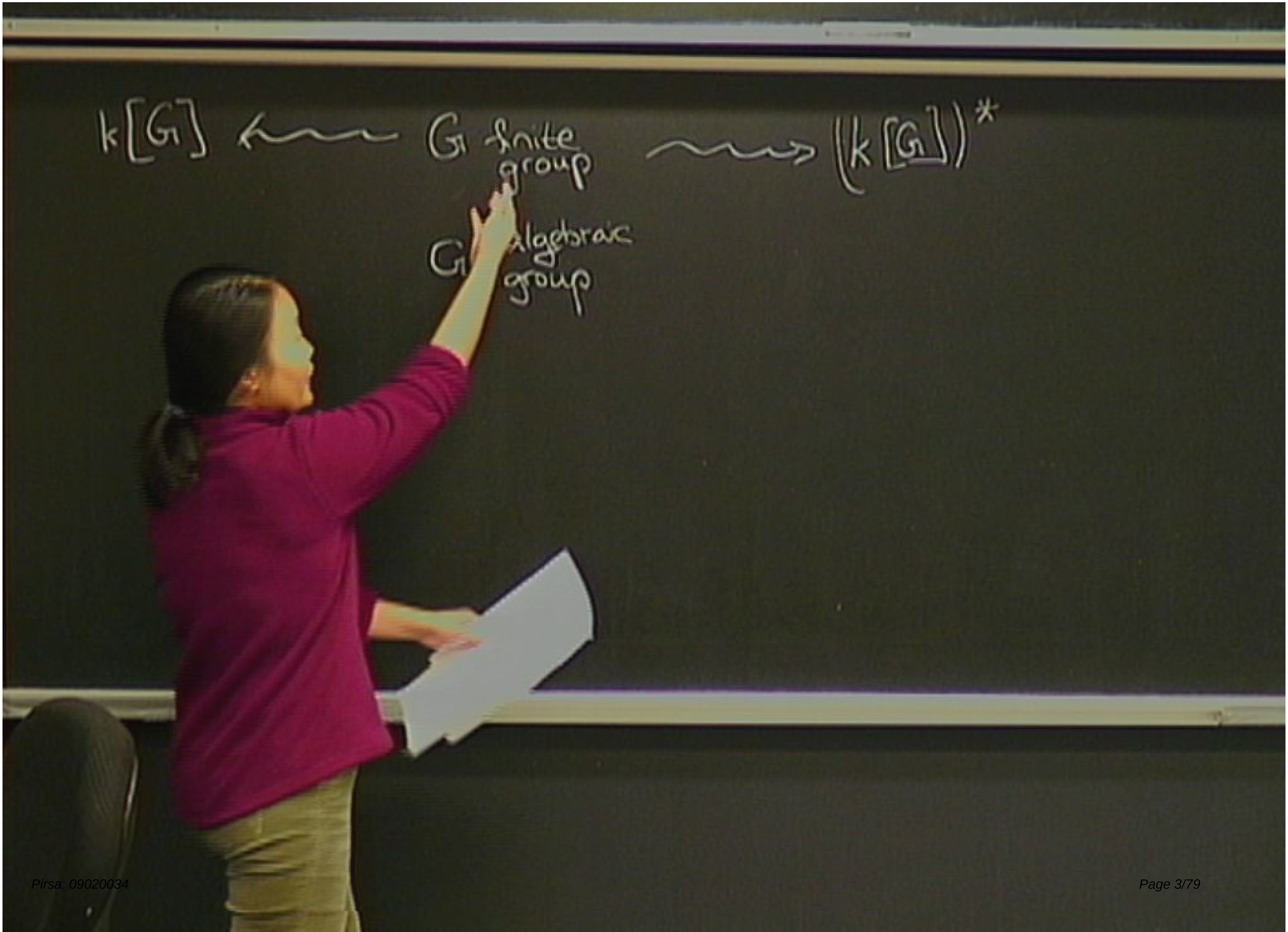
Title: Grad Talks 2

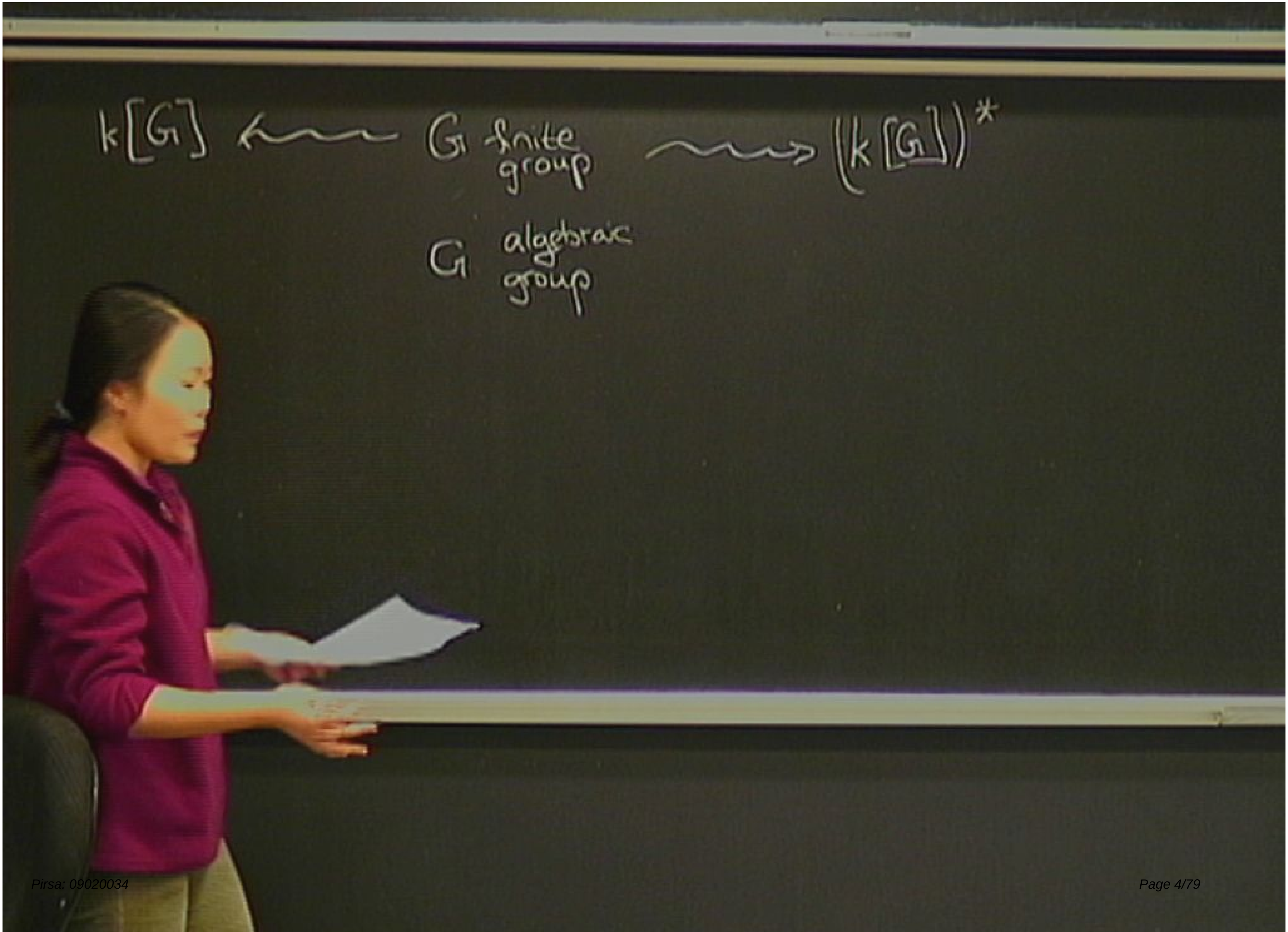
Date: Feb 09, 2009 11:00 AM

URL: http://pirsa.org/09020034

Abstract: Lecture on Quantum Groups by Lucy Zhang

$k[G] \leftsquigarrow G$ finite group $\rightsquigarrow (k[G])^*$


k[G]
G finite group
(k[G])*
G algebraic group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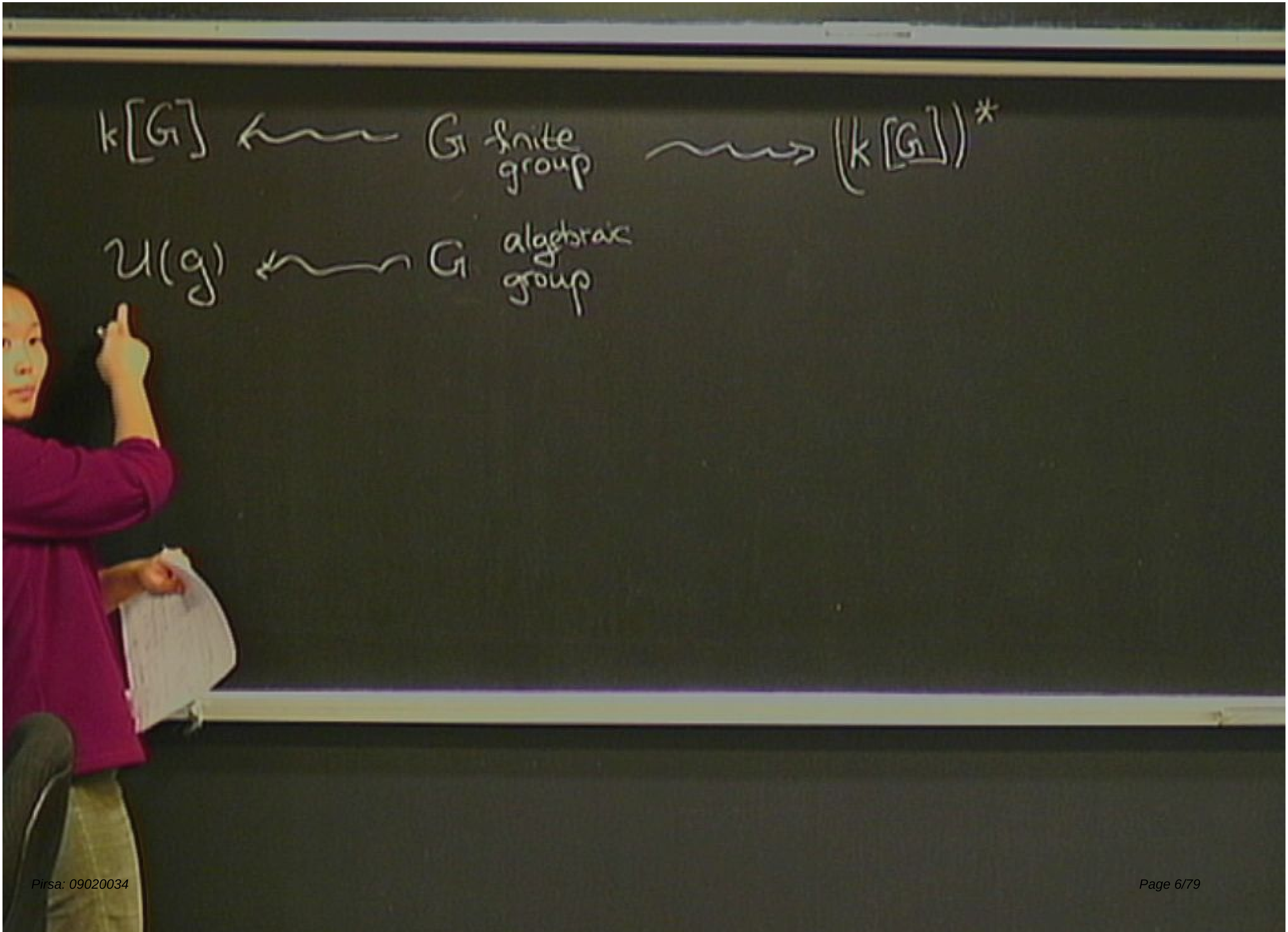
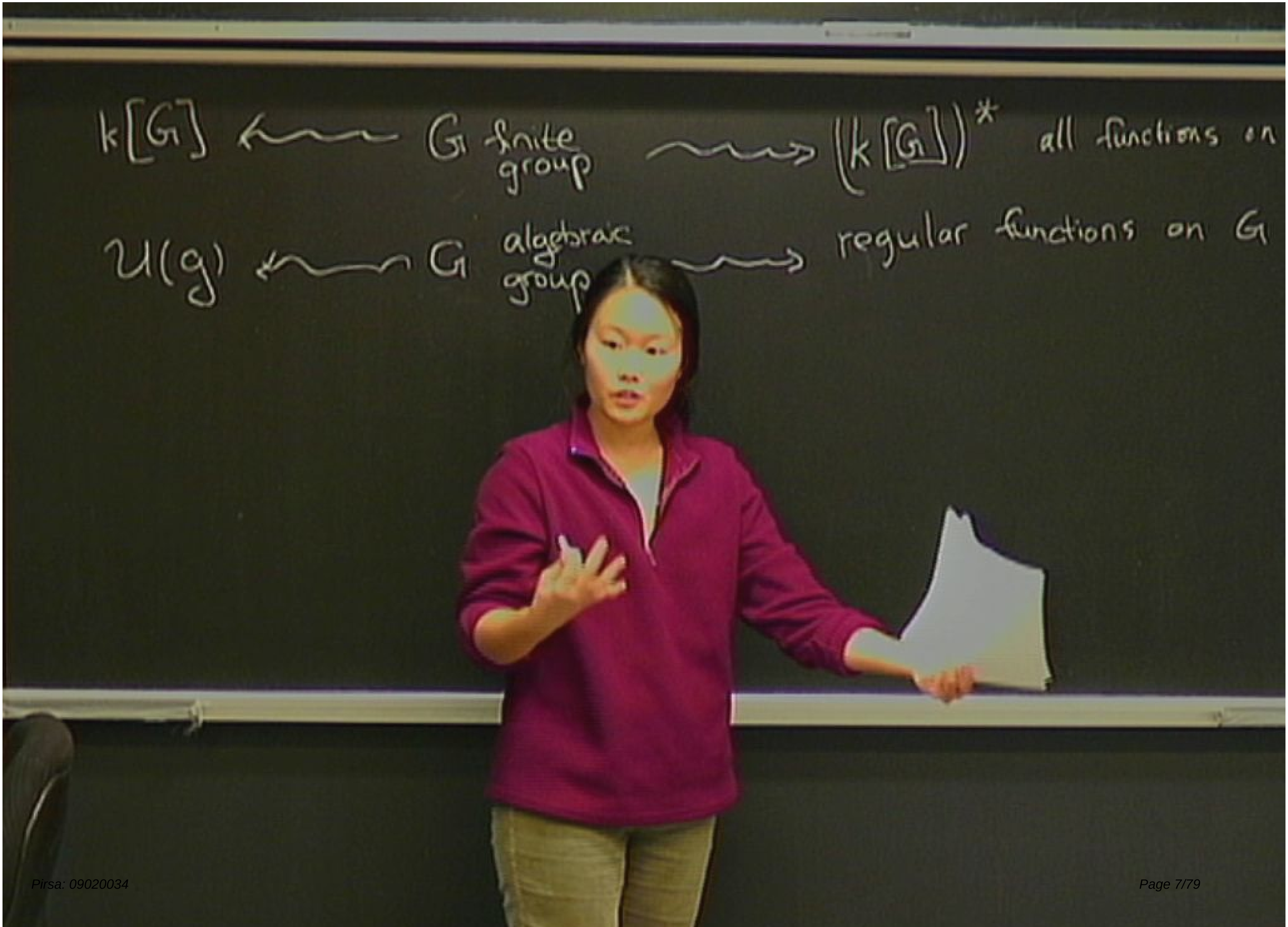
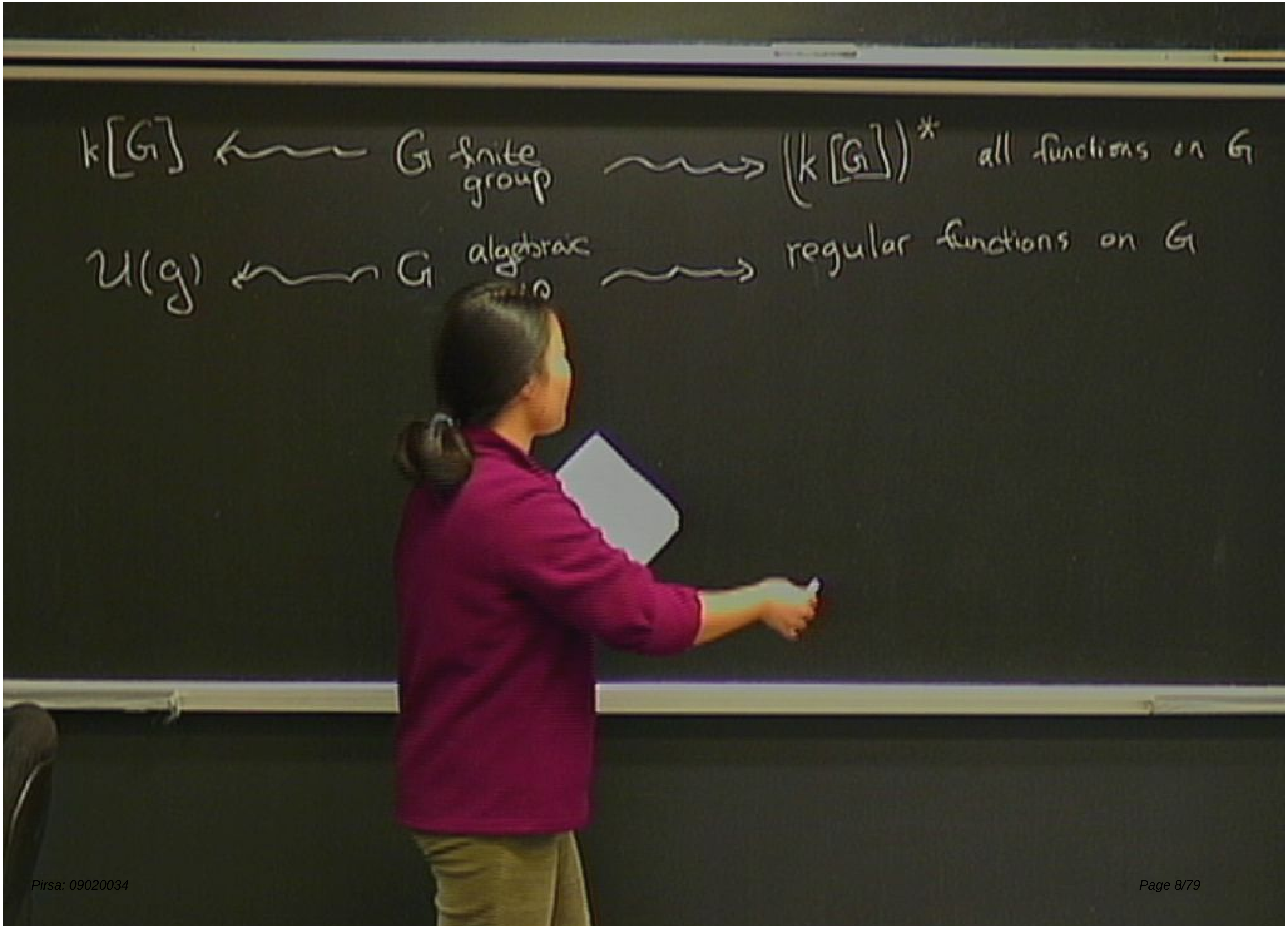
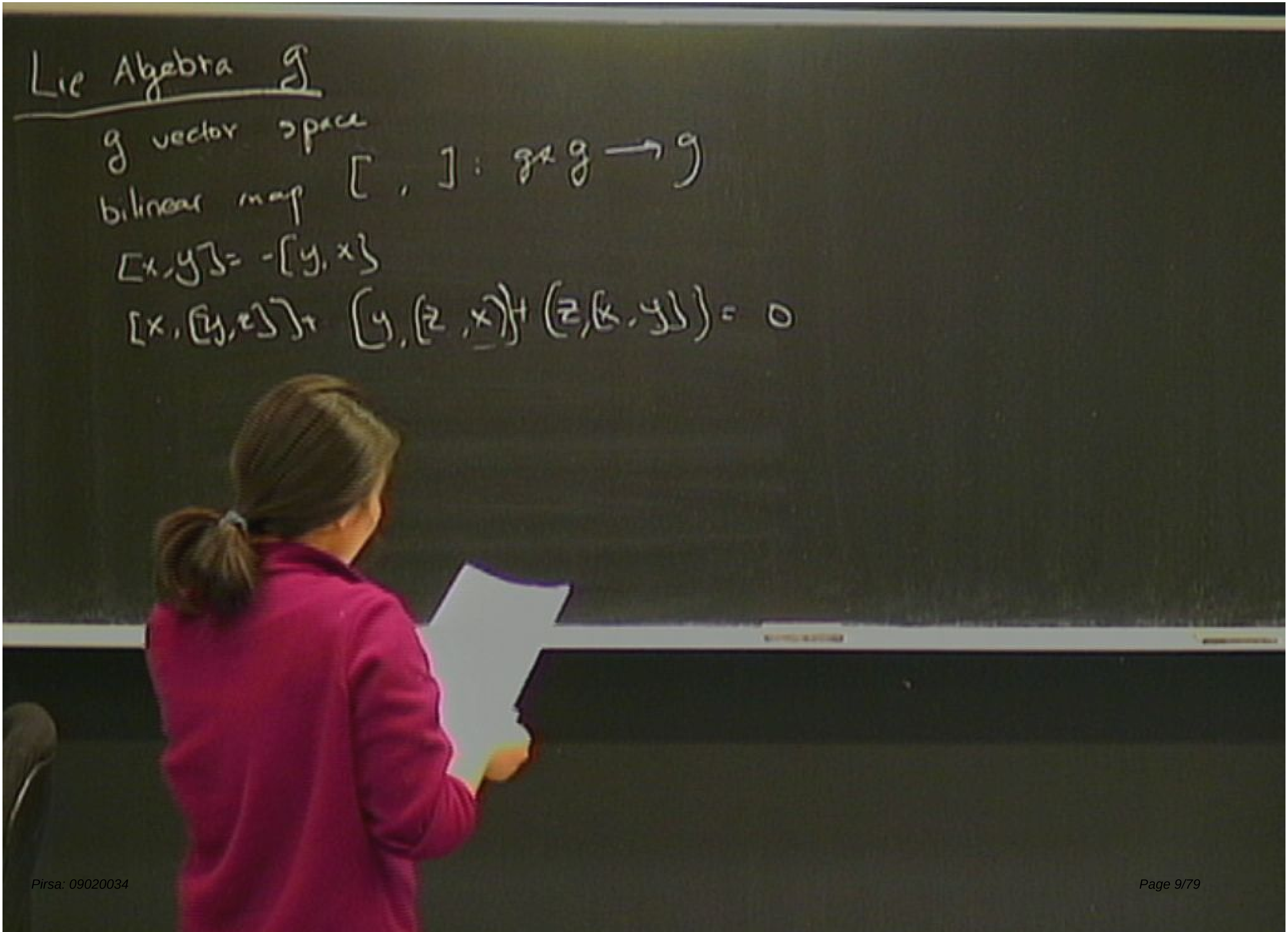
$\leftsquigarrow G$ algebraic group


k[G] ⟵ G finite group ⟶ (k[G])*
U(g) ⟵ G algebraic group


k[G] ⟵ G finite group ⟶ (k[G])* all functions on
U(g) ⟵ G algebraic group ⟶ regular functions on G


$k[G] \leftsquigarrow G$ finite group $\rightsquigarrow (k[G])^*$ all functions on G
$\mathcal{U}(\mathfrak{g}) \leftsquigarrow G$ algebraic $\rightsquigarrow$ regular functions on G


Lie Algebra $\mathfrak{g}$
$\mathfrak{g}$ vector space
bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$
$[x,y] = -[y,x]$
$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

Lie Algebra $\mathfrak{g}$

$\mathfrak{g}$ vector space

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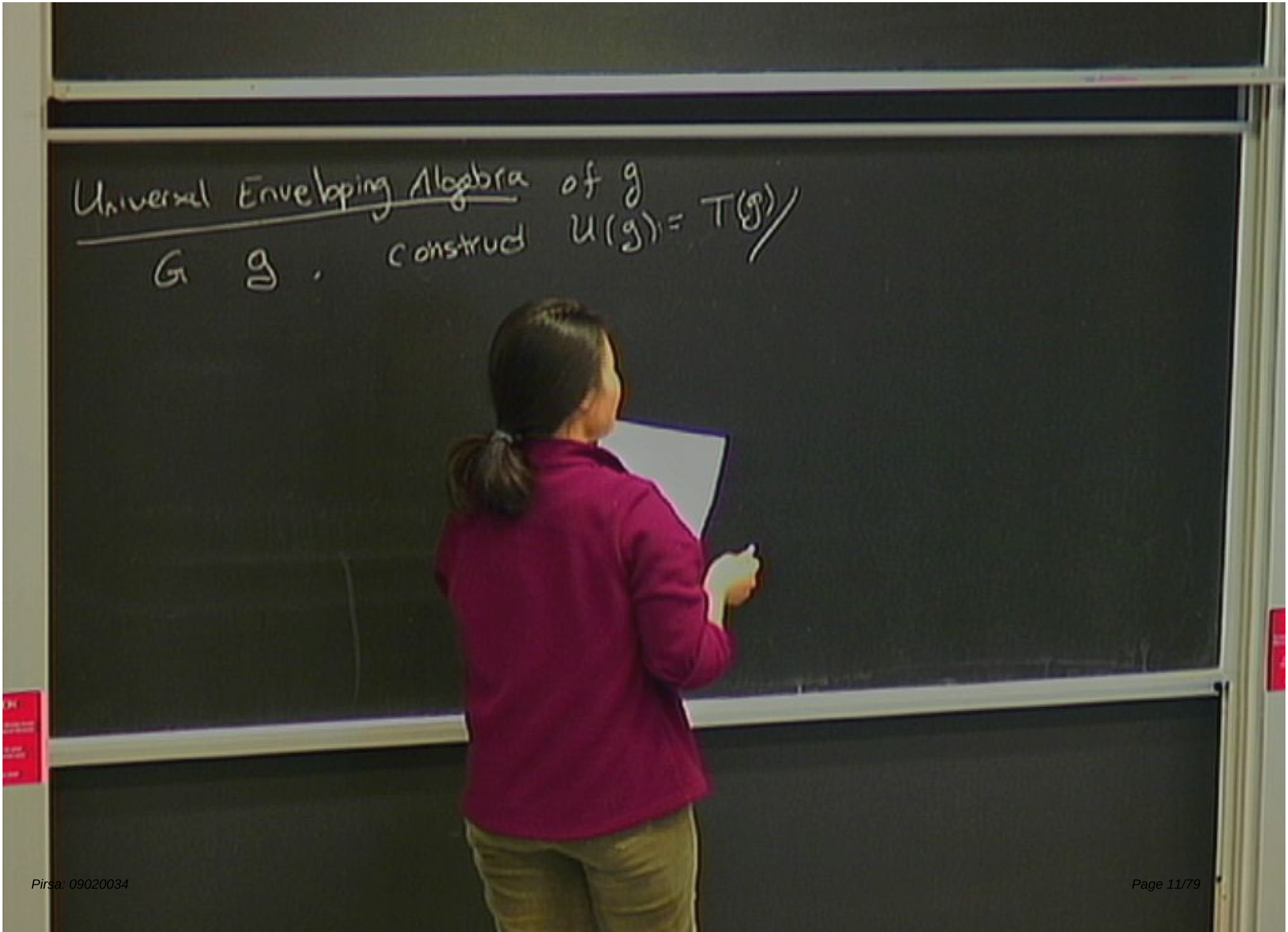
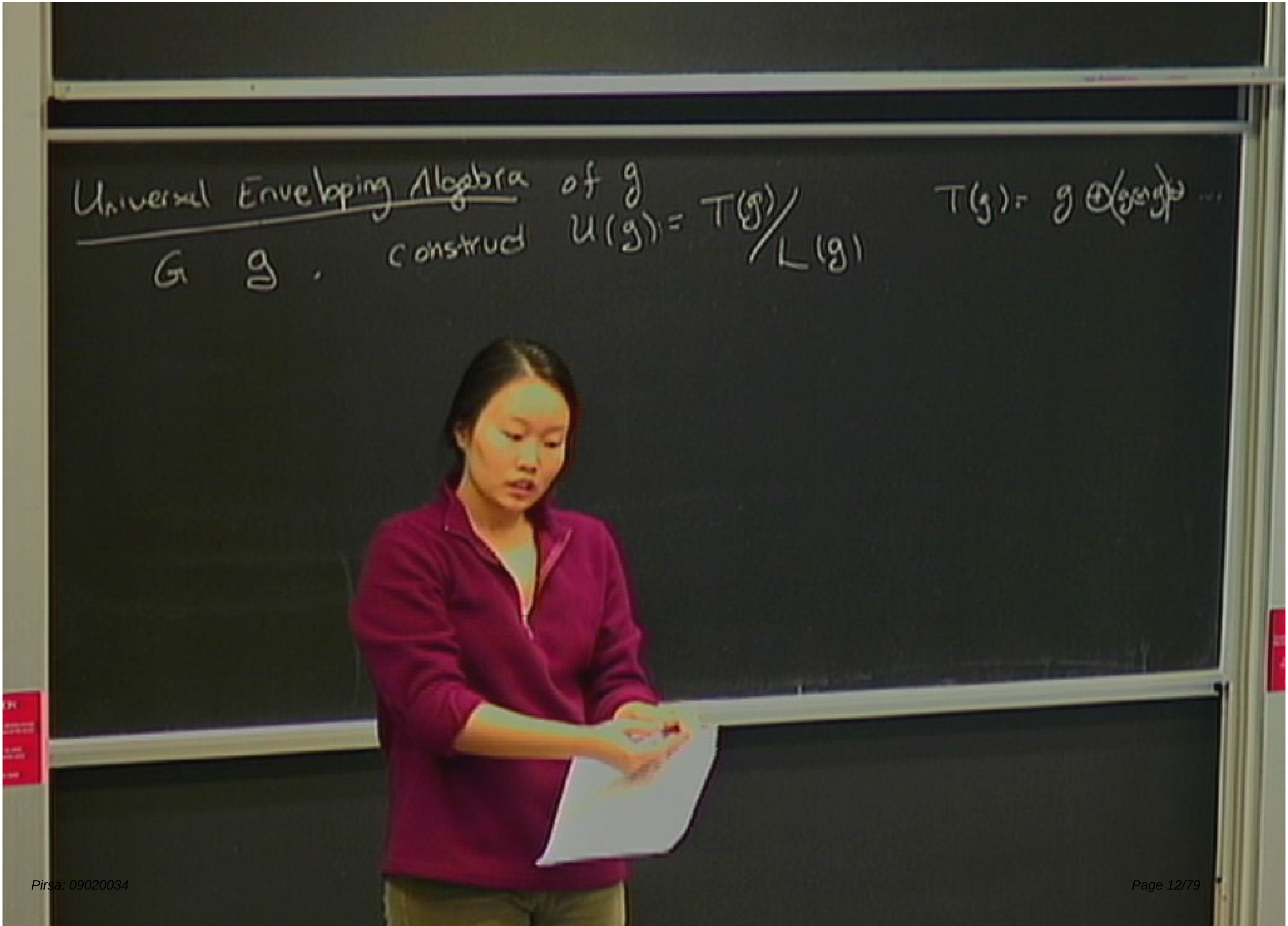
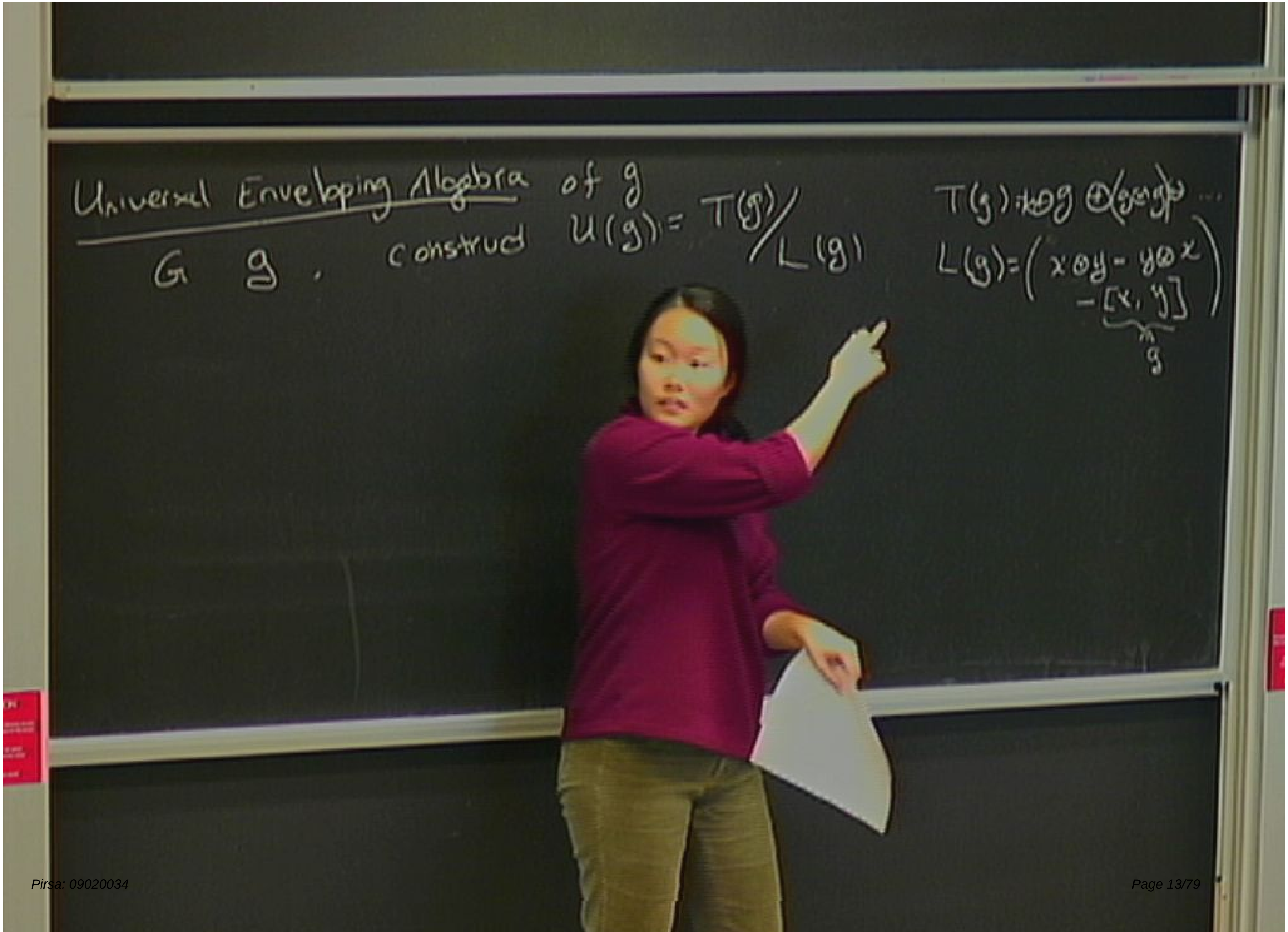
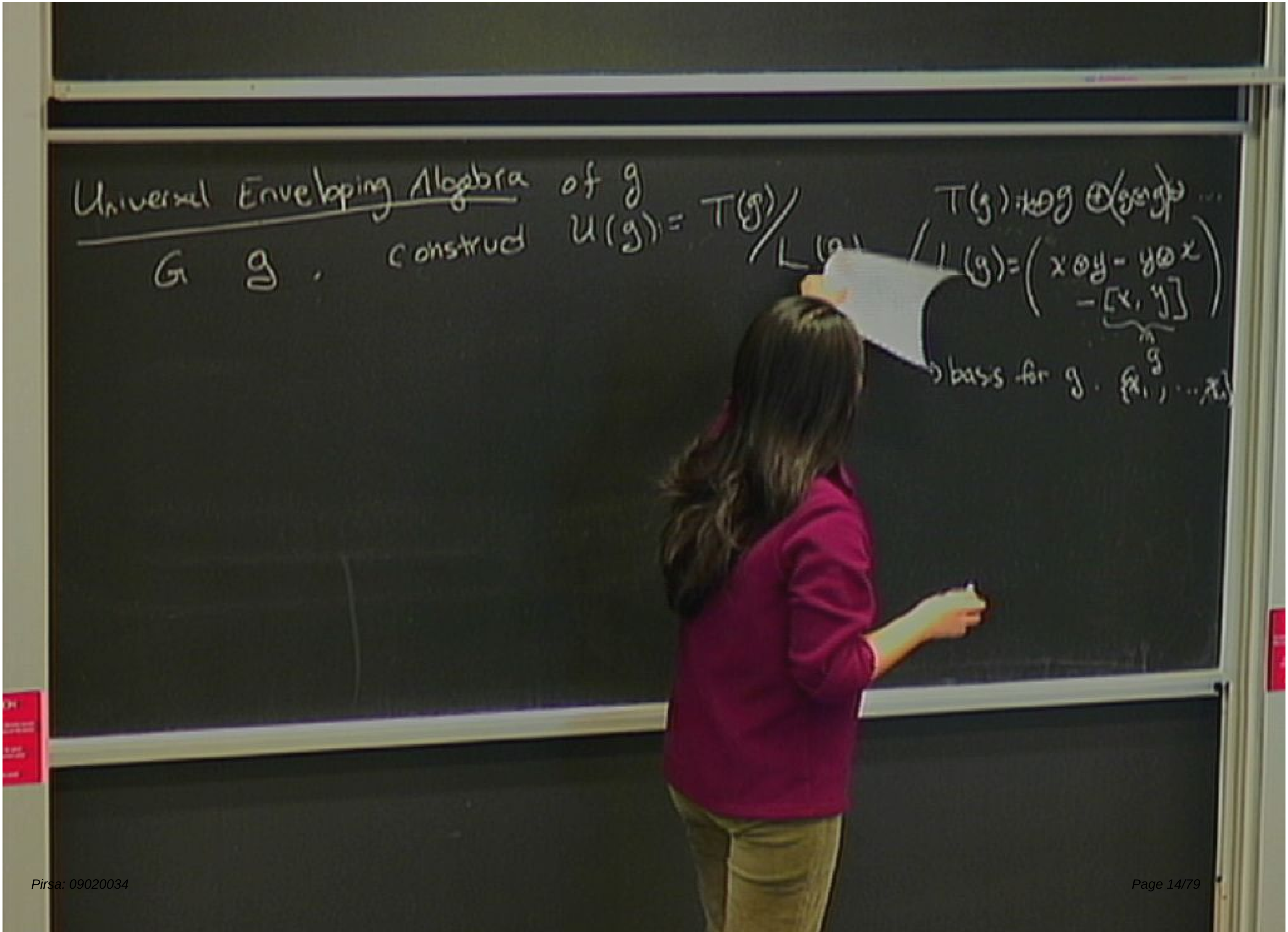

Universal Enveloping Algebra of g
G g . construct U(g) := T(g)/


Universal Enveloping Algebra of g
G g . construct U(g) := T(g)/L(g)
T(g) := g ⊕ (g⊗g) ⊕ ...


Universal Enveloping Algebra of g
G g . construct U(g) := T(g)/L(g)
T(g) := k ⊕ g ⊕ (g⊗g) ⊕ ...
L(g) = (x⊗y − y⊗x − [x, y])
g


Universal Enveloping Algebra of $\mathfrak{g}$
G $\mathfrak{g}$.
construct $U(\mathfrak{g}) = T(\mathfrak{g})/L(\mathfrak{g})$
$T(\mathfrak{g}) = k \oplus \mathfrak{g} \oplus (\mathfrak{g}\otimes\mathfrak{g}) \oplus \cdots$
$L(\mathfrak{g}) = (x\otimes y - y\otimes x - \underbrace{[x, y]}_{\mathfrak{g}})$
basis for $\mathfrak{g}$. $\{x_1, \ldots, x_n\}$

Universal Enveloping Algebra of $\mathfrak{g}$

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basis for $\mathfrak{g}$: $\{x_1, \ldots, x_n\}$

basis for $T(\mathfrak{g})$:

Universal Enveloping Algebra of $\mathfrak{g}$

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basis for $T(\mathfrak{g})$: words

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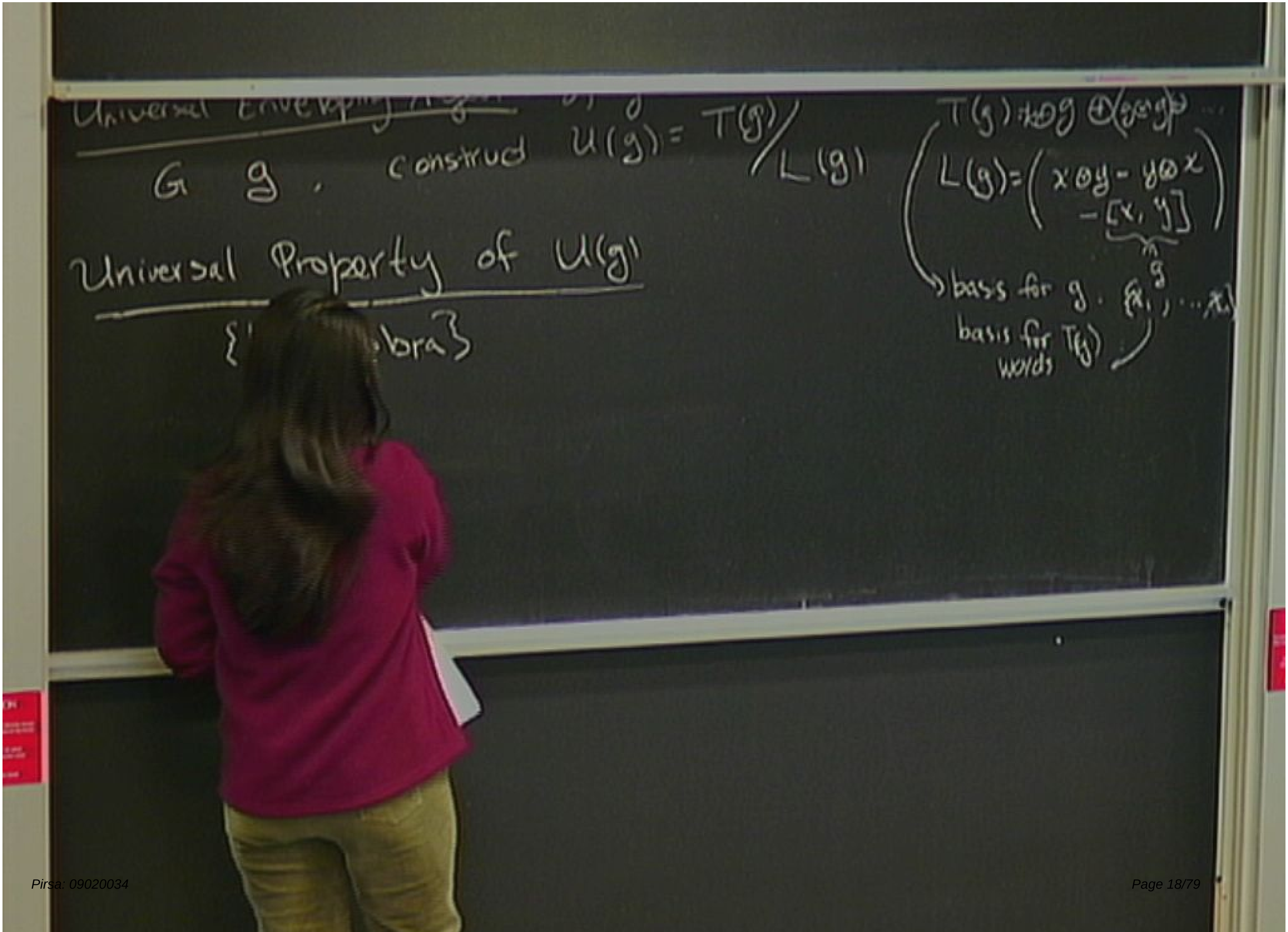
basis for $T(\mathfrak{g})$: words

Universal Property of $U(\mathfrak{g})$



construct U(g) = T(g)/L(g)
G g .
Universal Property of U(g)
bra}
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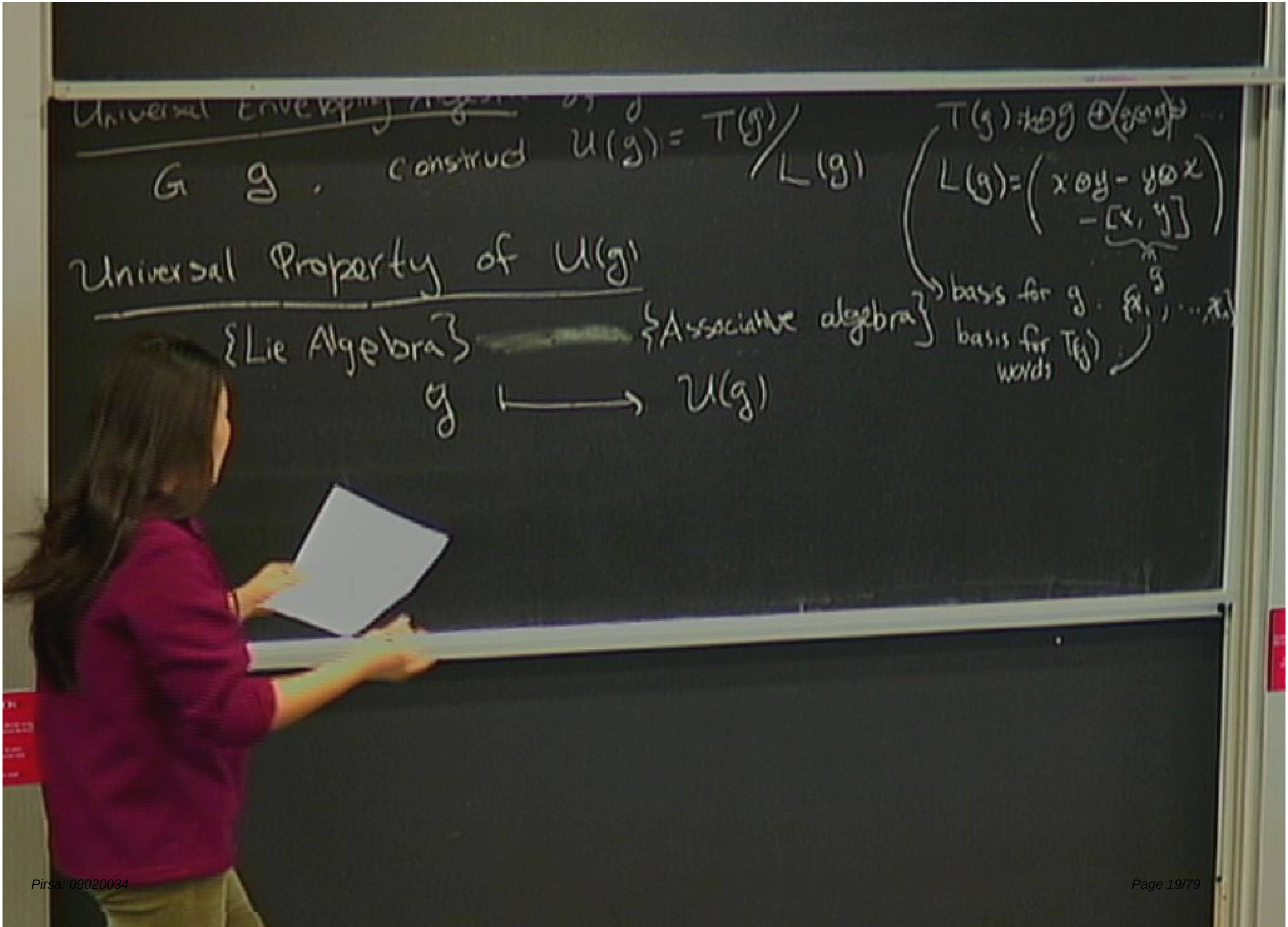
g
basis for g . {x1, ..., xn}
basis for T(g)
words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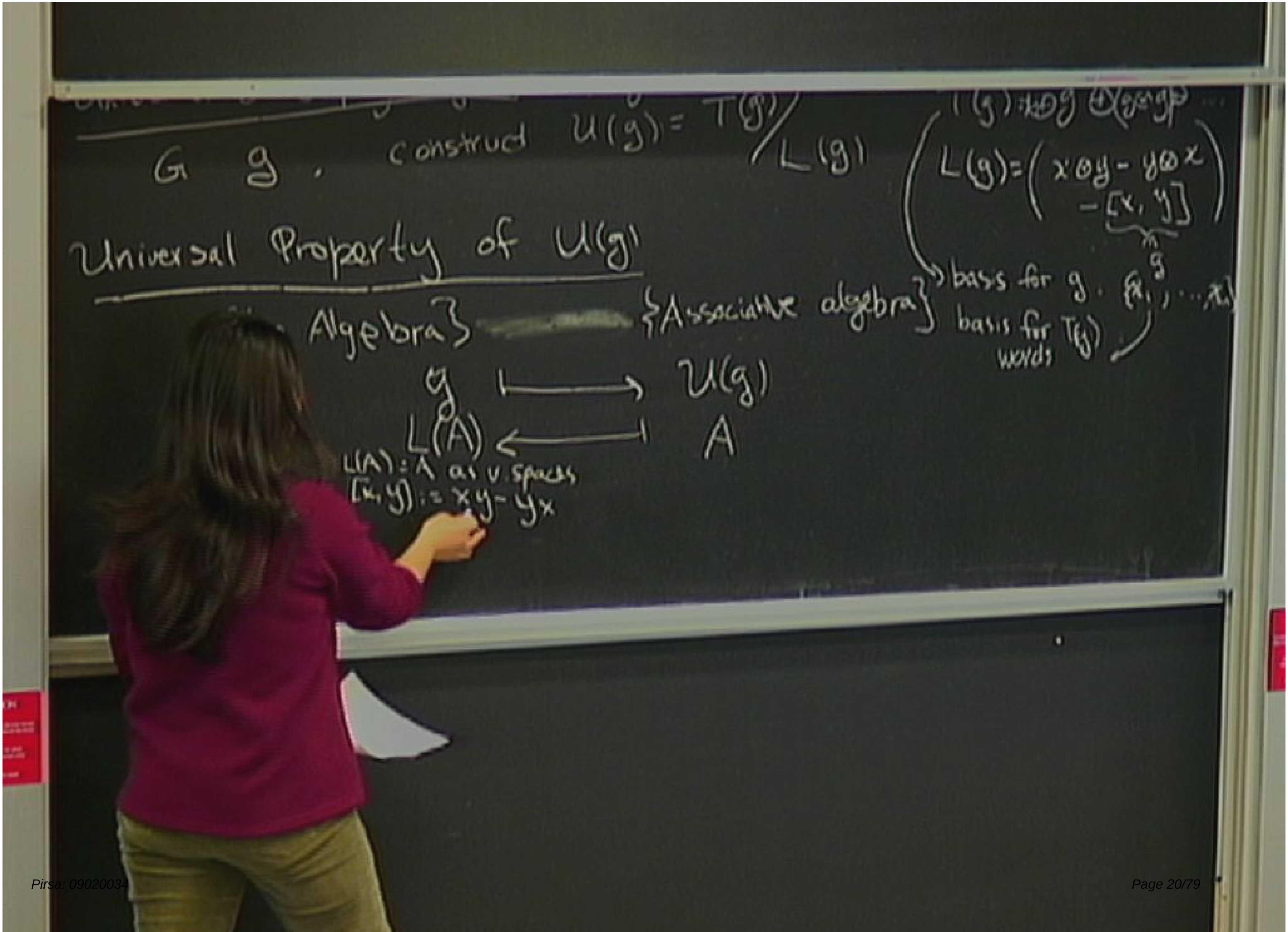
Universal Property of U(g)
{Lie Algebra}
{Associative algebra}
g ⟼ U(g)
basis for g . {x1, ..., xn}
basis for T(g)
words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g ⟼ U(g)
L(A) ⟻ A
L(A) = A as v spaces
[x, y] := xy - yx

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basis for $\mathfrak{g}$. $\{x_1, \ldots, x_n\}$

basis for $T(\mathfrak{g})$ words

Universal Property of $U(\mathfrak{g})$

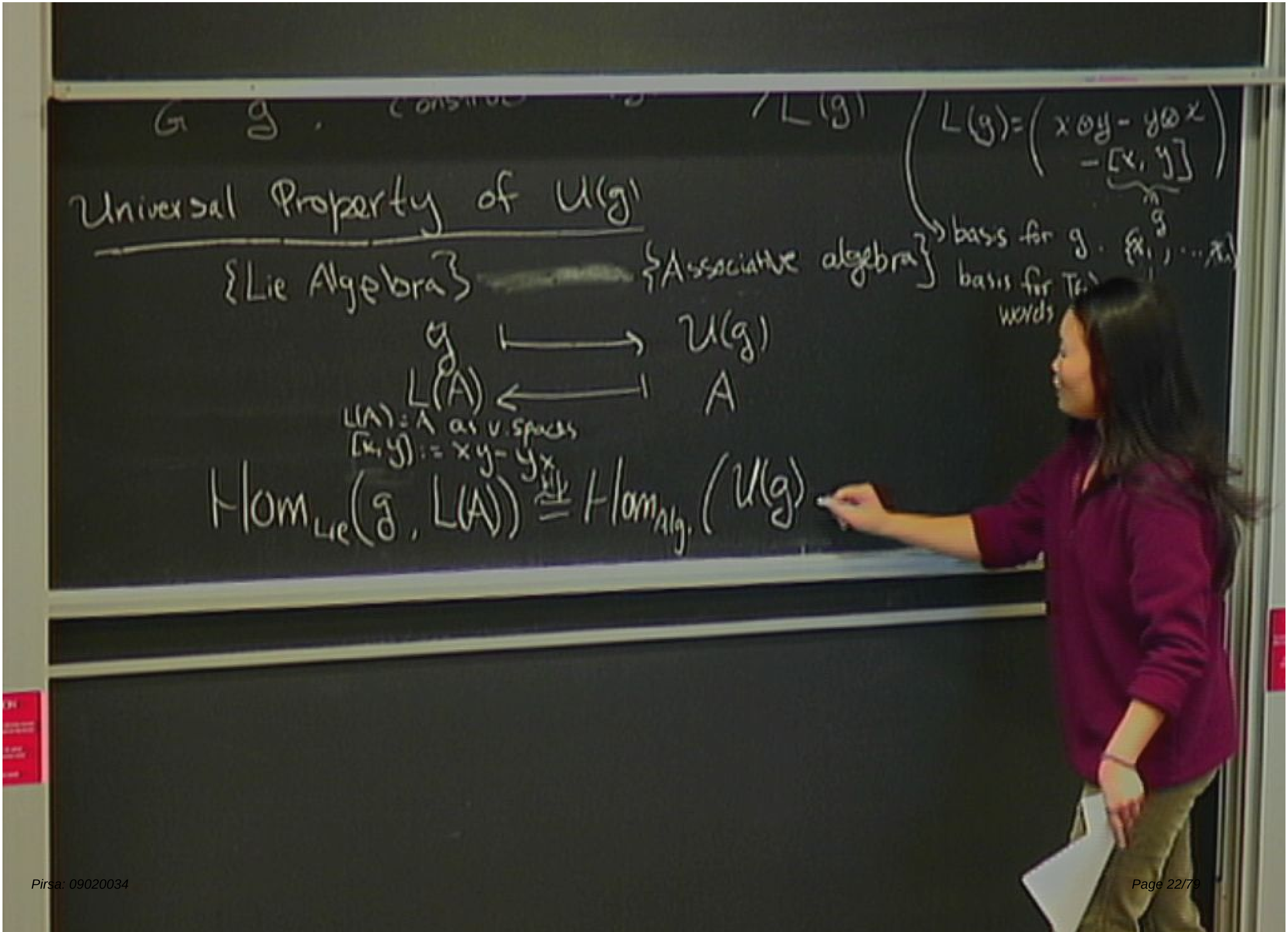
{Lie Algebra} {Associative algebra}

$$\mathfrak{g} \longmapsto U(\mathfrak{g})$$

$$L(A) \longleftarrow\!\shortmid A$$

$L(A) = A$ as v. spaces

$[x, y] := xy - yx$


Universal Property of U(g)
{Lie Algebra} {Associative algebra}
g ⟼ U(g)
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L(A): A as v. spaces
[x, y] := xy − yx
Hom_Lie(g, L(A)) ≅ Hom_Alg.(U(g)
L(g) = (x⊗y − y⊗x − [x, y])
basis for g. {x1, ..., xn}
basis for T(
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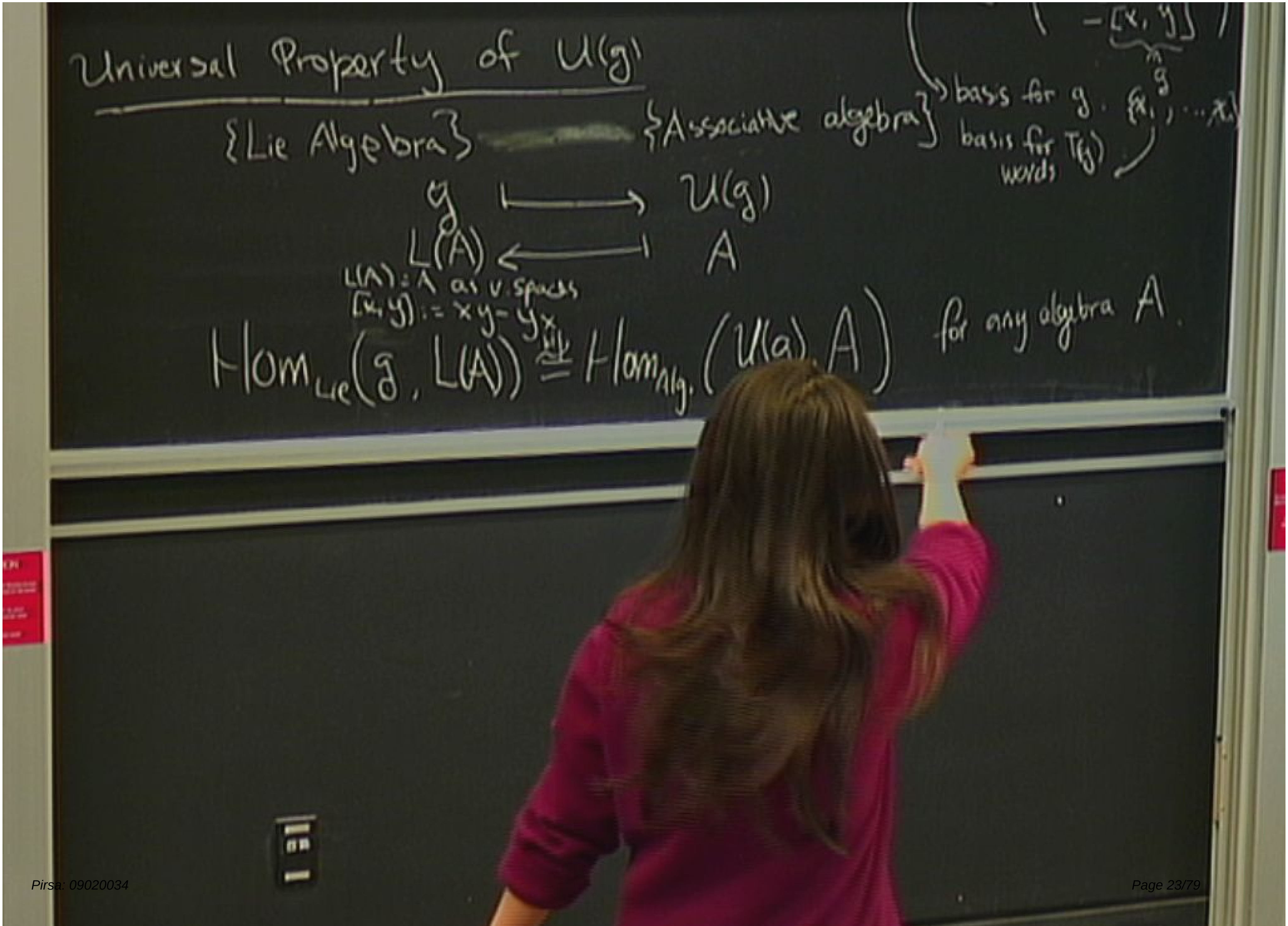
Universal Property of U(g)
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g ⟼ U(g)
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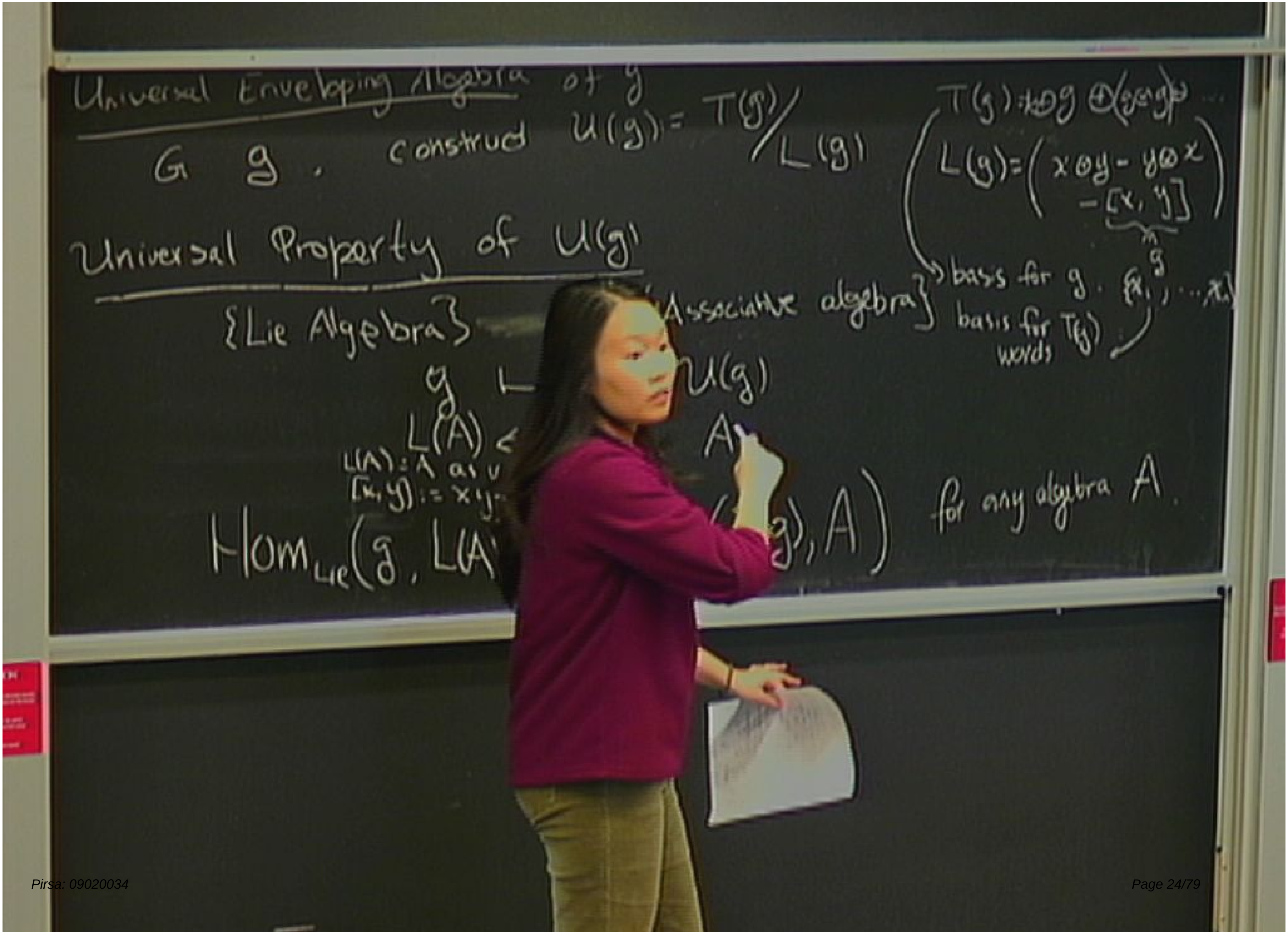
Hom_Lie(g, L(A)) ≅ Hom_Alg.(U(g), A) for any algebra A.
basis for g. {x_1, ..., x_n}
basis for T(g)
words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U(g)
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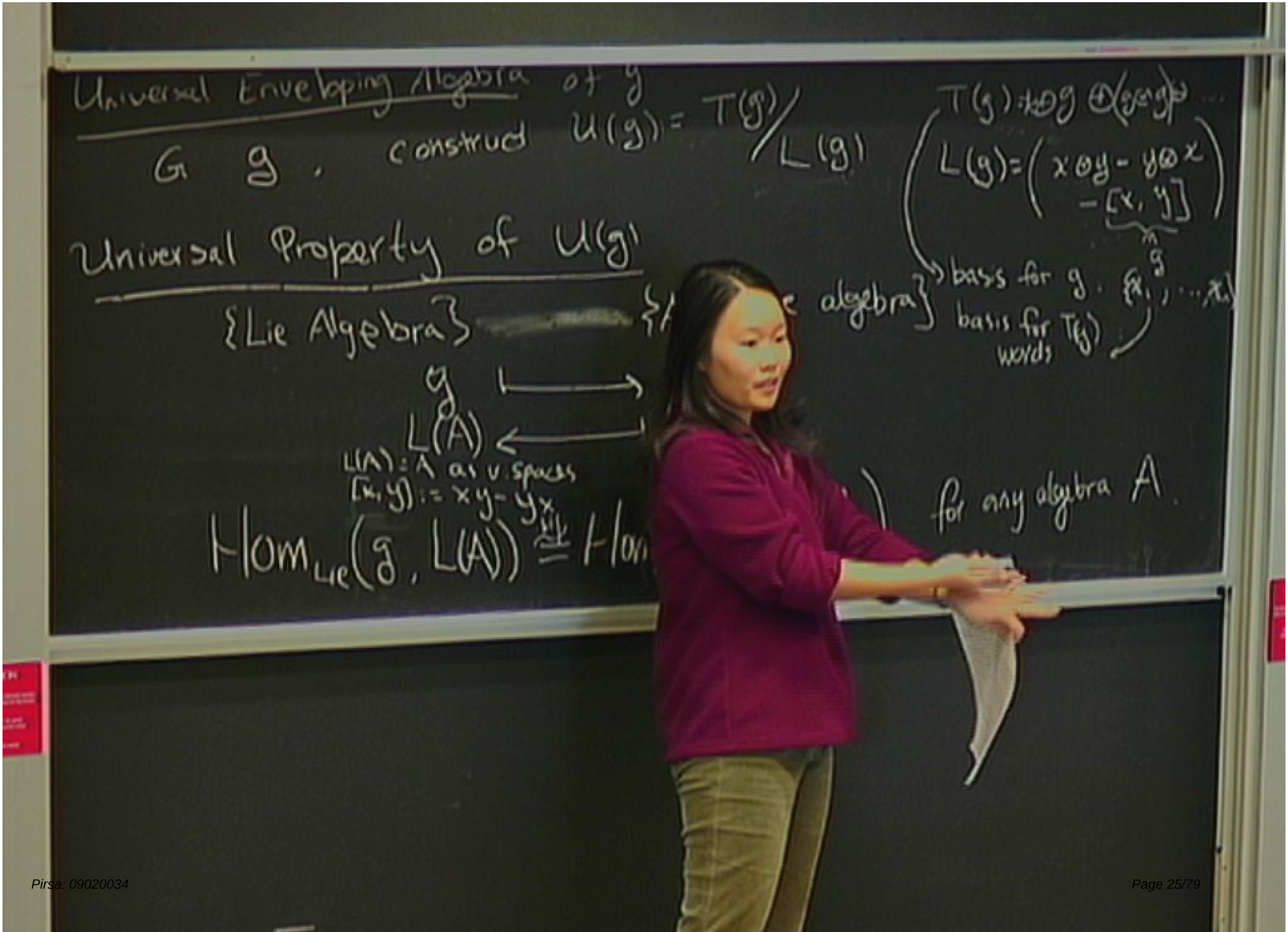
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basis for $\mathfrak{g}$. $\{x_1, \ldots, x_n\}$

basis for $T(\mathfrak{g})$ words

Universal Property of $\mathcal{U}(\mathfrak{g})$

$$\{\text{Lie Algebra}\} \longrightarrow \{\text{Associative algebra}\}$$

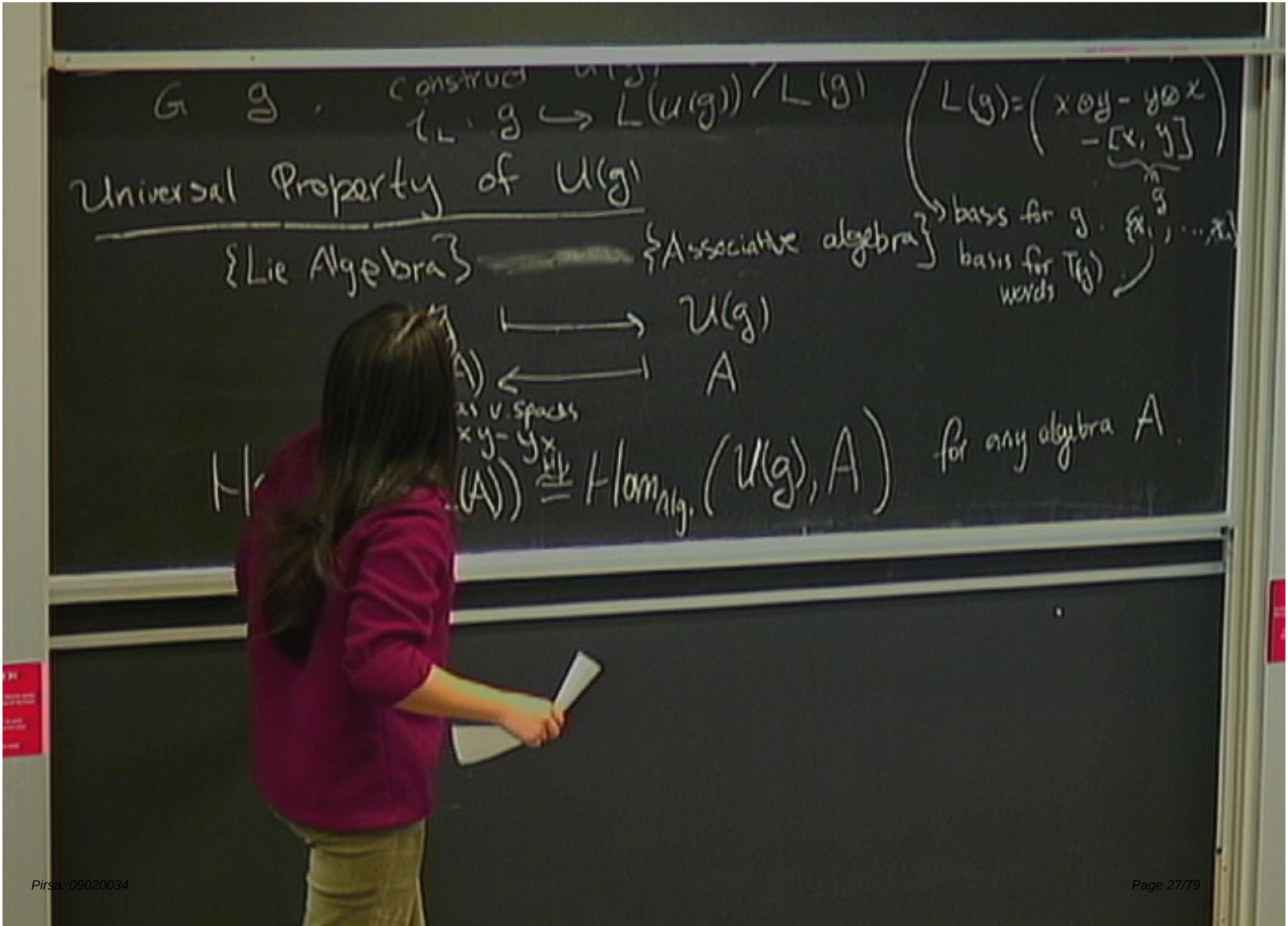
$$\mathfrak{g} \longmapsto \mathcal{U}(\mathfrak{g})$$

$$L(A) \longleftarrow A$$

$L(A) := A$ as v. spaces

$[x, y] := xy - yx$

$$\mathrm{Hom}_{Lie}(\mathfrak{g}, L(A)) \stackrel{nat}{\cong} \mathrm{Hom}_{Alg.}(\mathcal{U}(\mathfrak{g}), A)$$ for any algebra A.


G g . construct
$i_L : g \hookrightarrow L(U(g)) / L(g)$
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$$\{\text{Lie Algebra}\} \longrightarrow \{\text{Associative algebra}\}$$

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$L(A) := A$ as v. spaces

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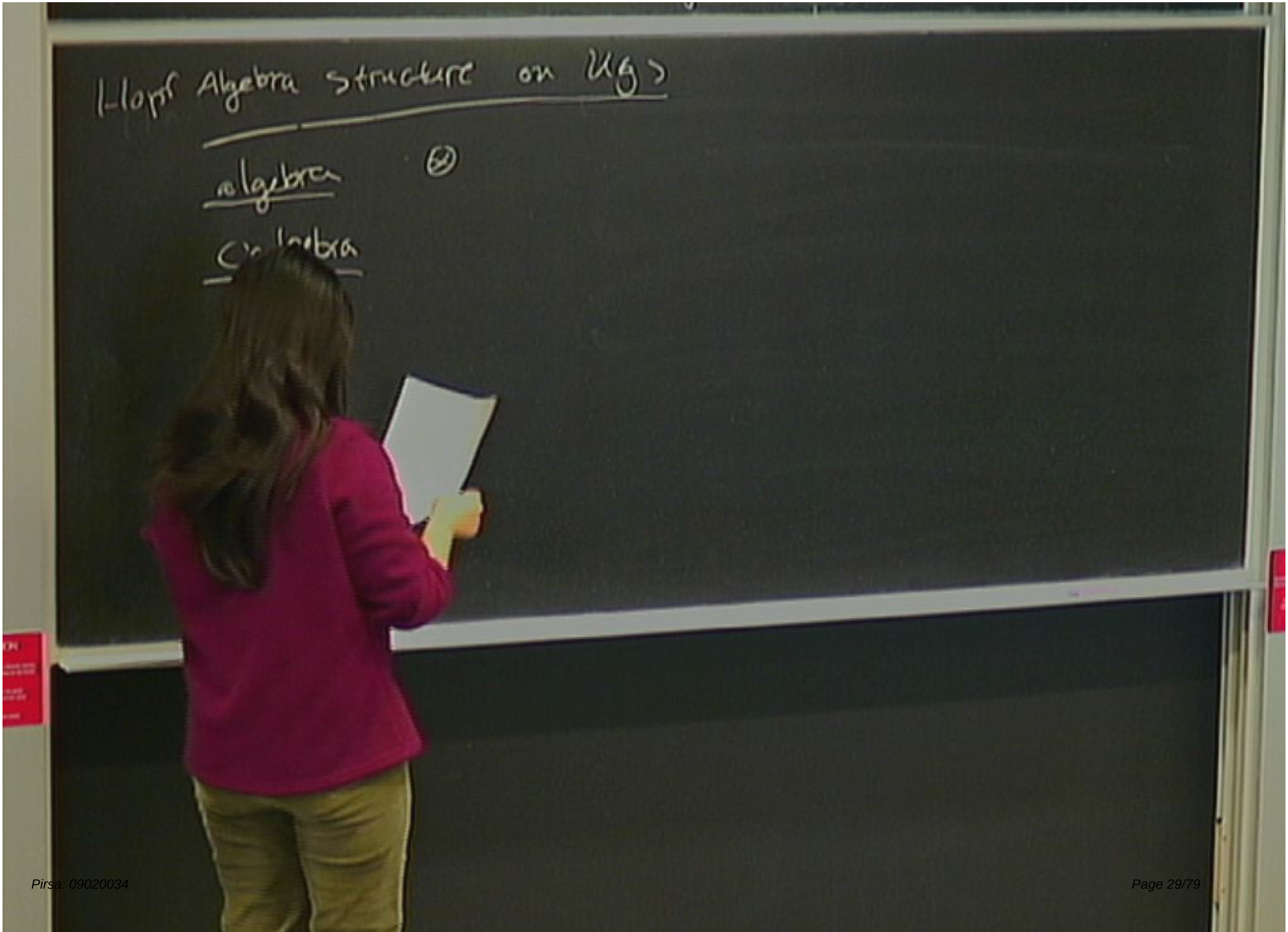
$$\mathrm{Hom}_{Lie}(\mathfrak{g}, L(A)) \cong \mathrm{Hom}_{Alg.}(\mathcal{U}(\mathfrak{g}), A) \quad \text{for any algebra } A$$

$\mathfrak{g}: \{x_1, \dots, x_n\}$

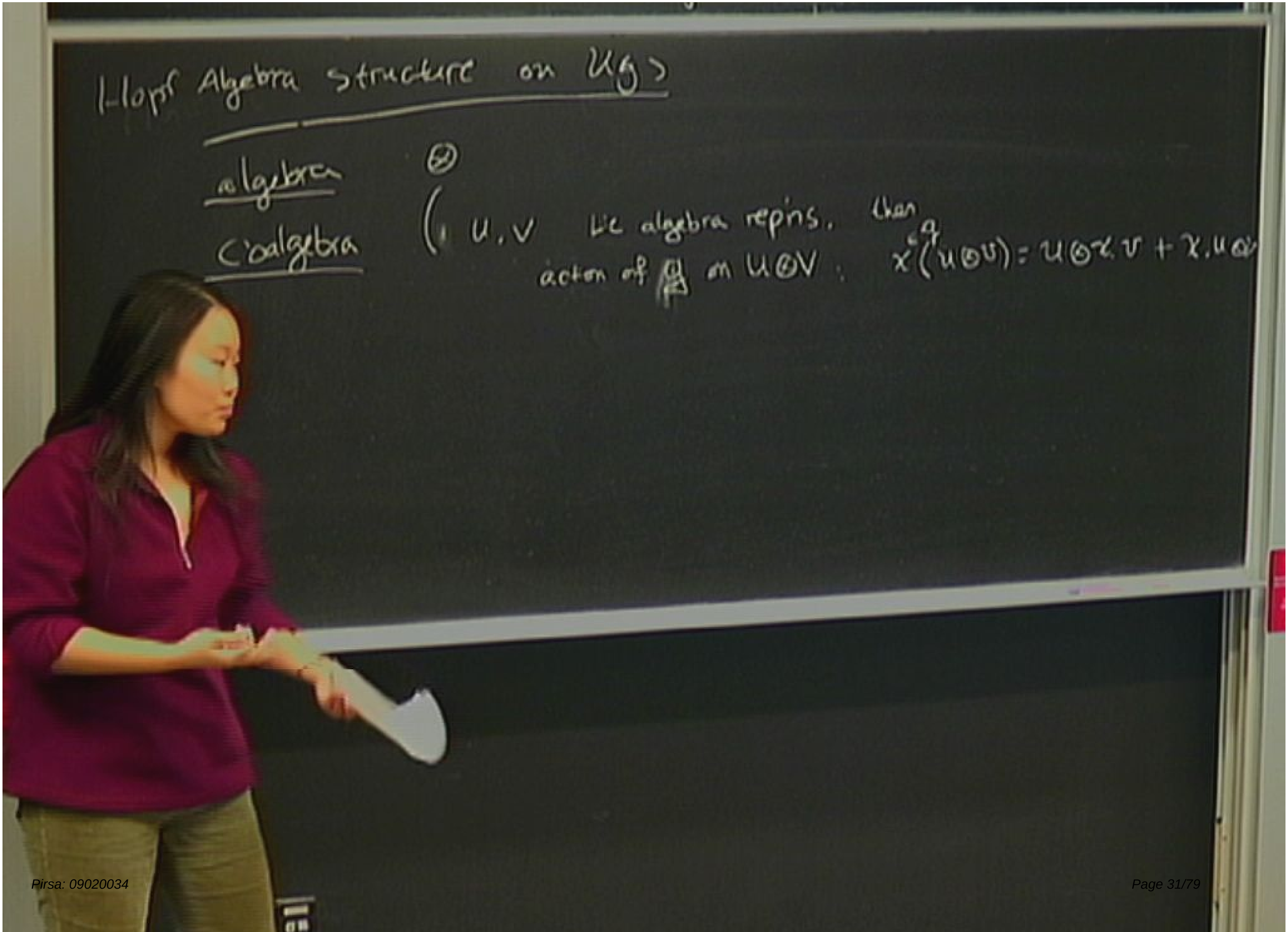
basis for $T(\mathfrak{g})$ words


Hopf Algebra structure on U(g)
algebra
⊗

Hopf Algebra structure on $U(g)$

algebra $\otimes$

Coalgebra


Hopf Algebra structure on U(g)
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Coalgebra
U, V Lie algebra rep'ns. then
action of g on U⊗V : x(u⊗v) = u⊗x.v + x.u⊗

Hopf Algebra structure on $U(\mathfrak{g})$

algebra $\otimes$

Coalgebra (U, V Lie algebra rep'ns. then

action of $\mathfrak{g}$ on $U \otimes V$: e.g. $x(u \otimes v) = u \otimes x.v + x.u \otimes v$

$$\Delta(x) = 1 \otimes x + x \otimes 1$$

Hopf Algebra structure on $U\mathfrak{g}$

algebra $\otimes$

Coalgebra (U, V Lie algebra rep'ns, then action of $\mathfrak{g}$ on $U \otimes V$: $x(u \otimes v) = u \otimes x.v + x.u \otimes$

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Hopf Algebra structure on $\mathcal{U}\mathfrak{g}$ >

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Hopf Algebra structure on $U(\mathfrak{g})$

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Coalgebra (u, v Lie algebra rep'ns, then

action of $\mathfrak{g}$ on $u \otimes v$: $x^{\mathfrak{g}}(u \otimes v) = u \otimes x.v + x.u \otimes$

$\Delta(x) = 1 \otimes x + x \otimes 1$

$\Delta(y$

Hopf Algebra structure on $\mathcal{U}(\mathfrak{g})$

algebra $\otimes$

Coalgebra (U, V Lie algebra rep'ns, then

action of $\mathfrak{g}$ on $U \otimes V$: $x \in \mathfrak{g}$ $(u \otimes v) = u \otimes x.v + x.u \otimes$

$x \in \mathfrak{g}$. $\Delta(x) = 1 \otimes x + x \otimes 1$

$x_1, \ldots x_n \in \mathfrak{g}$, $\Delta(x_1, \ldots x_n)$

Hopf Algebra structure on $U(\mathfrak{g})$

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Coalgebra (U, V Lie algebra rep'ns. then

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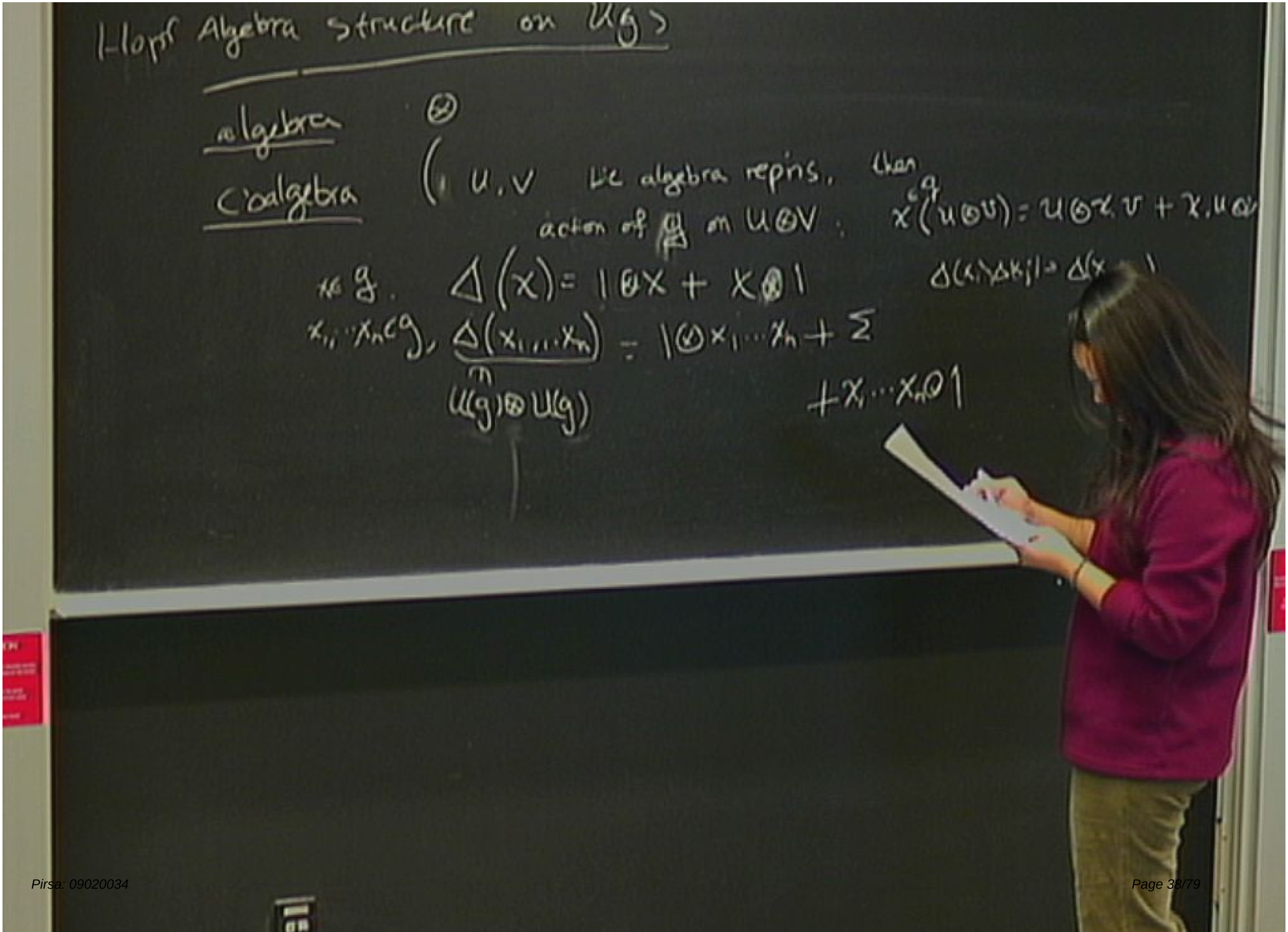
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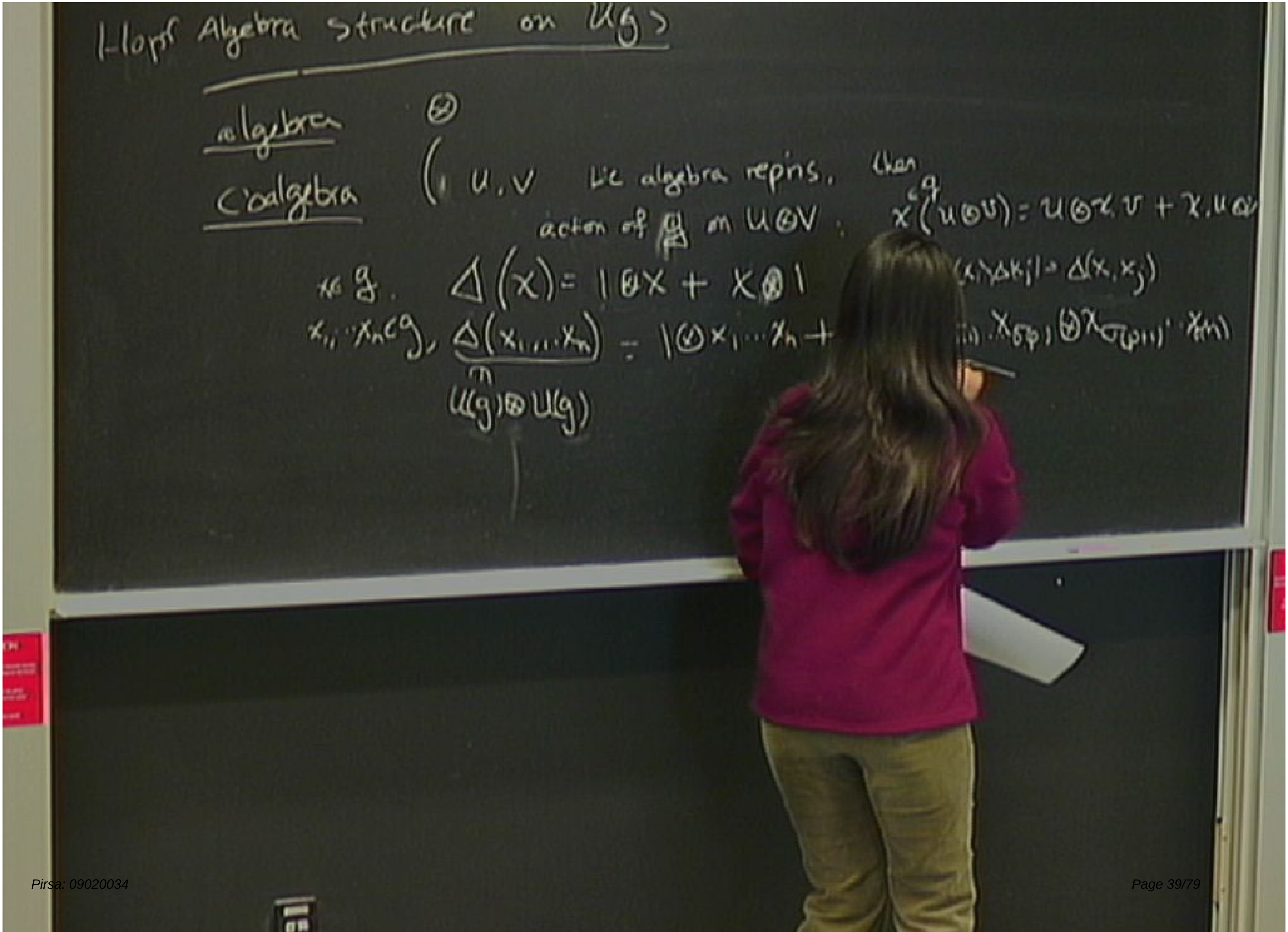
Δ(x_i)Δ(x_j) = Δ(x
x_1 ⋯ x_n ∈ g, Δ(x_1, ⋯ x_n) = 1⊗x_1 ⋯ x_n + Σ
∩
U(g)⊗U(g)
+ x_1 ⋯ x_n⊗1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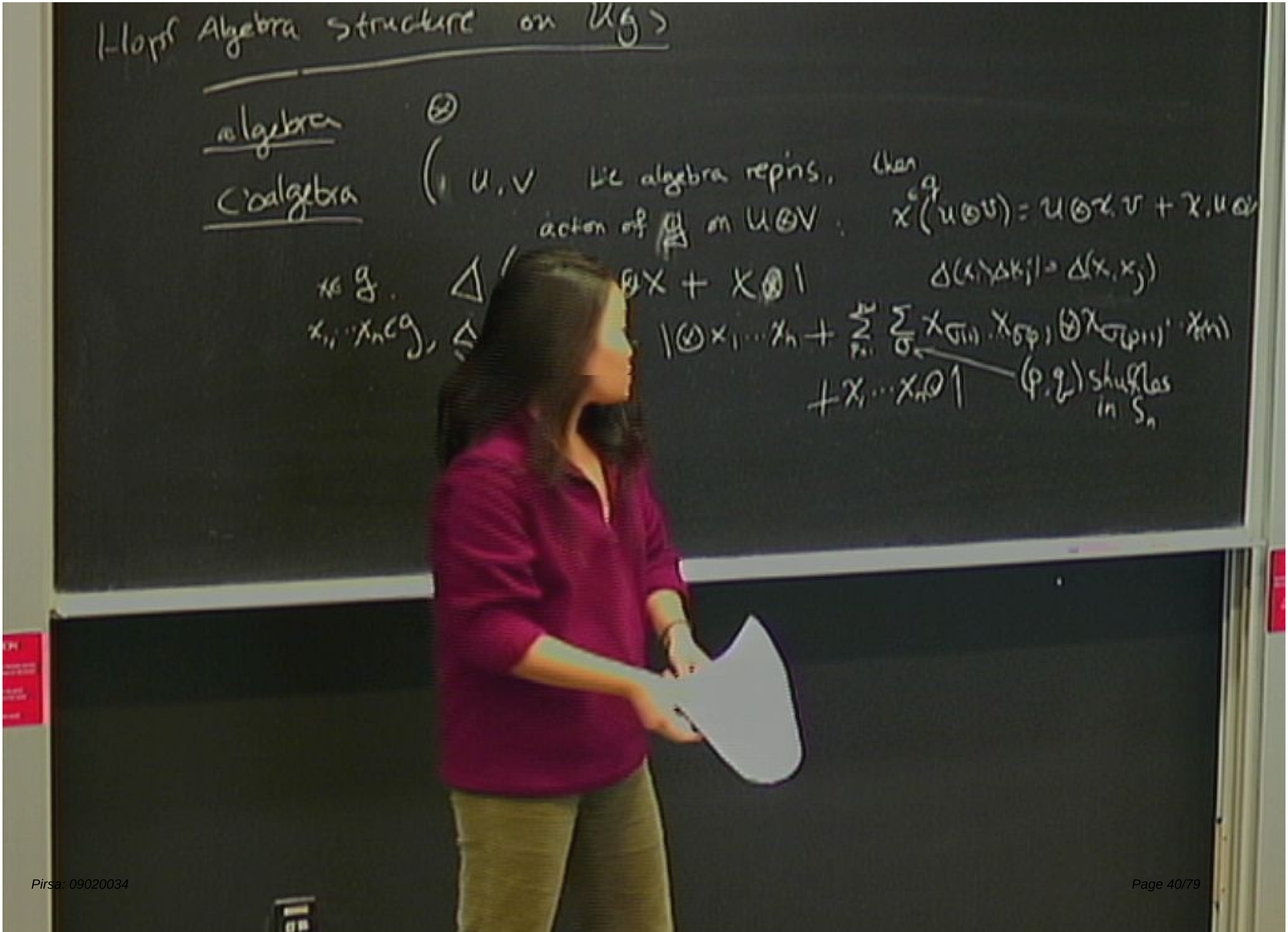
Δ(x_i)Δ(x_j) = Δ(x_i x_j)
x_1…x_n∈g, Δ(x_1…x_n) = 1⊗x_1…x_n + … x_σ(1)…x_σ(p) ⊗ x_σ(p+1)…x_σ(n)
∩
U(g)⊗U(g)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Δ(x_i)Δ(x_j) = Δ(x_i x_j)
x_1…x_n∈g, 1⊗x_1…x_n + Σ Σ x_σ(1)…x_σ(p) ⊗ x_σ(p+1)…x_σ(n)
+ x_1…x_n⊗1
(p,q) shuffles in S_n

Hopf Algebra structure on $U(g)$

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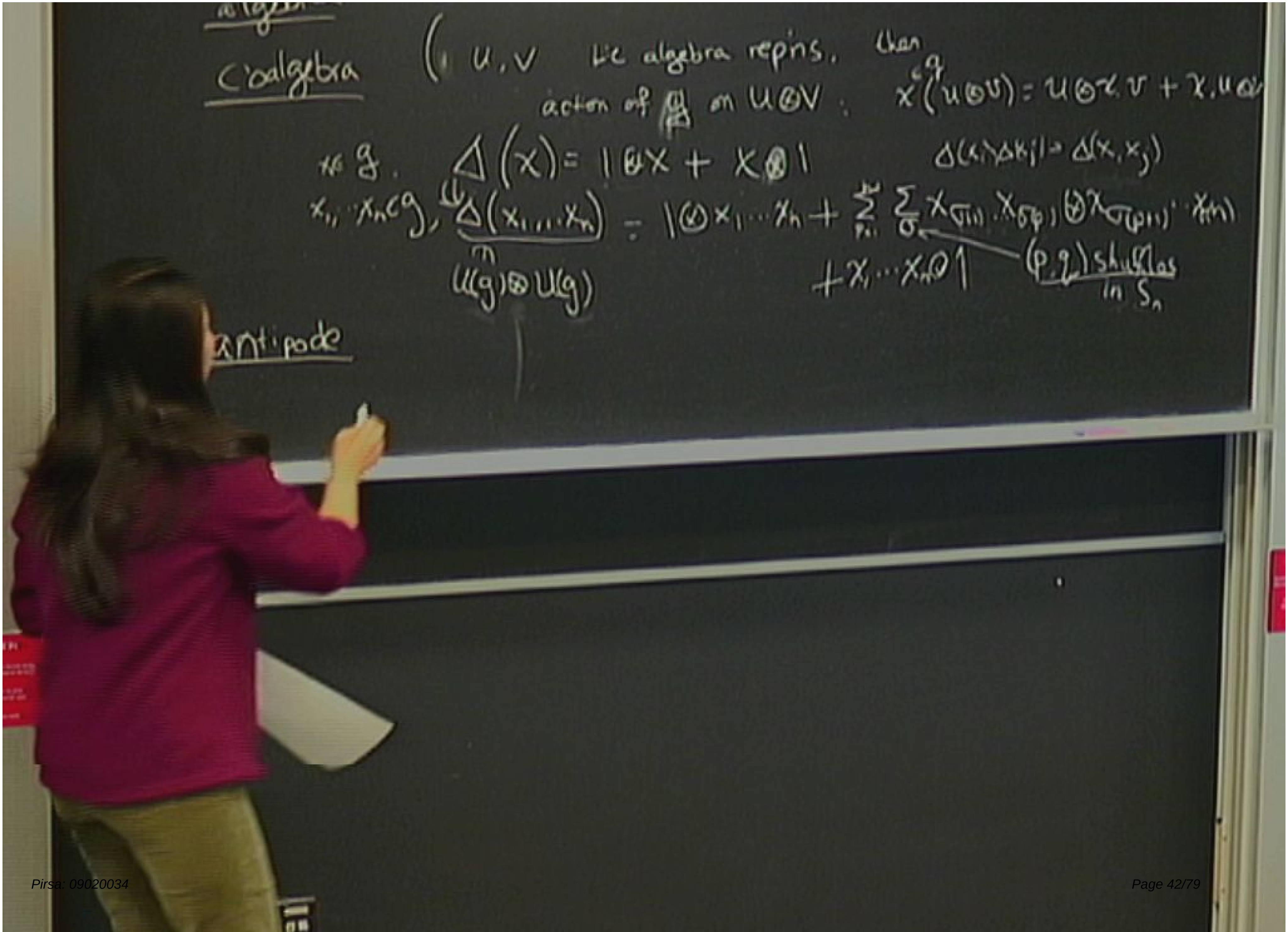
Coalgebra (U, V Lie algebra rep'ns, then

action of g on $U \otimes V$: $x^{\in g}(u \otimes v) = u \otimes x.v + x.u \otimes v$

$x \in g$. $\Delta(x) = 1 \otimes x + x \otimes 1$ $\qquad \Delta(x_i)\Delta(x_j) = \Delta(x_i x_j)$

$x_1 \cdots x_n \in g$, $\underset{\underset{U(g) \otimes U(g)}{\cap}}{\underline{\Delta(x_1 \cdots x_n)}} = 1 \otimes x_1 \cdots x_n + \sum_{p=1}^{n} \sum_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \cdots x_{\sigma(n)} + x_1 \cdots x_n \otimes 1$

(p,q) shuffles in S_n


Coalgebra
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action of $\mathfrak{g}$ on $U \otimes V$: $x \in \mathfrak{g}$, $x(u \otimes v) = u \otimes x.v + x.u \otimes v$
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$U(\mathfrak{g}) \otimes U(\mathfrak{g})$
(p,q) shuffles in S_n
antipode

Coalgebra (U, V Lie algebra rep'ns, then

action of $\mathfrak{g}$ on $U \otimes V$: $x^{\in \mathfrak{g}}(u \otimes v) = u \otimes x.v + x.u \otimes v$

$x \in \mathfrak{g}$. $\Delta(x) = 1 \otimes x + x \otimes 1$ $\Delta(x_i)\Delta(x_j) = \Delta(x_i x_j)$

$x_1, \ldots, x_n \in \mathfrak{g}$, $\underbrace{\Delta(x_1 \ldots x_n)}_{\substack{\cap \\ U(\mathfrak{g}) \otimes U(\mathfrak{g})}} = 1 \otimes x_1 \cdots x_n + \sum_{p=1}^{n} \sum_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \cdots x_{\sigma(n)} + x_1 \cdots x_n \otimes 1$

σ : (p,q) shuffles in S_n

antipode $S(x) = -x$


Coalgebra
U, V Lie algebra rep'ns. then
action of g on U⊗V: x(u⊗v) = u⊗x.v + x.u⊗
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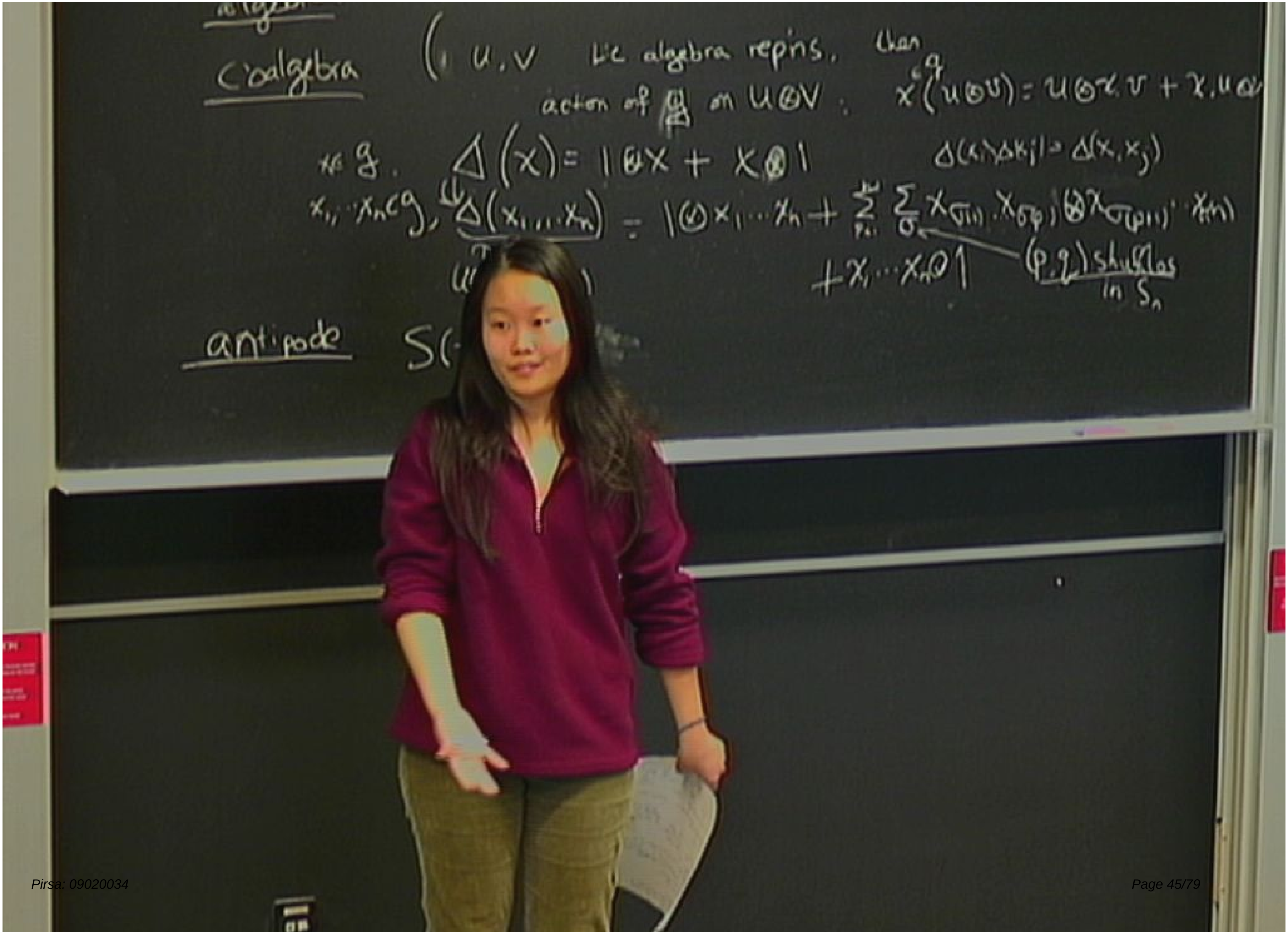
Δ(x_i)Δ(x_j) = Δ(x_i x_j)
x_1,...,x_n ∈ g, Δ(x_1...x_n) = 1⊗x_1...x_n + Σ Σ_σ x_σ(1)...x_σ(p) ⊗ x_σ(p+1)...x_σ(n)
+ x_1...x_n ⊗ 1
(p,q) shuffles in S_n
antipode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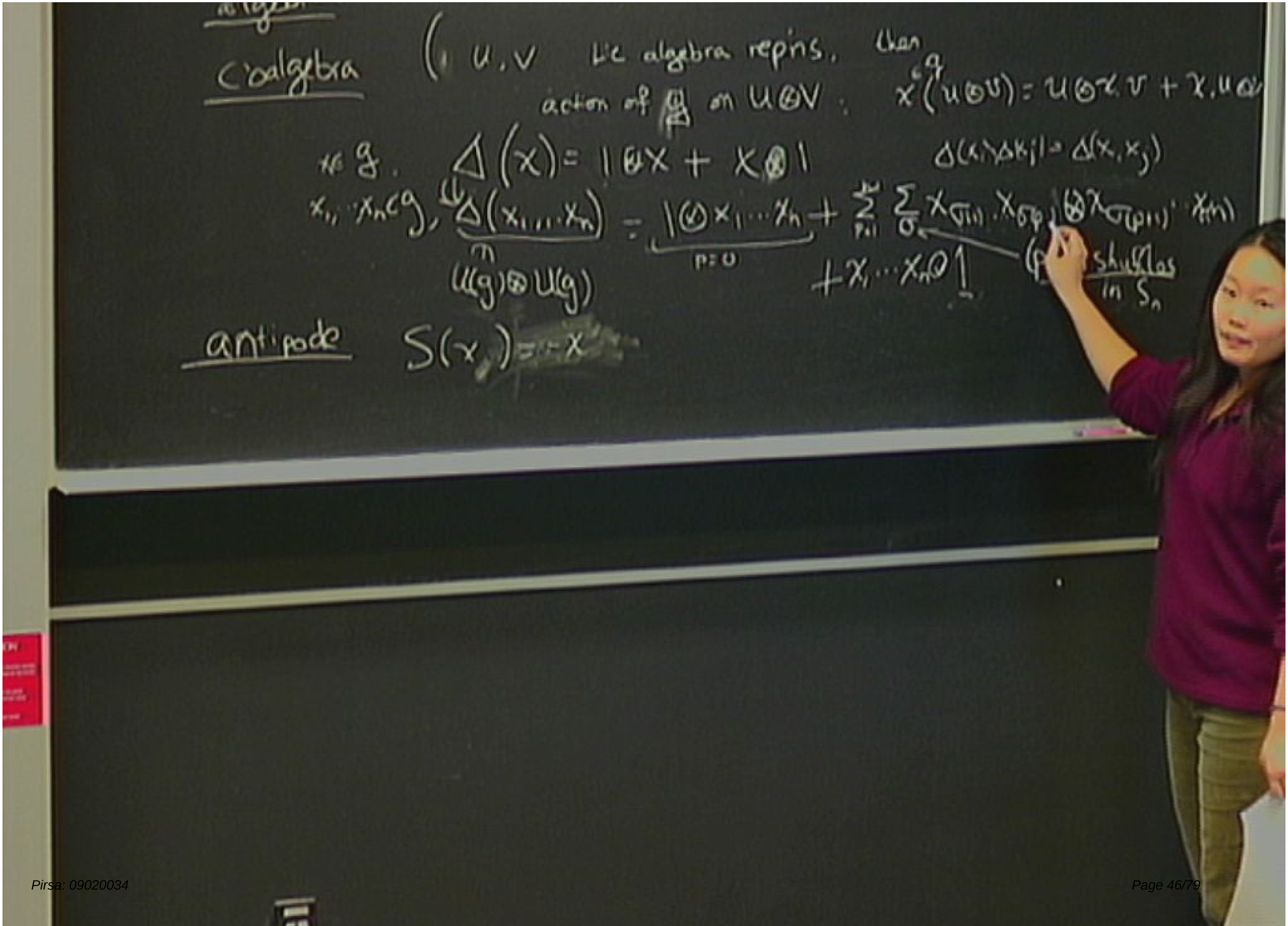
Δ(x_i)Δ(x_j) = Δ(x_i x_j)
x_1, ..., x_n ∈ g, Δ(x_1 ... x_n) = 1⊗x_1 ... x_n + Σ Σ x_σ(1) ... x_σ(p) ⊗ x_σ(p+1) ... x_σ(n)
+ x_1 ... x_n ⊗ 1
(p,q) shuffles in S_n
antipode



Coalgebra
u, v Lie algebra rep'ns, then
action of g on U⊗V: x(u⊗v) = u⊗x.v + x.u⊗v
x∈g. Δ(x) = 1⊗x + x⊗1
Δ([x_i, x_j]) = [Δ(x_i), Δ(x_j)]
x_1, ..., x_n ∈ g, Δ(x_1 ... x_n) = 1⊗x_1 ... x_n + Σ Σ x_σ(1) ... x_σ(p) ⊗ x_σ(p+1) ... x_σ(n) + x_1 ... x_n ⊗ 1
U(g)⊗U(g)
p=0
shuffles in S_n
antipode
S(x) = -x

algebra

Coalgebra $\left(u, v \right.$ Lie algebra rep'ns, then

action of $\mathfrak{g}$ on $U \otimes V$: $x^{\in \mathfrak{g}}(u \otimes v) = u \otimes x.v + x.u \otimes$

$x \in \mathfrak{g}$. $\Delta(x) = 1 \otimes x + x \otimes 1$ $\quad \Delta(x_i)\Delta(x_j) = \Delta(x_i x_j)$

$x_1, \dots x_n \in \mathfrak{g}$, $\underset{\underset{U(\mathfrak{g}) \otimes U(\mathfrak{g})}{\cap}}{\Delta(x_1, \dots x_n)} = \underbrace{1 \otimes x_1 \cdots x_n}_{p=0} + \sum_{p \geq 1}^{n} \sum_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \cdots x_{\sigma(n)} + x_1 \cdots x_n \otimes 1$

σ: (p,q) shuffles in S_n

antipode $S(x) = -x$

algebra

Coalgebra

U, V Lie algebra rep'ns, then

action of $\mathfrak{g}$ on $U\otimes V$: $\quad x(u\otimes v) = u\otimes x.v + x.u\otimes v$, $\quad x \in \mathfrak{g}$

$x \in \mathfrak{g}$. $\quad \Delta(x) = 1\otimes x + x\otimes 1 \qquad \Delta(x_i)\Delta(x_j) = \Delta(x_i x_j)$

$$x_1\cdots x_n \in \mathfrak{g}, \quad \underbrace{\Delta(x_1\cdots x_n)}_{\in\, U(\mathfrak{g})\otimes U(\mathfrak{g})} = \underbrace{1\otimes x_1\cdots x_n}_{p=0} + \sum_{p=1}^{n}\sum_{\sigma} x_{\sigma(1)}\cdots x_{\sigma(p)}\otimes x_{\sigma(p+1)}\cdots x_{\sigma(n)} + x_1\cdots x_n\otimes 1$$

σ: (p,q) shuffles in S_n

antipode S

$S(x) = -x$

$S(x_1\cdots x_n) = (-1)^n x_n\cdots x_1$

Coalgebra (u, v Lie algebra rep'ns, then

action of $\mathfrak{g}$ on $u \otimes v$: $x^{\in \mathfrak{g}}(u \otimes v) = u \otimes x.v + x.u \otimes v$

$x \in \mathfrak{g}$. $\Delta(x) = 1 \otimes x + x \otimes 1$ $\quad \Delta(x_i)\Delta(x_j) = \Delta(x_i x_j)$

$x_1, \dots, x_n \in \mathfrak{g}$, $\underbrace{\Delta(x_1 \cdots x_n)}_{\substack{\cap \\ U(\mathfrak{g}) \otimes U(\mathfrak{g})}} = \underbrace{1 \otimes x_1 \cdots x_n}_{p=0} + \sum_{p=1}^{n-1} \sum_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \cdots x_{\sigma(n)}$

$+ x_1 \cdots x_n \otimes 1$ $\quad$ (p,q) shuffles in S_n

antipode $S(x) = -x$

S algebra antimorphism $\boxed{S(x_1 \cdots x_n) = (-1)^n x_n \cdots x_1}$

Hopf Algebra structure on $U(g)$

algebra $\otimes$

Coalgebra U, V Lie algebra rep'ns, then

action of g on $U\otimes V$: $x\cdot(u\otimes v) = u\otimes x.v + x.u\otimes v$

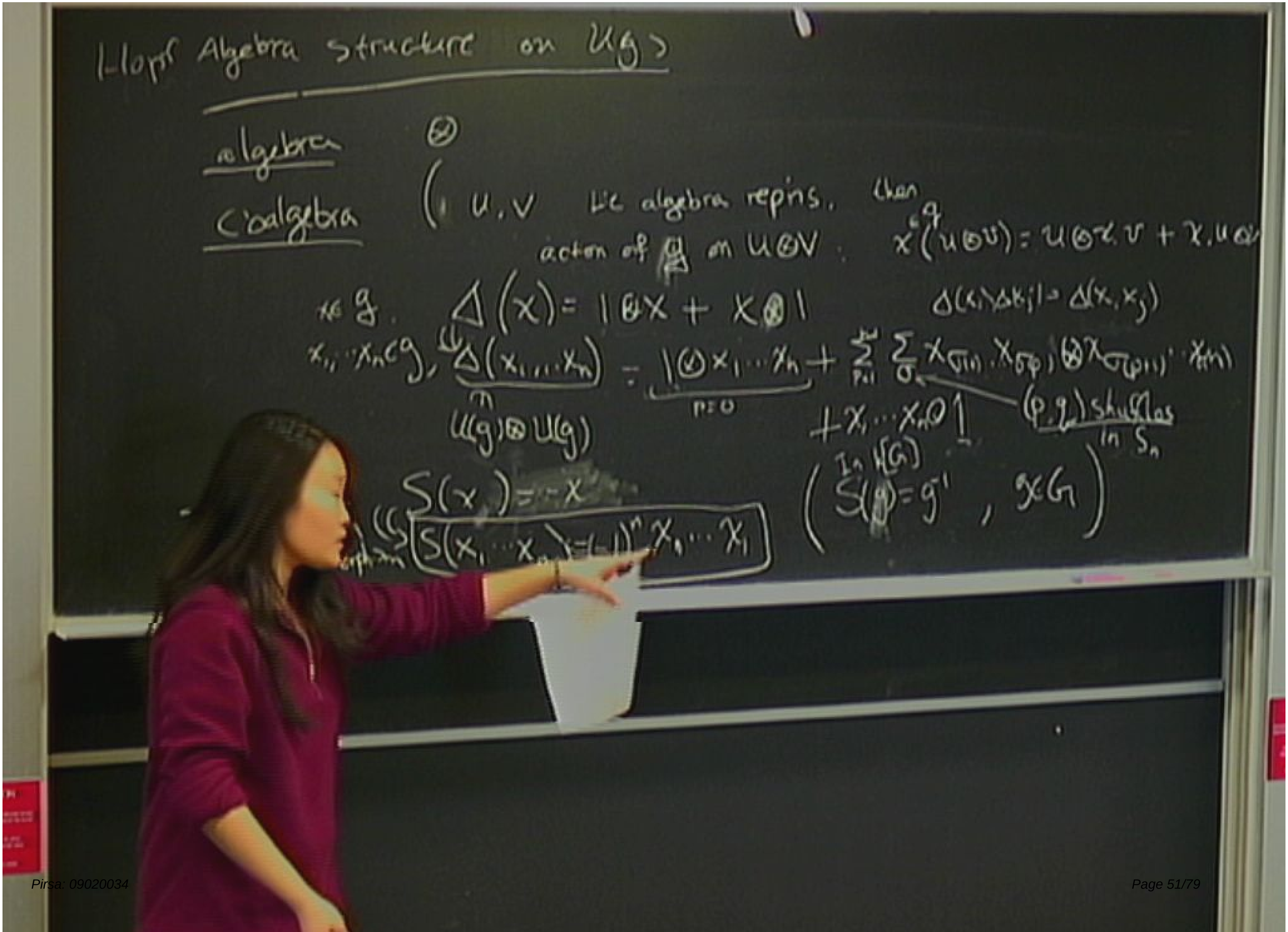
$x\in g$. $\Delta(x) = 1\otimes x + x\otimes 1$ $\quad \Delta(x_i)\Delta(x_j) = \Delta(x_i x_j)$

$x_1,\dots,x_n\in g$, $\underbrace{\Delta(x_1\cdots x_n)}_{\in\, U(g)\otimes U(g)} = \underbrace{1\otimes x_1\cdots x_n}_{p=0} + \sum_{p=1}^{n}\sum_{\sigma} x_{\sigma(1)}\cdots x_{\sigma(p)}\otimes x_{\sigma(p+1)}\cdots x_{\sigma(n)} + x_1\cdots x_n\otimes 1$

σ: (p,q) shuffles in S_n

antipode $S(x) = -x$

S algebra antihomomorphism $\boxed{S(x_1\cdots x_n) = (-1)^n x_n\cdots x_1}$


Hopf Algebra structure on U(g)
algebra ⊗
Coalgebra
U, V Lie algebra rep'ns. then
action of g on U⊗V : x(u⊗v) = u⊗x.v + x.u⊗v
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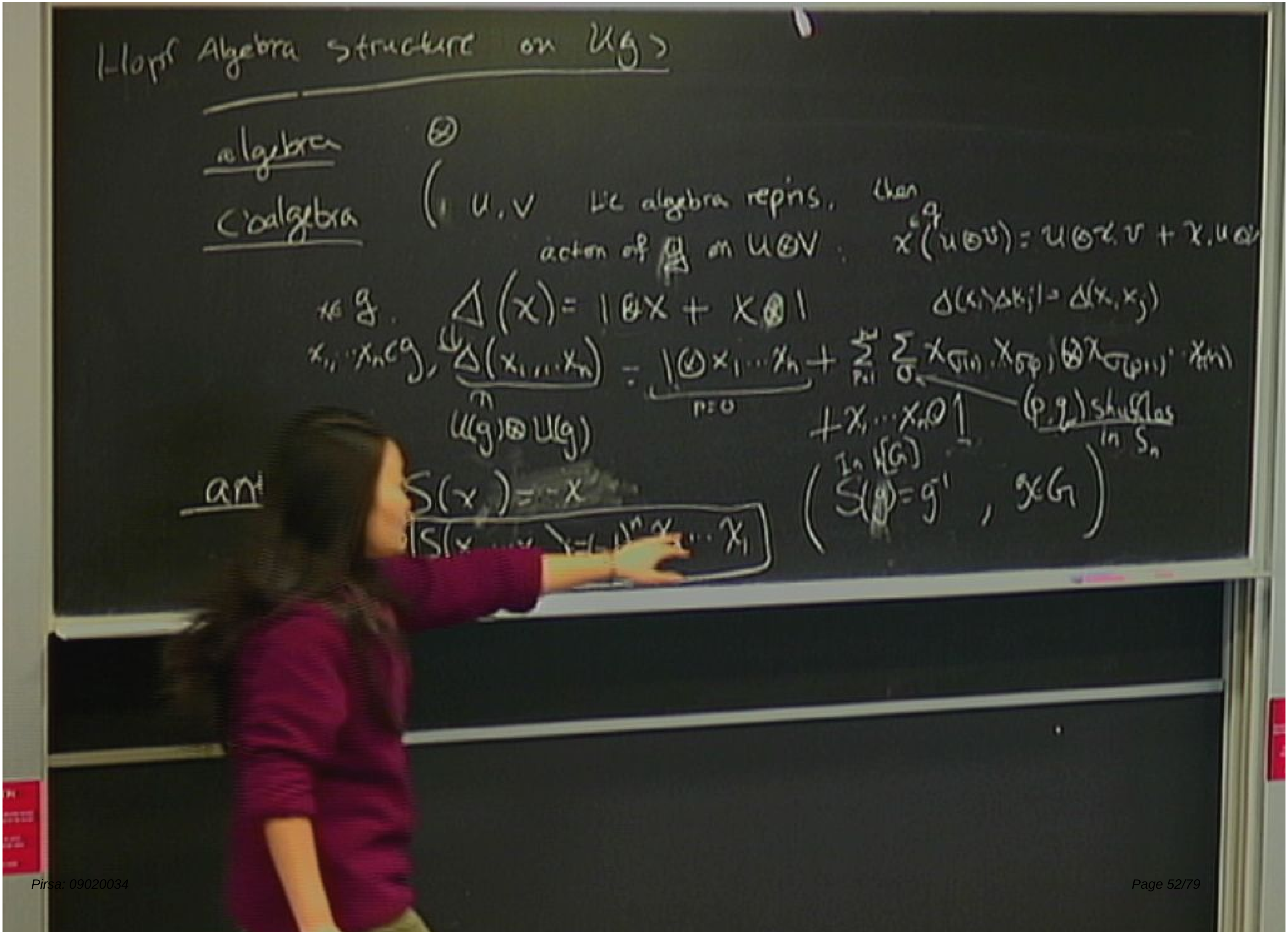
Δ(x_i)Δ(x_j) = Δ(x_i x_j)
x_1 ⋯ x_n ∈ g, Δ(x_1 ⋯ x_n) = 1⊗x_1 ⋯ x_n + Σ_{p≥1} Σ_σ x_σ(1) ⋯ x_σ(p) ⊗ x_σ(p+1) ⋯ x_σ(n) + x_1 ⋯ x_n ⊗ 1
U(g)⊗U(g)
p=0
(p,q) shuffles in S_n
S(x) = −x
S(x_1 ⋯ x_n) = (−1)^n x_n ⋯ x_1
(In k(G) S(g) = g^{-1}, g∈G)


Hopf Algebra structure on U(g)
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Coalgebra
U, V Lie algebra rep'ns, then
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Δ(x_i)Δ(x_j) = Δ(x_i x_j)
x_1,…,x_n∈g, Δ(x_1…x_n) = 1⊗x_1…x_n + Σ_{p=1}^{n} Σ_σ x_σ(1)…x_σ(p) ⊗ x_σ(p+1)…x_σ(n) + x_1…x_n⊗1
p=0
U(g)⊗U(g)
(p,q) shuffles in S_n
S(x) = −x
S(x_1…x_n) = (−1)^n x_n…x_1
In k(G)
S(g) = g^{-1}, g∈G

Hopf Algebra structure on $U(\mathfrak{g})$

algebra $\otimes$

Coalgebra (U, V Lie algebra rep'ns, then

action of $\mathfrak{g}$ on $U \otimes V$: $x^{\mathfrak{g}}(u \otimes v) = u \otimes x.v + x.u \otimes v$

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σ: (p,q) shuffles in S_n

antipode $S(x) = -x$

$$\boxed{S(x_1 \cdots x_n) = (-1)^n x_n \cdots x_1}$$

(In $k[G]$: $S(g) = g^{-1}$, $g \in G$)

$L(A) = A$ as v. spaces

$[x, y] := xy - yx$

$$\mathrm{Hom}_{Lie}(\mathfrak{g}, L(A)) \cong \mathrm{Hom}_{Alg.}(U(\mathfrak{g}), A) \quad \text{for any algebra } A$$

Hopf Algebra structure on $U\mathfrak{g}$ >

algebra $\otimes$

Coalgebra (U, V Lie algebra rep'ns. then

action of $\mathfrak{g}$ on $U \otimes V$: $x^{\mathfrak{g}}(u \otimes v) = u \otimes x.v + x.u \otimes v$

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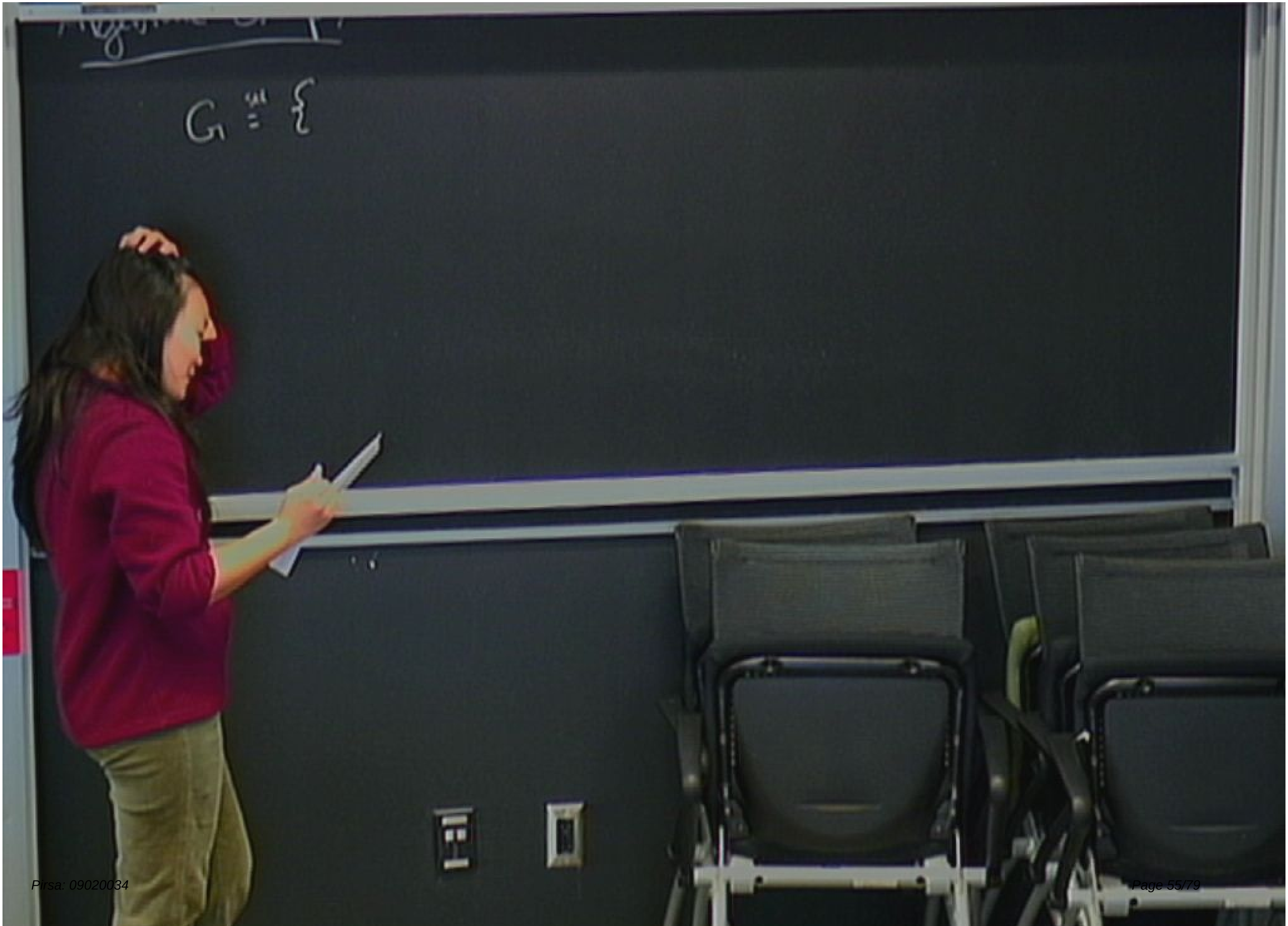
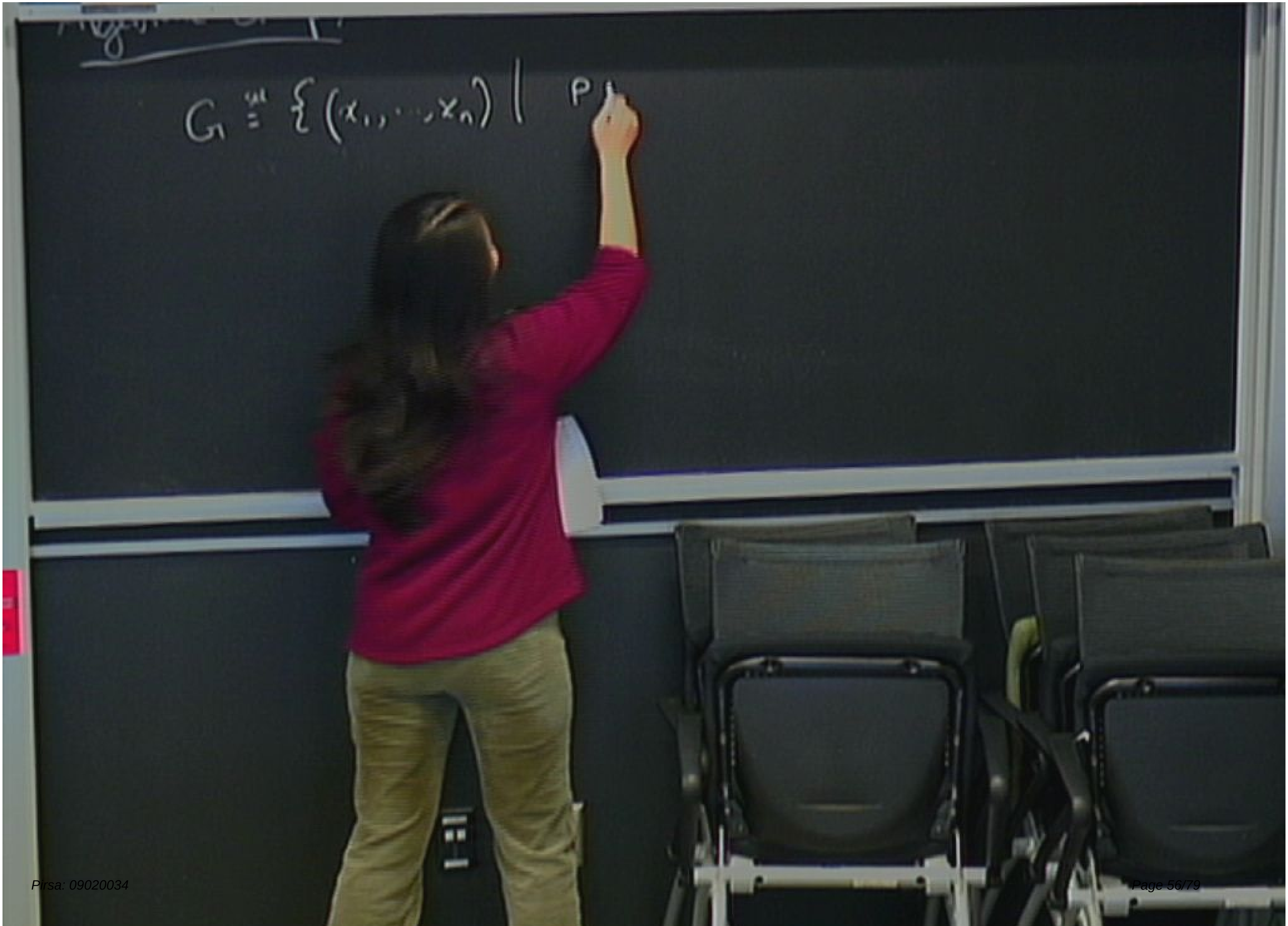
σ: (p, q) shuffles in S_n

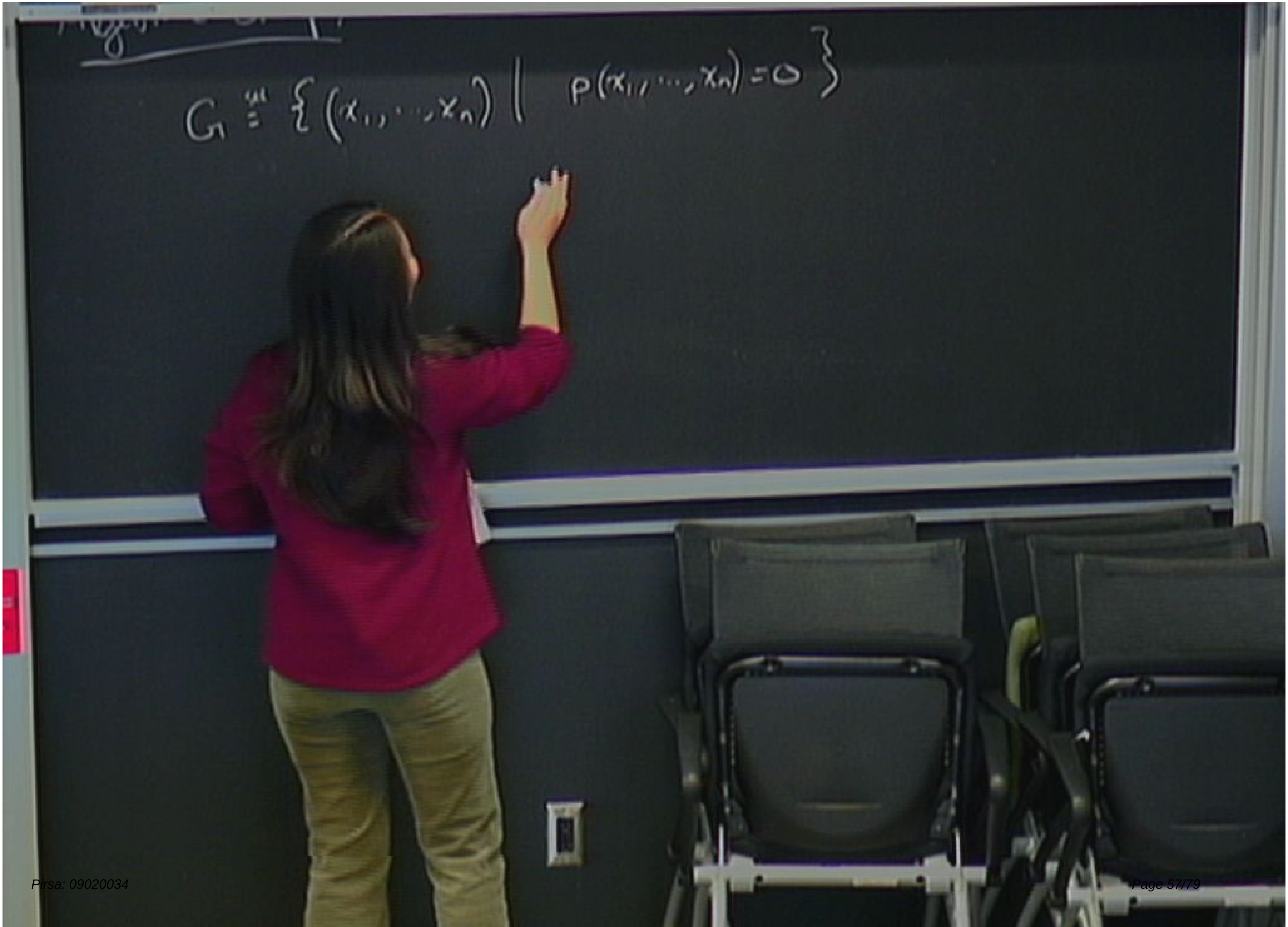
antipode $S(x) = -x$

S algebra antimorphism $\Leftarrow$ $\boxed{S(x_1 \cdots x_n) = (-1)^n x_n \cdots x_1}$

(In $k[G]$: $S(g) = g^{-1}$, $g \in G$)


G ≅ {


$G := \{(x_1, \dots, x_n) \mid P$


$G \stackrel{def}{=} \{ (x_1, \dots, x_n) \mid p(x_1, \dots, x_n) = 0 \}$

$$G \stackrel{\text{def}}{=} \{ (x_1, \dots, x_n) \mid p_j(x_1, \dots, x_n) = 0 \}$$

$$G \overset{\text{def}}{=} \{ (x_1, \dots, x_n) \mid p_j(x_1, \dots, x_n) = 0 \}$$

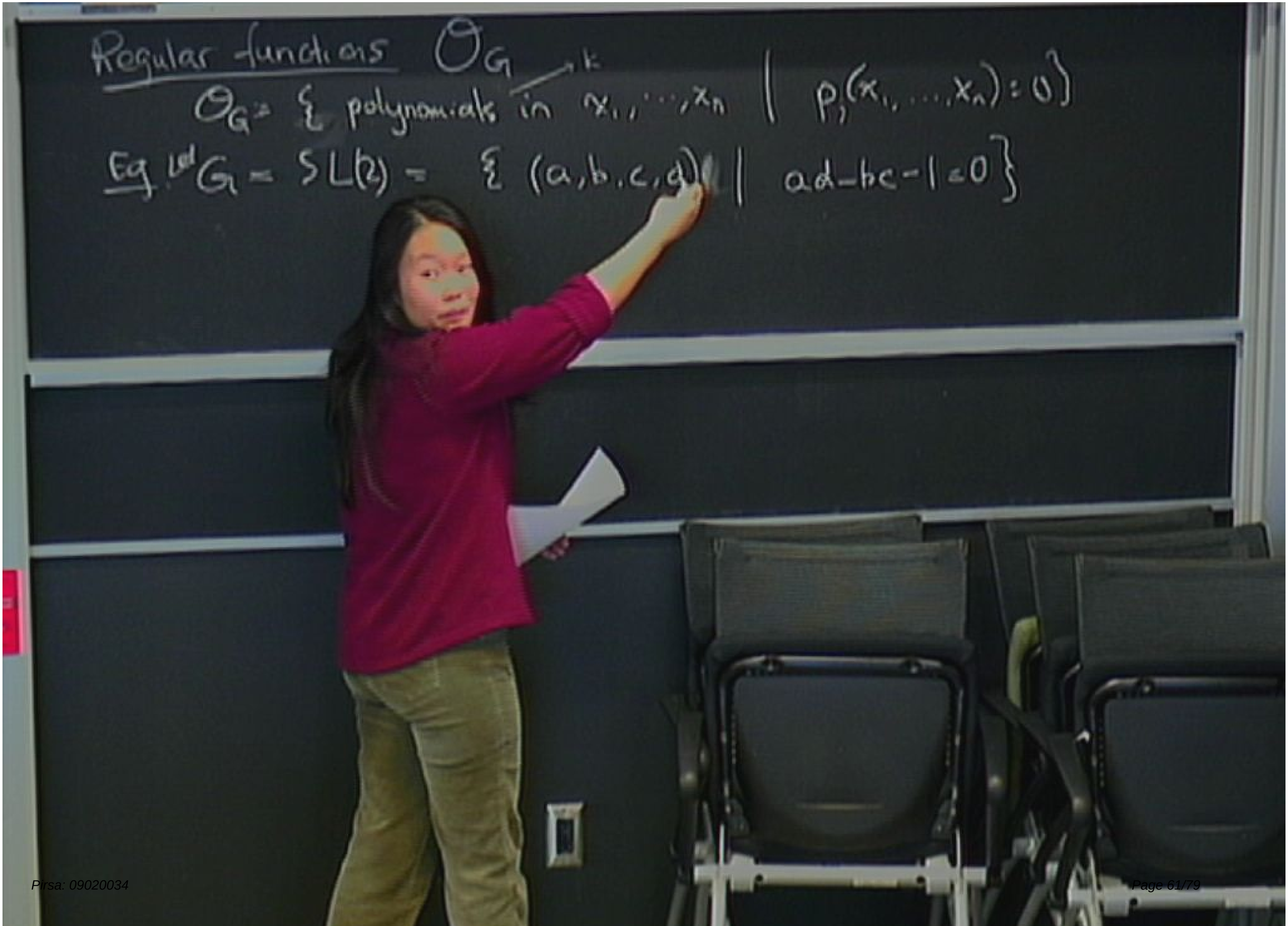
Regular functions $\mathcal{O}_G$

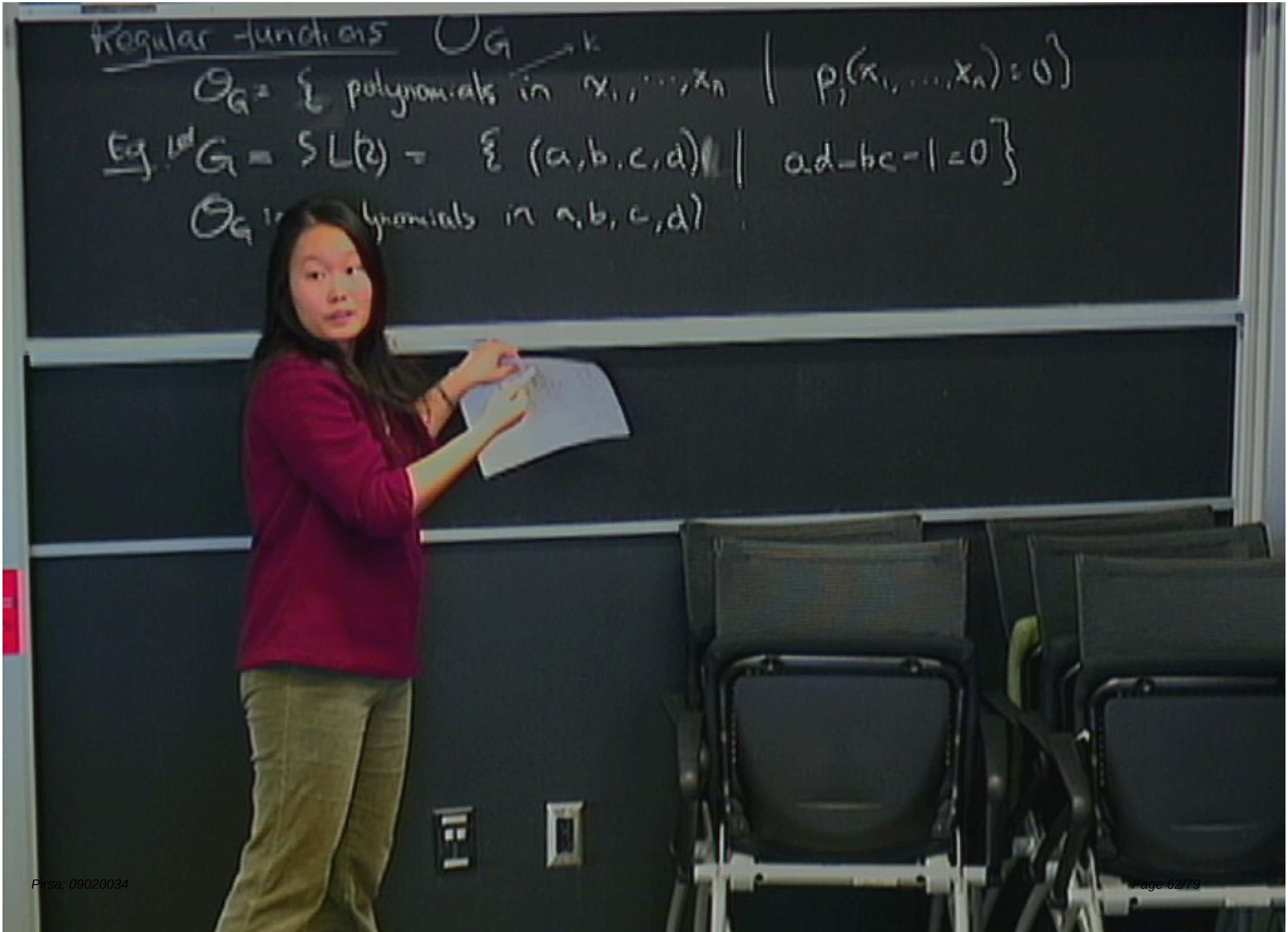
$$\mathcal{O}_G = \{$$

$$G := \{ (x_1, \dots, x_n) \mid p_j(x_1, \dots, x_n) = 0 \}$$

<u>Regular functions</u> $\mathcal{O}_G \to k$

$$\mathcal{O}_G = \{ \text{polynomials in } x_1, \dots, x_n \mid p(x_1, \dots, x_n) = 0 \}$$


Regular functions O_G
O_G = { polynomials in x_1, …, x_n | p_j(x_1, …, x_n) = 0 }
k
Eg 1st G = SL(2) = { (a,b,c,d) | ad-bc-1=0 }


Regular functions O_G
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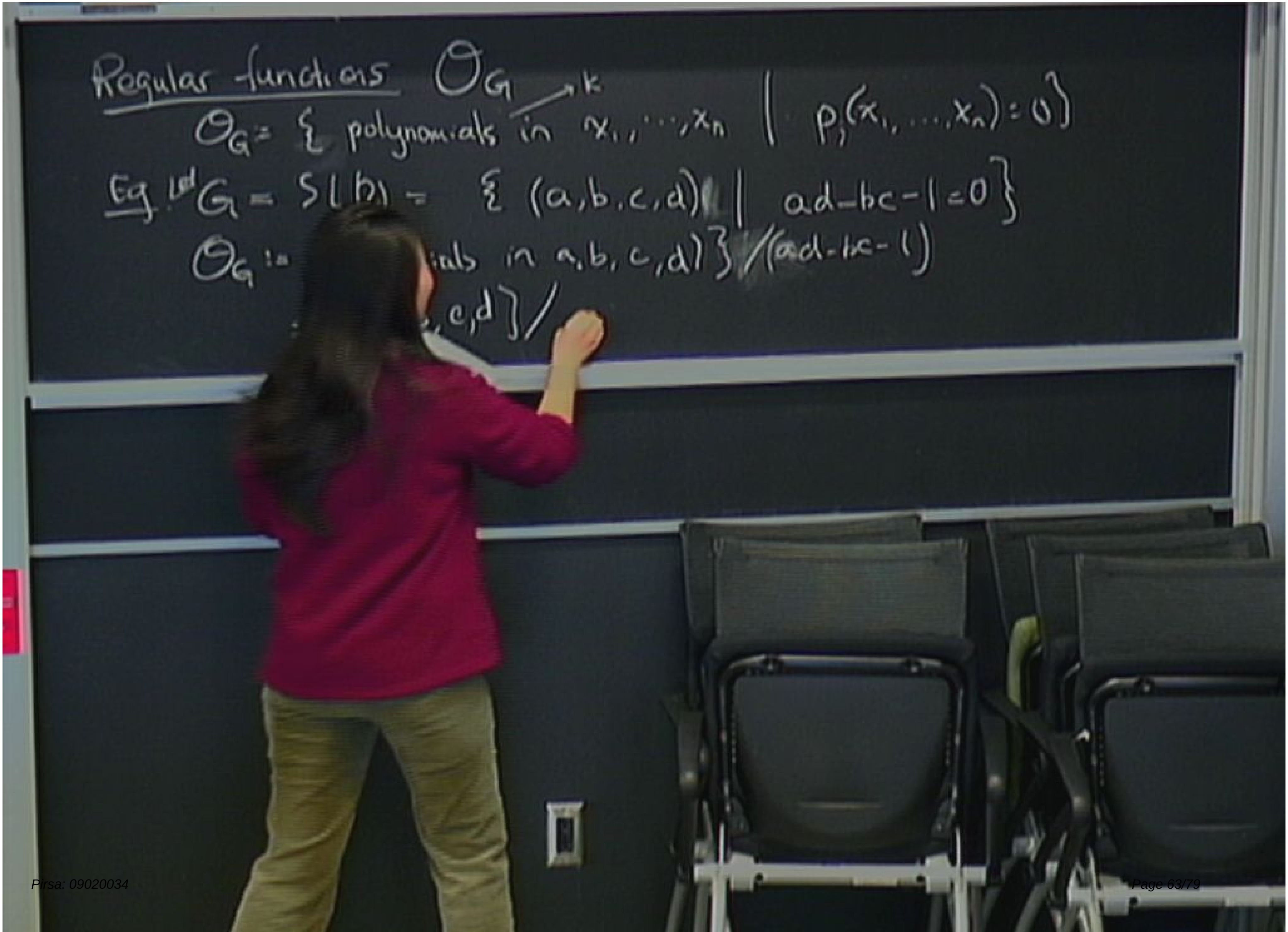
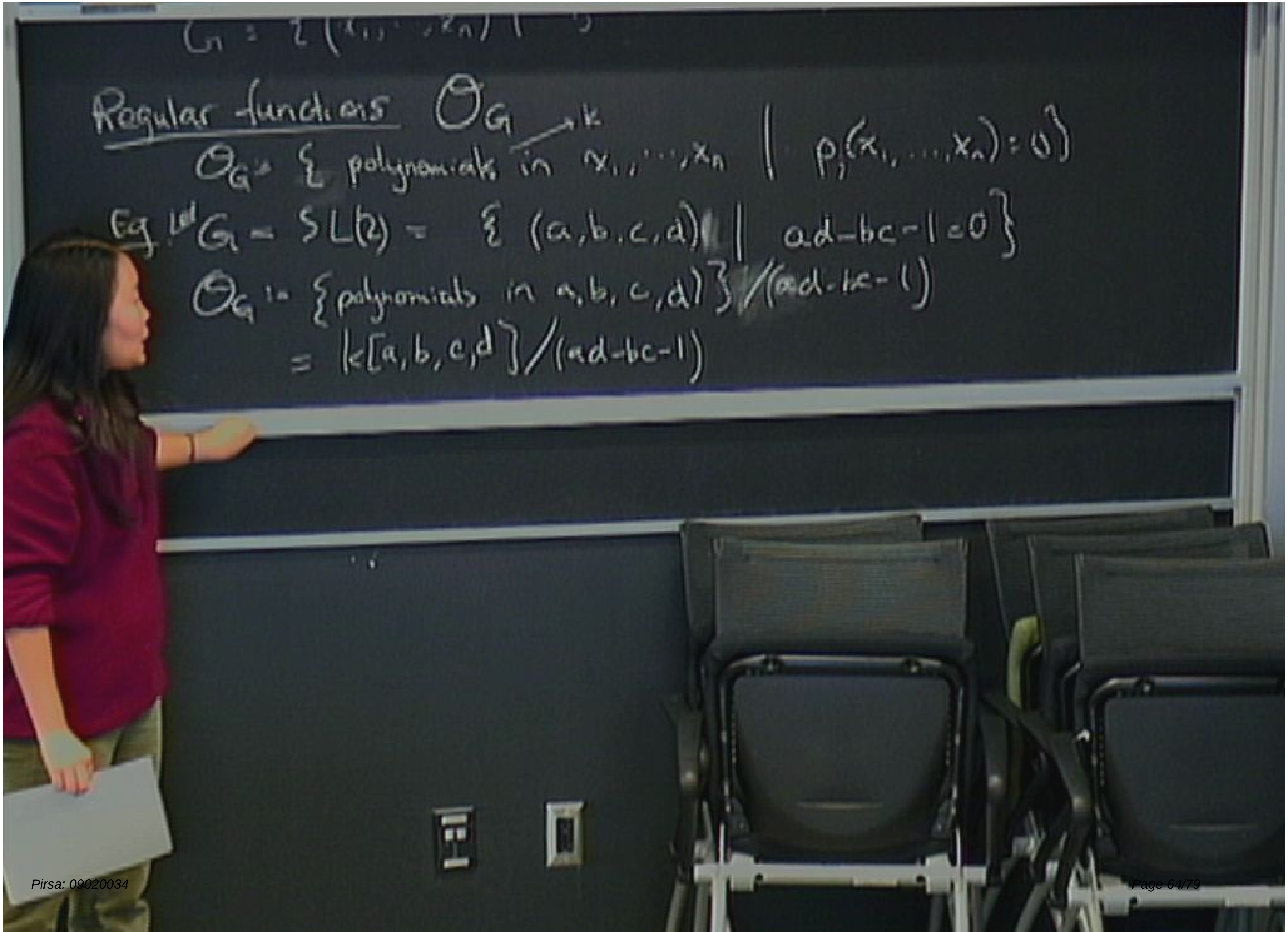
O_G = { polynomials in x_1, ..., x_n | p_j(x_1, ..., x_n) = 0 }
Eg. 1st G = SL(2) = { (a,b,c,d) | ad-bc-1=0 }
O_G
ynomials in a,b,c,d }


Regular functions $O_G \to k$
$O_G = \{$ polynomials in $x_1, \dots, x_n \mid p_i(x_1, \dots, x_n) = 0 \}$
Eg. 1st $G = SL(2) = \{ (a,b,c,d) \mid ad-bc-1=0 \}$
$O_G :=$ ials in $a,b,c,d\} / (ad-bc-1)$
$c,d] /$


Regular functions O_G
O_G = { polynomials in x_1, ..., x_n → k | p_i(x_1, ..., x_n) = 0 }
Eg. G = SL(2) = { (a, b, c, d) | ad − bc − 1 = 0 }
O_G := {polynomials in a, b, c, d} / (ad − bc − 1)
= k[a, b, c, d] / (ad − bc − 1)

$$G := \{ (x_1, \dots, x_n) \mid p_j(x_1, \dots, x_n) = 0 \}$$

Regular functions $\mathcal{O}_G \to k$

$$\mathcal{O}_G = \{ \text{polynomials in } x_1, \dots, x_n \} / (p_j(x_1, \dots, x_n) = 0)$$

Eg. 1st $G = SL(2) = \{ (a,b,c,d) \mid ad - bc - 1 = 0 \}$

$$\mathcal{O}_G := \{ \text{polynomials in } a, b, c, d \} / (ad - bc - 1)$$

$$= k[a,b,c,d] / (ad - bc - 1)$$

Hopf Algebra structure on $\mathcal{U}(g)$

algebra $\otimes$, $\eta : k \longrightarrow \mathcal{U}(g)$, $\eta(1) \longmapsto 1$

Coalgebra (U, V Lie algebra rep'ns, then action of g on $U \otimes V$: $x \in g$, $x(u \otimes v) = u \otimes x.v + x.u \otimes$

$x \in g$. $\Delta(x) = 1 \otimes x + x \otimes 1$ $\quad \Delta(x_i)\Delta(x_j) = \Delta(x_i x_j)$

$x_1 \cdots x_n \in g$, $\underset{\substack{\cap \\ \mathcal{U}(g) \otimes \mathcal{U}(g)}}{\Delta(x_1 \cdots x_n)} = \underbrace{1 \otimes x_1 \cdots x_n}_{p=0} + \sum_{p=1} \sum_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \cdots x_{\sigma(n)}$

$\varepsilon(x) = 0$

$\varepsilon(1) = 1$

$+ x_1 \cdots x_n \otimes 1$ $\quad$ (p,q) shuffles in S_n

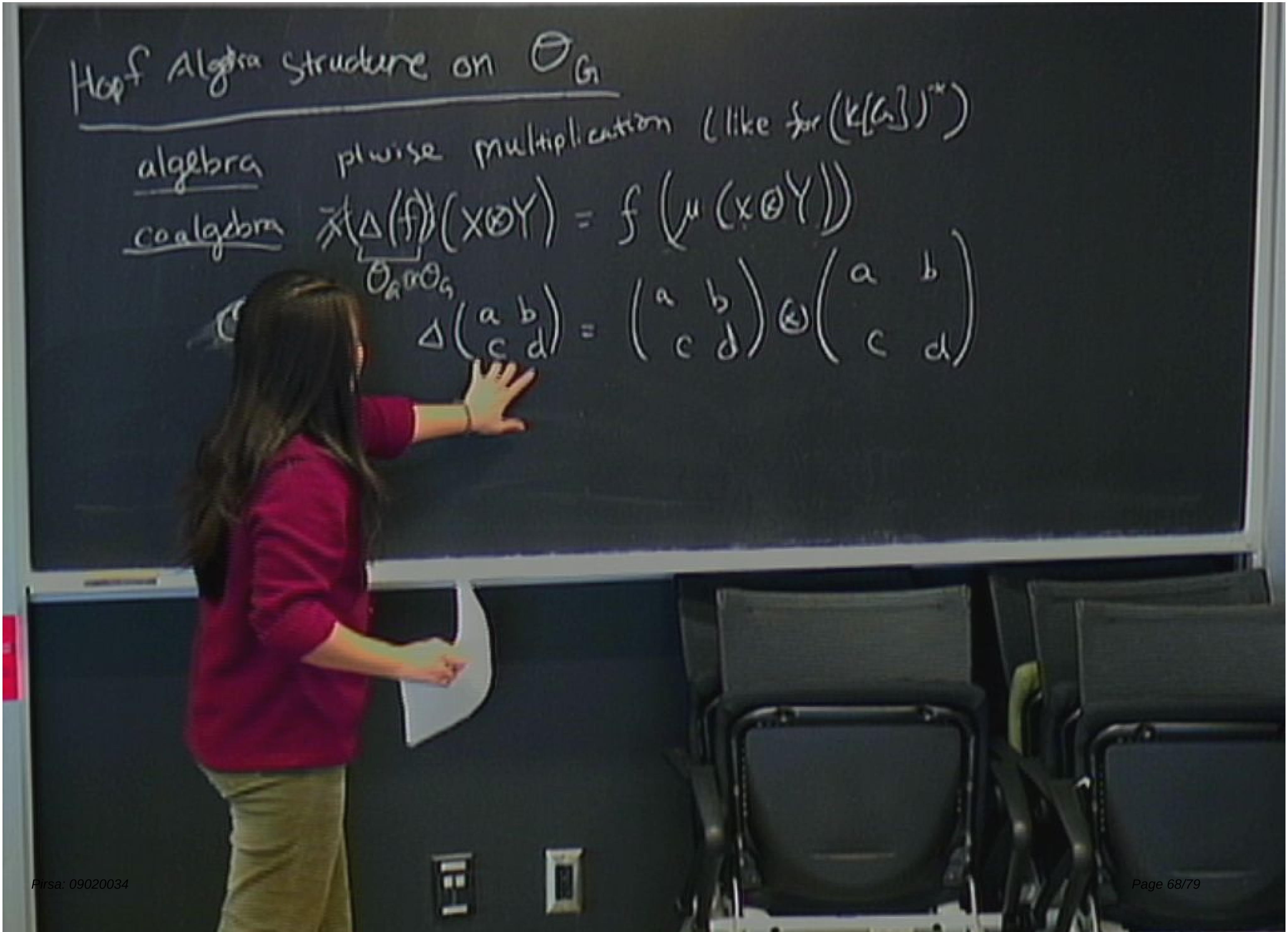
antipode $S(x) = -x$

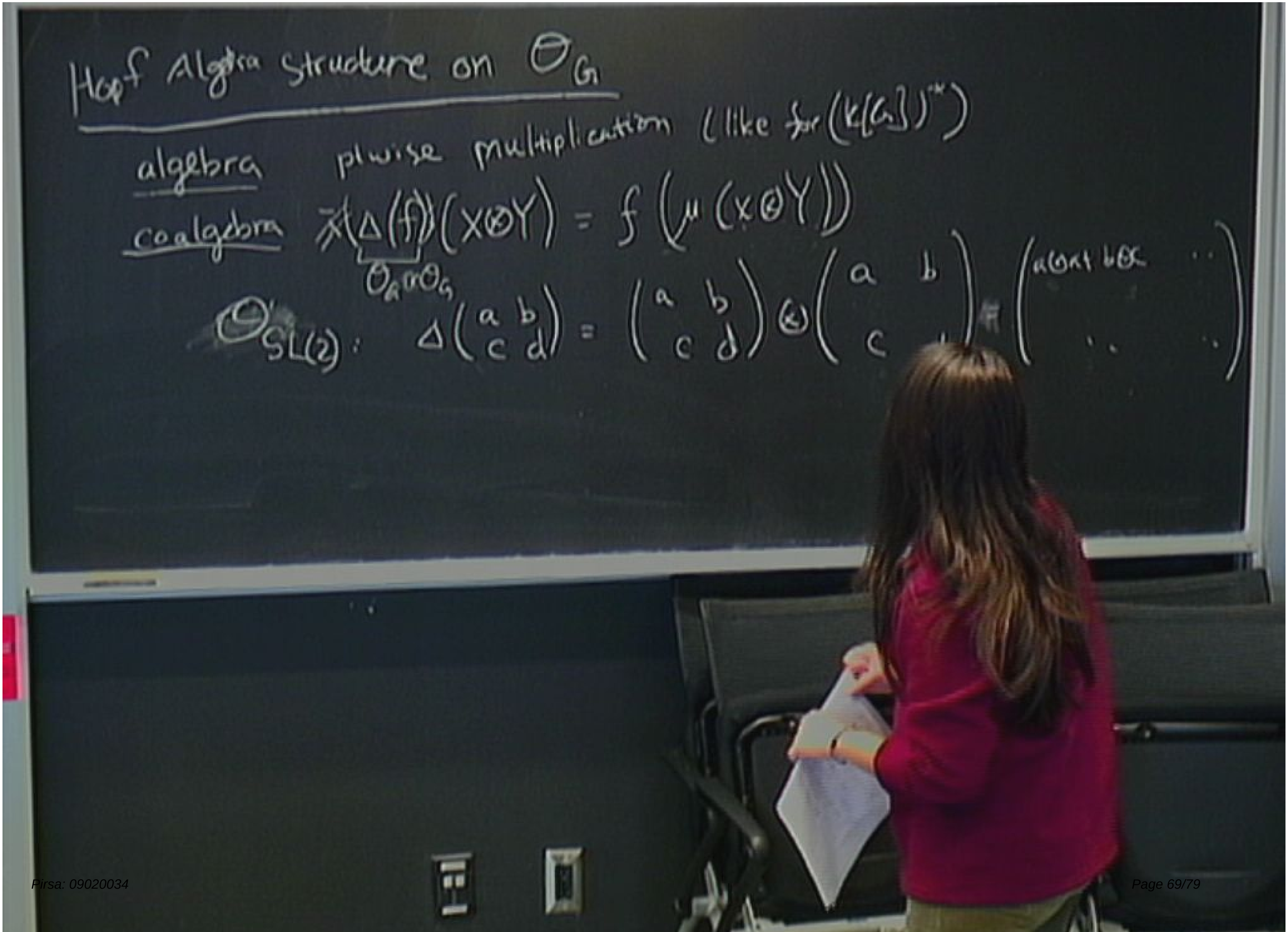
S algebra antimorphism $\boxed{S(x_1 \cdots x_n) = (-1)^n x_n \cdots x_1}$ $\quad$ (In $k[G]$ $S(g) = g^{-1}$, $g \in G$)

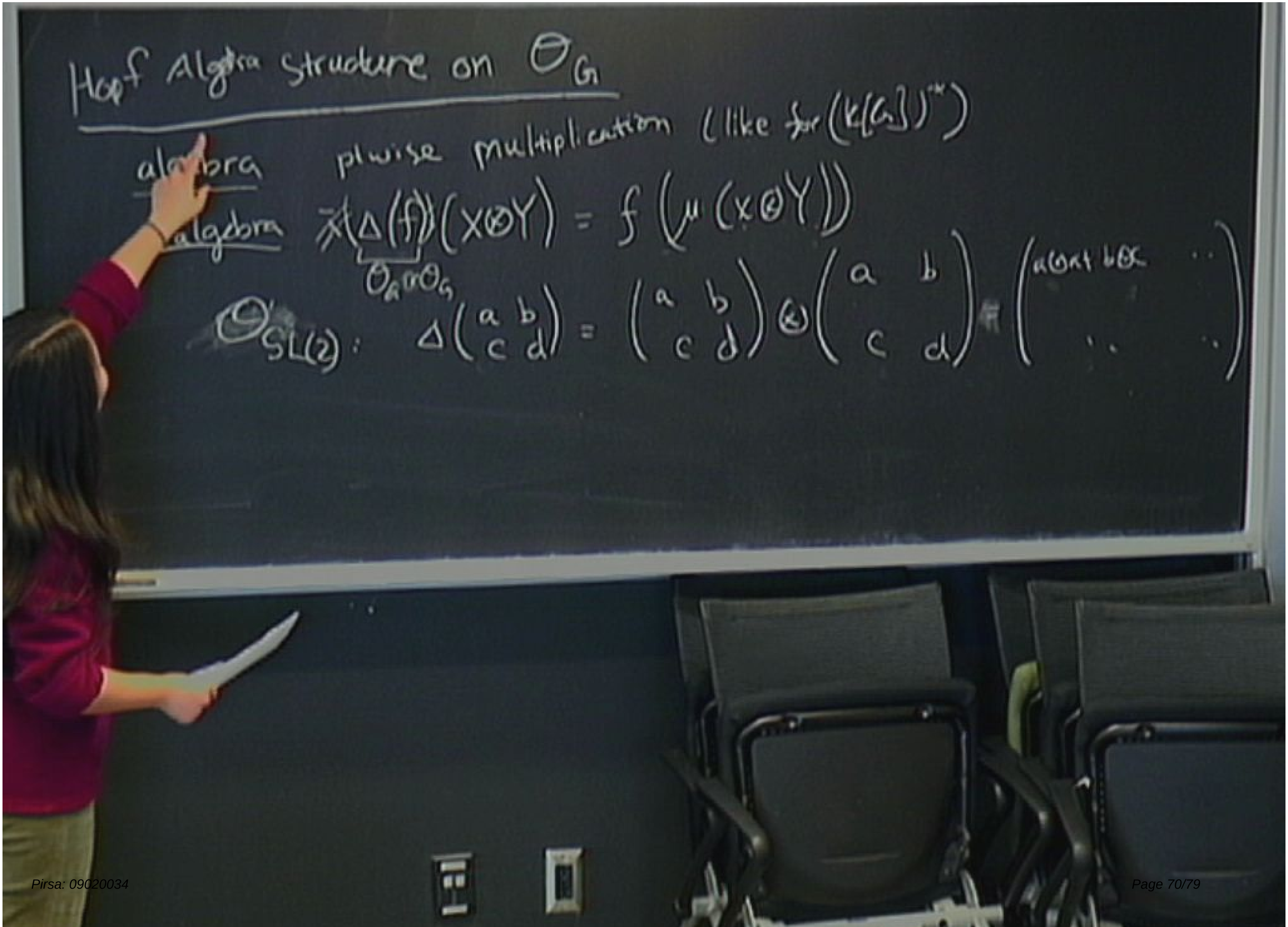
Hopf Algebra structure on $\mathcal{O}_G$

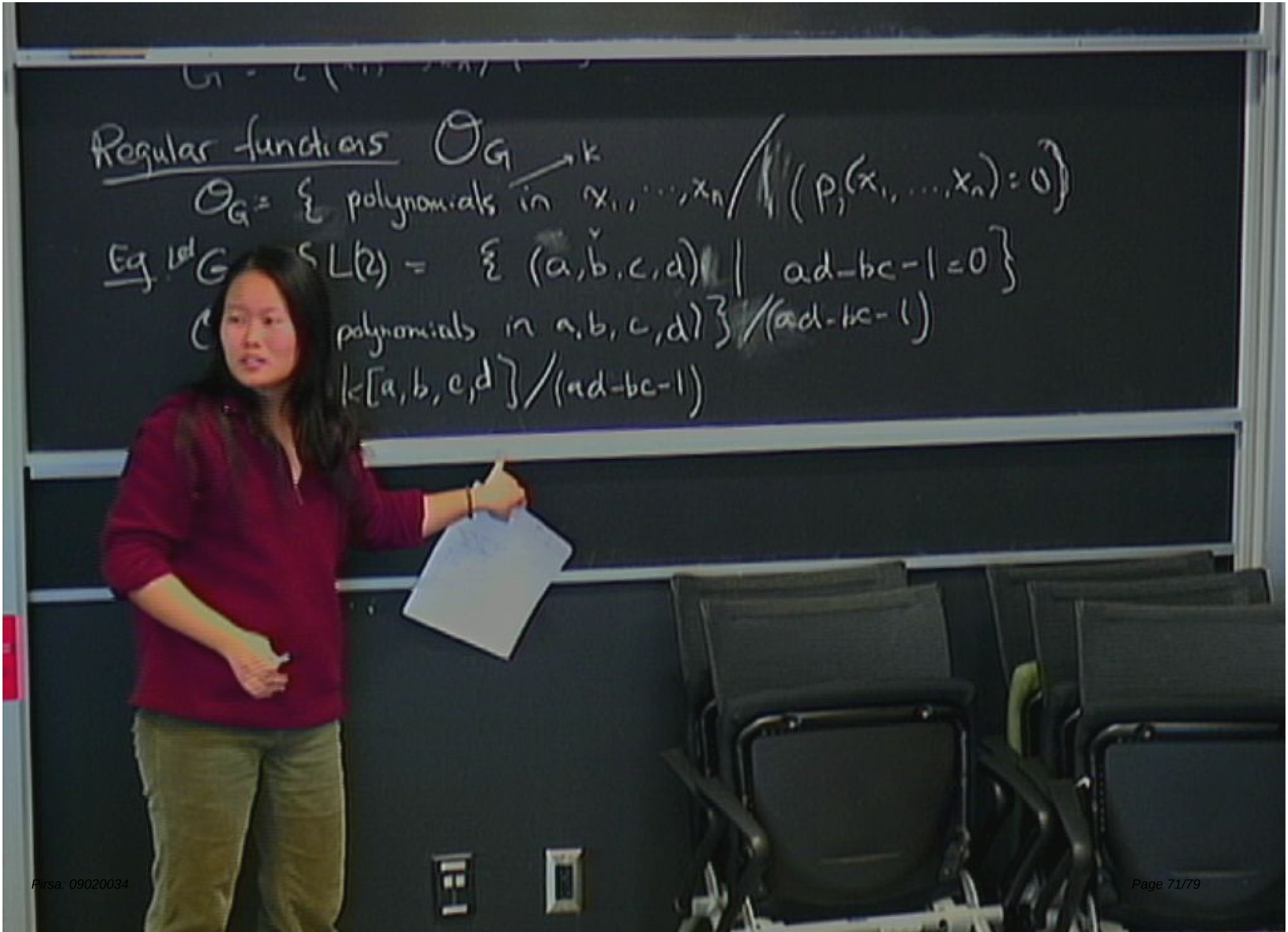
algebra ptwise multiplication (like for $(k[G])^{**}$)

coalgebra


Hopf Algebra structure on $\mathcal{O}_G$
algebra ptwise multiplication (like for $((k[G])^{**})$
coalgebra $\Delta(f)(X \otimes Y) = f(\mu(X \otimes Y))$
$\mathcal{O}_G \otimes \mathcal{O}_G$
$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$


Hopf Algebra structure on $\mathcal{O}_G$
algebra ptwise multiplication (like for $((k[G])^*)$)
coalgebra $\Delta(f)(X\otimes Y) = f(\mu(X\otimes Y))$
$\mathcal{O}_G\otimes\mathcal{O}_G$
$\mathcal{O}_{SL(2)}$: $\Delta\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}a&b\\c&d\end{pmatrix}\otimes\begin{pmatrix}a&b\\c\end{pmatrix} = \begin{pmatrix}a\otimes a+b\otimes c & \cdots\\ \cdots & \cdots\end{pmatrix}$


Hopf Algebra structure on $\mathcal{O}_G$
algebra ptwise multiplication (like for $(k[G])^*$)
coalgebra $(\Delta(f))(X \otimes Y) = f(\mu(X \otimes Y))$
$\mathcal{O}_G \otimes \mathcal{O}_G$
$\mathcal{O}_{SL(2)}$: $\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\otimes a + b\otimes c & \cdots \\ \cdots & \cdots \end{pmatrix}$


Regular functions O_G
O_G = { polynomials in x_1, ..., x_n / (p_i(x_1, ..., x_n) = 0)
k
Eg. 1st
L(2) = { (a, b, c, d) | ad-bc-1=0 }
polynomials in a, b, c, d } / (ad-bc-1)
k[a, b, c, d] / (ad-bc-1)

Hopf Algebra structure on $\mathcal{O}_G$

algebra: p'wise multiplication (like for $((k[G])^*)$)

coalgebra: $\underbrace{\Delta(f)}_{\mathcal{O}_G\otimes\mathcal{O}_G}(X\otimes Y) = f(\mu(X\otimes Y))$

$\mathcal{O}_{SL(2)}$: $\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\otimes a + b\otimes c & \cdots \\ \vdots & \ddots \end{pmatrix}$

Hopf Algebra structure on $\mathcal{O}_G$

$\underline{\text{algebra}}$ ptwise multiplication (like for $((k[G])^*)$)

$$\begin{pmatrix}\Delta(a) & \Delta(b)\\ \Delta(c) & \Delta(d)\end{pmatrix}$$

$\underline{\text{coalgebra}}$ $\underbrace{\Delta(f)}_{\mathcal{O}_G\otimes\mathcal{O}_G}(X\otimes Y) = f(\mu(X\otimes Y))$

$\mathcal{O}_{SL(2)}$: $\Delta\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\otimes\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a\otimes a + b\otimes c & \cdots\\ \cdots & \cdots\end{pmatrix}$

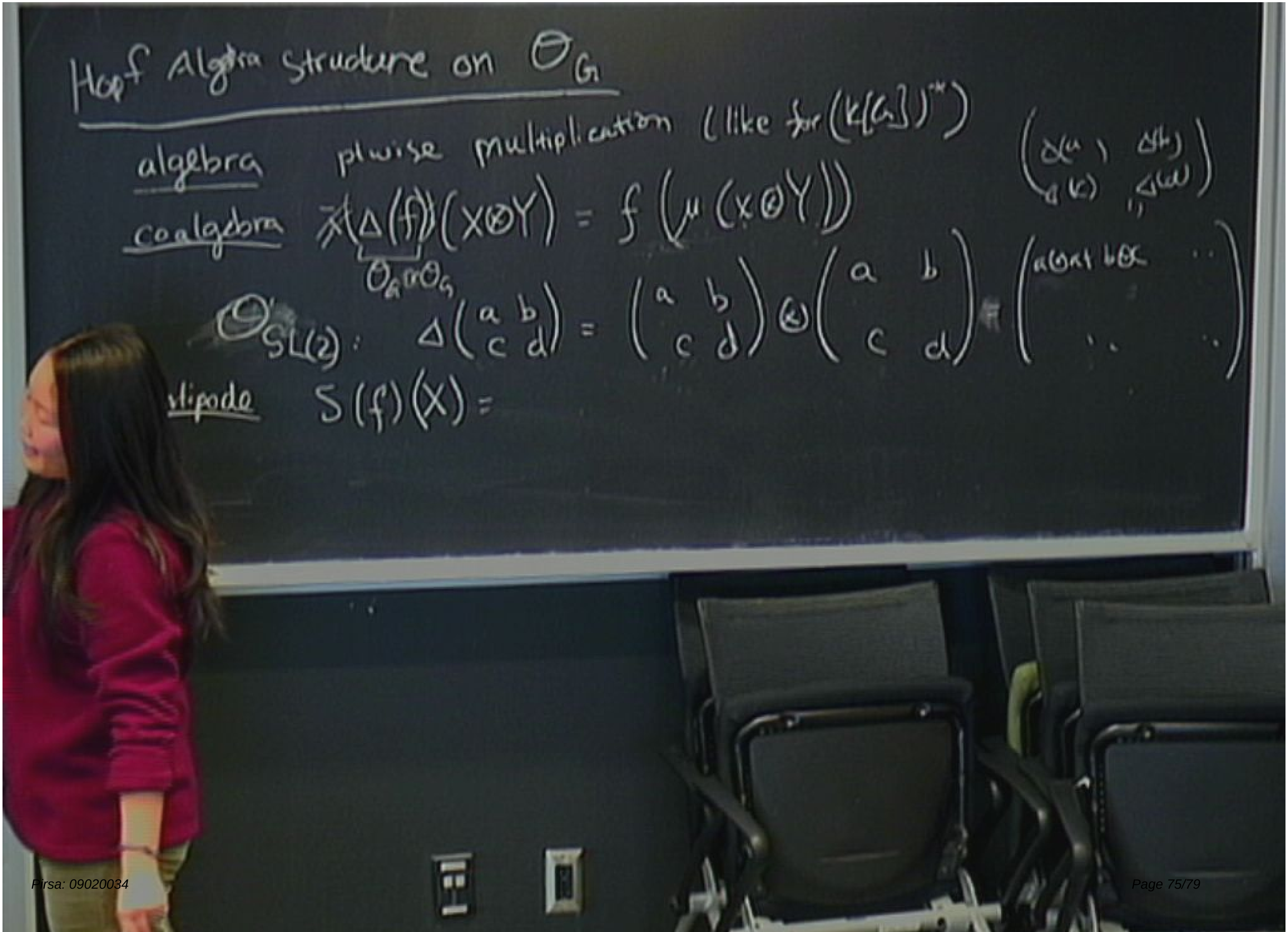
Hopf Algebra structure on $\mathcal{O}_G$

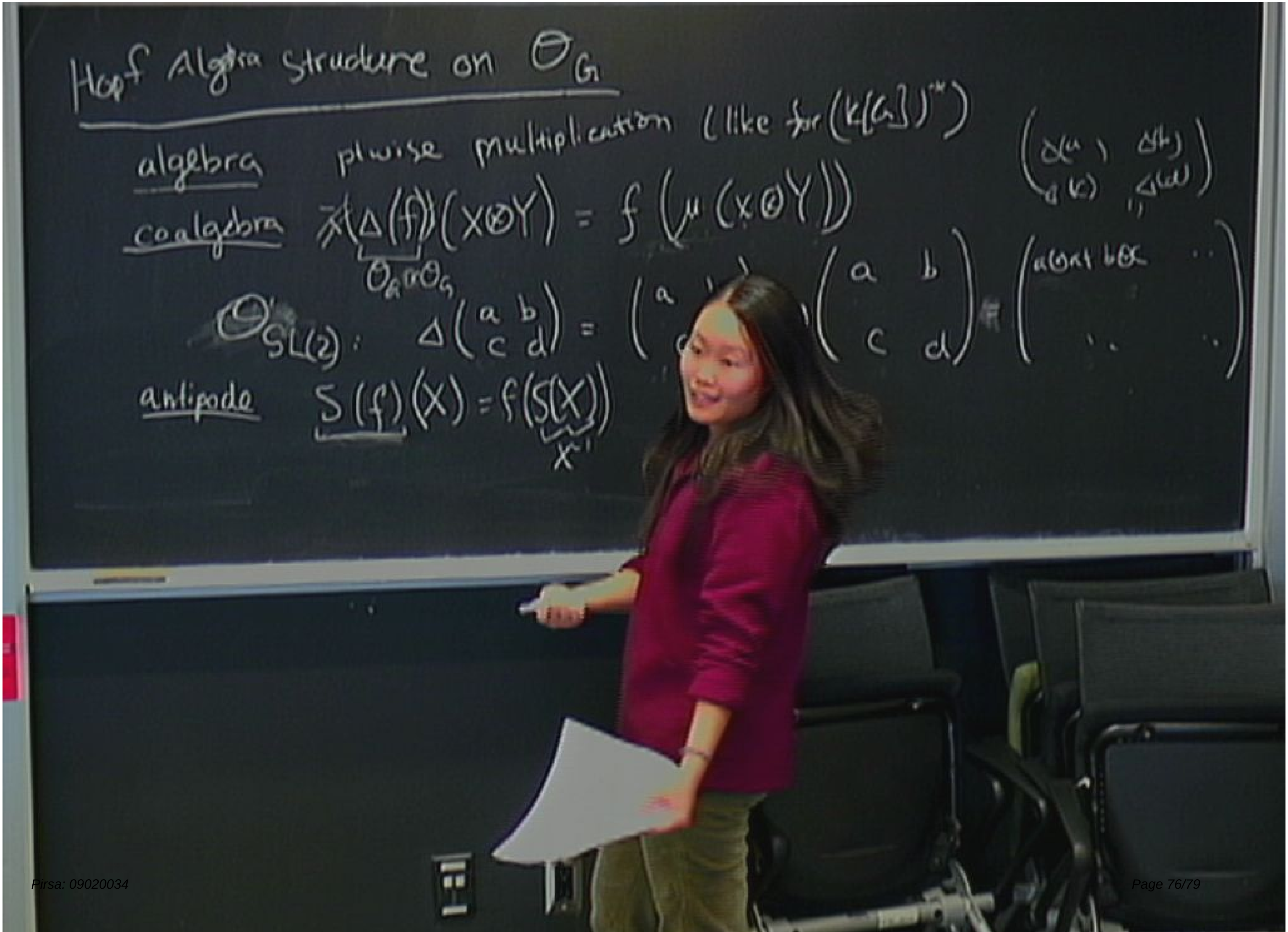
algebra ptwise multiplication (like for $((k[G])^*)$)

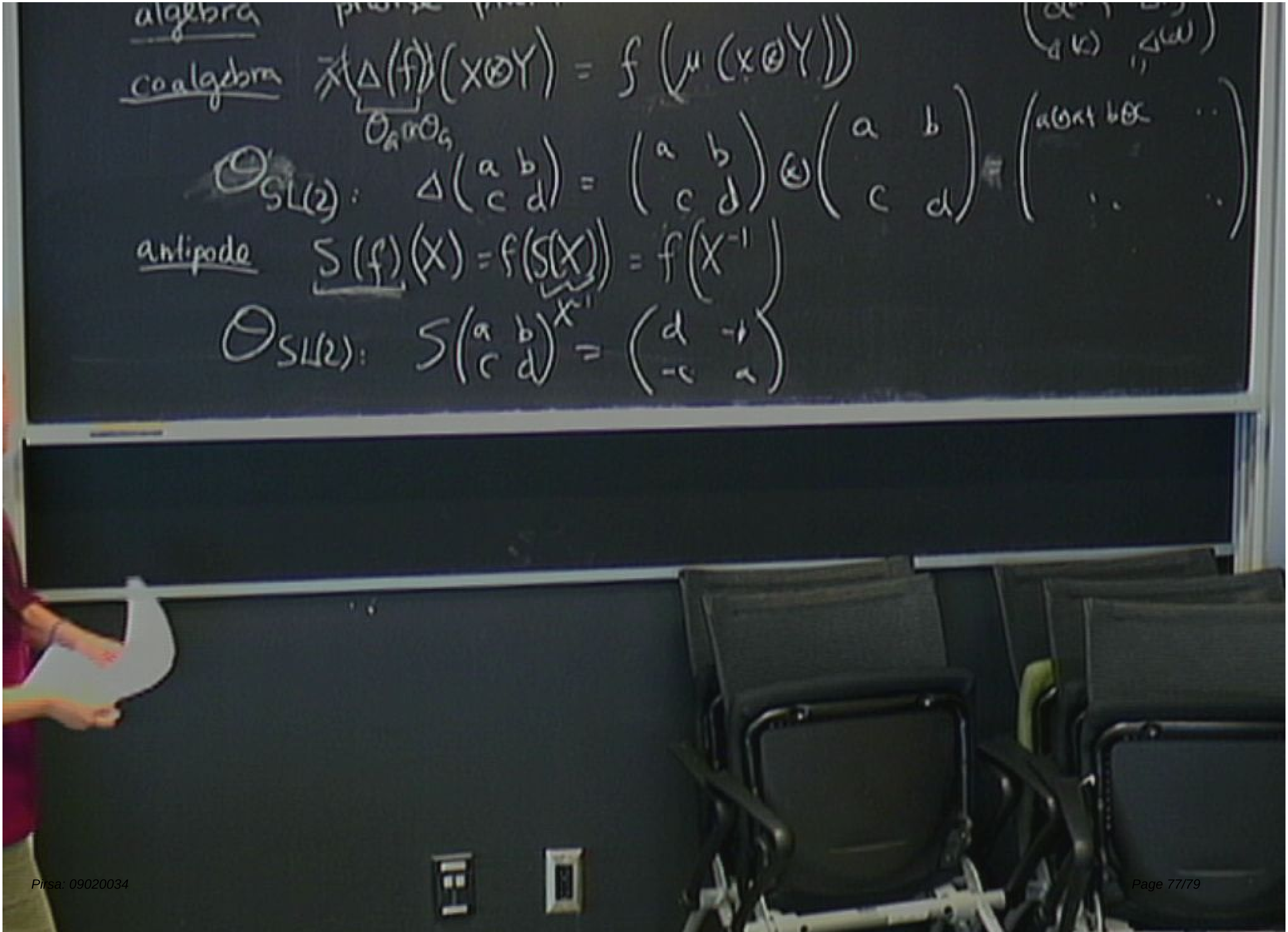
$$\begin{pmatrix} \Delta(a), & \Delta(b) \\ \Delta(c), & \Delta(d) \end{pmatrix}$$

coalgebra $\underbrace{\Delta(f)}_{\mathcal{O}_G \otimes \mathcal{O}_G}(X \otimes Y) = f(\mu(X \otimes Y))$

$\mathcal{O}_{SL(2)}$: $\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\otimes a + b\otimes c & \cdots \\ \cdots & \cdots \end{pmatrix}$


Hopf Algebra structure on O_G
algebra pt-wise multiplication (like for (k[G])*)
coalgebra Δ(f)(X⊗Y) = f(μ(X⊗Y))
O_G ⊗ O_G
O_SL(2): Δ(a b / c d) = (a b / c d) ⊗ (a b / c d) = (a⊗a + b⊗c ...)
antipode S(f)(X) =


Hopf Algebra structure on O_G
algebra ptwise multiplication
coalgebra Δ(f)(X⊗Y) = f(μ(X⊗Y))
O_G⊗O_G
O_SL(2): Δ(a b / c d) =
antipode S(f)(X) = f(S(X))
X^-1


algebra
coalgebra $\Delta(f)(X \otimes Y) = f(\mu(X \otimes Y))$
$\Theta_G \otimes \Theta_G$
$\Theta_{SL(2)}$: $\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
antipode $S(f)(X) = f(S(X)) = f(X^{-1})$
$\Theta_{SL(2)}$: $S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

algebra

coalgebra $\underbrace{\Delta(f)}_{\mathcal{O}_G \otimes \mathcal{O}_G}(X \otimes Y) = f(\mu(X \otimes Y))$

$\mathcal{O}_{SL(2)}$: $\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a\otimes a + b\otimes c & \cdots \\ \cdots & \cdots \end{pmatrix}$

antipode $\underline{S(f)}(X) = f(\underbrace{S(X)}) = f(X^{-1})$

$\mathcal{O}_{SL(2)}$: $S\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{X^{-1}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

algebra   pointwise product ...

$$\begin{pmatrix}\Delta(a) & \Delta(b)\\ \Delta(c) & \Delta(d)\end{pmatrix}$$

coalgebra $$\underbrace{\Delta(f)}_{\mathcal{O}_G\otimes\mathcal{O}_G}(X\otimes Y) = f(\mu(X\otimes Y))$$

$$\mathcal{O}_{SL(2)}: \quad \Delta\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\otimes\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a\otimes a + b\otimes c & \cdots\\ \cdots & \cdots\end{pmatrix}$$

antipode $$\underline{S(f)}(X) = f(\underbrace{S(X)}_{X^{-1}}) = f(X^{-1})$$

$$\mathcal{O}_{SL(2)}: \quad S\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$$