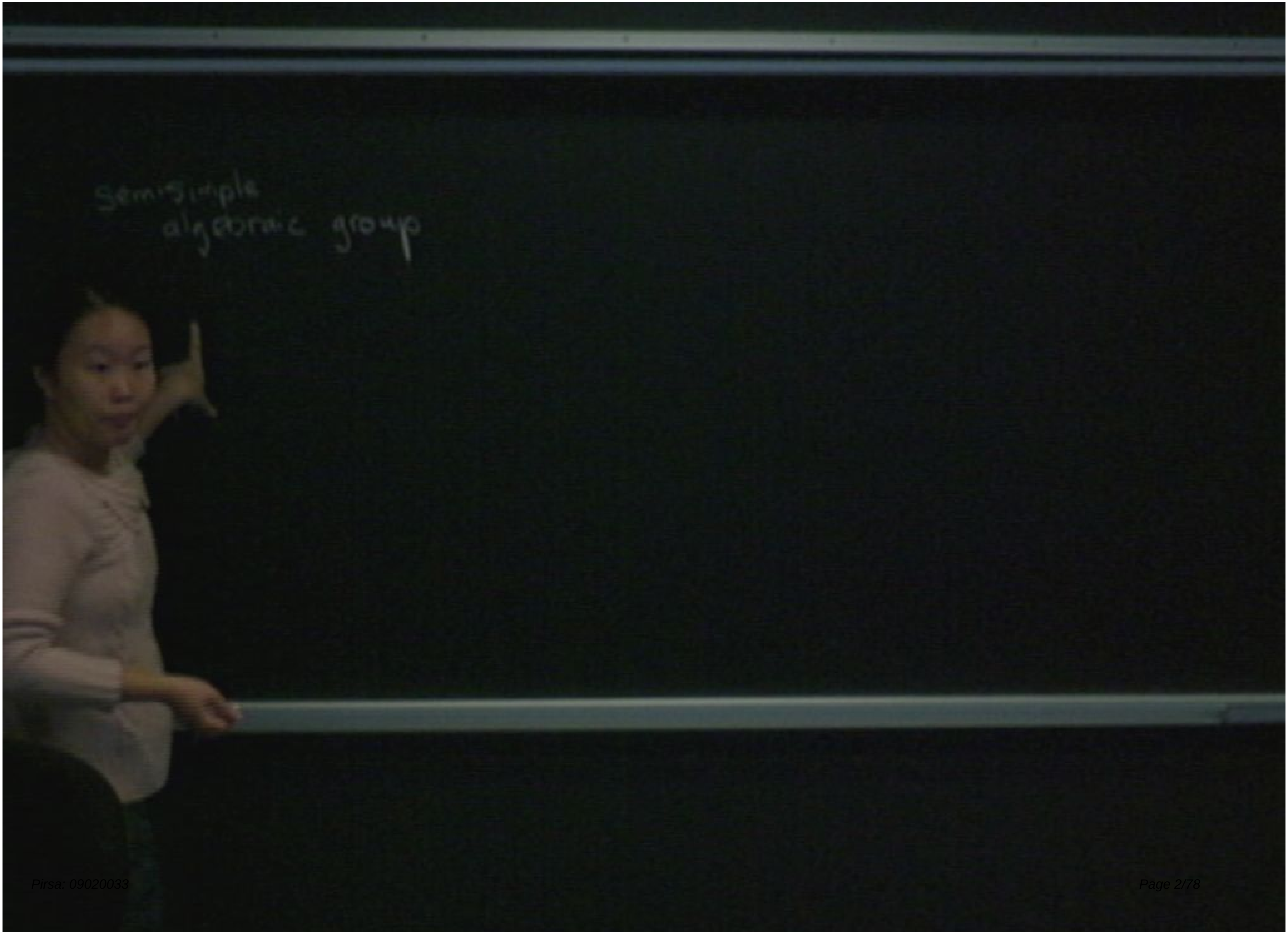
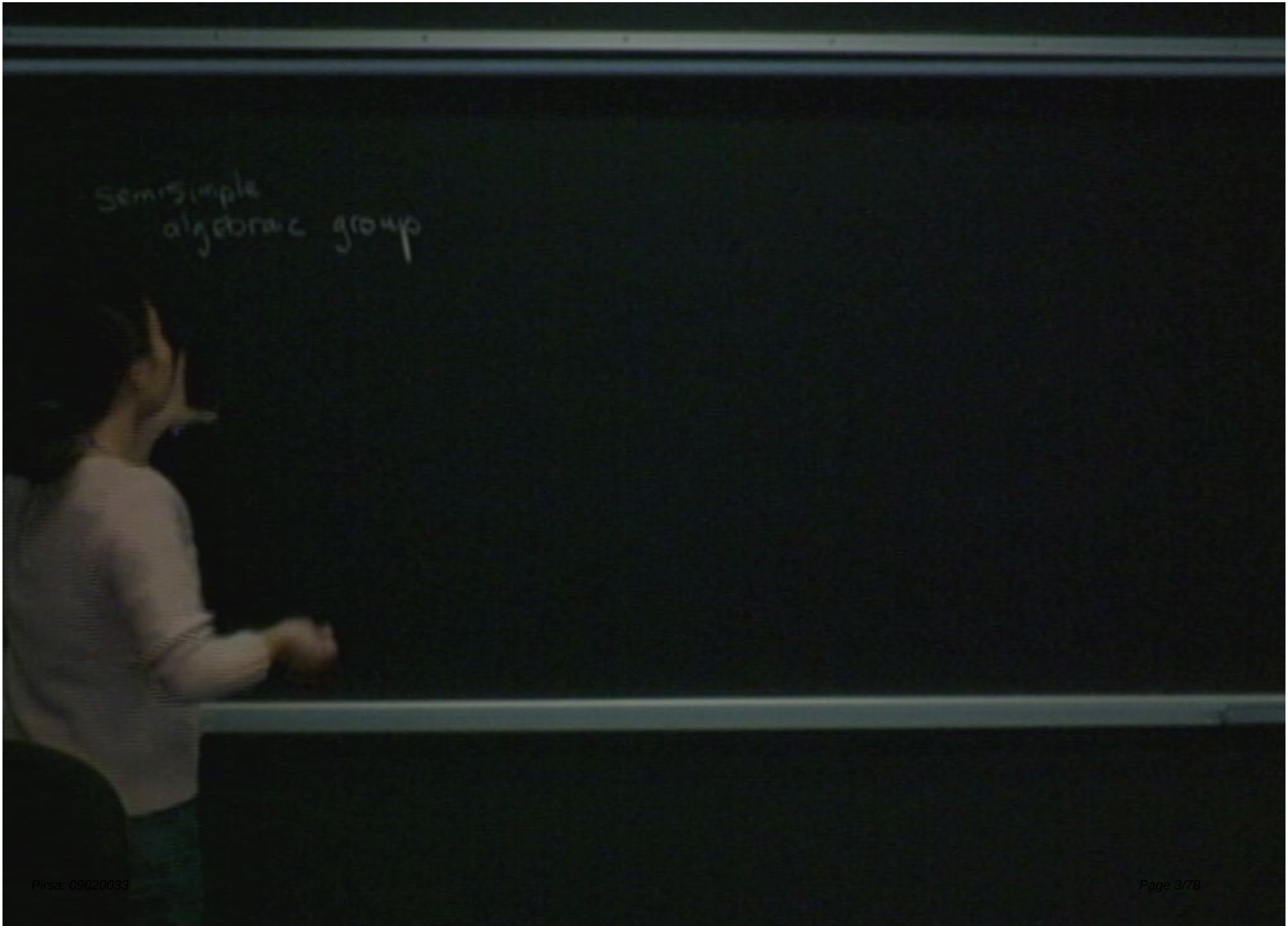
Title: Grad Talks 1

Date: Feb 02, 2009 11:00 AM

URL: http://pirsa.org/09020033

Abstract: Lecture on Quantum Groups by Lucy Zhang


semisimple
algebraic group


semisimple
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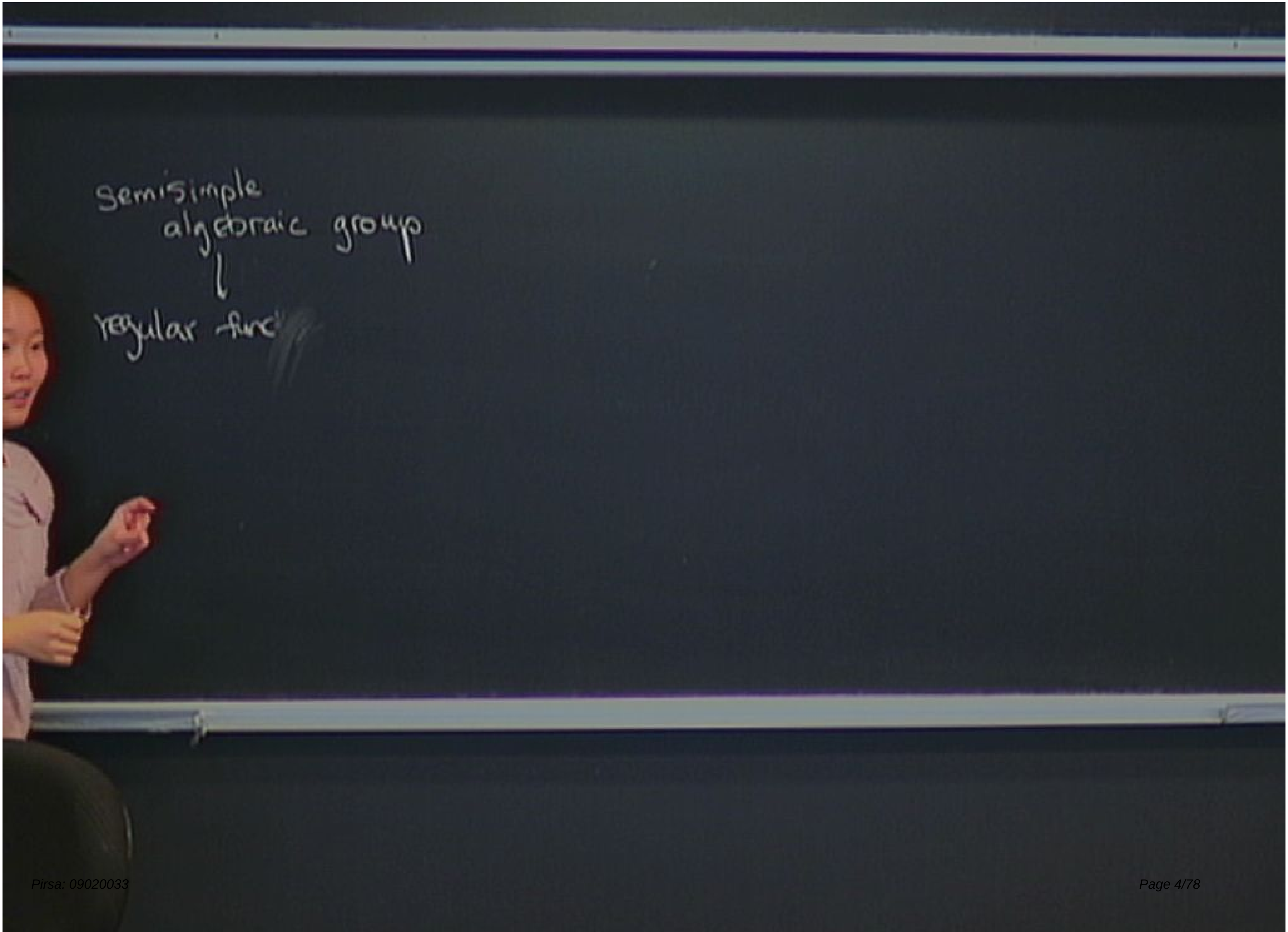

semisimple
algebraic group
regular func

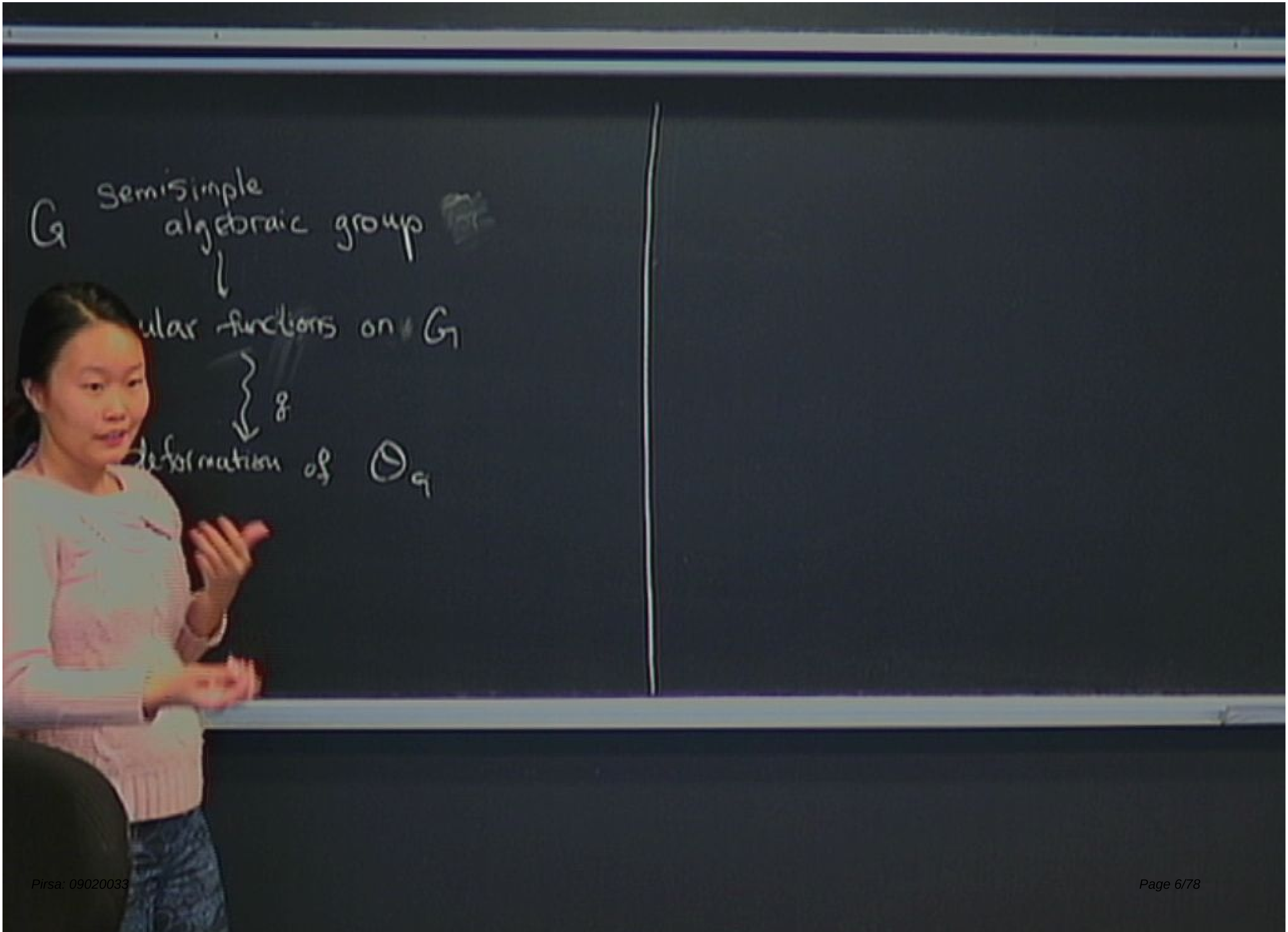
G semisimple algebraic group

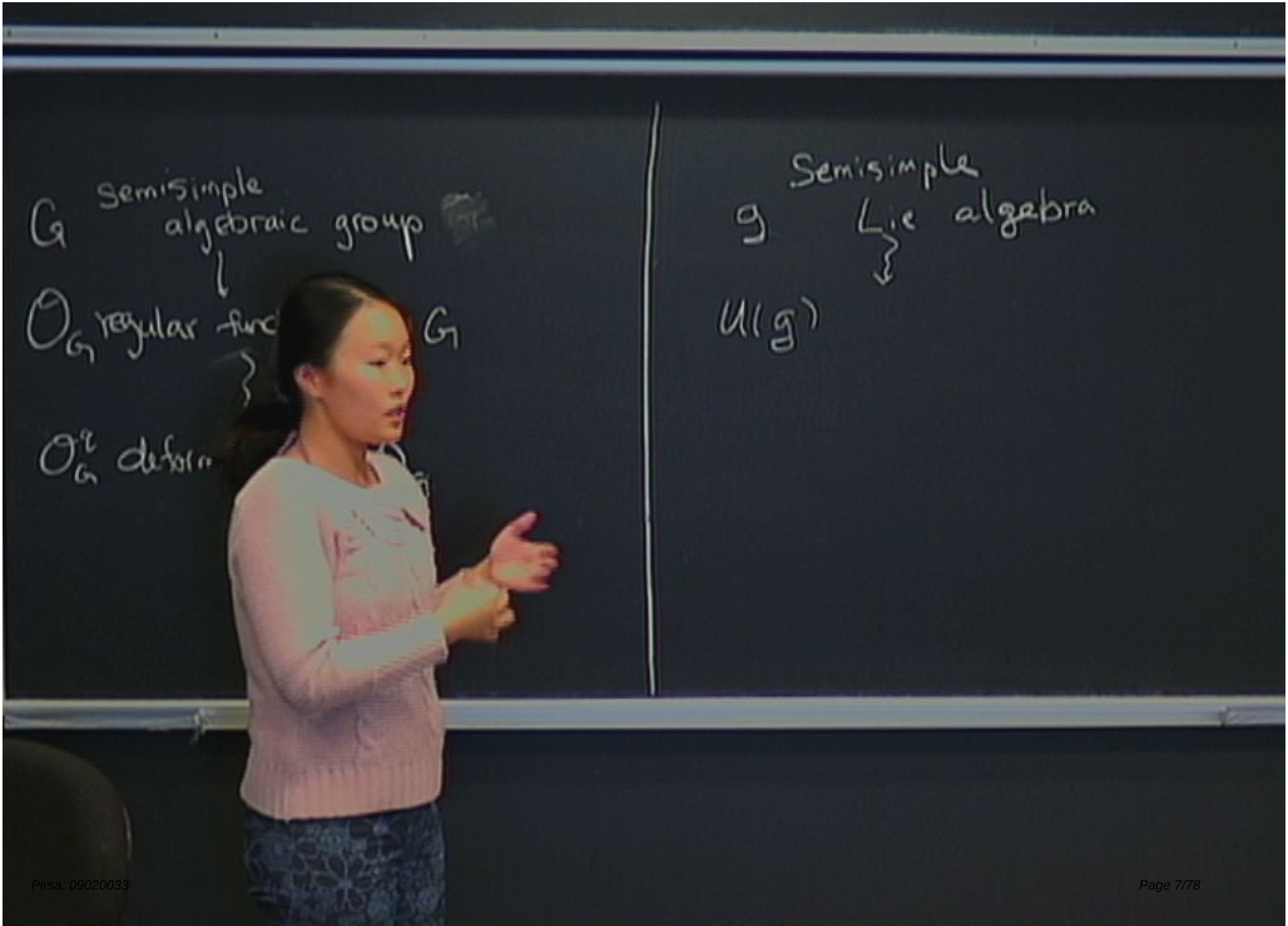
$\downarrow$

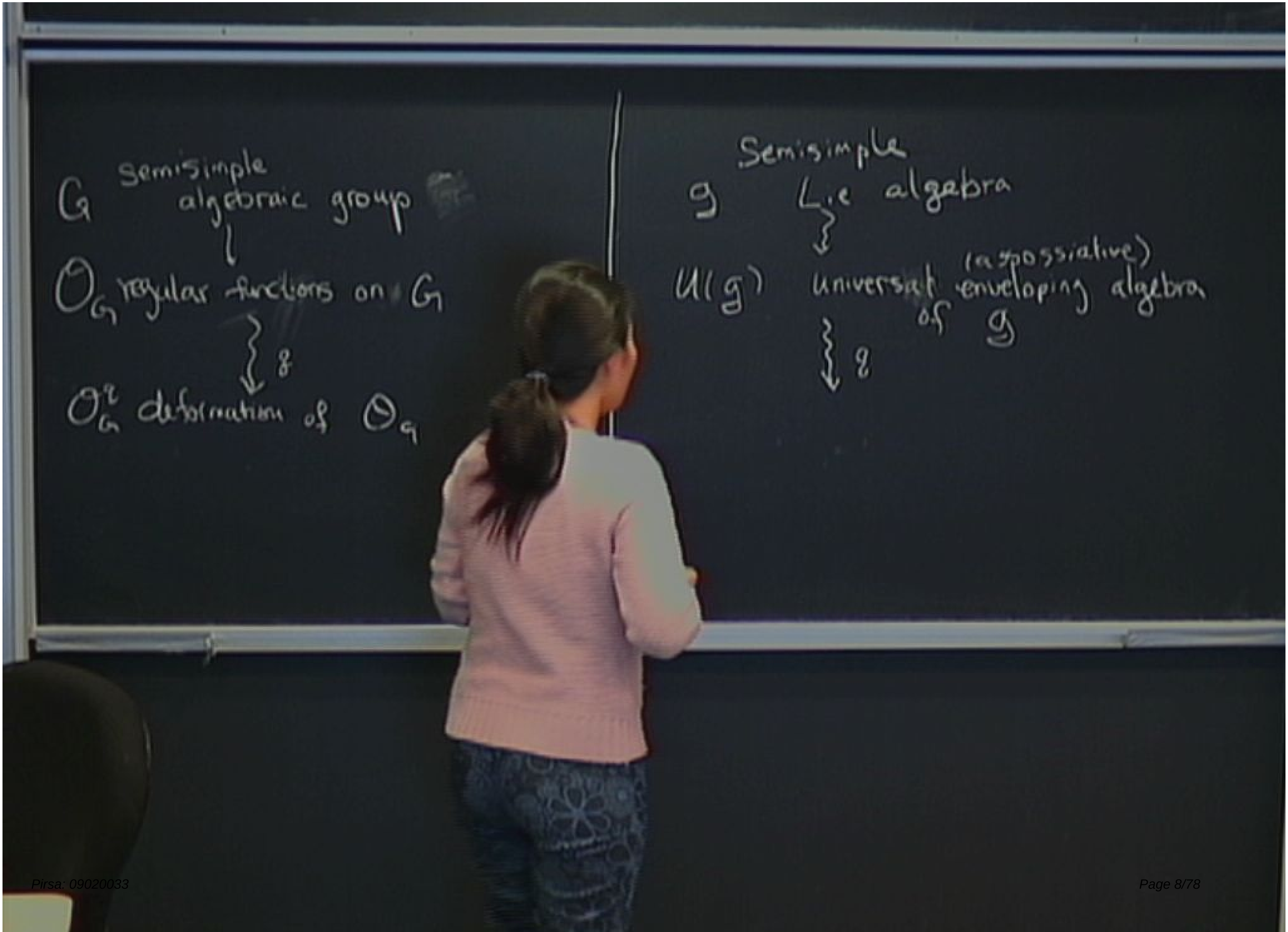
$\mathcal{O}_G$ regular functions on G

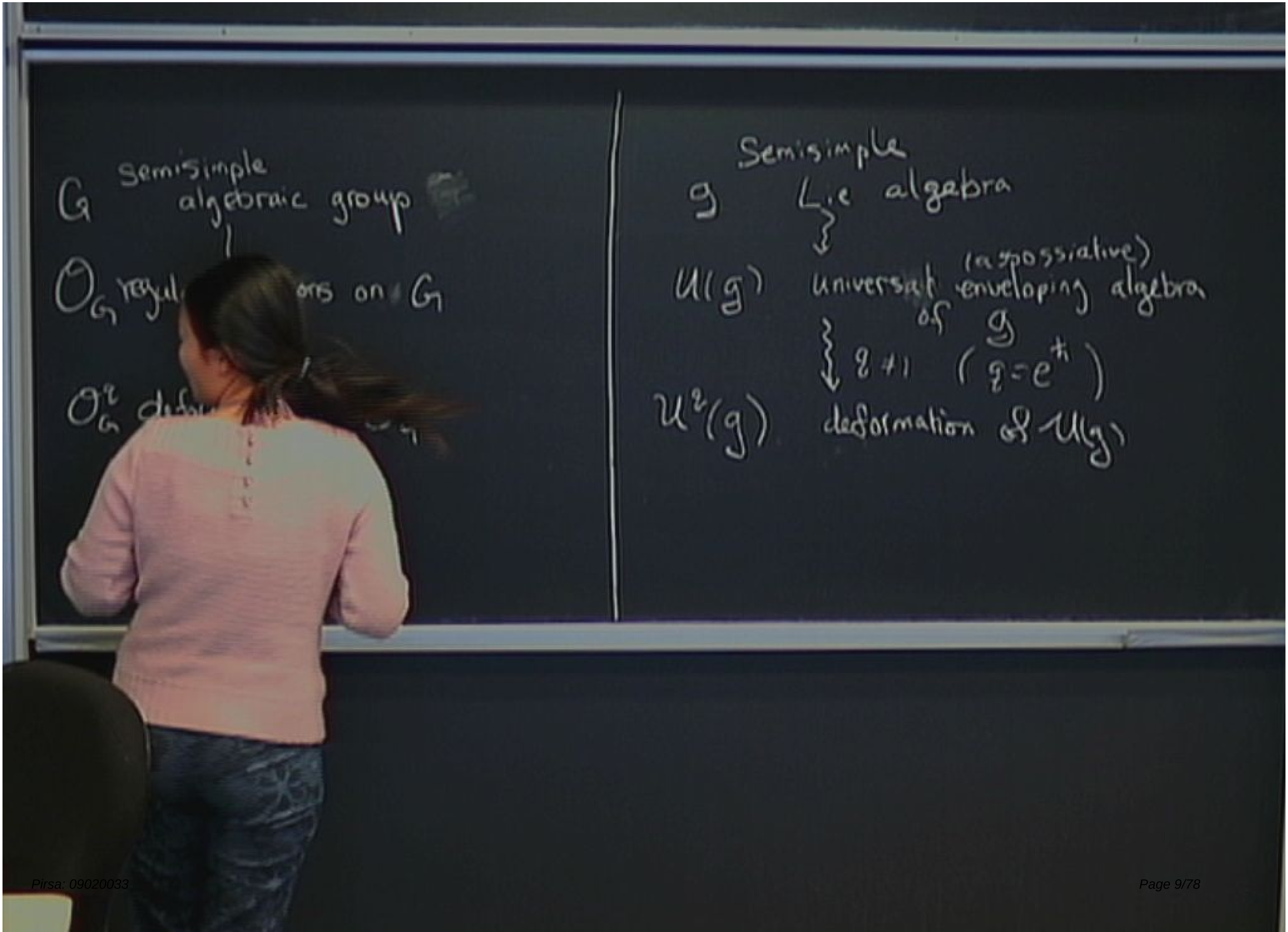
$\rightsquigarrow \hbar$

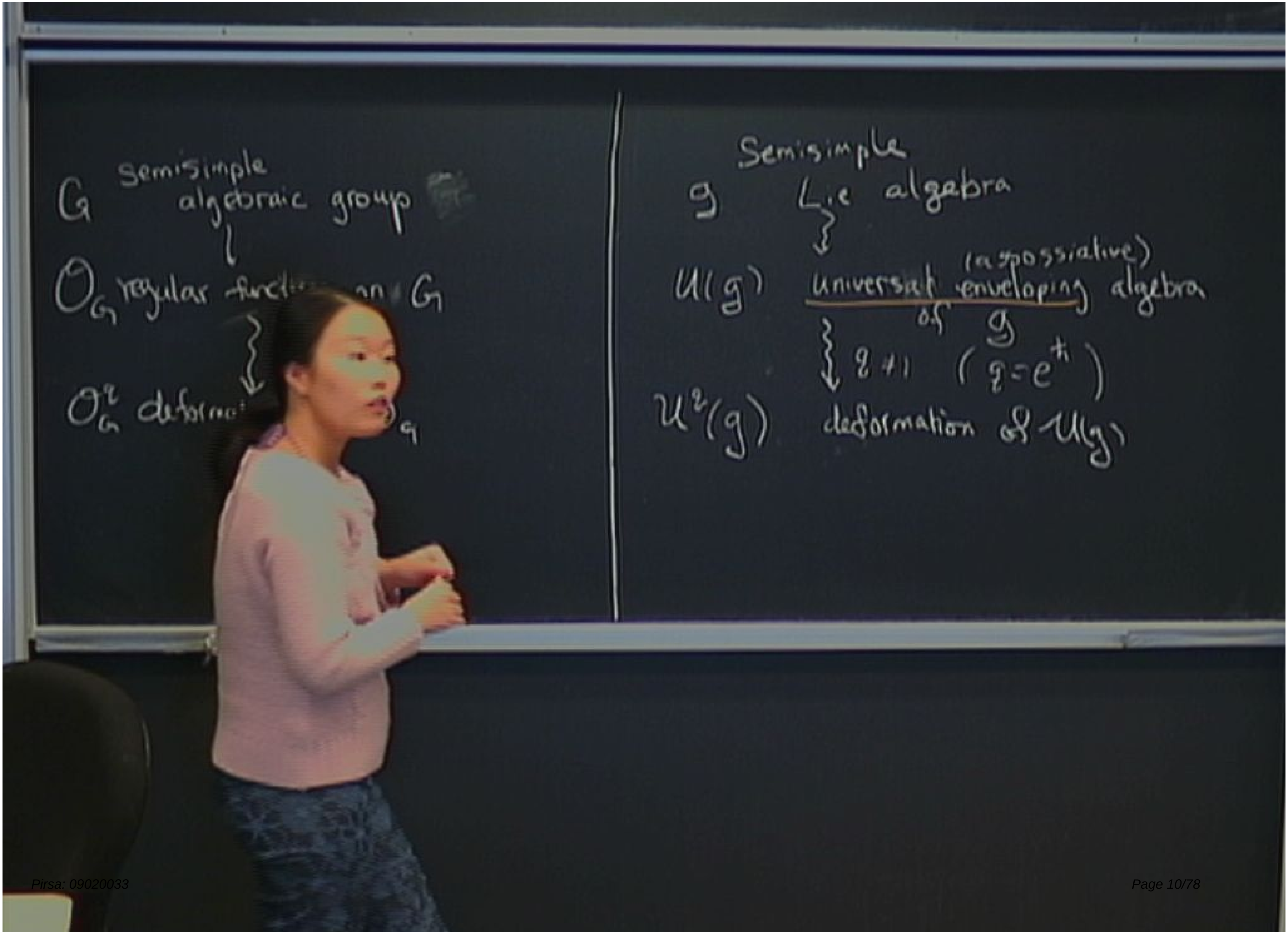
deformation of


G semisimple
algebraic group
ular functions on G
q
eformation of O_q


G semisimple
algebraic group
$\mathcal{O}_G$ regular
$\mathcal{O}_G^q$
Semisimple
g Lie algebra
U(g)


G semisimple algebraic group
O_G regular functions on G
q
O_G^q deformation of O_G
g Semisimple Lie algebra
U(g) universal (associative) enveloping algebra of g
q


G semisimple algebraic group
on G
Semisimple
g Lie algebra
(associative)
U(g) universal enveloping algebra of g
q ≠ 1 (q = e^ħ)
U^q(g) deformation of U(g)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U^q(g) deformation of U(g)

G semisimple algebraic group

$\downarrow$

$\mathcal{O}_G$ regular functions on G

$\Big\updownarrow q$

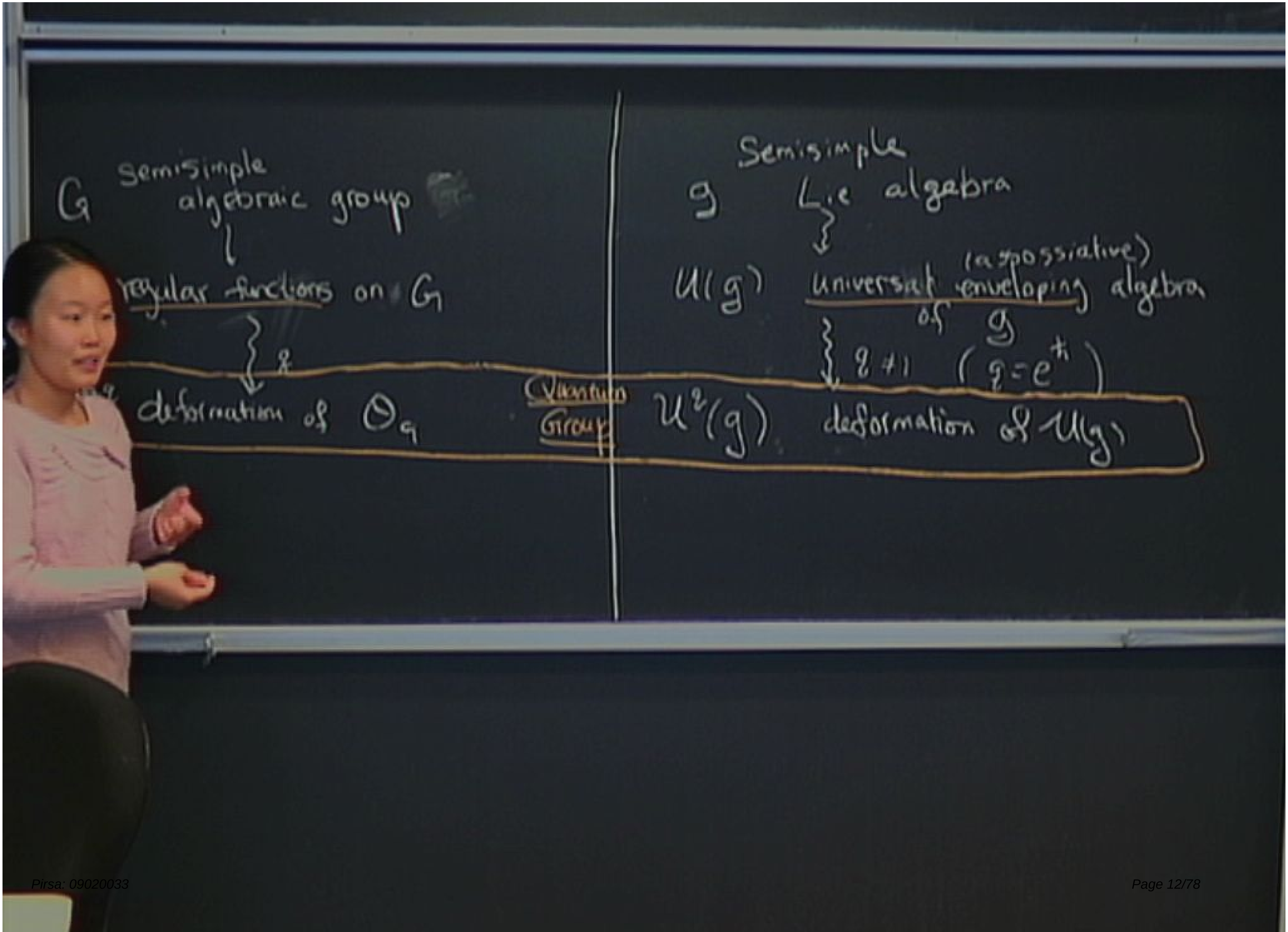
$\boxed{\mathcal{O}^q_G \text{ deformation of } \mathcal{O}_G \qquad \mathcal{U}^q(\mathfrak{g}) \text{ deformation of } \mathcal{U}(\mathfrak{g})}$

$\mathfrak{g}$ Semisimple Lie algebra

$\Big\downarrow$

$\mathcal{U}(\mathfrak{g})$ universal enveloping algebra (associative) of $\mathfrak{g}$

$\Big\downarrow q \neq 1 \quad (q = e^{\hbar})$


G semisimple algebraic group
regular functions on G
q
deformation of $\mathcal{O}_q$
Quantum Group
Semisimple
g Lie algebra
(associative)
U(g) universal enveloping algebra of g
$q \neq 1$ ($q = e^{\hbar}$)
$U^q(g)$ deformation of U(g)

G semisimple algebraic group

$\downarrow$

$\mathcal{O}_G$ regular functions on G

$\Big\} q$

$\mathcal{O}_G^q$ deformation of $\mathcal{O}_G$

Quantum Group

$\mathfrak{g}$ Semisimple Lie algebra

$\Big\}$

$U(\mathfrak{g})$ universal (associative) enveloping algebra of $\mathfrak{g}$

$\Big\} q \neq 1 \quad (q = e^{\hbar})$

$U^q(\mathfrak{g})$ deformation of $U(\mathfrak{g})$

Quantum group is certain class of <u>Hopf algebras</u>

↳ bialgebra

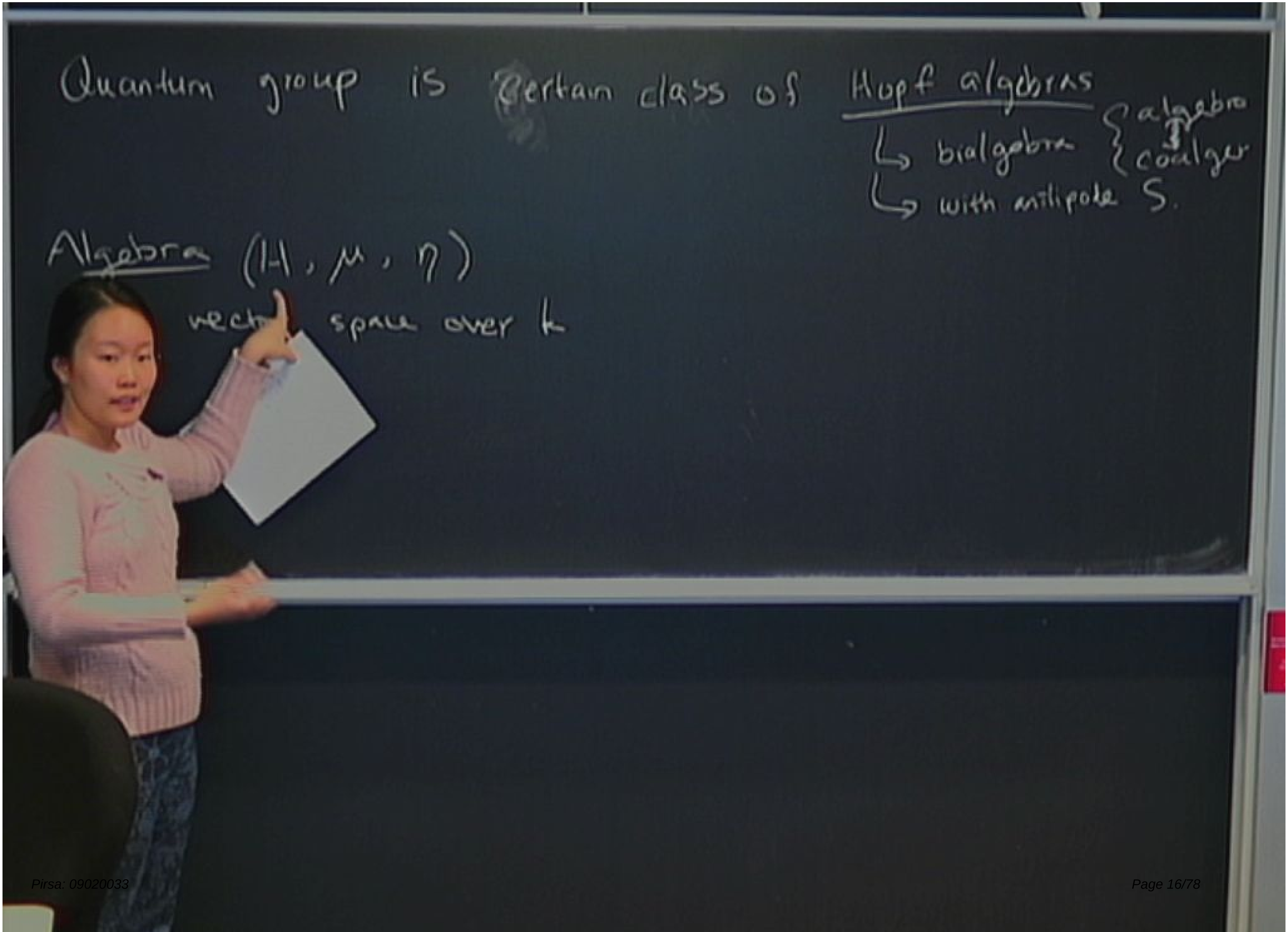
↳ with antipode S.

Quantum group is certain class of Hopf algebras

↳ bialgebra $\left\{ \begin{array}{l} \text{algebra} \\ \& \\ \text{coalgebra} \end{array} \right.$

↳ with antipode S.

Algebra


Quantum group is certain class of Hopf algebras
↳ bialgebra { algebra
{ coalgebra
↳ with antipode S.
Algebra (H, μ, η)
vector space over k

Quantum group is certain class of Hopf algebras

↳ bialgebra { algebra & coalgebra

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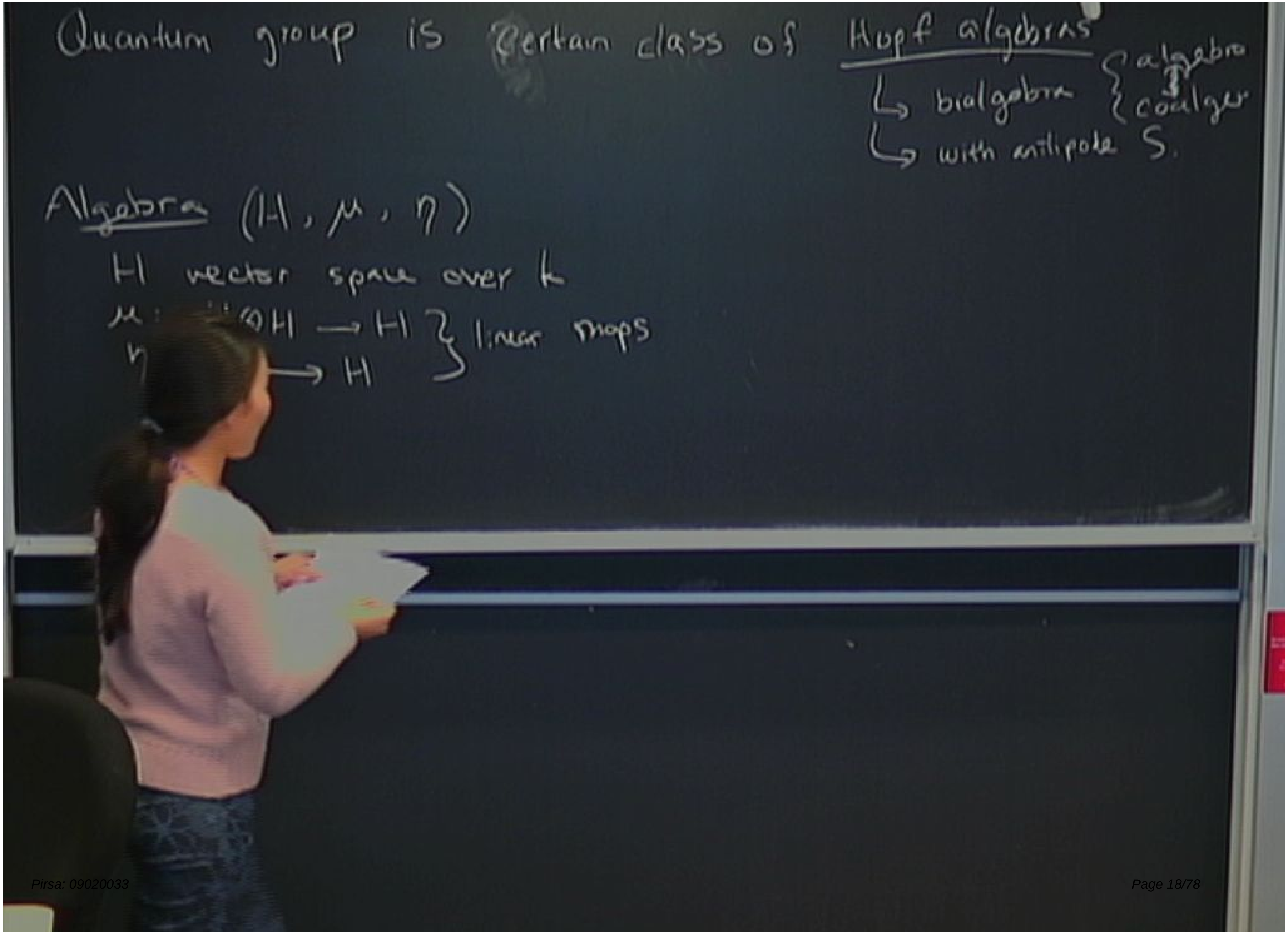
Algebra (H, μ, η)

H vector space over k

$\mu : H \otimes H \to H$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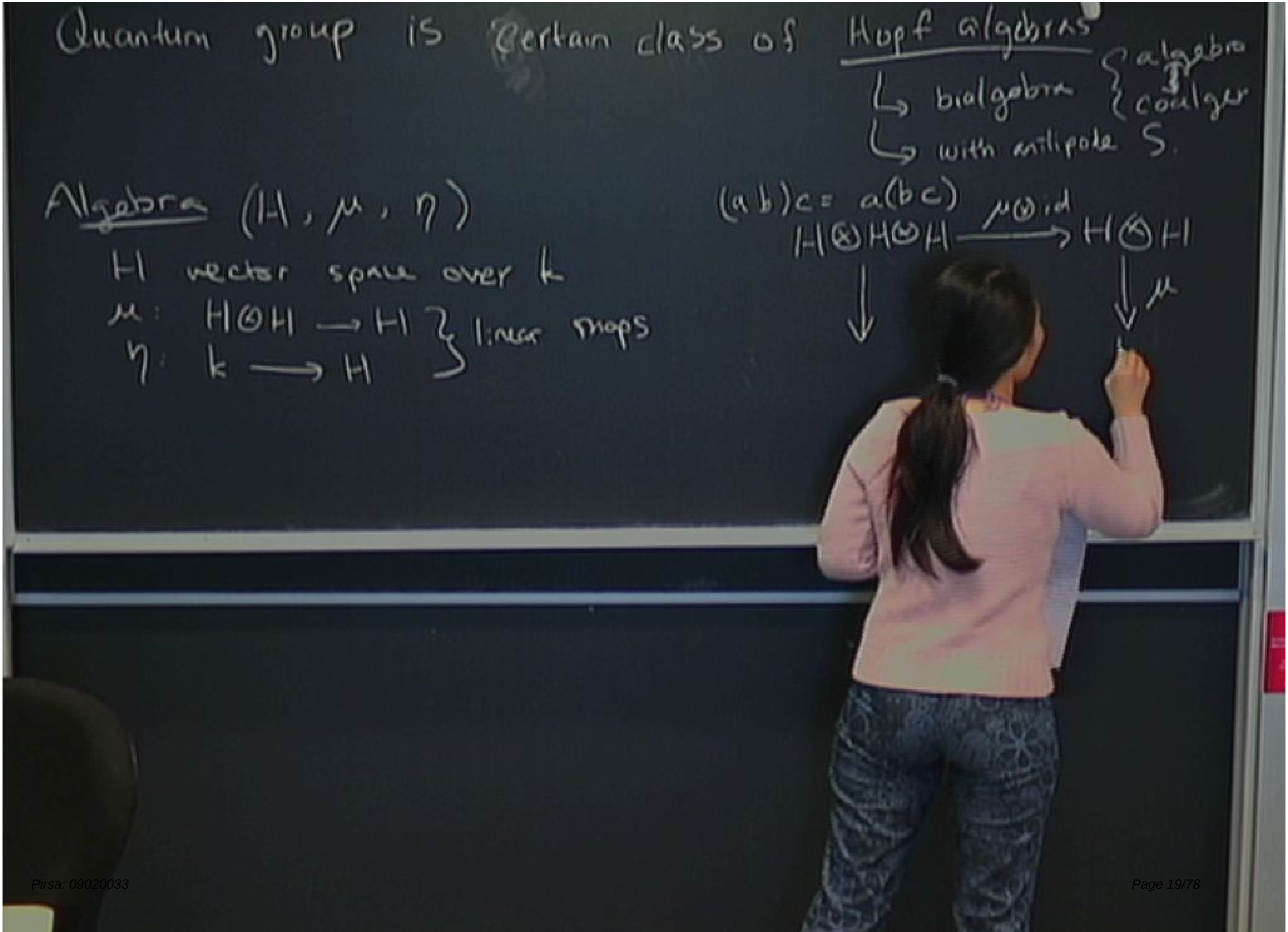
μ : ⊗H → H } linear maps
η → H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Algebra (H, μ, η)
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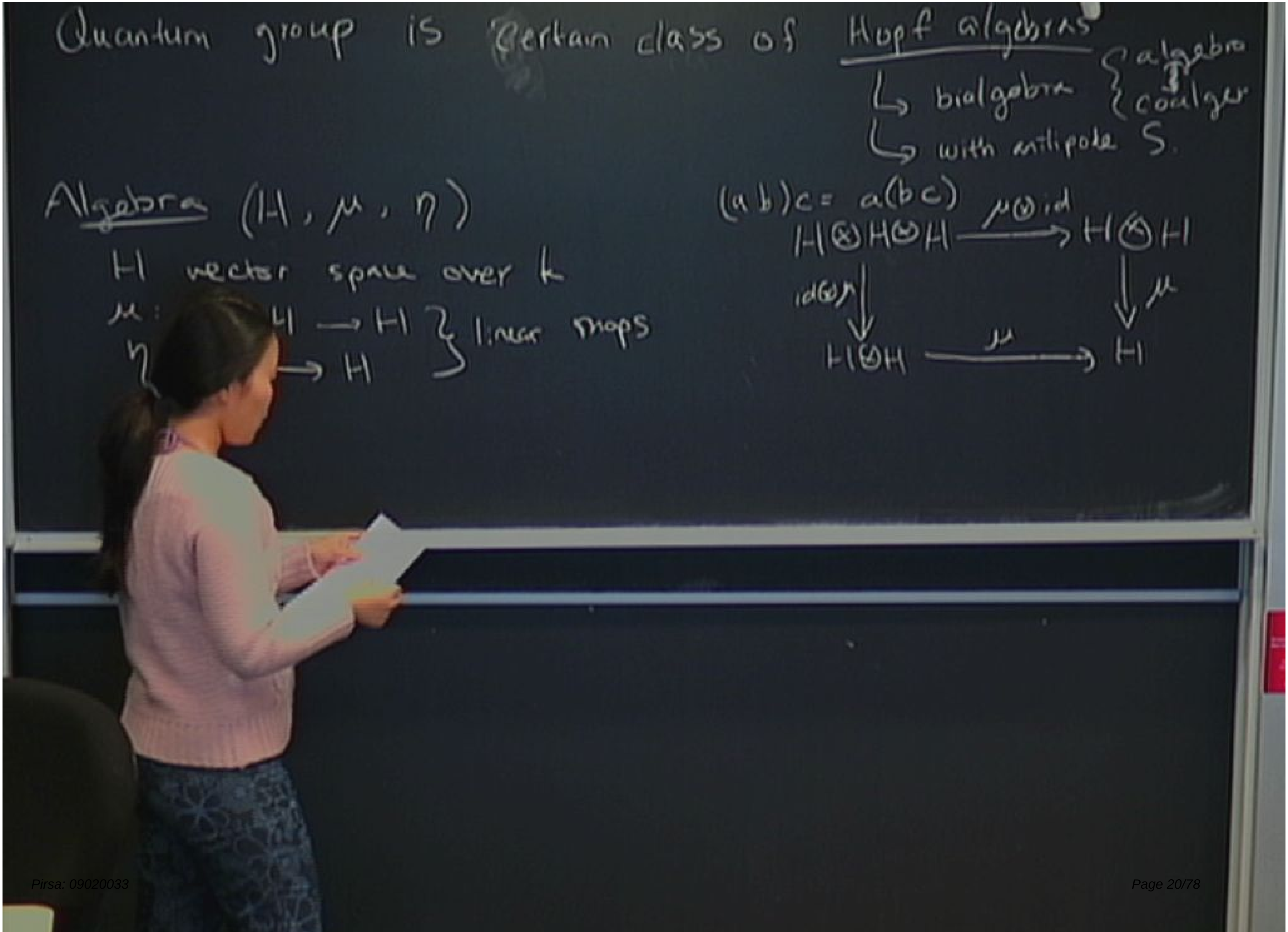
μ: H⊗H → H
η: k → H
linear maps
(ab)c = a(bc)
H⊗H⊗H → H⊗H
μ⊗id
μ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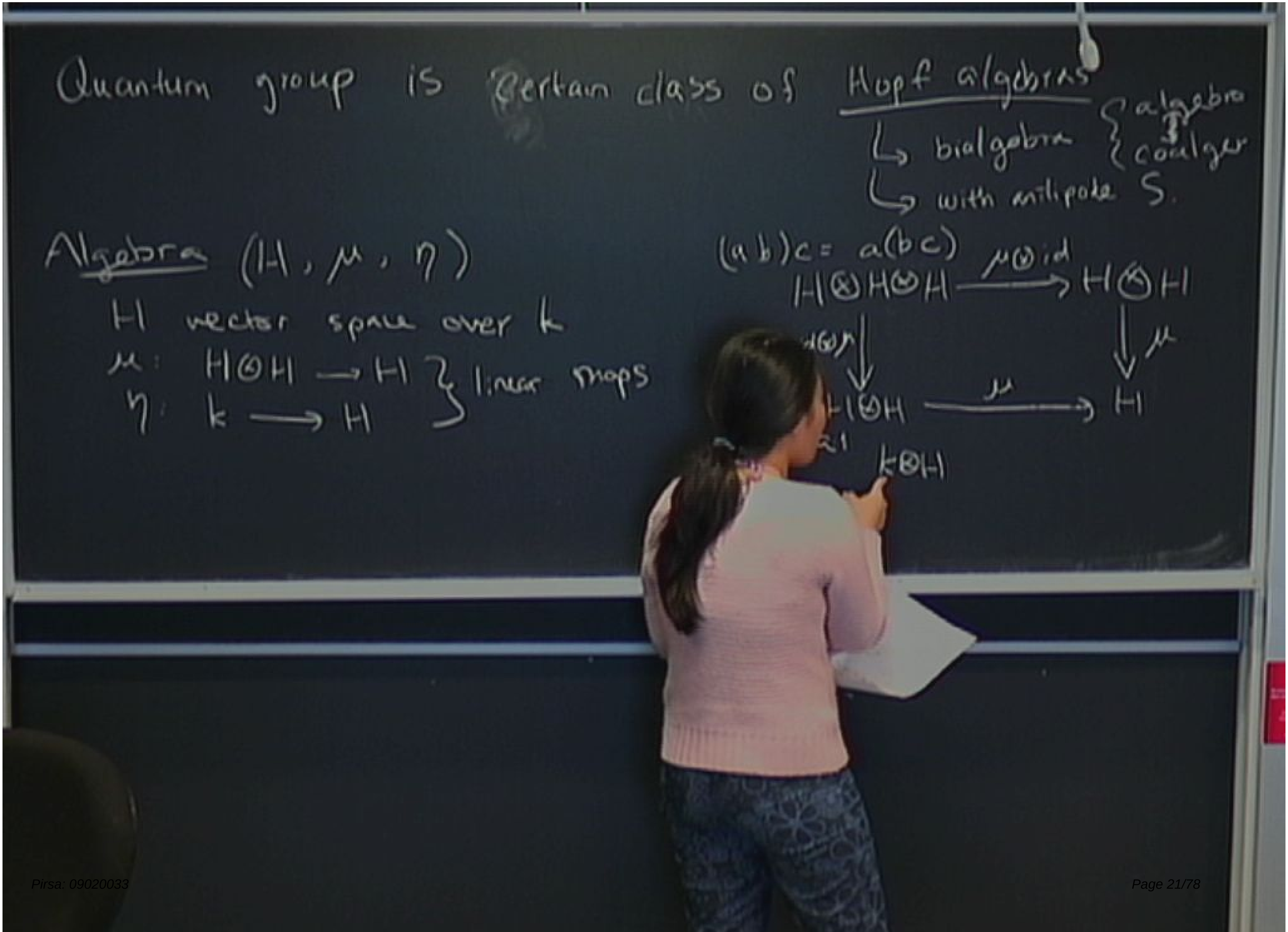
H⊗H⊗H → H⊗H (μ⊗id)
id⊗μ
μ
H⊗H → H (μ)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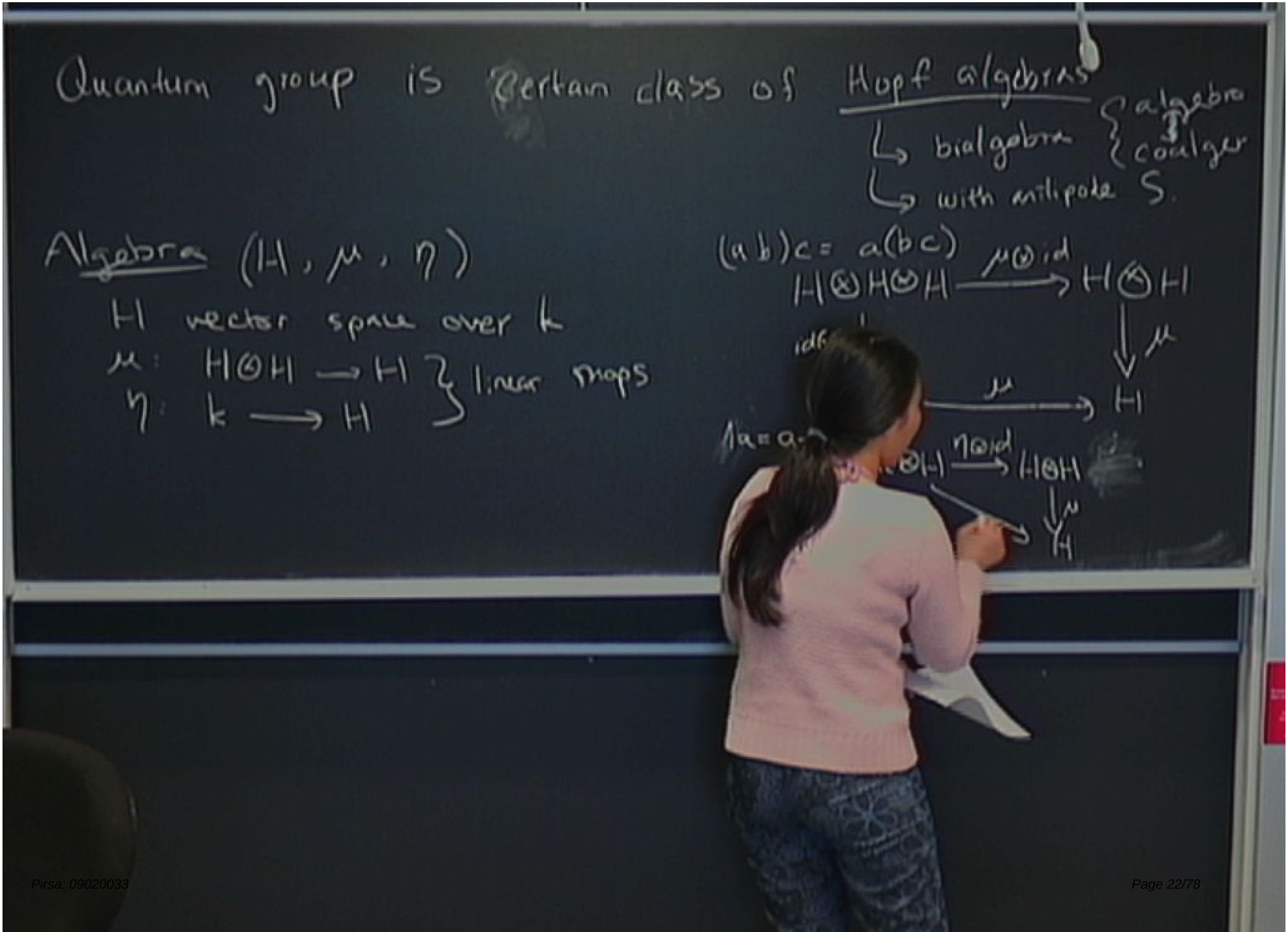
H
k⊗H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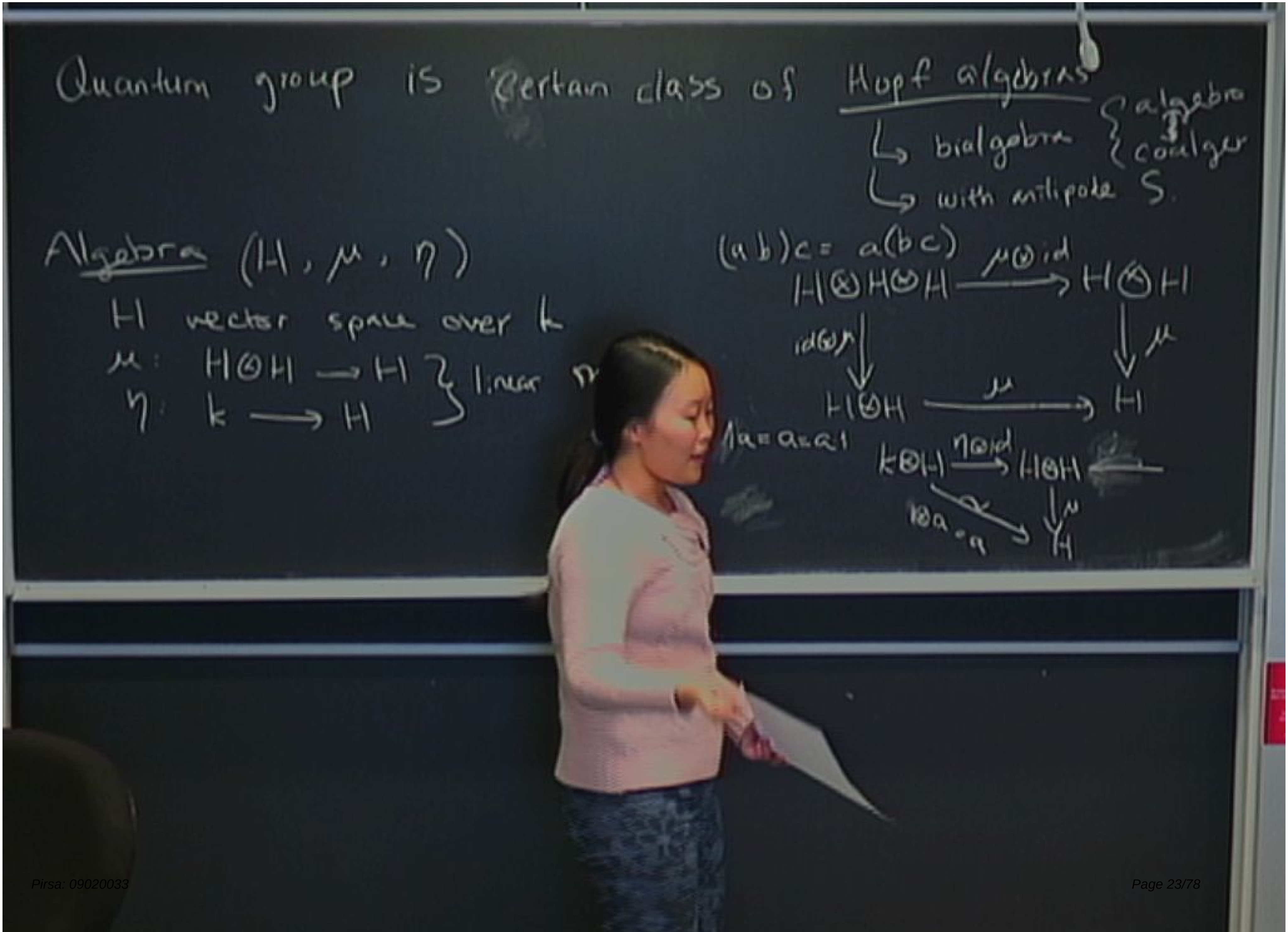
↓μ
id⊗
μ → H
1a = a
⊗H →η⊗id H⊗H
↓μ
H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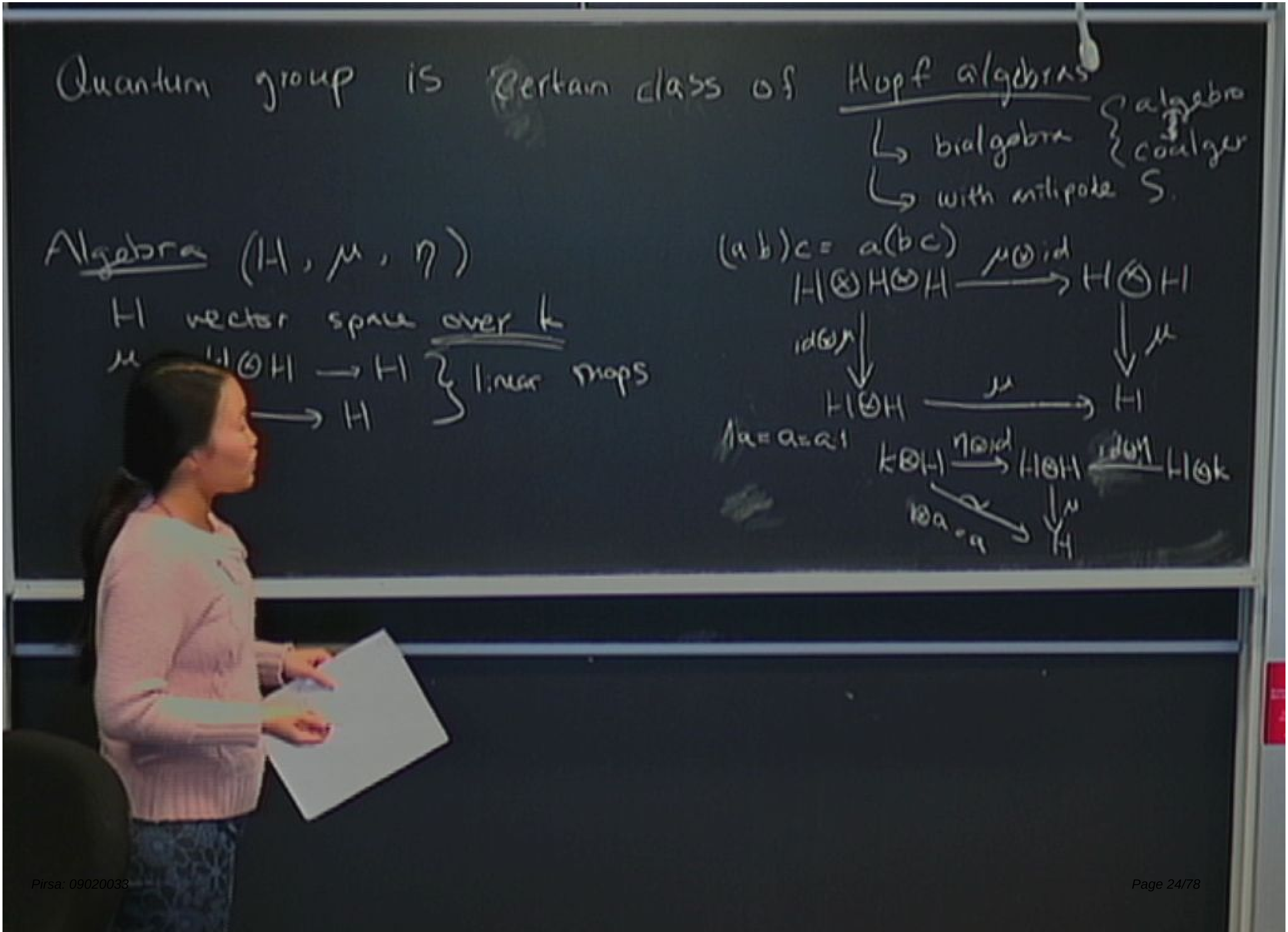
μ
H⊗H → H
1a = a = a1
k⊗H → H⊗H
η⊗id
μ
1⊗a ↦ a
H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μ
H⊗H
μ
H
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η⊗id
H⊗H
id⊗η
H⊗k
1⊗a ↦ a
μ
H

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Algebra (H, μ, η)

H vector space over k

$\mu: H \otimes H \to H$ } linear maps

$\eta: k \longrightarrow H$

$(ab)c = a(bc)$

$$\begin{array}{ccc} H\otimes H\otimes H & \xrightarrow{\mu\otimes id} & H\otimes H \\ \downarrow{\scriptstyle id\otimes\mu} & & \downarrow{\scriptstyle \mu} \\ H\otimes H & \xrightarrow{\mu} & H \end{array}$$

$1a = a = a1$

$$\begin{array}{ccccc} k\otimes H & \xrightarrow{\eta\otimes id} & H\otimes H & \xleftarrow{id\otimes\eta} & H\otimes k \\ & \searrow{\scriptstyle \cong} & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \sim} & \\ & & H & & \end{array}$$

$1\otimes a \mapsto a$

Quantum group is certain class of Hopf algebras

$\hookrightarrow$ bialgebra $\left\{ \begin{array}{l} \text{algebra} \\ \text{coalgebra} \end{array} \right.$

$\hookrightarrow$ with antipode S.

Coalgebra (H, Δ, ε)

Algebra (H, μ, η)

H vector space over k

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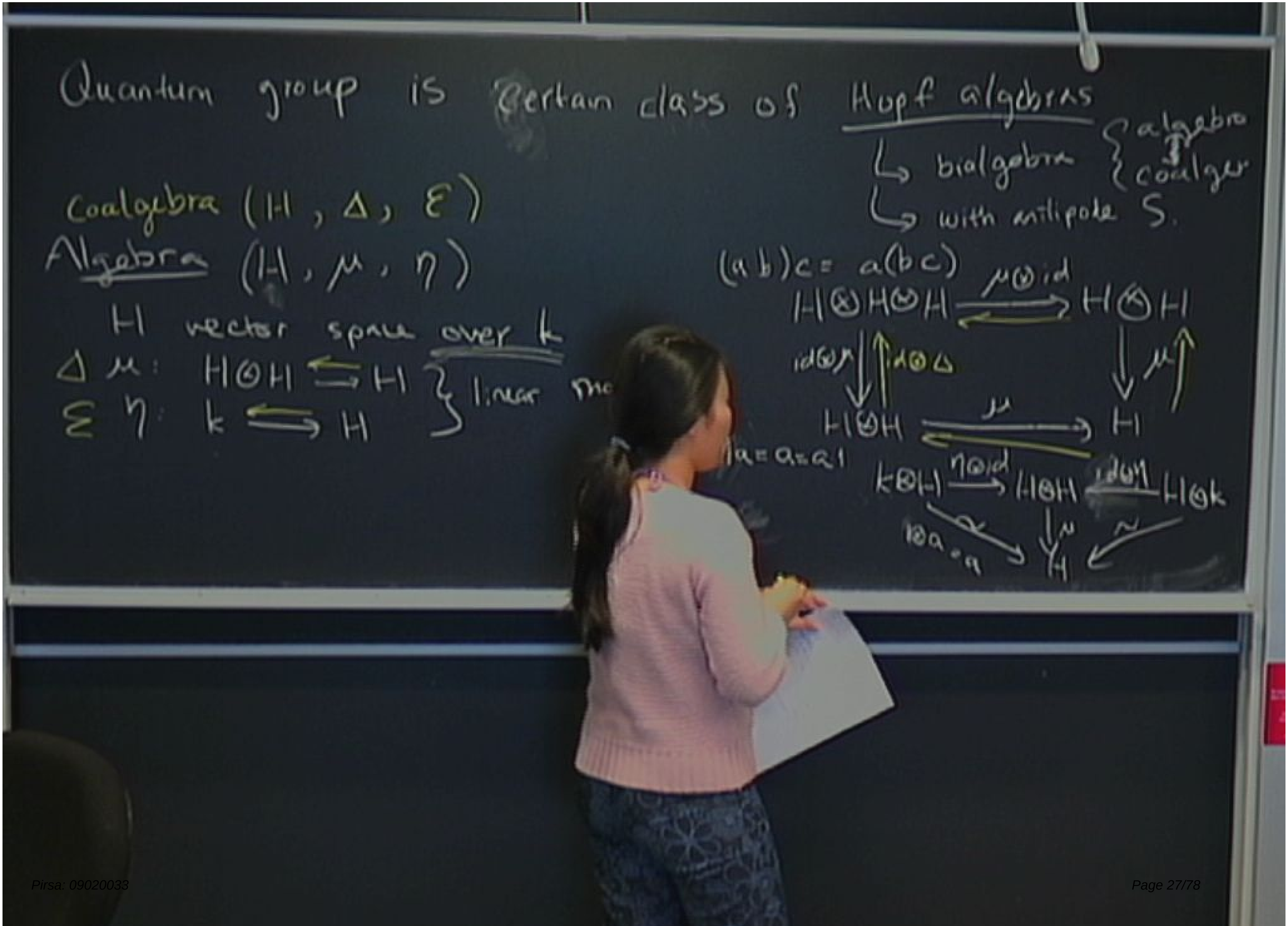
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id⊗Δ
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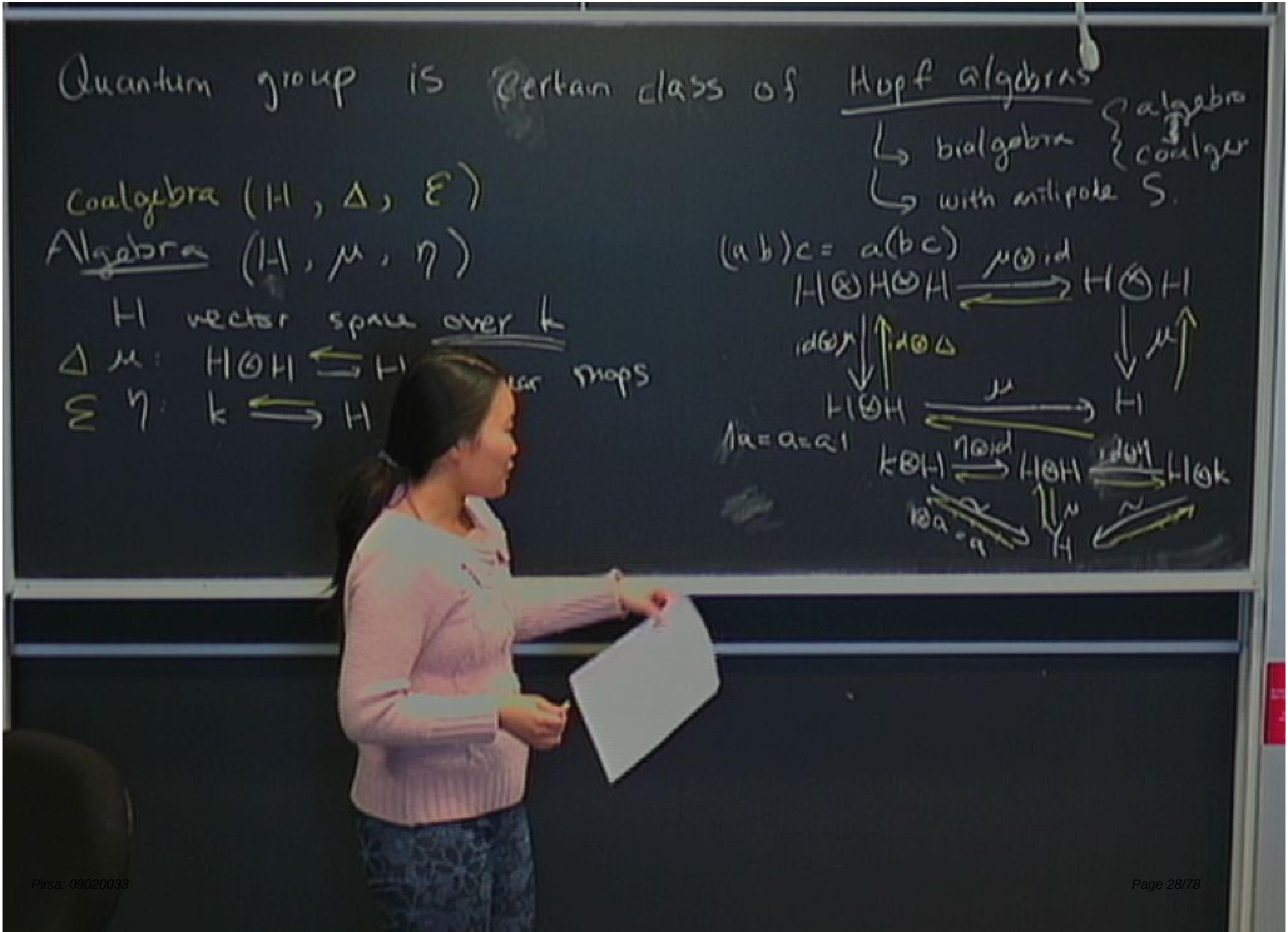
k⊗H
η⊗id
id⊗η
H⊗k
1⊗a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H⊗H⊗H
H⊗H
id⊗μ
id⊗Δ
μ
H⊗H
H
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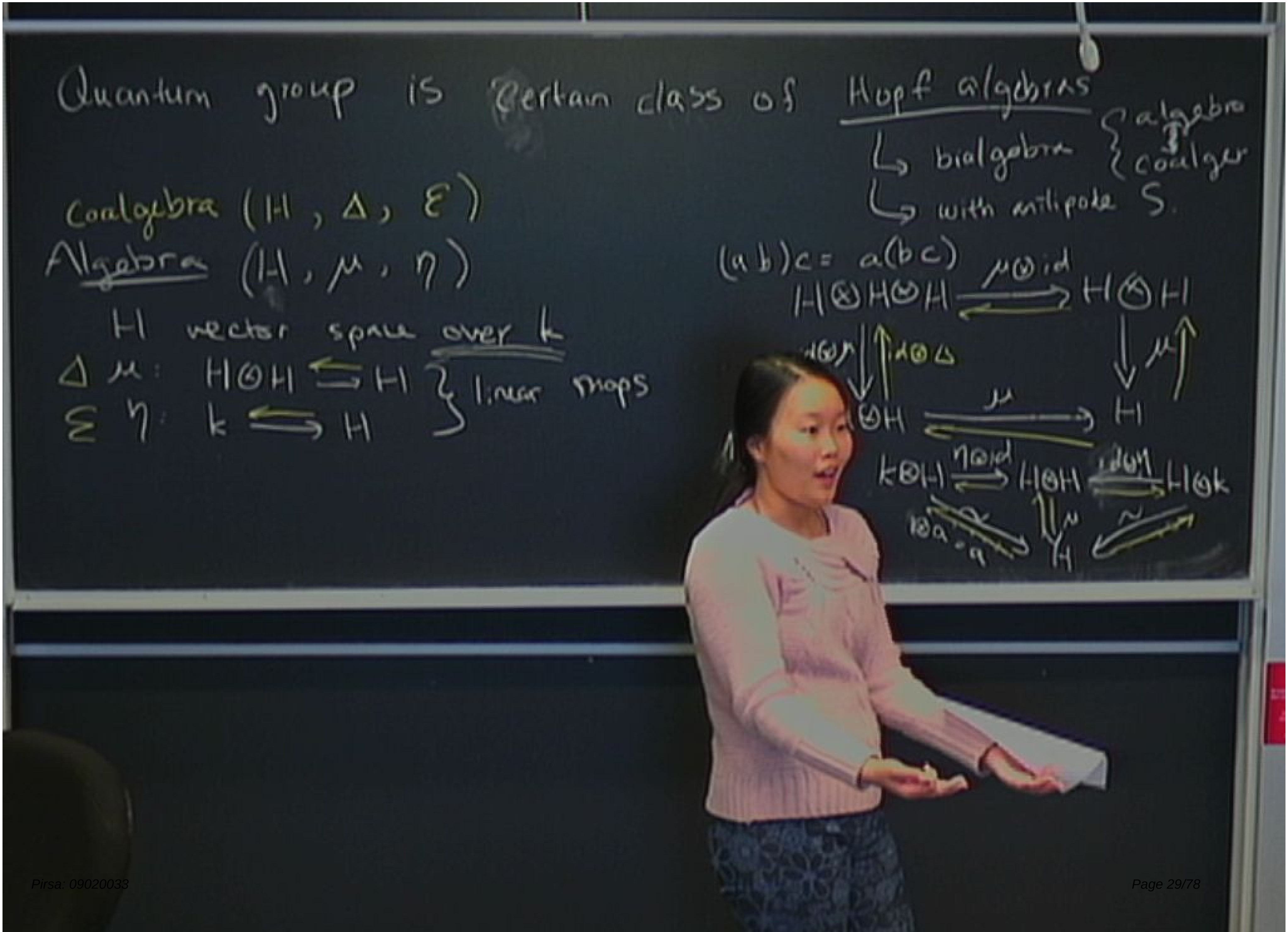
η⊗id
H⊗H
id⊗η
H⊗k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k⊗H → H⊗H ← H⊗k
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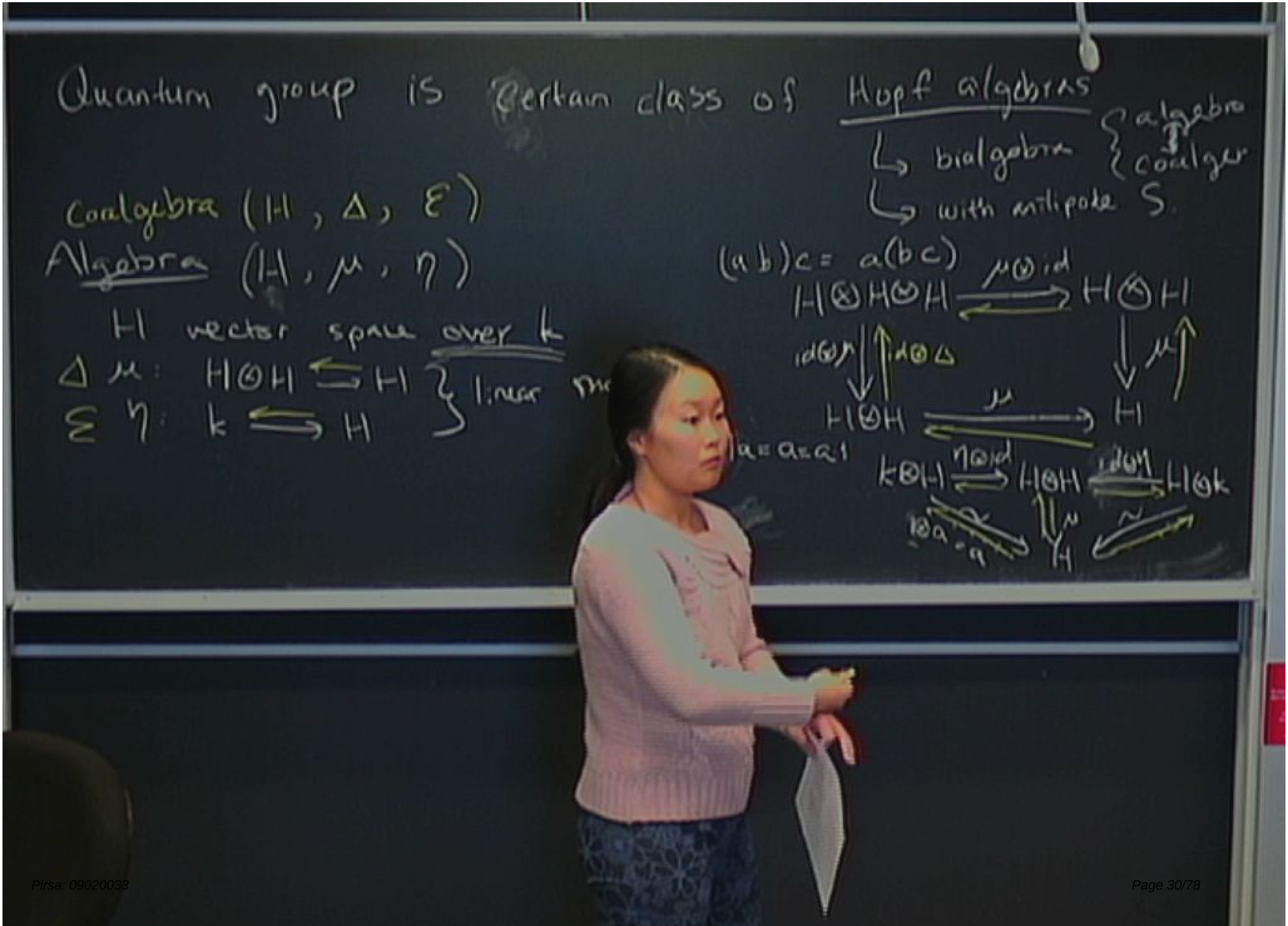
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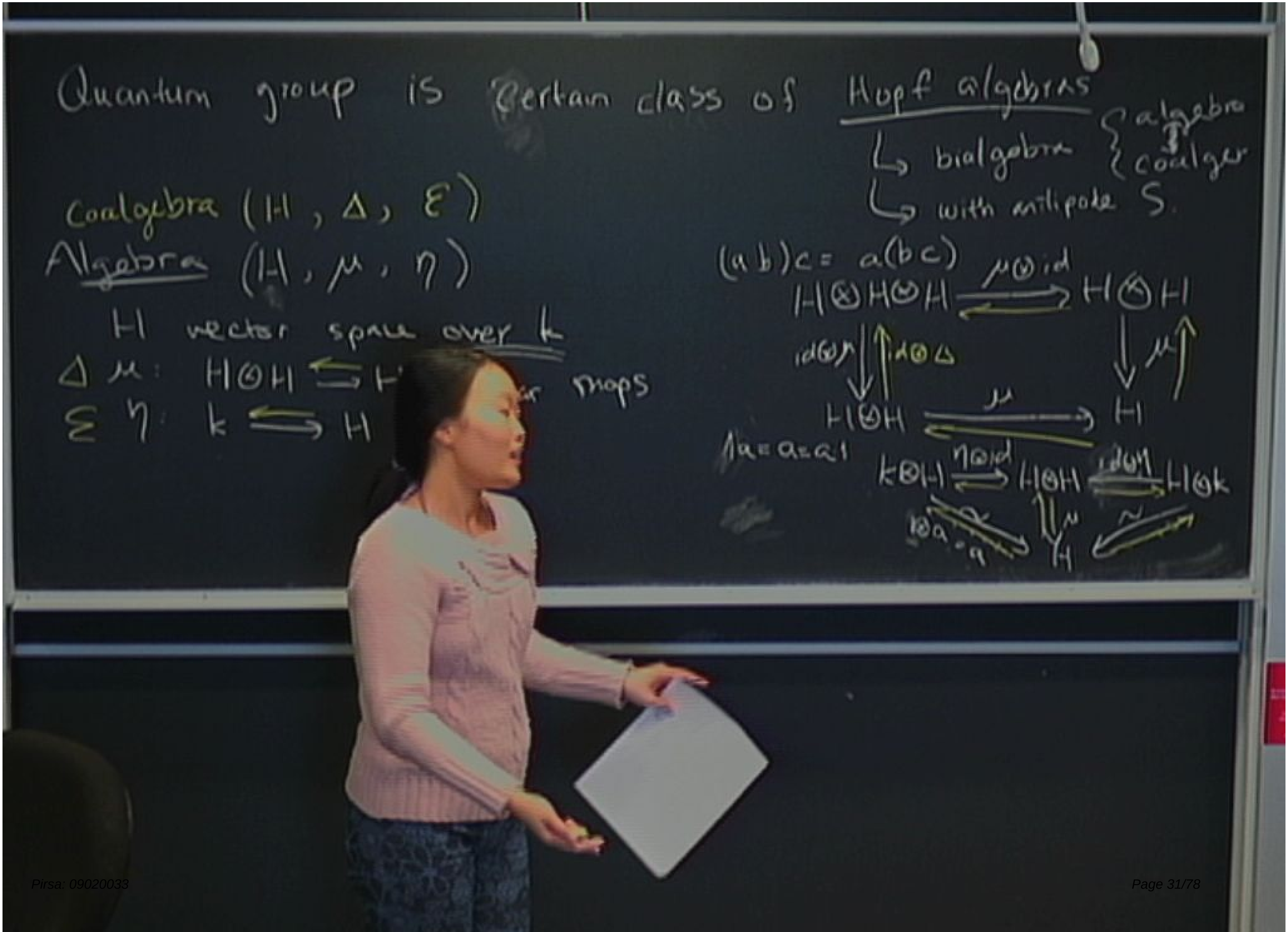
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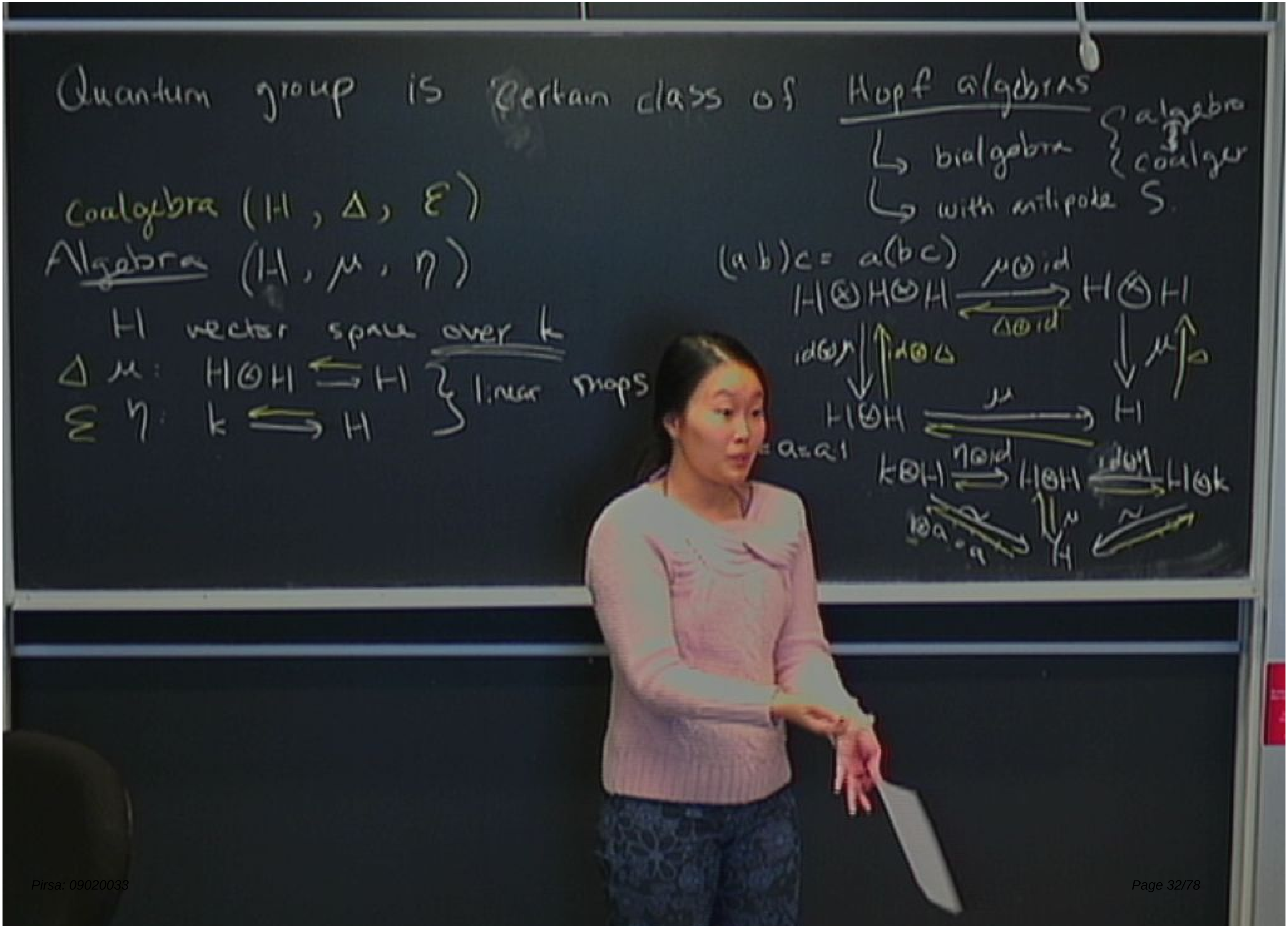
k⊗H ⇆ H⊗H ⇆ H⊗k
id⊗η
μ
H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μ⊗id
Δ⊗id
id⊗μ
id⊗Δ
H⊗H ⇆ H
μ
1·a=a·1
k⊗H ⇆ H⊗H ⇆ H⊗k
η⊗id
id⊗η
H
μ

Quantum group is certain class of Hopf algebras

↳ bialgebra $\begin{cases} \text{algebra} \\ \& \\ \text{coalgebra} \end{cases}$

↳ with antipode S.

Coalgebra (H, Δ, ε)

Algebra (H, μ, η)

H vector space over k

$$\left.\begin{array}{l} \Delta\ \mu : H \otimes H \rightleftarrows H \\ \varepsilon\ \eta : k \rightleftarrows H \end{array}\right\} \text{linear maps}$$

$(ab)c = a(bc)$

$$\begin{tikzcd}
H \otimes H \otimes H \arrow[r, shift left, "\mu \otimes id"] \arrow[d, shift right, "id \otimes \mu"'] & H \otimes H \arrow[l, shift left, "\Delta \otimes id"] \arrow[d, shift right] \\
H \otimes H \arrow[u, shift right, "id \otimes \Delta"'] \arrow[r, shift left, "\mu"] & H \arrow[l, shift left, "\Delta"] \arrow[u, shift right, "\mu \ \Delta"']
\end{tikzcd}$$

$1a = a = a1$

$$\begin{tikzcd}
k \otimes H \arrow[r, shift left, "\eta \otimes id"] \arrow[dr, "\cong"'] & H \otimes H \arrow[l, shift left] \arrow[d, "\mu"] & H \otimes k \arrow[l, shift left, "id \otimes \eta"'] \arrow[dl, "\sim"] \\
& H &
\end{tikzcd}$$

$1 \otimes a \mapsto a$

Quantum group is certain class of Hopf algebras

↳ bialgebra $\left\{\begin{array}{l}\text{algebra} \\ \& \\ \text{coalgebra}\end{array}\right.$

↳ with antipode S.

Coalgebra (H, Δ, ε)

Algebra (H, μ, η)

H vector space over k

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H\otimes H \arrow[u, shift right, "id\otimes\Delta"'] \arrow[r, shift left, "\mu"] & H \arrow[l, shift left, "\Delta"] \arrow[u, shift right, "\Delta"']
\end{tikzcd}$$

$1a = a = a1$

$$\begin{tikzcd}
k\otimes H \arrow[r, shift left, "\eta\otimes id"] \arrow[dr, "1\otimes a \mapsto a"'] & H\otimes H \arrow[d, "\mu"] & H\otimes k \arrow[l, shift left, "id\otimes\eta"'] \arrow[dl, "\sim"] \\
 & H &
\end{tikzcd}$$

Eg of algebra / coalgebra

G finite group

Eg of algebra / coalgebra

G finite group

$k[G]$ group algebra

Eg of algebra / coalgebra

G finite group

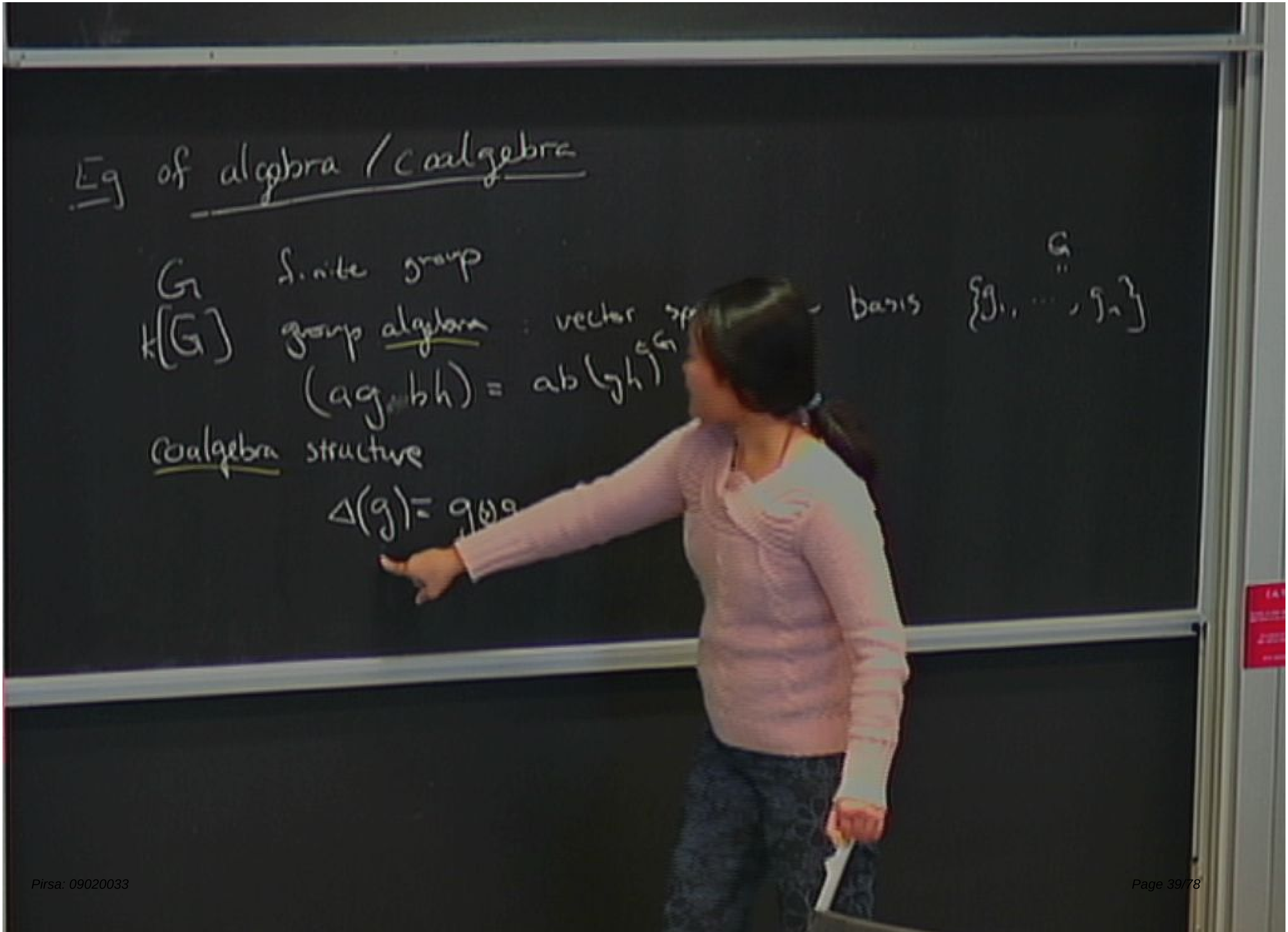
$k[G]$ group algebra : vector space with basis $\{g_1, \cdots, g_n\}$ ($G = \{g_1, \cdots, g_n\}$)

Eg of algebra / coalgebra

G finite group

$k[G]$ group algebra : vector space with basis $\overset{G}{\overbrace{\{g_1, \dots, g_n\}}}$

$(ag \cdot bh) = ab\,(gh)^{\in G}$



Eg of algebra / coalgebra
G finite group
k[G] group algebra : vector sp
basis {g1, ..., gn}
(ag, bh) = ab(gh)
coalgebra structure
Δ(g) = g⊗g

Eg of algebra / coalgebra

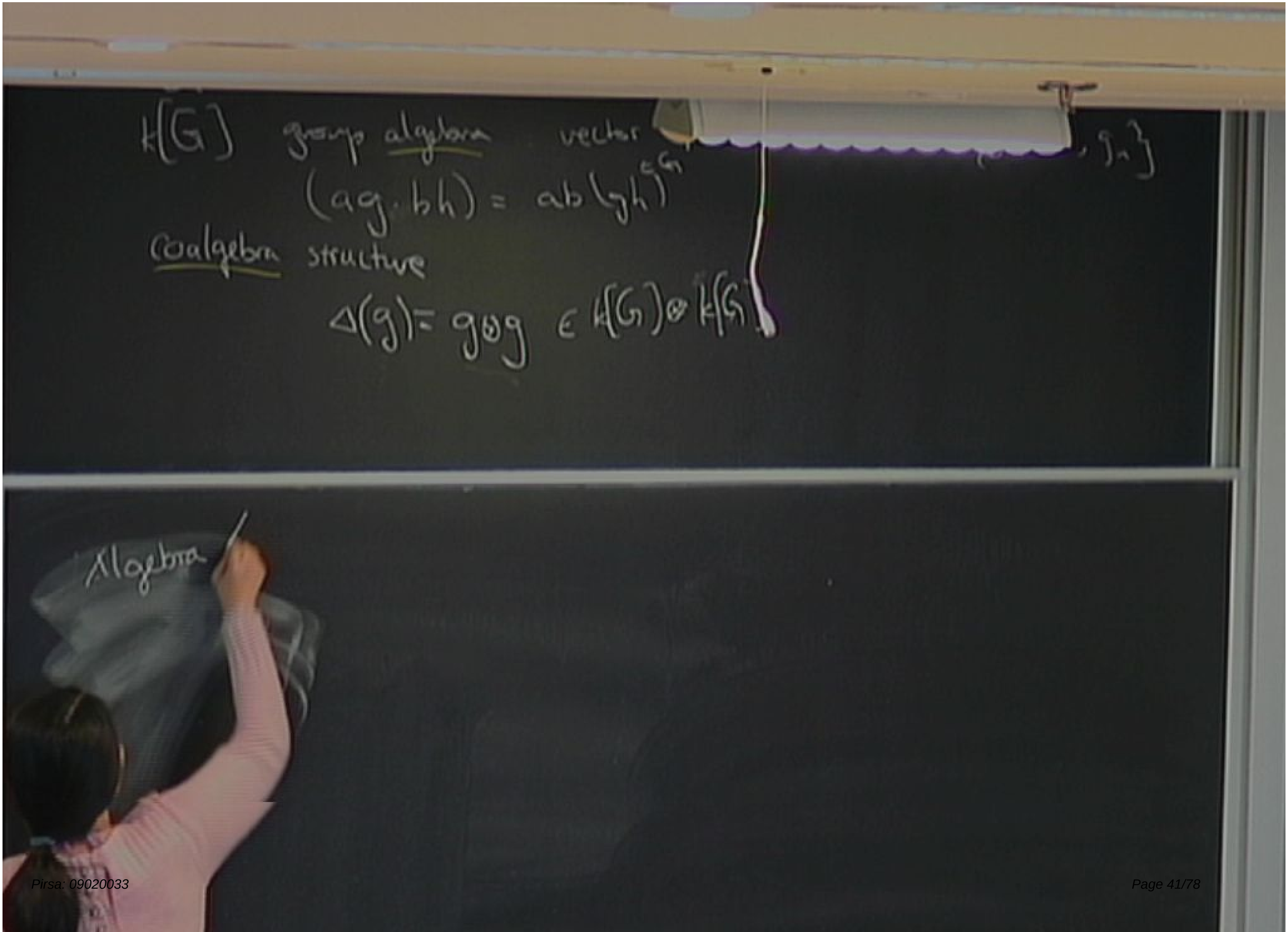
G finite group

$k[G]$ group algebra : vector space with basis $\{g_1, \cdots, g_n\}$ ($= G$)

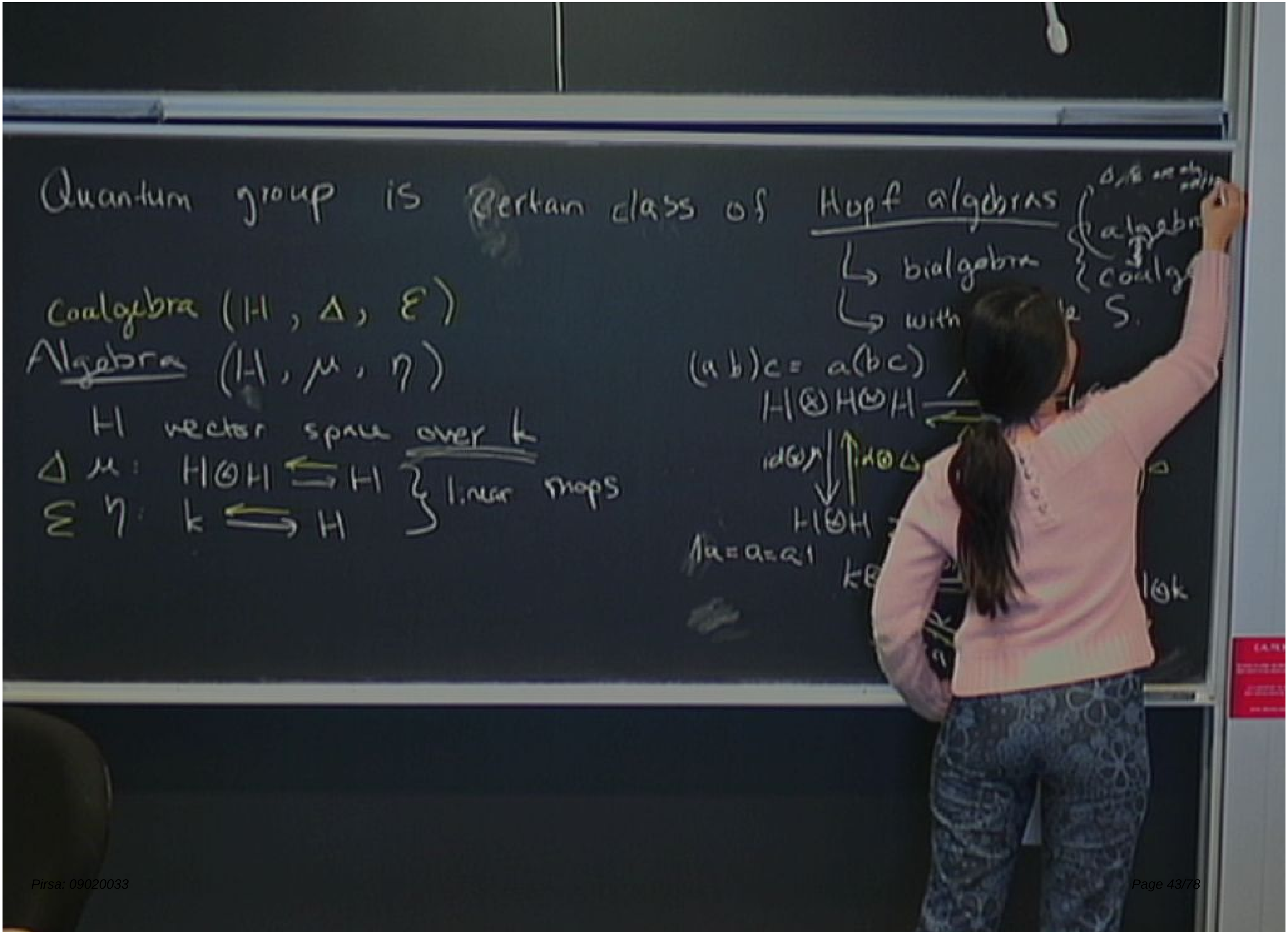
$(ag \cdot bh) = ab(gh)^{G}$

coalgebra structure

$\Delta(g) = g \otimes g \in k[G] \otimes k[G]$


k[G] group algebra vector
, g_n}
(ag · bh) = ab (gh) ∈ G
Coalgebra structure
Δ(g) = g⊗g ∈ k[G]⊗k[G]
Algebra

Algebra / Coalgebra Duality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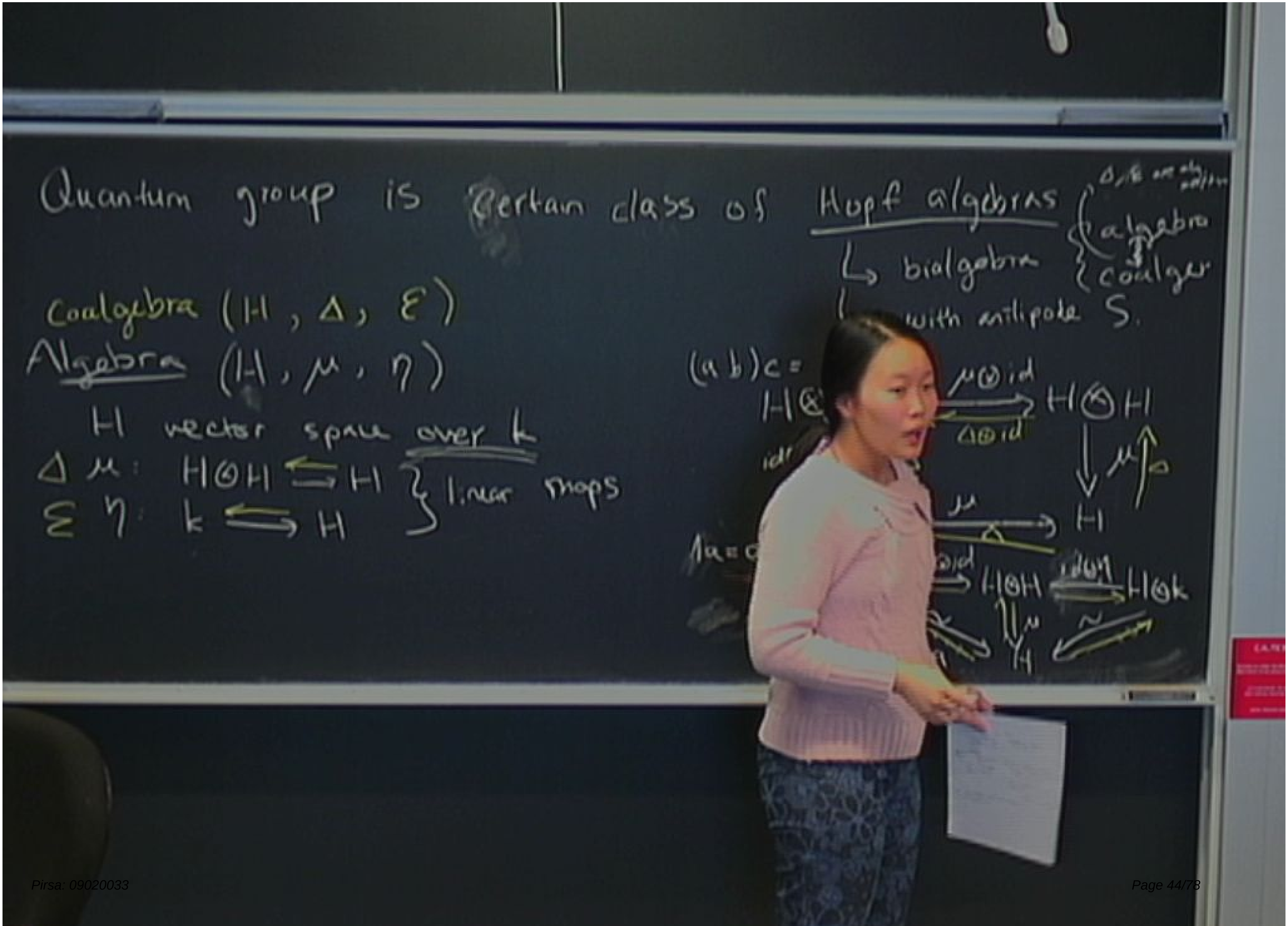
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Δ

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H \otimes H \arrow[r, shift left, "\mu"] \arrow[u, shift right, "id \otimes \Delta"'] & H \arrow[l, shift left, "\Delta"] \arrow[u, shift right, "\Delta"']
\end{tikzcd}$$

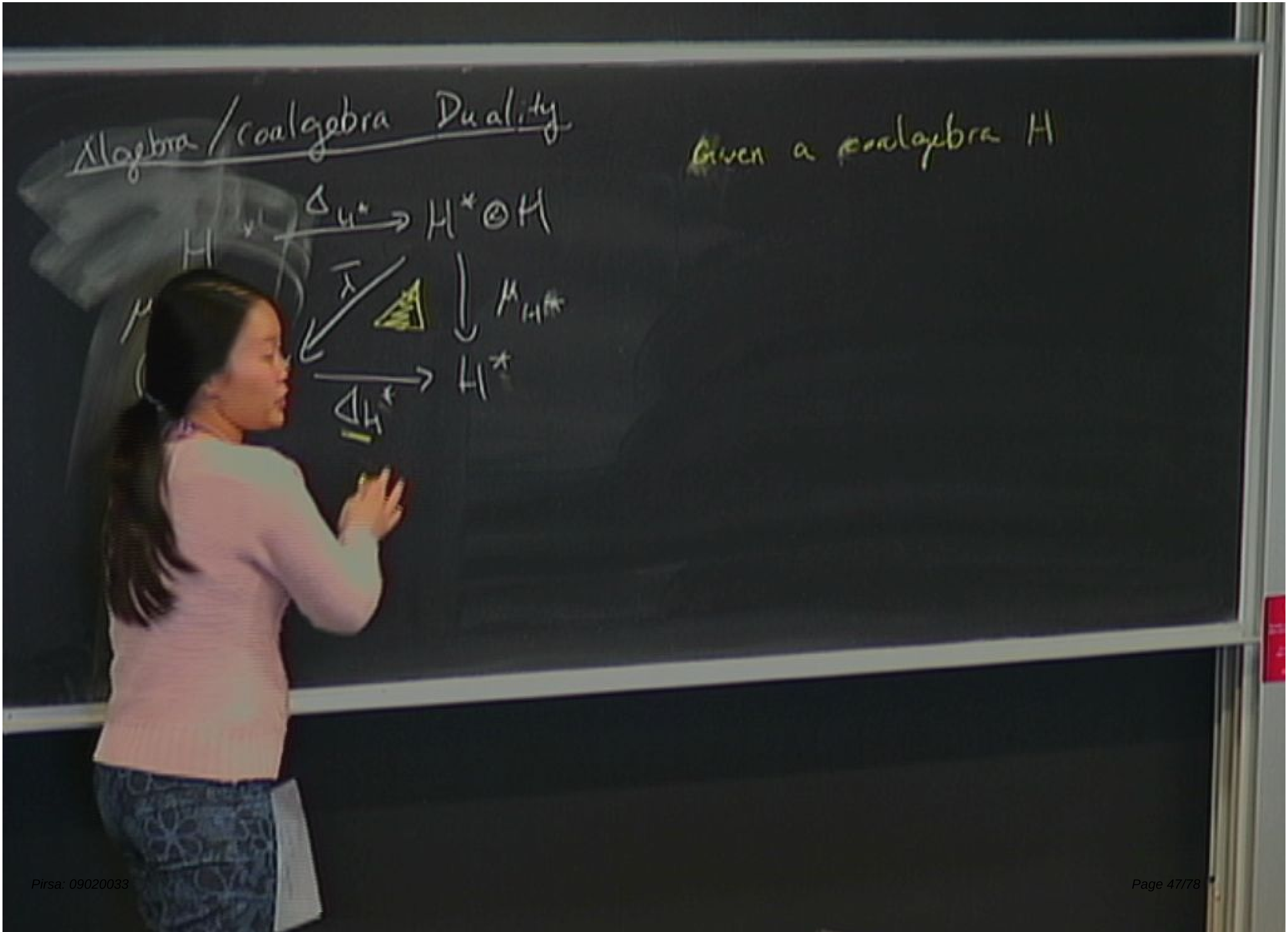
$1a = a = a1$

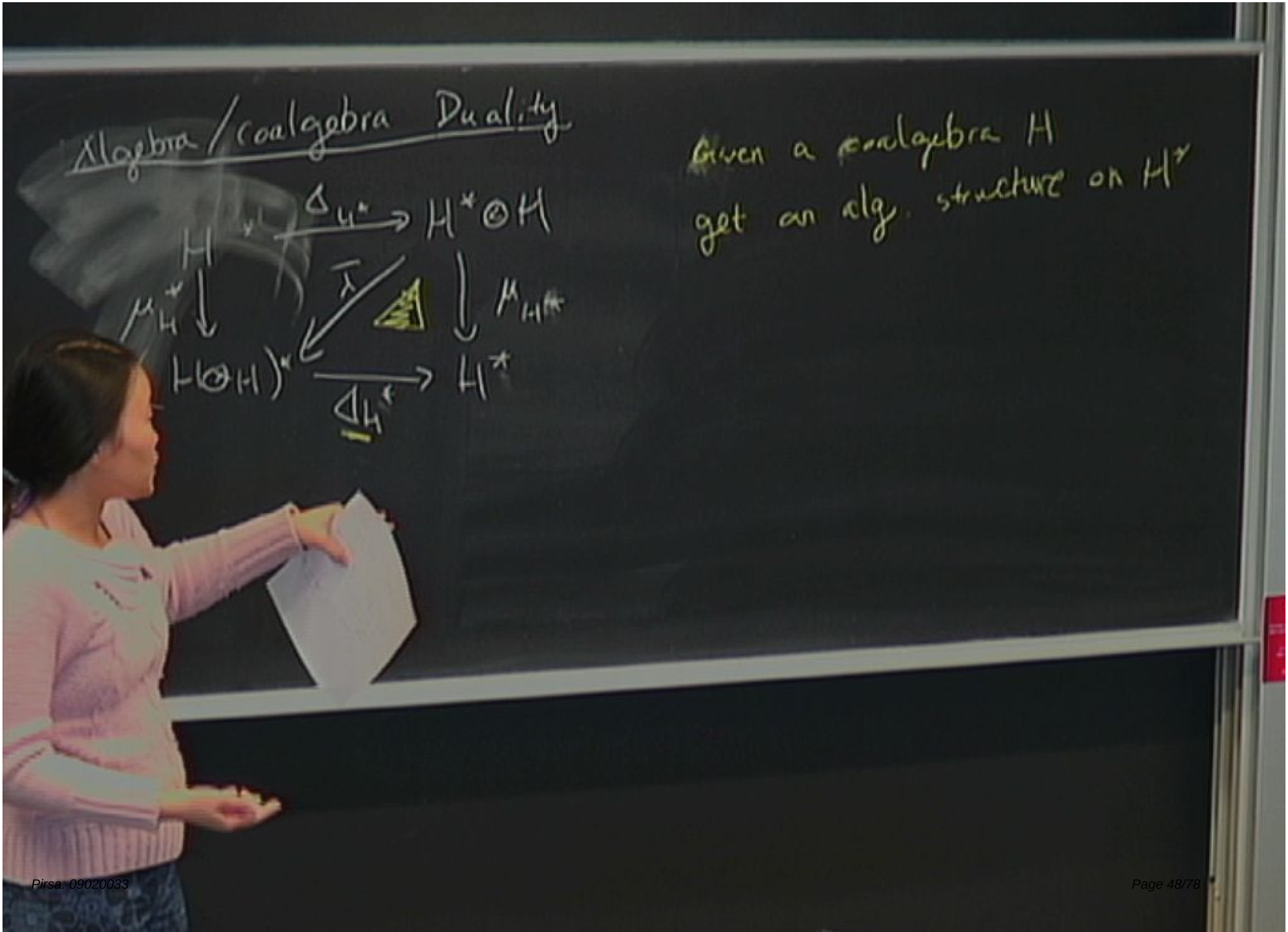
$$\begin{tikzcd}
k \otimes H \arrow[r, shift left, "\eta \otimes id"] \arrow[dr, "\sim"'] & H \otimes H \arrow[l, shift left] \arrow[d, shift left, "\mu"] & H \otimes k \arrow[l, shift right, "id \otimes \eta"'] \arrow[dl, "\sim"] \\
& H \arrow[u, shift left] &
\end{tikzcd}$$

$1 \otimes a \mapsto a$

Algebra / coalgebra Duality

$$\begin{tikzcd} H^* \arrow[r, "\Delta_{H^*}"] \arrow[d, "\mu_H^*"'] & H^* \otimes H \arrow[dl, "\bar{\lambda}"'] \arrow[d, "\mu_{H^*}"] \\ (H \otimes H)^* \arrow[r, "\Delta_H^*"'] & H^* \end{tikzcd}$$


Algebra / coalgebra Duality
Given a coalgebra H
$H^* \otimes H$
μ_{H^*}
H^*
Δ_{H^*}


Algebra / coalgebra Duality
Given a coalgebra H
get an alg. structure on H*

Algebra / coalgebra Duality

$$
\begin{tikzcd}
H^* \arrow[r, "\Delta_{H^*}"] \arrow[d, "\mu_H^*"'] & H^* \otimes H \arrow[dl, "\bar{\lambda}"'] \arrow[d, "\mu_{H^*}"] \\
(H \otimes H)^* \arrow[r, "\Delta_H^*"'] & H^*
\end{tikzcd}
$$

Given a coalgebra (H, μ, η) get an alg. structure on H^*

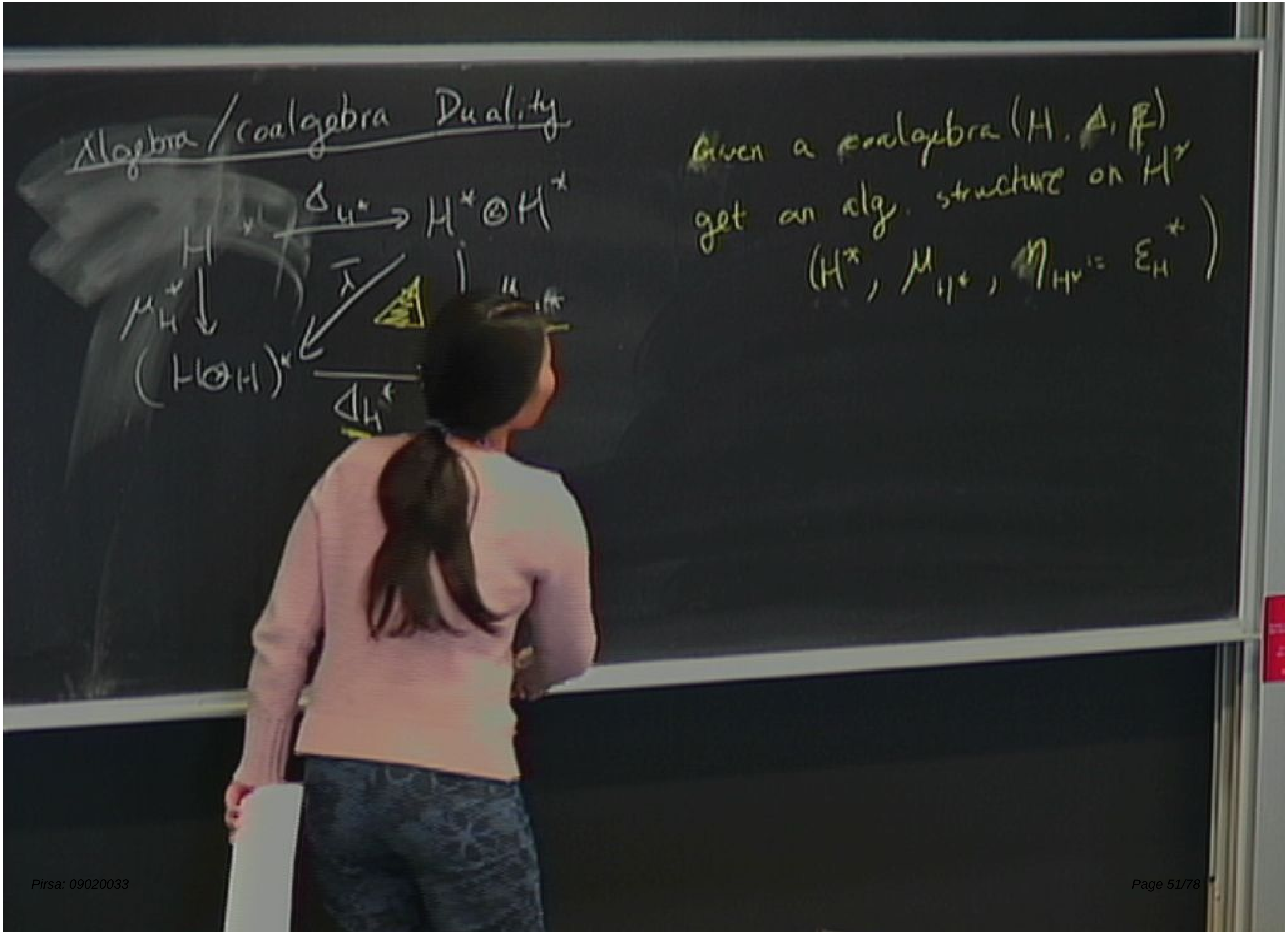
$$(H^*, \mu_{H^*}, \eta_{H^*} := \varepsilon_H^*)$$

Algebra / coalgebra Duality

$$\begin{tikzcd} H^* \arrow[r, "\Delta_{H^*}"] \arrow[d, "\mu_H^*"'] & H^* \otimes H^* \arrow[d, "\mu_{H^*}"] \arrow[dl, "\bar{\lambda}"'] \\ (H \otimes H)^* \arrow[r, "\Delta_H^*"'] & H^* \end{tikzcd}$$

Given a coalgebra (H, μ, η) get an alg. structure on H^*

$$(H^*, \mu_{H^*}, \eta_{H^*} := \varepsilon_H^*)$$


Algebra / coalgebra Duality
$H^* \xrightarrow{\Delta_{H^*}} H^* \otimes H^*$
μ_H^*
$\bar{\lambda}$
$(H \otimes H)^*$
Δ_H^*
Given a coalgebra (H, Δ, ε)
get an alg. structure on H^*
$(H^*, \mu_{H^*}, \eta_{H^*} := \varepsilon_H^*)$

Algebra / coalgebra Duality

$$
\begin{tikzcd}
H^* \arrow[r, "\Delta_{H^*}"] \arrow[d, "\mu_H^*"'] & H^* \otimes H^* \arrow[dl, "\bar{\lambda}"'] \arrow[d, "\mu_{H^*}"] \\
(H \otimes H)^* \arrow[r, "\Delta_H^*"'] & H^*
\end{tikzcd}
$$

Given a coalgebra (H, Δ, ε) get an alg. structure on H^*

$$\left(H^*, \mu_{H^*}, \eta_{H^*} := \varepsilon_H^*\right)$$

Algebra / coalgebra Duality

$$\begin{tikzcd}
H^* \arrow[r, "\Delta_{H^*}"] \arrow[d, "\mu_H^*"'] & H^* \otimes H^* \arrow[d, "\mu_{H^*}"] \arrow[dl, "\bar{\lambda}"'] \\
(H \otimes H)^* \arrow[r, "\Delta_H^*"'] & H^*
\end{tikzcd}$$

$\bar{\lambda}$

Given a coalgebra (H, Δ, ε)
get an alg. structure on H^*

$$\left(H^*,\ \mu_{H^*},\ \eta_{H^*} := \varepsilon_H^*\right)$$

$$\mu_{H^*} = \Delta_H^* \circ \bar{\lambda}$$

Algebra / coalgebra Duality

$$\begin{tikzcd} H^* \arrow[r, "\Delta_{H^*}"] \arrow[d, "\mu_H^*"'] & H^* \otimes H^* \arrow[dl, "\bar{\lambda}"'] \arrow[d, "\mu_{H^*}"] \\ (H \otimes H)^* \arrow[r, "\Delta_H^*"'] & H^* \end{tikzcd}$$

$\bar{\lambda}(\alpha \otimes \beta)$

Given a coalgebra (H, Δ, ε) get an alg. structure on H^*

$$\left(H^*,\ \mu_{H^*},\ \eta_{H^*} := \varepsilon_H^*\right)$$

$$\mu_{H^*} = \Delta_H^* \circ \bar{\lambda}$$

Algebra / coalgebra Duality

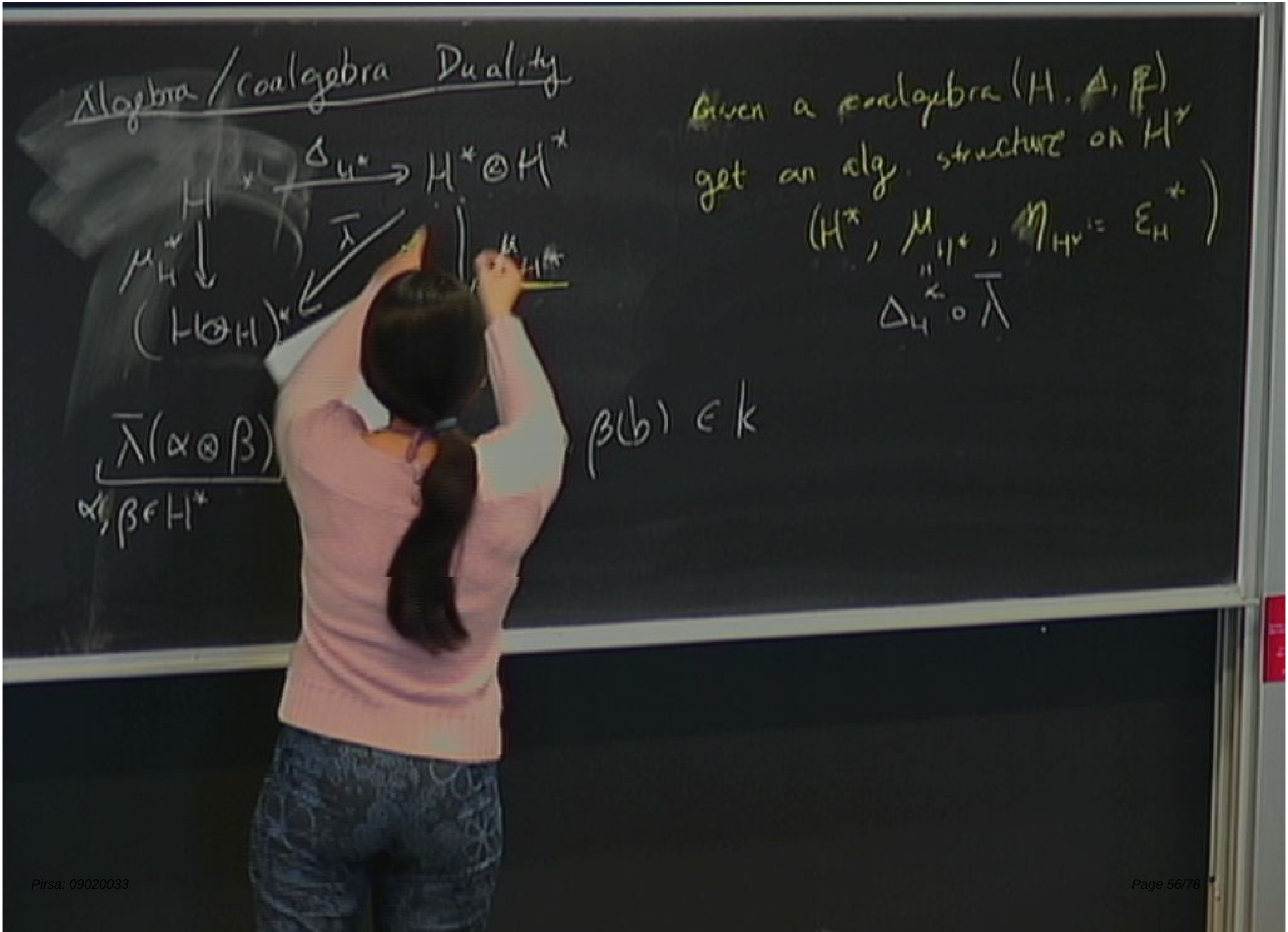
$$\begin{tikzcd}
H^* \arrow[r, "\Delta_{H^*}"] \arrow[d, "\mu_H^*"'] & H^* \otimes H^* \arrow[dl, "\bar{\lambda}"'] \arrow[d, "\mu_{H^*}"] \\
(H \otimes H)^* \arrow[r, "\Delta_H^*"'] & H^*
\end{tikzcd}$$

$$\underbrace{(\alpha \otimes \beta)}_{H^*}(a \otimes b) = \qquad a, b \in H$$

Given a coalgebra (H, Δ, ε) get an alg. structure on H^*

$$\left(H^*, \mu_{H^*}, \eta_{H^*} := \varepsilon_H^*\right)$$

$$\mu_{H^*} = \Delta_H^* \circ \bar{\lambda}$$


Algebra / coalgebra Duality
Given a coalgebra (H, Δ, ε)
get an alg. structure on H*
(H*, μ_{H*}, η_{H*} := ε_H*)
= Δ_H* ∘ λ̄
H* ⊗ H*
(H ⊗ H)*
λ̄(α ⊗ β)
α, β ∈ H*
β(b) ∈ k

Algebra / Coalgebra Duality

$$
\begin{tikzcd}
H^* \arrow[r, "\Delta_{H^*}"] \arrow[d, "\mu_H^*"'] & H^* \otimes H^* \arrow[dl, "\bar{\lambda}"'] \arrow[d, "\mu_{H^*}"] \\
(H \otimes H)^* \arrow[r, "\Delta_H^*"'] & H^*
\end{tikzcd}
$$

Given a coalgebra (H, Δ, ε) get an alg. structure on H^*

$$\left(H^*, \underset{\substack{\parallel \\ \Delta_H^* \circ \bar{\lambda}}}{\mu_{H^*}}, \eta_{H^*} := \varepsilon_H^*\right)$$

$$\underbrace{\bar{\lambda}(\alpha \otimes \beta)}_{\alpha, \beta \in H^*}\underset{a,b\in H}{(a \otimes b)} = \alpha(a)\,\beta(b) \in k$$

Algebra/coalgebra Duality

$$\begin{tikzcd}
H^* \arrow[r, "\Delta_{H^*}"] \arrow[d, "\mu_H^*"'] & H^* \otimes H^* \arrow[dl, "\bar{\lambda}"'] \arrow[d, "\mu_{H^*}"] \\
(H \otimes H)^* \arrow[r, "\Delta_H^*"'] & H^*
\end{tikzcd}$$

$$\underbrace{\bar{\lambda}(\alpha \otimes \beta)}_{\alpha, \beta \in H^*}\underbrace{(a \otimes b)}_{a, b \in H} = \alpha(a)\,\beta(b) \in k$$

Given a coalgebra (H, Δ, ε) get an alg. structure on H^*

$$\left(H^*,\ \mu_{H^*},\ \eta_{H^*} := \varepsilon_H^*\right)$$

$$\mu_{H^*} = \Delta_H^* \circ \bar{\lambda}$$

Algebra / coalgebra Duality

$$\begin{tikzcd}
H^* \arrow[r, "\Delta_{H^*}"] \arrow[d, "\mu_H^*"'] & H^* \otimes H^* \arrow[dl, "\bar{\lambda}"'] \arrow[d, "\mu_{H^*}"] \\
(H \otimes H)^* \arrow[r, "\Delta_H^*"'] & H^*
\end{tikzcd}$$

Given a coalgebra (H, Δ, ε) get an alg. structure on H^*

$$\left(H^*, \underset{\Delta_H^* \circ \bar{\lambda}}{\mu_{H^*}} , \eta_{H^*} := \varepsilon_H^*\right)$$

$$\underbrace{\bar{\lambda}(\alpha \otimes \beta)}_{}(a \otimes b) = \alpha(a)\,\beta(b) \in k$$

$\alpha, \beta \in H^*$ $\qquad a, b \in H$

$$f: A \to B \;; \quad f^*: B^* \to A^*$$

$$\underbrace{f^*(\beta)}_{\in A^*}(\hat{a}) =$$

structure

$g \in G \quad \Delta(g) = g \otimes g \in k[G] \otimes k[G]$

Algebra/coalgebra Duality

$$
\begin{tikzcd}
H^* \arrow[r, "\Delta_{H^*}"] \arrow[d, "\mu_H^*"'] & H^* \otimes H^* \arrow[dl, "\bar{\lambda}"'] \arrow[d, "\mu_{H^*}"] \\
(H \otimes H)^* \arrow[r, "\Delta_H^*"'] & H^*
\end{tikzcd}
$$

Given a coalgebra (H, Δ, ε) get an alg. structure on H^*

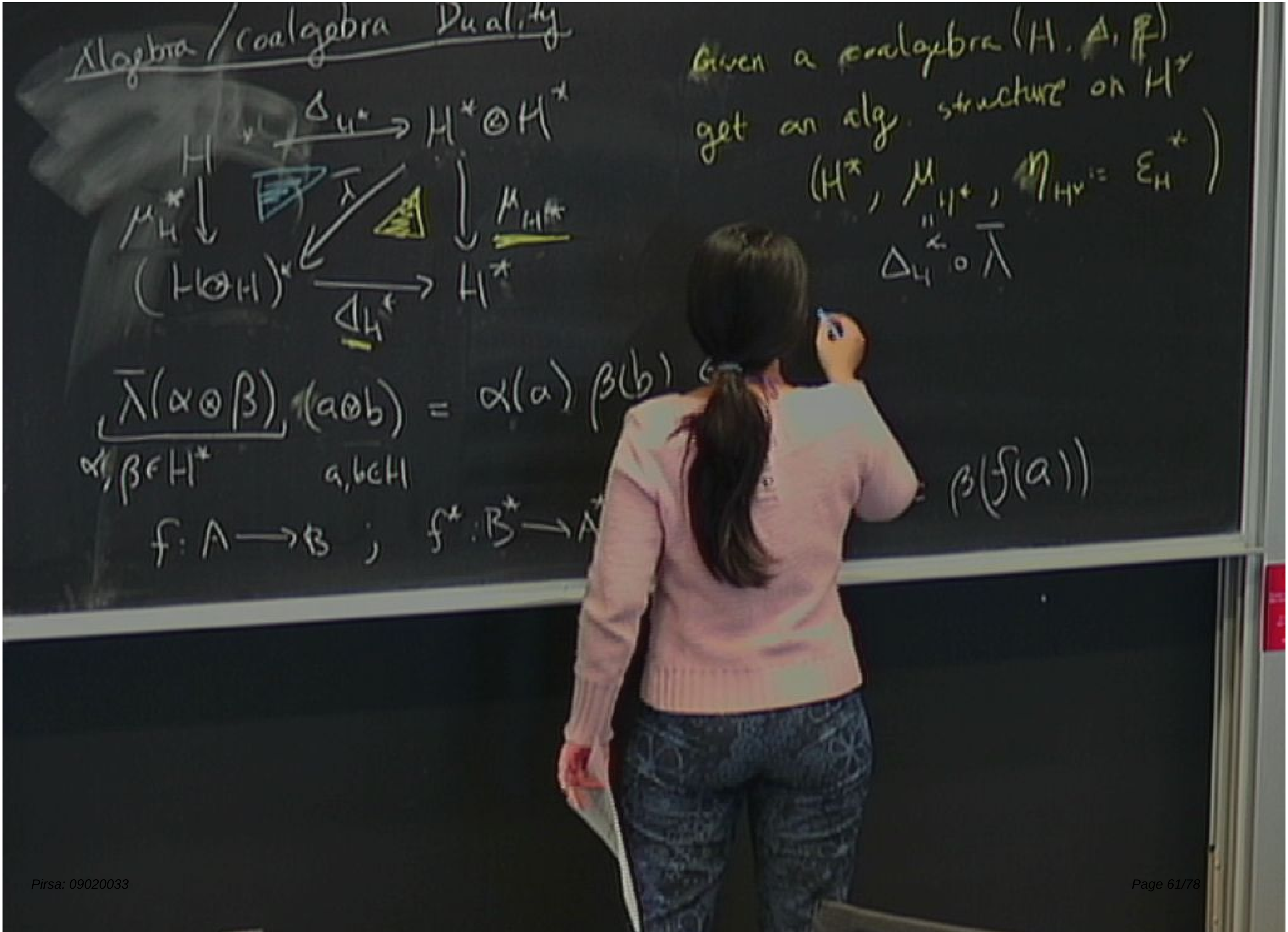
$$(H^*, \mu_{H^*}, \eta_{H^*} := \varepsilon_H^*)$$

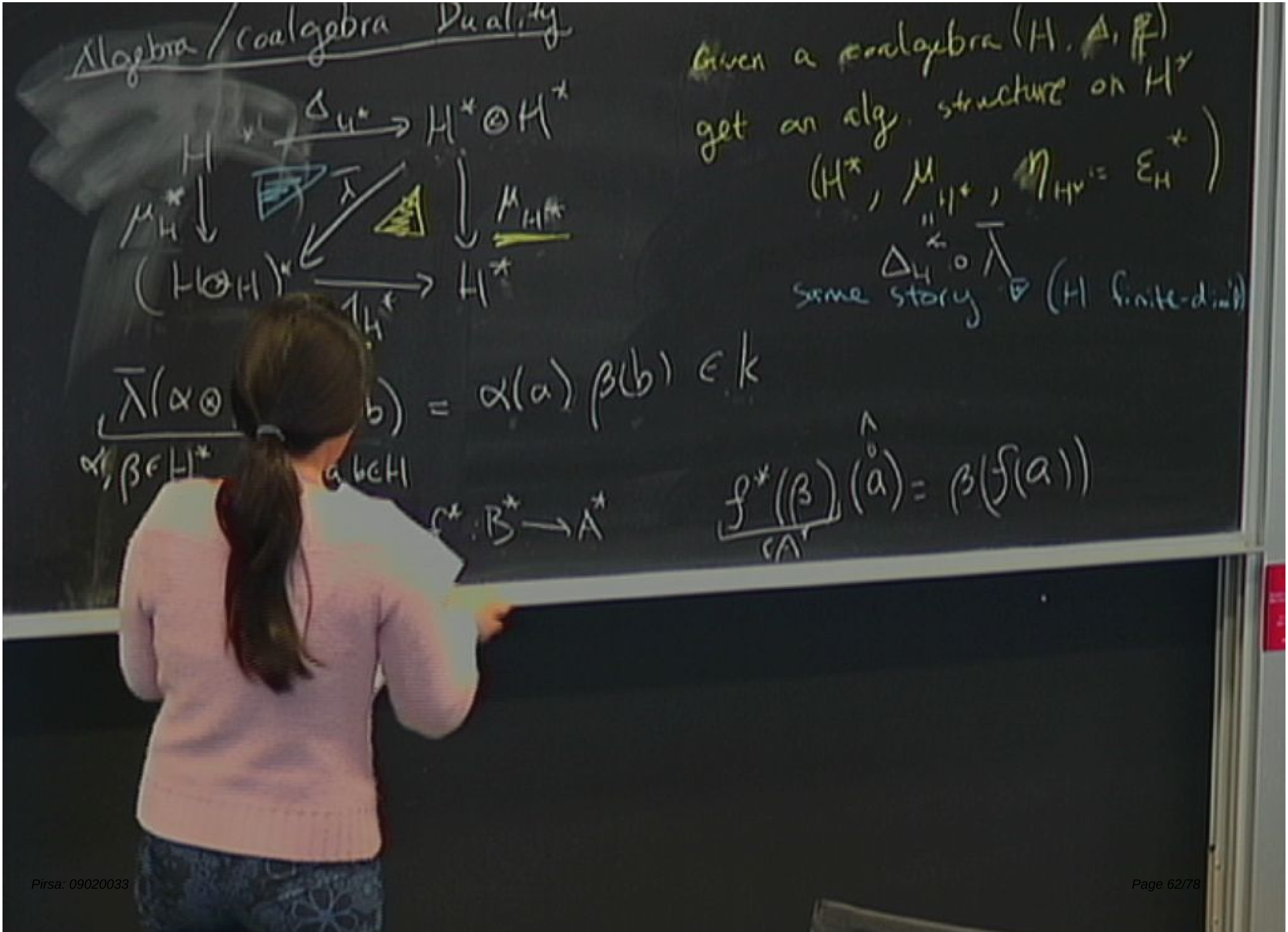
$$\mu_{H^*} = \Delta_H^* \circ \bar{\lambda}$$

$$\underbrace{\bar{\lambda}(\alpha \otimes \beta)}_{\alpha, \beta \in H^*}(a \otimes b) = \alpha(a)\beta(b) \in k \qquad a, b \in H$$

$f : A \longrightarrow B \; ; \; f^* : B^* \longrightarrow A^*$

$$\underbrace{f^*(\beta)}_{\in A^*}(\overset{\hat{a}}{a}) = \beta(f(a))$$


Algebra / Coalgebra Duality
$H^* \xrightarrow{\Delta_{H^*}} H^* \otimes H^*$
$\mu_{H^*} \downarrow$ $\bar{\lambda}$ $\downarrow \mu_{H^*}$
$(H \otimes H)^* \xrightarrow{\Delta_H^*} H^*$
$\bar{\lambda}(\alpha \otimes \beta)(a \otimes b) = \alpha(a)\,\beta(b)$
$\alpha, \beta \in H^*$ $a, b \in H$
$f: A \longrightarrow B$; $f^*: B^* \longrightarrow A^*$
$\beta(f(a))$
Given a coalgebra (H, Δ, ε)
get an alg. structure on H^*
$(H^*, \mu_{H^*}, \eta_{H^*} := \varepsilon_H^*)$
$\mu_{H^*} = \Delta_H^* \circ \bar{\lambda}$


Algebra / Coalgebra Duality
Given a coalgebra (H, Δ, ε) get an alg. structure on H*
(H*, μ_H*, η_H* := ε_H*)
Δ_H* ∘ λ̄
Same story (H finite-dim'l)
= α(a) β(b) ∈ k
f*: B* → A*
f*(β)(a) = β(f(a))

Algebra / Coalgebra Duality

$$\begin{tikzcd}
H^* \arrow[r, "\Delta_{H^*}"] \arrow[d, "\mu_H^*"'] & H^* \otimes H^* \arrow[d, "\mu_{H^*}"] \arrow[dl, "\bar{\lambda}"'] \\
(H \otimes H)^* \arrow[r, "\Delta_H^*"'] & H^*
\end{tikzcd}$$

Given a coalgebra (H, Δ, ε) get an alg. structure on H^*

$$\left(H^*, \mu_{H^*}, \eta_{H^*} := \varepsilon_H^*\right)$$

$$\mu_{H^*} = \Delta_H^* \circ \bar{\lambda}$$

Same story for (H finite-dim'l)

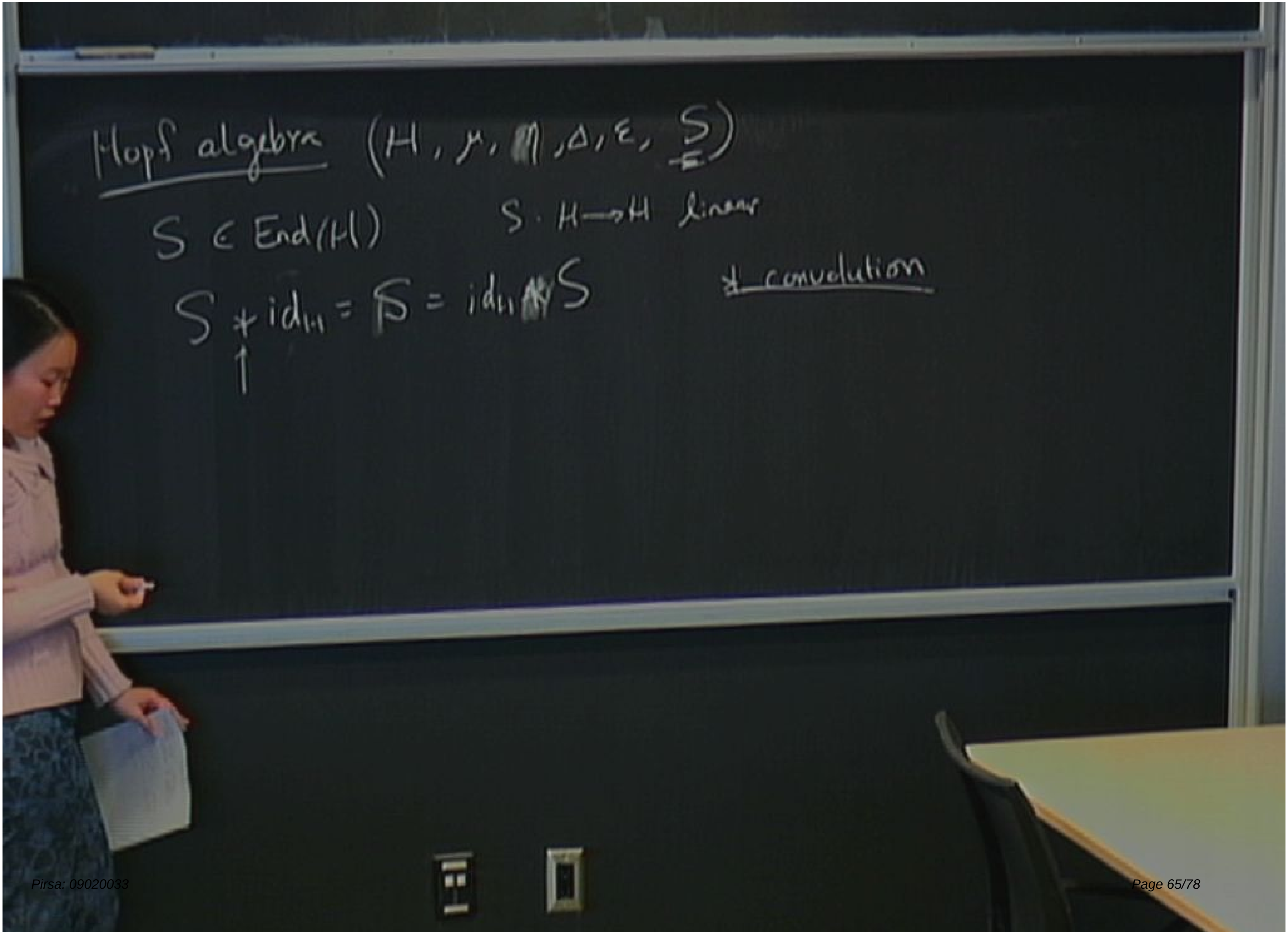
$$\underbrace{\bar{\lambda}(\alpha \otimes \beta)}_{\alpha, \beta \in H^*} \underbrace{(a \otimes b)}_{a, b \in H} = \alpha(a)\, \beta(b) \in k$$

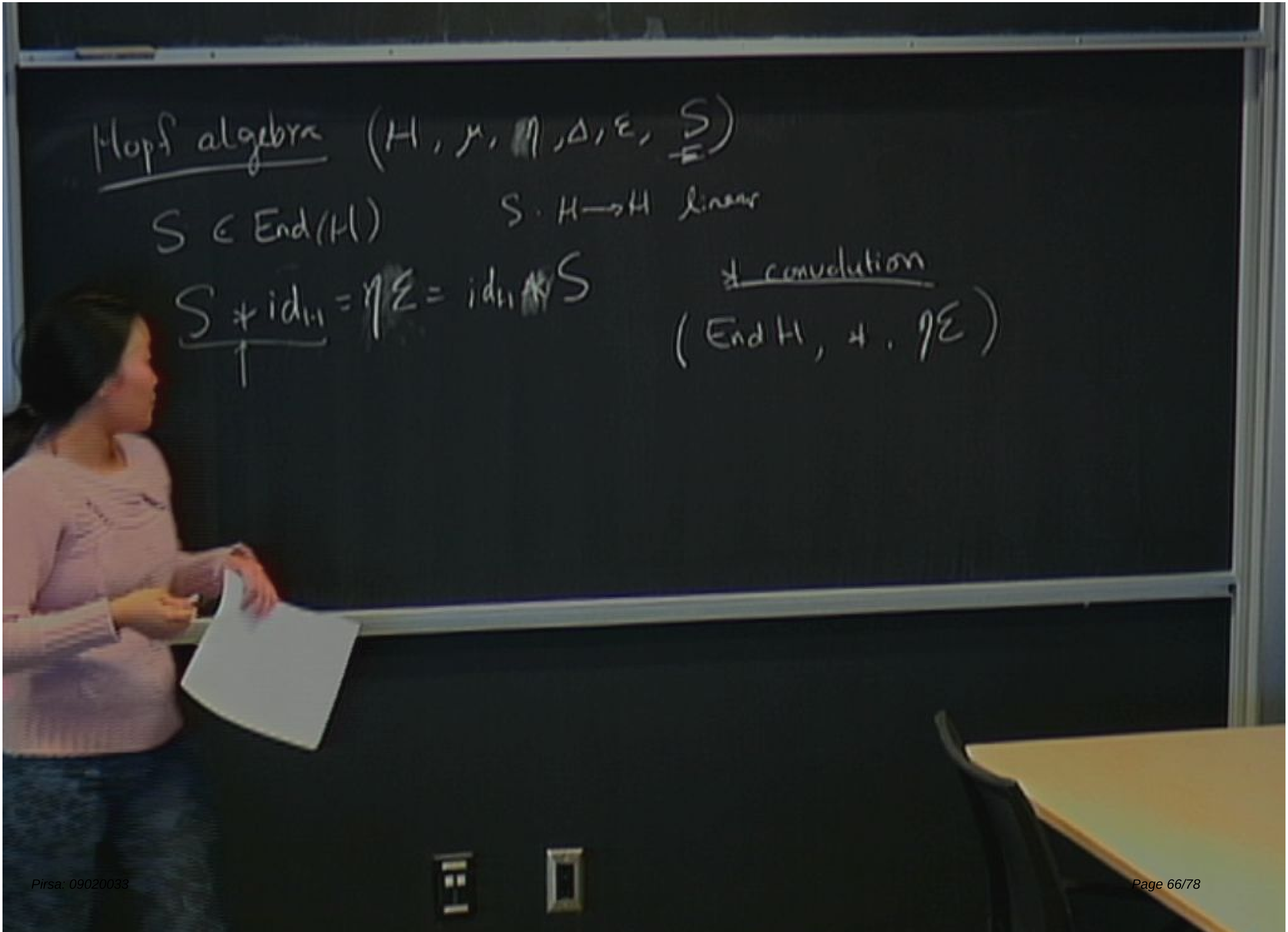
$$f : A \longrightarrow B \; ; \quad f^* : B^* \longrightarrow A^*$$

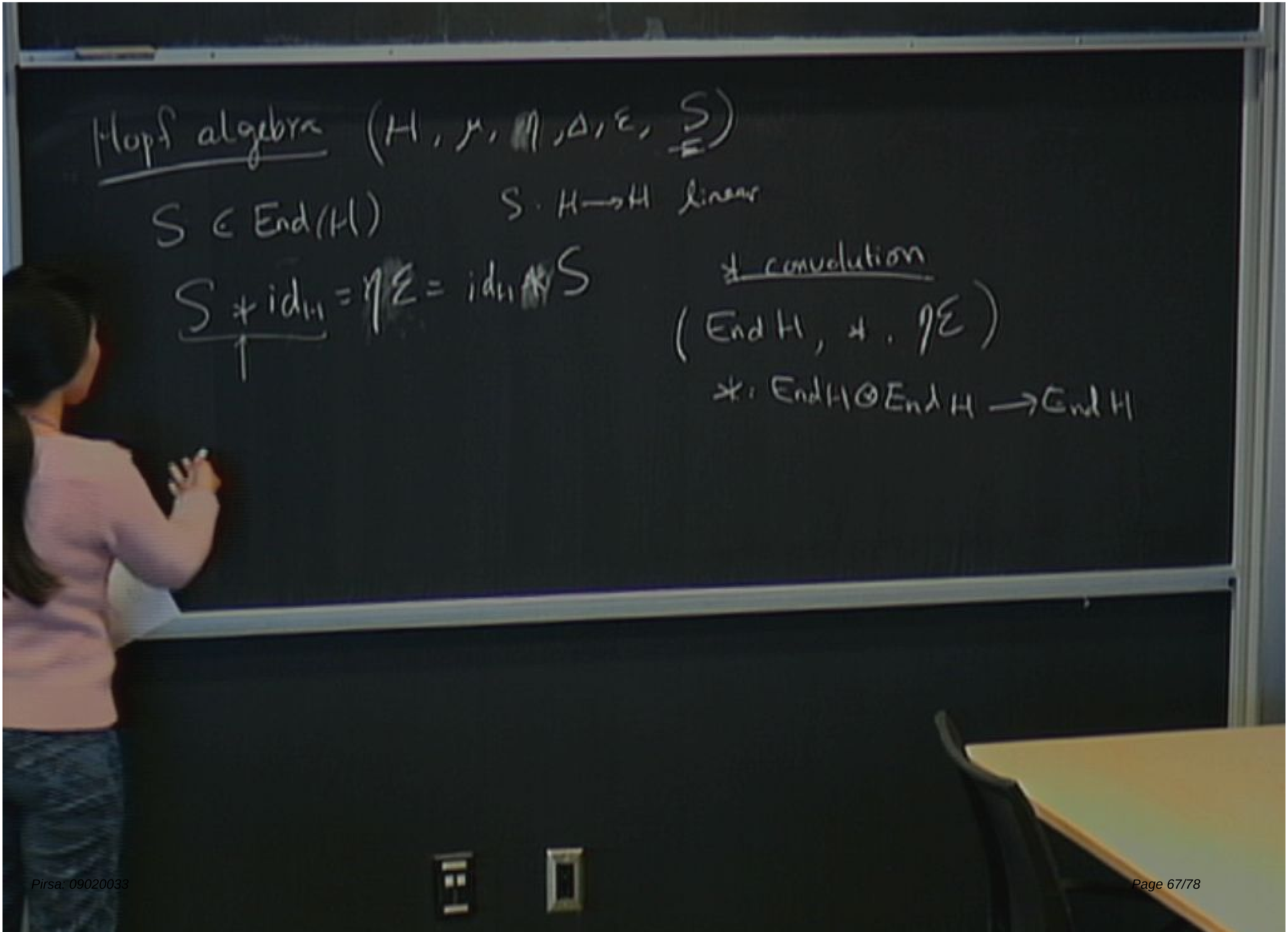
$$\underbrace{f^*(\beta)}_{\in A^*}(a) = \beta(f(a))$$

Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, S)$

$S \in \mathrm{End}(H)$ $\quad$ $S : H \to H$ linear


Hopf algebra (H, μ, , Δ, ε, S)
S ∈ End(H)
S : H → H linear
S * id_H = S = id_H * S
* convolution


Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, S)$
$S \in \mathrm{End}(H)$
$S: H \to H$ linear
$S * id_H = \eta\varepsilon = id_H * S$
$*$ convolution
$(\mathrm{End}\, H, *, \eta\varepsilon)$


Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, S)$
$S \in End(H)$
$S: H \to H$ linear
$S * id_H = \eta\varepsilon = id_H * S$
$*$ convolution
$(End\,H, *, \eta\varepsilon)$
$*: End\,H \otimes End\,H \to End\,H$

Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, \underline{S})$

$S \in End(H)$ $\quad S : H \to H$ linear

$$\underbrace{S * id_H}_{\uparrow} = \eta\varepsilon = id_H * S$$

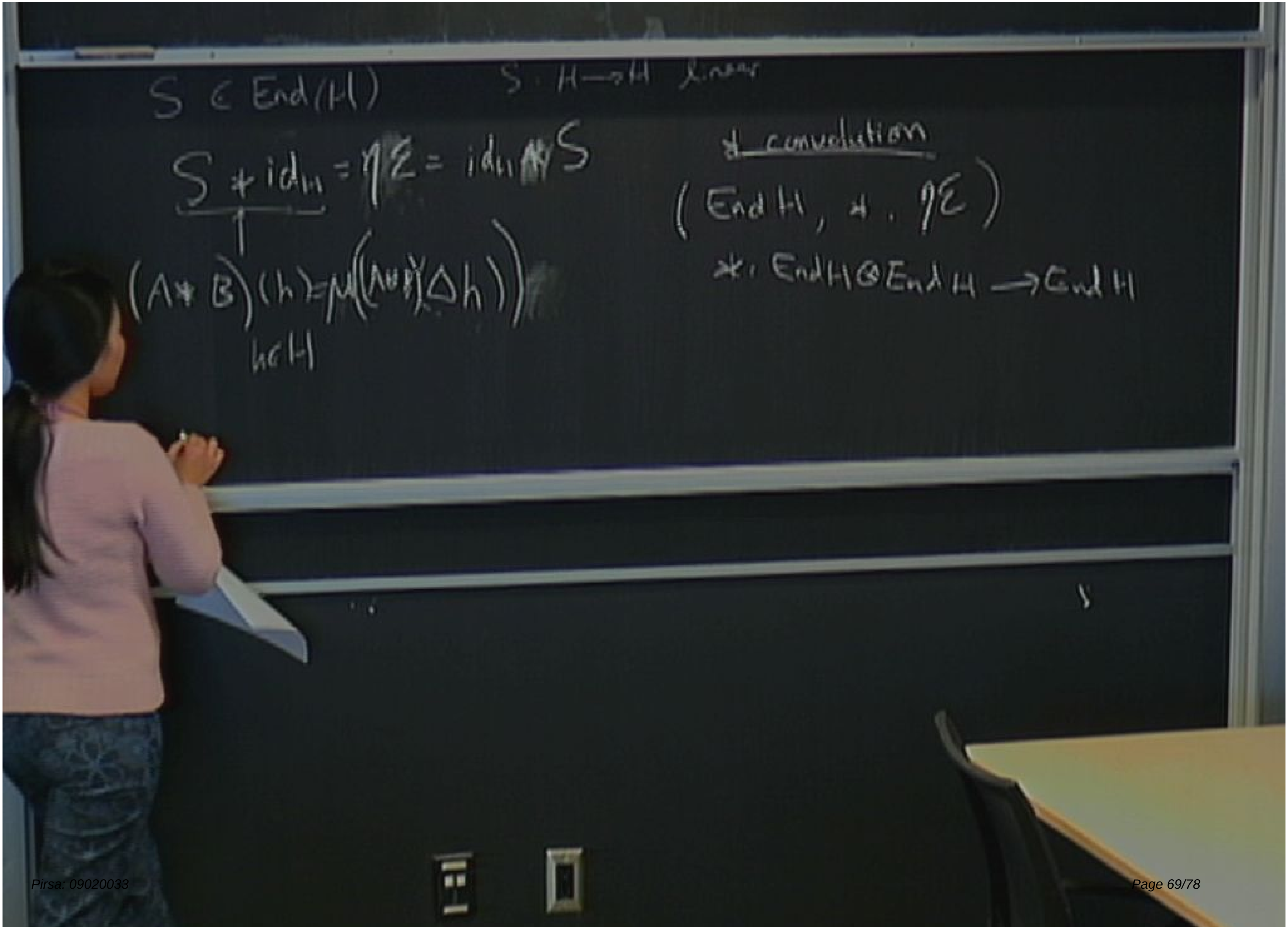
$(A * B)(h)$

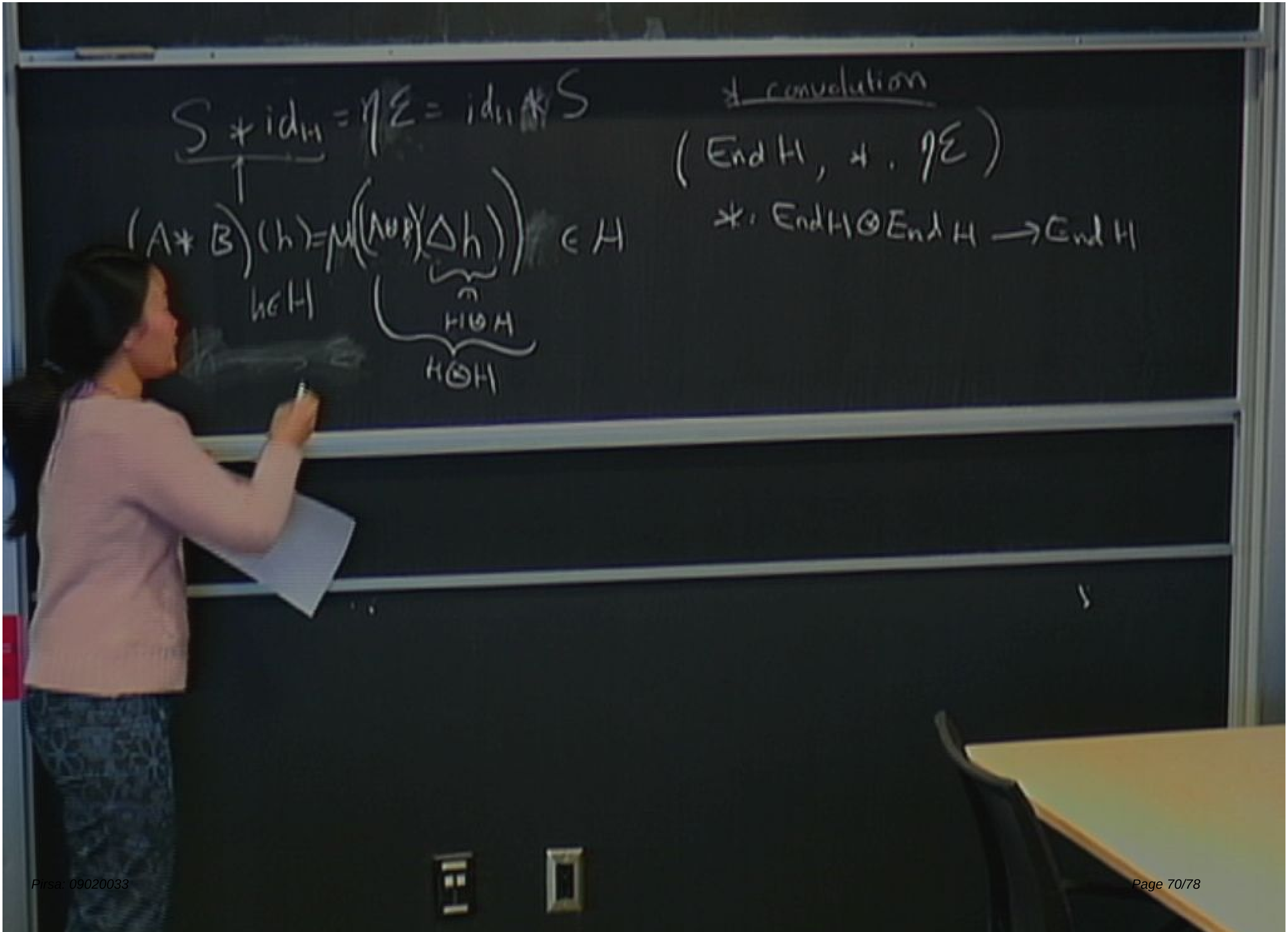
$h \in H$

$*$ convolution

$(End\, H, *, \eta\varepsilon)$

$* : End\, H \otimes End\, H \to End\, H$


S ∈ End(H)
S: H → H linear
S * id_H = ηε = id_H * S
* convolution
(End H, *, ηε)
*: End H ⊗ End H → End H
(A * B)(h) = m((A ⊗ B)(Δh))
h ∈ H


$S * id_H = \eta\varepsilon = id_H * S$
$(A * B)(h) = \mu((A \otimes B)(\Delta h)) \in H$
$h \in H$
$H \otimes H$
$H \otimes H$
* convolution
$(\mathrm{End}\,H, *, \eta\varepsilon)$
$*: \mathrm{End}\,H \otimes \mathrm{End}\,H \to \mathrm{End}\,H$

Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, S)$

$S \in \mathrm{End}(H)$ $\qquad$ $S : H \to H$ linear

$$\underline{S * id_H} = \eta\varepsilon = id_H * S$$

<u>$*$ convolution</u>

$(\mathrm{End}\, H, *, \eta\varepsilon)$

$$(A * B)(h) = \mu\Big(\underbrace{(A \otimes B)\big(\underbrace{\Delta h}_{H \otimes H}\big)}_{H \otimes H}\Big) \in H$$

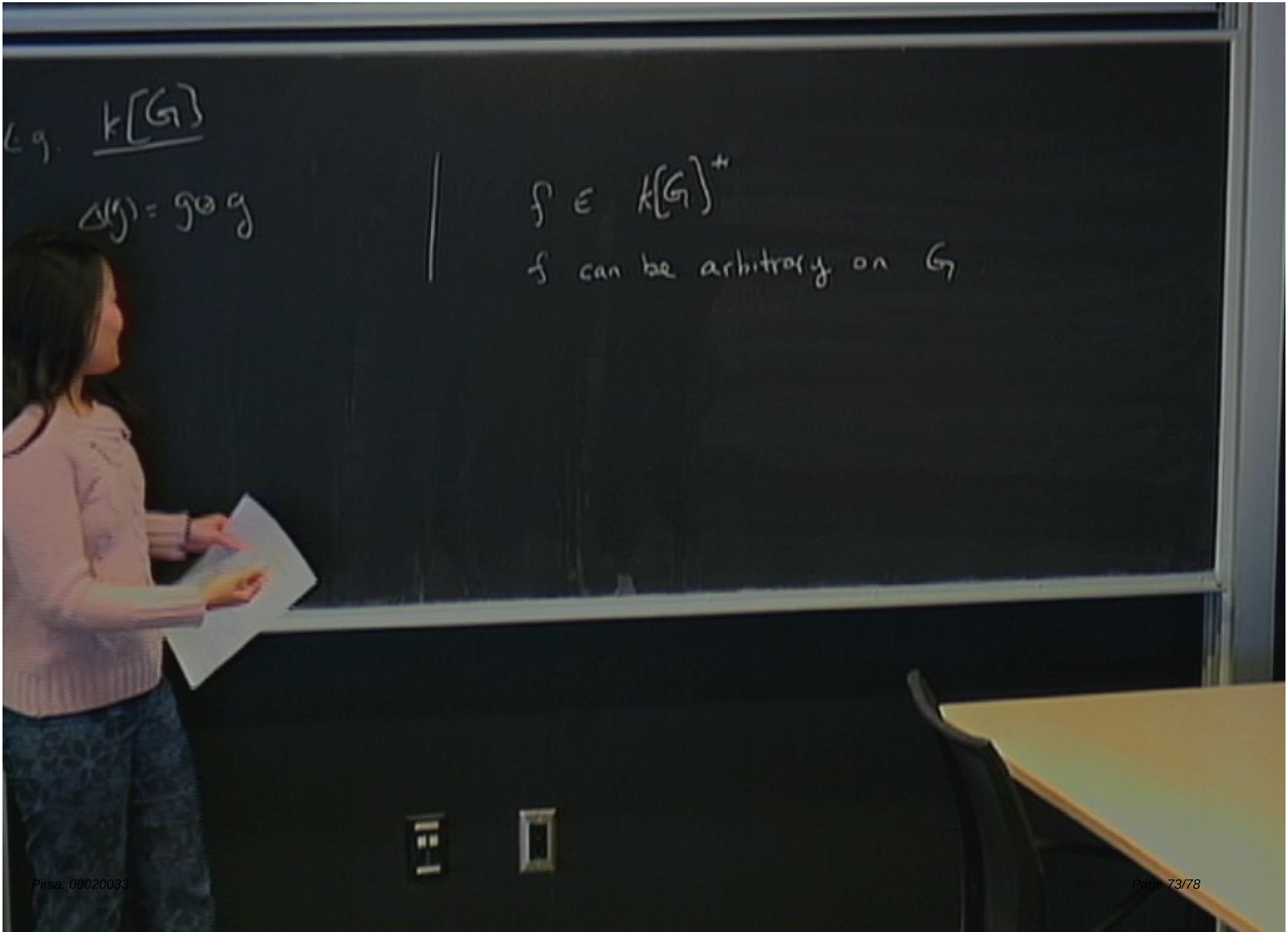
$h \in H$

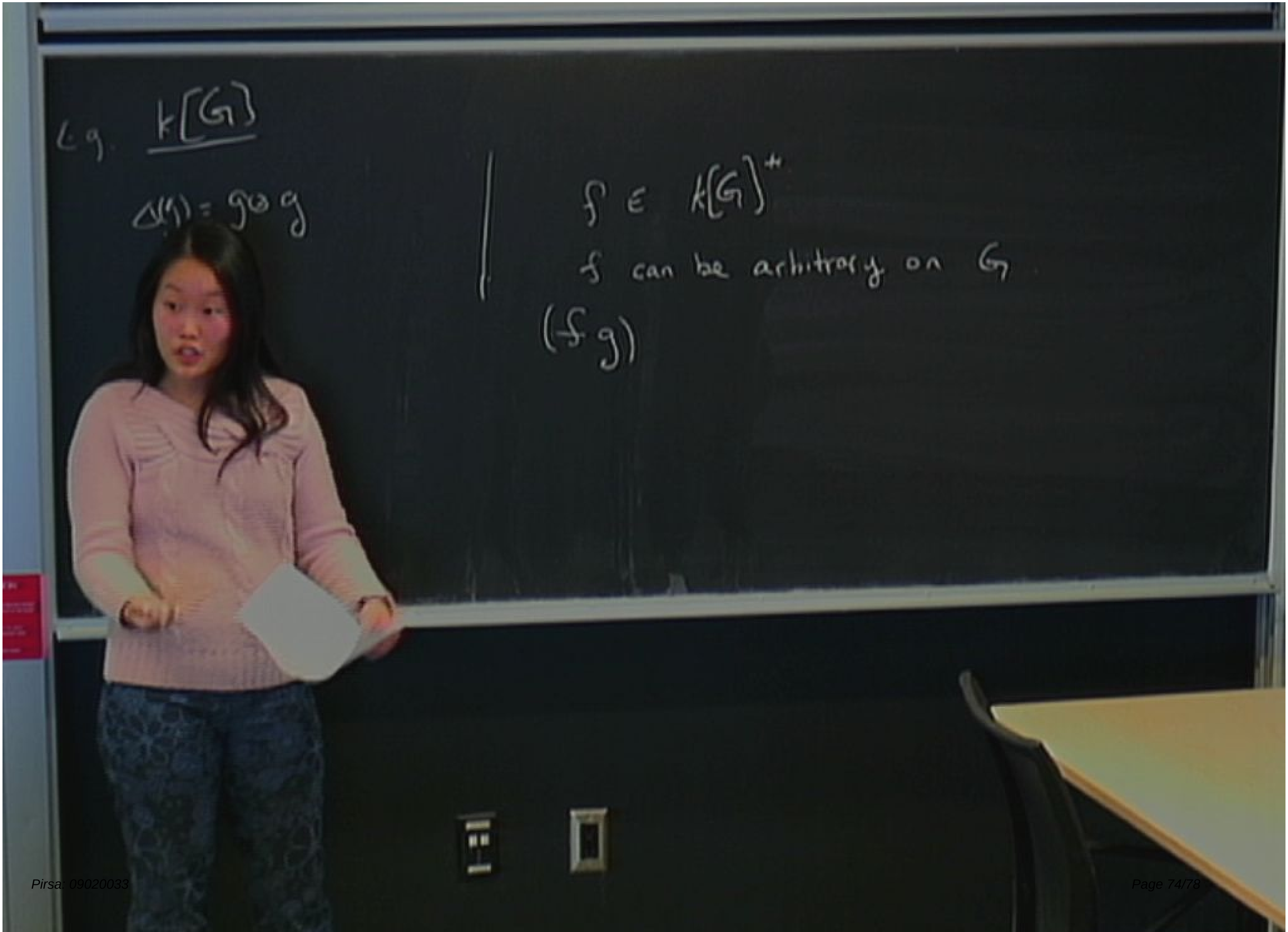
$* : \mathrm{End}\, H \otimes \mathrm{End}\, H \to \mathrm{End}\, H$

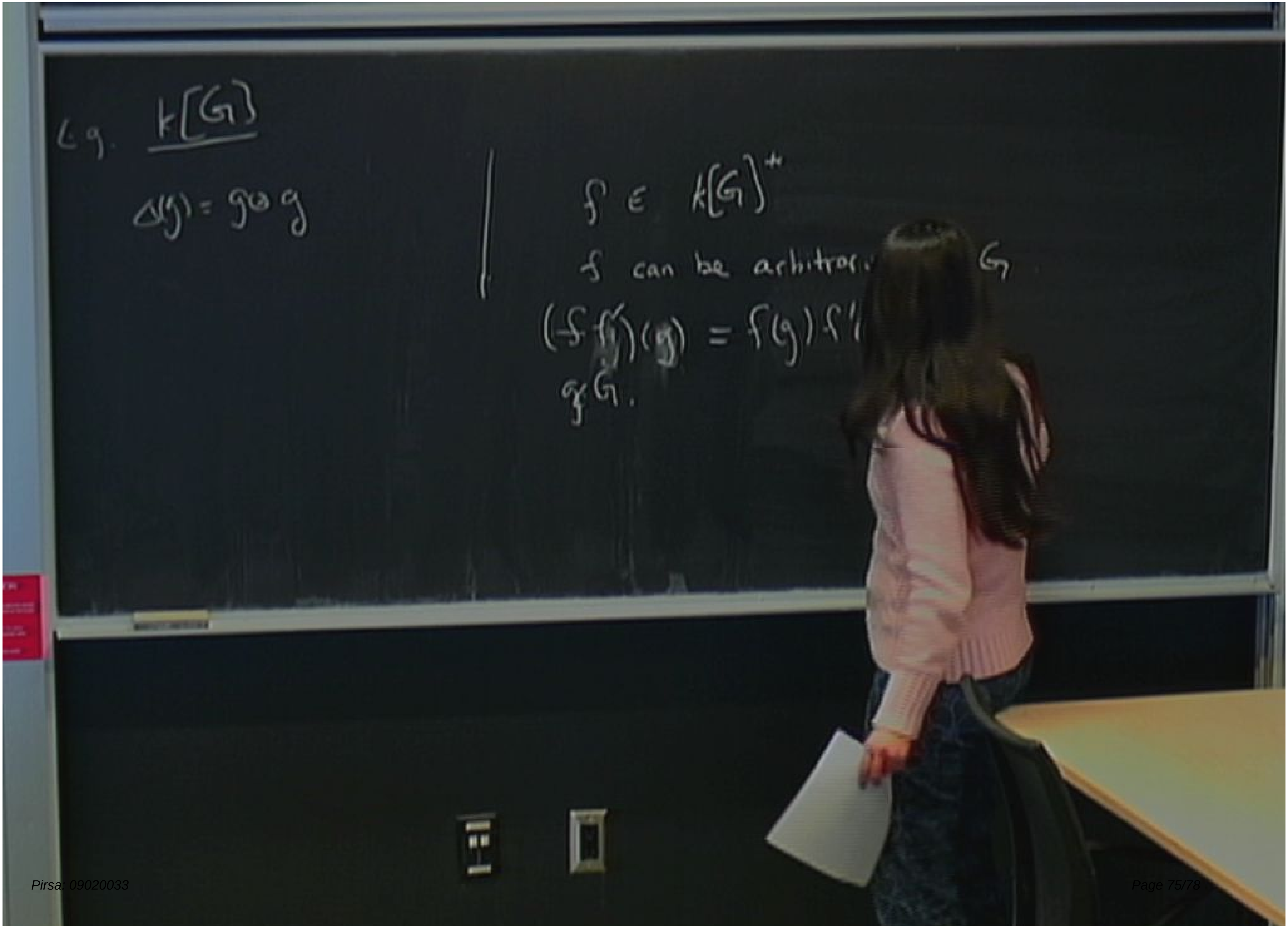
Δ comultiplication of H

μ multiplication of H

$k[G]$


e.g. $k[G]$
$\Delta(g) = g \otimes g$
$f \in k[G]^{**}$
f can be arbitrary on G


e.g. k[G]
Δ(g) = g⊗g
f ∈ k[G]*
f can be arbitrary on G
(f, g)



e.g. k[G]
Δ(g) = g⊗g
f ∈ k[G]*
f can be arbitrar
G
(f f')(g) = f(g) f'(
g∈G.

e.g. $\underline{k[G]}$

$\Delta(g) = g \otimes g$

$f \in k[G]^*$

f can be arbitrary on G

$(f f')(g) = f(g) f'(g)$

$g \in G$.

e.g. $\underline{k[G]}$

$\Delta(g) = g \otimes g$

$f \in k[G]^*$

f can be arbitrary on G

$(f f')(g) = f(g) f'(g)$

$g \in G$. $\quad g \to g \otimes g$

e.g. $\underline{k[G]}$

$\Delta(g) = g \otimes g$

$f, f' \in k[G]^*$

f can be arbitrary on G

$(f f')(g) = f(g) f'(g)$

$g \in G$.

$g \to g \otimes g$