Title: Quantum computation with Topological Lattice Field Theories

Date: Feb 04, 2009 04:00 PM

URL: http://pirsa.org/09020026

Abstract: Recent results have shown that quantum computers can approximate the value of a tensor network efficiently. These results have initiated a search for tensor networks which contract to computationally interesting quantities. Topological Lattice Field Theories (TLFTs) are one source of such networks; when defined appropriately, networks arising from TLFTs contract to give topological invariants. In this elementary talk, we will define and classify TLFTs which lead to invariants of surfaces, and sketch out the corresponding quantum algorithm. Our exposition will be targetted at a general mathematically-inclined audience; no previous knowledge of field theories is required.

A LEISURELY STROLL THROUGH TLFTs*

(and how they may be useful in quantum computation)

Gorjan Alagic
IQC / C&O U.Waterloo

Other people thinking about this stuff include:

Alexander Russell (Uconn)
Cris Moore (UNM/SFI)
Jon Yard (LANL)
Zeph Landau (CUNY)
... and others.

*Topological Lattice Field Theories, of course!

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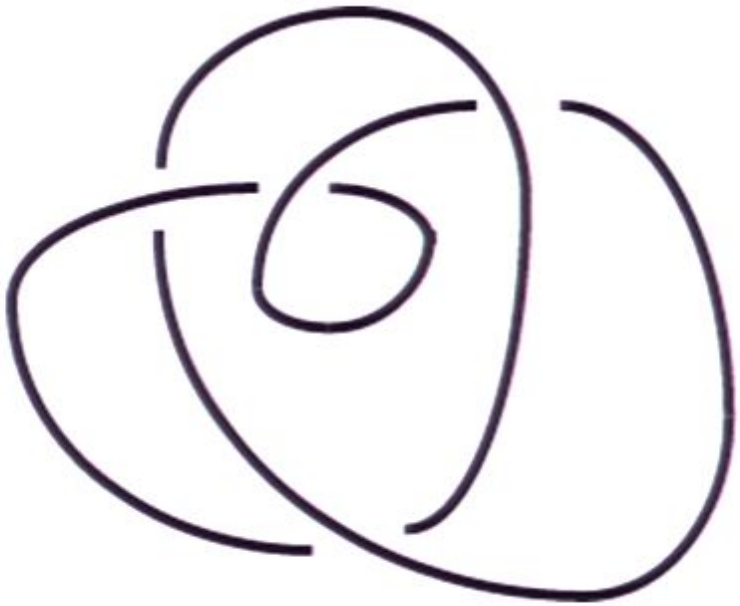
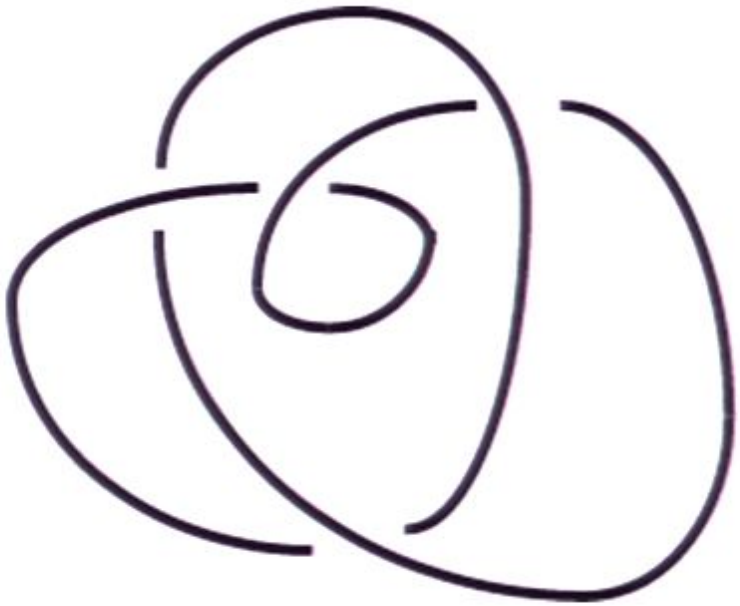
Knots



Knot diagrams



Knot and link diagrams

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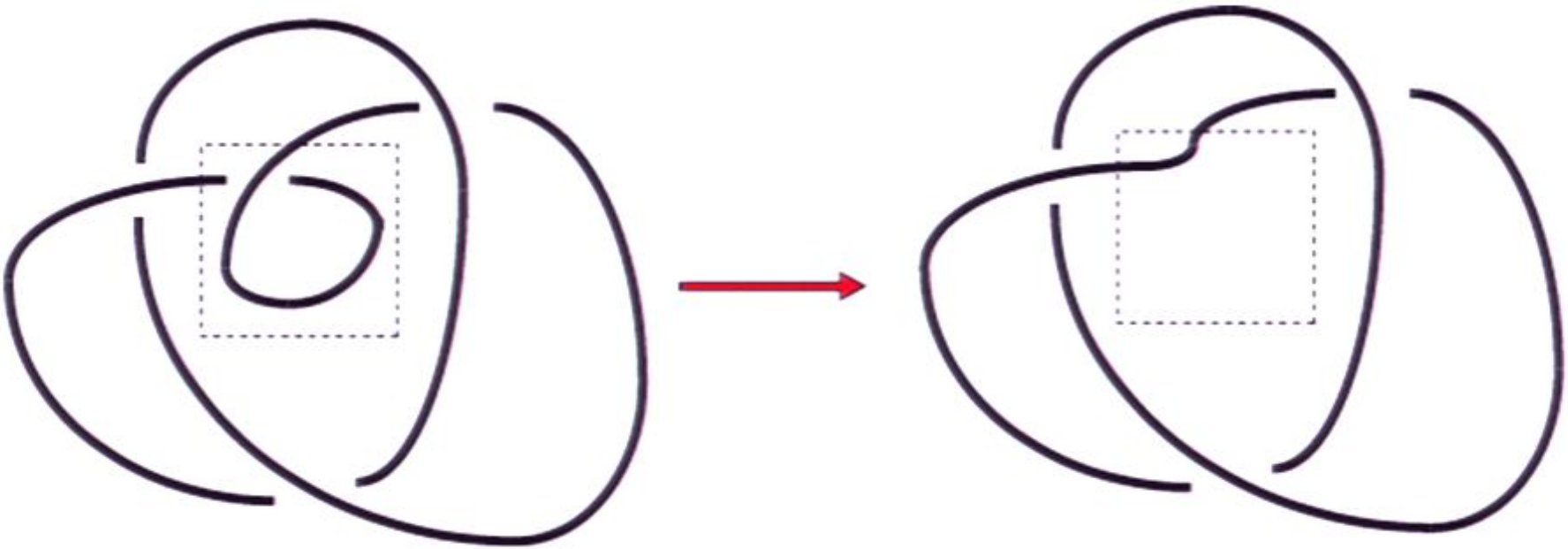

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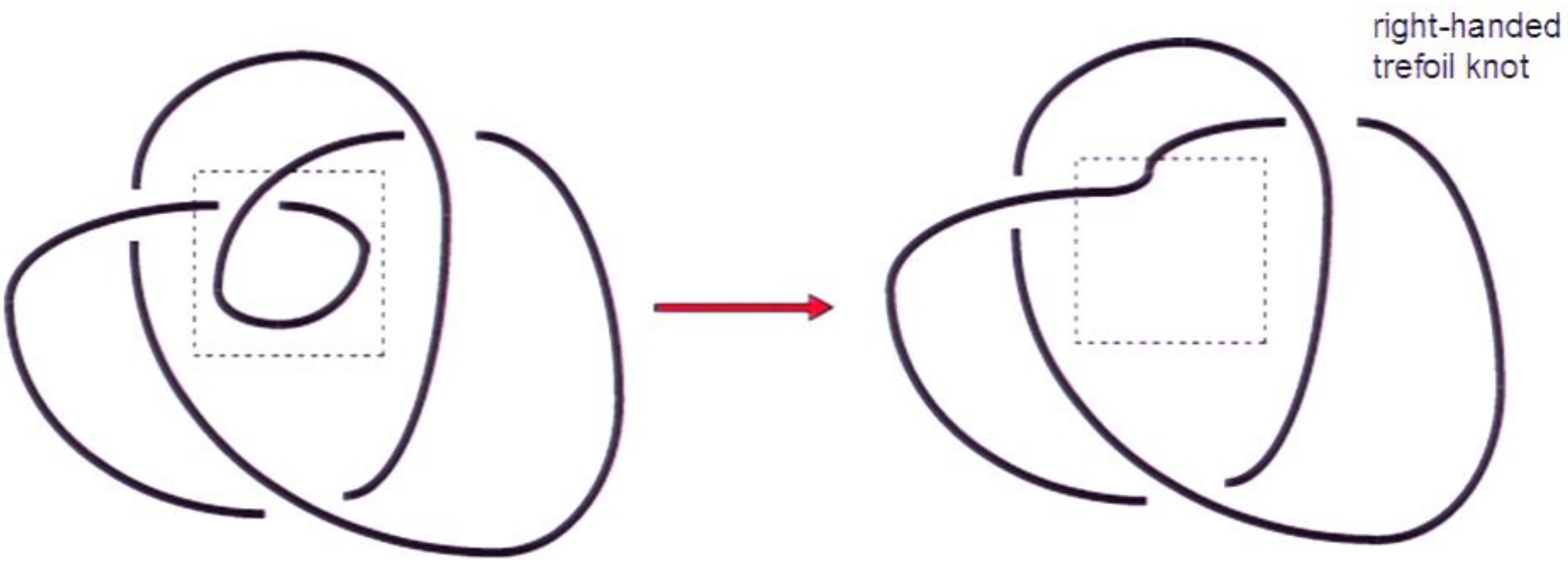
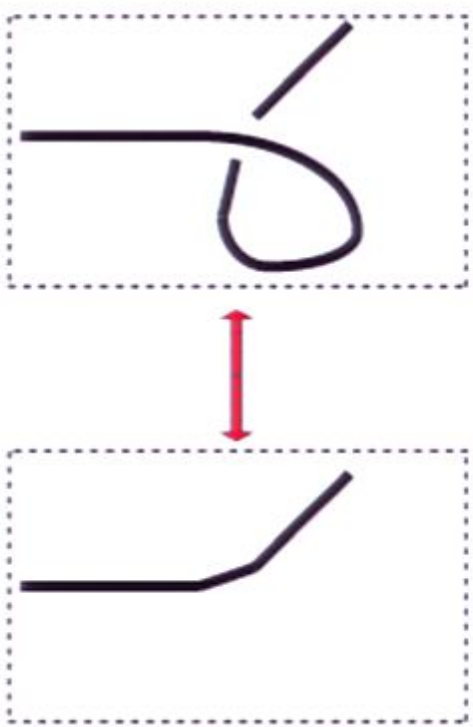


Reidemeister Moves



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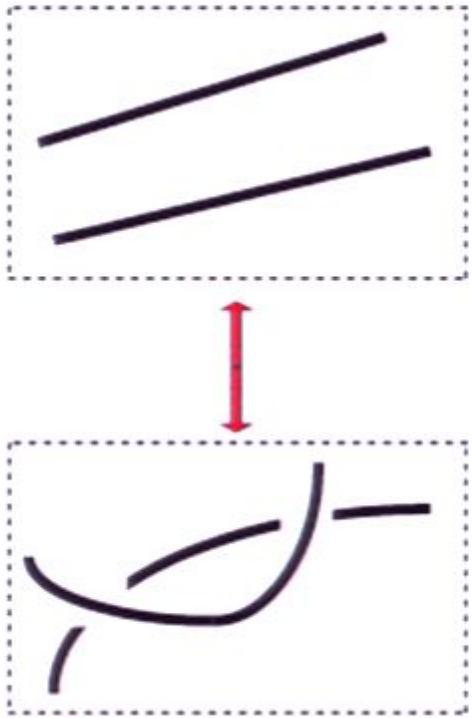

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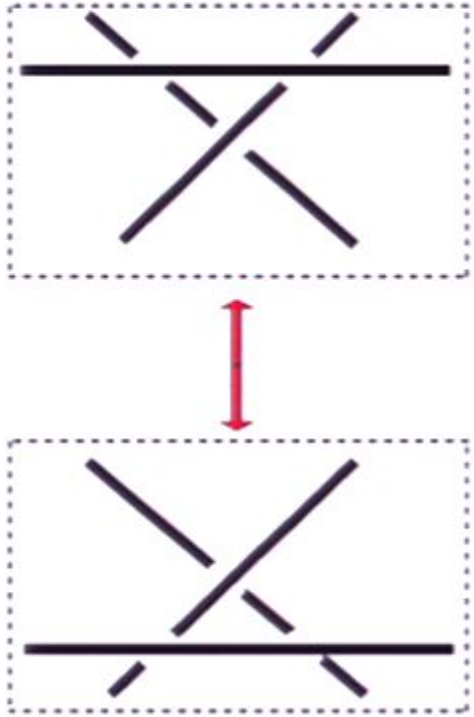
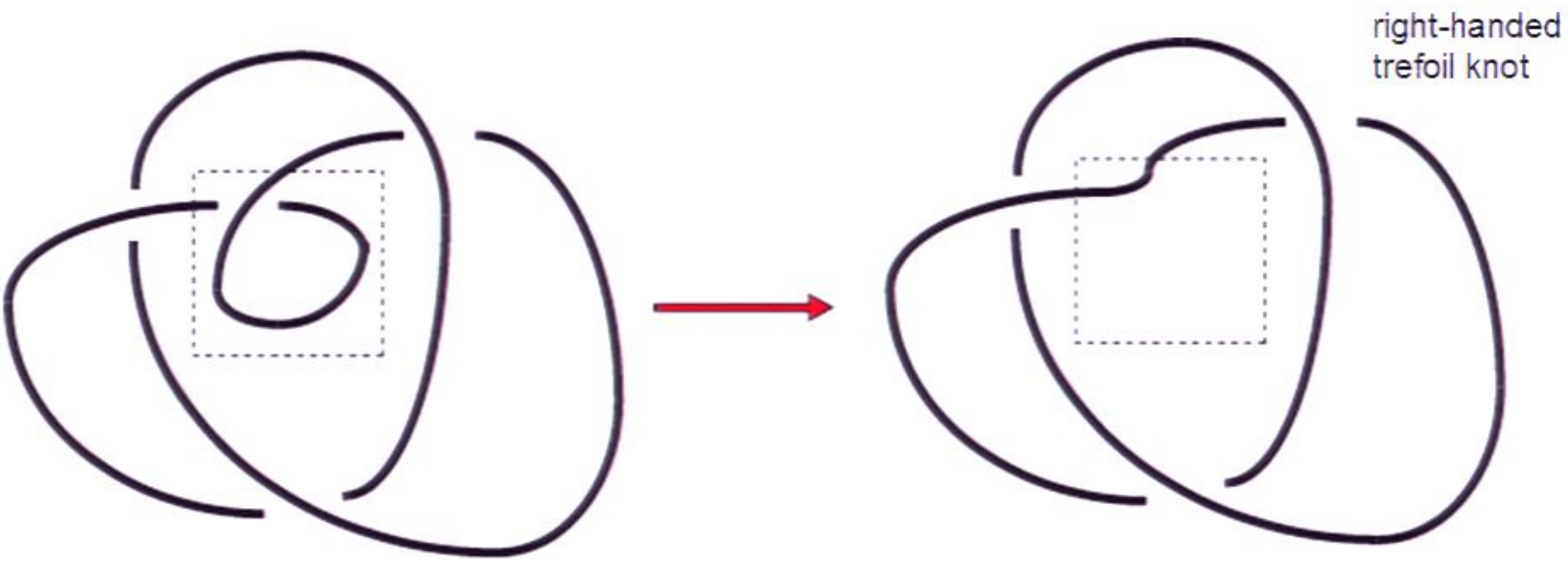


Type III

Purely combinatorial rules!

Knot and link diagrams



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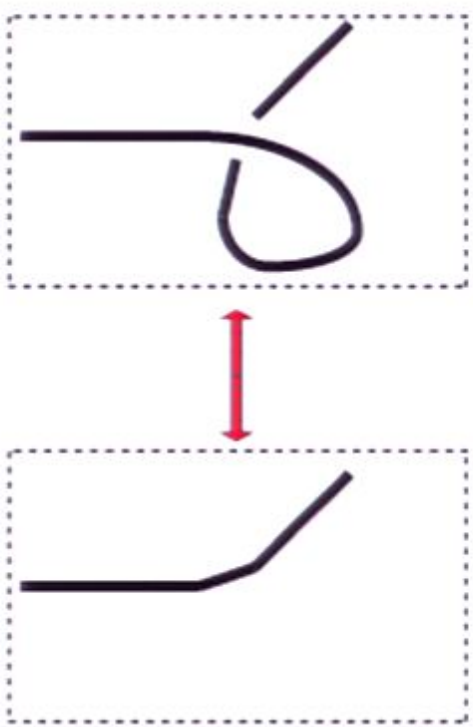

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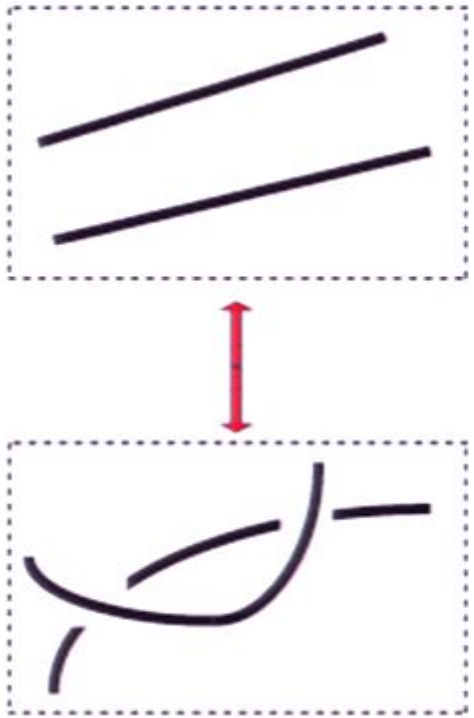

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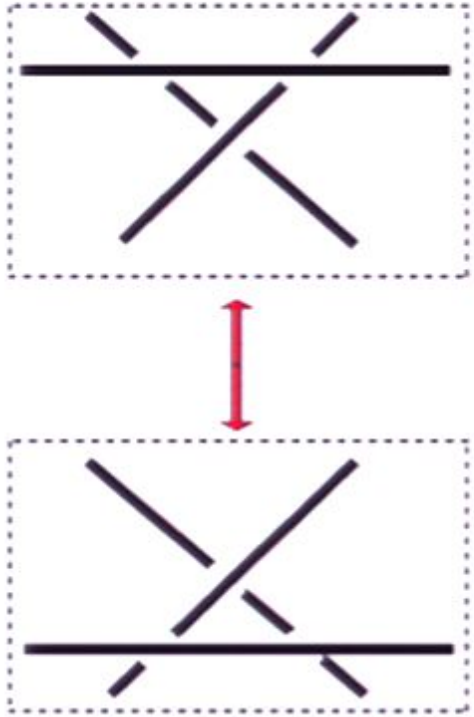

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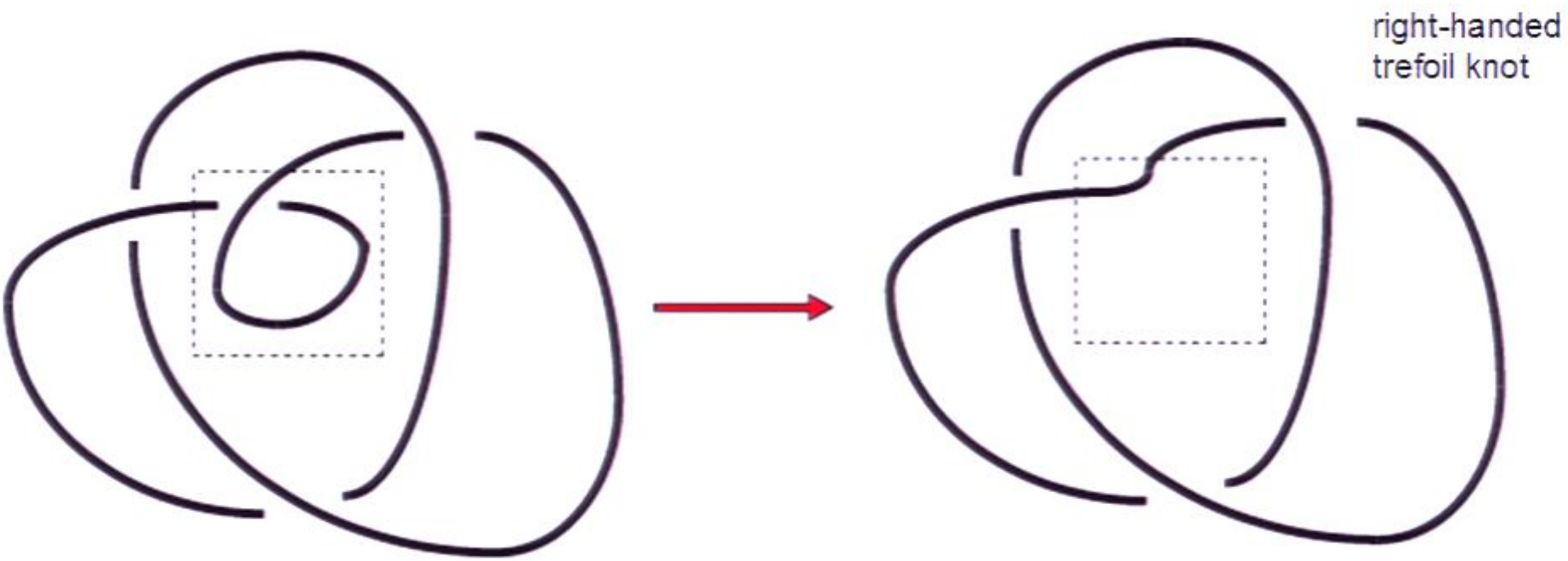

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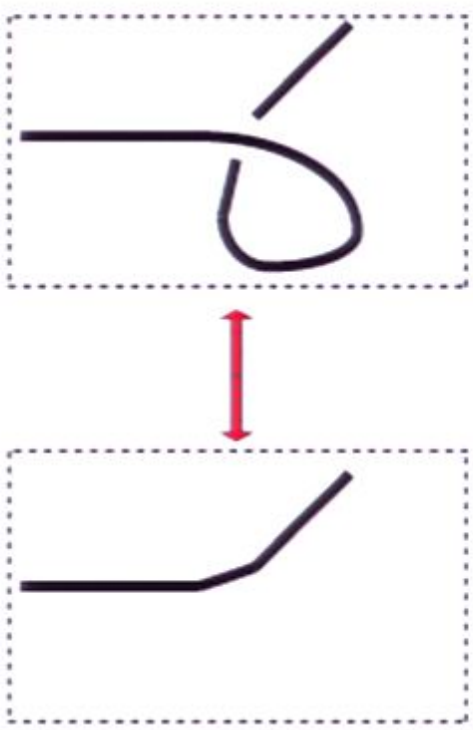

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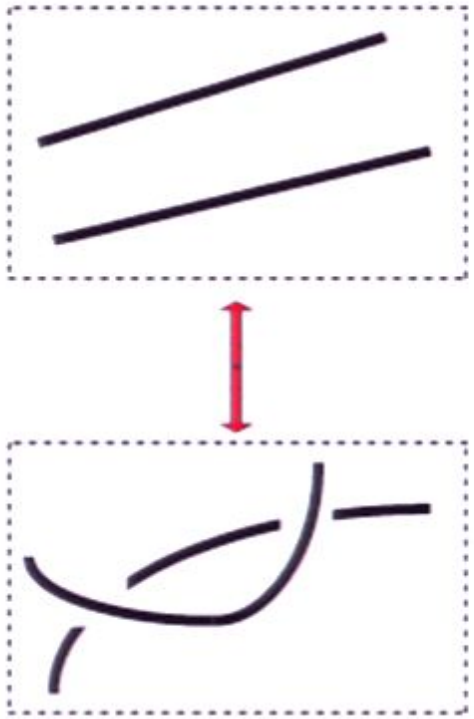

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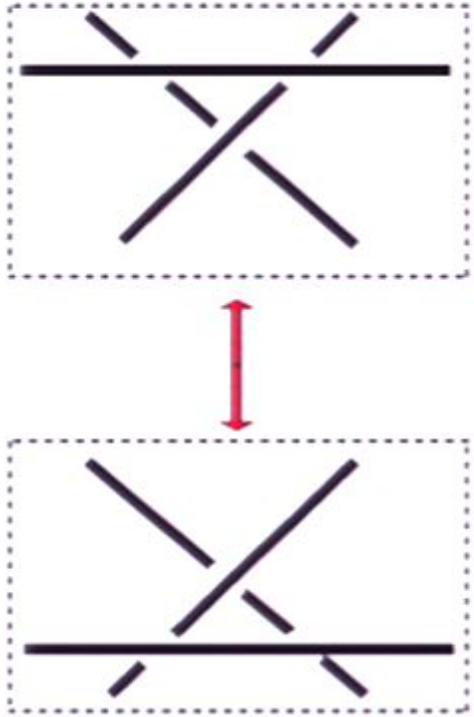

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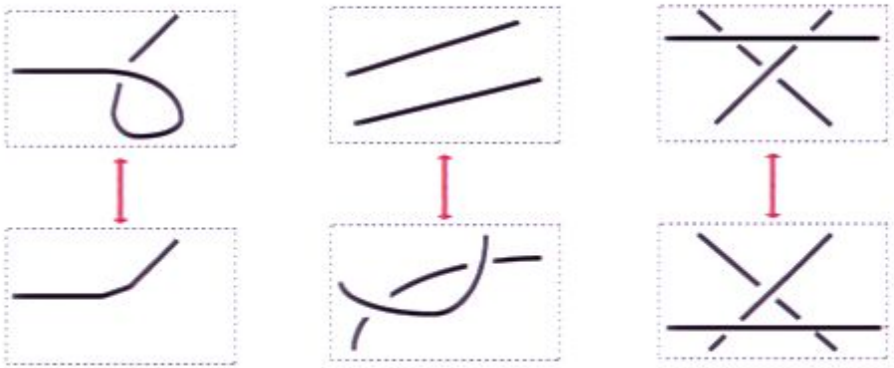

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Any two diagrams of the same link differ by a finite number of these moves.

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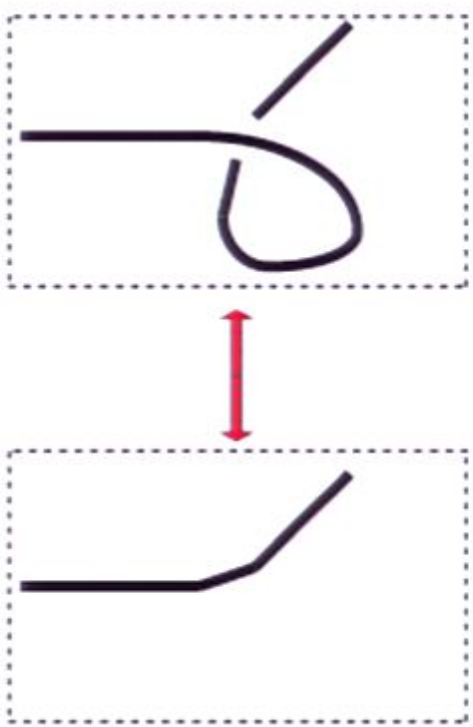

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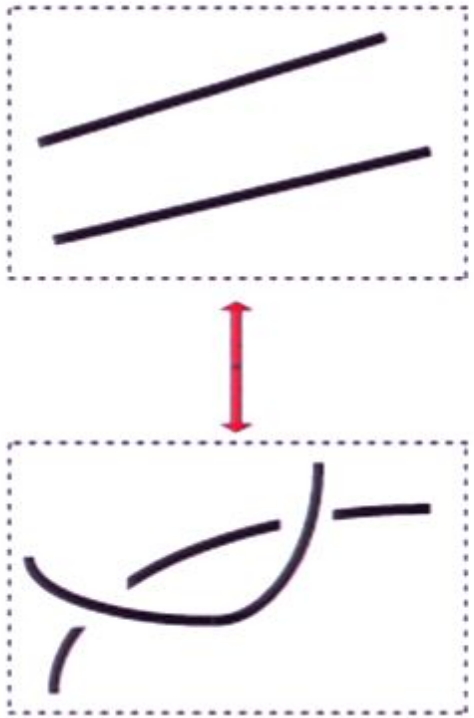

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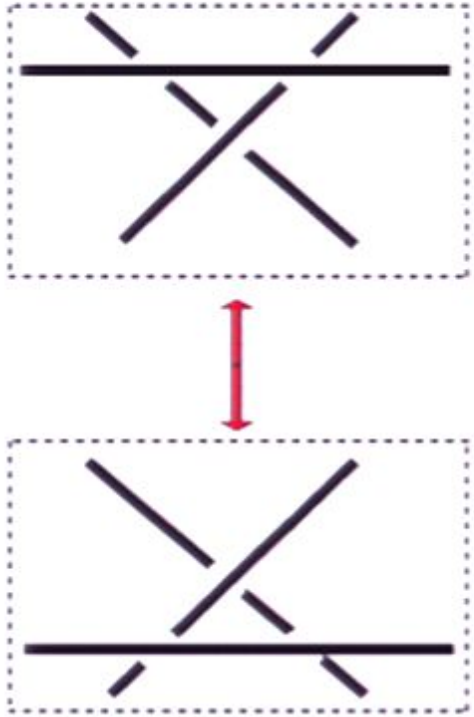

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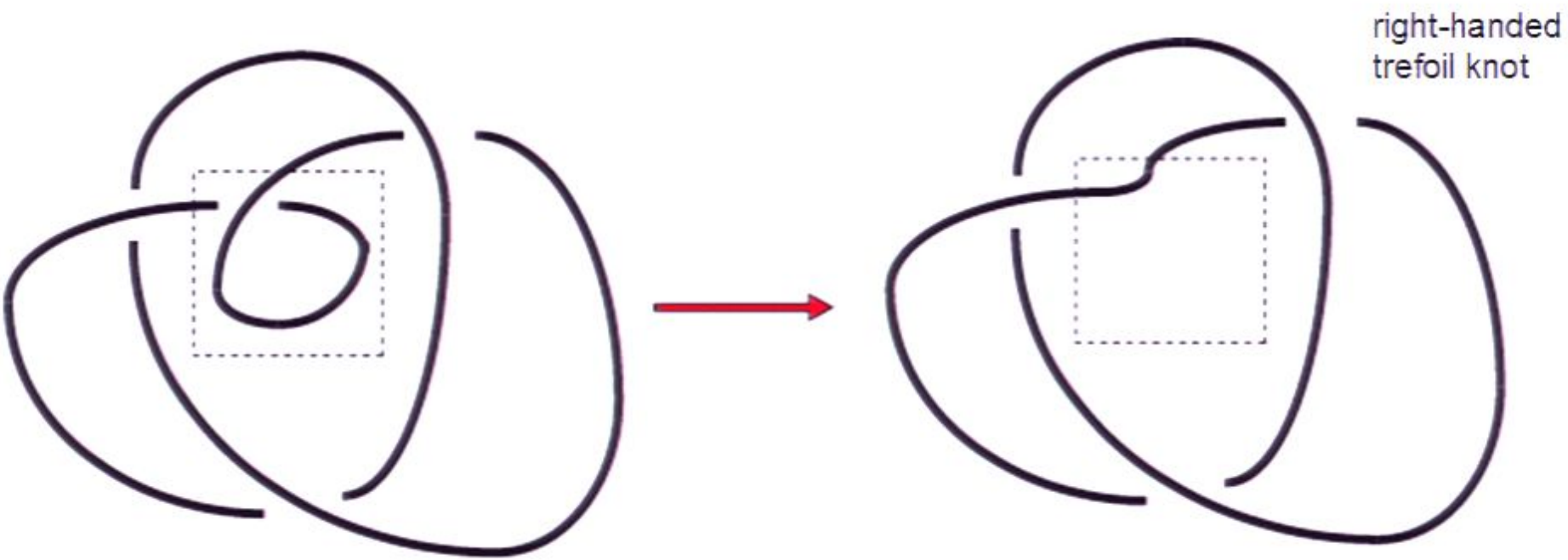

"Type I Reidemeister Move"

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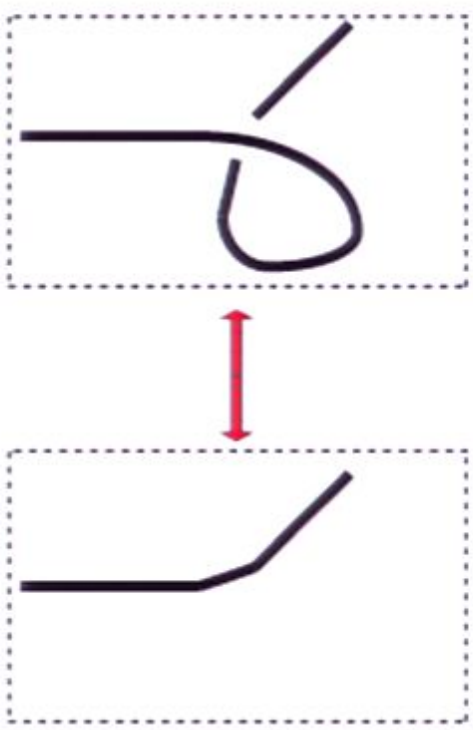

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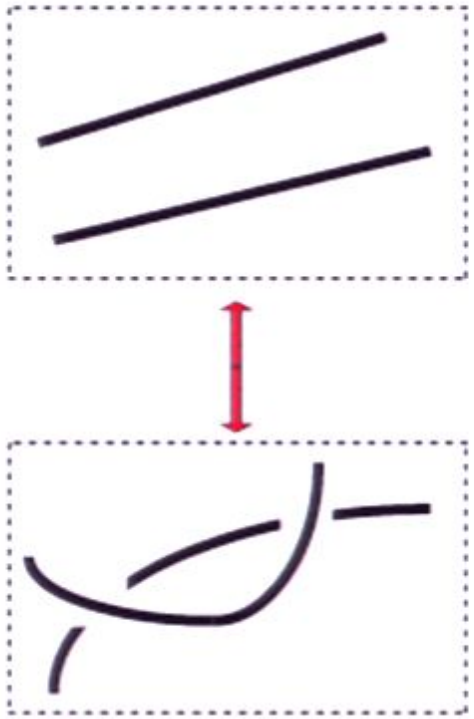

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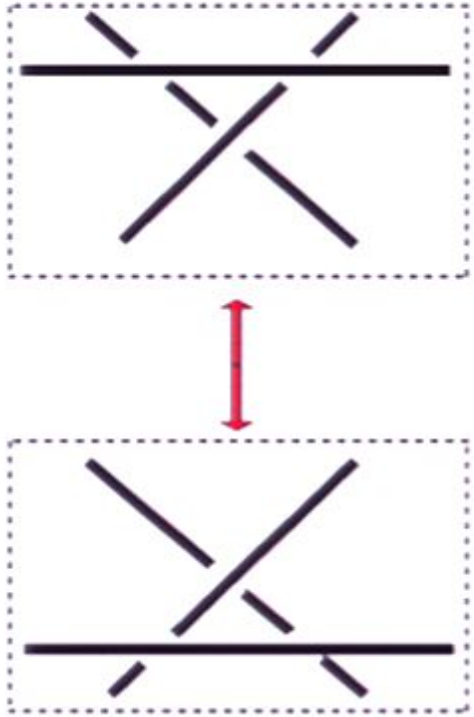

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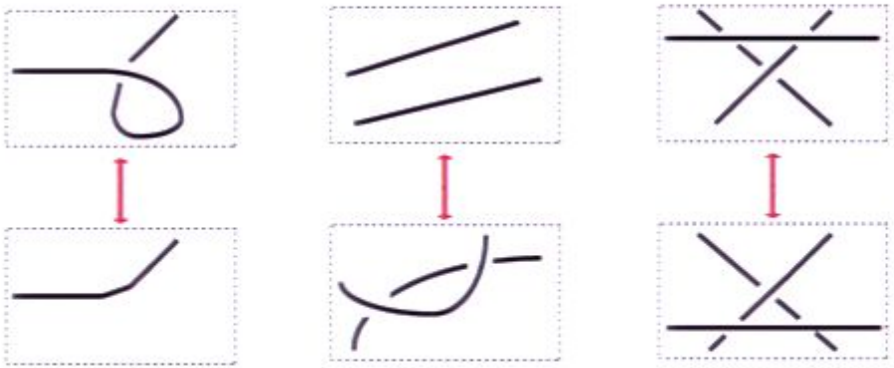

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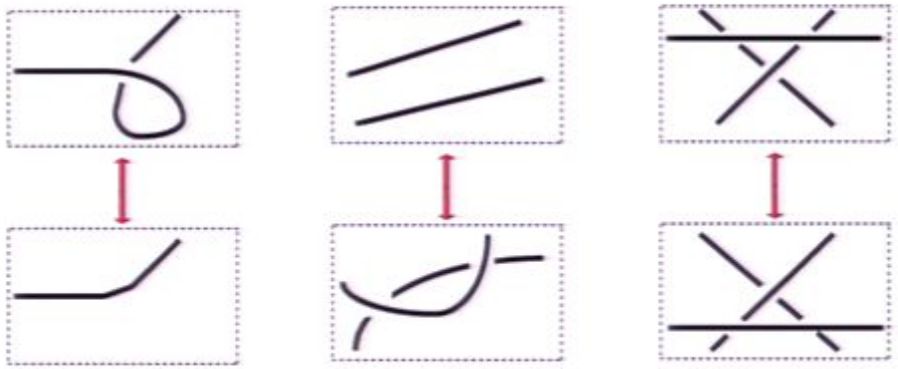

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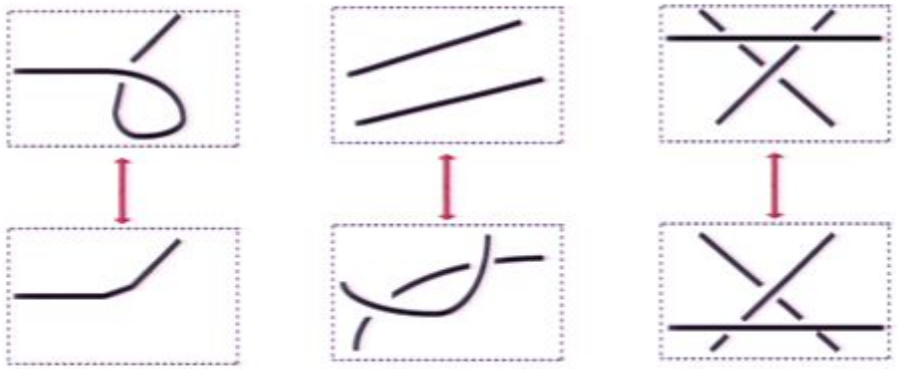

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Does something analogous happen in higher dimensions?

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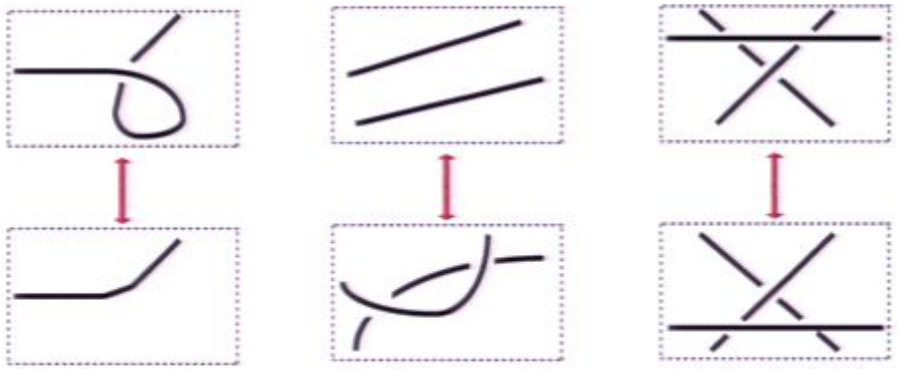

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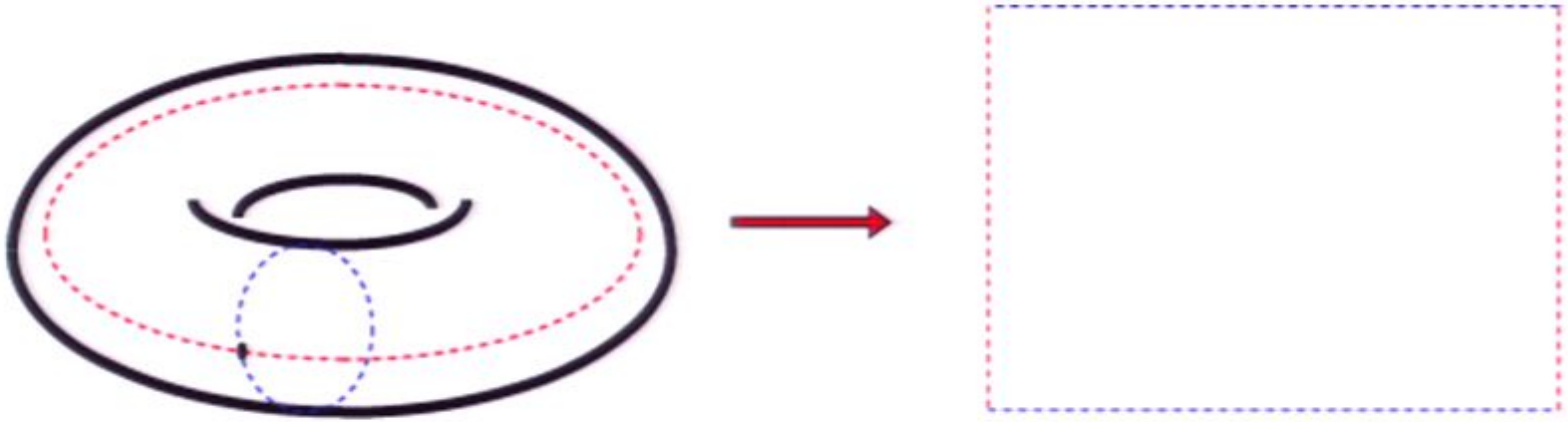

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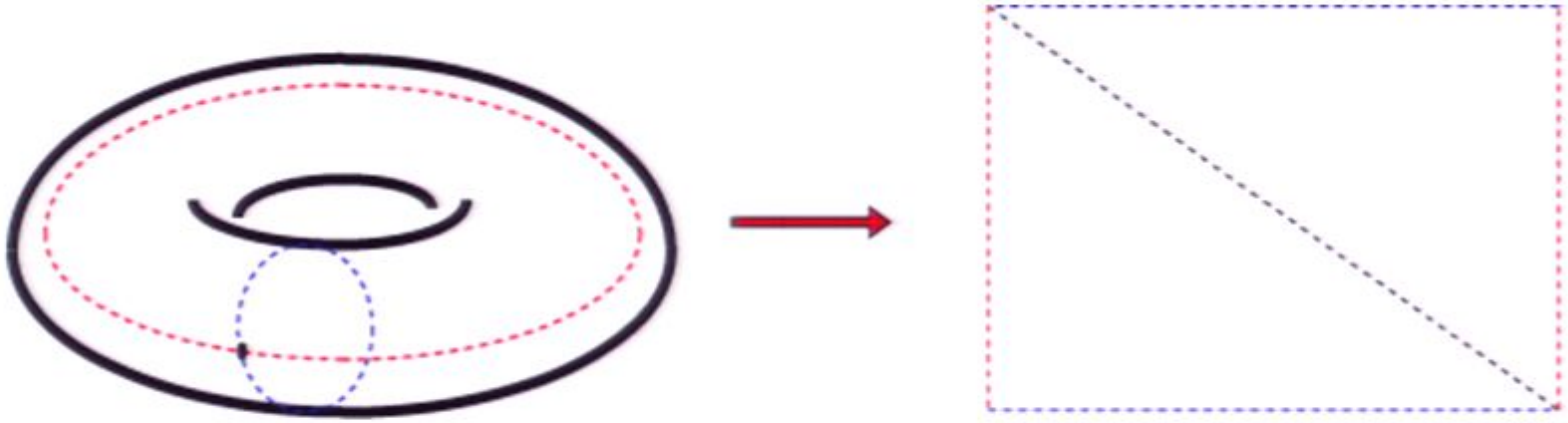
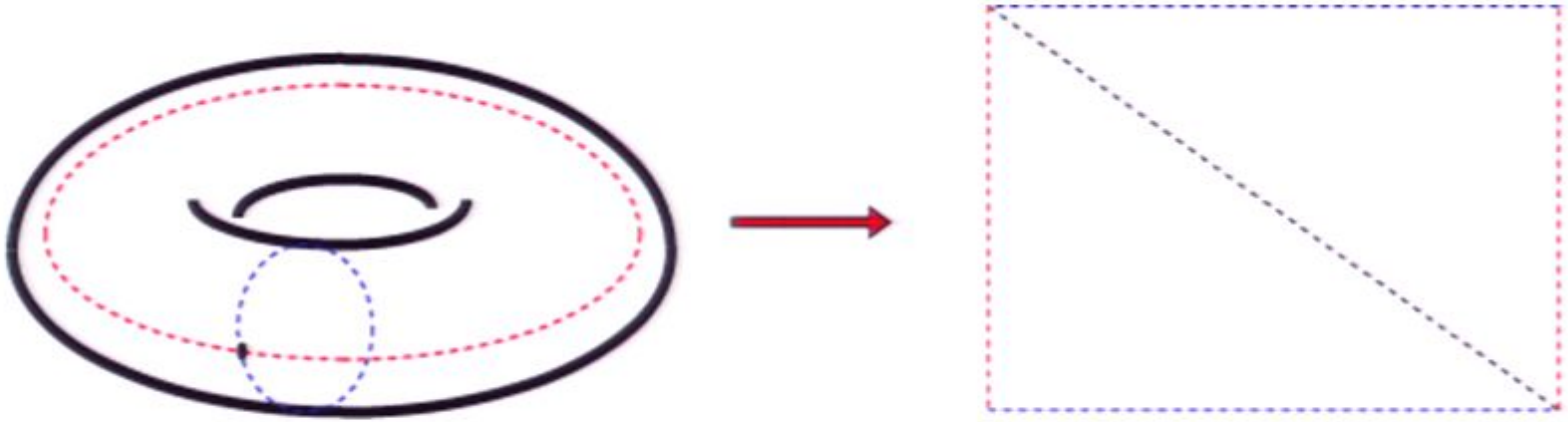


surface

triangulation

Orientable Surfaces



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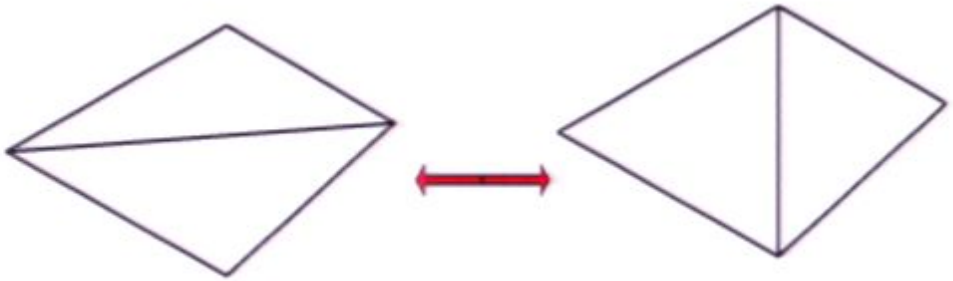
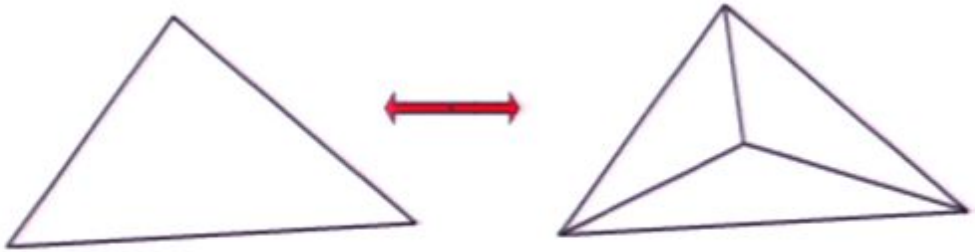


2-2 move



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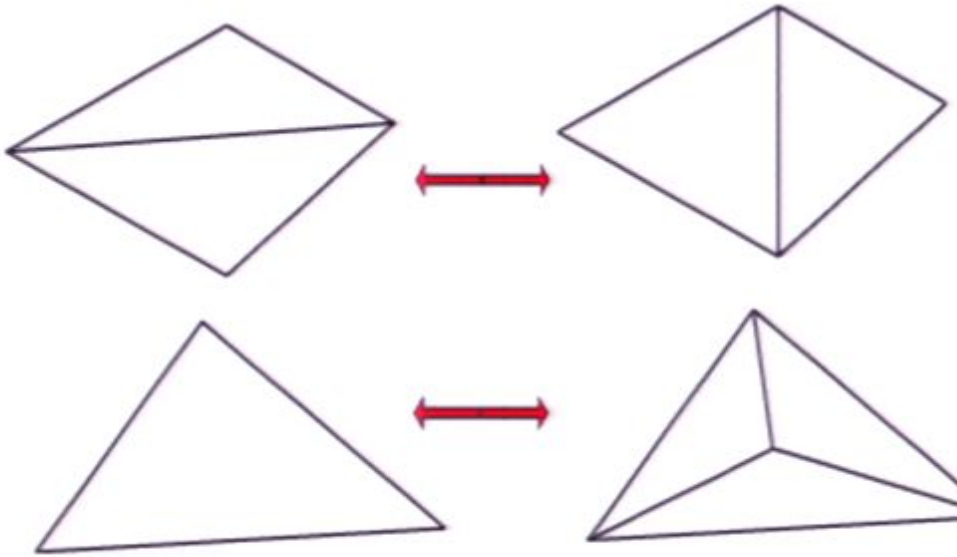


2-2 move

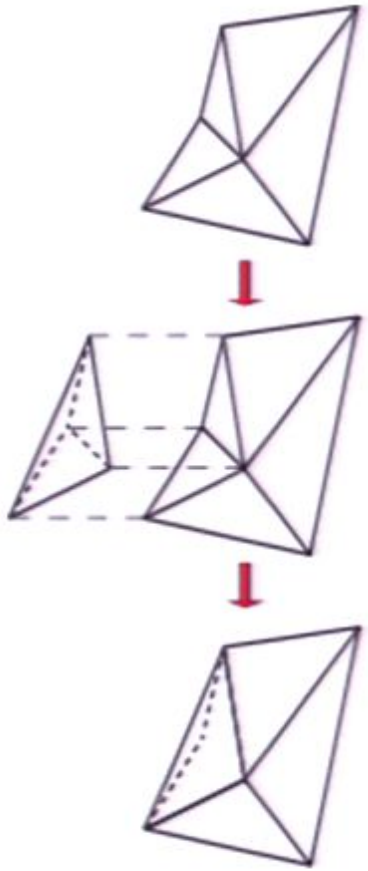
1-3 move

Theorem (Pachner) [1991]:
Any two triangulations of the same surface differ by a finite number of these moves.

Manifolds

What about higher dimensions still? Picture the Pachner moves like this:

2-2



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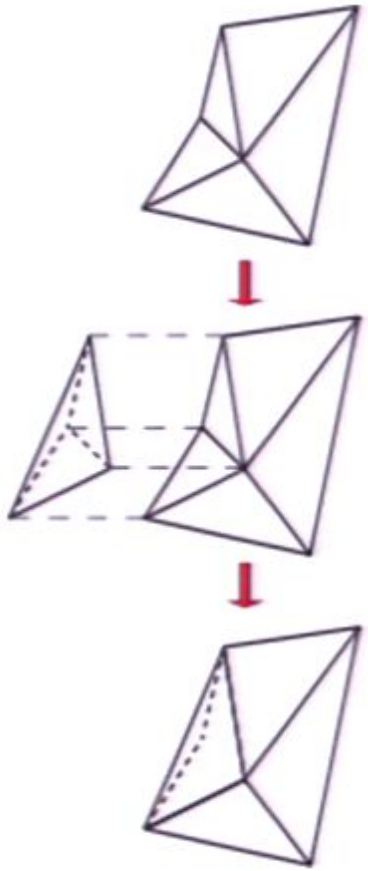

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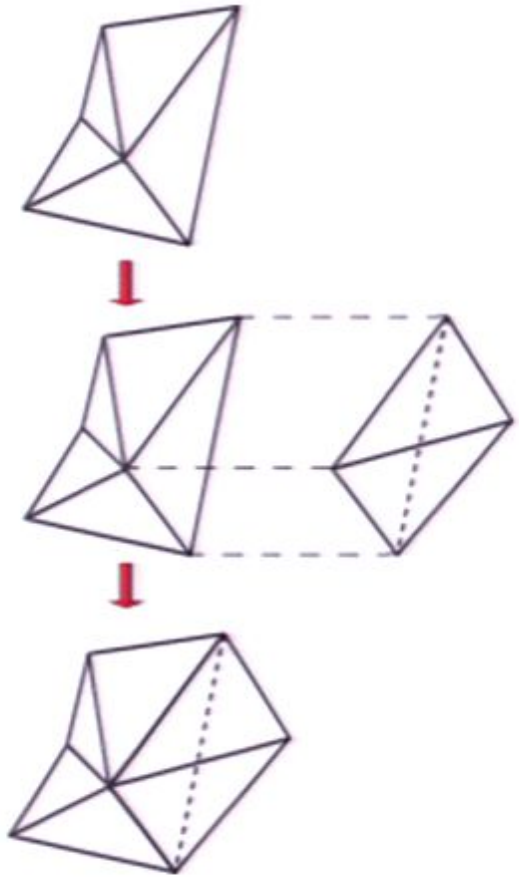

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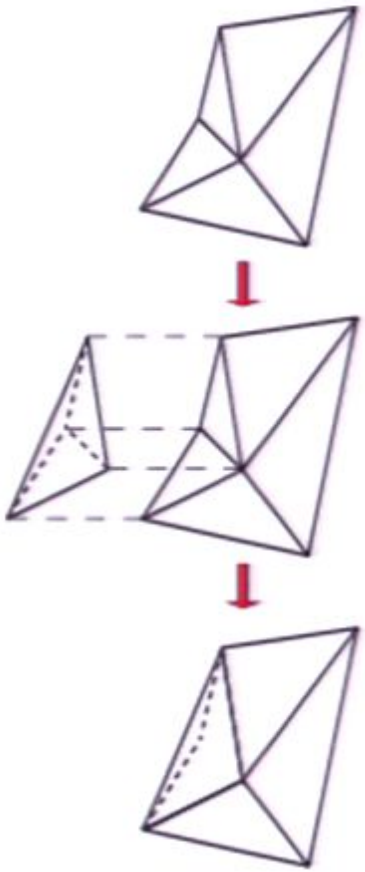

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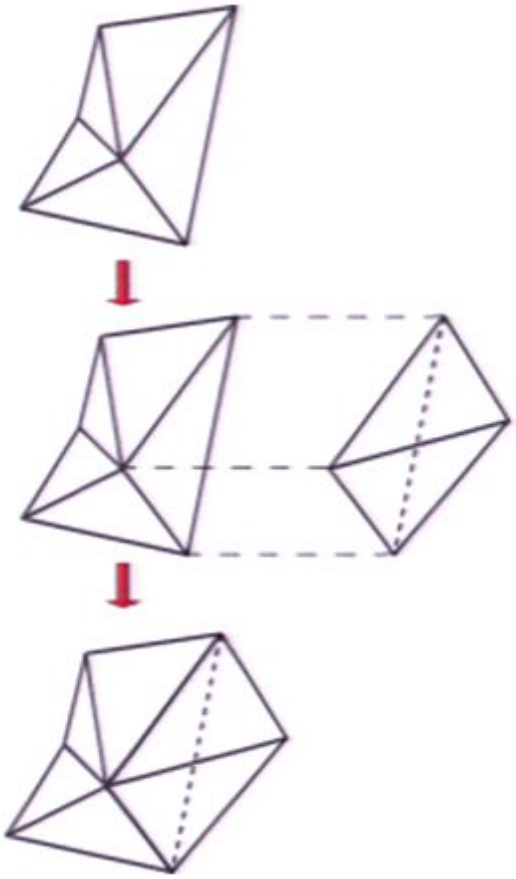
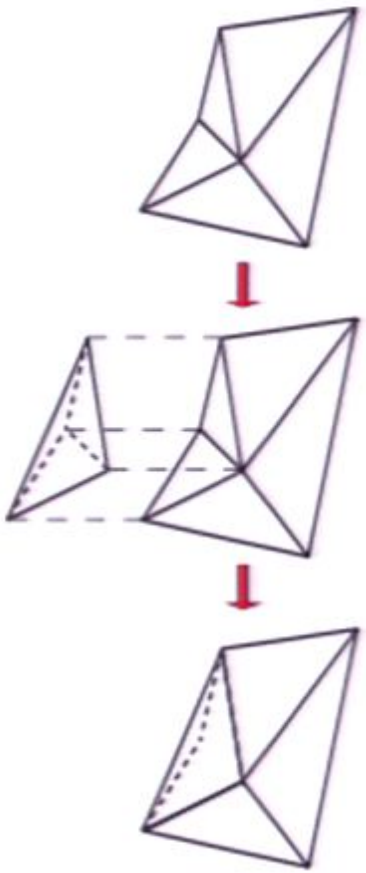


1-3

The n-dimensional Pachner moves are in one-to-one correspondence with the ways of cutting the boundary of an n+1 simplex into two connected pieces.

Manifolds



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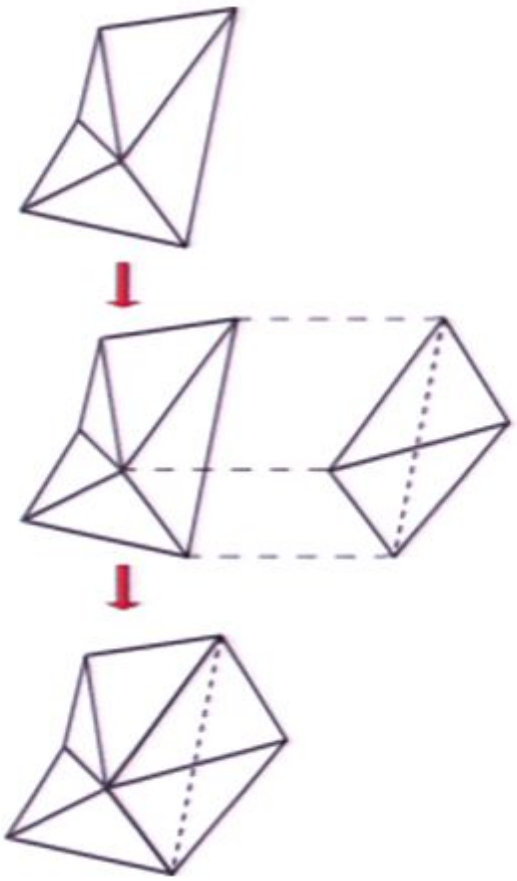

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2. find **tensors** which satisfy these legal moves;
3. decorate the combinatorial structure with the tensors;
4. profit!

This is good for a lot of things.

We'll just focus on the fact that it gives us **topological invariants!**

Tensors: definition

Tensors are a convenient way to do multilinear algebra.
Pick a Hilbert space V. Here's an (l, k)-tensor:

$$T \in \underbrace{V \otimes V \otimes \cdots \otimes V}_{l} \otimes \underbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}_{k}$$

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If we had a basis for V:

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… but we will just denote it by $T^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_k}$.

By following some basic conventions, we can use this notation in a basis-independent way.

Tensors: operations

We can perform two operations on tensors:

1) Tensor product (notice no repeated indices)

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Tensors: examples

Here are some basic examples of tensors, in tensor notation:

- A vector: $\mathbf{v}^i$
- A dual vector: $\mathbf{u}_j$
- A linear operator: $\mathbf{A}_i^j$
- Inner product (by contraction): $\mathbf{v}_i\mathbf{u}^i$
- Applying an operator to a vector: $\mathbf{A}_i^j\mathbf{v}^i$
- Matrix product: $\mathbf{A}_i^j\mathbf{B}_j^k$
- Hilbert-Schmidt norm: $\mathbf{A}_i^j\mathbf{A}_j^i$

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Tensors are a convenient way to do multilinear algebra.
Pick a Hilbert space V. Here's an (l, k)-tensor:

$$T \in \underbrace{V \otimes V \otimes \cdots \otimes V}_{l} \otimes \underbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}_{k}$$

T has contravariant rank l and covariant rank k.
If we had a basis for V:

$$T = \sum T^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_k} \; b_{i_1} \otimes b_{i_2} \otimes \cdots \otimes b_{i_l} \otimes b^*_{j_1} \otimes b^*_{j_2} \otimes \cdots b^*_{j_k}$$

… but we will just denote it by $T^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_k}$.

By following some basic conventions, we can use this notation in a basis-independent way.

Tensors: examples

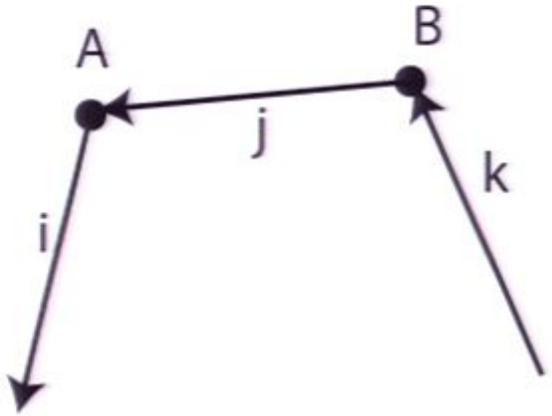
Here are some basic examples of tensors, in tensor notation:

- A vector: $\mathbf{v}^i$
- A dual vector: $\mathbf{u}_j$
- A linear operator: $\mathbf{A}_i^j$
- Inner product (by contraction): $\mathbf{v}_i\mathbf{u}^i$
- Applying an operator to a vector: $\mathbf{A}_i^j\mathbf{v}^i$
- Matrix product: $\mathbf{A}_i^j\mathbf{B}_j^k$
- Hilbert-Schmidt norm: $\mathbf{A}_i^j\mathbf{A}_j^i$

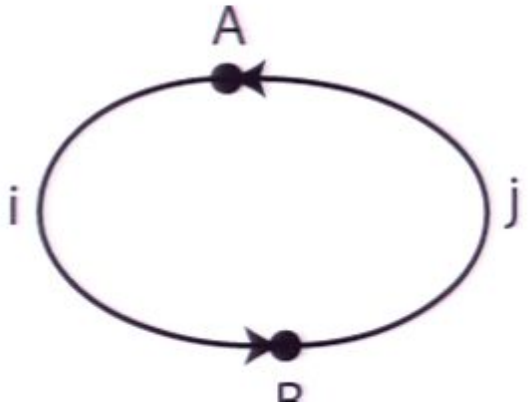
Tensor networks

A **tensor network** is a graph decorated with tensors in the natural way:

- Matrix product: $\mathbf{A}_i^j\mathbf{B}_j^k$



- Trace of a product: $\mathbf{A}_i^j\mathbf{B}_j^i$



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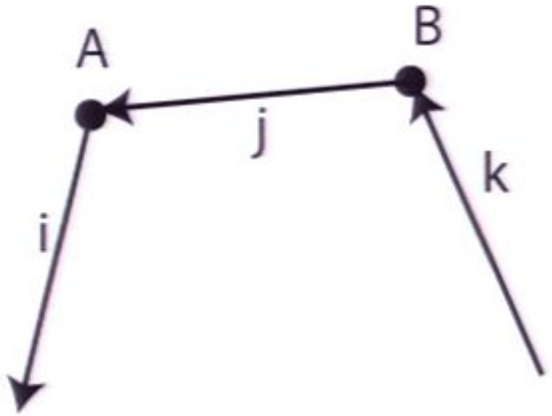

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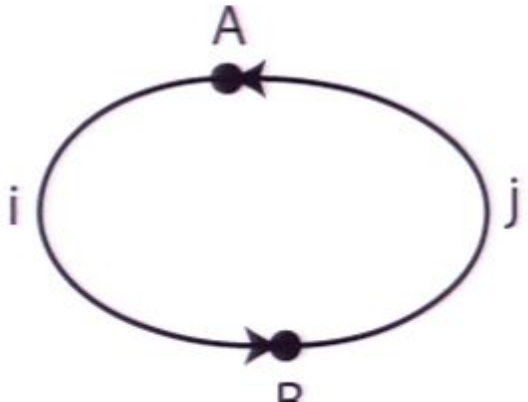
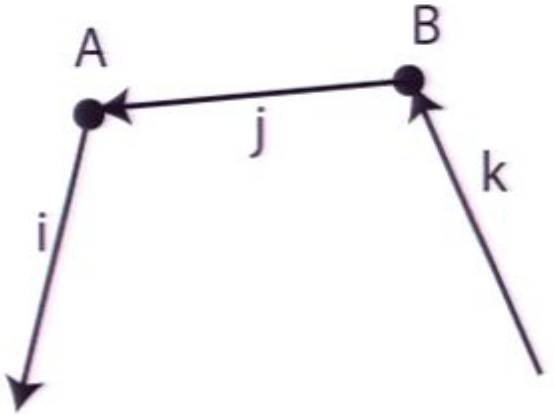
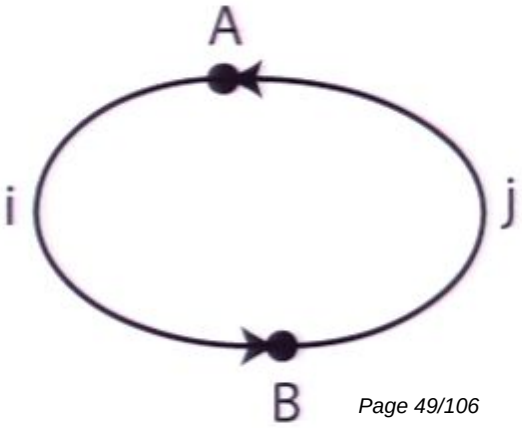


Tensor networks

Two facts about tensor networks:

1) Every network corresponds to a single tensor; rank = (# of incoming edges, # of outgoing edges.)





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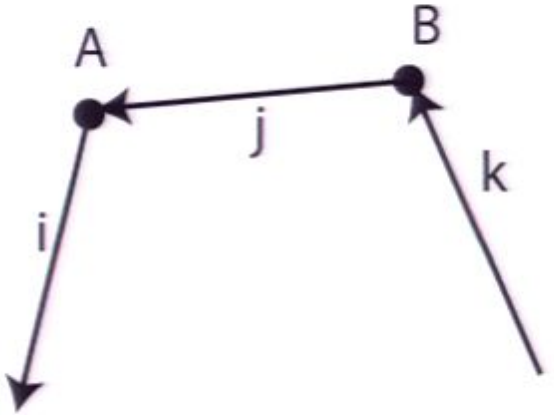
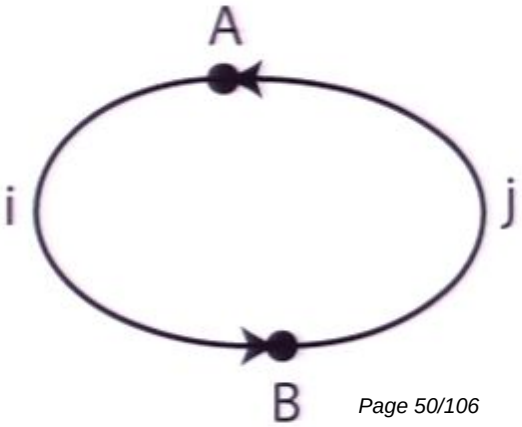


2) A network with no loose edges corresponds to a single complex number:

- ✓ label each edge with a basis vector;
- ✓ each tensor is now a complex number;
- ✓ multiply these numbers together;
- ✓ sum over all possible labelings.

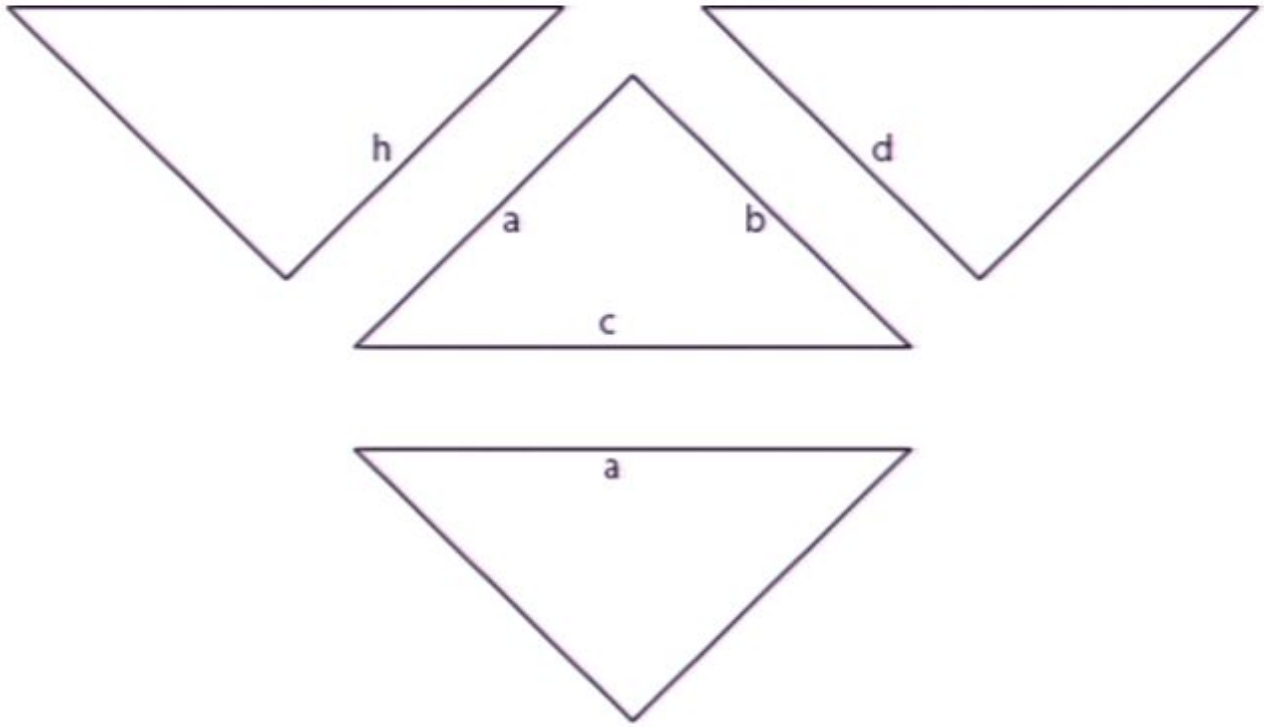


This is called the **contraction** or the **value** of the network.

Decorating triangulations

Decorating a triangulation:

1. label each edge with an index;



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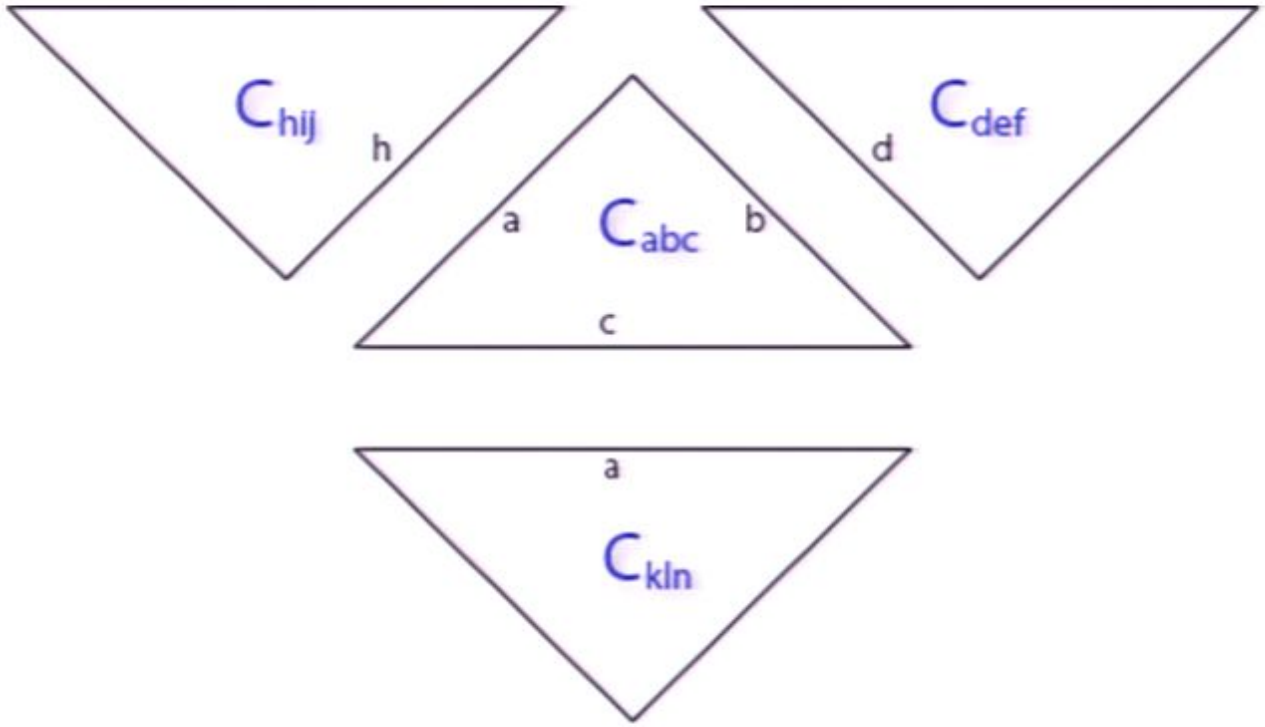

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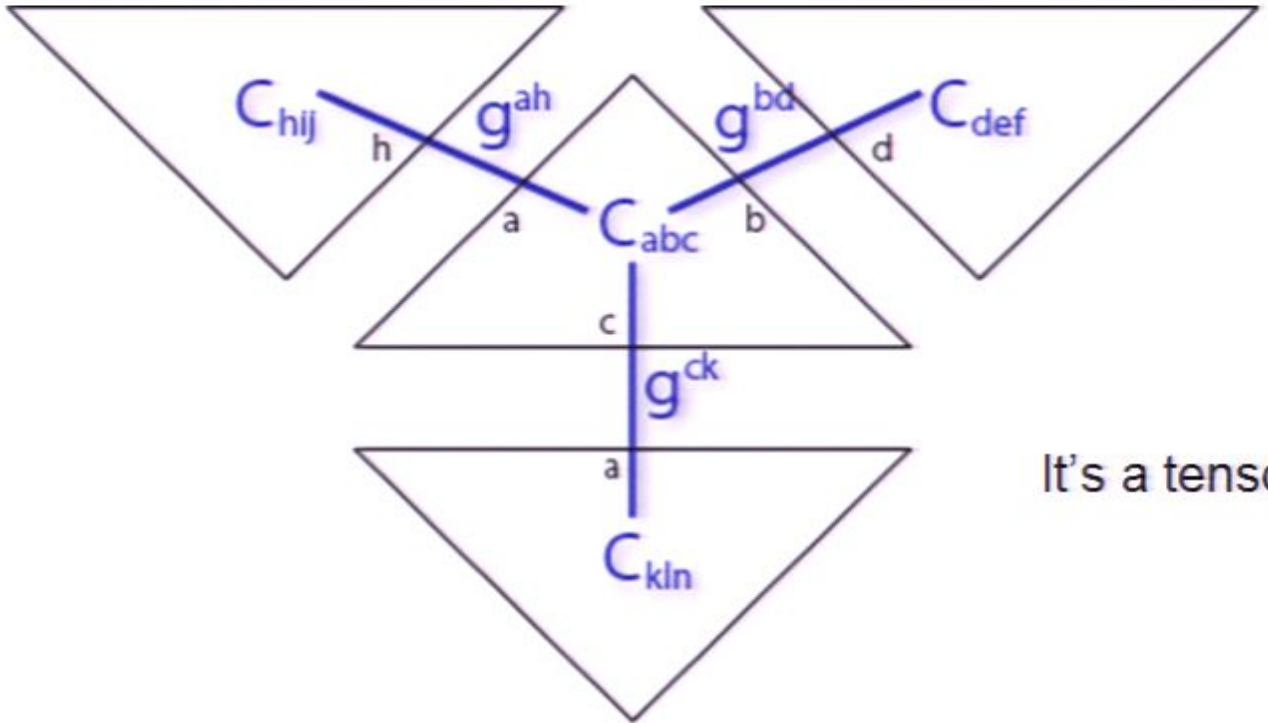
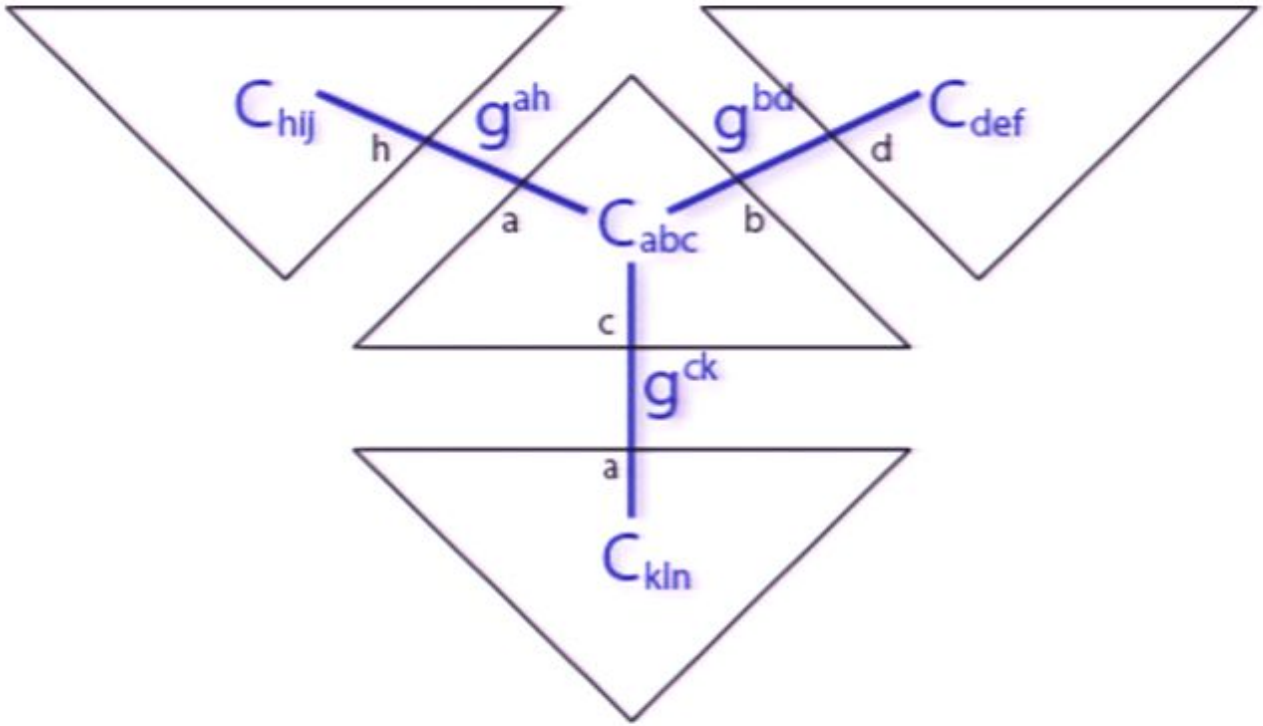


It's a tensor network!

Topological Lattice Field Theories



We want this theory to be topological, so we require that the tensors satisfy the Pachner moves.

Definition: A 2-D Topological Lattice Field Theory (TLFT) is just a choice of 3-tensor **C** and 2-tensor **g** which satisfy the 2-2 and 1-3 Pachner moves.

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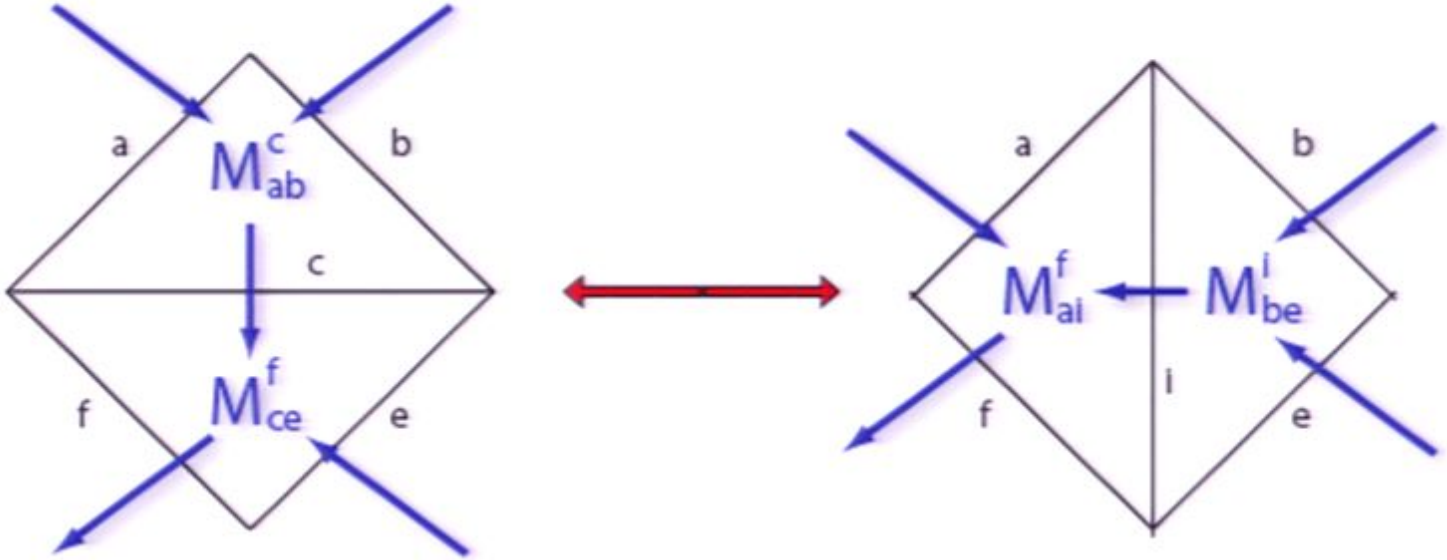

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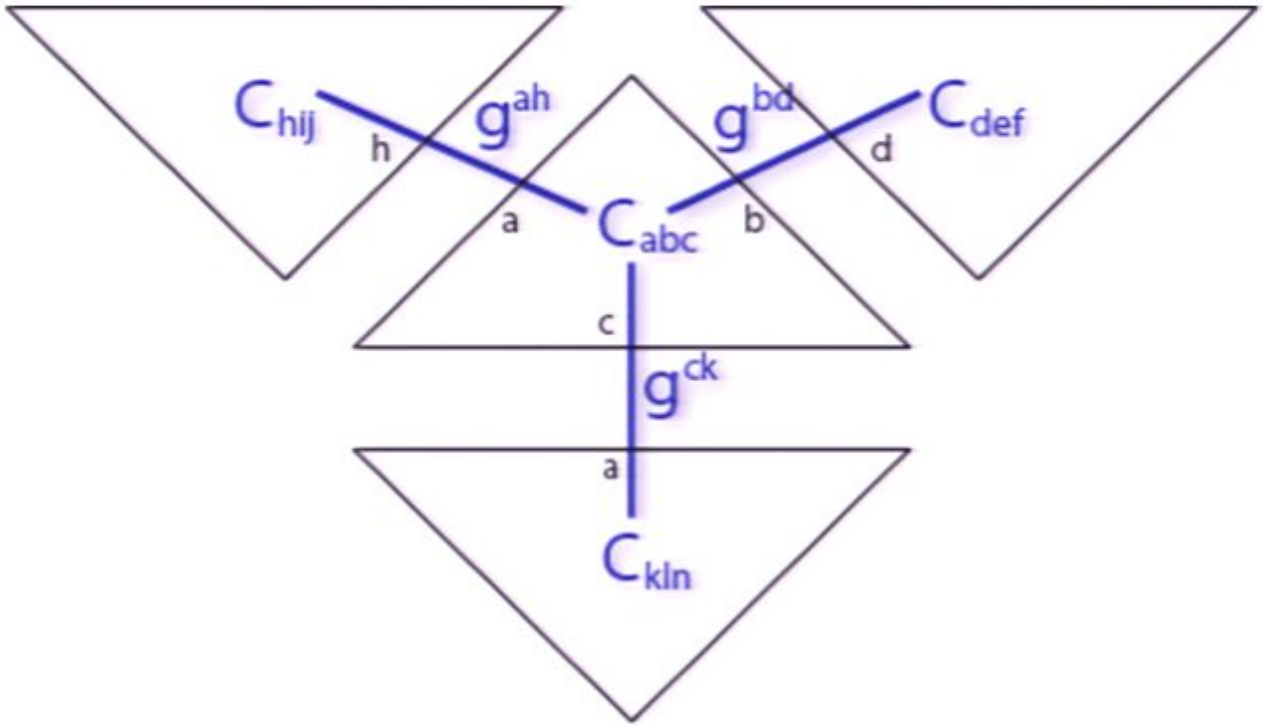

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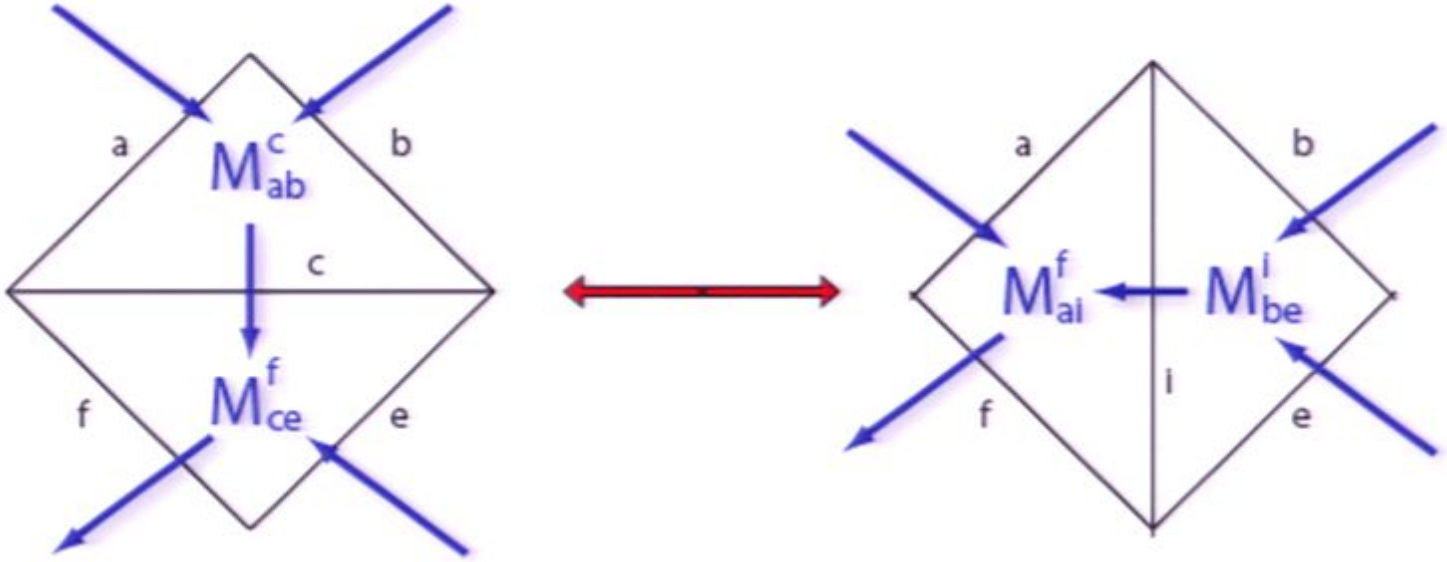

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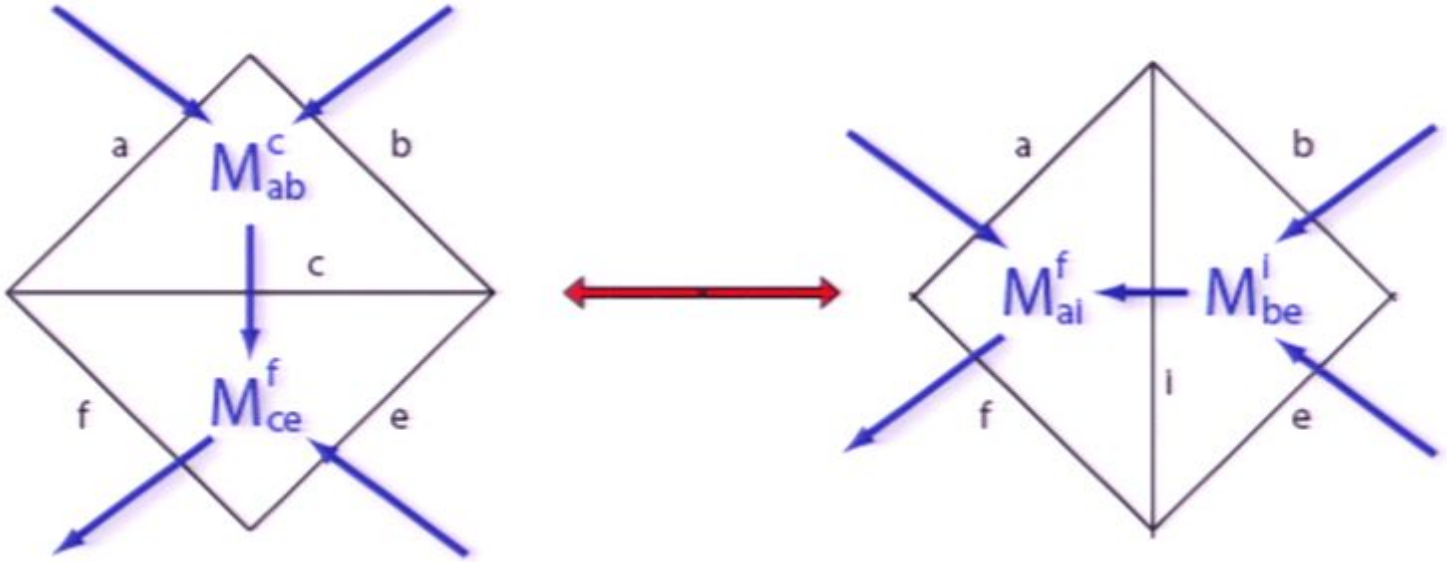

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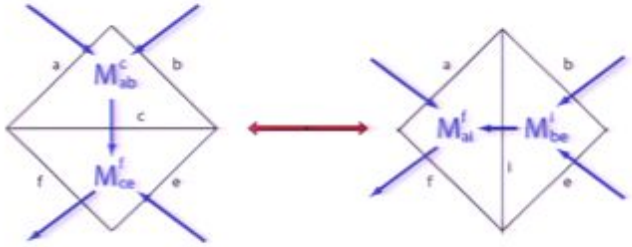

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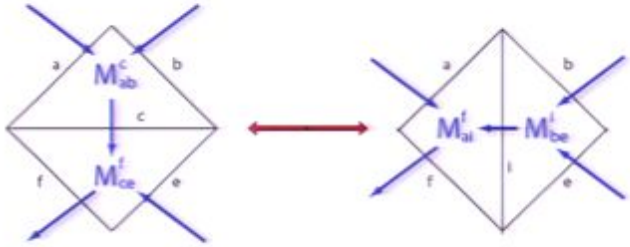

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Theorem (Fukuma, Hosono, Kawai) [1992]:
2-D Topological Lattice Field Theories are in one-to-one correspondence with semisimple associative algebras.

Group Algebras

What associative, semisimple algebras do we know?

Group algebras!

- **G** : finite group
- **CG** : maps from **G** to **C**
- multiplication on **CG** = convolution of functions
- **CG** is associative and semisimple; what are the simple modules?
- They are the **irreducible representations** of **G**.

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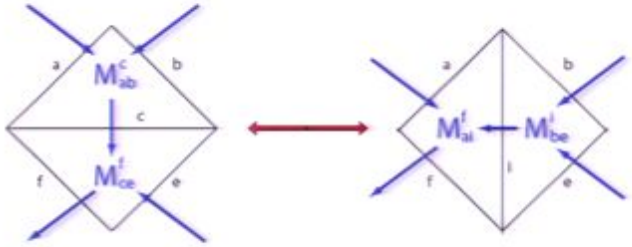

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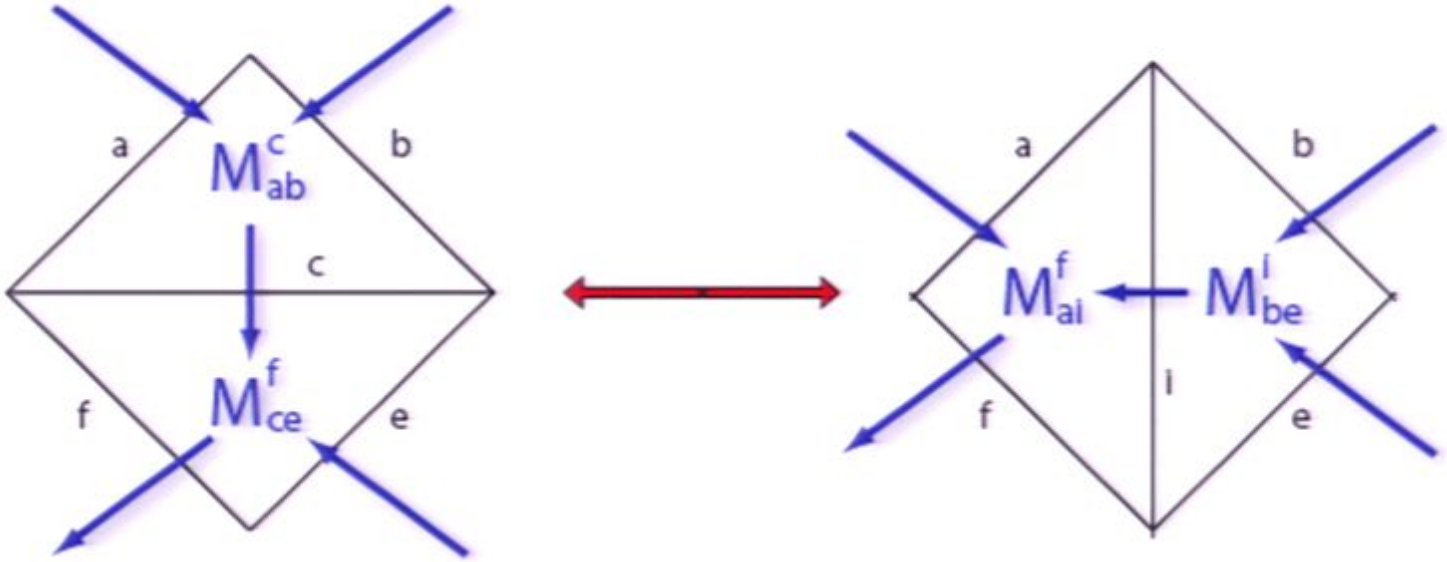

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CG is a representation: $\mathbf{x[\ f\ ](y) = f(x^{-1}y)}$.

Every irrep of **G** appears in **CG**, and it appears its dimension many times.

Note: group basis ➔ representation basis map is the **Fourier Transform**.

2d TLFTs with Group Algebras

What happens when we build a TLFT using a group algebra?

3-tensor: $$a \otimes b \otimes c \mapsto \begin{cases} |G| & \text{if } abc = 1 \\ 0 & \text{otherwise.} \end{cases}$$

2-tensor: $$1 \mapsto \frac{1}{|G|} \sum_{g \in G} g \otimes g^{-1}$$

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So pick a surface (without boundary) **S** and a group **G**. Take a triangulation of **S**, decorate it with the above tensors of **G**, and contract the whole thing to get a number in **C**...

What in blazes do you get!?

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Snyder (2007):

Euler characteristic of **S** = 2-2(genus)

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RHS = computed in group basis; LHS = computed in Fourier basis.
This is a famous fact called **Mednykh's equality.** Snyder proved it using TLFTs!

Quantum computation

I promised that all of this would have something to do with quantum computation!

Theorem (Arad, Landau)[2008]: Quantum computers can approximate the value of tensor networks efficiently.

The catch:

- the approximation is additive;
- the approximation scale might be so large that the output is useless;
- it's not always easy to tell how bad the scale is.

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Implement a linear operator **L** like this:

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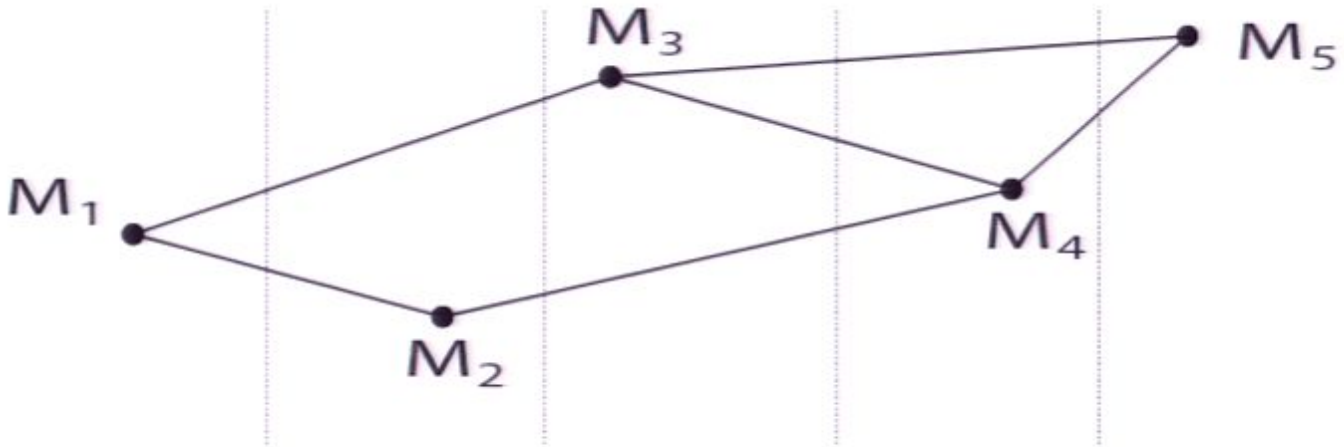

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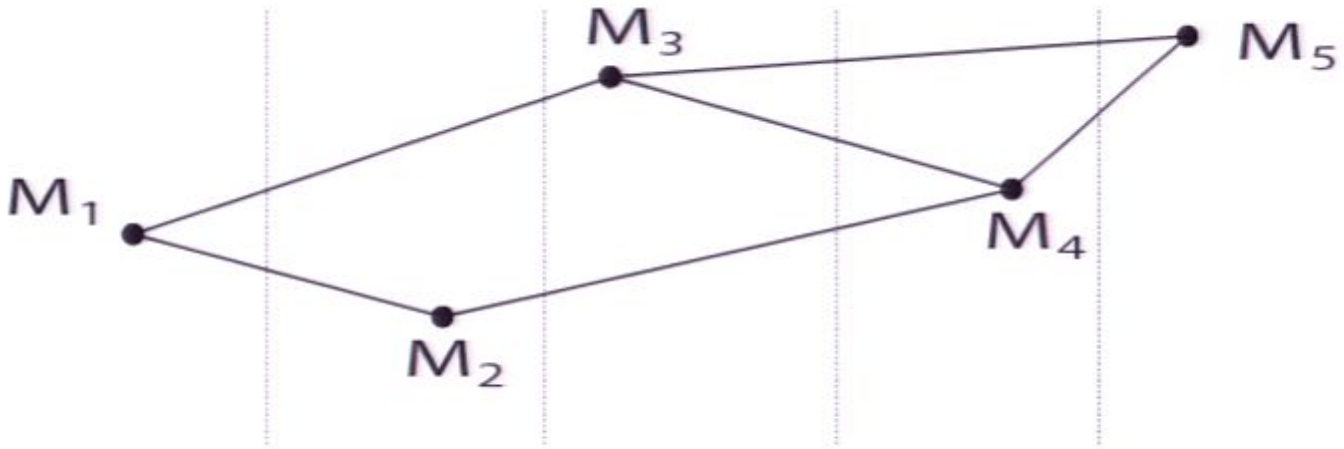

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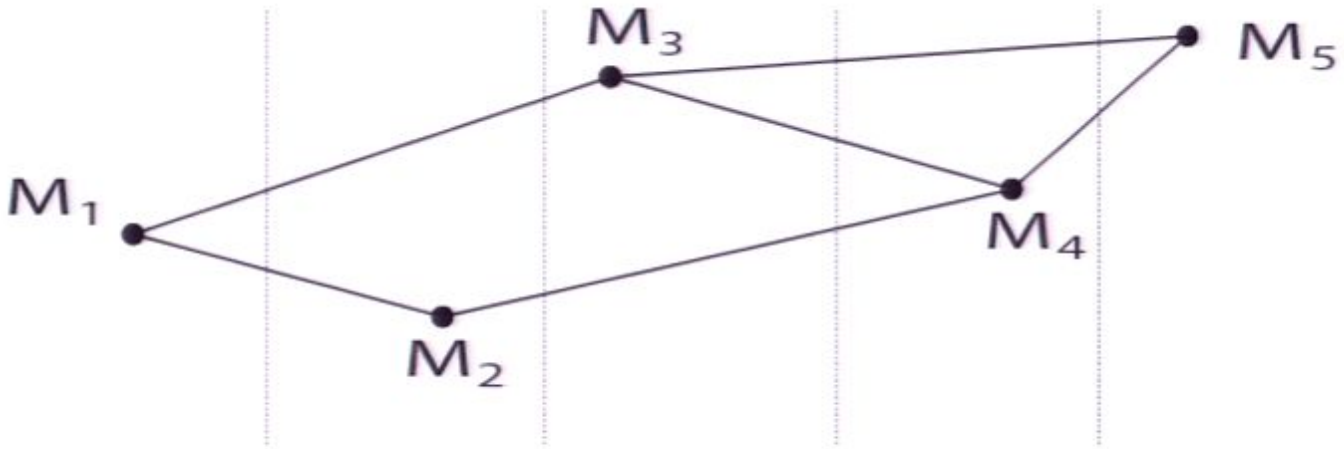

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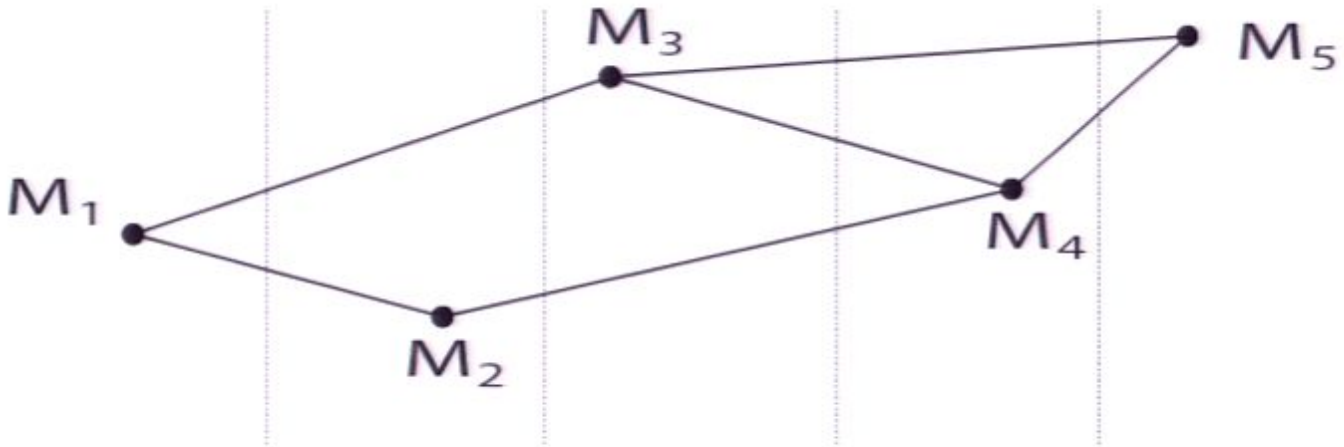

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Input: a triangulation of a surface **S**; a group **G**.

Output: an approximation of $\sum_{V \in \hat{G}} \dim(V)^{\chi(S)} = |G|^{\chi(S)-1} |\mathrm{Hom}(\pi_1(S), G)|$

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4. Output the result.

Quantum algorithm for TLFTs

The good news:

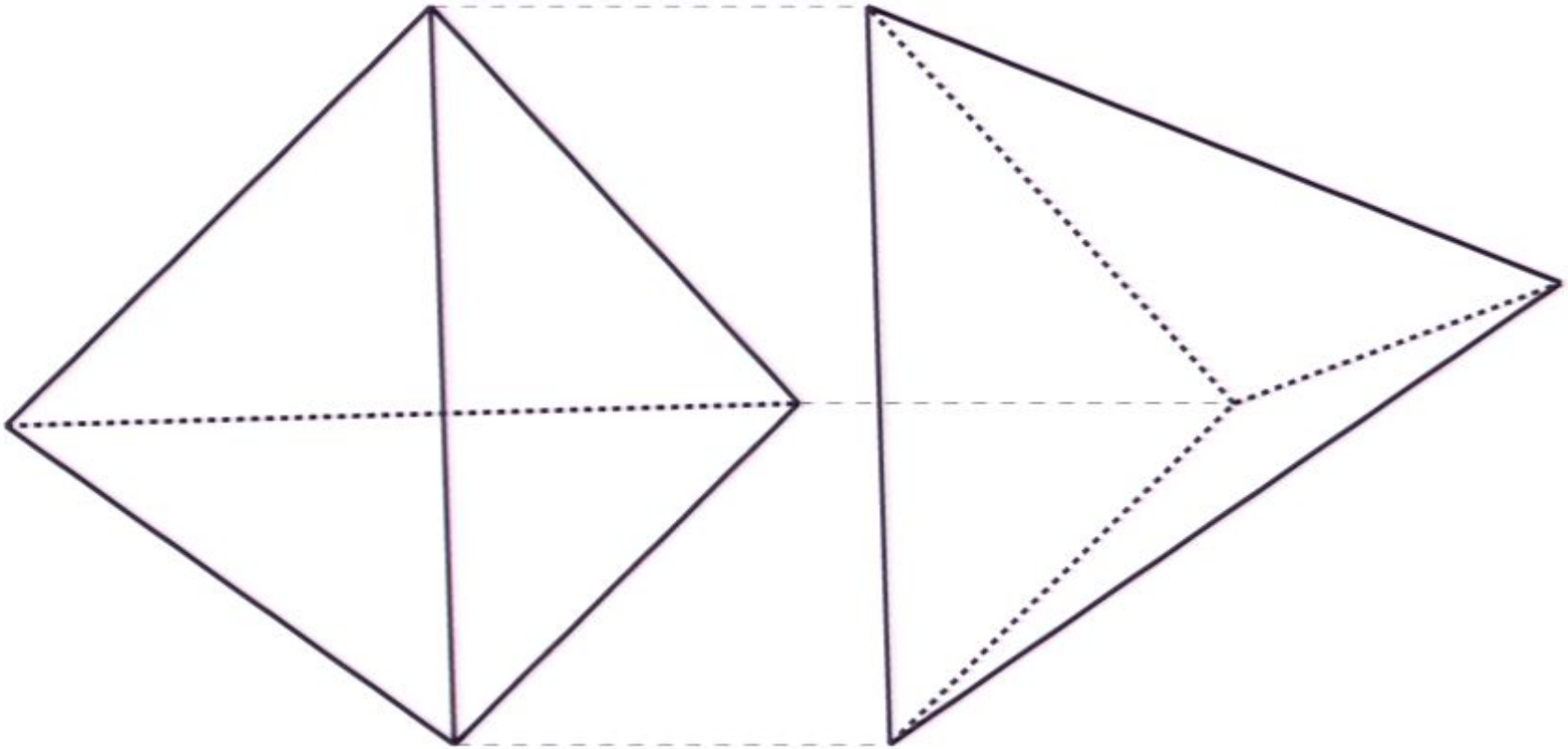
- can implement these particular tensors in polylog(|G|) time;
- the basic strategy applies to the 3-D case, which is what we really care about!
- the 3-D case involves much harder math!

The bad news:

- easy classical algorithm for computing Euler characteristic;
- the speed of the algorithm depends on the quality of the triangulation;
- "obvious" implementation leads to a bad approximation scale;

3-D TLFTs

This time, the triangulation will be into tetrahedra...



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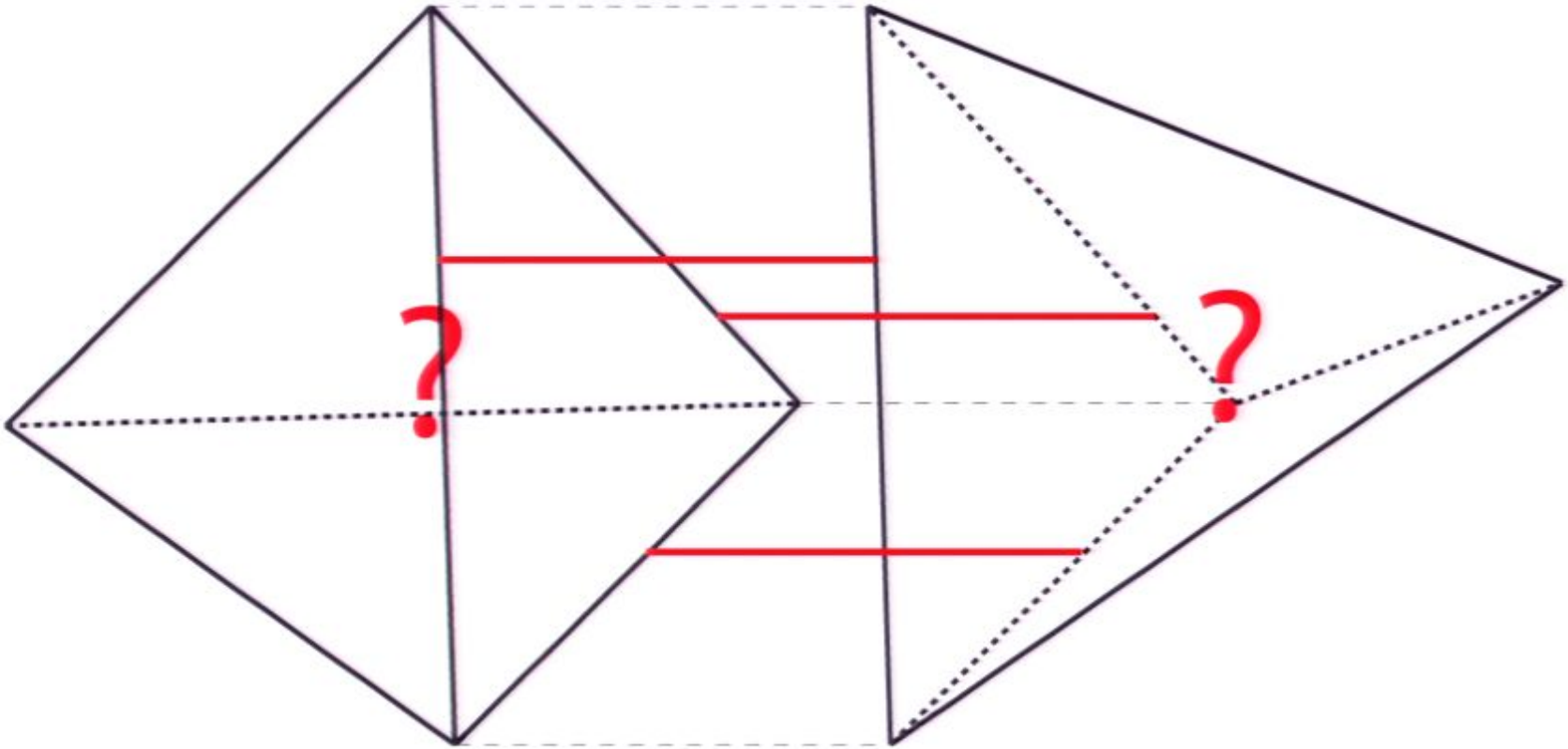

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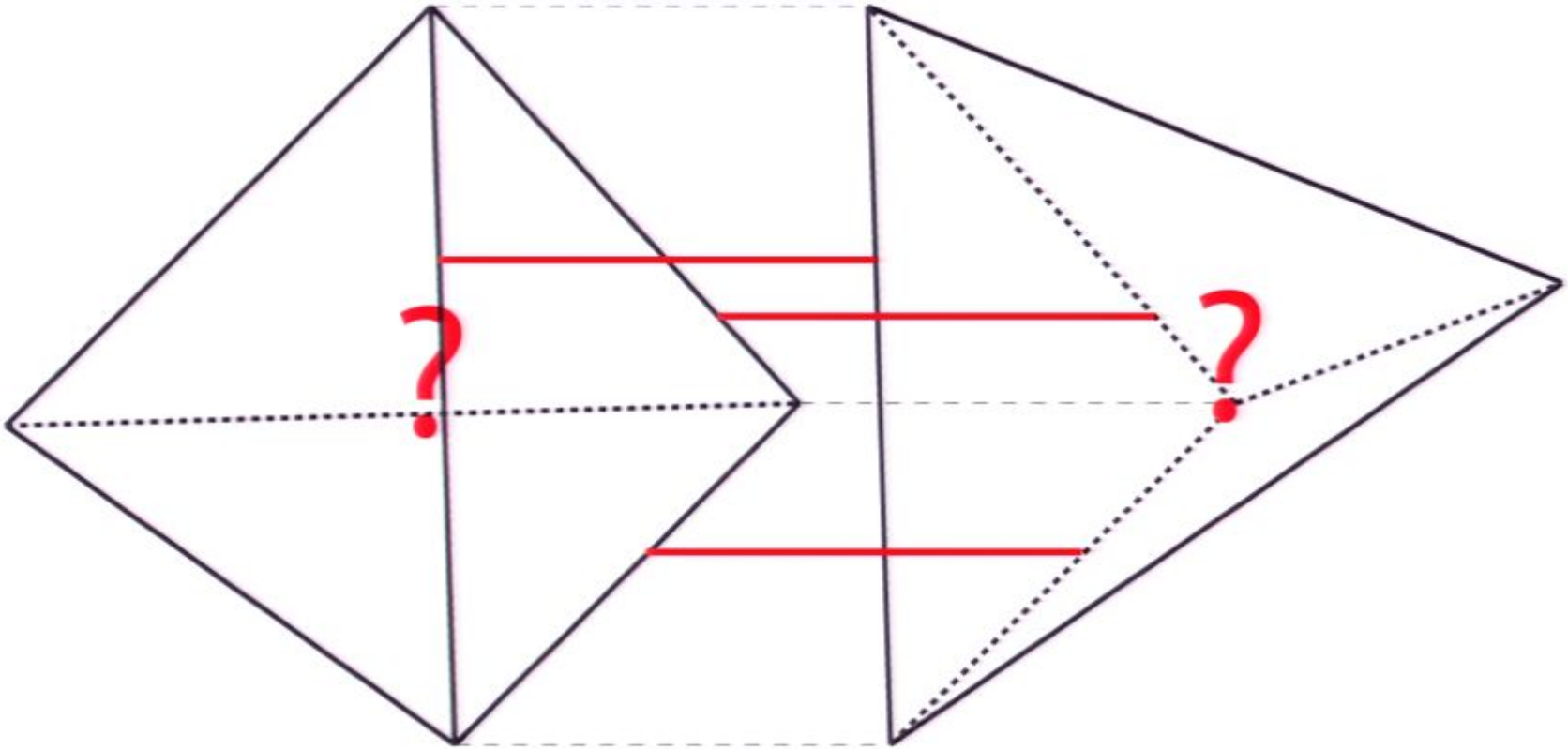

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3-D TLFTs: 6j symbols

It turns out that one should glue along edges; the right 6-tensor comes from the representation theory of **SU(2)** and its quantum variants.

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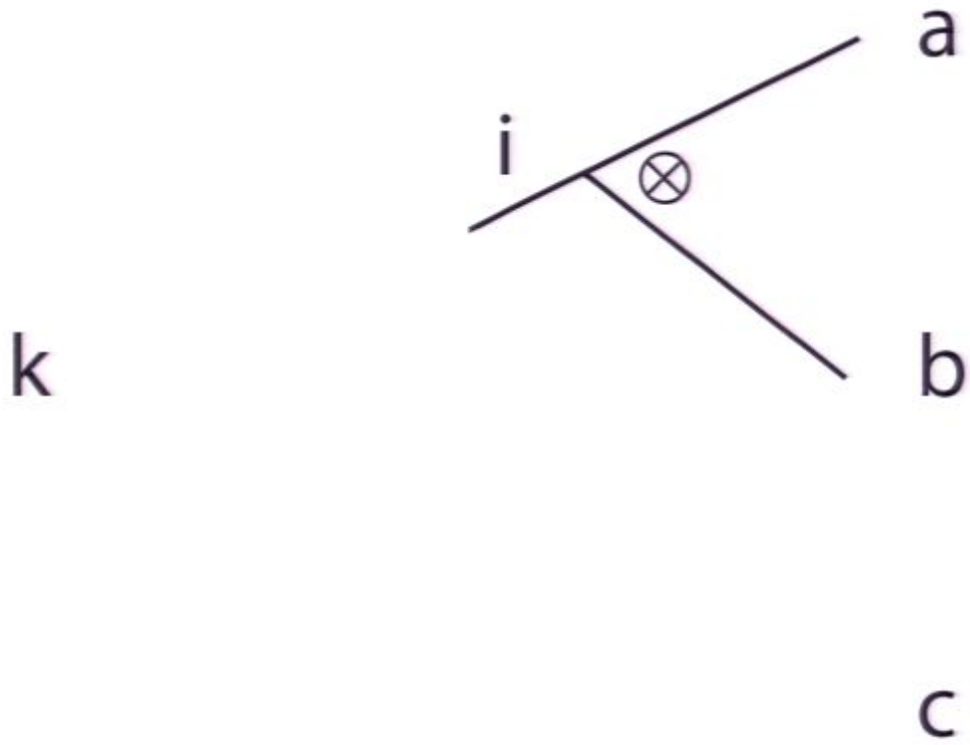

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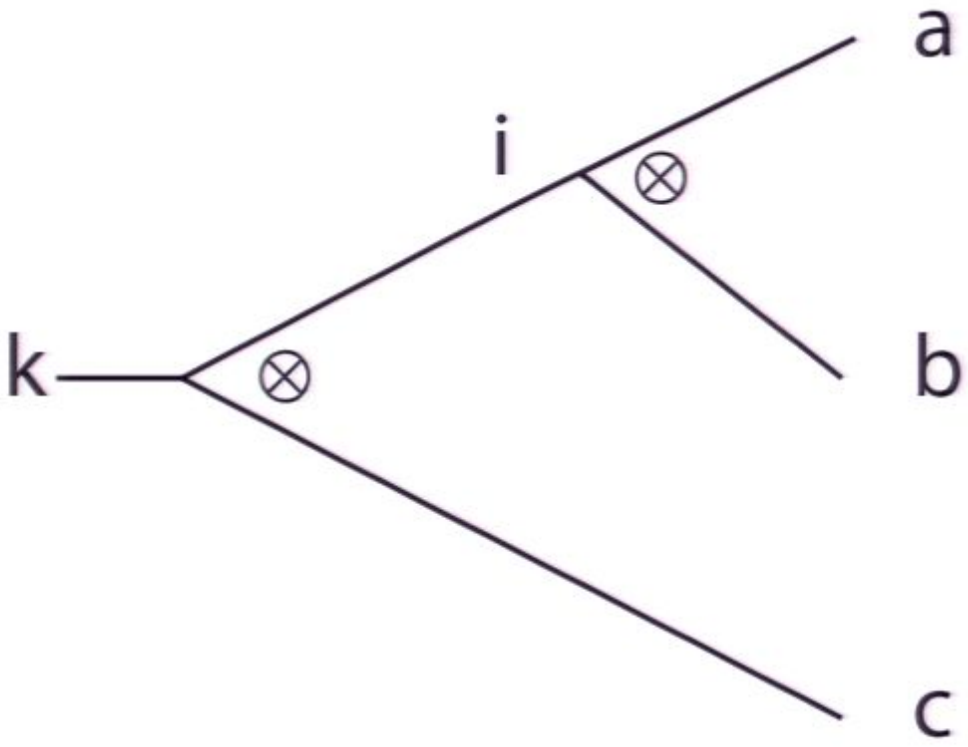

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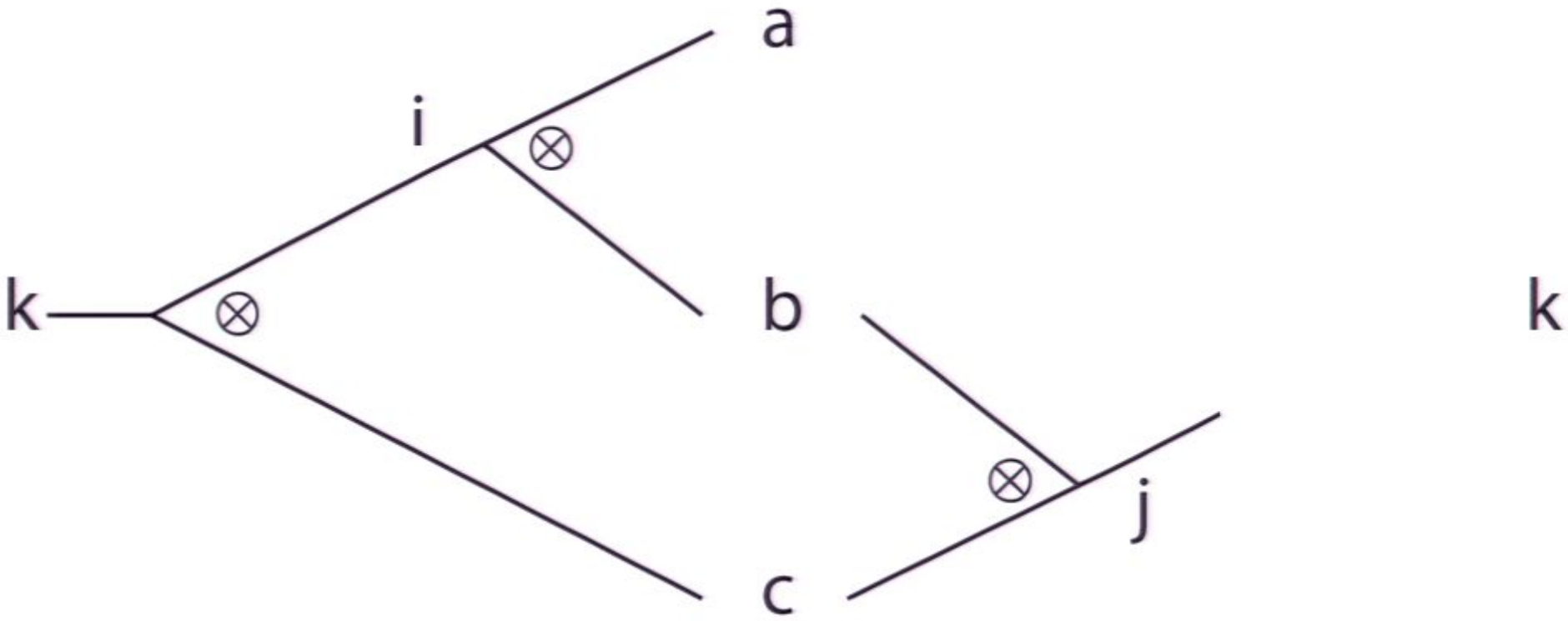

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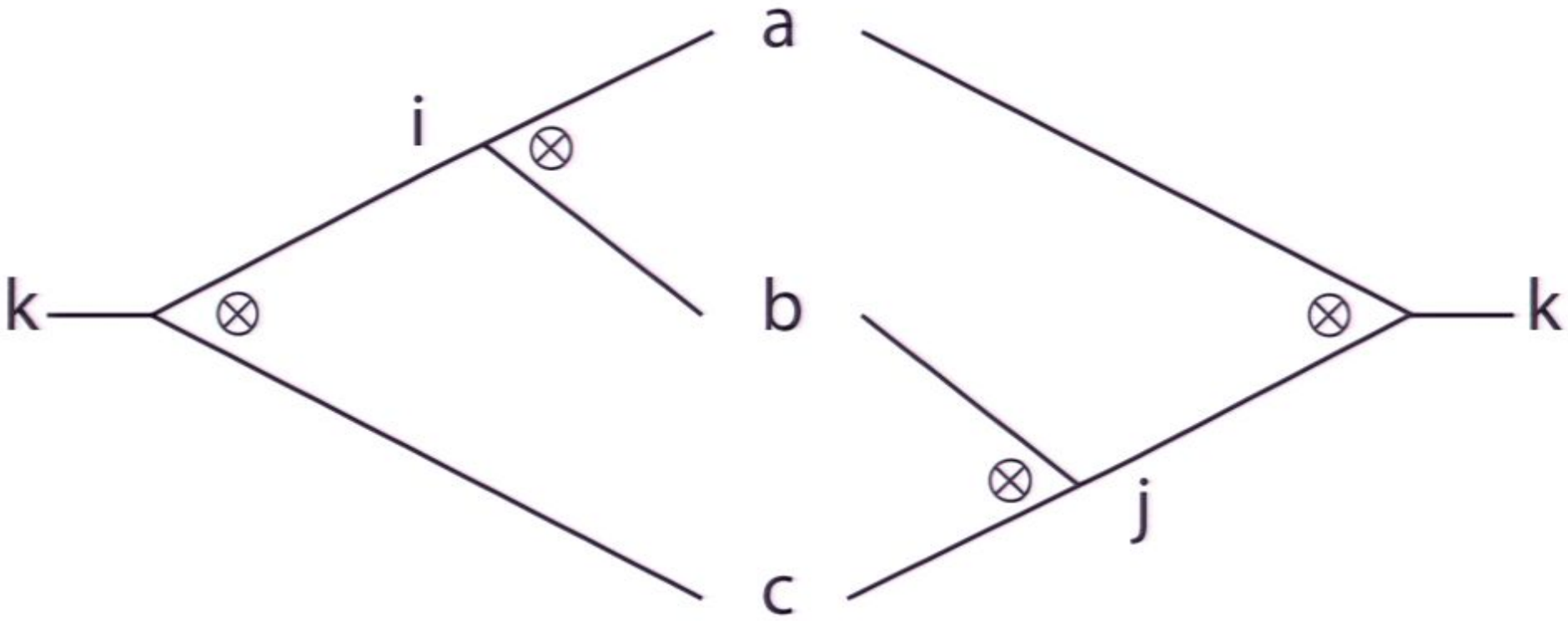

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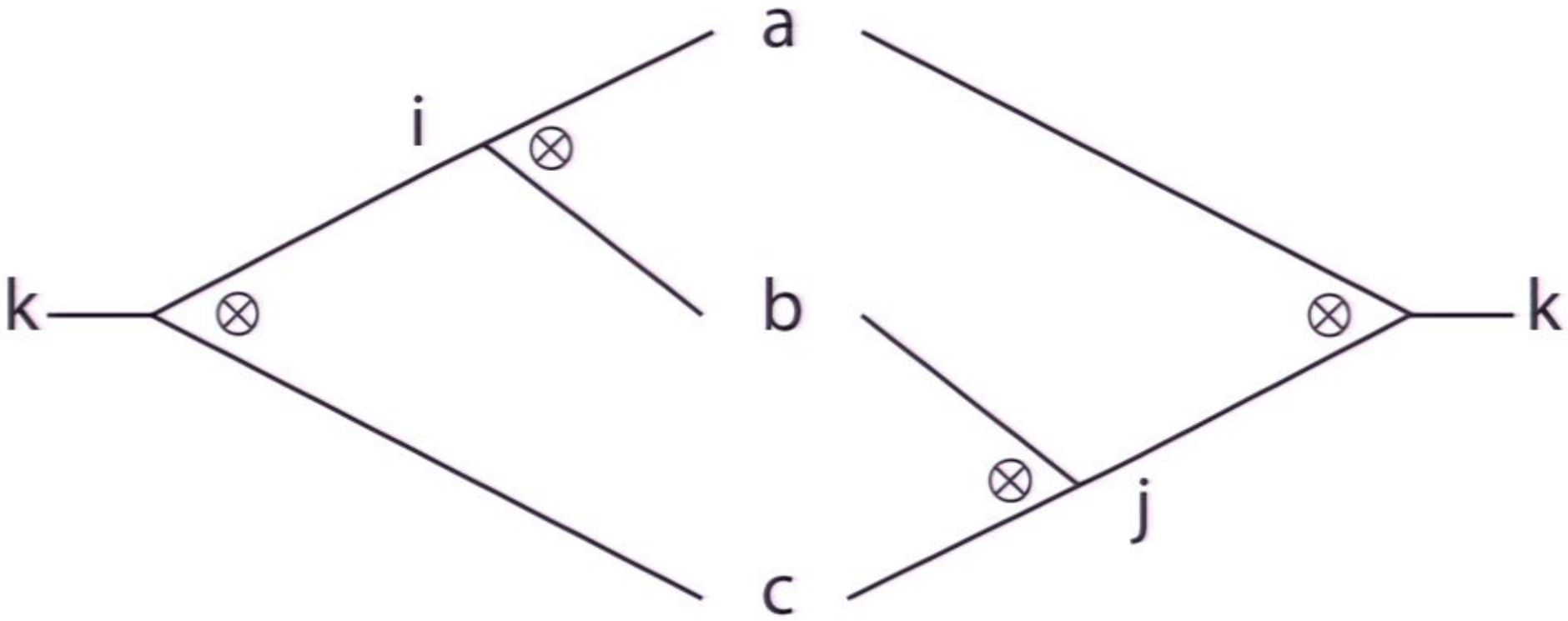

Schur's Lemma: Any **G**-invariant map from an irrep to itself is scalar multiplication.

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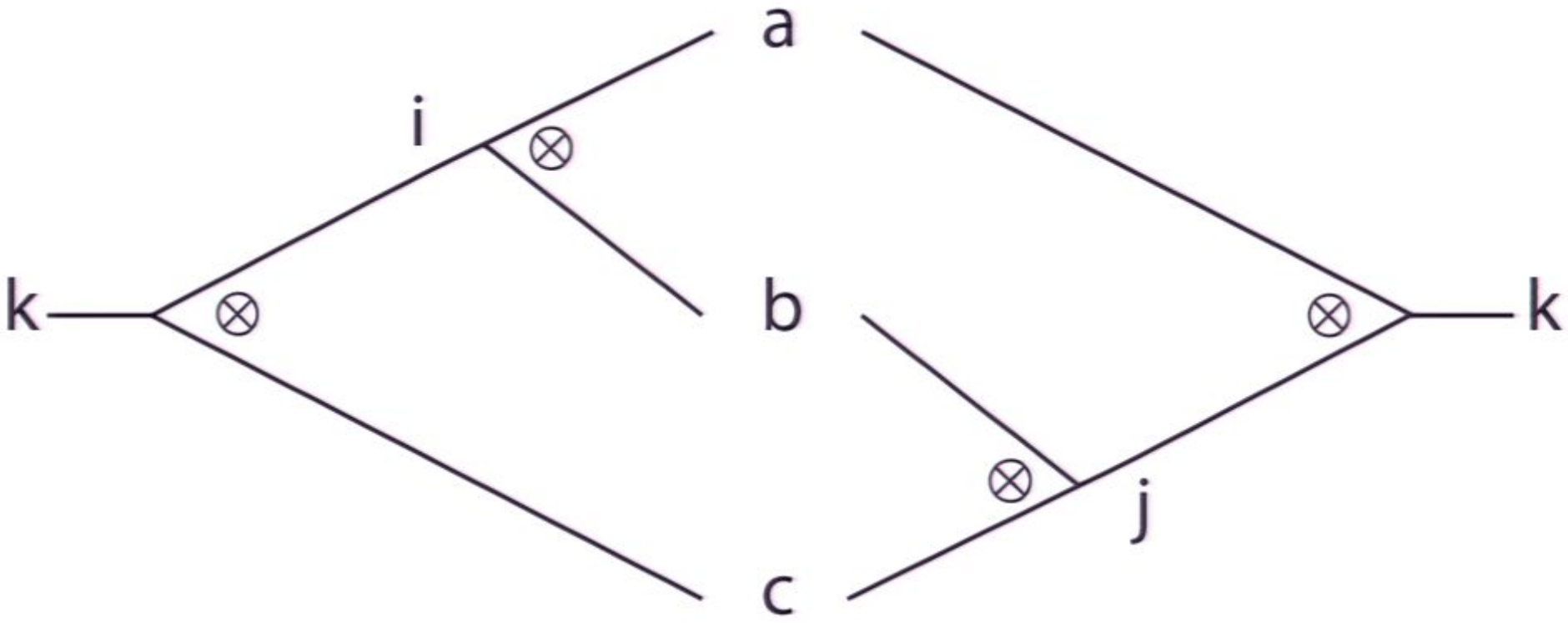

Biedenharn-Elliott identity: These 6j symbols satisfy the 3-D Pachner moves.

3-D TLFTs: 6j symbols

So now we…

- start with a tetrahedral triangulation of a 3-manifold;
- label the edges of all the tetrahedra with **SU(2)** representation names (spins);
- assign a 6j symbol to every tetrahedron;

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So the situation is more complicated than 2-D, but it can be worked out…

The result is the Turaev-Viro 3-manifold invariant.

3-D TLFTs: 6j symbols

Open: Is there an efficient quantum algorithm for the Turaev-Viro invariant?

THANKS!

Quantum algorithm for TLFTs

Input: a triangulation of a surface **S**; a group **G**.

Output: an approximation of $\sum_{V \in \hat{G}} \dim(V)^{\chi(S)} = |G|^{\chi(S)-1} |\mathrm{Hom}(\pi_1(S), G)|$

Algorithm:

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