

Title: Grad Talks 2

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URL: <http://pirsa.org/09020034>

Abstract: Lecture on Quantum Groups by Lucy Zhang

$$k[G] \xleftarrow{\quad} G \text{ finite group} \xrightarrow{\quad} (k[G])^*$$

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G algebraic group

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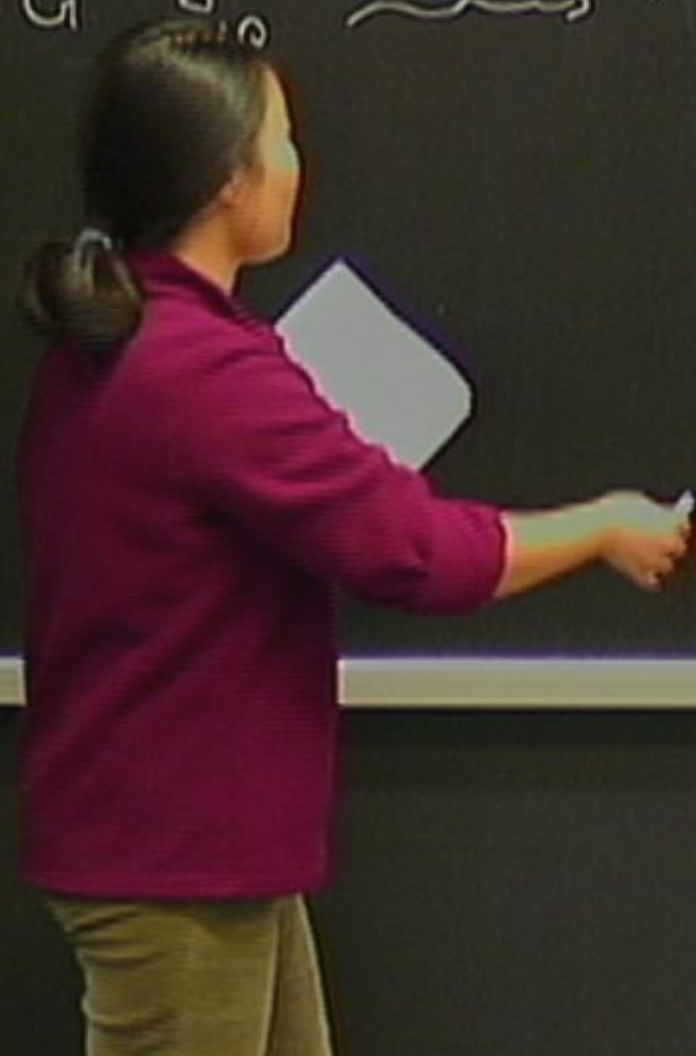
$$k[G] \leftarrow G \text{ finite group} \rightsquigarrow (k[G])^*$$

$$U(\mathfrak{g}) \leftarrow G \text{ algebraic group}$$

$k[G] \leftarrow G \text{ finite group} \rightarrow (k[G])^*$ all functions on

$U(\mathfrak{g}) \leftarrow G \text{ algebraic group} \rightarrow$ regular functions on G

$k[G] \leftarrow G \text{ finite group} \rightarrow (k[G])^*$ all functions on G
 $U(G) \leftarrow G \text{ algebraic} \rightarrow$ regular functions on G



Lie Algebra $\mathfrak{g}$

$\mathfrak{g}$ vector space

bilinear map $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$

$$[x, y] = -[y, x]$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

Lie Algebra $\mathfrak{g}$

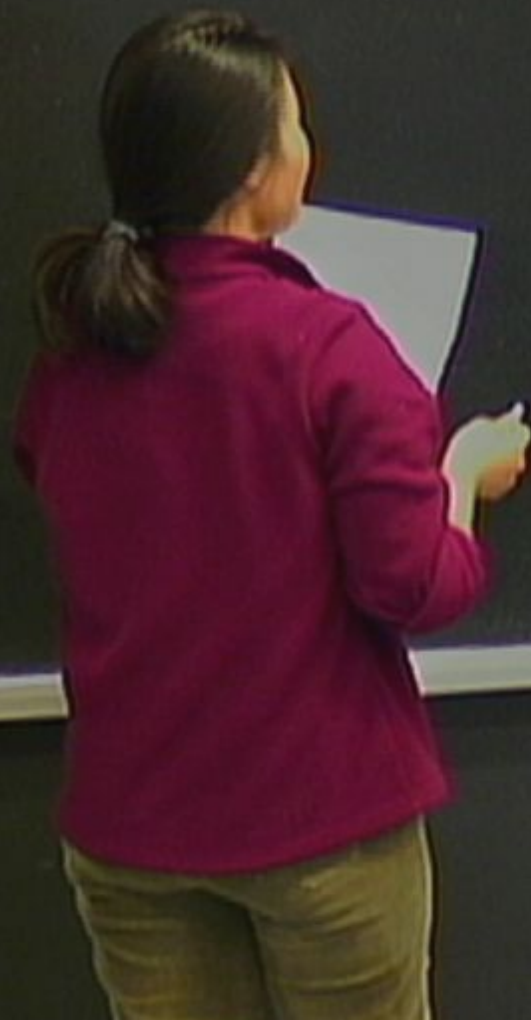
$\mathfrak{g}$ vector space

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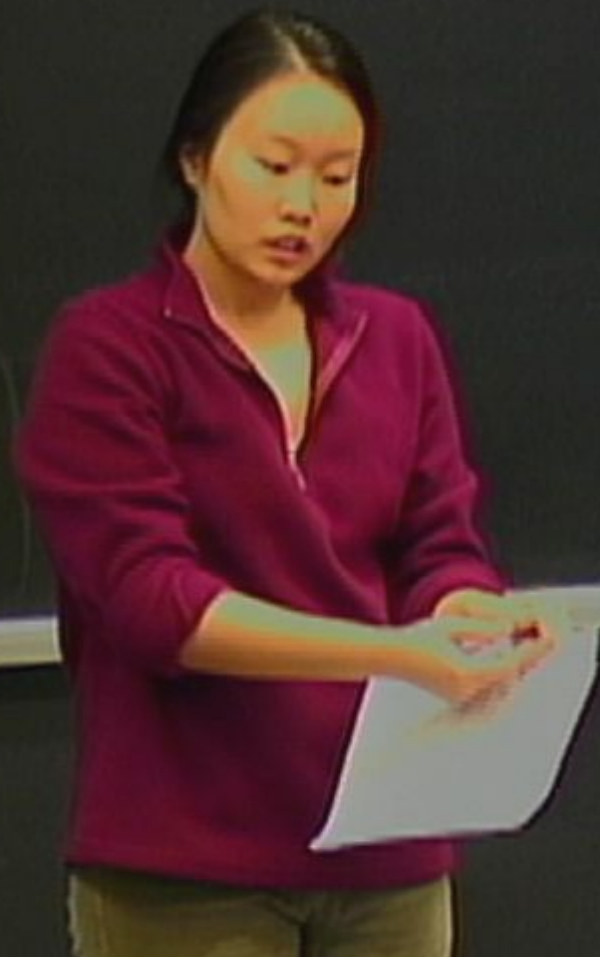
$$[x, y] = -[y, x]$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

Universal Enveloping Algebra of $\mathfrak{g}$
 $G \quad \mathfrak{g} \quad \text{construct} \quad U(\mathfrak{g}) = T(\mathfrak{g}) /$



Universal Enveloping Algebra of $\mathfrak{g}$
 $G \quad \mathfrak{g}$, construct $U(\mathfrak{g}) = T(\mathfrak{g}) / L(\mathfrak{g})$ $T(\mathfrak{g}) = \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \dots$



Universal Enveloping Algebra of $\mathfrak{g}$
 $G \quad \mathfrak{g}$, construct $U(\mathfrak{g}) = T(\mathfrak{g}) / I(\mathfrak{g})$

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}$$

$$I(\mathfrak{g}) = \left(x \otimes y - y \otimes x - \underbrace{[x, y]}_{\in \mathfrak{g}} \right)$$

Universal Enveloping Algebra of $\mathfrak{g}$

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$\rightarrow$ basis for $\mathfrak{g}$: $\{x_1, \dots, x_n\}$

Universal Enveloping Algebra of $\mathfrak{g}$
 $G \quad \mathfrak{g}$. construct $U(\mathfrak{g}) = T(\mathfrak{g}) / L(\mathfrak{g})$

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}$$

$$L(\mathfrak{g}) = \left(\begin{array}{c} x \otimes y - y \otimes x \\ - \underbrace{[x, y]}_{\in \mathfrak{g}} \end{array} \right)$$

→ basis for $\mathfrak{g}$: $\{x_1, \dots, x_n\}$
 basis for $T(\mathfrak{g})$:

Universal Enveloping Algebra of $\mathfrak{g}$
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$\rightarrow$ basis for $\mathfrak{g}$: $\{x_1, \dots, x_n\}$
 basis for $T(\mathfrak{g})$ words

Universal Enveloping Algebra of $\mathfrak{g}$
 $G \quad \mathfrak{g}$. construct $U(\mathfrak{g}) = T(\mathfrak{g}) / L(\mathfrak{g})$

Universal Property of $U(\mathfrak{g})$

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n} \quad \dots$$

$$L(\mathfrak{g}) = \left(\begin{array}{c} x \otimes y - y \otimes x \\ -[x, y] \end{array} \right)$$

$\rightarrow$ basis for $\mathfrak{g}$: $\{x_1, \dots, x_n\}$
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Universal Enveloping Algebra

$G = \mathfrak{g}$, construct $U(\mathfrak{g}) = T(\mathfrak{g}) / L(\mathfrak{g})$

Universal Property of $U(\mathfrak{g})$

$\{ \text{bra} \}$

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}$$
$$L(\mathfrak{g}) = \left(\begin{array}{c} x \otimes y - y \otimes x \\ -[x, y] \end{array} \right)$$

↘ basis for $\mathfrak{g}$: $\{x_1, \dots, x_n\}$
basis for words $T(\mathfrak{g})$

Universal Enveloping Algebra

Let $\mathfrak{g}$ be a Lie algebra. Construct $U(\mathfrak{g}) = T(\mathfrak{g}) / L(\mathfrak{g})$

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}$$

$$L(\mathfrak{g}) = \left(\begin{array}{c} x \otimes y - y \otimes x \\ -[x, y] \end{array} \right)$$

Universal Property of $U(\mathfrak{g})$

{Lie Algebra}

{Associative algebra}

$$\mathfrak{g} \longmapsto U(\mathfrak{g})$$

basis for $\mathfrak{g}$: $\{x_1, \dots, x_n\}$
 basis for $T(\mathfrak{g})$: words

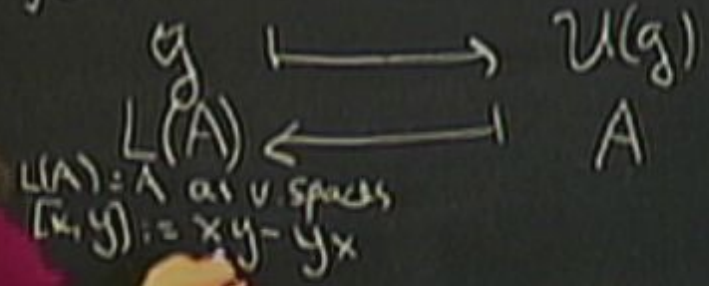
Given $\mathfrak{g}$, construct $U(\mathfrak{g}) = T(\mathfrak{g}) / L(\mathfrak{g})$

$$L(\mathfrak{g}) = \left(\begin{array}{c} x \otimes y - y \otimes x \\ -[x, y] \end{array} \right)$$

basis for $\mathfrak{g}$: $\{x_1, \dots, x_n\}$
 basis for $T(\mathfrak{g})$ words

Universal Property of $U(\mathfrak{g})$

Algebra $\{ \}$ $\{ \}$ Associative algebra



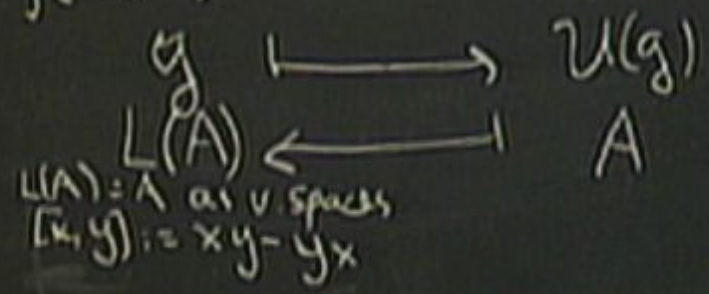
Given $\mathfrak{g}$, construct $U(\mathfrak{g}) = T(\mathfrak{g}) / L(\mathfrak{g})$

$$L(\mathfrak{g}) = \left(\begin{array}{c} x \otimes y - y \otimes x \\ -[x, y] \end{array} \right)$$

$L(\mathfrak{g}) = \mathfrak{g} \otimes \mathfrak{g} + \langle [x, y] \rangle$
 basis for $\mathfrak{g}$: $\{x_1, \dots, x_n\}$
 basis for $T(\mathfrak{g})$: words

Universal Property of $U(\mathfrak{g})$

$\{\text{Lie Algebra}\}$ $\longleftrightarrow$ $\{\text{Associative algebra}\}$



$\mathfrak{g}$, construct

$L(\mathfrak{g})$

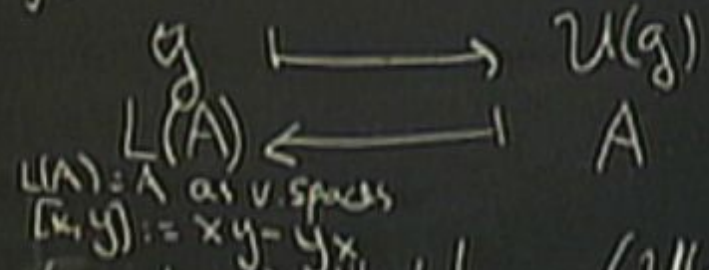
$$L(\mathfrak{g}) = \begin{pmatrix} x \otimes y - y \otimes x \\ -[x, y] \end{pmatrix}$$

Universal Property of $U(\mathfrak{g})$

{Lie Algebra}

{Associative algebra}

basis for $\mathfrak{g}$: $\{x_1, \dots, x_n\}$
 basis for $T(\mathfrak{g})$ words



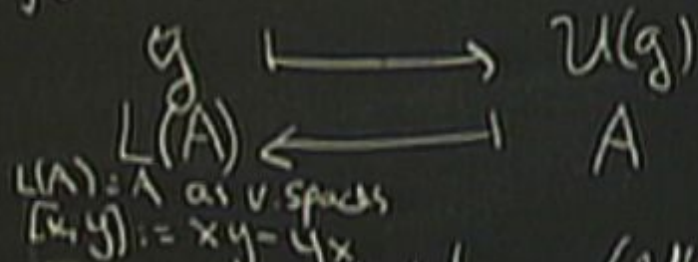
$$\text{Hom}_{\text{Lie}}(\mathfrak{g}, L(A)) \cong \text{Hom}_{\text{Alg.}}(U(\mathfrak{g}), A)$$

Universal Property of $U(\mathfrak{g})$

{Lie Algebra}

{Associative algebra}

basis for $\mathfrak{g}$: $\{x_1, \dots, x_n\}$
 basis for $T(\mathfrak{g})$ words



$$\text{Hom}_{\text{Lie}}(\mathfrak{g}, L(A)) \cong \text{Hom}_{\text{Alg.}}(U(\mathfrak{g}), A) \text{ for any algebra } A.$$

Universal Enveloping Algebra of $\mathfrak{g}$
 $\mathfrak{g} \rightarrow \mathfrak{g}$, construct $U(\mathfrak{g}) = T(\mathfrak{g}) / L(\mathfrak{g})$

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}$$

$$L(\mathfrak{g}) = \left(\begin{array}{c} x \otimes y - y \otimes x \\ -[x, y] \end{array} \right)$$

basis for $\mathfrak{g}$: $\{x_1, \dots, x_n\}$
 basis for $T(\mathfrak{g})$ words

Universal Property of $U(\mathfrak{g})$

{Lie Algebra}

{Associative algebra}

$$\begin{array}{c} \mathfrak{g} \hookrightarrow U(\mathfrak{g}) \\ L(A) \hookrightarrow A \\ L(A) := A \text{ as } V \\ [x, y] := x \cdot y - y \cdot x \end{array}$$

$$\text{Hom}_{\text{Lie}}(\mathfrak{g}, L(A))$$

A

$(\mathfrak{g}, A)$ for any algebra A

Universal Enveloping Algebra of $\mathfrak{g}$
 $\mathfrak{g} \rightarrow \mathfrak{g}$, construct $U(\mathfrak{g}) = T(\mathfrak{g}) / L(\mathfrak{g})$

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}$$

$$L(\mathfrak{g}) = \left(\begin{array}{c} x \otimes y - y \otimes x \\ -[x, y] \end{array} \right)$$

$\left. \begin{array}{l} \text{basis for } \mathfrak{g}: \{x_1, \dots, x_n\} \\ \text{basis for } T(\mathfrak{g}) \text{ words} \end{array} \right\}$

Universal Property of $U(\mathfrak{g})$

$\{\text{Lie Algebra}\}$

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\quad} & U(\mathfrak{g}) \\ L(A) & \xleftarrow{\quad} & U(\mathfrak{g}) \end{array}$$

$L(A) = A$ as v. spaces
 $[x, y] := xy - yx$

$$\text{Hom}_{\text{Lie}}(\mathfrak{g}, L(A)) \cong \text{Hom}_{\text{alg}}(U(\mathfrak{g}), A)$$

for any algebra A

Universal Enveloping Algebra of $\mathfrak{g}$

$\mathfrak{g}$, construct $U(\mathfrak{g}) = T(\mathfrak{g}) / L(\mathfrak{g})$

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}$$

$$L(\mathfrak{g}) = \left(\begin{array}{c} x \otimes y - y \otimes x \\ -[x, y] \end{array} \right)$$

Universal Property of $U(\mathfrak{g})$

{Lie Algebra}

{Associative algebra}

$$\mathfrak{g} \longrightarrow U(\mathfrak{g})$$

$$L(A) \longleftarrow A$$

$$L(A) = A \text{ as v. spaces}$$

$$[x, y] := xy - yx$$

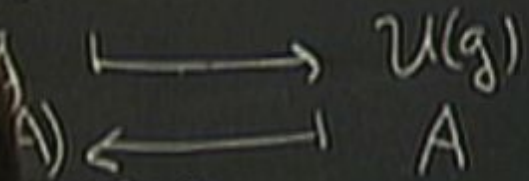
$$\text{Hom}_{\text{Lie}}(\mathfrak{g}, L(A)) \stackrel{\text{is}}{=} \text{Hom}_{\text{Alg}}(U(\mathfrak{g}), A) \text{ for any algebra } A$$

basis for $\mathfrak{g}$: $\{x_1, \dots, x_n\}$
basis for $T(\mathfrak{g})$ words

$G \cong \mathfrak{g}$, construct $U(\mathfrak{g})$
 $i_L: \mathfrak{g} \hookrightarrow L(U(\mathfrak{g})) / L(\mathfrak{g})$ $\left(L(\mathfrak{g}) = \begin{pmatrix} x \otimes y - y \otimes x \\ -[x, y] \end{pmatrix} \right)$

Universal Property of $U(\mathfrak{g})$

$\{\text{Lie Algebra}\} \xrightarrow{\quad} \{\text{Associative algebra}\}$
 basis for $\mathfrak{g}$: $\{x_1, \dots, x_n\}$
 basis for $T(\mathfrak{g})$ words

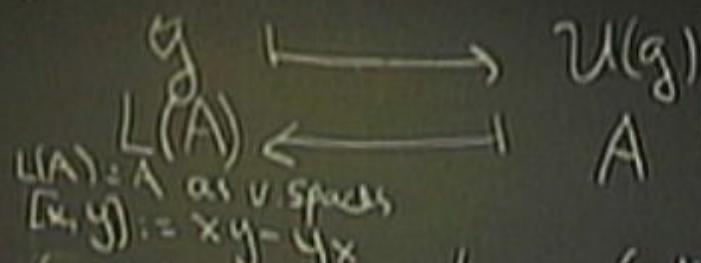


$H(A) \stackrel{\text{def}}{=} \text{Hom}_{\text{Alg.}}(U(\mathfrak{g}), A)$ for any algebra A

{Lie Algebra}

{Associative algebra}

basis for $T(g)$ words $\{x_1, \dots, x_n\}$



$$\text{Hom}_{\text{Lie}}(g, L(A)) \cong \text{Hom}_{\text{Alg.}}(U(g), A) \text{ for any algebra } A$$

Hopf Algebra structure on $U(g)$

algebra (2)

coalgebra

Hopf Algebra structure on $U(g)$

algebra ②

coalgebra

Hopf Algebra structure on $U(g)$

algebra

coalgebra

②

(u, v Lie algebra reps. then
action of $\mathfrak{g}$ on $U \otimes V$: $\chi^g(u \otimes v) = u \otimes \chi.v + \chi.u \otimes v$)

Hopf Algebra structure on $U(g)$

algebra

coalgebra

②

(u, v) Lie algebra reps.

then

action of $\mathfrak{g}$ on $U \otimes V$:

$$x(u \otimes v) = u \otimes xv + x.u \otimes v$$

$$\Delta(x) = 1 \otimes x + x \otimes 1$$

Hopf Algebra structure on $U(g)$

algebra

coalgebra

②

$(u, v \in \text{algebra reps.})$

action of $\mathfrak{g}$ on $U \otimes V$:

then

$$x(u \otimes v) = u \otimes x.v + x.u \otimes v$$

$$\Delta(x) = 1 \otimes x + x \otimes 1$$

Hopf Algebra structure on $U(g)$

algebra

coalgebra

②

$(u, v \in \text{algebra reps.})$

action of $\mathfrak{g}$ on $U \otimes V$:

then

$$x^a(u \otimes v) = u \otimes x^a v + x^a u \otimes v$$

$$\Delta(x) = 1 \otimes x + x \otimes 1$$

Hopf Algebra structure on $U(g)$

algebra

②

coalgebra

$(u, v$

$u, v \in$ algebra reps,

then

action of g on $u \otimes v$:

$$x(u \otimes v) = u \otimes xv + x.u \otimes v$$

$$\Delta(x) = 1 \otimes x + x \otimes 1$$

$$\Delta(1)$$

Hopf Algebra structure on $U(g)$

algebra

②

Coalgebra

$(u, v$

$\in$ algebra reps.,

then

action of g on $U \otimes V$:

$$x^{(g)}(u \otimes v) = u \otimes x.v + x.u \otimes v$$

$$x \in g, \Delta(x) = 1 \otimes x + x \otimes 1$$

$$x_1, \dots, x_n \in g, \Delta(x_1, \dots, x_n)$$

Hopf Algebra structure on $U(g)$

algebra

②

Coalgebra

(u, v

$\in$ algebra reps.,

then

action of g on $U \otimes V$:

$$\chi^g(u \otimes v) = u \otimes \chi v + \chi u \otimes v$$

$$\chi \in g, \Delta(\chi) = 1 \otimes \chi + \chi \otimes 1$$

$$\chi_1, \dots, \chi_n \in g, \Delta(\chi_1 \dots \chi_n)$$

Hopf Algebra structure on $U(g)$

algebra

②

Coalgebra

(u, v

u, v are algebra reps., then

action of g on $u \otimes v$:

then

$$x^g(u \otimes v) = u \otimes x.v + x.u \otimes v$$

$$x \in g, \Delta(x) = 1 \otimes x + x \otimes 1$$

$$\Delta(x_i) \Delta(x_j) = \Delta(x_i x_j)$$

$$x_1, \dots, x_n \in g, \Delta(x_1 \dots x_n) = 1 \otimes x_1 \dots x_n + \sum_{i=1}^n (x_1 \dots x_{i-1} \otimes x_i \dots x_n) + x_1 \dots x_n \otimes 1$$

$$+ x_1 \dots x_n \otimes 1$$

Hopf Algebra structure on $U(g)$

algebra

②

Coalgebra

(

u, v are algebra reps.

then

action of g on $U \otimes V$:

$$x^{(g)}(u \otimes v) = u \otimes x.v + x.u \otimes v$$

$$x \in g, \Delta(x) = 1 \otimes x + x \otimes 1$$

$$x_1, \dots, x_n \in g, \Delta(x_1 \dots x_n) = 1 \otimes x_1 \dots x_n + \sum_{i=1}^n (x_1 \dots x_{i-1} \otimes x_i \dots x_n)$$

$$(x_i \Delta x_j) = \Delta(x_i, x_j)$$

$$(x_i \otimes x_j) \otimes x_k = x_i \otimes (x_j \otimes x_k)$$

Hopf Algebra structure on $U(g)$

algebra

coalgebra

②

(u, v)

u, v are algebra reps., then
action of g on $u \otimes v$:

then

$$x(u \otimes v) = u \otimes xv + x.u \otimes v$$

$x \in g$

Δ

$$\Delta(x) = 1 \otimes x + x \otimes 1$$

$$\Delta(x_i \Delta x_j) = \Delta(x_i, x_j)$$

$x_1, \dots, x_n \in g$

Δ

$$\begin{aligned} & 1 \otimes x_1 \dots x_n + \sum_{p=1}^n \sum_{\sigma} x_{\sigma(1)} \dots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \dots x_n \\ & + x_1 \dots x_n \otimes 1 \end{aligned}$$

(p, q) shuffles in S_n

Hopf Algebra structure on $U(g)$

algebra

②

Coalgebra

(

u, v are algebra reps.

then

action of g on $U \otimes V$:

$$x^g(u \otimes v) = u \otimes x^g v + x^g u \otimes v$$

$$x \in g, \Delta(x) = 1 \otimes x + x \otimes 1$$

$$\Delta(x_i \Delta x_j) = \Delta(x_i, x_j)$$

$$x_1, \dots, x_n \in g, \Delta(x_1, \dots, x_n) = 1 \otimes x_1 \dots x_n + \sum_{p=1}^n \sum_{\sigma} x_{\sigma(1)} \dots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \dots x_n$$

$(\underbrace{\quad}_{U(g) \otimes U(g)})$ (p.g.) shuffles in S_n

algebra
C-algebra

(U, V Lie algebra reps. then
 action of $\mathfrak{g}$ on $U \otimes V$: $x^{\mathfrak{g}}(u \otimes v) = u \otimes x.v + x.u \otimes v$

$x \in \mathfrak{g}$, $\Delta(x) = 1 \otimes x + x \otimes 1$ $\Delta(x_i) \Delta(x_j) = \Delta(x_i x_j)$
 $x_1, \dots, x_n \in \mathfrak{g}$, $\Delta(x_1, \dots, x_n) = 1 \otimes x_1 \dots x_n + \sum_{p=1}^n \sum_{\sigma} x_{\sigma(1)} \dots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \dots x_n$
 $\underbrace{\quad}_{U(\mathfrak{g}) \otimes U(\mathfrak{g})} \quad + x_1 \dots x_n \otimes 1$ (p,q) shuffles in S_n

antipode

10/10/20

Coalgebra

(U, V Lie algebra reps, then
action of $\mathfrak{g}$ on $U \otimes V$: $x(u \otimes v) = u \otimes xv + x.u \otimes v$)

$$\begin{aligned} x \in \mathfrak{g}, \quad \Delta(x) &= 1 \otimes x + x \otimes 1 & \Delta(x_i \Delta x_j) &= \Delta(x_i, x_j) \\ x_1, \dots, x_n \in \mathfrak{g}, \quad \Delta(x_1 \dots x_n) &= 1 \otimes x_1 \dots x_n + \sum_{p=1}^n \sum_{\sigma \in S_n} x_{\sigma(1)} \dots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \dots x_n \\ &\quad + x_1 \dots x_n \otimes 1 \quad \text{(p.g.) shuffles in } S_n \\ &\quad \underbrace{\qquad\qquad\qquad}_{U(\mathfrak{g}) \otimes U(\mathfrak{g})} \end{aligned}$$

antipode

$$S(x) = -x$$

coalgebra

(u, v Lie algebra reps, then
action of $\mathfrak{g}$ on $u \otimes v$: $x(u \otimes v) = u \otimes x.v + x.u \otimes v$)

$x \in \mathfrak{g}$, $\Delta(x) = 1 \otimes x + x \otimes 1$ $\Delta(x_i) \Delta(x_j) = \Delta(x_i x_j)$

$x_1, \dots, x_n \in \mathfrak{g}$, $\Delta(x_1 \dots x_n) = 1 \otimes x_1 \dots x_n + \sum_{p=1}^n \sum_{\sigma \in S_n} x_{\sigma(1)} \dots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \dots x_n + x_1 \dots x_n \otimes 1$ (p.q) shuffles in S_n

antipode

$S(-)$



algebra

Coalgebra

(U, V Lie algebra reps, then
action of $\mathfrak{g}$ on $U \otimes V$: $x^{\mathfrak{g}}(u \otimes v) = u \otimes x.v + x.u \otimes v$)

$x \in \mathfrak{g}$, $\Delta(x) = 1 \otimes x + x \otimes 1$ $\Delta(x_i) \Delta(x_j) = \Delta(x_i x_j)$

$x_1, \dots, x_n \in \mathfrak{g}$, $\Delta(x_1, \dots, x_n) = 1 \otimes x_1 \dots x_n + \sum_{p=1}^n \sum_{\sigma \in S_n} x_{\sigma(1)} \dots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \dots x_n + x_1 \dots x_n \otimes 1$ (p.q) shuffles in S_n

antipode

$S(-)$



algebra

Coalgebra

(U, V Lie algebra reps, then
action of $\mathfrak{g}$ on $U \otimes V$: $x^{\mathfrak{g}}(u \otimes v) = u \otimes x.v + x.u \otimes v$)

$$x \in \mathfrak{g}, \Delta(x) = 1 \otimes x + x \otimes 1$$

$$x_1, \dots, x_n \in \mathfrak{g}, \Delta(x_1, \dots, x_n) = \underbrace{1 \otimes x_1 \dots x_n}_{p=0} + \sum_{p=1}^n \sum_{\sigma} x_{\sigma(1)} \dots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \dots x_n$$

(p shuffles in S_n)

antipode

$$S(x) = -x$$

algebra

Coalgebra

(u, v Lie algebra reps, then
action of $\mathfrak{g}$ on $u \otimes v$: $x(u \otimes v) = u \otimes xv + x.u \otimes v$)

$$x \in \mathfrak{g}, \quad \Delta(x) = 1 \otimes x + x \otimes 1$$
$$x_1, \dots, x_n \in \mathfrak{g}, \quad \Delta(x_1, \dots, x_n) = \underbrace{1 \otimes x_1 \dots x_n}_{\substack{n \\ U(\mathfrak{g}) \otimes U(\mathfrak{g})}} + \sum_{p=1}^n \sum_{\sigma} x_{\sigma(1)} \dots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \dots x_n$$

$\Delta(x_i) \Delta(x_j) = \Delta(x_i x_j)$
(p.q) shuffles in S_n

antipode

$$S(x) = -x$$

algebra

C' algebra

(u, v Lie algebra reps, then

action of $\mathfrak{g}$ on $u \otimes v$:

$$x^g(u \otimes v) = u \otimes x.v + x.u \otimes v$$

$$x \in \mathfrak{g}, \Delta(x) = 1 \otimes x + x \otimes 1$$

$$\Delta(x_i) \Delta(x_j) = \Delta(x_i x_j)$$

$$x_1, \dots, x_n \in \mathfrak{g}, \Delta(x_1 \dots x_n) = \underbrace{1 \otimes x_1 \dots x_n}_{n=0} + \sum_{p=1}^n \sum_{\sigma} x_{\sigma(1)} \dots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \dots x_n$$

(p,q) shuffles in S_n

antipode
S

$$S(x) = -x$$

$$S(x_1 \dots x_n) = (-1)^n x_1 \dots x_n$$

10/10/20

Coalgebra

(U, V Lie algebra reps, then
action of $\mathfrak{g}$ on $U \otimes V$: $x^{\mathfrak{g}}(u \otimes v) = u \otimes x.v + x.u \otimes v$)

$$x \in \mathfrak{g}, \quad \Delta(x) = 1 \otimes x + x \otimes 1 \quad \Delta(x_i) \Delta(x_j) = \Delta(x_i x_j)$$
$$x_1, \dots, x_n \in \mathfrak{g}, \quad \underbrace{\Delta(x_1 \dots x_n)}_{U(\mathfrak{g}) \otimes U(\mathfrak{g})} = \underbrace{1 \otimes x_1 \dots x_n}_{n=0} + \sum_{p=1}^n \sum_{\sigma} x_{\sigma(1)} \dots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \dots x_n$$

(p.q) shuffles in S_n

antipode

S-algebra antihomomorphism

$$S(x) = -x$$
$$S(x_1 \dots x_n) = (-1)^n x_1 \dots x_n$$

Hopf Algebra structure on $U(g)$

algebra

⊗

coalgebra

(

u, v Lie algebra reps.

then

action of g on $U \otimes V$:

$$x^g(u \otimes v) = u \otimes x.v + x.u \otimes v$$

$$x \in g, \Delta(x) = 1 \otimes x + x \otimes 1$$

$$\Delta(x_i \Delta x_j) = \Delta(x_i, x_j)$$

$$x_1, \dots, x_n \in g, \Delta(x_1, \dots, x_n) = \underbrace{1 \otimes x_1 \dots x_n}_{\substack{n \\ U(g) \otimes U(g)}} + \sum_{p=1}^n \sum_{\sigma} x_{\sigma(1)} \dots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \dots x_n$$

(p, q) shuffles in S_n

antipode

$$S(x) = -x$$

$$S(x_1 \dots x_n) = (-1)^n x_1 \dots x_n$$

Hopf Algebra structure on $U(g)$

algebra

②

C' algebra

(u, v) Lie algebra reps.

then

action of g on $U \otimes V$

$$x^g(u \otimes v) = u \otimes x.v + x.u \otimes v$$

$$x \in g, \Delta(x) = 1 \otimes x + x \otimes 1$$

$$\Delta(x_i \Delta x_j) = \Delta(x_i, x_j)$$

$$x_1, \dots, x_n \in g, \Delta(x_1, \dots, x_n) = \underbrace{1 \otimes x_1 \dots x_n}_{U(g) \otimes U(g)} + \sum_{p=1}^n \sum_{\sigma} x_{\sigma(1)} \dots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \dots x_n$$

(p, q) shuffles in S_n

$$S(x) = -x$$

$$S(x_1 \dots x_n) = (-1)^n x_1 \dots x_n$$

$$\left(\begin{matrix} \text{In } U(g) \\ S(g) = g^{-1}, g \in G \end{matrix} \right)$$

Hopf Algebra structure on $U(g)$

algebra

②

C' algebra

(u, v Lie algebra reps., then

action of $U(g)$ on $U \otimes V$: $x^g(u \otimes v) = u \otimes x.v + x.u \otimes v$

$$x \in g, \Delta(x) = 1 \otimes x + x \otimes 1$$

$$\Delta(x_i \Delta x_j) = \Delta(x_i, x_j)$$

$$x_1, \dots, x_n \in g, \Delta(x_1, \dots, x_n) = \underbrace{1 \otimes x_1 \dots x_n}_{U(g) \otimes U(g)} + \sum_{p=1}^n \sum_{\sigma} x_{\sigma(1)} \dots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \dots x_n$$

$+ x_1 \dots x_n \otimes 1$ (p, q) shuffles in S_n

ant

$$S(x) = -x$$

$$S(x_1 \dots x_n) = (-1)^n x_n \dots x_1$$

$$\left(\begin{matrix} \text{In } U(g) \\ S(g) = g^{-1}, g \in G \end{matrix} \right)$$

Hopf Algebra structure on $U(\mathfrak{g})$

algebra

⊗

coalgebra

(u, v Lie algebra reps., then

action of $\mathfrak{g}$ on $U \otimes V$:

$$x^{\mathfrak{g}}(u \otimes v) = u \otimes x.v + x.u \otimes v$$

$$x \in \mathfrak{g}, \Delta(x) = 1 \otimes x + x \otimes 1$$

$$\Delta(x_1 \Delta x_2) = \Delta(x_1, x_2)$$

$$x_1, \dots, x_n \in \mathfrak{g}, \Delta(x_1, \dots, x_n) = \underbrace{1 \otimes x_1 \dots x_n}_{\substack{\in U(\mathfrak{g}) \otimes U(\mathfrak{g}) \\ n=0}} + \sum_{p=1}^n \sum_{\sigma} x_{\sigma(1)} \dots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \dots x_n$$

$+ x_1 \dots x_n \otimes 1$ $\xrightarrow{(p,q) \text{ shuffles in } S_n}$

antipode

$$S(x) = -x$$

$$S(x_1 \dots x_n) = (-1)^n x_1 \dots x_n$$

$$\left(\begin{matrix} \text{In } U(\mathfrak{g}) \\ S(g) = g^{-1}, g \in G \end{matrix} \right)$$

$[x, y] := xy - yx$
 $L(A) := A$ as v. spaces

$$\text{Hom}_{\text{Lie}}(\mathfrak{g}, L(A)) \cong \text{Hom}_{\text{Alg}}(U(\mathfrak{g}), A) \text{ for any algebra } A$$

Hopf Algebra structure on $U(\mathfrak{g})$

algebra

coalgebra

⊗

u, v Lie algebra reps.

then

action of $\mathfrak{g}$ on $U \otimes V$:

$$x(u \otimes v) = u \otimes x.v + x.u \otimes v$$

$$x \in \mathfrak{g}, \Delta(x) = 1 \otimes x + x \otimes 1$$

$$\Delta(x_i) \Delta(x_j) = \Delta(x_i x_j)$$

$$x_1, \dots, x_n \in \mathfrak{g}, \Delta(x_1, \dots, x_n) = \underbrace{1 \otimes x_1 \dots x_n}_{n=0} + \sum_{p=1}^n \sum_{\sigma \in S_n} x_{\sigma(1)} \dots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \dots x_n$$

$U(\mathfrak{g}) \otimes U(\mathfrak{g})$

$+ x_1 \dots x_n \otimes 1$

$(p, q) \text{ shuffles in } S_n$

antipode

$$S(x) = -x$$

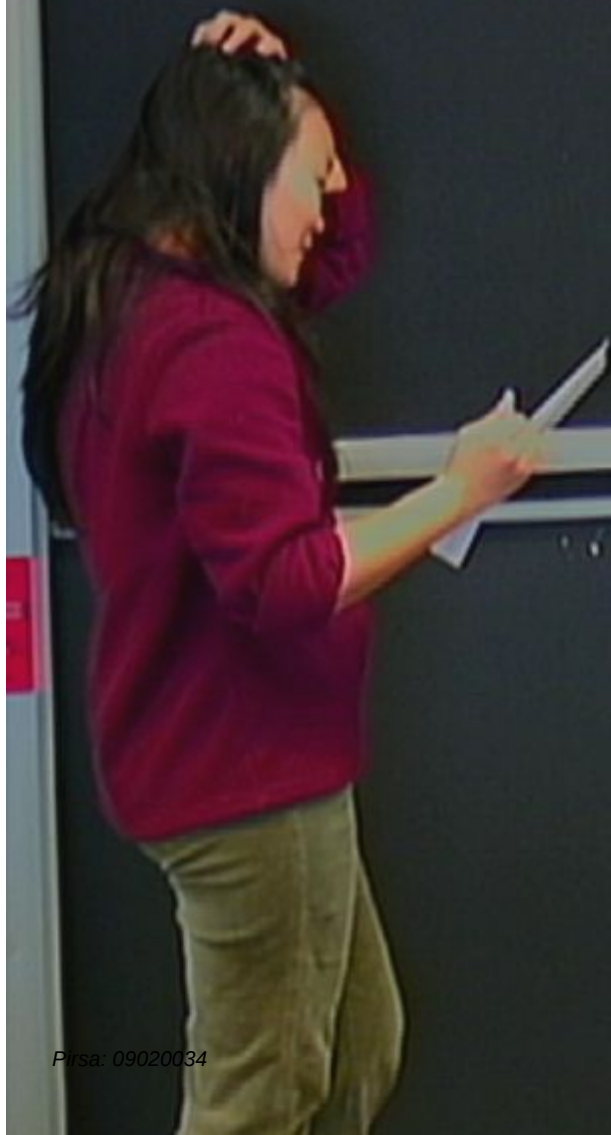
$$S(x_1 \dots x_n) = (-1)^n x_1 \dots x_n$$

S is an algebra antihomomorphism

$$\left(\begin{matrix} 1_n \\ S(g) = g^{-1} \end{matrix}, x \in \mathfrak{g} \right)$$

Definition 1.1

$$G = \{$$



Definition 1

$$G \stackrel{\text{def}}{=} \{ (x_1, \dots, x_n) \mid P \}$$

$$G \stackrel{\text{def}}{=} \{ (x_1, \dots, x_n) \mid p(x_1, \dots, x_n) = 0 \}$$

$$G \stackrel{\text{def}}{=} \{ (x_1, \dots, x_n) \mid p_j(x_1, \dots, x_n) = 0 \}$$

$$G \stackrel{\text{def}}{=} \{ (x_1, \dots, x_n) \mid p_j(x_1, \dots, x_n) = 0 \}$$

Regular functions $\mathcal{O}_G$

$$\mathcal{O}_G = \{$$

$$G \stackrel{\text{def}}{=} \{ (x_1, \dots, x_n) \mid p_j(x_1, \dots, x_n) = 0 \}$$

Regular functions $\mathcal{O}_G \xrightarrow{k}$
 $\mathcal{O}_G = \{ \text{polynomials in } x_1, \dots, x_n \mid p(x_1, \dots, x_n) = 0 \}$

Regular functions $\mathcal{O}_G \xrightarrow{k}$
 $\mathcal{O}_G = \{ \text{polynomials in } x_1, \dots, x_n \mid p_i(x_1, \dots, x_n) = 0 \}$
Ex. $G = SL(2) = \{ (a, b, c, d) \mid ad - bc - 1 = 0 \}$

Regular functions $\mathcal{O}_G \xrightarrow{k}$

$$\mathcal{O}_G = \{ \text{polynomials in } x_1, \dots, x_n \mid p_i(x_1, \dots, x_n) = 0 \}$$

Ex. 1 $G = SL(2) = \{ (a, b, c, d) \mid ad - bc - 1 = 0 \}$

$$\mathcal{O}_G = \{ \text{polynomials in } a, b, c, d \}$$

Regular functions $\mathcal{O}_G \xrightarrow{\quad} k$
 $\mathcal{O}_G = \{ \text{polynomials in } x_1, \dots, x_n \mid p_i(x_1, \dots, x_n) = 0 \}$

Ex. $G = \text{SL}(2) = \{ (a, b, c, d) \mid ad - bc - 1 = 0 \}$

$\mathcal{O}_G = \{ \text{polynomials in } a, b, c, d \} / (ad - bc - 1)$
 $\{ a, b, c, d \} /$

$$G = \{ (x_1, \dots, x_n) \mid \dots \}$$

Regular functions $\mathcal{O}_G \xrightarrow{\quad} k$

$\mathcal{O}_G = \{ \text{polynomials in } x_1, \dots, x_n \mid p_i(x_1, \dots, x_n) = 0 \}$

Eg. let $G = SL(2) = \{ (a, b, c, d) \mid ad - bc - 1 = 0 \}$

$\mathcal{O}_G = \{ \text{polynomials in } a, b, c, d \} / (ad - bc - 1)$
 $= k[a, b, c, d] / (ad - bc - 1)$

$$G \stackrel{\text{def}}{=} \{ (x_1, \dots, x_n) \mid p_j(x_1, \dots, x_n) = 0 \}$$

Regular functions $\mathcal{O}_G \xrightarrow{k} k$
 $\mathcal{O}_G = \{ \text{polynomials in } x_1, \dots, x_n \} / (p_j(x_1, \dots, x_n) = 0)$

Ex. 1st $G = SL(2) = \{ (a, b, c, d) \mid ad - bc - 1 = 0 \}$

$$\begin{aligned} \mathcal{O}_G &:= \{ \text{polynomials in } a, b, c, d \} / (ad - bc - 1) \\ &= k[a, b, c, d] / (ad - bc - 1) \end{aligned}$$

Hopf Algebra structure on $U(g)$

algebra $\otimes$, $\eta: k \rightarrow U(g)$
 $\eta(1) \mapsto 1$

coalgebra $(u, v \text{ Lie algebra reps.})$ then
 action of $U(g)$ on $U \otimes V$: $x^{(q)}(u \otimes v) = u \otimes x.v + x.u \otimes v$

$x \in g$
 $x_1, \dots, x_n \in g$
 $E(x) = 0$
 $E(1) = 1$

$$\Delta(x) = 1 \otimes x + x \otimes 1$$

$$\Delta(x_1, \dots, x_n) = \underbrace{1 \otimes x_1 \dots x_n}_{p=0} + \sum_{p=1}^n \sum_{\sigma} x_{\sigma(1)} \dots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \dots x_n$$

$U(g) \otimes U(g)$

$\Delta(x_i) \Delta(x_j) = \Delta(x_i, x_j)$

$(p, q) \text{ shuffles in } S_n$

antipode

$S(x) = -x$

$S(x_1 \dots x_n) = (-1)^n x_n \dots x_1$

$\left(\begin{matrix} \text{In } H(G) \\ S(g) = g^{-1}, g \in G \end{matrix} \right)$

Hopf Algebra structure on $\mathcal{O}_G$

algebra

coalgebra

pointwise multiplication (like for $(k[G])^*$)

Hopf Algebra structure on $\mathcal{O}_G$

algebra

pointwise multiplication (like for $(k[G])^*$)

coalgebra

$$\pi(\Delta(f))(x \otimes y) = f(\mu(x \otimes y))$$

$\mathcal{O}_G \otimes \mathcal{O}_G$

$$\Delta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Hopf Algebra structure on $\mathcal{O}_G$

algebra

pointwise multiplication (like for $(k[G])^*$)

coalgebra

$$\sum_{\alpha \in \mathcal{O}_A \otimes \mathcal{O}_G} \alpha(\Delta(f))(X \otimes Y) = f(\mu(X \otimes Y))$$

$$\textcircled{1} \text{ }_{SL(2)}: \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes 1 & b \otimes 1 \\ c \otimes 1 & d \otimes 1 \end{pmatrix}$$

Hopf Algebra structure on $\mathcal{O}_G$

algebra

pointwise multiplication (like for $(k[G])^{**}$)

algebra

$$\underbrace{\Delta(f)}_{\mathcal{O}_A \otimes \mathcal{O}_A}(X \otimes Y) = f(\mu(X \otimes Y))$$

$$\textcircled{1} \text{ }_{SL(2)}: \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes 1 & b \otimes 1 \\ c \otimes 1 & d \otimes 1 \end{pmatrix}$$

Regular functions $\mathcal{O}_G \xrightarrow{k}$
 $\mathcal{O}_G = \{ \text{polynomials in } x_1, \dots, x_n \} / \{ (p_i(x_1, \dots, x_n) = 0) \}$

Ex. let $G = SL(2) = \{ (a, b, c, d) \mid ad - bc - 1 = 0 \}$
 $\{ \text{polynomials in } a, b, c, d \} / (ad - bc - 1)$
 $k[a, b, c, d] / (ad - bc - 1)$

Hopf Algebra structure on $\mathcal{O}_G$

algebra

pointwise multiplication (like for $(k[G])^*$)

coalgebra

$$\sum_{\theta \in \mathcal{O}_A \otimes \mathcal{O}_G} \Delta(f)(x \otimes y) = f(\mu(x \otimes y))$$

$$\mathcal{O}_{SL(2)}: \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes 1 & b \otimes 1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Hopf Algebra structure on $\mathcal{O}_G$

algebra

pointwise multiplication (like for $(k[G])^*$)

coalgebra

$$\sum_{\theta \in \mathcal{O}_G \otimes \mathcal{O}_G} \langle \Delta(f), \theta \rangle (X \otimes Y) = f(\mu(X \otimes Y))$$

$$\begin{pmatrix} \Delta(u) & \Delta(h) \\ \Delta(k) & \Delta(d) \end{pmatrix}$$

$$\textcircled{C} \text{ }_{SL(2)}: \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes 1 & b \otimes 1 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Hopf Algebra structure on $\mathcal{O}_G$

algebra

pointwise multiplication (like for $(k[G])^*$)

coalgebra

$$\sum_{\theta \in \mathcal{O}_G} \Delta(f)(\theta) (\theta \otimes Y) = f(\mu(X \otimes Y))$$

$$\begin{pmatrix} \Delta(u) & \Delta(h) \\ \Delta(k) & \Delta(w) \end{pmatrix}$$

$$\mathcal{O}_{SL(2)}: \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes 1 & b \otimes 1 & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

Hopf Algebra structure on $\mathcal{O}_G$

algebra

pointwise multiplication (like for $(k[G])^*$)

coalgebra

$$\sum_{\theta \in \mathcal{O}_G \otimes \mathcal{O}_G} \Delta(f)(x \otimes y) = f(\mu(x \otimes y))$$

$$\begin{pmatrix} \Delta(u) & \Delta(v) \\ \Delta(w) & \Delta(x) \end{pmatrix}$$

$\mathcal{O}_{SL(2)}$

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes 1 & b \otimes 1 \\ c \otimes 1 & d \otimes 1 \end{pmatrix}$$

antipode

$$S(f)(x) =$$

Hopf Algebra structure on $\mathcal{O}_G$

algebra

pointwise multiplication (like for $(k[G])^*$)

coalgebra

$$\sum_{\theta \in \mathcal{O}_G \otimes \mathcal{O}_G} \Delta(f)(x \otimes y) = f(\mu(x \otimes y))$$

$$\begin{pmatrix} \Delta(u) & \Delta(h) \\ \Delta(k) & \Delta(d) \end{pmatrix}$$

$\mathcal{O}_{SL(2)}$

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 1 \\ c & 1 \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{pmatrix}$$

antipode

$$\sum_i f(x_i) S(x_i) = f(S(x))$$

algebra $\mu(\Delta(f))(X \otimes Y) = f(\mu(X \otimes Y))$
coalgebra $\Delta(\Delta(f)) = f(\mu(X \otimes Y))$
 $\Theta_{SL(2)}: \Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes a & b \otimes a & b \otimes c & b \otimes d \\ a \otimes c & b \otimes c & c \otimes a & c \otimes b \\ a \otimes d & b \otimes d & c \otimes c & c \otimes d \\ d \otimes a & d \otimes b & d \otimes c & d \otimes d \end{pmatrix}$
antipode $S(f)(X) = f(S(X)) = f(X^{-1})$
 $\Theta_{SL(2)}: S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

algebra $\mu(\Delta(f))(X \otimes Y) = f(\mu(X \otimes Y))$
coalgebra $\Delta(\Delta(f)) = f(\mu(X \otimes Y))$
 $\Theta_{SL(2)}: \Delta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes a & a \otimes b & b \otimes a & b \otimes b \\ c \otimes a & c \otimes b & d \otimes a & d \otimes b \end{pmatrix}$
antipode $S(f)(X) = f(S(X)) = f(X^{-1})$
 $\Theta_{SL(2)}: S\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^{X^{-1}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

algebra coalgebra antipode

$$\begin{aligned}
 & \underbrace{\Delta(f)}_{\theta_A \otimes \theta_B}(X \otimes Y) = f(\mu(X \otimes Y)) \\
 & \textcircled{O}_{SL(2)}: \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes a & b \otimes a & b \otimes c & b \otimes d \\ c \otimes a & c \otimes b & c \otimes c & c \otimes d \\ d \otimes a & d \otimes b & d \otimes c & d \otimes d \end{pmatrix} \\
 & \underbrace{S(f)}(X) = f(\underbrace{S(X)}) = f(X^{-1}) \\
 & \textcircled{O}_{SL(2)}: S \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{X^{-1}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
 \end{aligned}$$