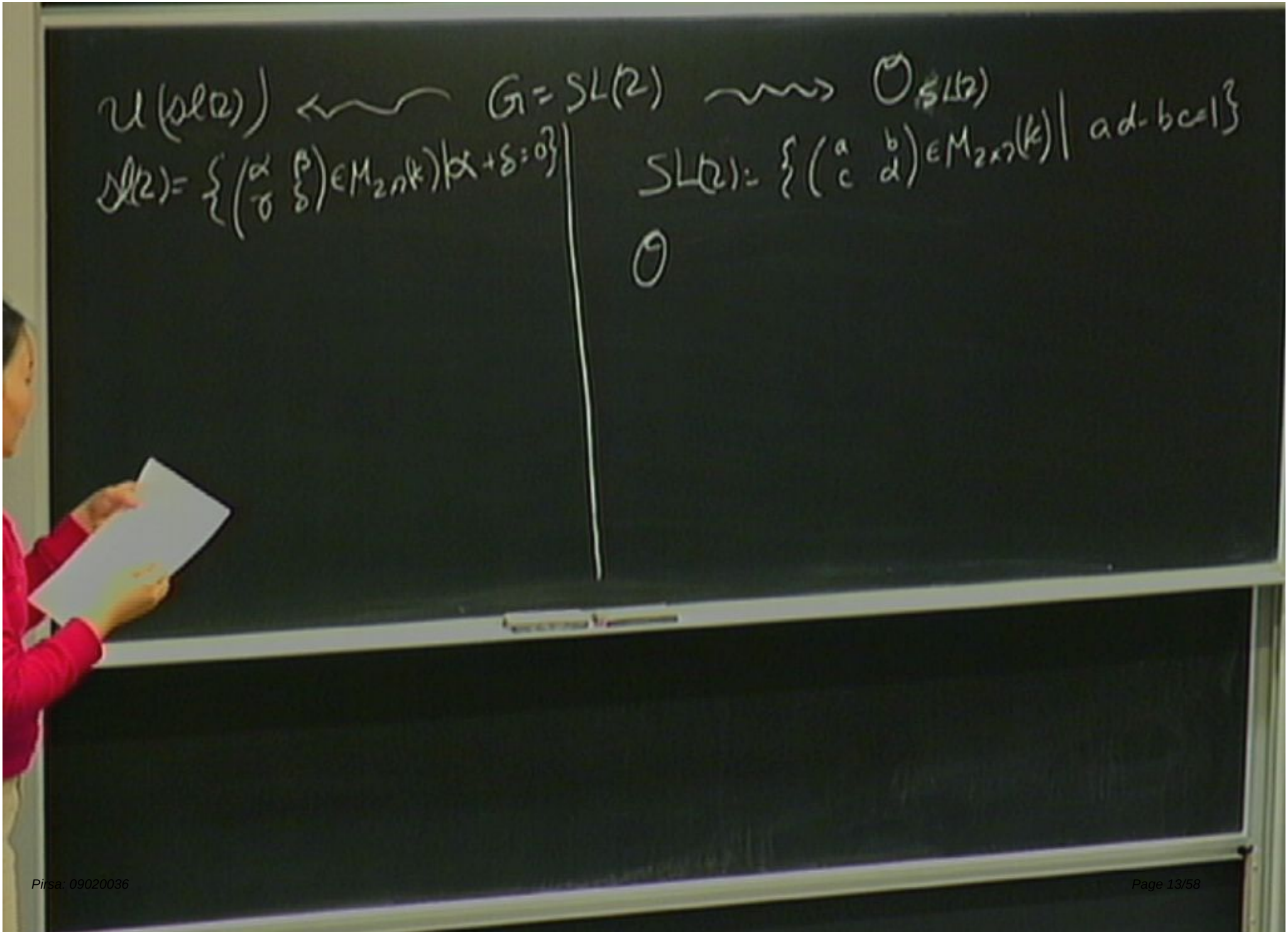

Regular Functions on G

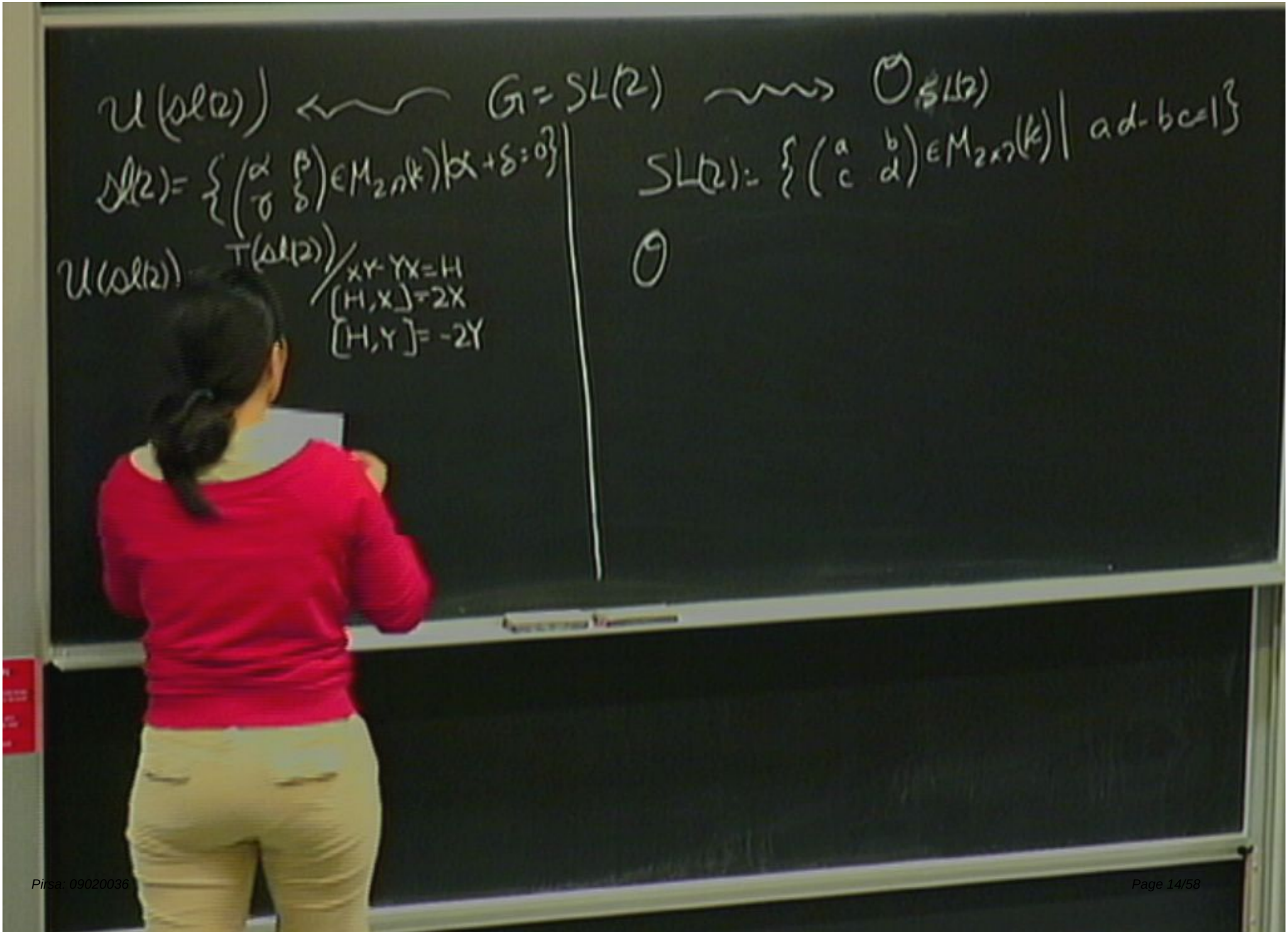
## Regular Functions on $G$

$$\mathcal{O}_G := k[z_1, \dots, z_n] \Big/ \{ f \in k[z_1, \dots, z_n] : f|_G \equiv 0 \}$$


G = SL(2)
O_SL(2)


U(sl(2)) ⟷ G = SL(2) ⇝ O_SL(2)
SL(2) := {(a b; c d) ∈ M_{2×2}(k) | ad − bc = 1}


$\mathcal{U}(\mathfrak{sl}(2))$ $G = SL(2)$ $\mathcal{O}_{SL(2)}$
$\mathfrak{sl}(2) = \{\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in M_{2\times 2}(k) \mid \alpha + \delta = 0\}$
$SL(2) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(k) \mid ad - bc = 1\}$
0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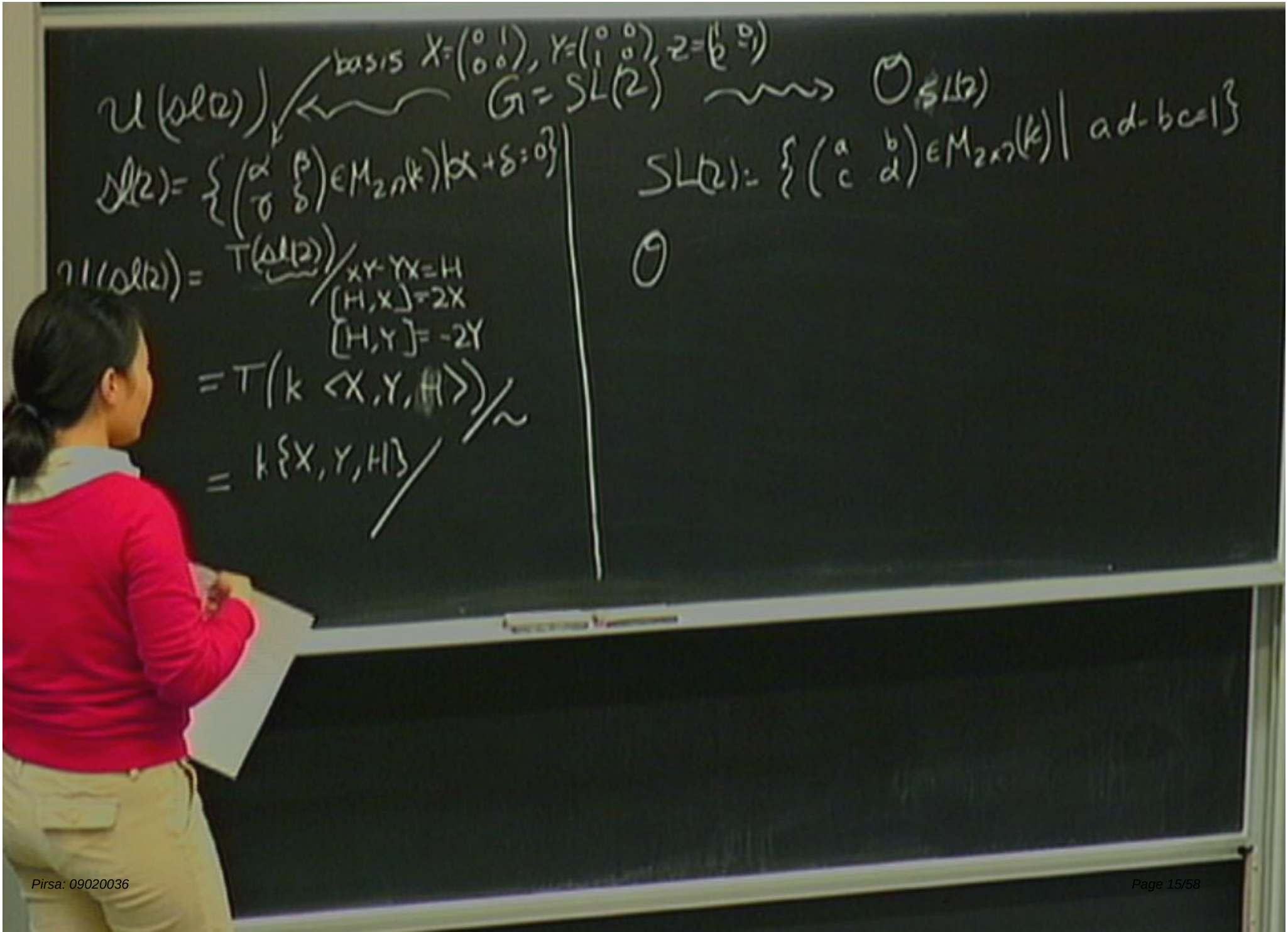
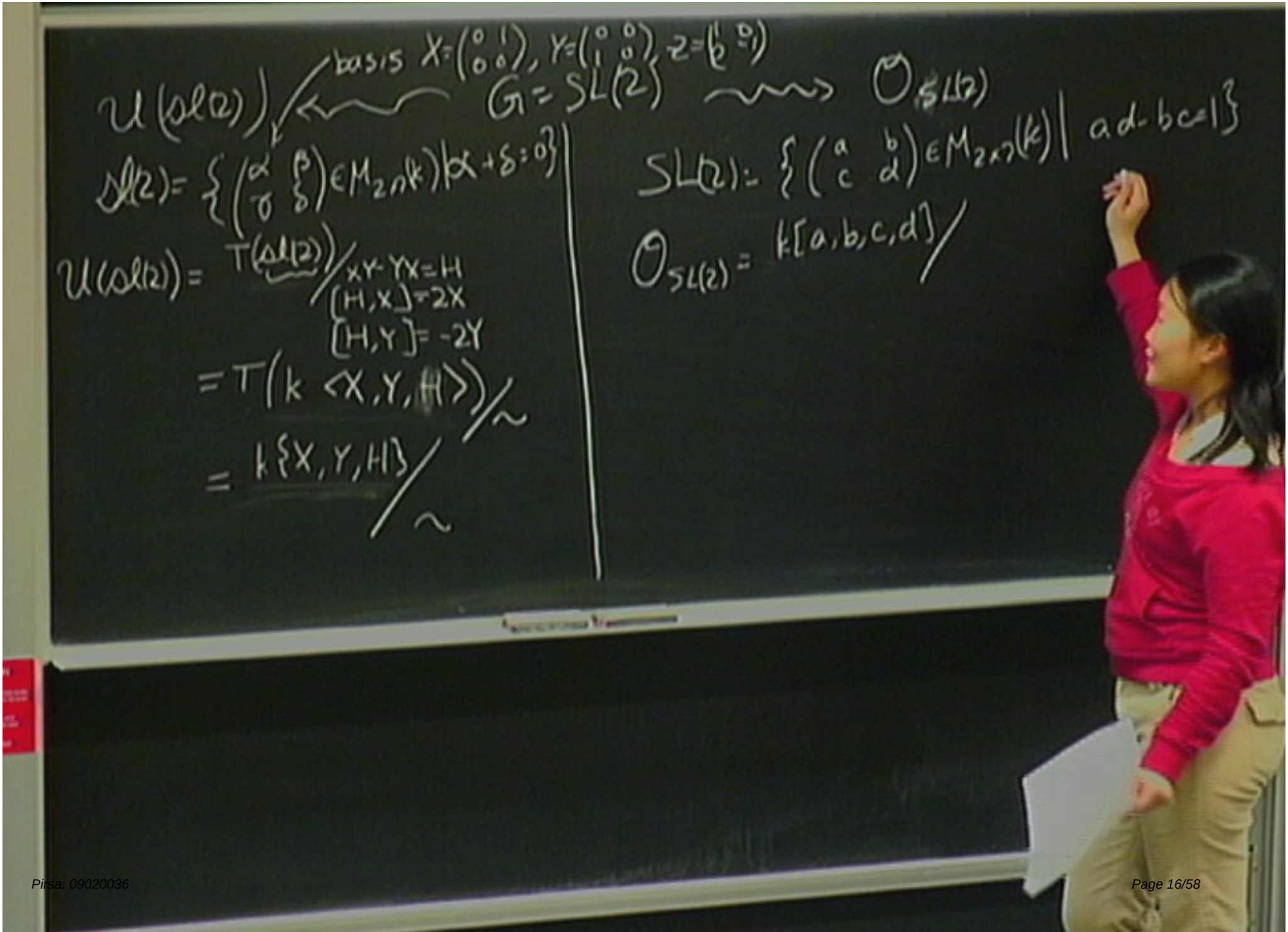
$\mathfrak{sl}(2) = \{\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in M_{2\times 2}(k) \mid \alpha + \delta = 0\}$
$SL(2) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(k) \mid ad - bc = 1\}$
$\mathcal{U}(\mathfrak{sl}(2)) \quad T(\mathfrak{sl}(2))/$
$xy - yx = H$
$[H, x] = 2x$
$[H, y] = -2y$
$0$


basis $X=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $Y=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $Z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$
$\mathcal{U}(\mathfrak{sl}(2))$
$G=SL(2)$
$\mathcal{O}_{SL(2)}$
$\mathfrak{sl}(2)=\{\begin{pmatrix}\alpha&\beta\\0&\delta\end{pmatrix}\in M_{2\times 2}(k) \mid \alpha+\delta=0\}$
$SL(2):=\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in M_{2\times 2}(k) \mid ad-bc=1\}$
$\mathcal{U}(\mathfrak{sl}(2))=T(\mathfrak{sl}(2))/$
$xy-yx=H$
$[H,X]=2X$
$[H,Y]=-2Y$
$=T(k\langle X,Y,H\rangle)/\sim$
$=k\{X,Y,H\}$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$\mathcal{U}(\mathfrak{sl}(2))$
$G=SL(2)$
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$\mathfrak{sl}(2)=\{\begin{pmatrix}\alpha&\beta\\0&\delta\end{pmatrix}\in M_{2\times 2}(k) \mid \alpha+\delta=0\}$
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$\mathcal{U}(\mathfrak{sl}(2))=T(\mathfrak{sl}(2))/$
$XY-YX=H$
$[H,X]=2X$
$[H,Y]=-2Y$
$=T(k\langle X,Y,H\rangle)/\sim$
$=k\{X,Y,H\}/\sim$
$\mathcal{O}_{SL(2)}=k[a,b,c,d]/$

$\mathcal{U}(\mathfrak{sl}(2))$ ← basis $X=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $Y=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $Z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ — $G=SL(2)$ ⇝ $\mathcal{O}_{SL(2)}$

$\mathfrak{sl}(2)=\left\{\begin{pmatrix}\alpha&\beta\\0&\delta\end{pmatrix}\in M_{2\times 2}(k) \mid \alpha+\delta=0\right\}$

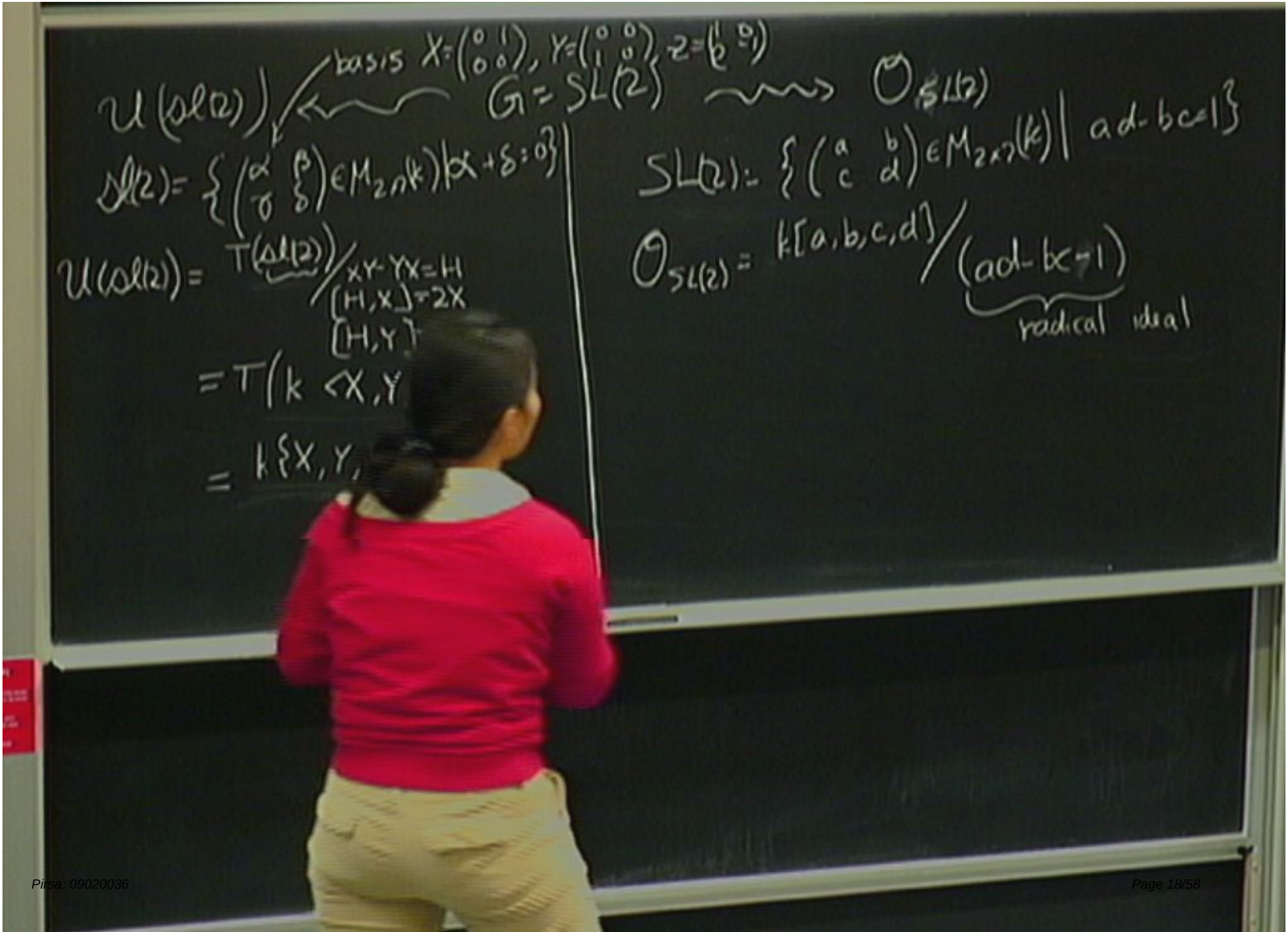
$SL(2):=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in M_{2\times 2}(k) \mid ad-bc=1\right\}$

$\mathcal{U}(\mathfrak{sl}(2))=T(\mathfrak{sl}(2))/\langle XY-YX=H,\ [H,X]=2X,\ [H,Y]=-2Y\rangle$

$=T(k\langle X,Y,H\rangle)/\sim$

$=k\{X,Y,H\}/\sim$

$\mathcal{O}_{SL(2)}=k[a,b,c,d]/\ldots$


basis $X=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $Y=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $Z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$
$\mathcal{U}(\mathfrak{sl}(2))$
$G=SL(2)$
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$\mathfrak{sl}(2)=\{\begin{pmatrix}\alpha&\beta\\0&\delta\end{pmatrix}\in M_{2\times 2}(k) \mid \alpha+\delta=0\}$
$SL(2):=\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in M_{2\times 2}(k) \mid ad-bc=1\}$
$\mathcal{U}(\mathfrak{sl}(2))=T(\mathfrak{sl}(2))/$
$XY-YX=H$
$[H,X]=2X$
$[H,Y]$
$=T(k\langle X,Y$
$=k\{X,Y,$
$\mathcal{O}_{SL(2)}=k[a,b,c,d]/(ad-bc=1)$
radical ideal

$\mathcal{U}(\mathfrak{sl}(2))$ ← basis $X=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $Y=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $Z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ ← $G=SL(2)$ ⇝ $\mathcal{O}_{SL(2)}$

$\mathfrak{sl}(2)=\left\{\begin{pmatrix}\alpha&\beta\\0&\delta\end{pmatrix}\in M_{2\times 2}(k) \mid \alpha+\delta=0\right\}$

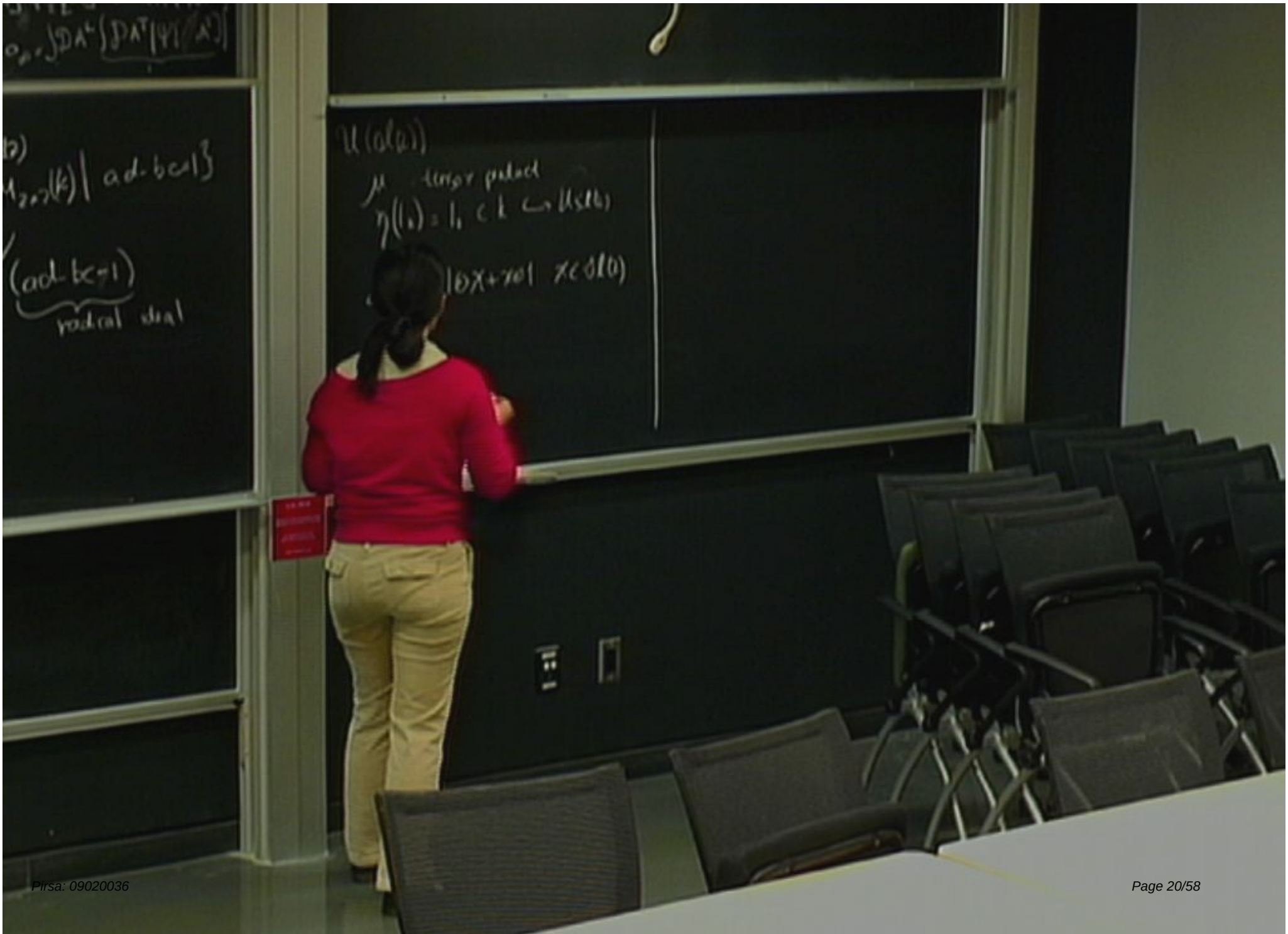
$SL(2):=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in M_{2\times 2}(k) \mid ad-bc=1\right\}$

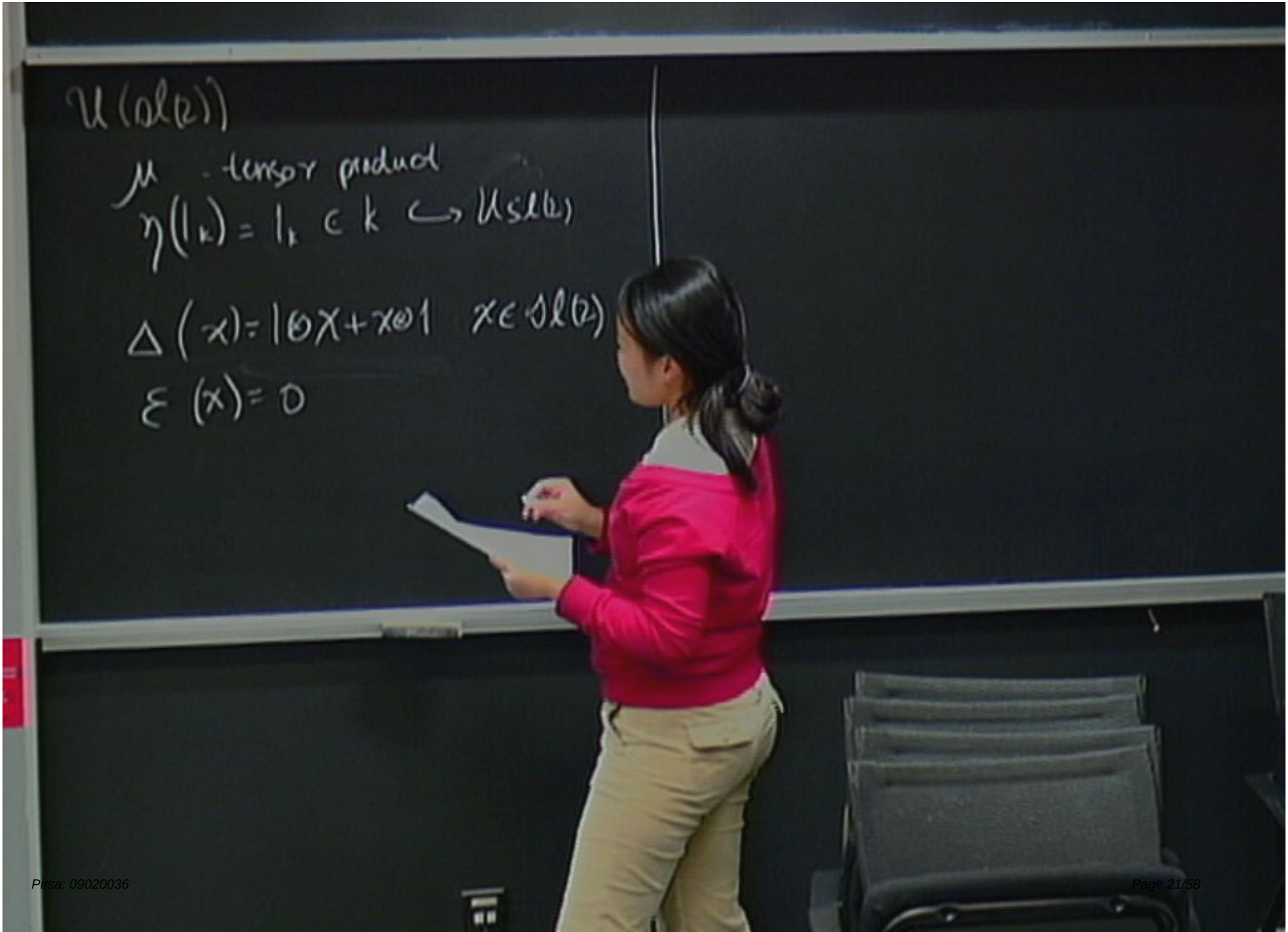
$\mathcal{U}(\mathfrak{sl}(2))=T(\mathfrak{sl}(2))/\begin{matrix}XY-YX=H\\[H,X]=2X\\[H,Y]=-2Y\end{matrix}$

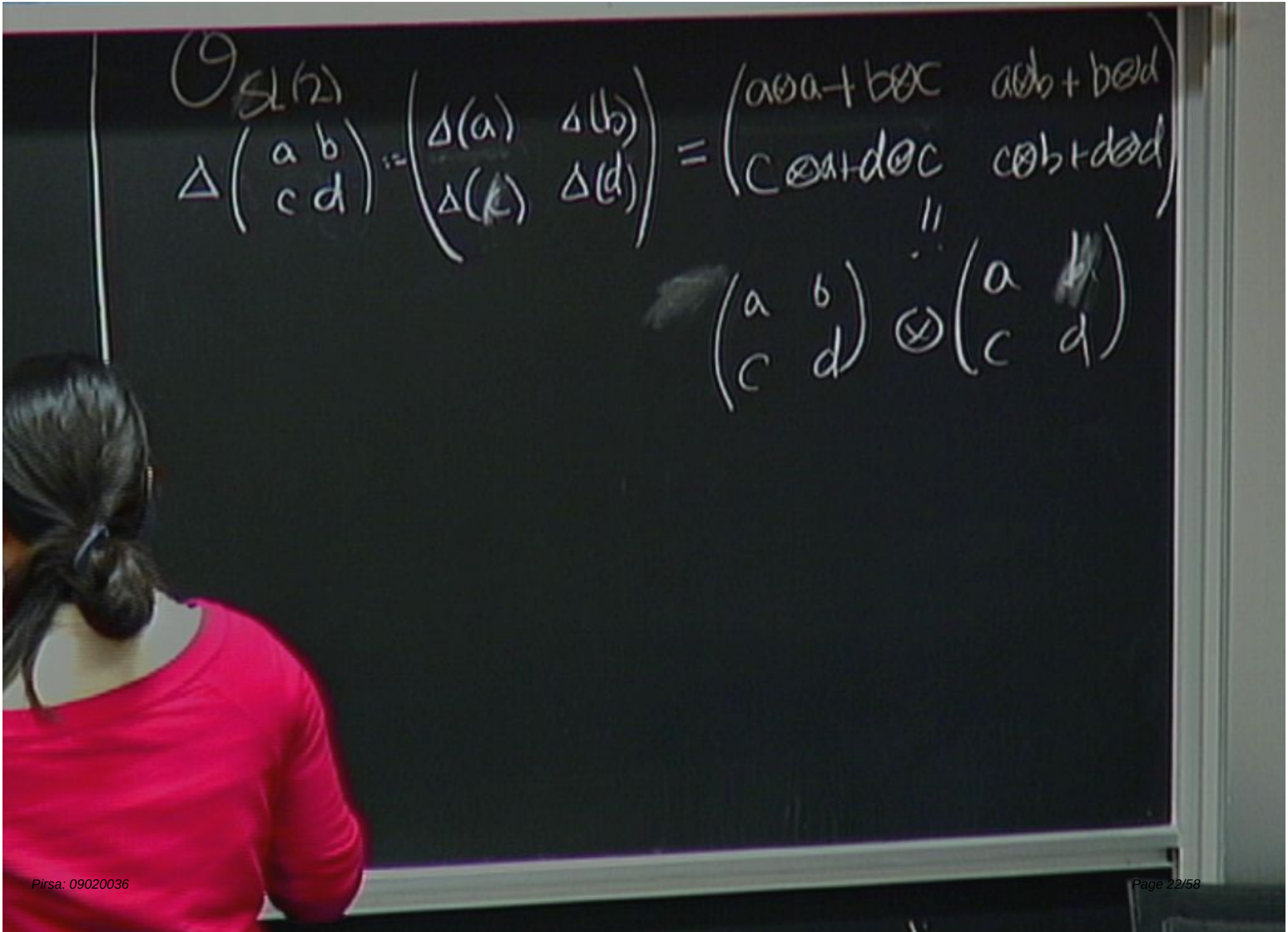
$=T(k\langle X,Y,H\rangle)/\sim$

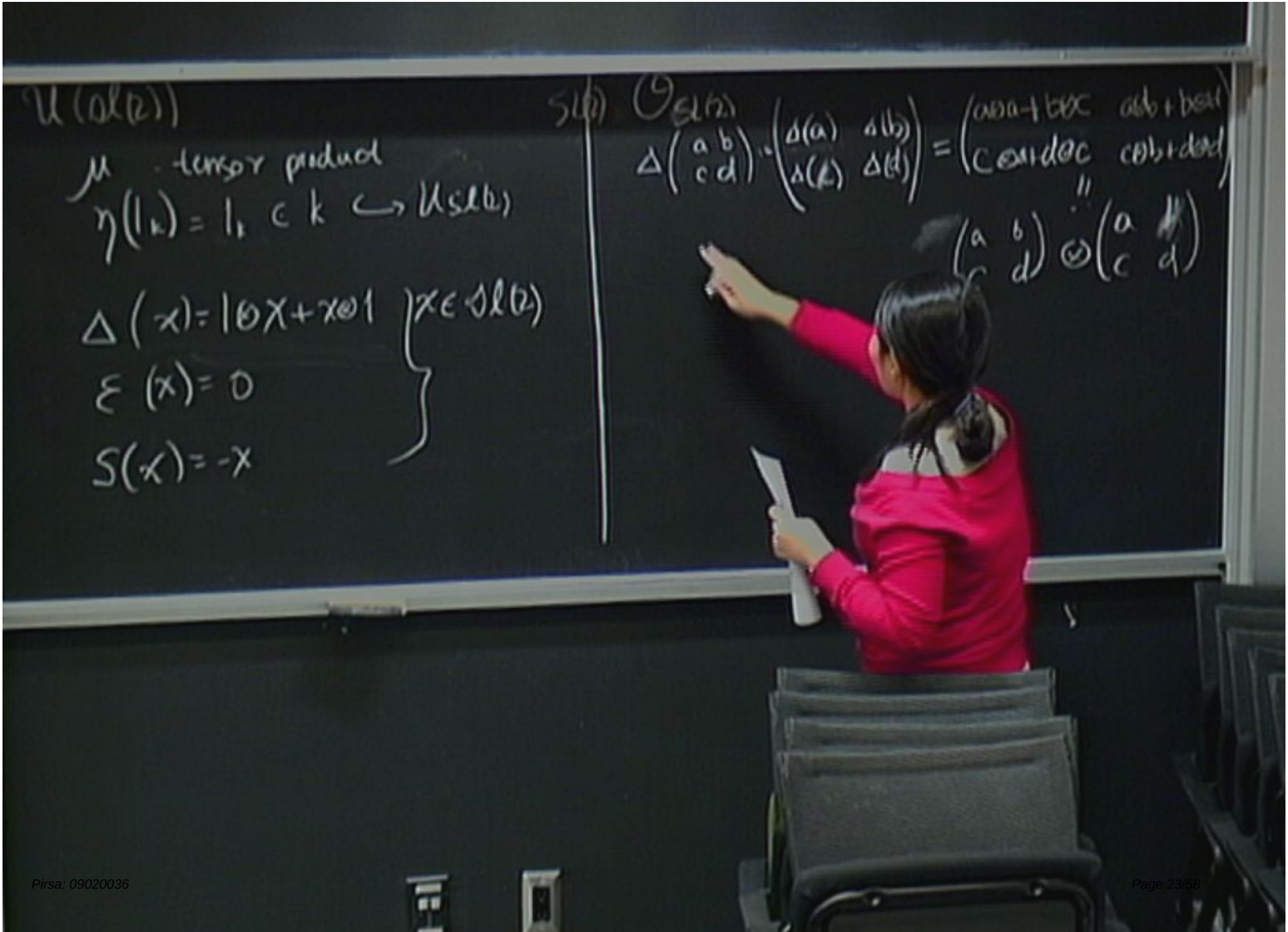
$=k\{X,Y,H\}/\sim$

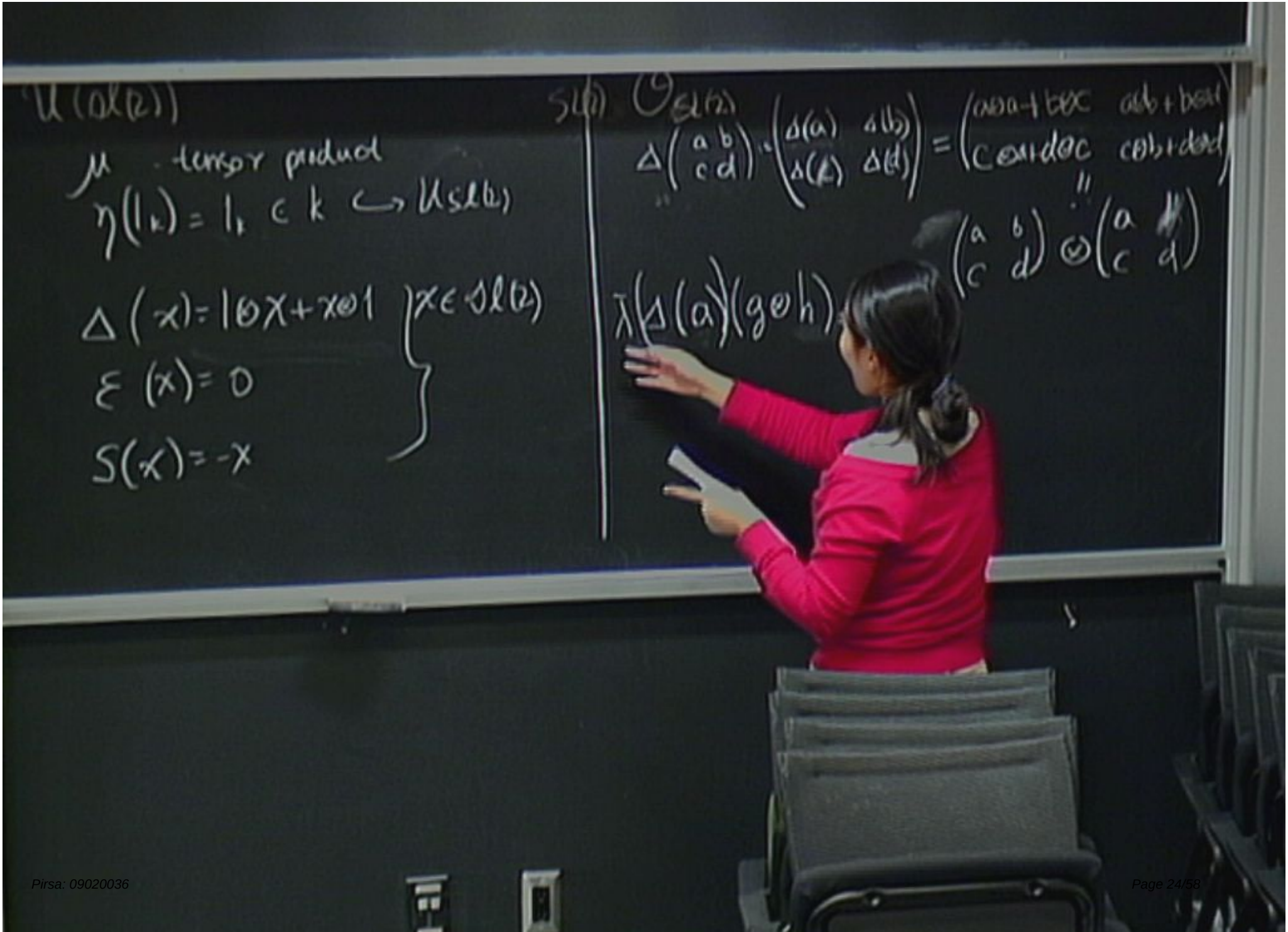
$\mathcal{O}_{SL(2)}=k[a,b,c,d]/\underbrace{(ad-bc-1)}_{\text{radical ideal}}$


(ad-bc=1)
radical ideal


U(sl(2))
μ tensor product
η(1_k) = 1_k ∈ k ↪ U(sl(2))
Δ(x) = 1⊗x + x⊗1 x ∈ sl(2)
ε(x) = 0


$\Theta_{SL(2)}$
$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} \Delta(a) & \Delta(b) \\ \Delta(c) & \Delta(d) \end{pmatrix} = \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{pmatrix}$
!!
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$


U(sl(2))
μ tensor product
η(1_k) = 1_k ∈ k ↪ Usl(2)
Δ(x) = 1⊗x + x⊗1
ε(x) = 0
S(x) = -x
x ∈ sl(2)
S(1)
sl(2)
Δ(a b / c d) = (Δ(a) Δ(b) / Δ(c) Δ(d)) = (a⊗a + b⊗c a⊗b + b⊗d / c⊗a + d⊗c c⊗b + d⊗d)
(a b / c d) ⊗ (a b / c d)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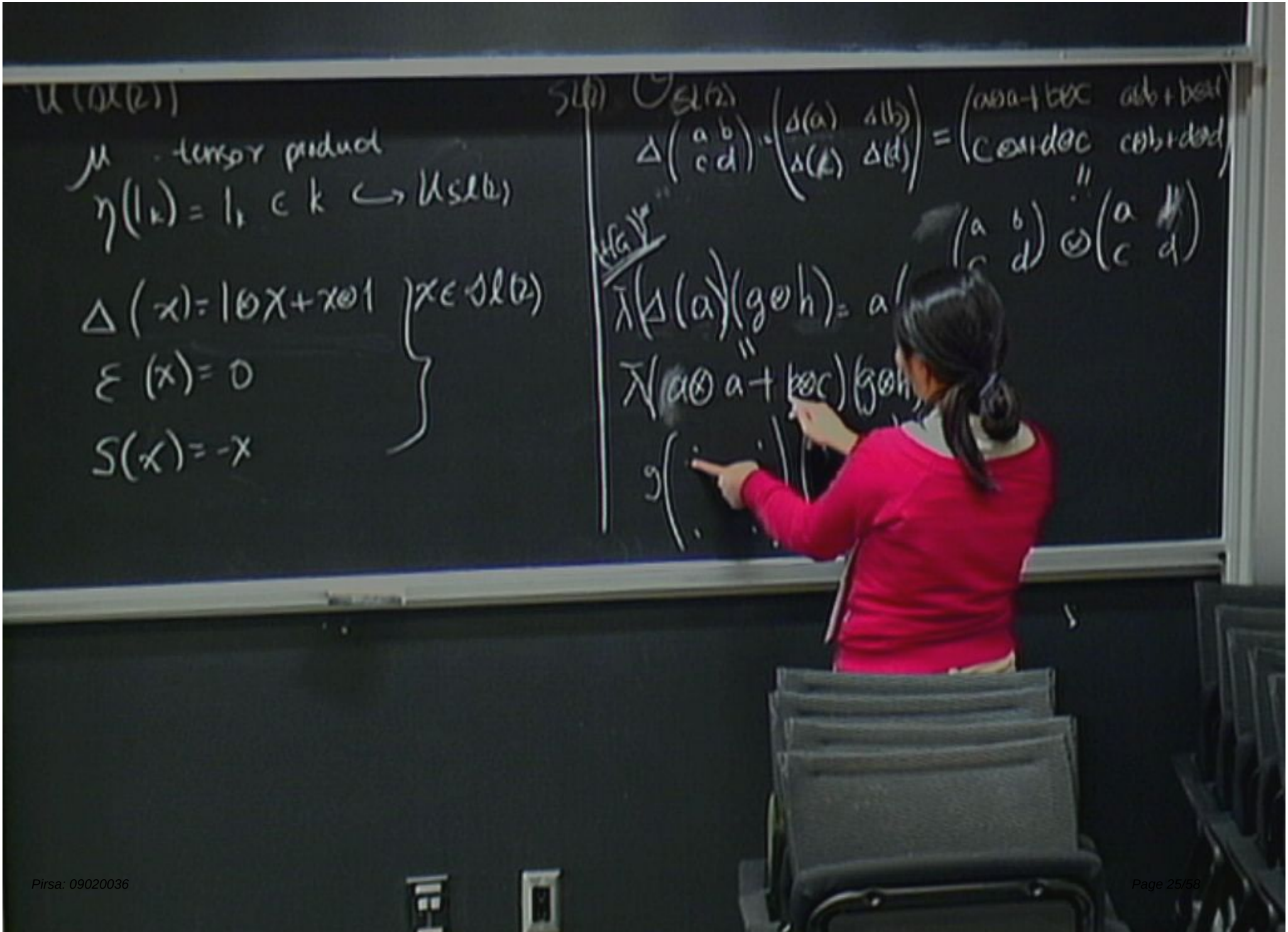
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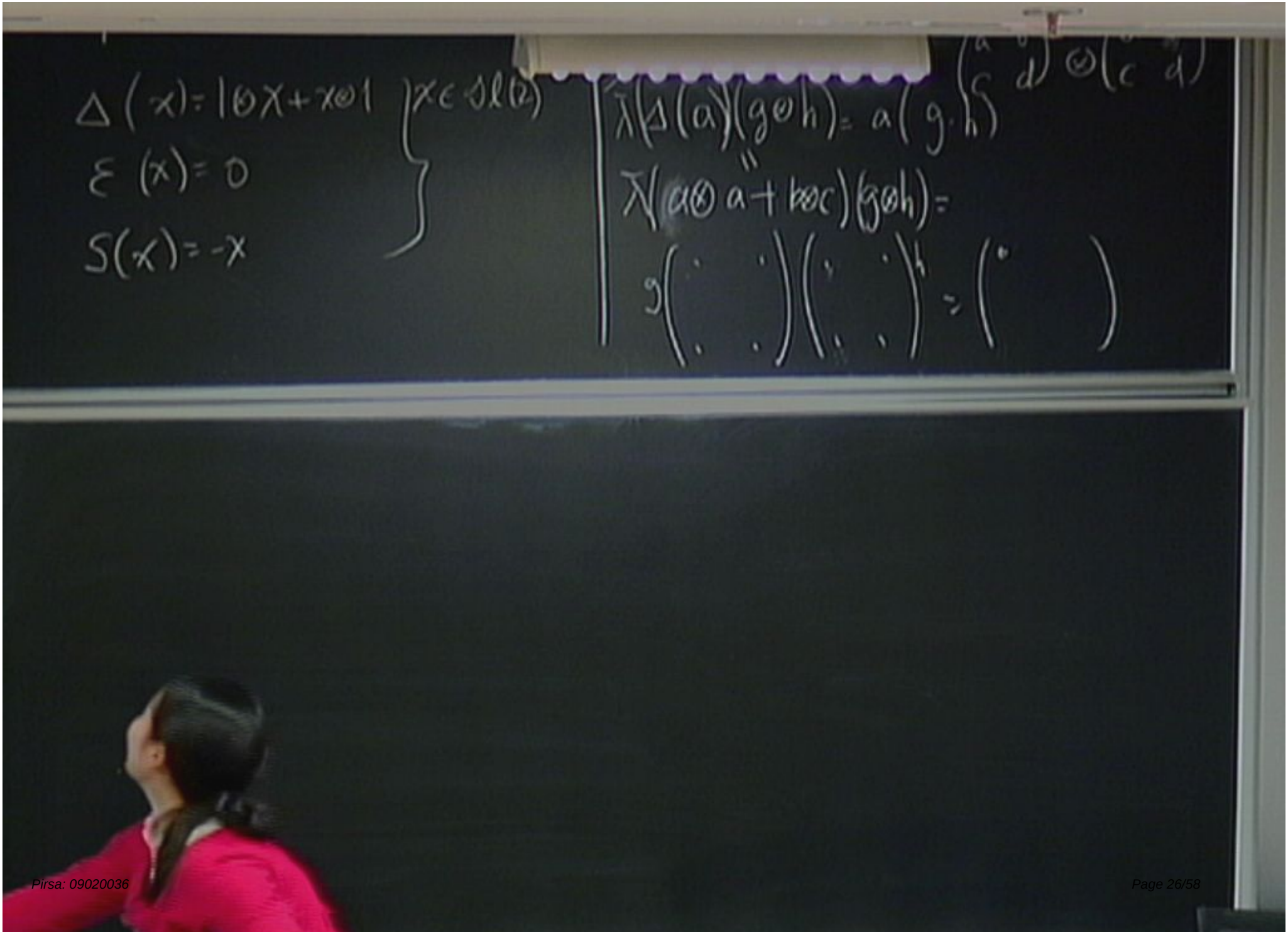
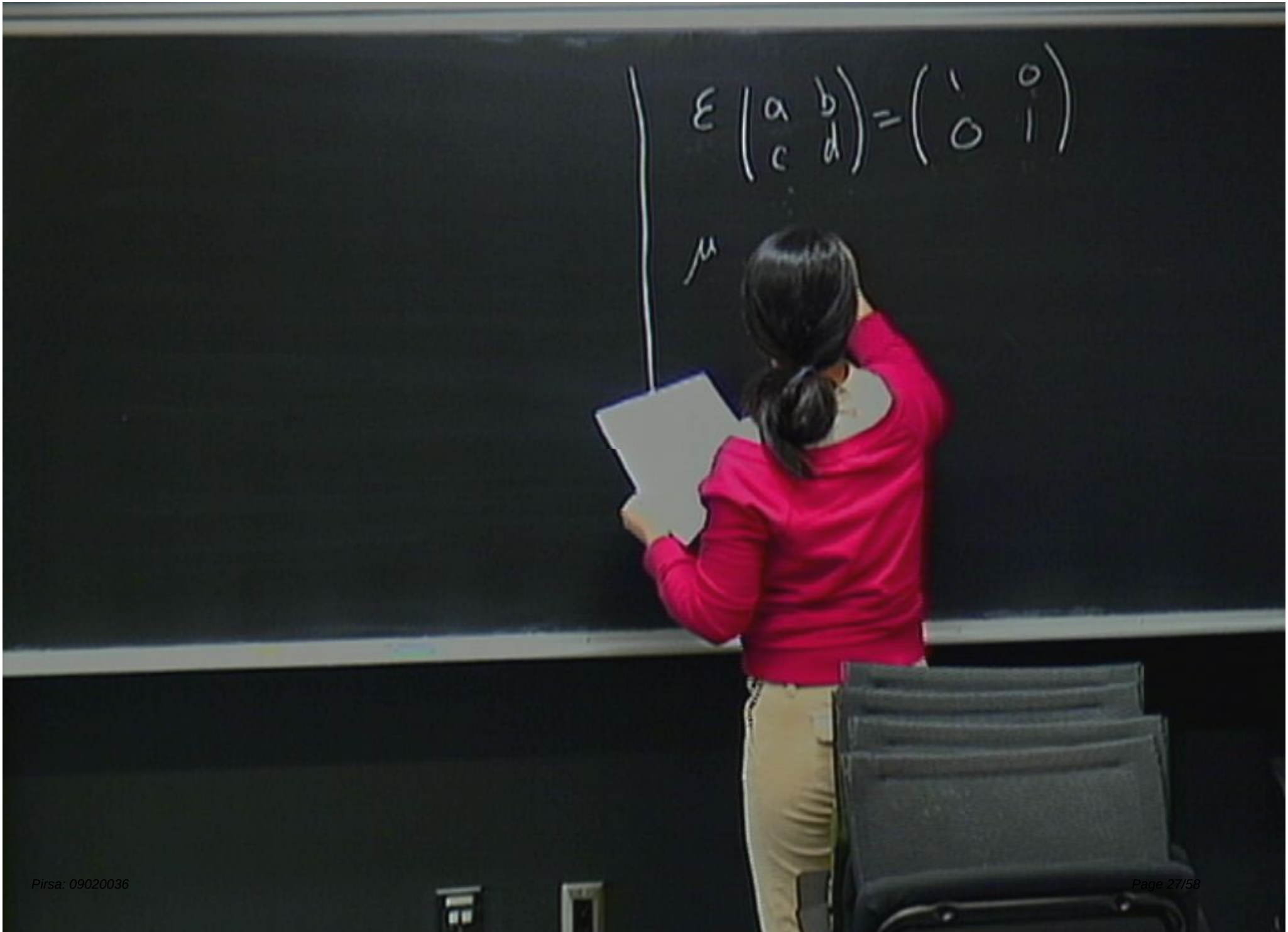
S(x) = -x
x ∈ sl(2)


Δ(x) = 1⊗x + x⊗1
ε(x) = 0
S(x) = -x
x ∈ sl(2)
λ̄(Δ(a))(g⊗h) = a(g·h)
λ̄(a⊗a + b⊗c)(g⊗h) =

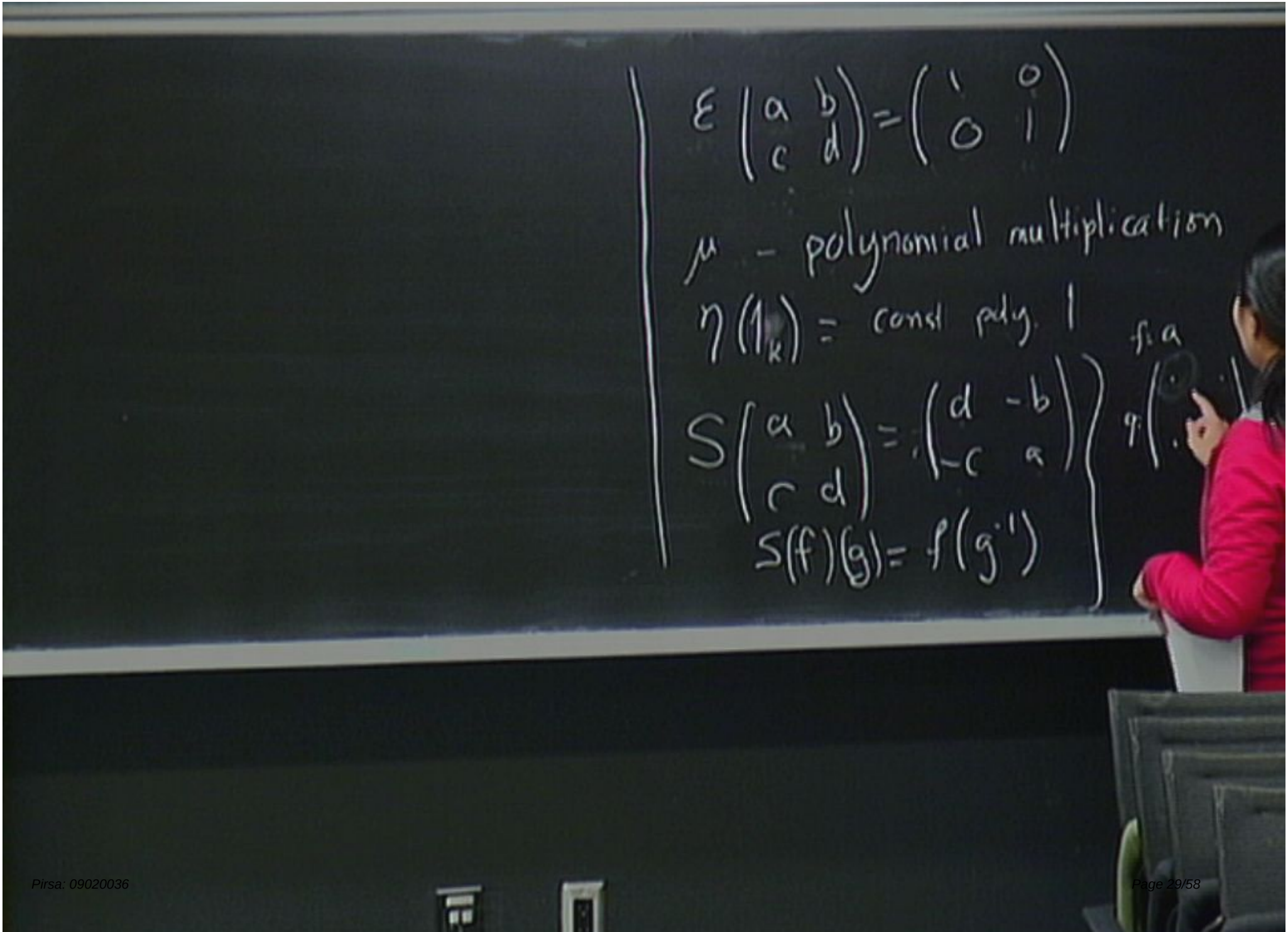

$\varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\mu$

$\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\mu$ - polynomial multiplication

$\eta(1_k) =$ const poly. 1

$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} =$


$\varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\mu$ - polynomial multiplication
$\eta(1_k)$ = const poly. 1
$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
$S(f)(g) = f(g^{-1})$
s.a

$$\varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

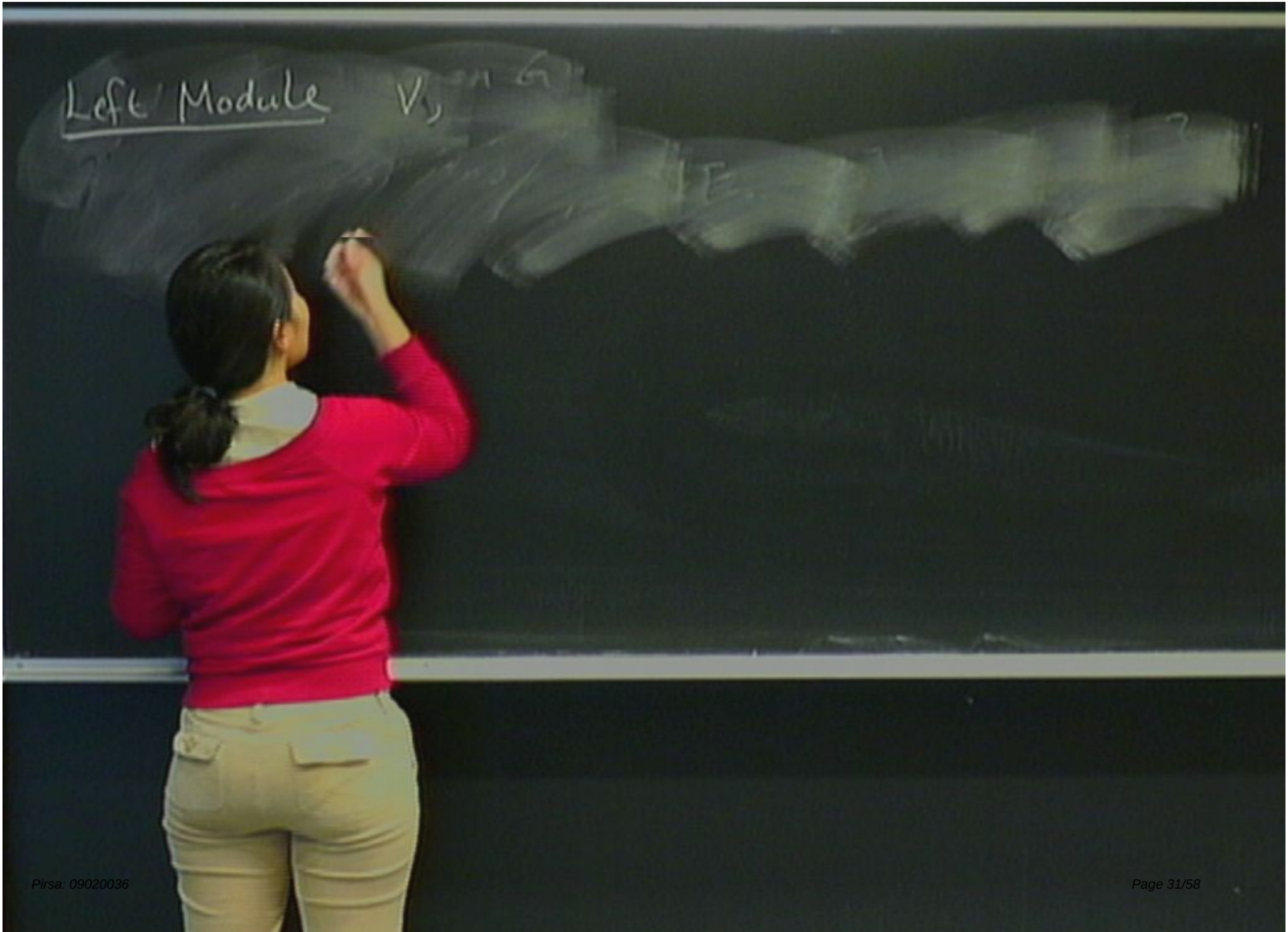
$\mu$ - polynomial multiplication

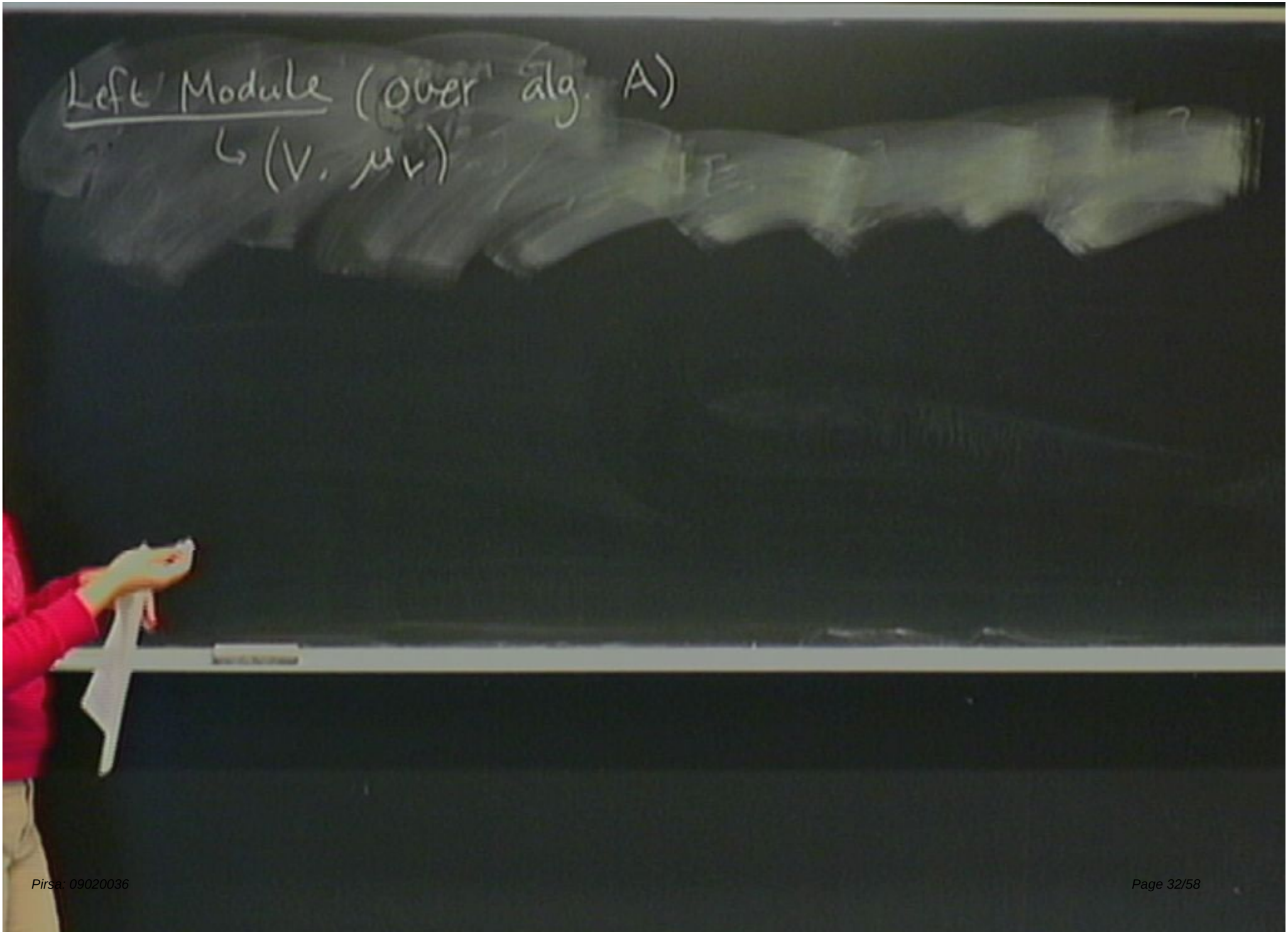
$\eta(1_k) =$ const poly. 1

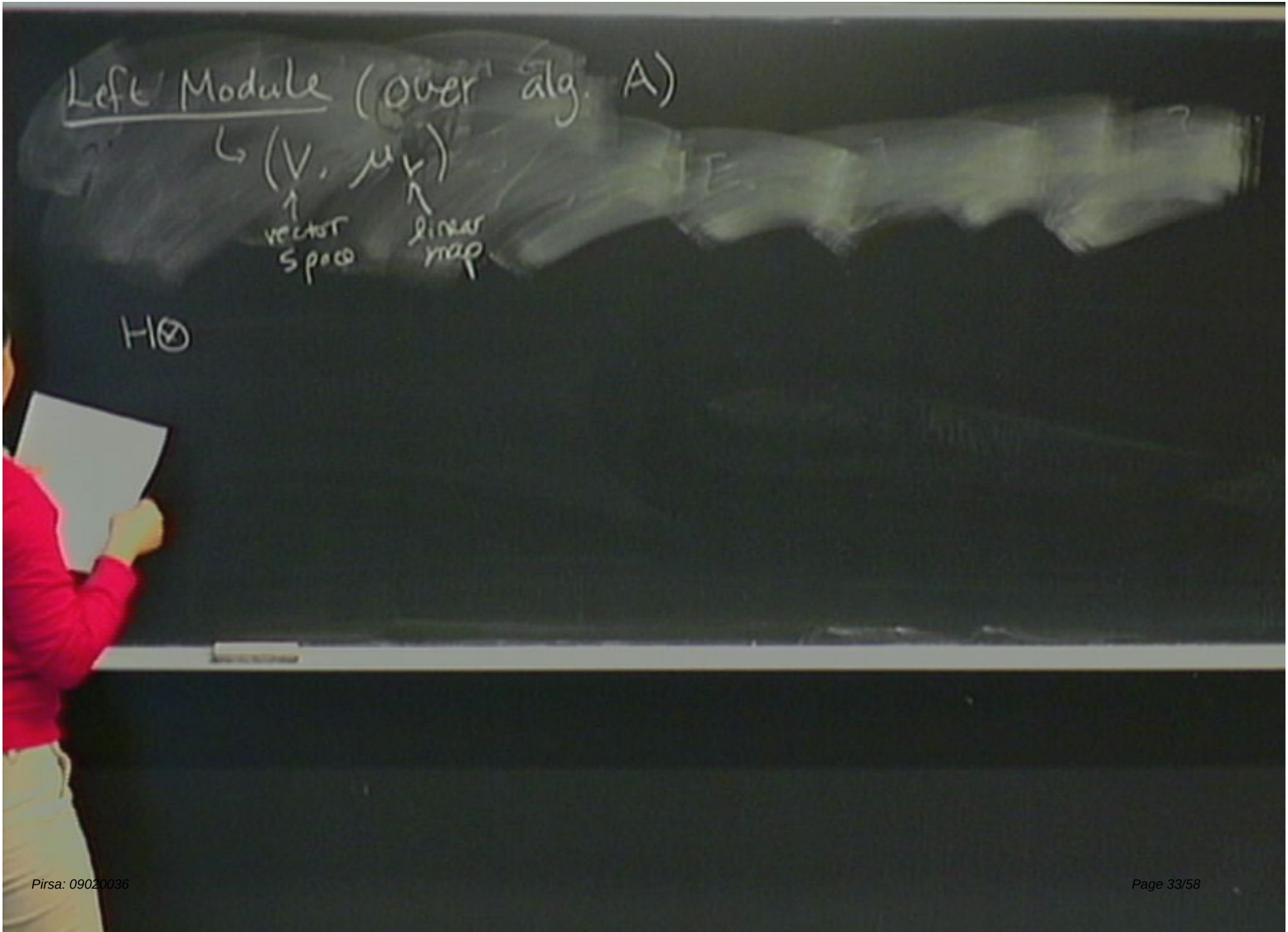
$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

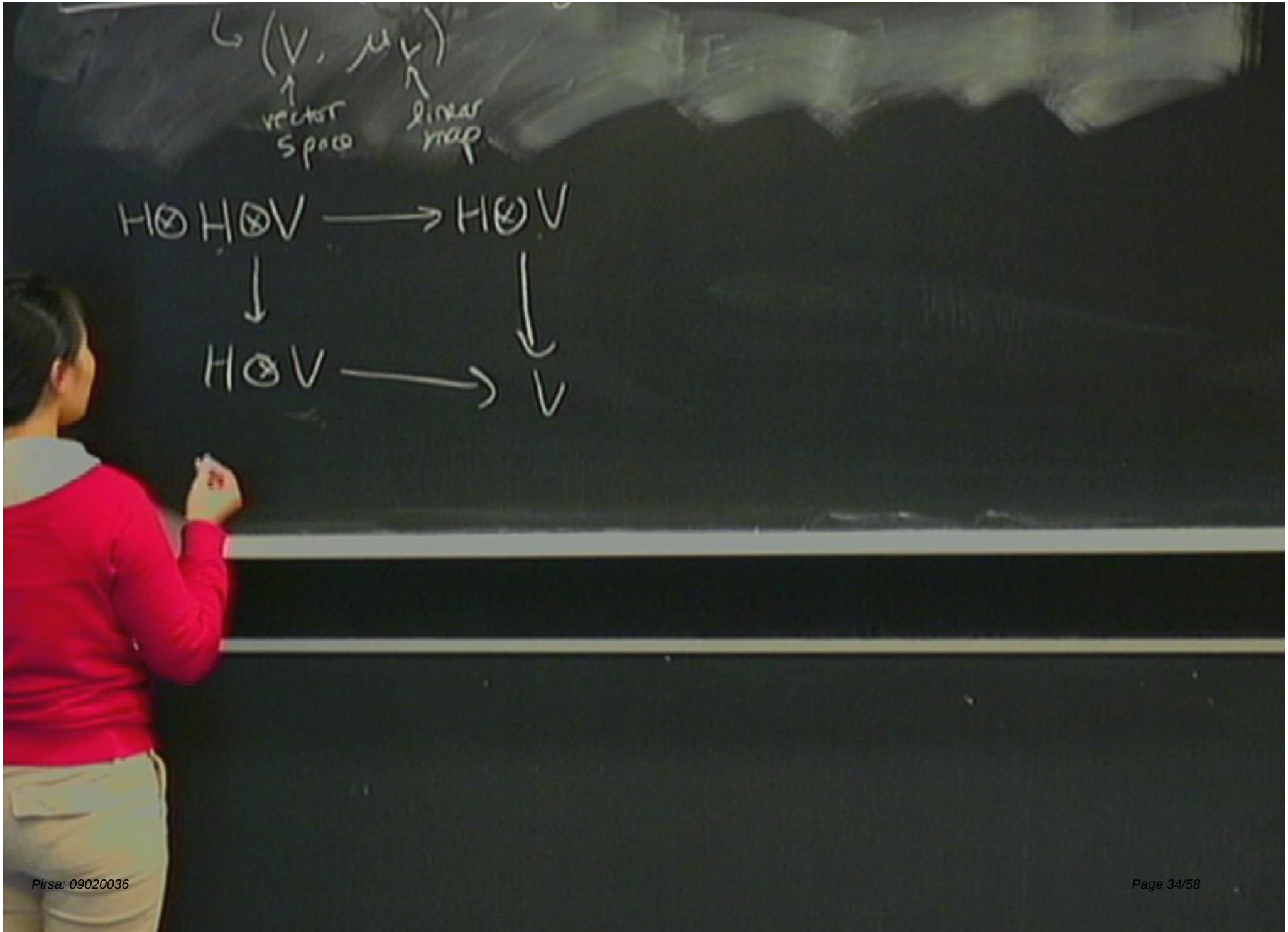
s.a $\eta\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$

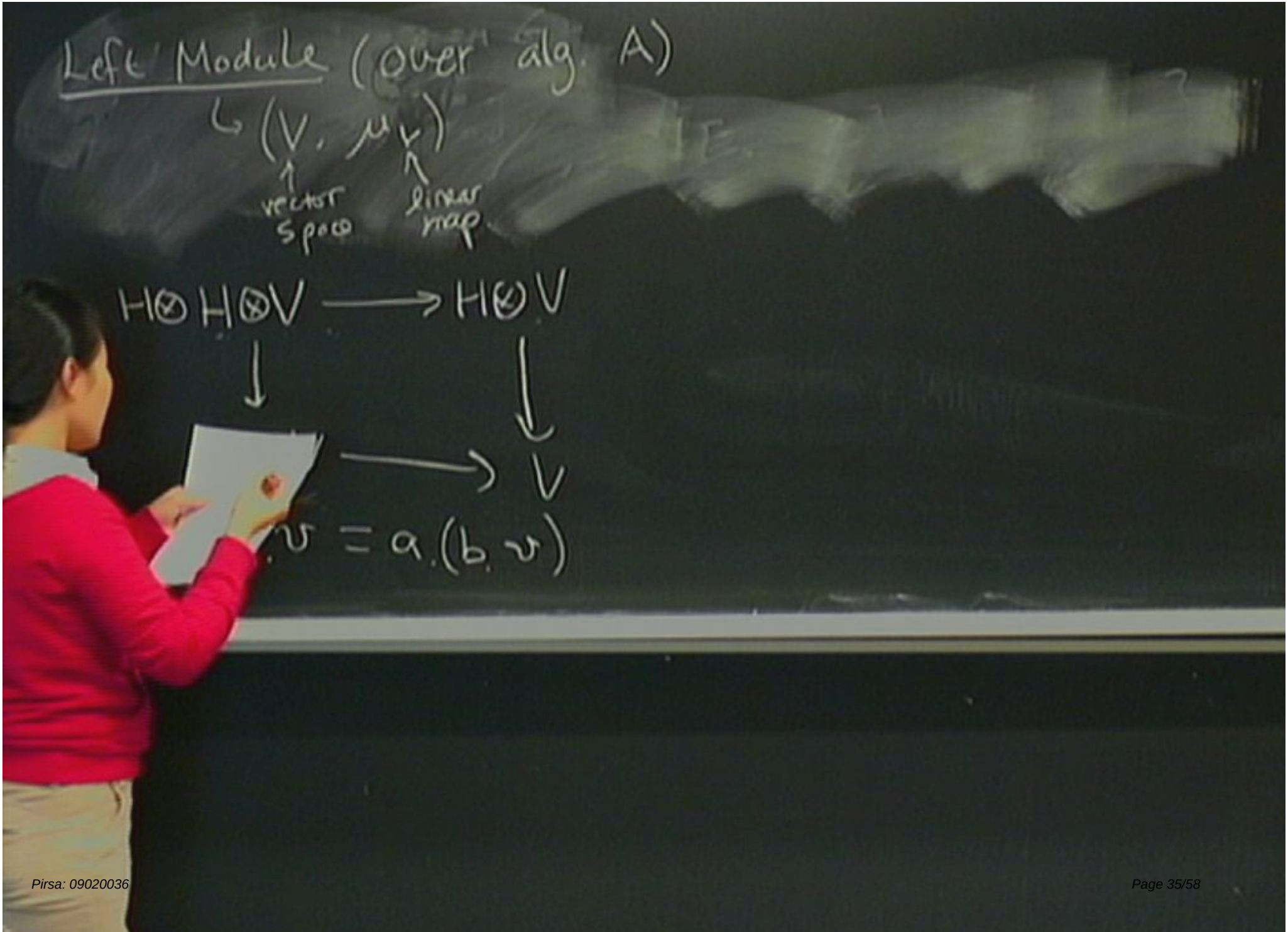
$S(f)(g) = f(g^{-1})$

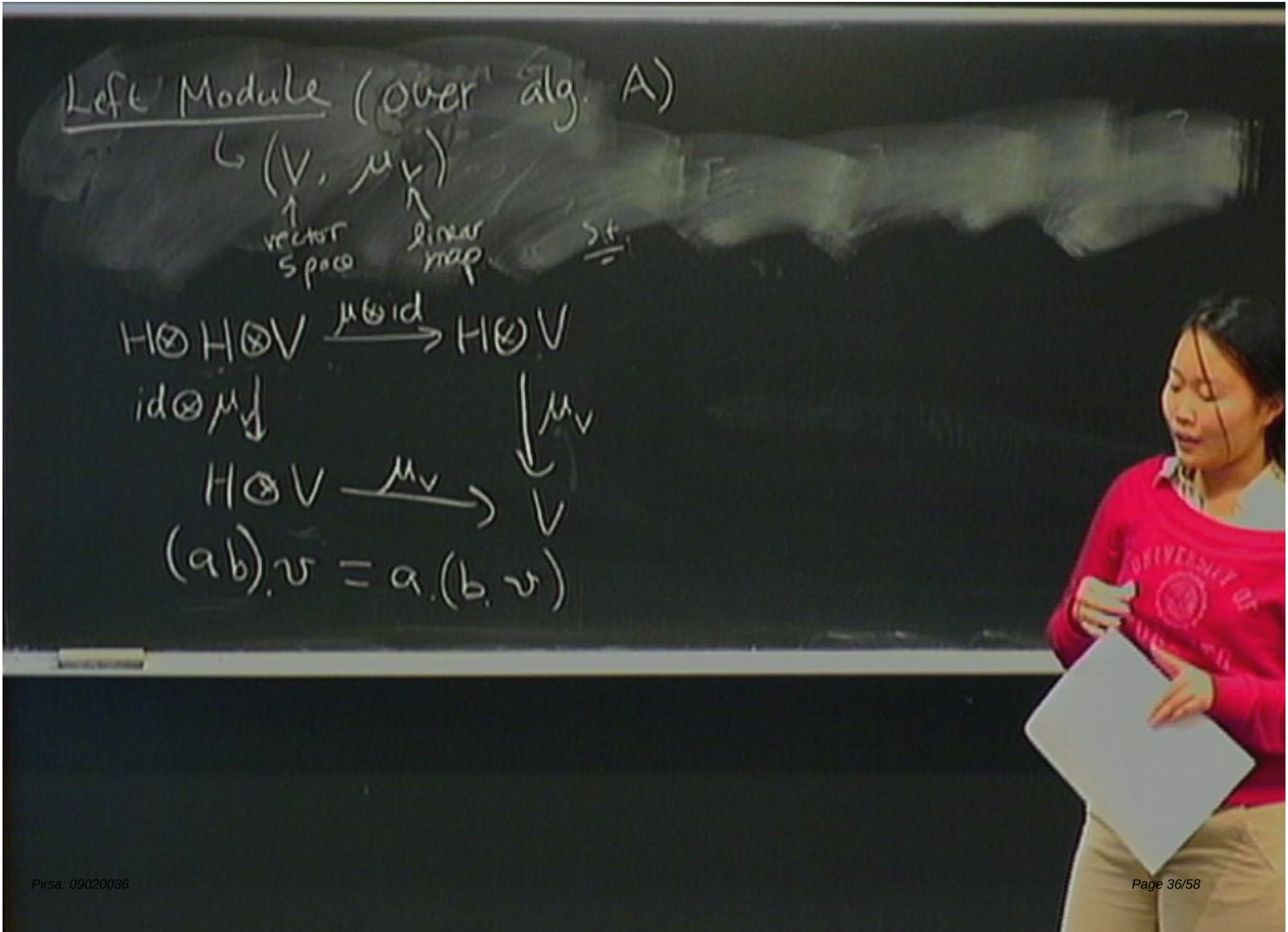

Left Module

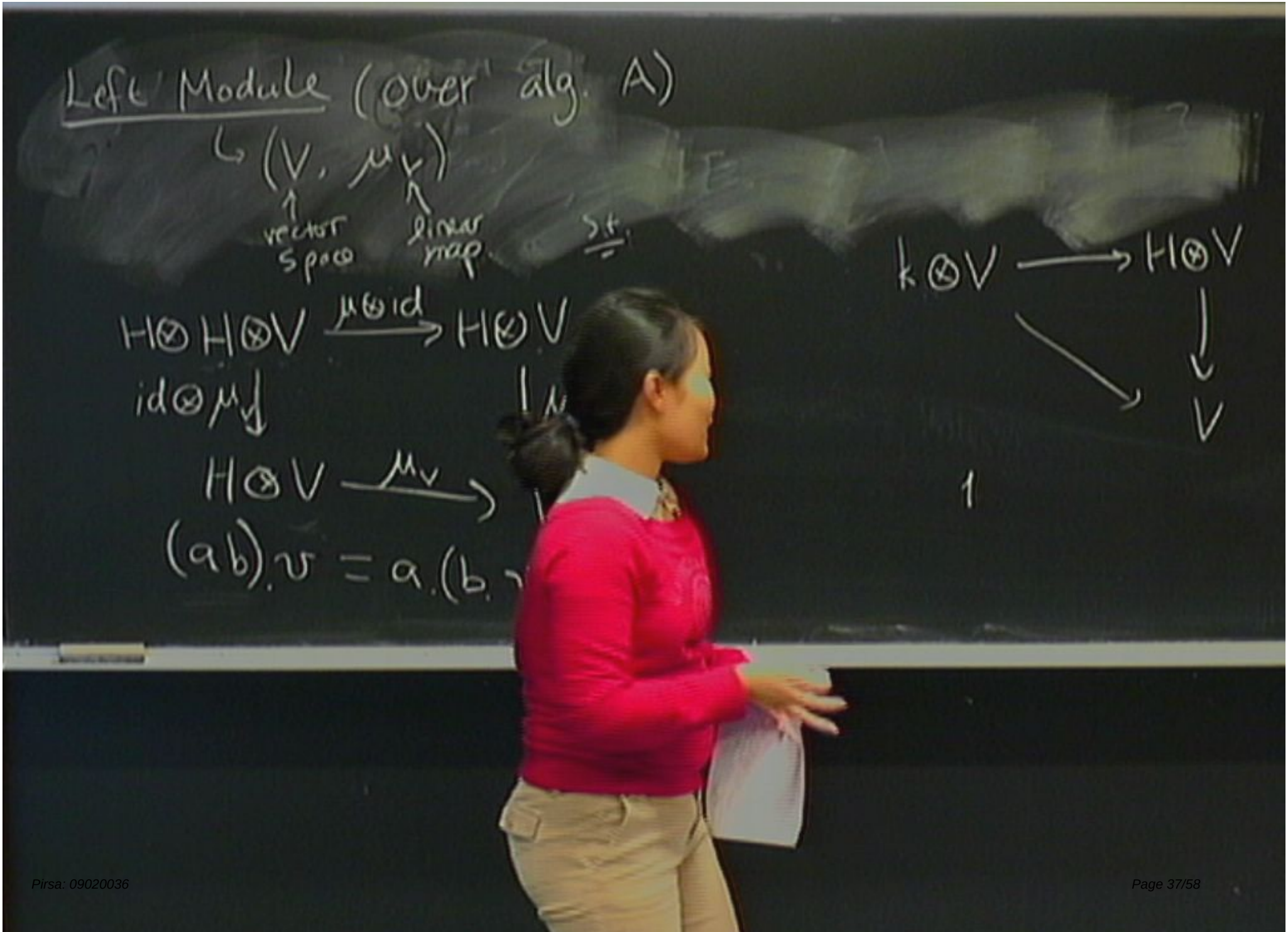

Left Module (over alg. A)
↳ (V, $\mu_V$)

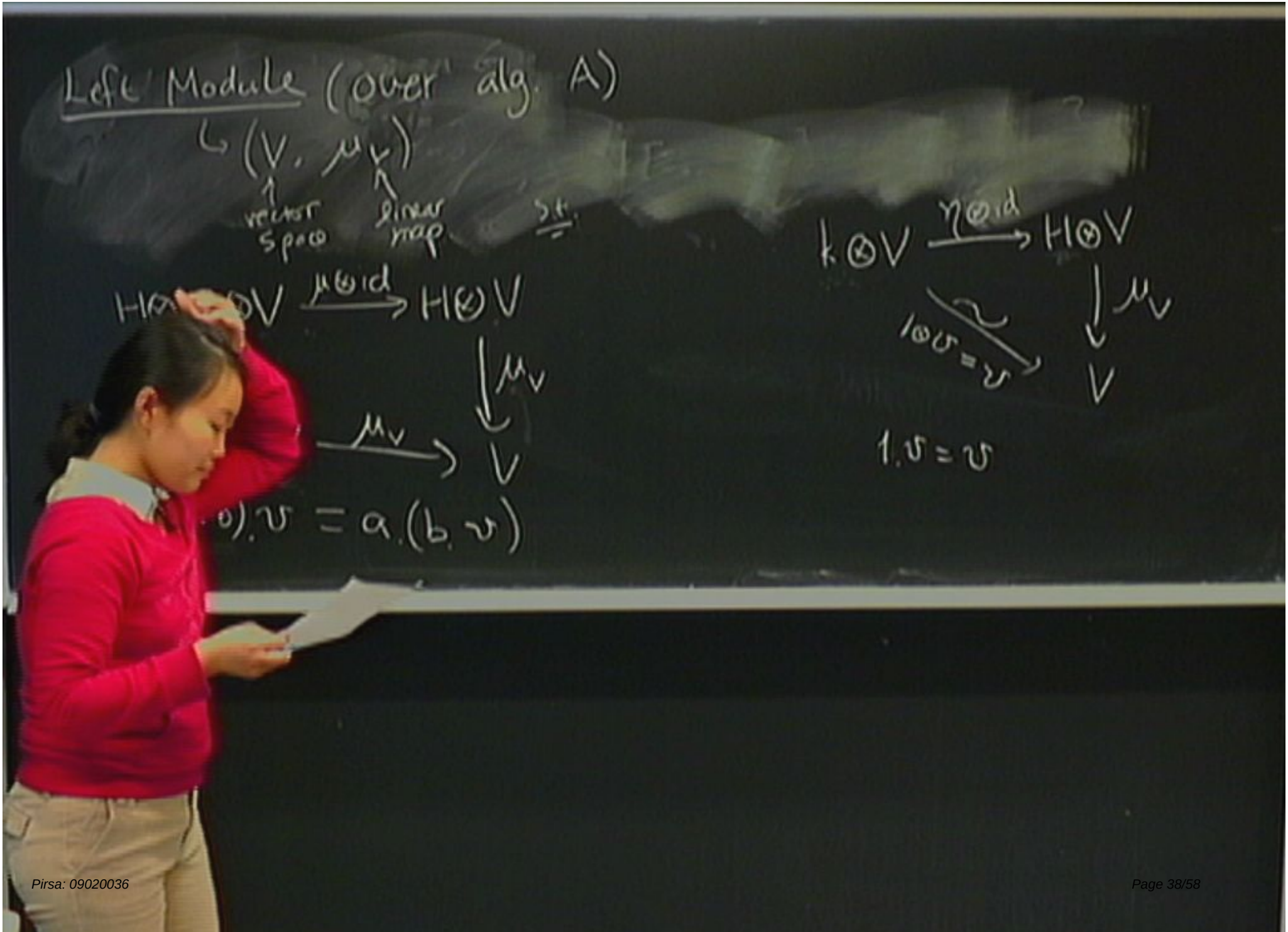

Left Module (over alg. A)
(V, μ)
vector space
linear map
H⊗

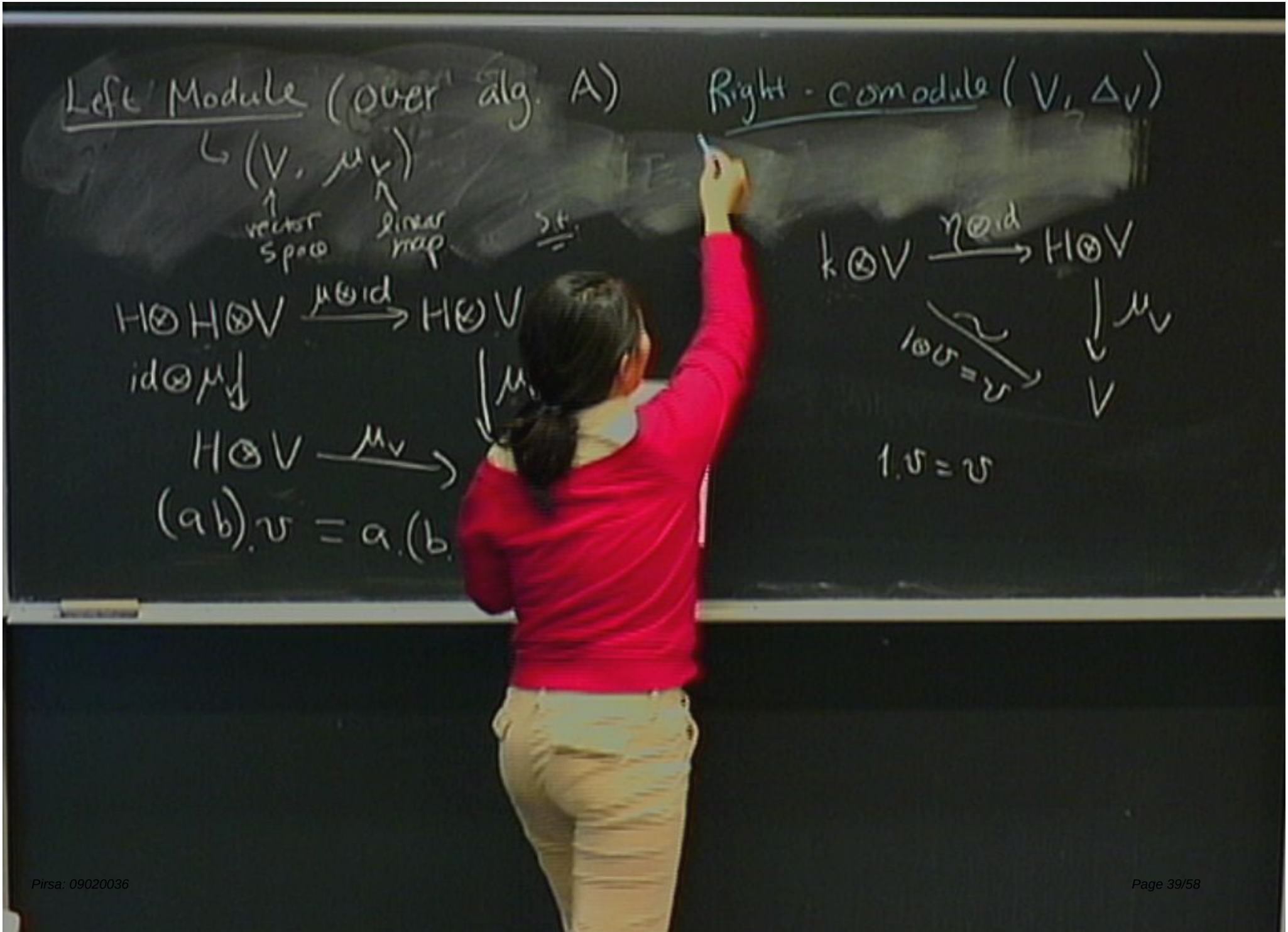

(V, μ)
vector space
linear map
H⊗H⊗V
H⊗V
H⊗V
V

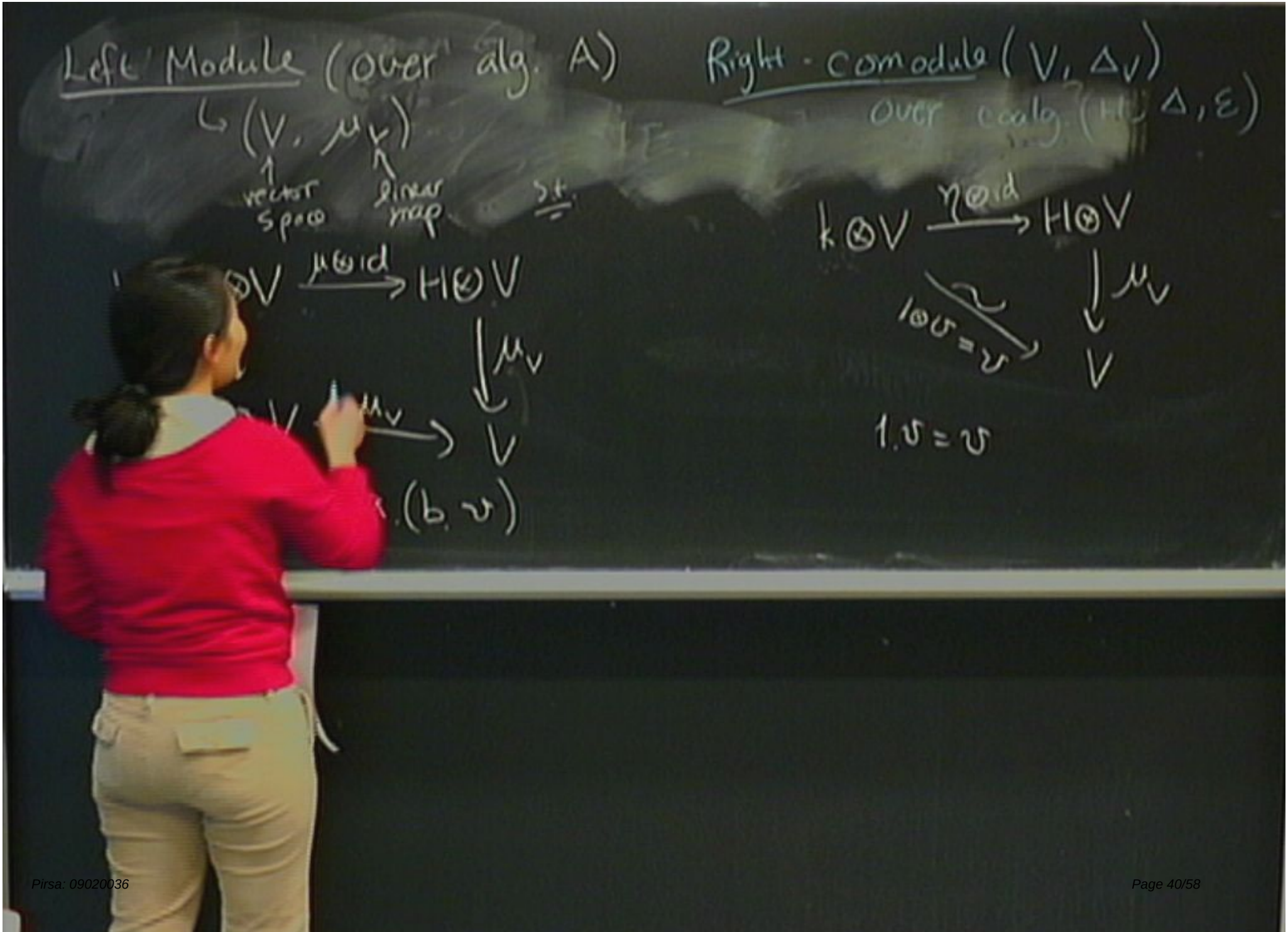

Left Module (over alg. A)
(V, μ, ν)
vector space
linear map
H⊗H⊗V
H⊗V
V
a.(b.v)

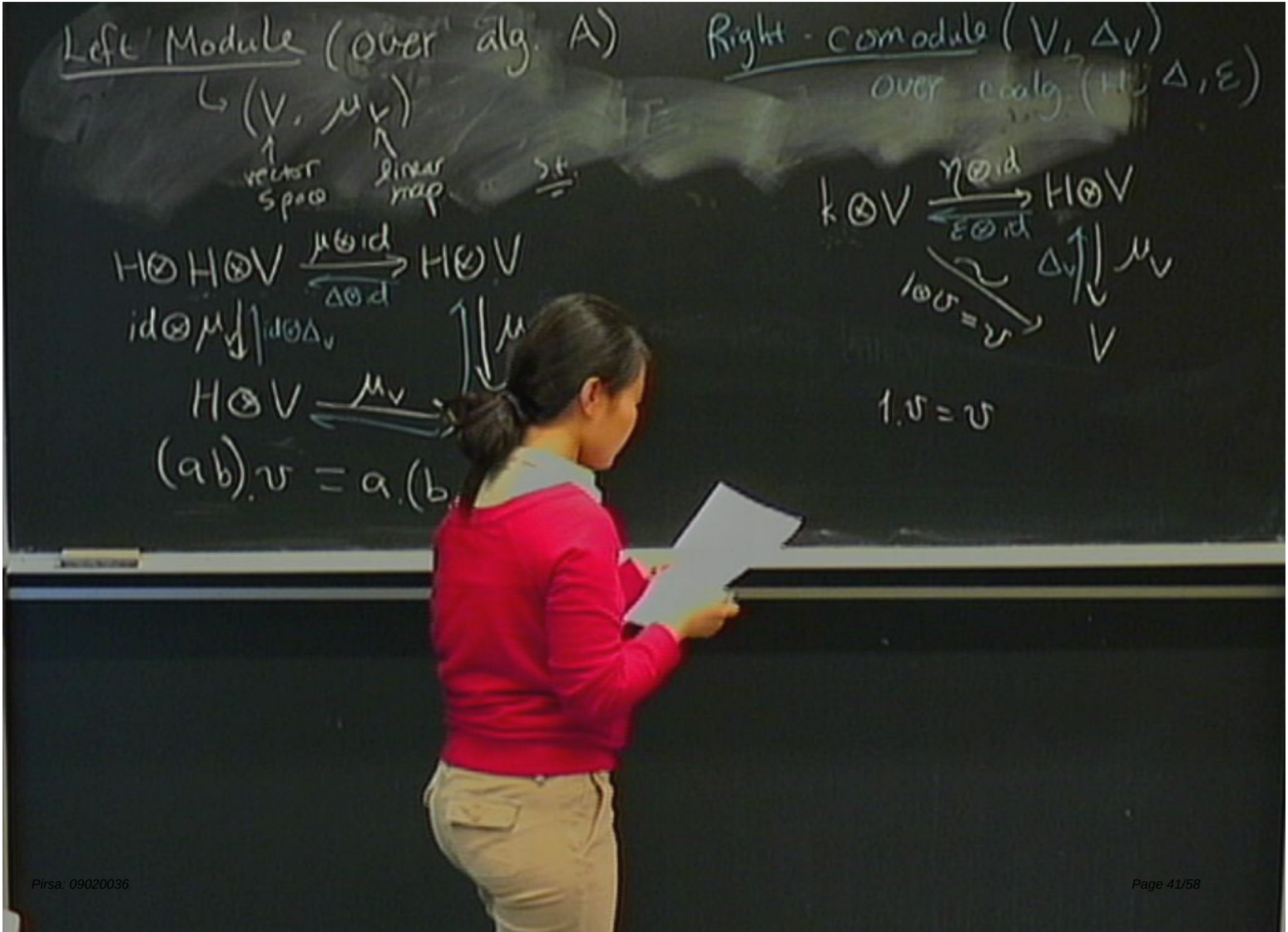

Left Module (over alg. A)
(V, μV)
vector space
linear map
s.t.
H⊗H⊗V
μ⊗id
H⊗V
id⊗μV
μV
H⊗V
μV
V
(ab).v = a.(b.v)

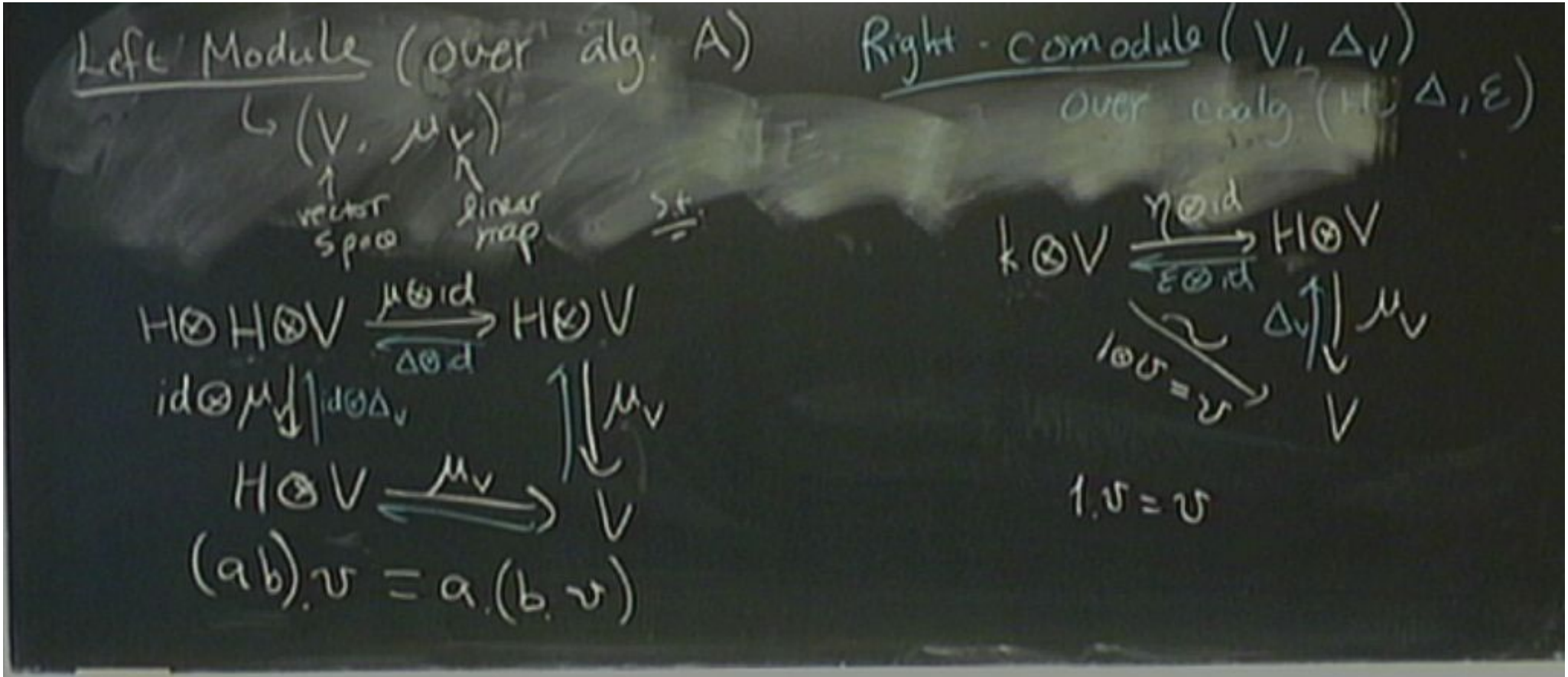

Left Module (over alg. A)
(V, μ_V)
vector space
linear map
s.t.
H⊗H⊗V
μ⊗id
H⊗V
id⊗μ_V
H⊗V
μ_V
(ab).v = a.(b.
k⊗V
H⊗V
V
1

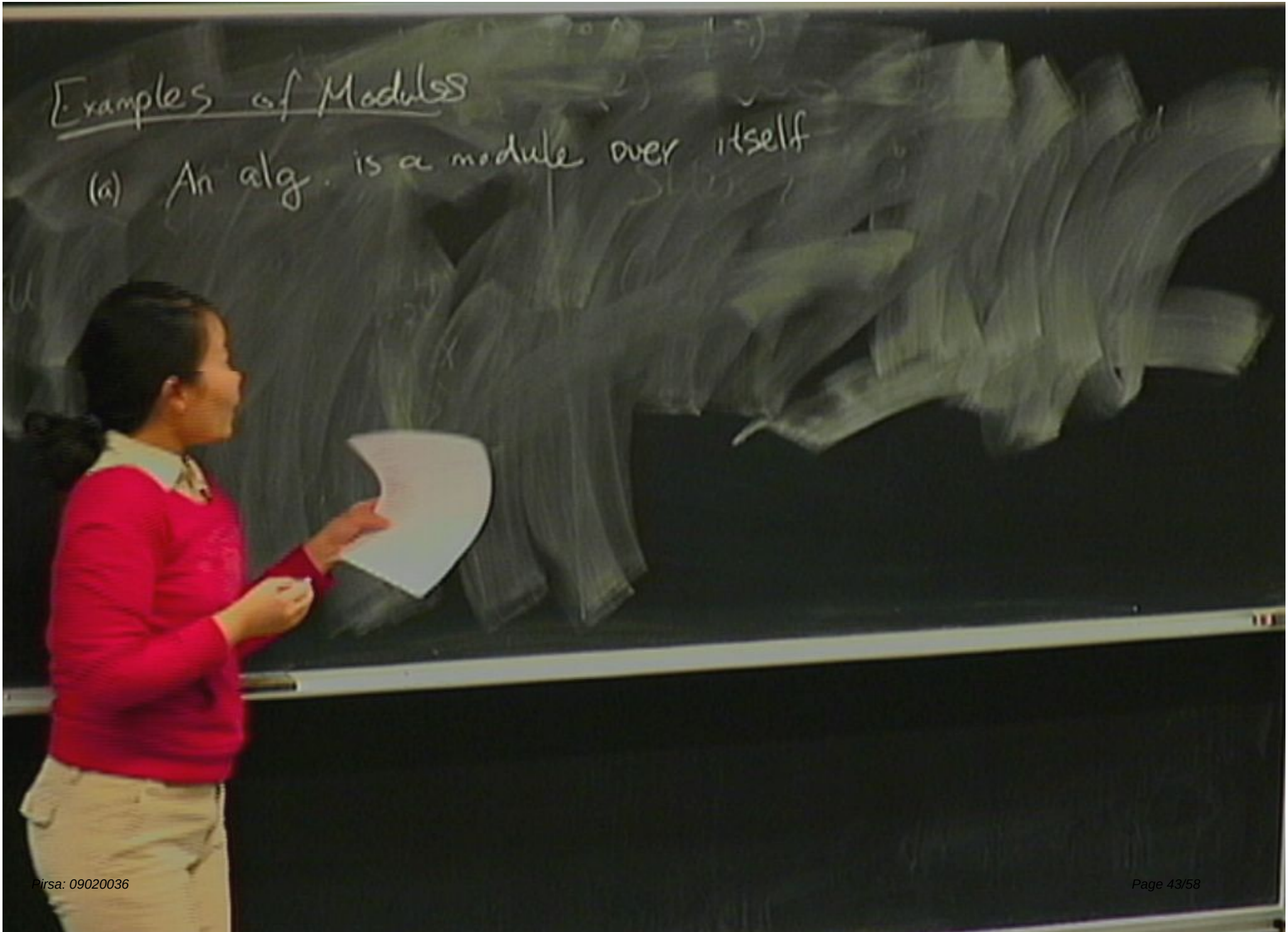

Left Module (over alg. A)
↳ (V, μ_V)
vector space
linear map
s.t.
μ⊗id
H⊗V
μ_V
V
k⊗V
η⊗id
1⊗v = v
1.v = v
a.(b.v)

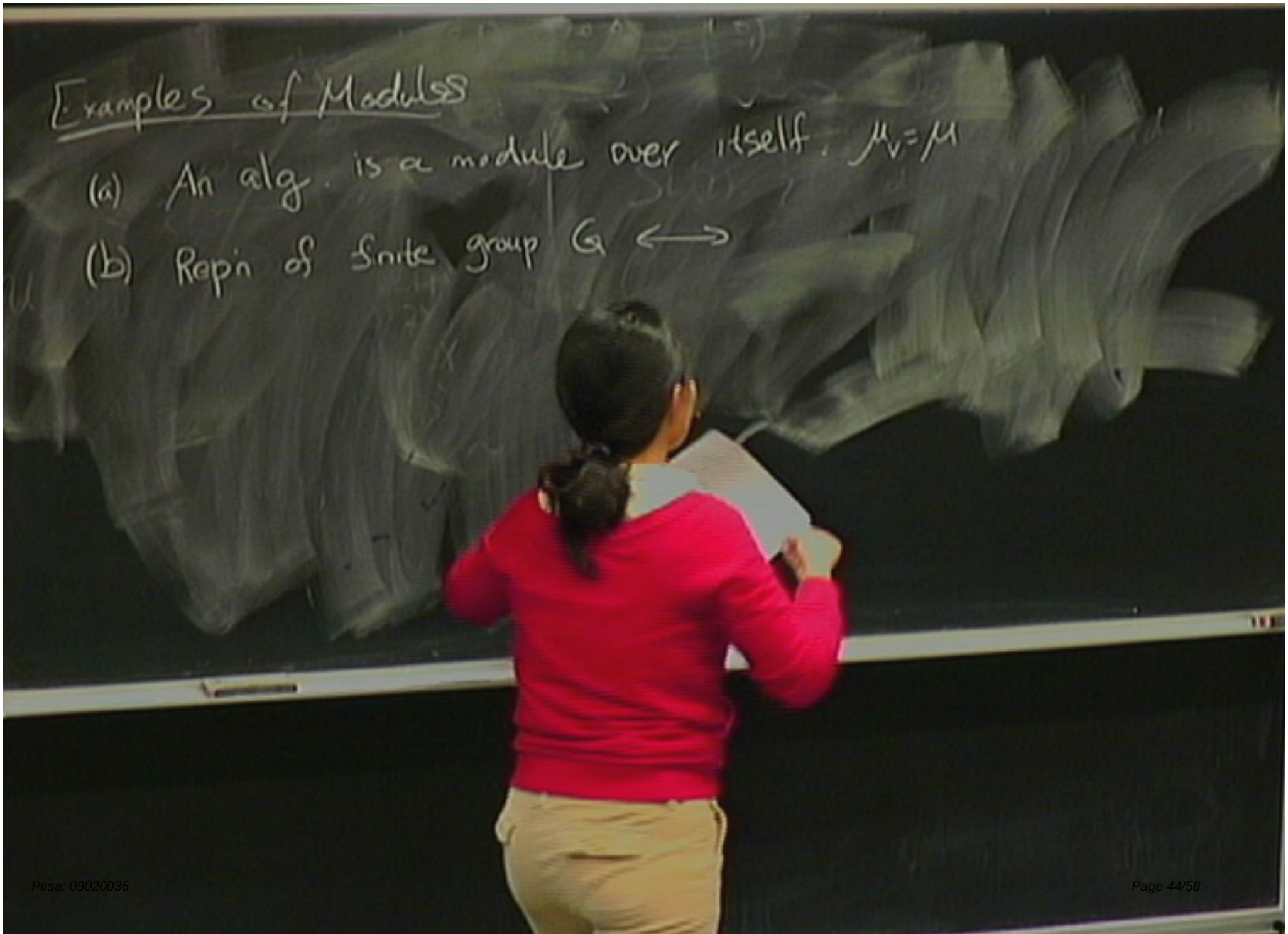

Left Module (over alg. A)
(V, μV)
vector space
linear map
s.t.
H⊗H⊗V
μ⊗id
H⊗V
id⊗μV
μV
H⊗V
μV
(ab).v = a.(b.
Right - comodule (V, ΔV)
k⊗V
η⊗id
H⊗V
ΔV
1⊗v = v
V
1.v = v


Left Module (over alg. A)
(V, μ_V)
vector space
linear map
s.t.
μ⊗id
1⊗v = v
1.v = v
(b.v)
Right - comodule (V, Δ_V)
over coalg.
Δ, ε)
η⊗id


Left Module (over alg. A)
↳ (V, μ_V)
vector space
linear map
s.t.
H⊗H⊗V
μ⊗id
Δ⊗id
H⊗V
id⊗μ
id⊗Δ_V
μ
H⊗V
μ_V
(ab).v = a.(b
Right - comodule (V, Δ_V)
over coalg (H, Δ, ε)
k⊗V
η⊗id
ε⊗id
H⊗V
Δ_V
μ_V
1⊗v = v
V
1.v = v


Left Module (over alg. A)
↳ (V, $\mu_V$)
vector space
linear map
s.t.
$H \otimes H \otimes V \rightleftarrows H \otimes V$ ($\mu \otimes id$, $\Delta \otimes id$)
$id \otimes \mu_V$, $id \otimes \Delta_V$, $\mu_V$
$H \otimes V \rightrightarrows V$ ($\mu_V$)
$(ab).v = a.(b.v)$
Right - comodule $(V, \Delta_V)$
over coalg $(H, \Delta, \varepsilon)$
$k \otimes V \rightleftarrows H \otimes V$ ($\eta \otimes id$, $\varepsilon \otimes id$)
$1 \otimes v = v$
$\Delta_V$, $\mu_V$
$V$
$1.v = v$


Examples of Modules
(a) An alg. is a module over itself


Examples of Modules
(a) An alg. is a module over itself. $M_V = M$
(b) Rep'n of finite group G ⟷

## Examples of Modules

(a) An alg. is a module over itself. $M_V = M$

(b) Rep'n of finite group $G \longleftrightarrow$ module over $k[G]$

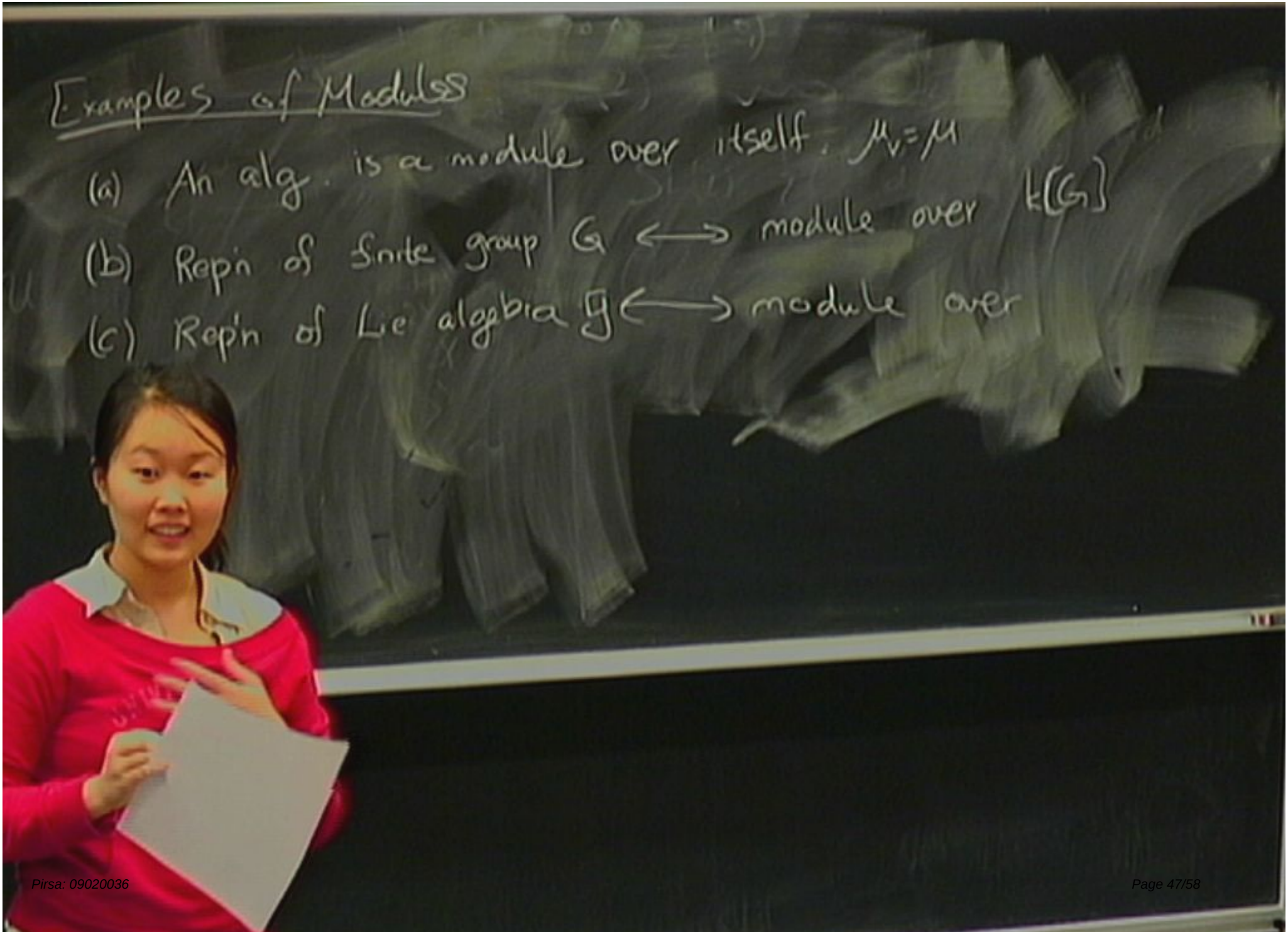
(c)

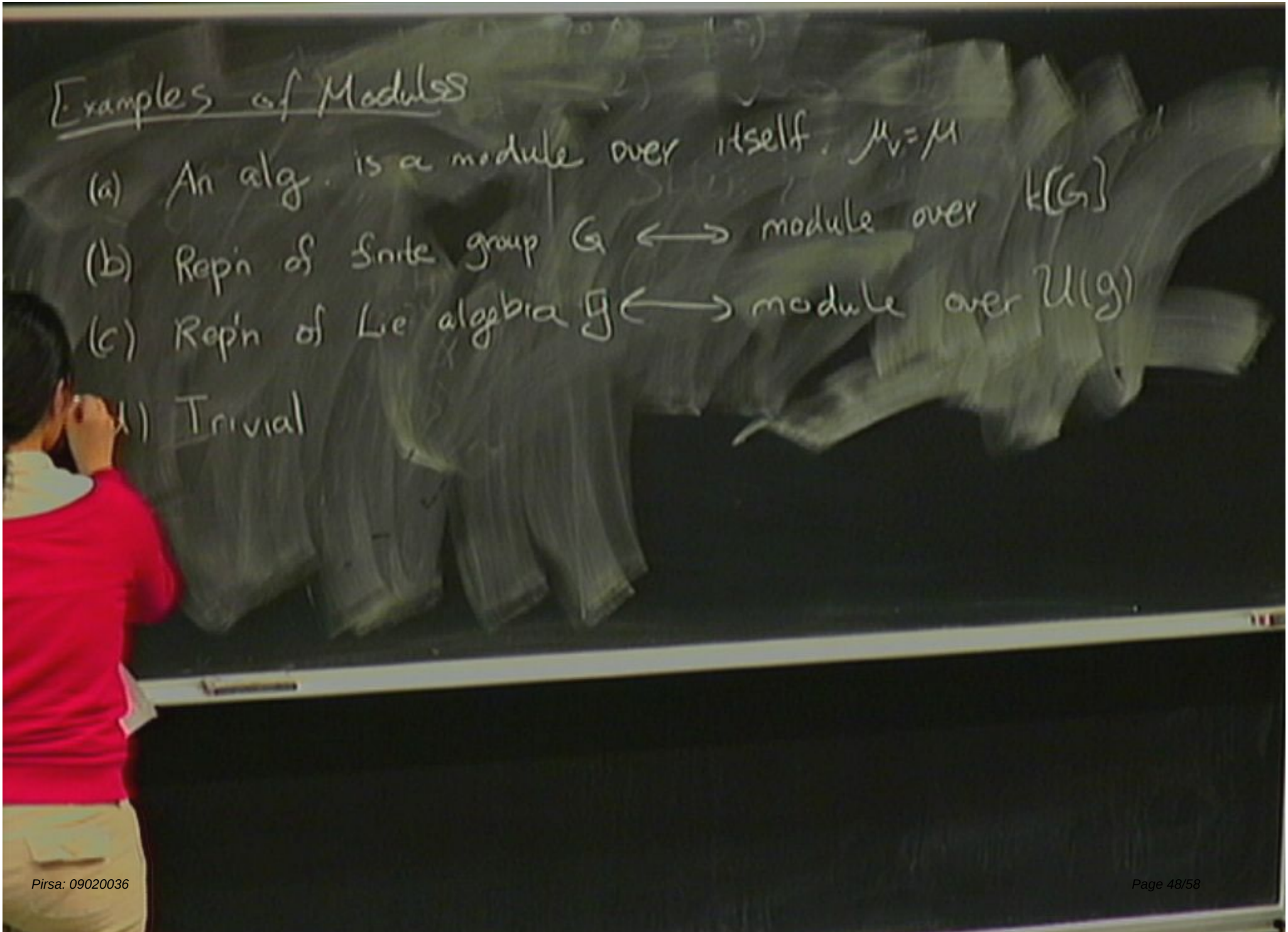
## Examples of Modules

(a) An alg. is a module over itself. $\mu_V = \mu$

(b) Rep'n of finite group $G \longleftrightarrow$ module over $k[G]$

(c) Rep'n of Lie algebra $\longleftrightarrow$ module over


Examples of Modules
(a) An alg. is a module over itself. $M_V = M$
(b) Rep'n of finite group G $\longleftrightarrow$ module over $k[G]$
(c) Rep'n of Lie algebra $\mathfrak{g}$ $\longleftrightarrow$ module over


Examples of Modules
(a) An alg. is a module over itself. $M_V = M$
(b) Rep'n of finite group G $\longleftrightarrow$ module over $k[G]$
(c) Rep'n of Lie algebra $\mathfrak{g}$ $\longleftrightarrow$ module over $U(\mathfrak{g})$
(d) Trivial

## Examples of Modules

(a) An alg. is a module over itself. $\mu_V = \mu$

(b) Rep'n of finite group $G \longleftrightarrow$ module over $k[G]$

(c) Rep'n of Lie algebra $\mathfrak{g} \longleftrightarrow$ module over $\mathcal{U}(\mathfrak{g})$

(d) Trivial $H$-module structure on any v. sp. $V$: $\mu_V(a \otimes v) = \varepsilon(a)v$

$a \cdot v$

$H$ bialg

(e)

## Examples of Modules

(a) An alg. is a module over itself. $\mu_V = \mu$

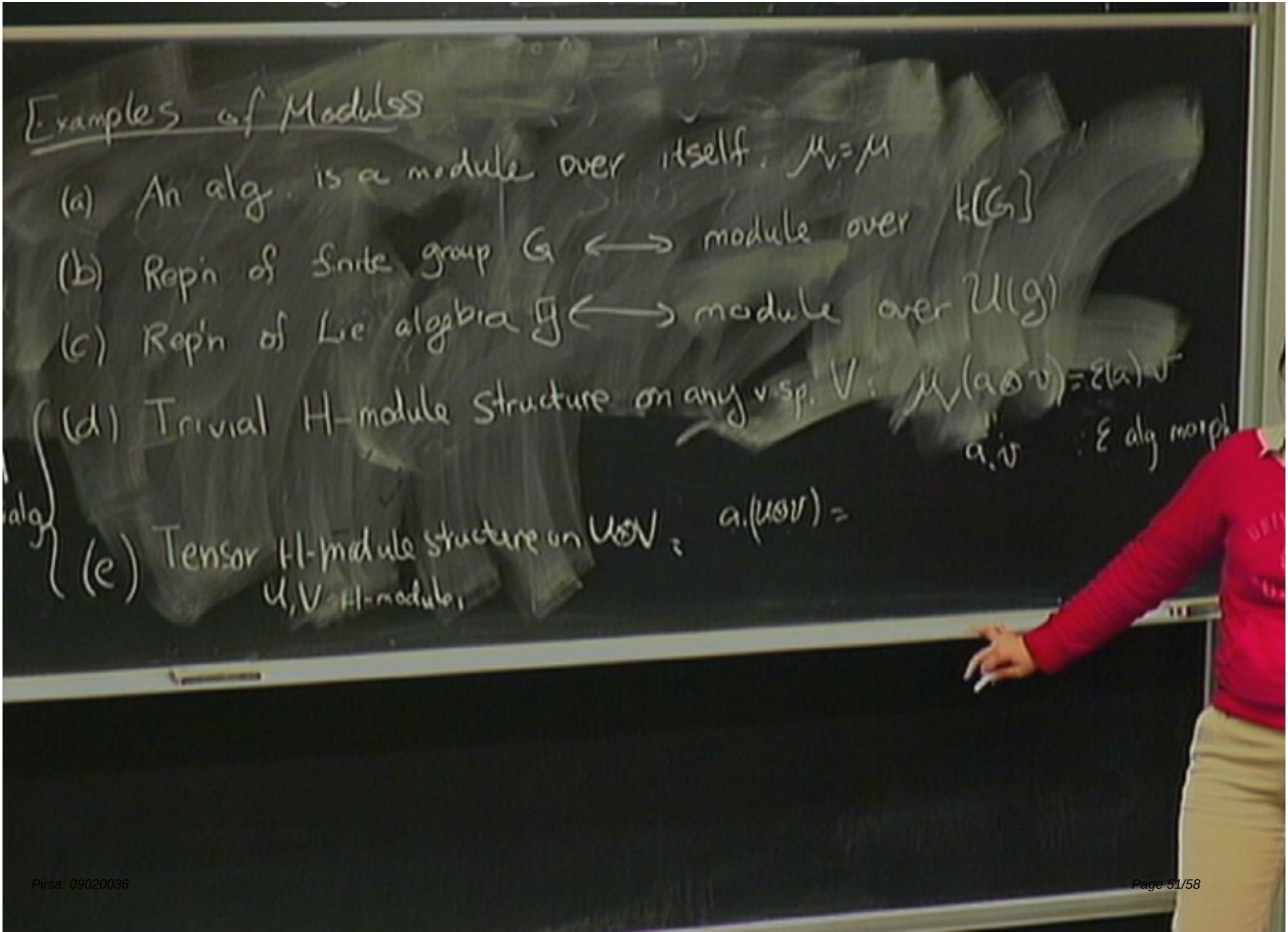
(b) Rep'n of finite group $G$ $\longleftrightarrow$ module over $k[G]$

(c) Rep'n of Lie algebra $\mathfrak{g}$ $\longleftrightarrow$ module over $U(\mathfrak{g})$

(d) Trivial $H$-module structure on any v. sp. $V$: $\mu_V(a \otimes v) = \varepsilon(a)v$

$a, v$ : $\varepsilon$ alg morph

(e)


Examples of Modules
(a) An alg. is a module over itself. $\mu_V = \mu$
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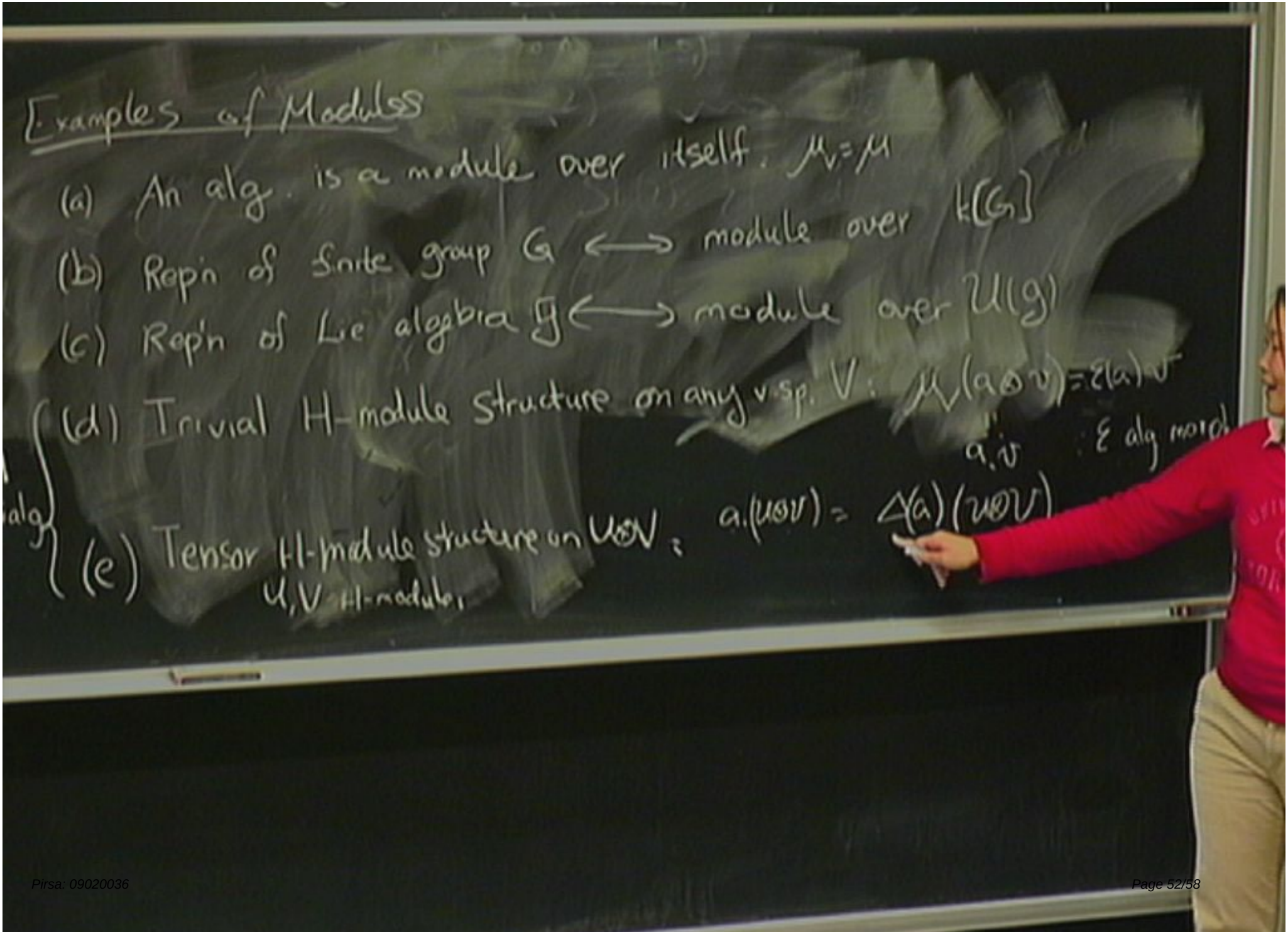
(d) Trivial $H$-module structure on any v. sp. $V$: $a \cdot v = \varepsilon(a)v$
$\varepsilon$ alg morph
(e) Tensor $H$-module structure on $U \otimes V$: $a \cdot (u \otimes v) =$
$U, V$ $H$-modules
alg


Examples of Modules
(a) An alg. is a module over itself. $\mu_V = \mu$
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(d) Trivial $H$-module structure on any v. sp. $V$: $a \cdot v = \varepsilon(a)v$ $\varepsilon$ alg morph
(e) Tensor $H$-module structure on $U \otimes V$: $a \cdot (u \otimes v) = \Delta(a)(u \otimes v)$
$U, V$ $H$-modules
alg

## Examples of Modules

(a) An alg. is a module over itself. $\mu_V = \mu$

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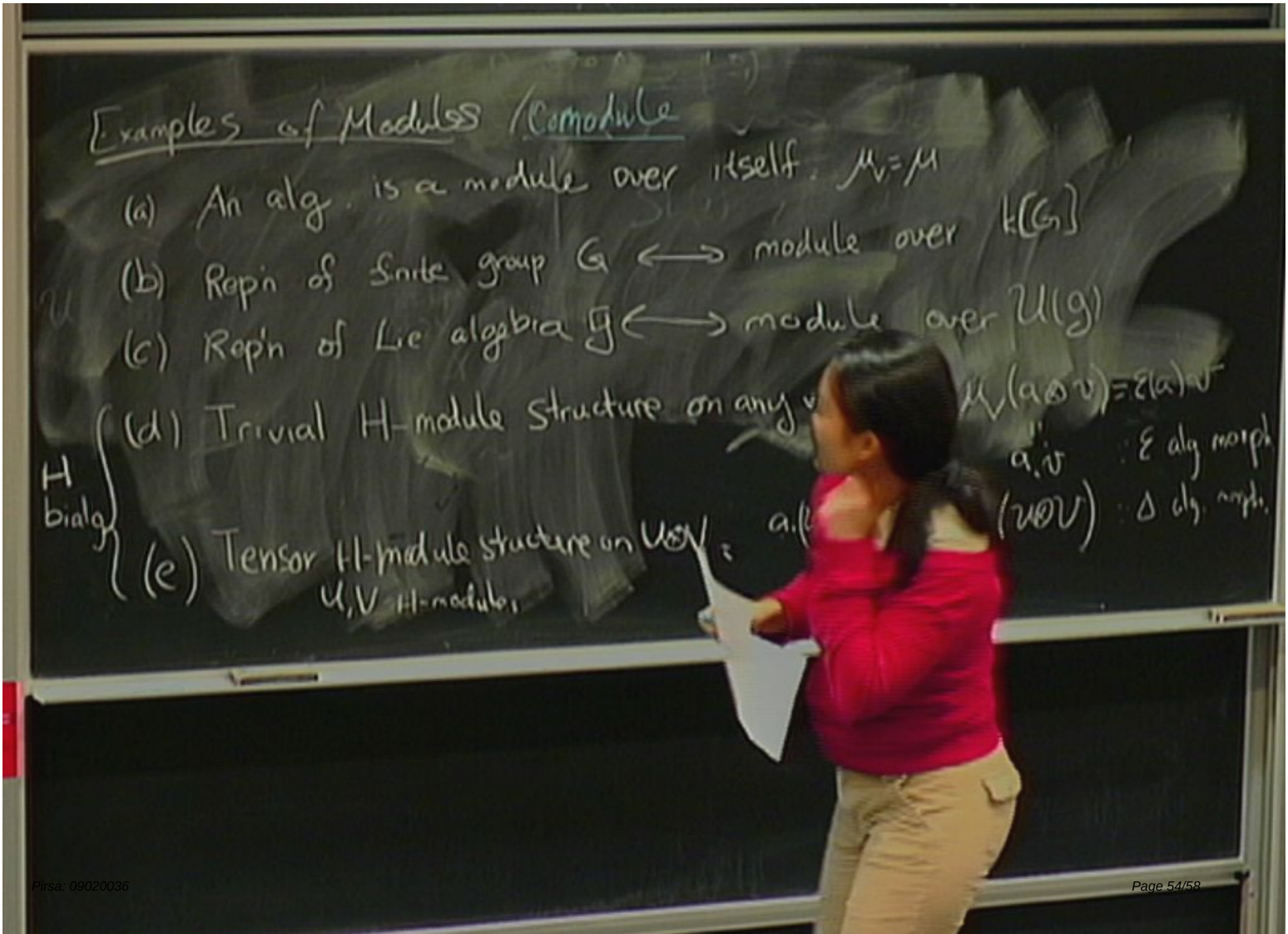
(c) Rep'n of Lie algebra $\mathfrak{g}$ $\longleftrightarrow$ module over $\mathcal{U}(\mathfrak{g})$

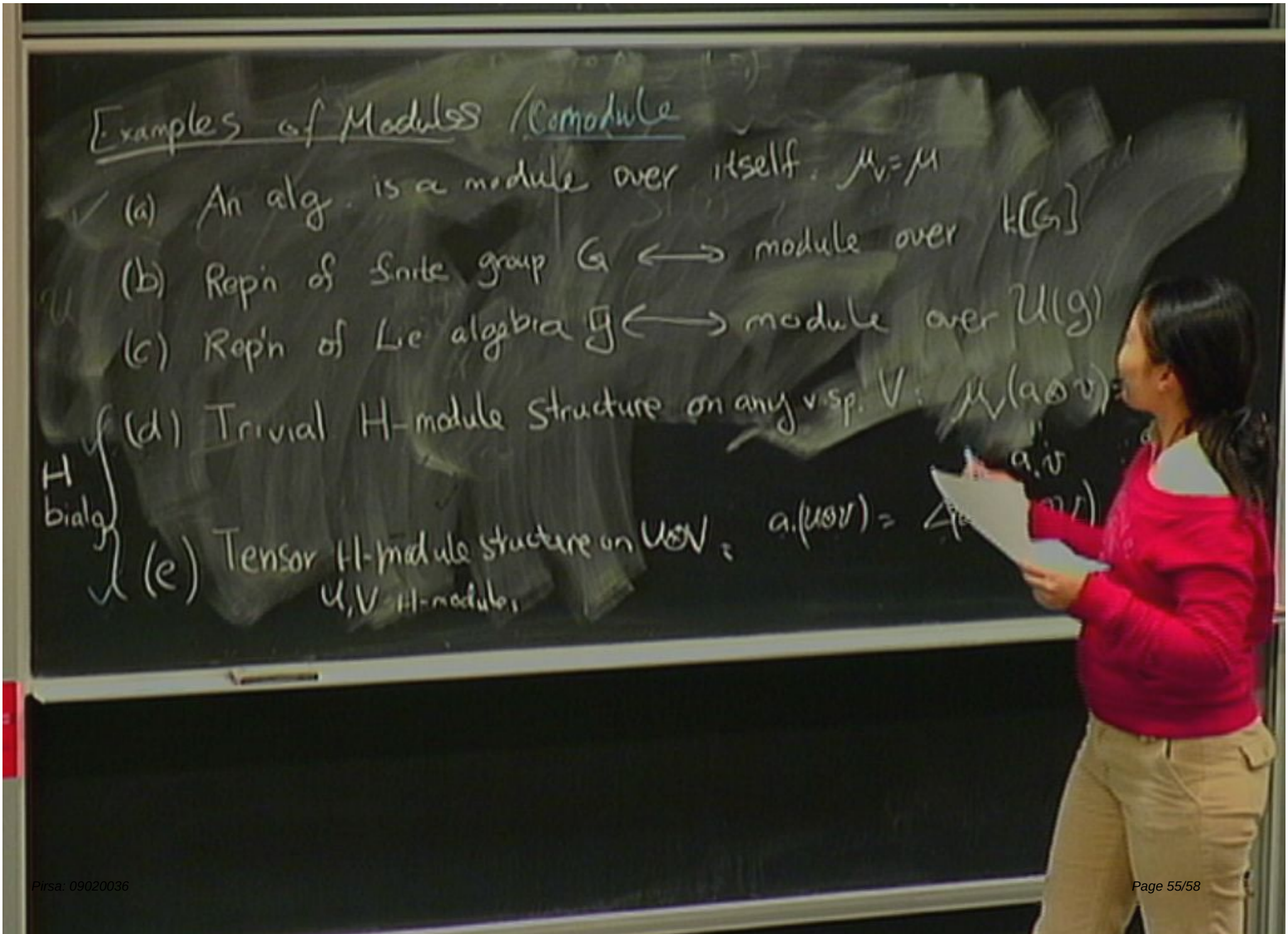
(d) Trivial $H$-module structure on any v. sp. $V$: $a.v = \varepsilon(a)v$ ($\varepsilon$ alg morph)

(e) Tensor $H$-module structure on $U \otimes V$: $a.(u \otimes v) = \Delta(a)(u \otimes v)$ ($\Delta$ alg. morph.)

alg

$U, V$ $H$-modules


Examples of Modules / Comodule
(a) An alg. is a module over itself. $\mu_V = \mu$
(b) Rep'n of finite group $G \longleftrightarrow$ module over $k[G]$
(c) Rep'n of Lie algebra $\mathfrak{g} \longleftrightarrow$ module over $\mathcal{U}(\mathfrak{g})$
H bialg
(d) Trivial H-module structure on any
$\mu_V(a \otimes v) = \varepsilon(a) \cdot v$
$a \cdot v$ : $\varepsilon$ alg morph
$(v \otimes V)$ : $\Delta$ alg. morph.
(e) Tensor H-module structure on $U \otimes V$ :
$U, V$ H-modules


Examples of Modules / Comodule
(a) An alg. is a module over itself. $\mu_V = \mu$
(b) Rep'n of finite group $G \longleftrightarrow$ module over $k[G]$
(c) Rep'n of Lie algebra $\mathfrak{g} \longleftrightarrow$ module over $U(\mathfrak{g})$
H bialg
(d) Trivial H-module structure on any v. sp. $V$: $a.v$
(e) Tensor H-module structure on $U \otimes V$: $a.(u \otimes v)$
$U, V$ H-modules

## Examples of Modules / Comodule

(a) An alg. is a module over itself. $\mu_V = \mu$ $\quad \Delta_V = \Delta$

(b) Rep'n of finite group $G \longleftrightarrow$ module over $k[G]$

(c) Rep'n of Lie algebra $\mathfrak{g} \longleftrightarrow$ module over $\mathcal{U}(\mathfrak{g})$

$H$ bialg

(d) Trivial $H$-module structure on any v. sp. $V$: $\mu_V(a \otimes v) = \varepsilon(a) \cdot v$

$a.v$ : $\varepsilon$ alg morph

(e) Tensor $H$-module structure on $U \otimes V$: $a.(u \otimes v) = \Delta(a)(u \otimes v)$ : $\Delta$ alg. morph.

$U, V$ $H$-modules

## Examples of Modules / Comodule

(a) An alg. is a module over itself. $\mu_V = \mu$ $\quad \Delta_V = \Delta$

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(c) Rep'n of Lie algebra $\mathfrak{g} \longleftrightarrow$ module over $U(\mathfrak{g})$

(d) Trivial $H$-module structure on any v. sp. $V$: $a \cdot v = \varepsilon(a)v$ ($\varepsilon$ alg m)

(e) Tensor $H$-module structure on $U \otimes V$: $a.(u \otimes v) = \Delta(a)(u \otimes v)$ ($\Delta$ alg.)

$U, V$ $H$-modules $\quad \Delta_{U \otimes V}(u \otimes v) =$

$H$ bialg

## Examples of Modules / Comodule

(a) An alg. is a module over itself. $\mu_V = \mu$ $\quad \Delta_V \; \Delta$

(b) Rep'n of finite group $G \longleftrightarrow$ module over $k[G]$

(c) Rep'n of Lie algebra $\mathfrak{g} \longleftrightarrow$ module over $U(\mathfrak{g})$

H bialg

(d) Trivial $H$-module structure on any v. sp. $V$: $\mu_V(a \otimes v) = \varepsilon(a)\, v$

$a.v$ : $\varepsilon$ alg morph

(e) Tensor $H$-module structure on $U \otimes V$ ($U, V$ $H$-modules): $a.(u \otimes v) = \Delta(a)(u \otimes v)$ : $\Delta$ alg. morph.

$\Delta_{U \otimes V}(u \otimes v) = (\mathrm{id} \otimes T_{U,H} \otimes \mathrm{id})(\Delta \otimes \Delta)(u \otimes v)$