Title: Crystal Melting and Toric Calabi-Yau Manifolds

Date: Feb 10, 2009 11:00 AM

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Abstract: Motivated by recent mathematical developements in non-commutative Donaldson-Thomas theory, we construct a new statistical mechanicalmodel of crystal melting to count BPS bound states of D0 and D2 branes on a single D6 brane wrapping an arbitrary toric Calabi-Yau threefold. We also discuss the wall crossing phenomena, which are crucial for the proper understanding of the relation between the crystal melting and the topological string theory.

Crystal Melting and Toric Calabi-Yau Manifolds

Masahito Yamazaki

Univ. Tokyo (Hongo&IPMU), Caltech

Based on arXiv:0811.2801[hep-th]
with Hirosi Ooguri

Feb 10th, 2009 @ Perimeter

Counting of BPS States

- **Counting BPS bound states of D-branes wrapping holomorphic cycles of Calabi-Yau**

Relation with various topics

- BH microstate counting
- Gauge theory (quiver, instanton…)
- Variations of AdS/CFT
- Wall crossing
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There are two different connections…..

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$$Z_{\mathrm{BH}} = |Z_{\mathrm{top}}|^2$$

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Donaldson-Thomas theory

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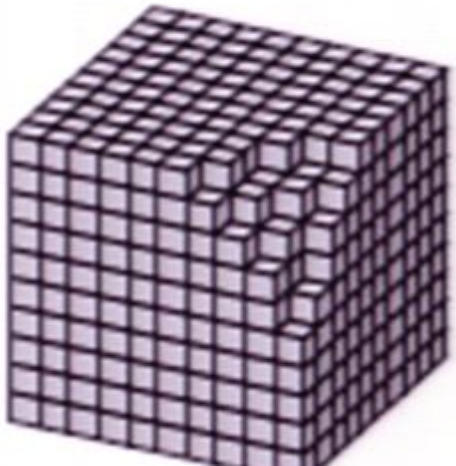

They in general hold in different region of validities

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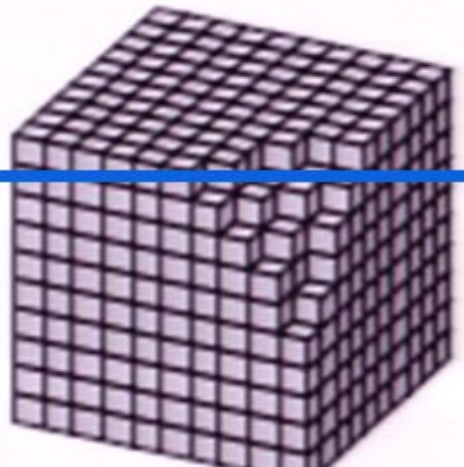

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We have generalized the crystal melting to an ***arbitrary*** toric Calabi-Yau manifold!

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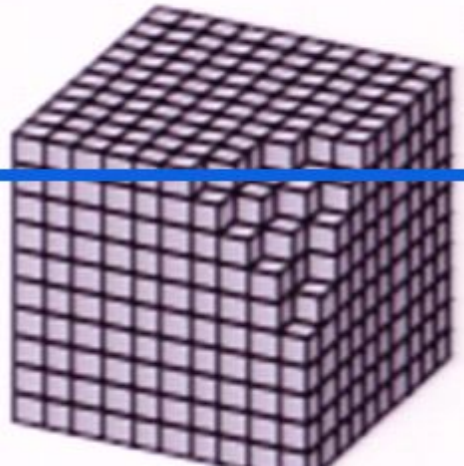

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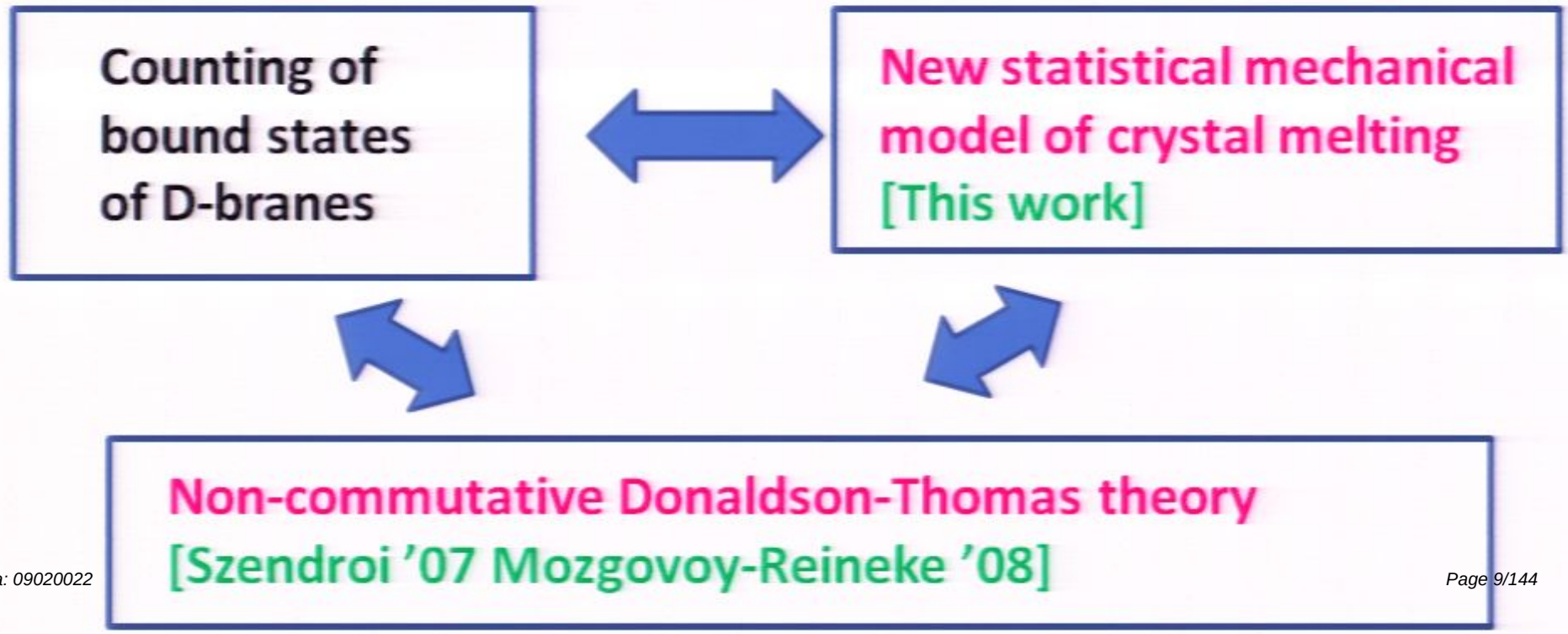

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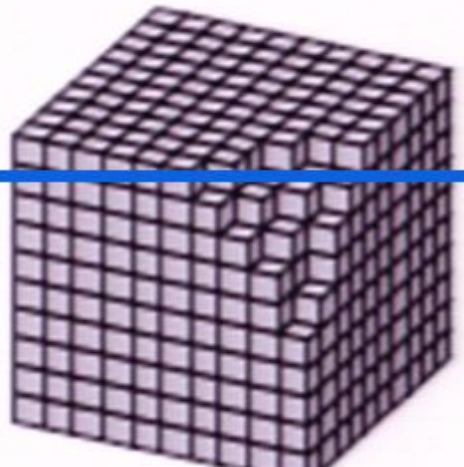

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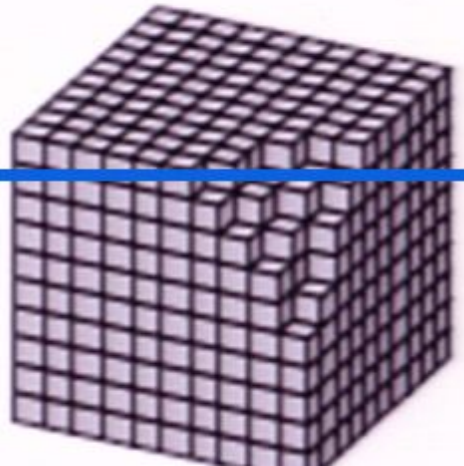

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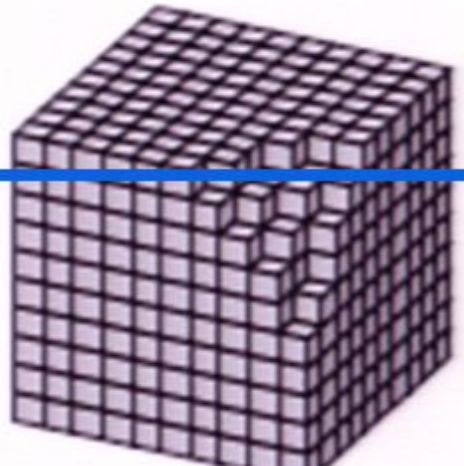

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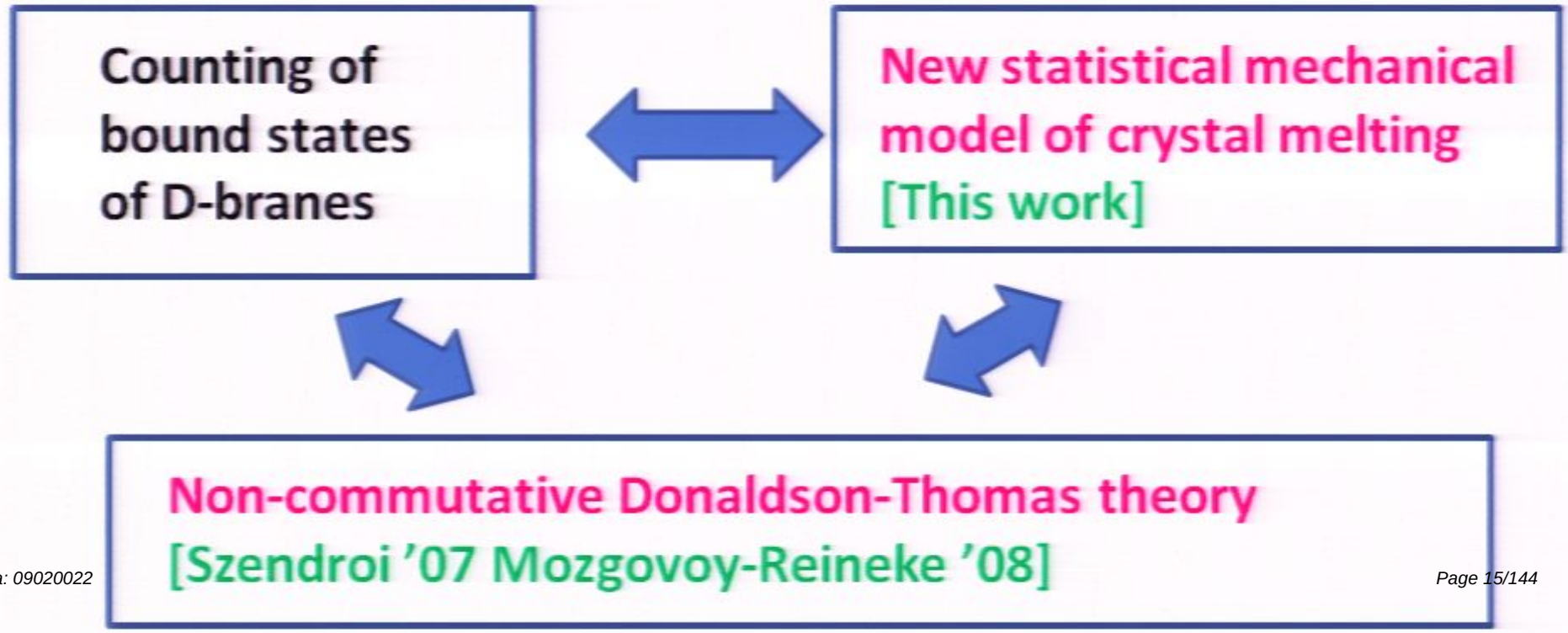

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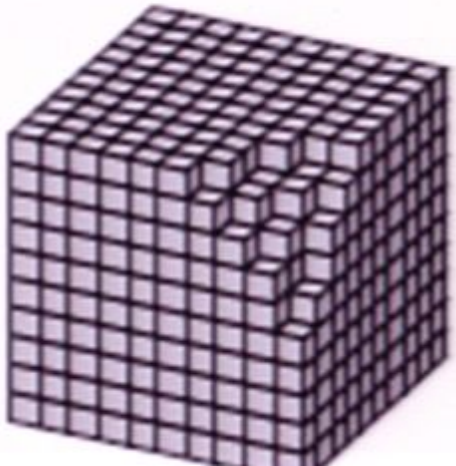

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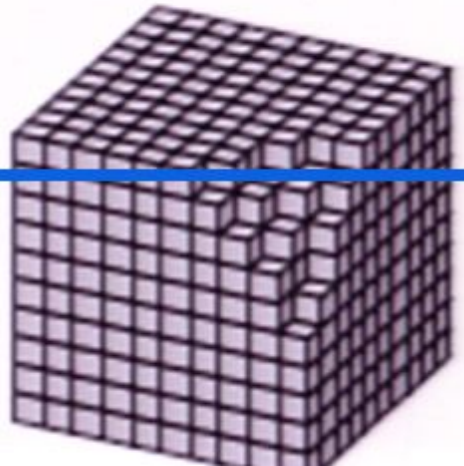

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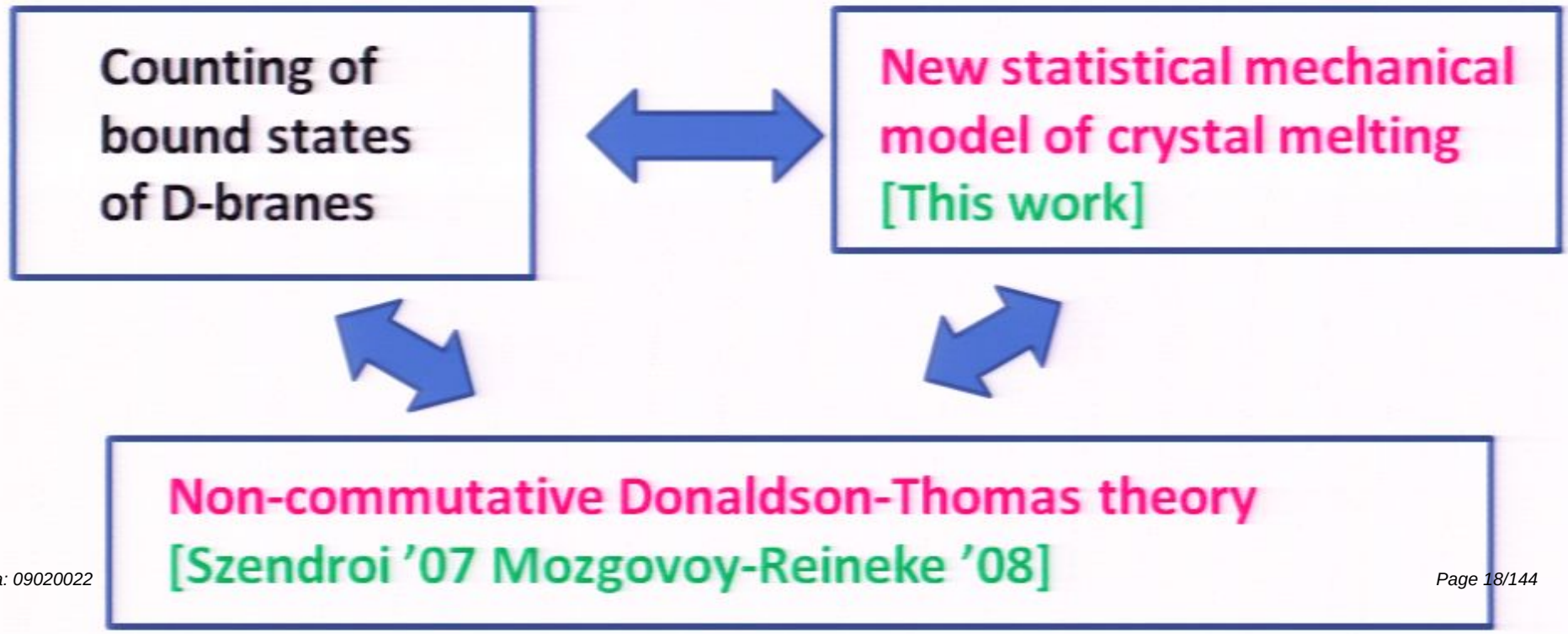

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② Supersymmetric quiver quantum mechanics

③ Our new crystal melting model

④ Proof via non-commutative Donaldson-Thoma theory

⑤ Discussions on wall crossing

⑥ Summary and discussions

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② Supersymmetric Quiver Quatum Mechanics

Quiver Quantum Mechanics

- Consider IIA on toric CY_3 with a single D6 and D0/D2 branes bound on it

  Bound states of BPS particles in 4d viewpoint

- The theory on D0/D2/D6-branes (in 4d) is the quiver quantum mechanics

  ✓ First discussed by [Douglas-Moore '96]

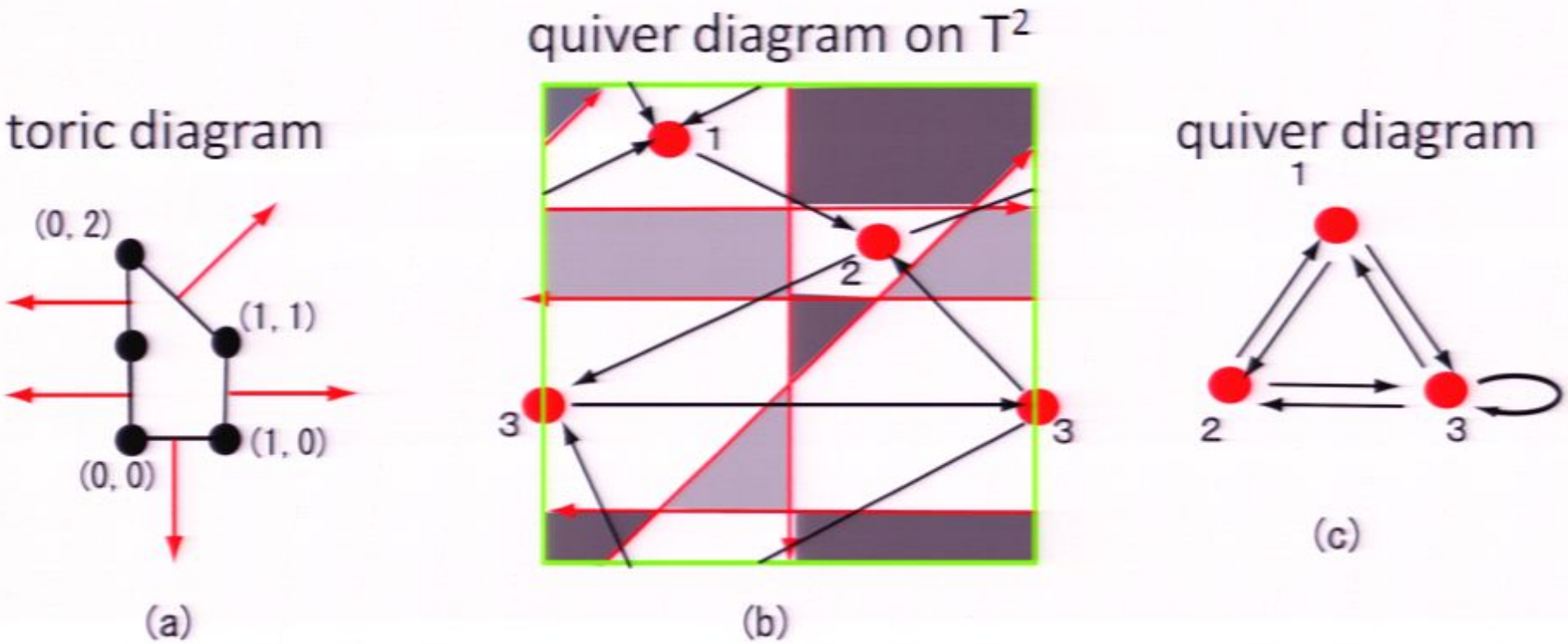
  ✓ Generalized to an arbitrary toric Calabi-Yau:

  brane tiling [Hanany and collaborators '05-]

  (see my review arXiv:0803.4474 for details)

- Brane tiling techniques gives a quiver diagram and a superpotential for an ***arbitrary*** toric CY_3

Example: Suspended Pinched Point ($xy=zw^2$)



These graphs represent physical configuration of D-branes!

[Feng-He-Kennaway-Vafa '06; Imamura-Kimura-Isono-M.Y. '07]

- **The upshot: the quiver diagram is written on T^2 , encoding the superpotential**

Each face of the quiver on T^2 corresponds to a term in the superpotential, e.g.

$$W = \mathrm{tr}(X_{21}X_{12}X_{23}X_{32} - X_{23}X_{33}X_{32} + X_{33}X_{31}X_{13} - X_{31}X_{12}X_{21}X_{13})$$

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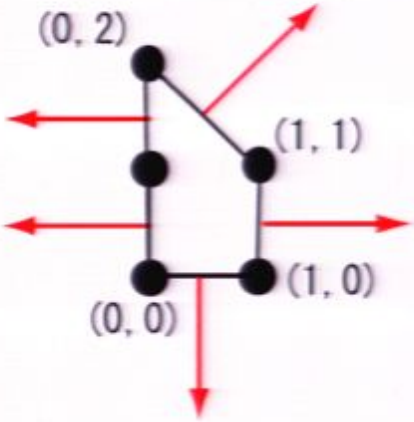

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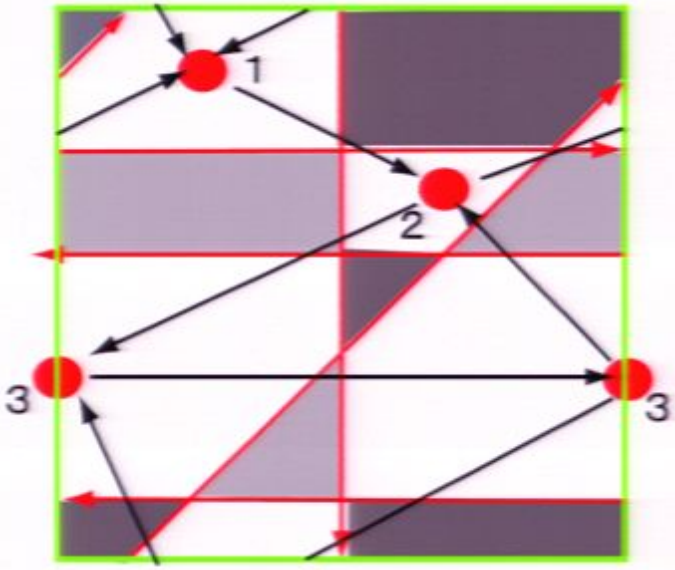
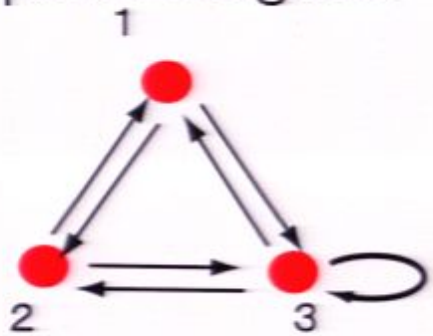
quiver diagram on T^2



(b)

quiver diagram



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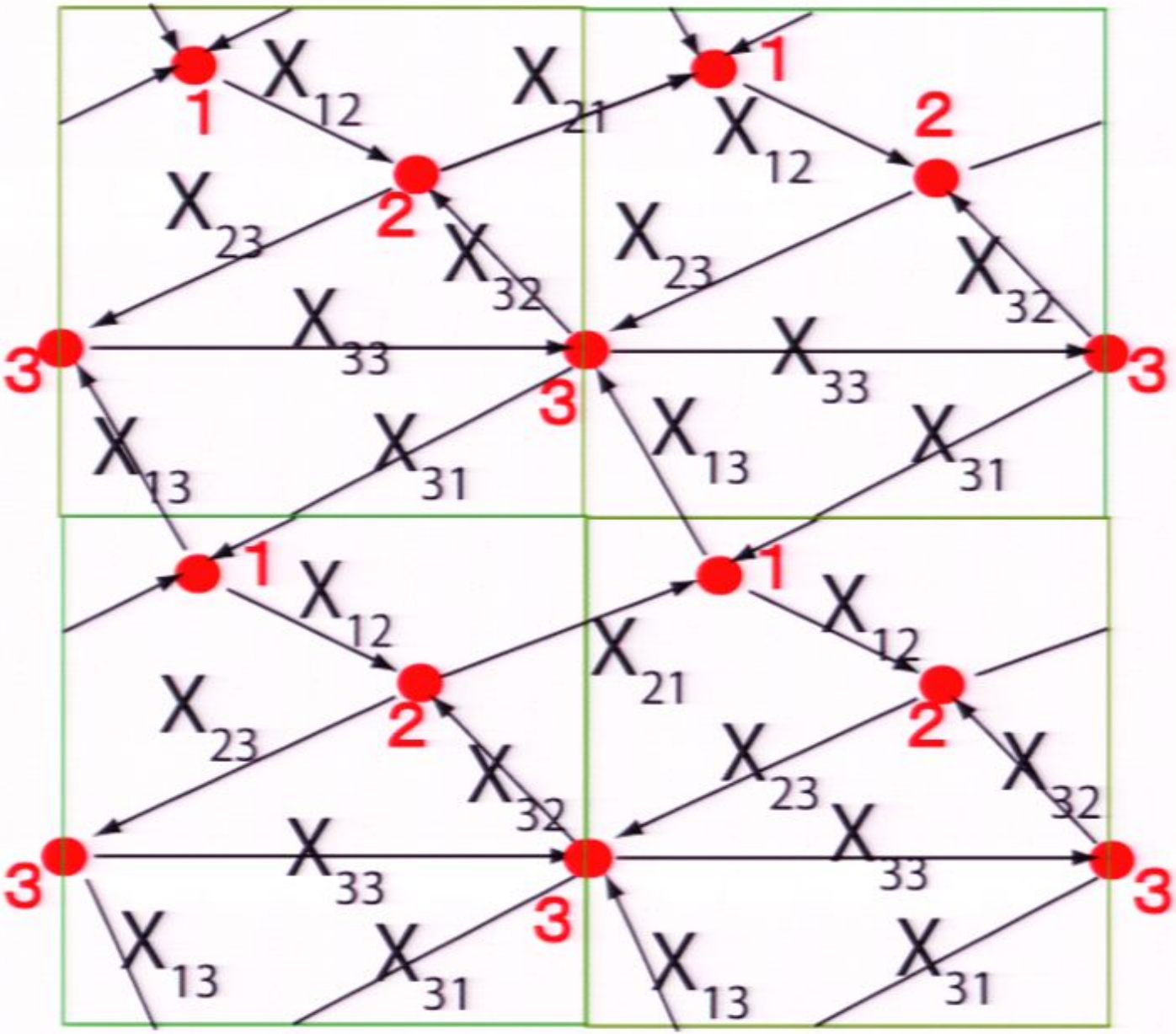
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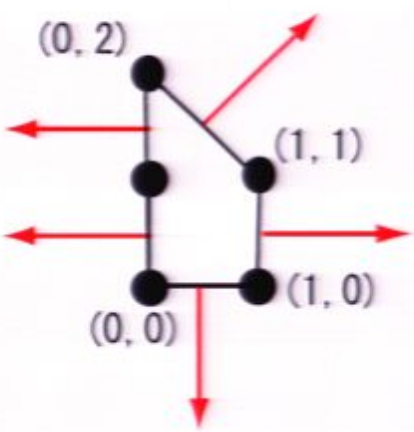

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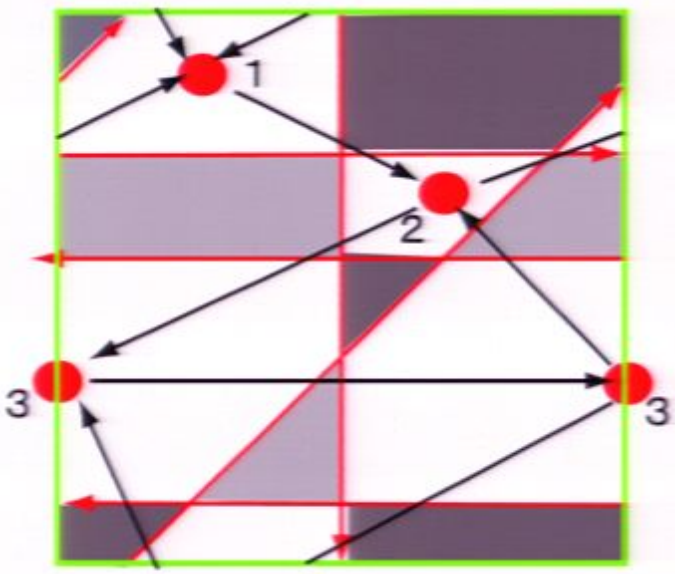

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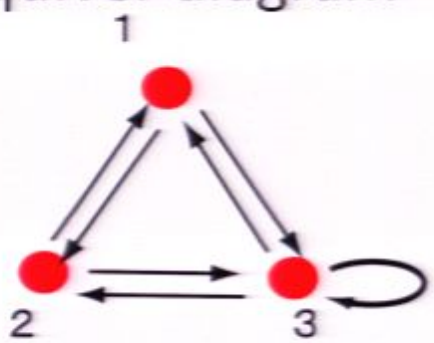

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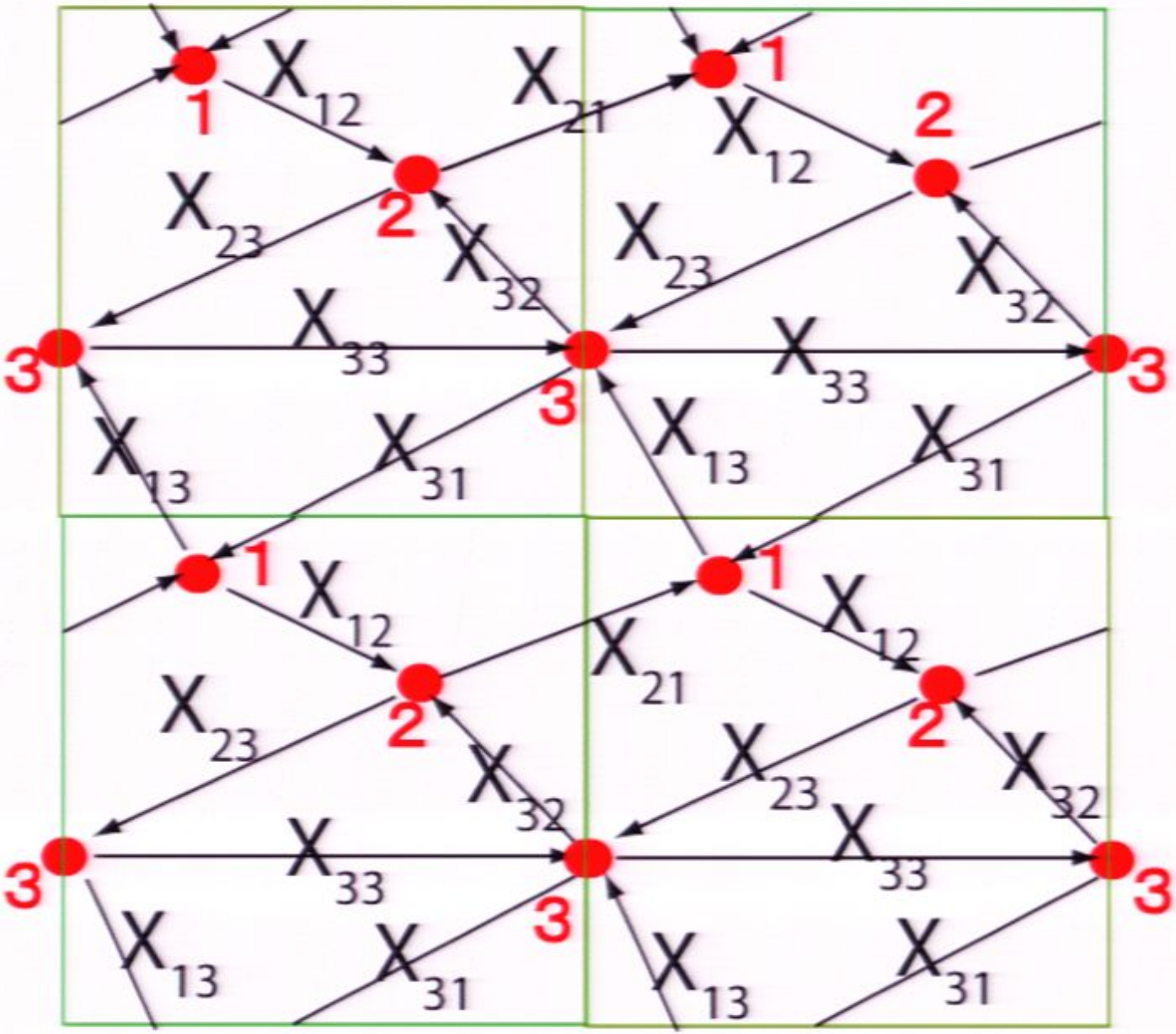
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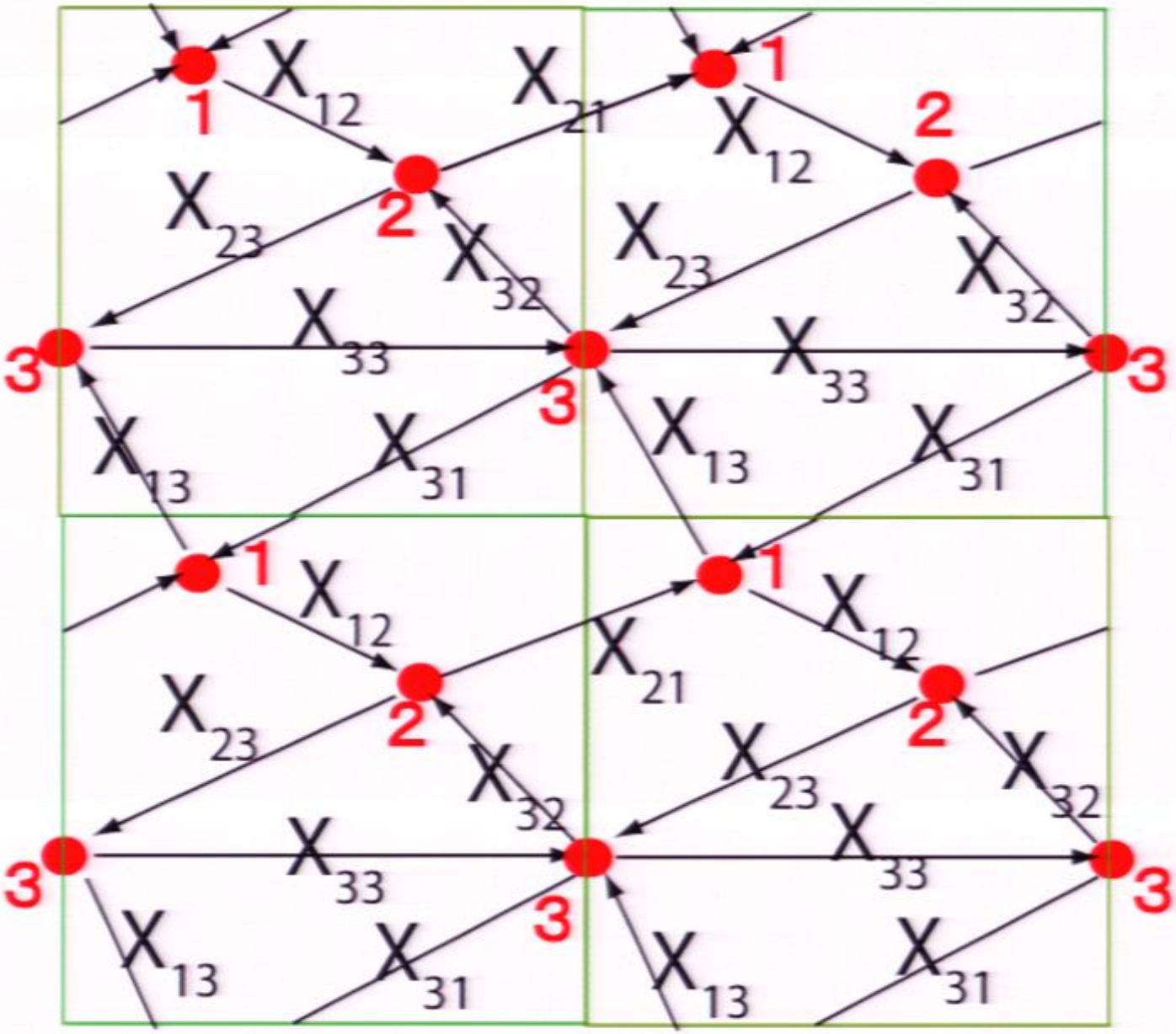
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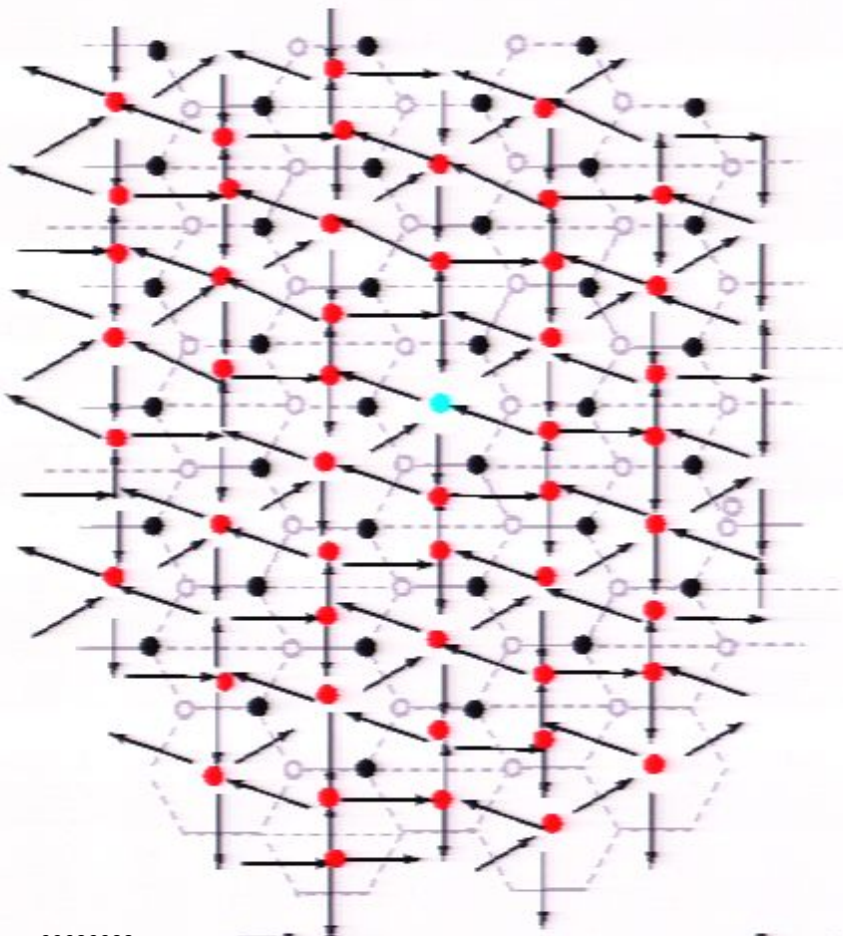
③New Crystal Melting Model

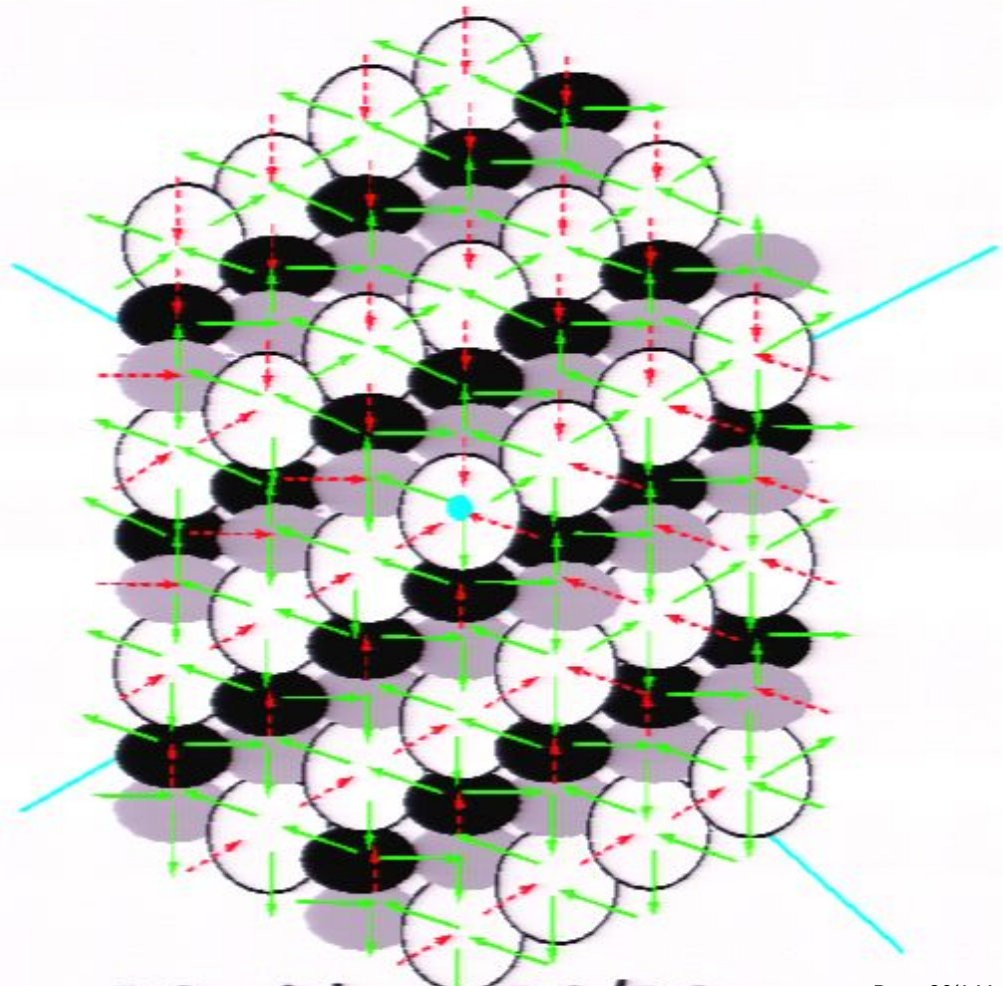
BPS state= a configuration of molten crystal

An example of a crystal

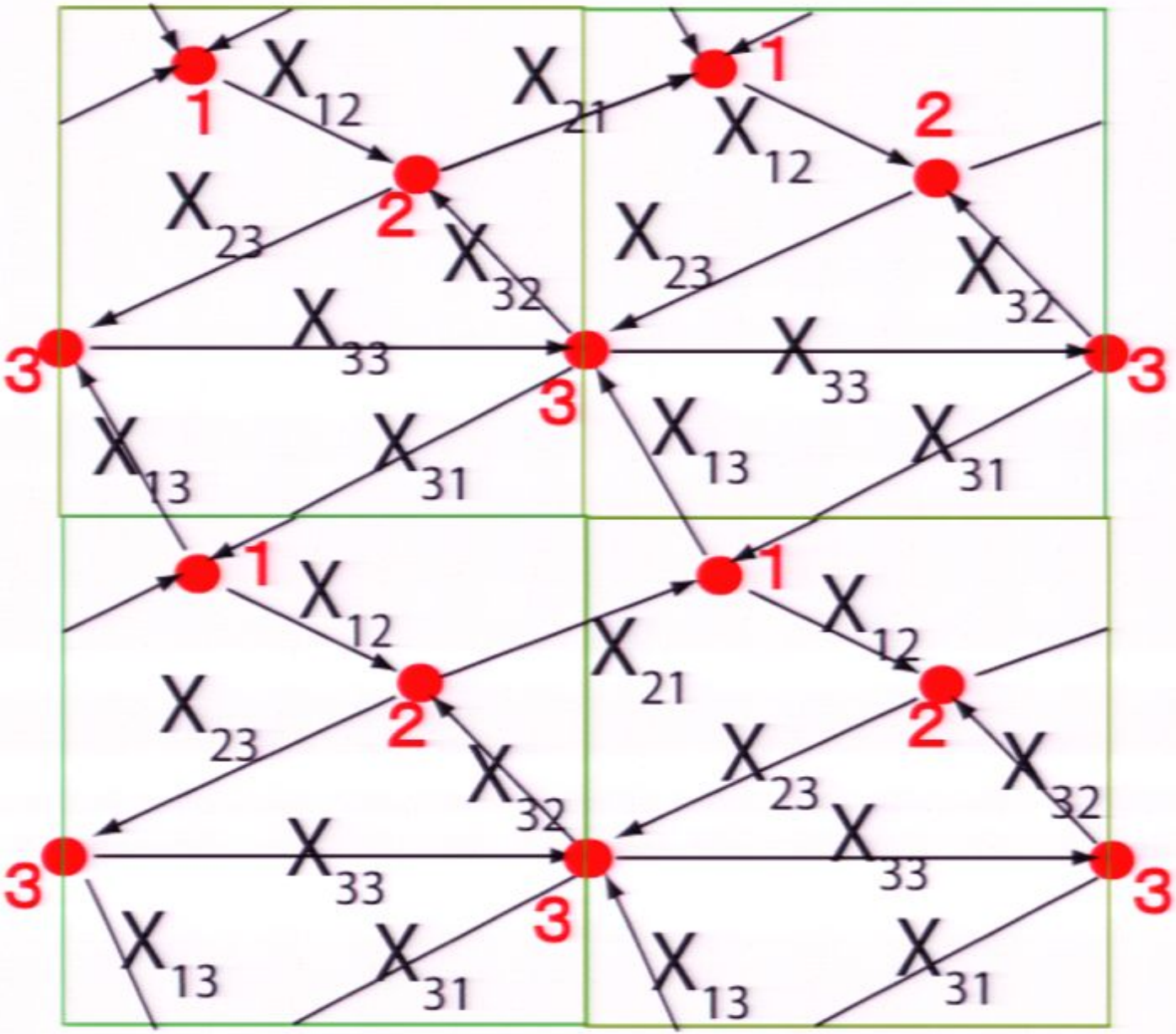
Lift of quiver diagram on T^2 to universal cover



Unmolten crystal (ground state; pyramid)



This corresponds to a D6 without D0/D2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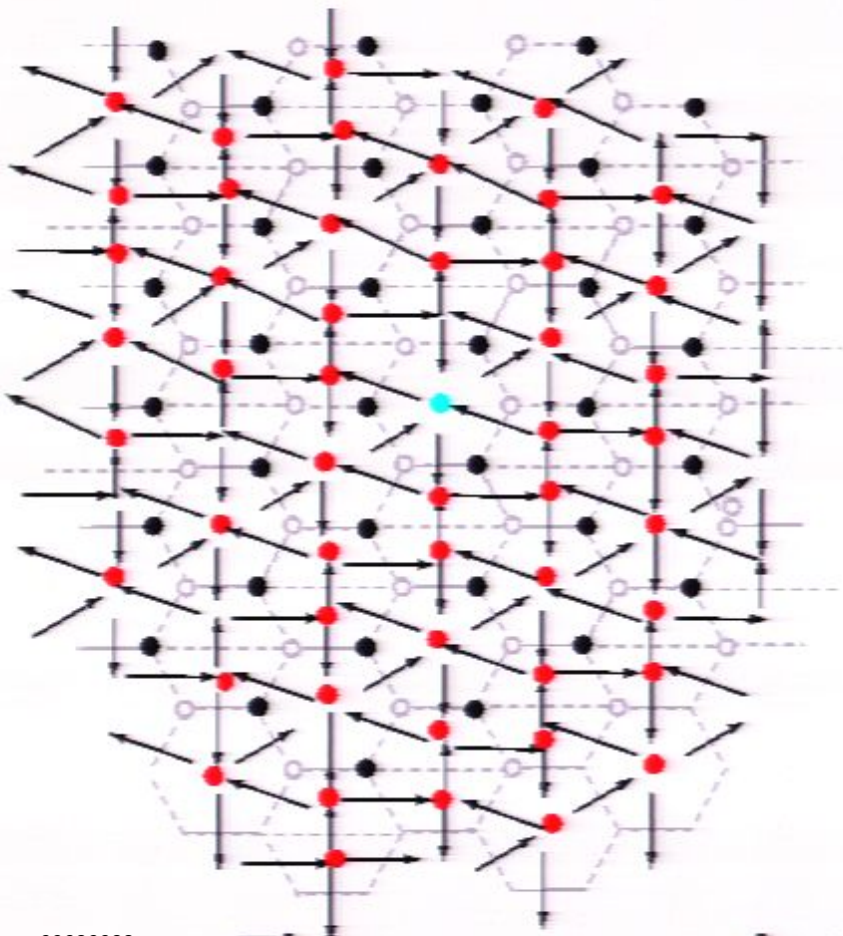

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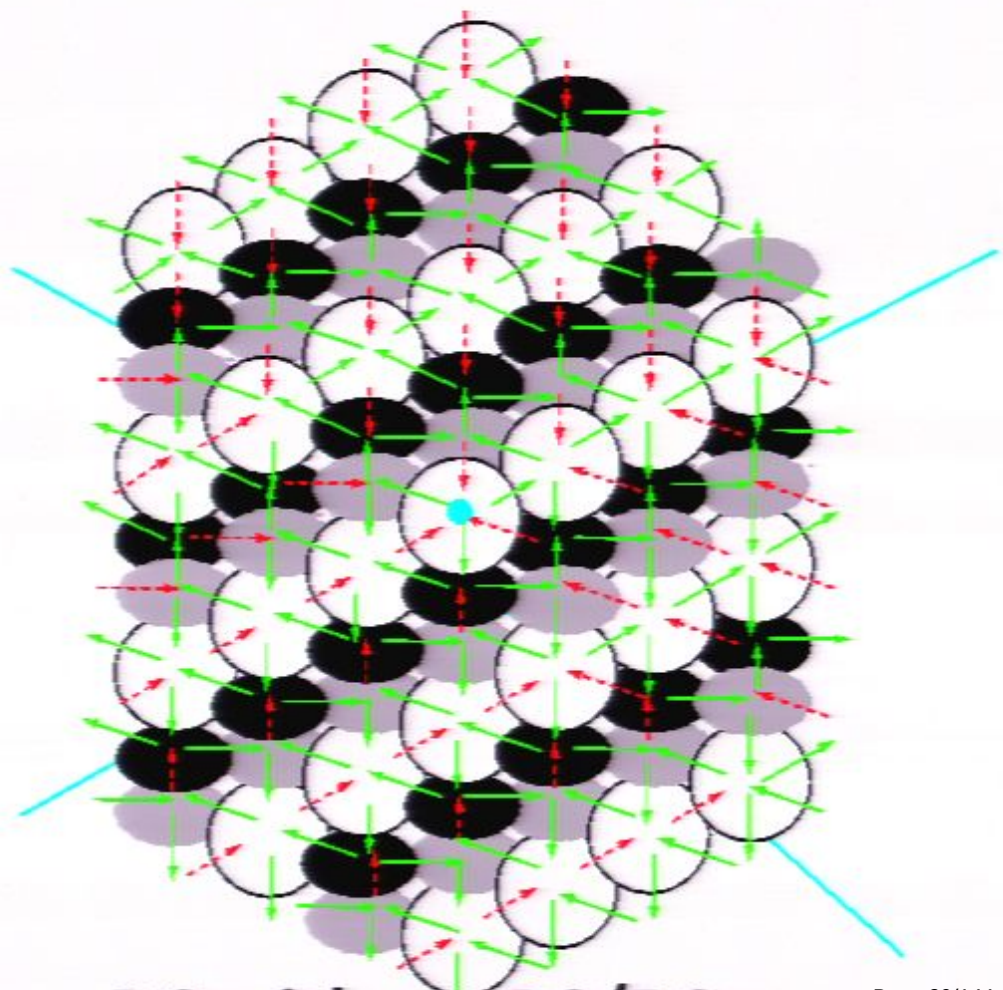

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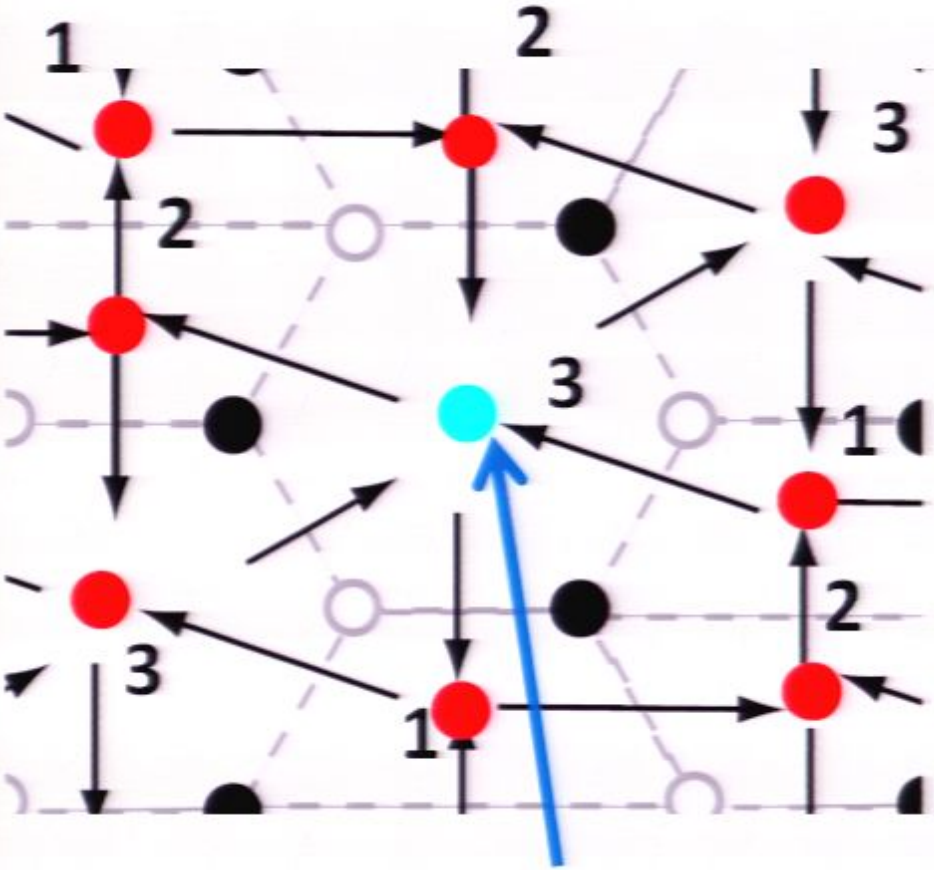

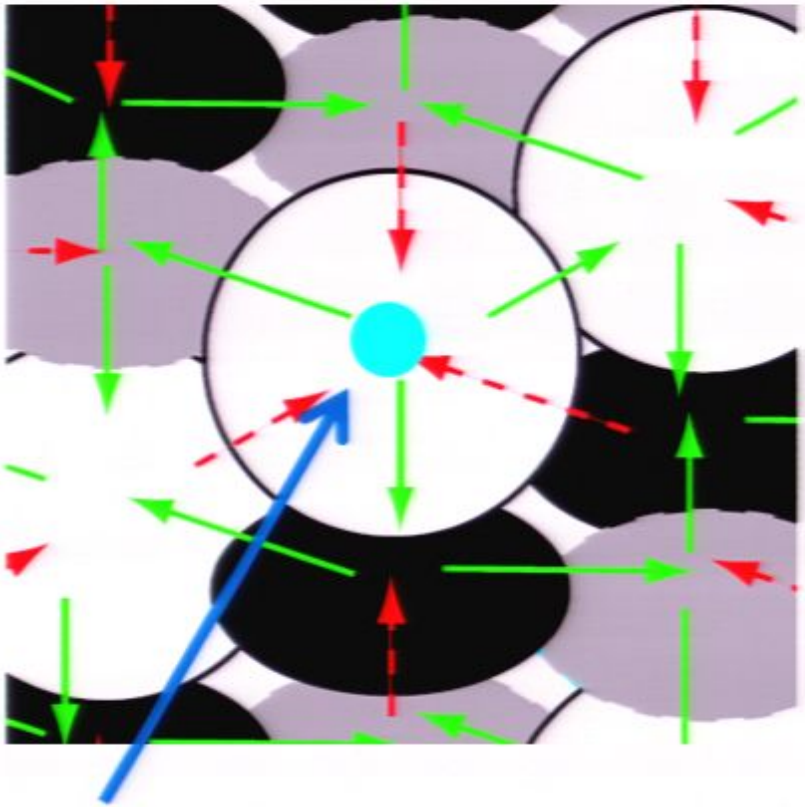

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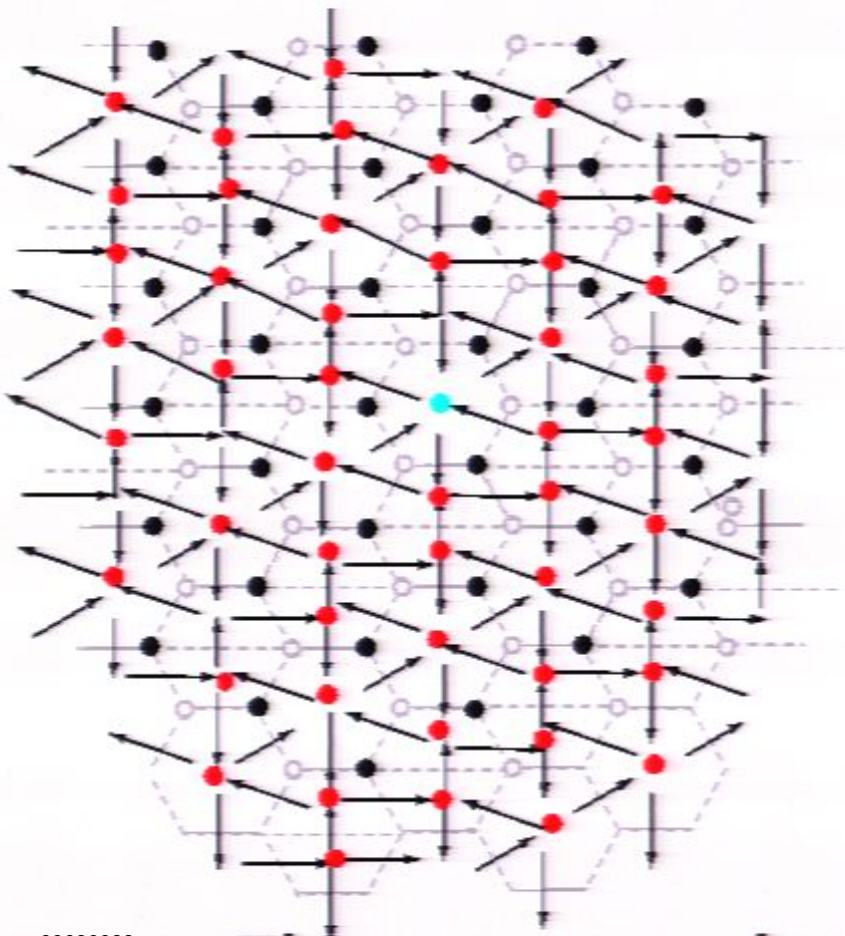

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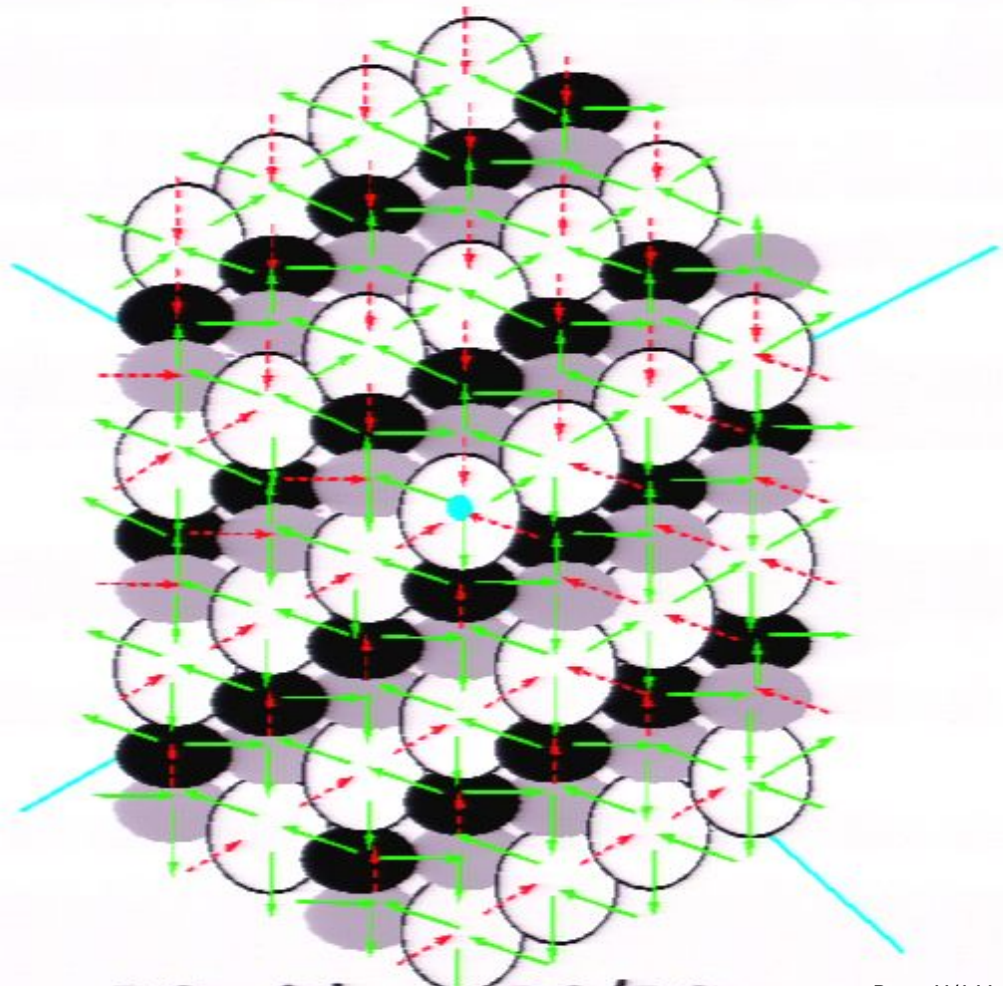

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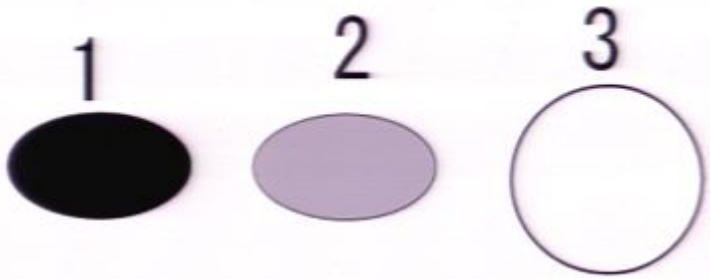
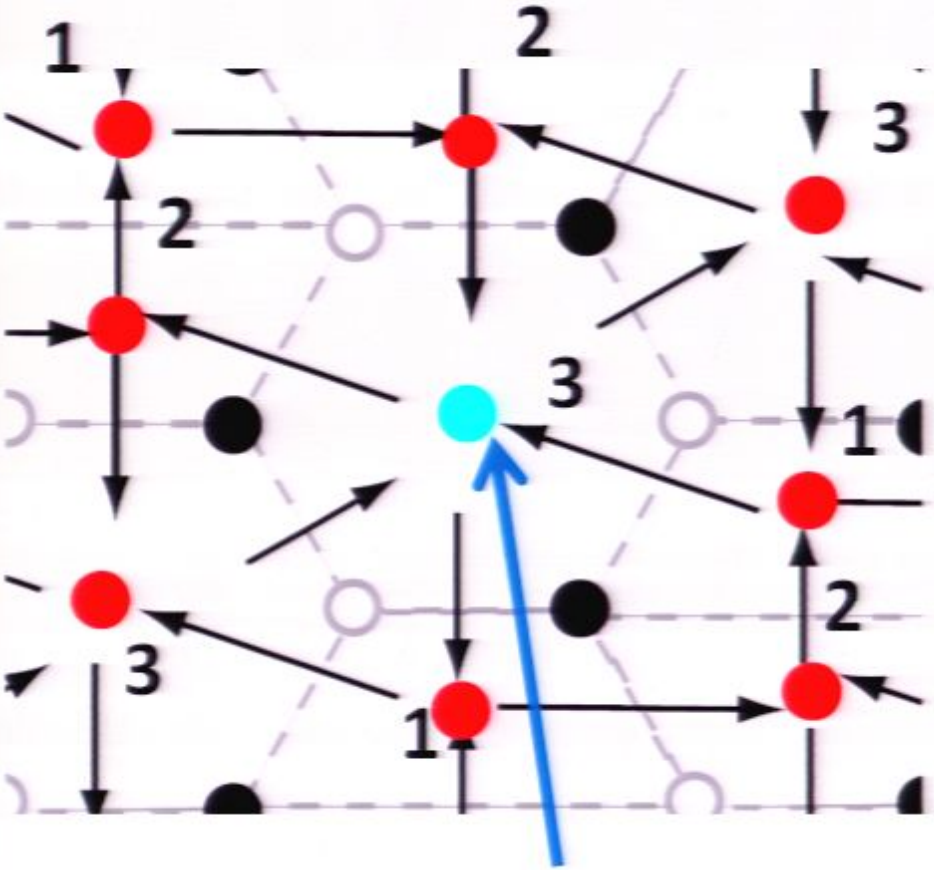
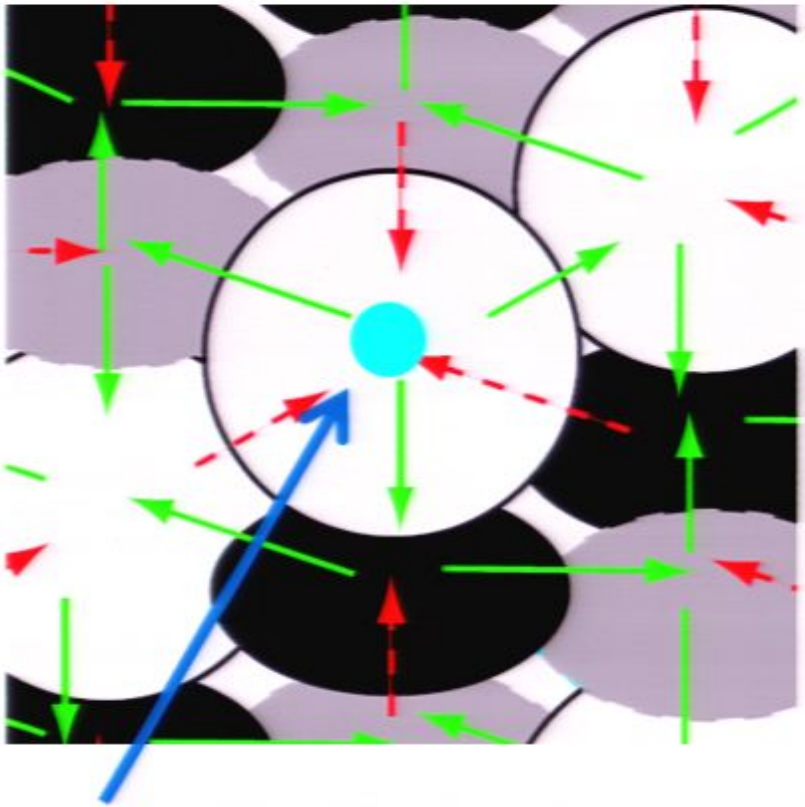


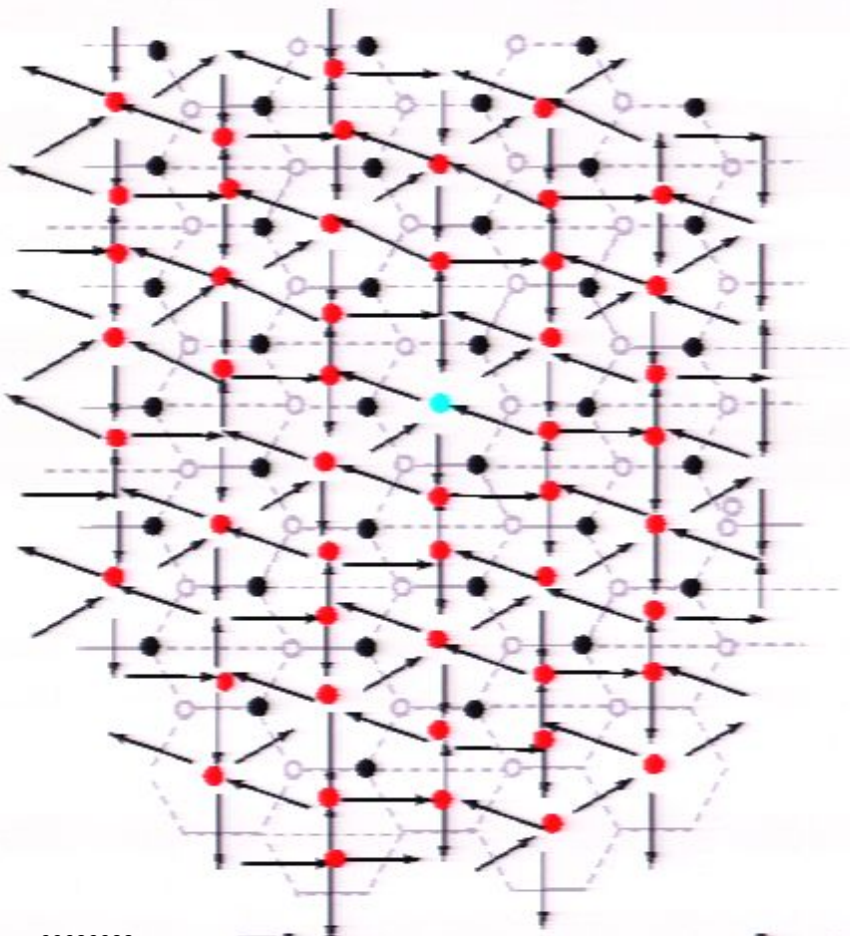




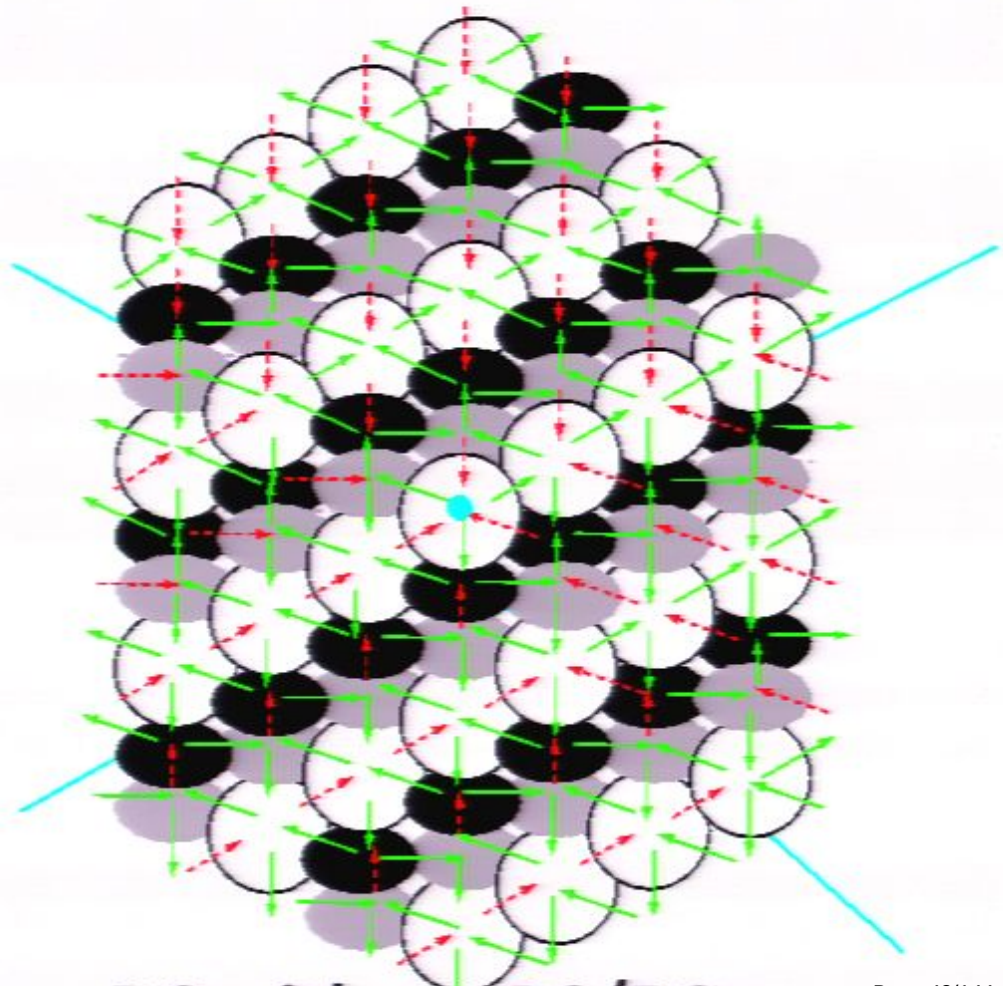
We specify the reference point

An example of a crystal

Lift of quiver diagram on T^2 to universal cover

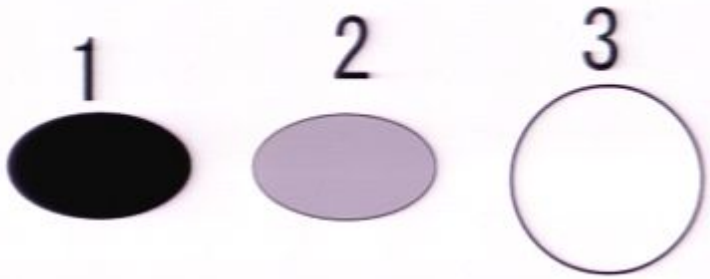


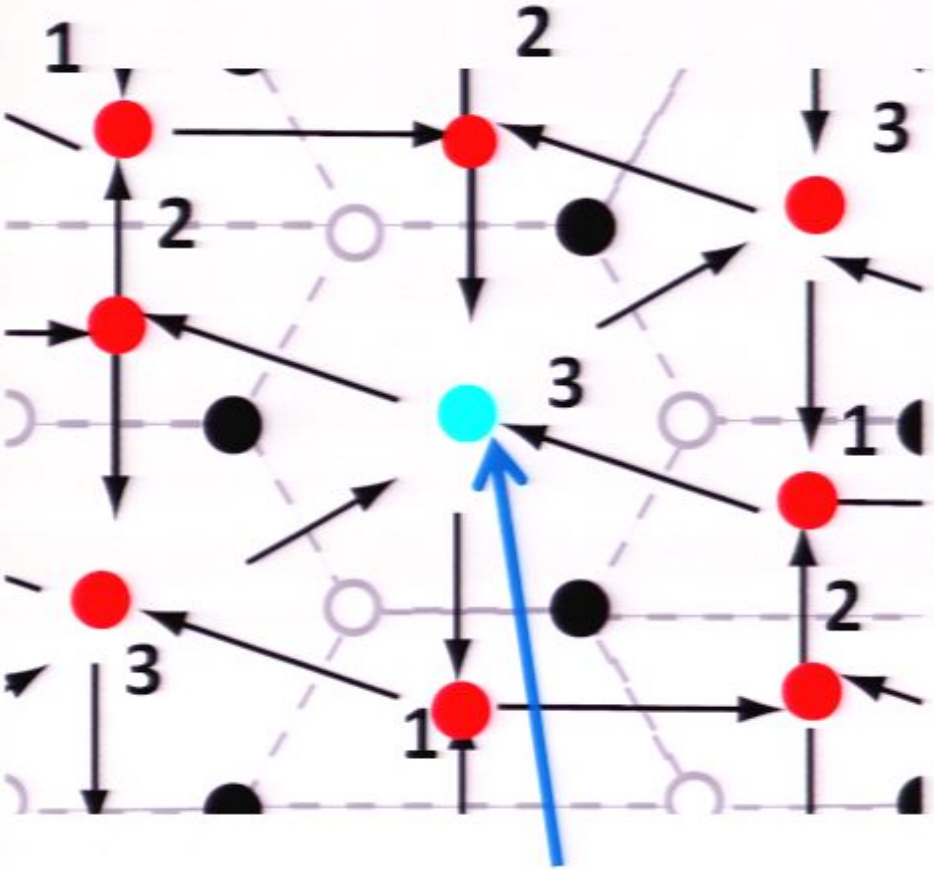
Unmolten crystal (ground state; pyramid)

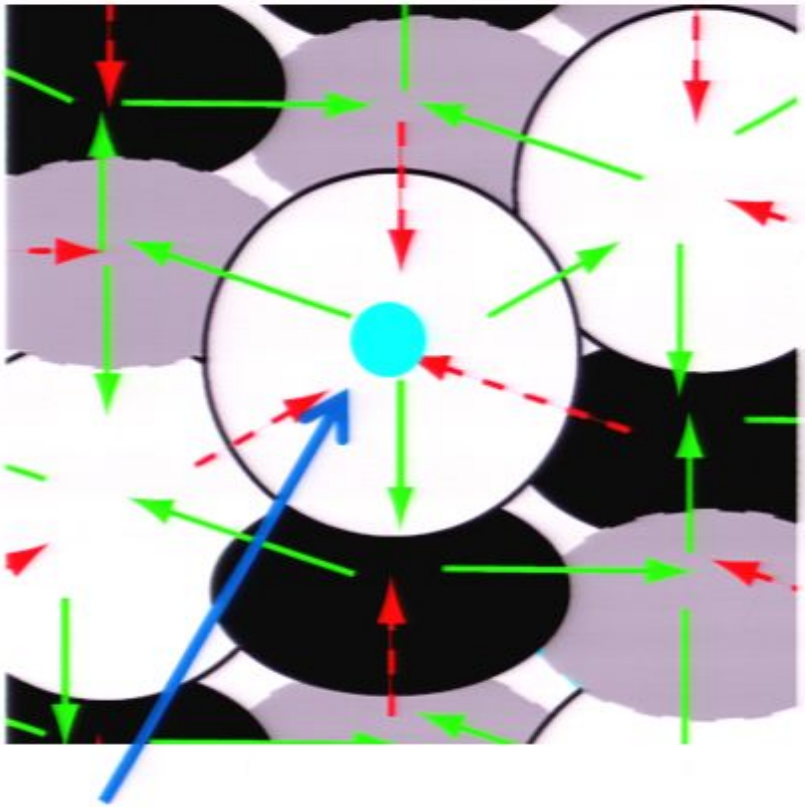


This corresponds to a D6 without D0/D2

The atoms are colored according to the position at original quiver diagram







We specify the reference point

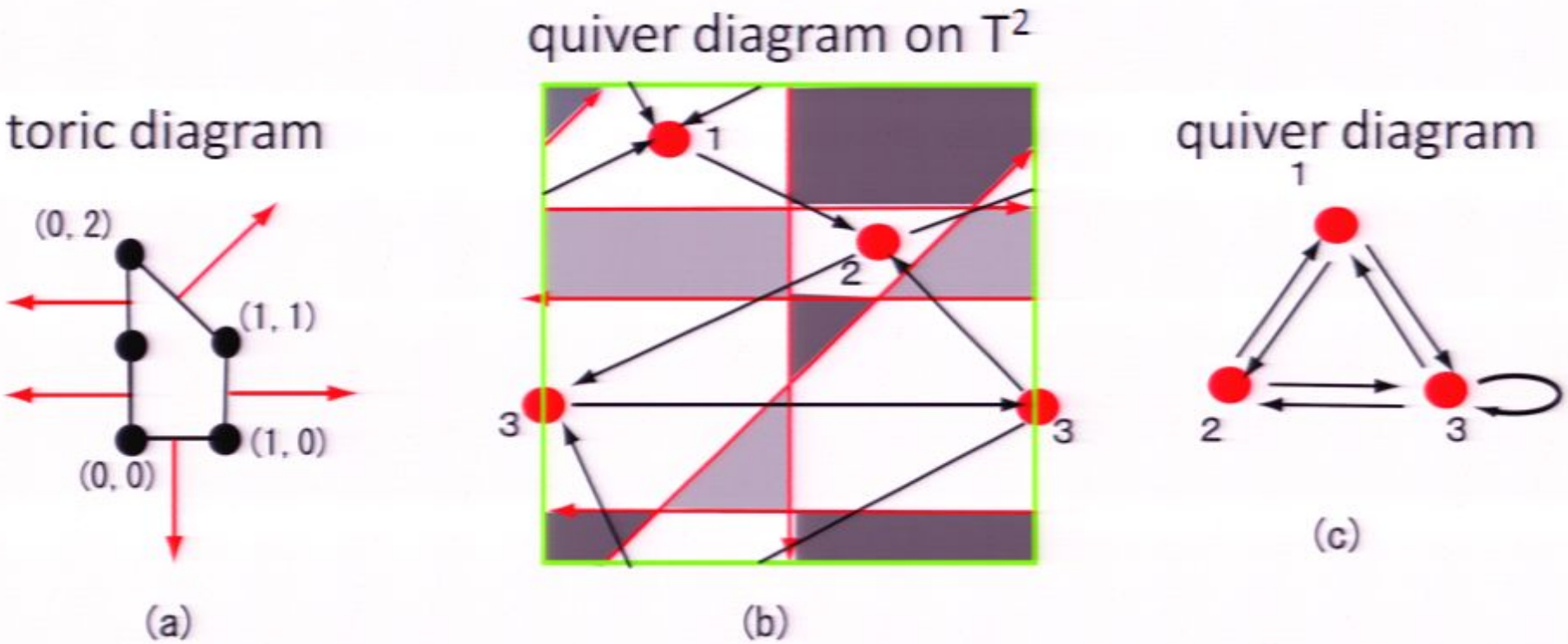
- **The upshot: the quiver diagram is written on T^2 , encoding the superpotential**

Each face of the quiver on T^2 corresponds to a term in the superpotential, e.g.

$$W = \mathrm{tr}(X_{21}X_{12}X_{23}X_{32} - X_{23}X_{33}X_{32} + X_{33}X_{31}X_{13} - X_{31}X_{12}X_{21}X_{13})$$

- Brane tiling techniques gives a quiver diagram and a superpotential for an ***arbitrary*** toric CY_3

Example: Suspended Pinched Point ($xy=zw^2$)



These graphs represent physical configuration of D-branes!

[Feng-He-Kennaway-Vafa '06; Imamura-Kimura-Isono-M.Y. '07]

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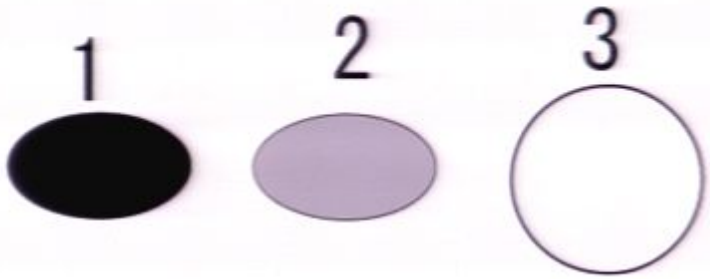
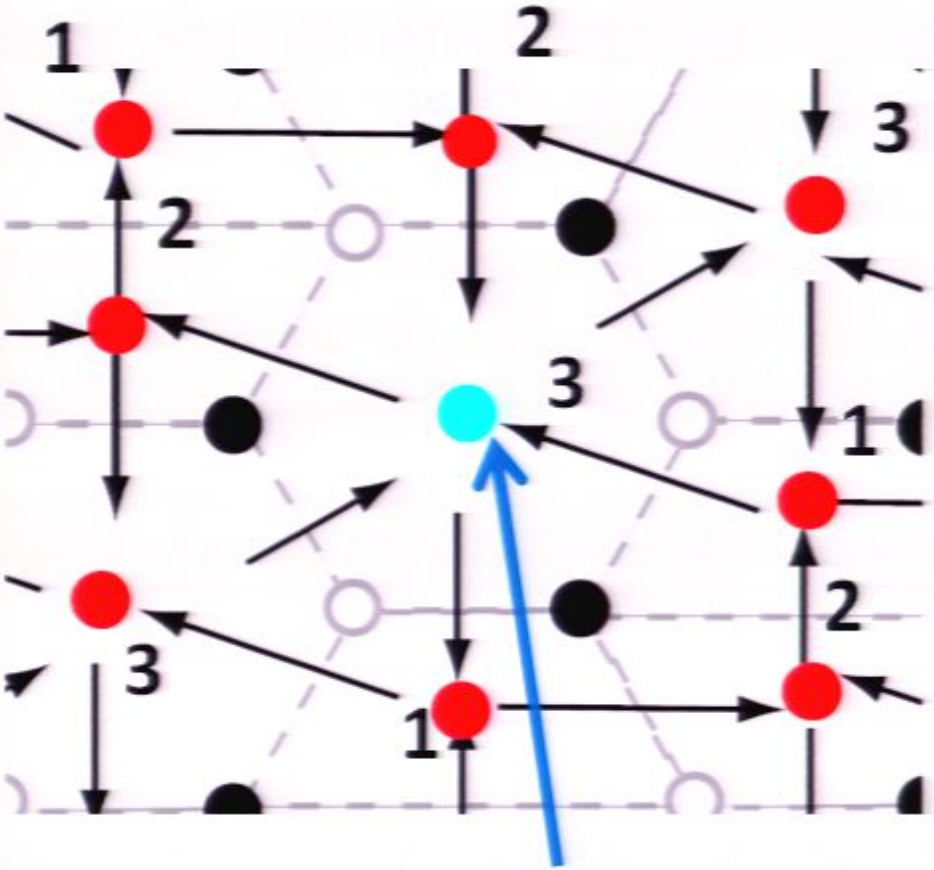
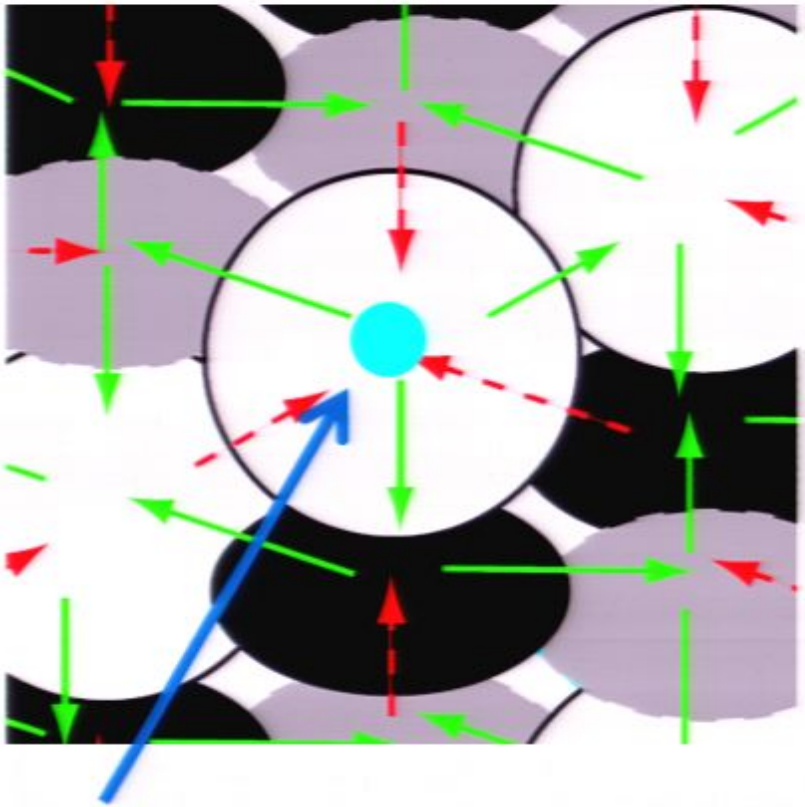


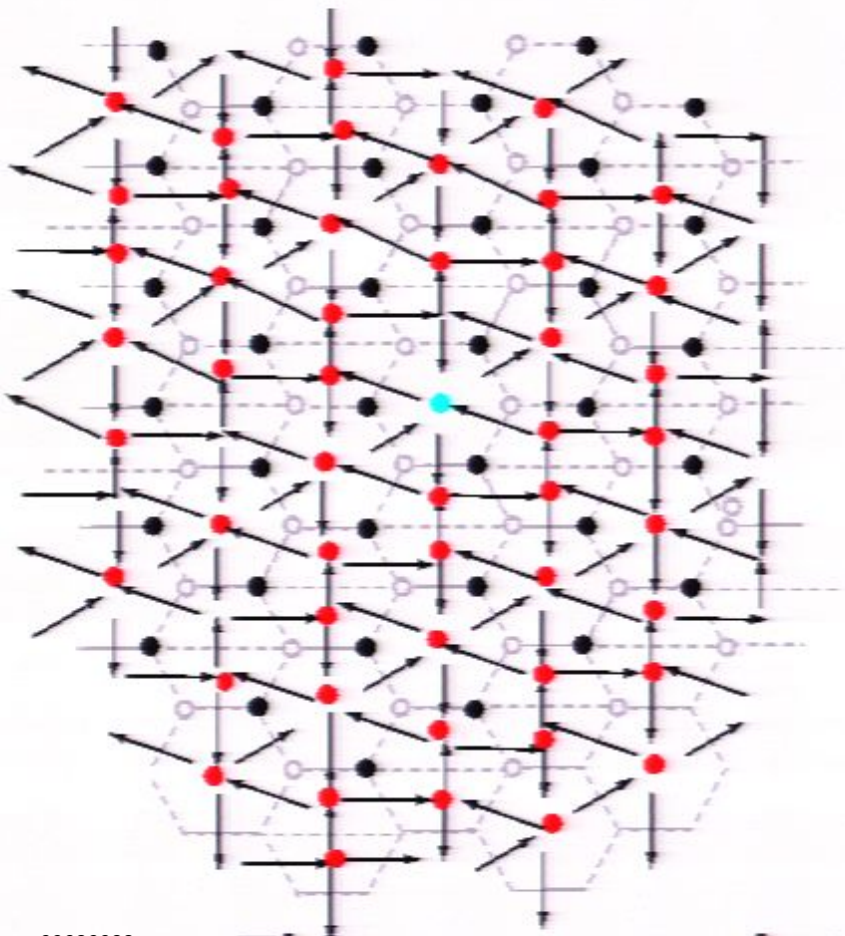




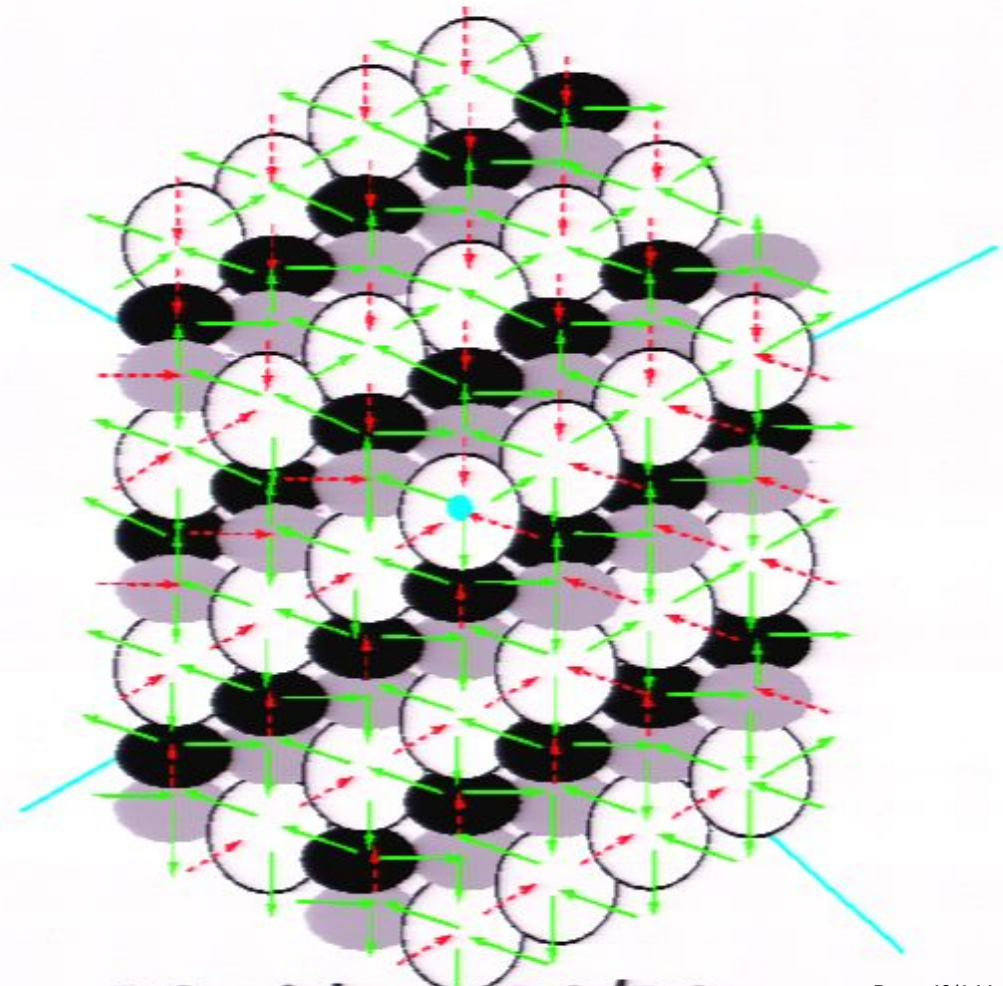
We specify the reference point

An example of a crystal

Lift of quiver diagram on T^2 to universal cover



Unmolten crystal (ground state; pyramid)



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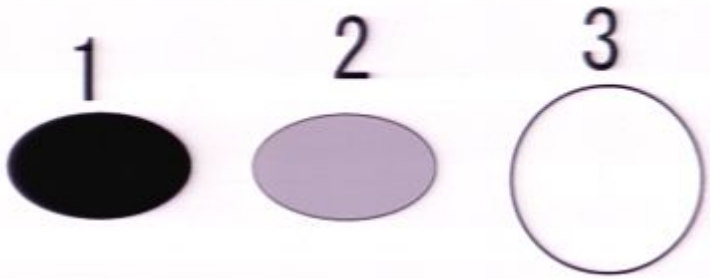

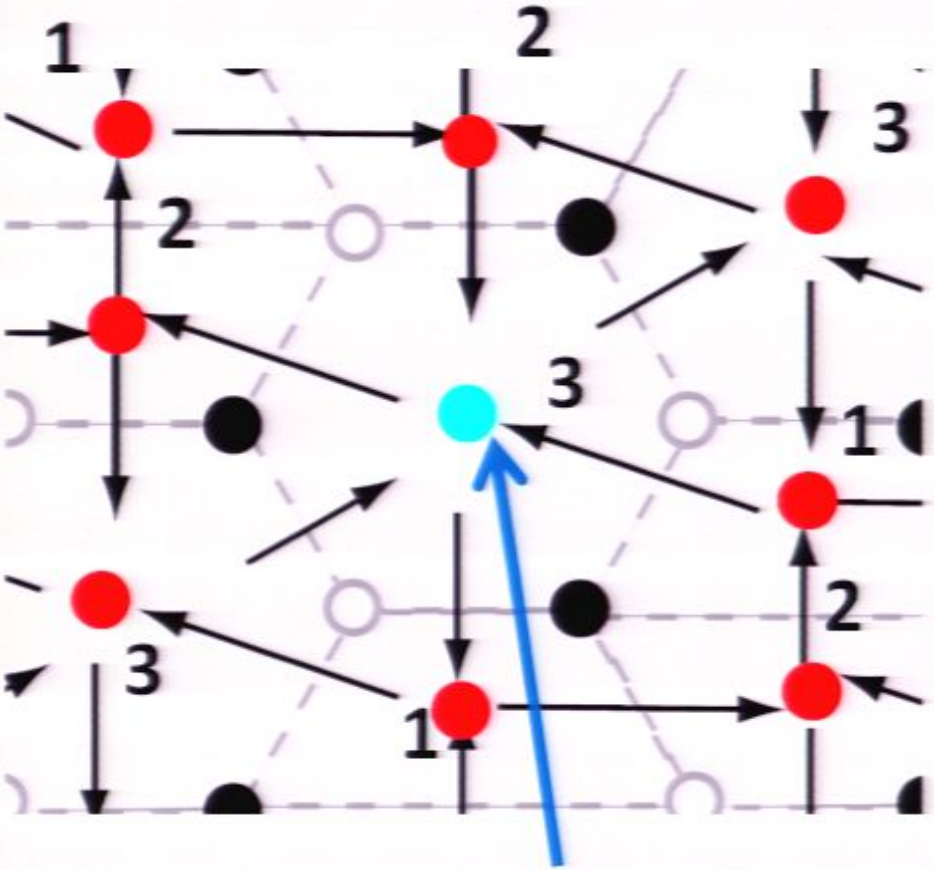

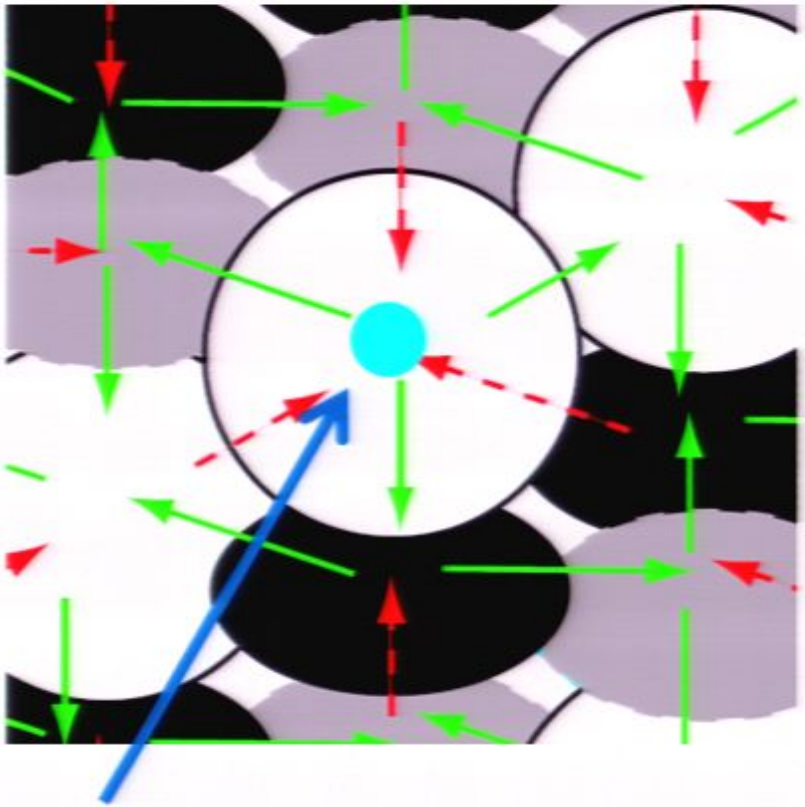

We specify the reference point

- Each atom corresponds to an equivalence class of paths starting at the reference point i_0 modded out by F-term conditions

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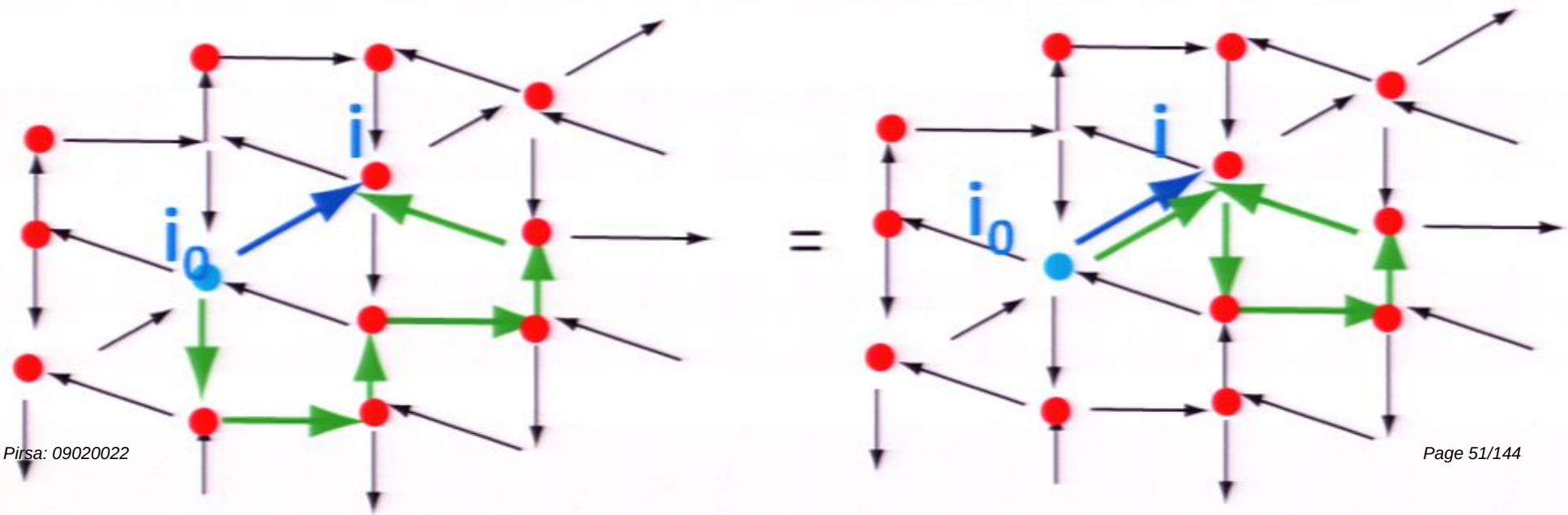

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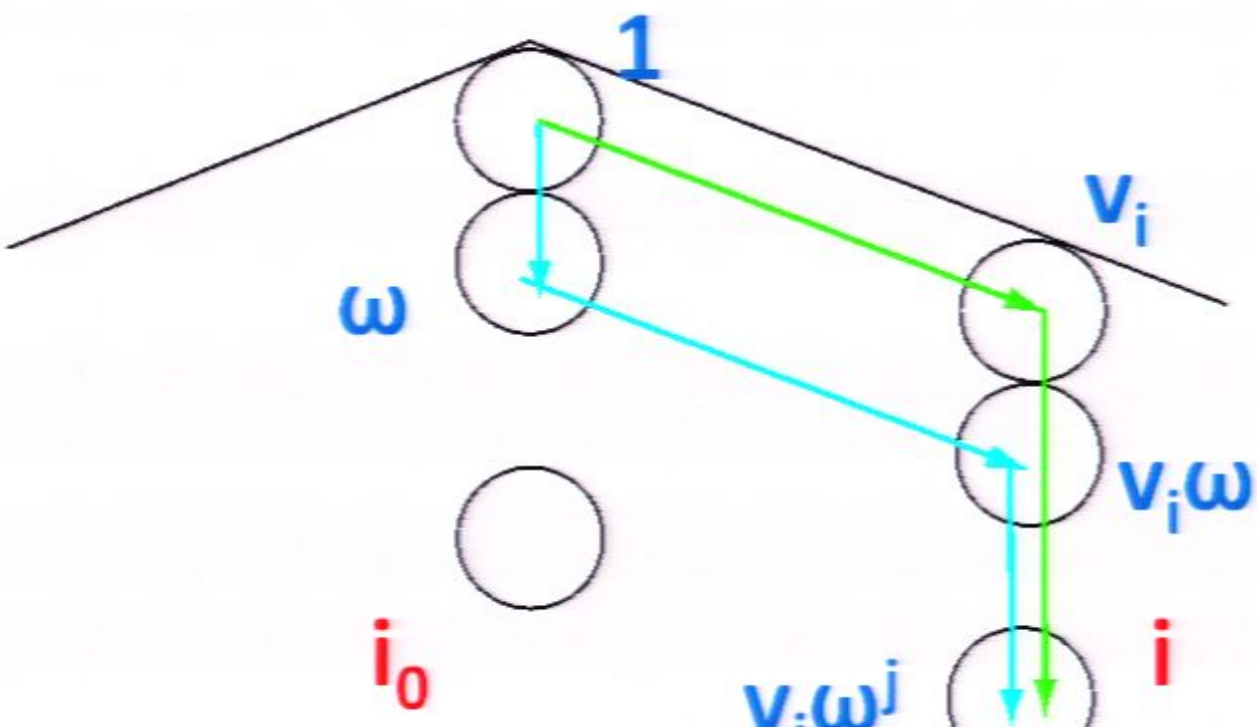

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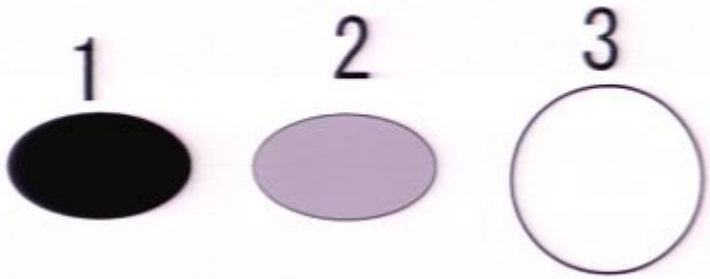

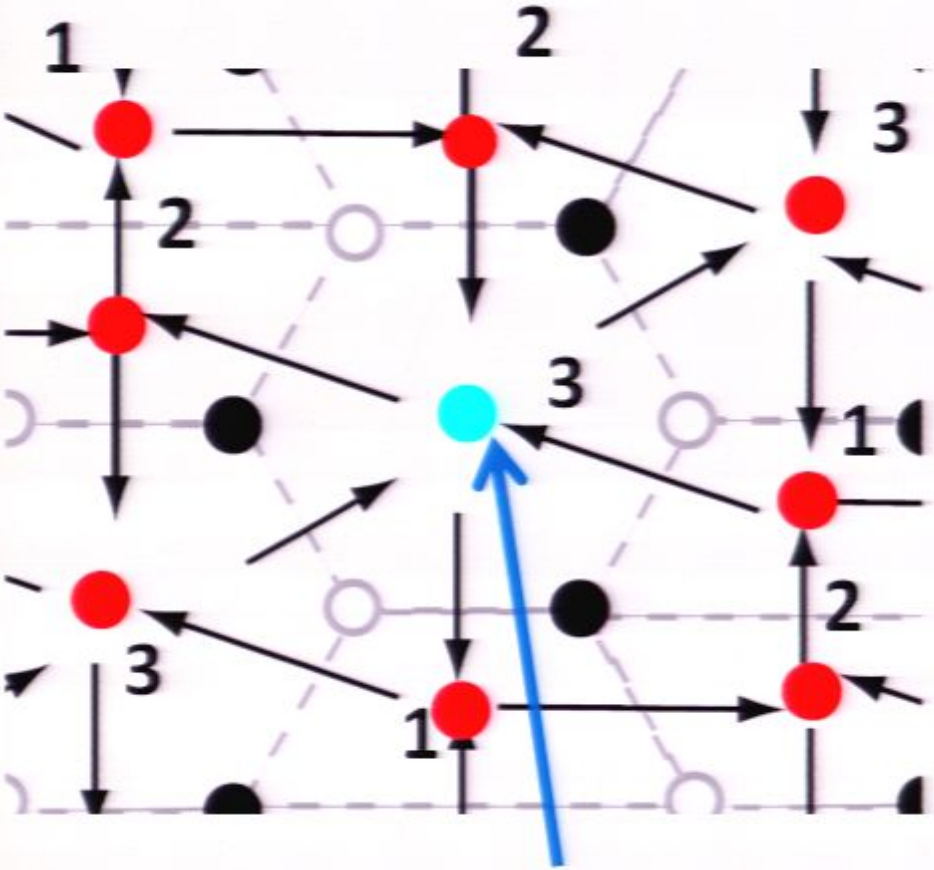

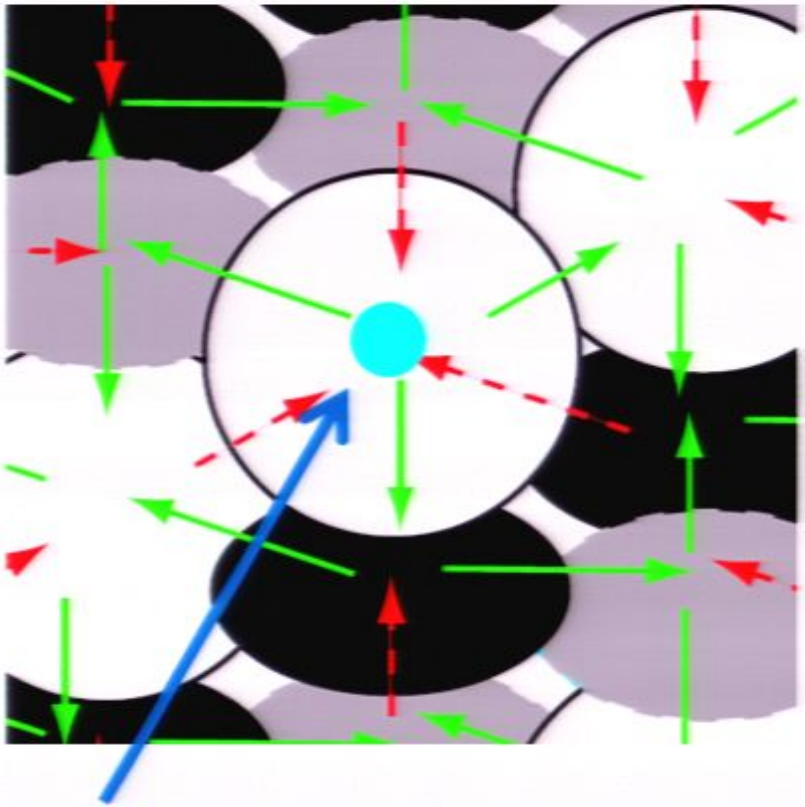

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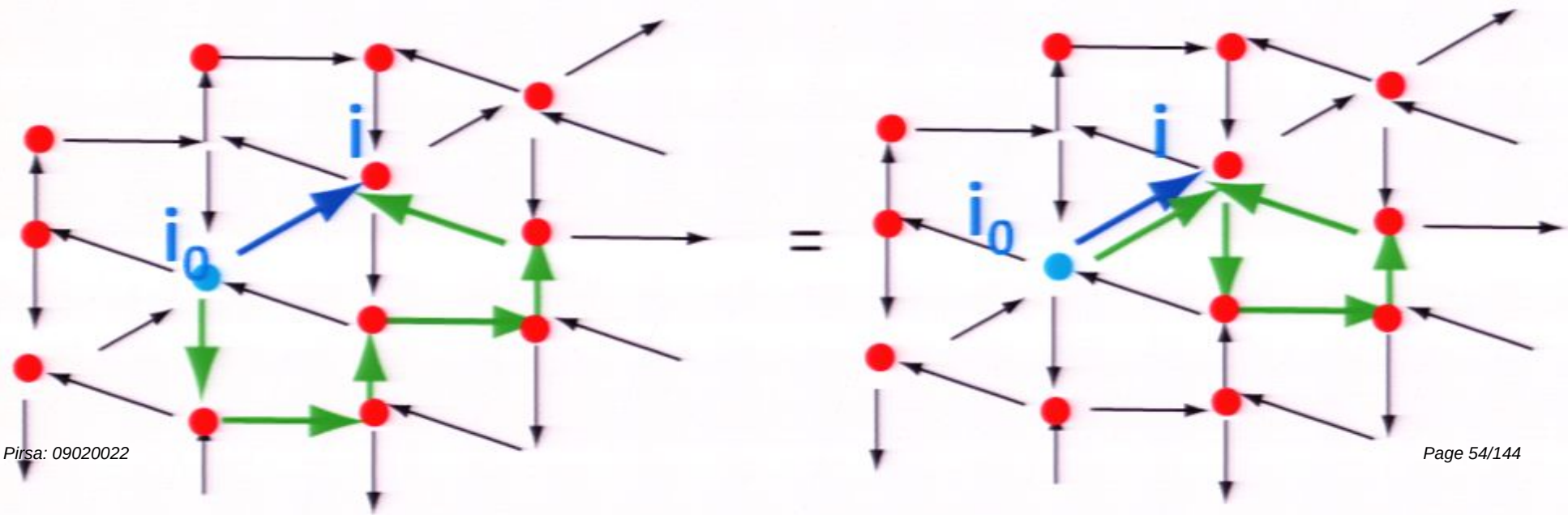

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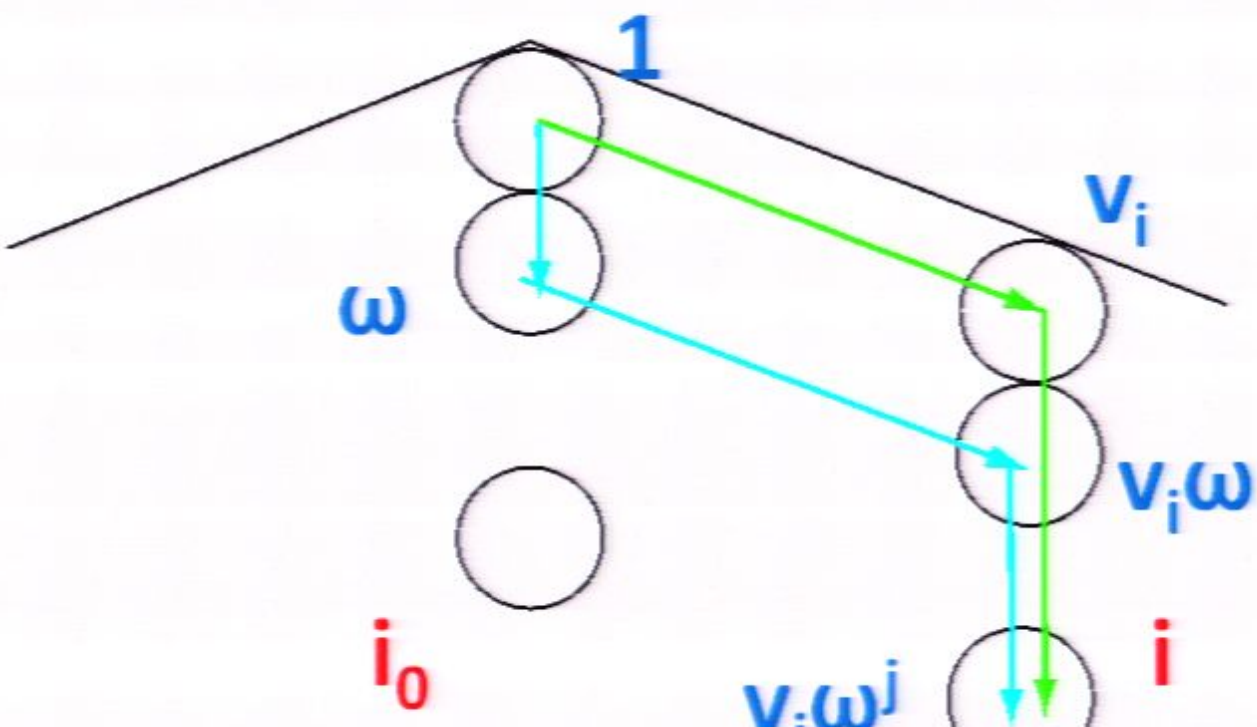
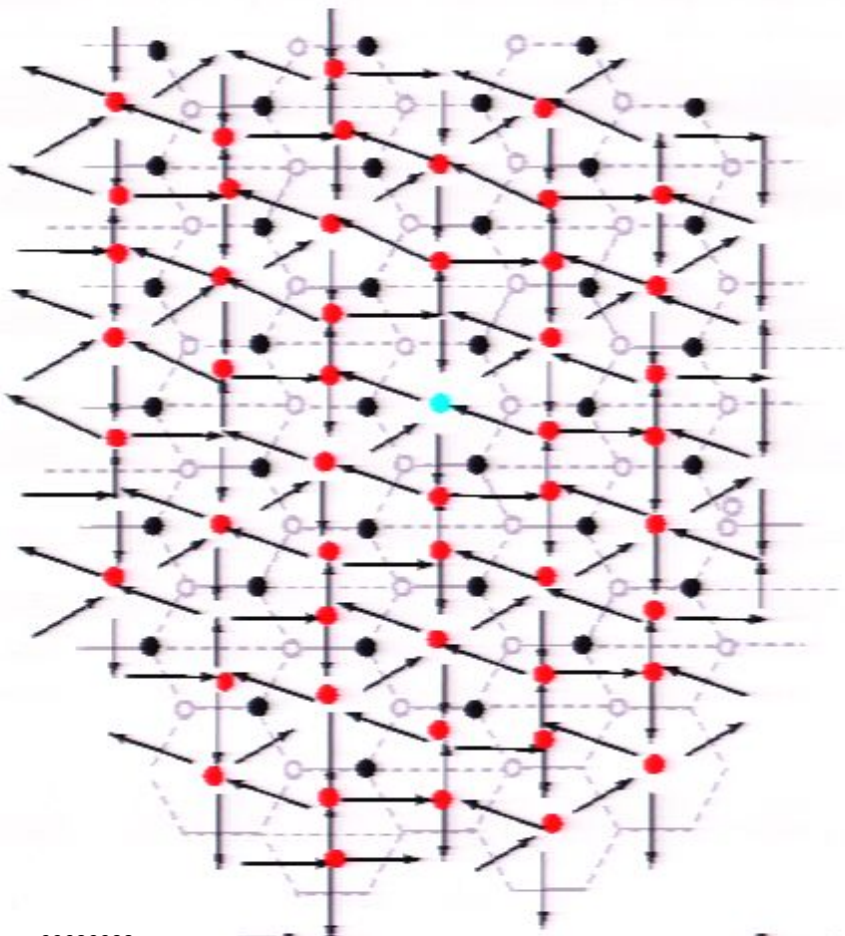
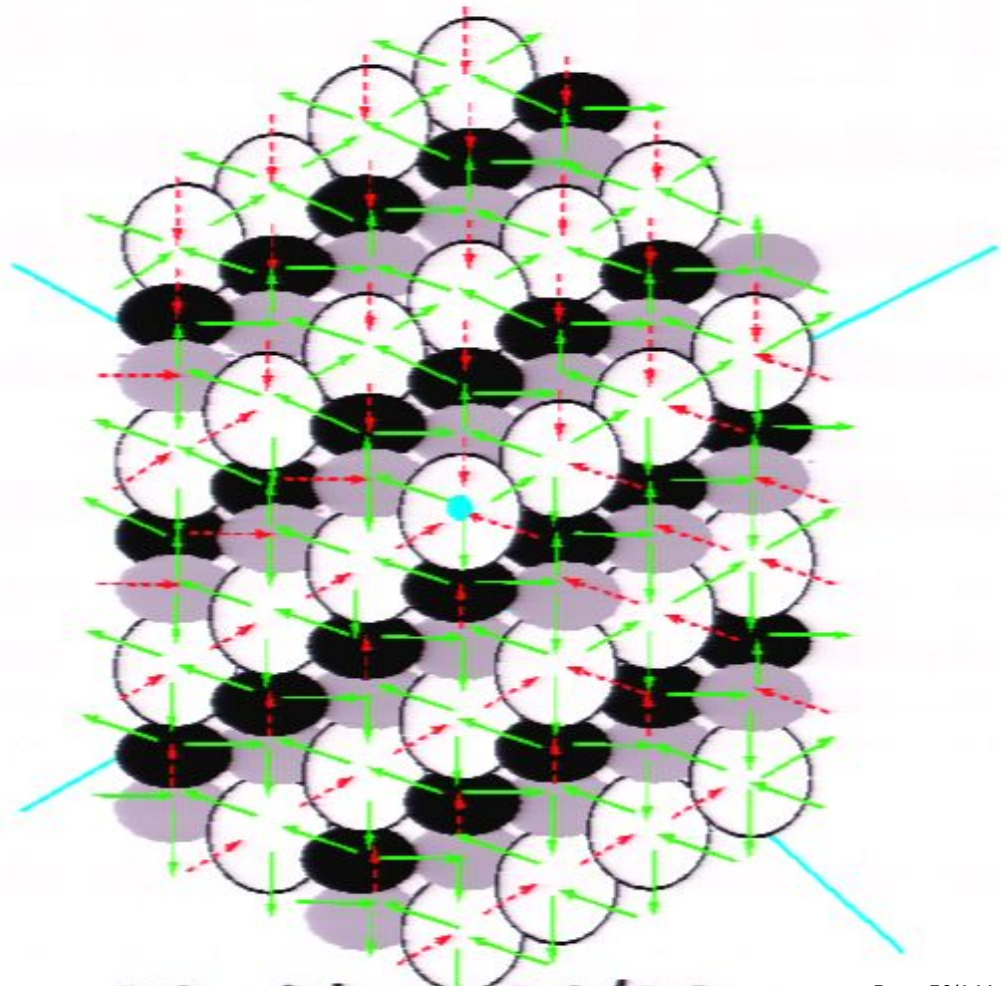


An example of a crystal

Lift of quiver diagram on T^2 to universal cover



Unmolten crystal (ground state; pyramid)



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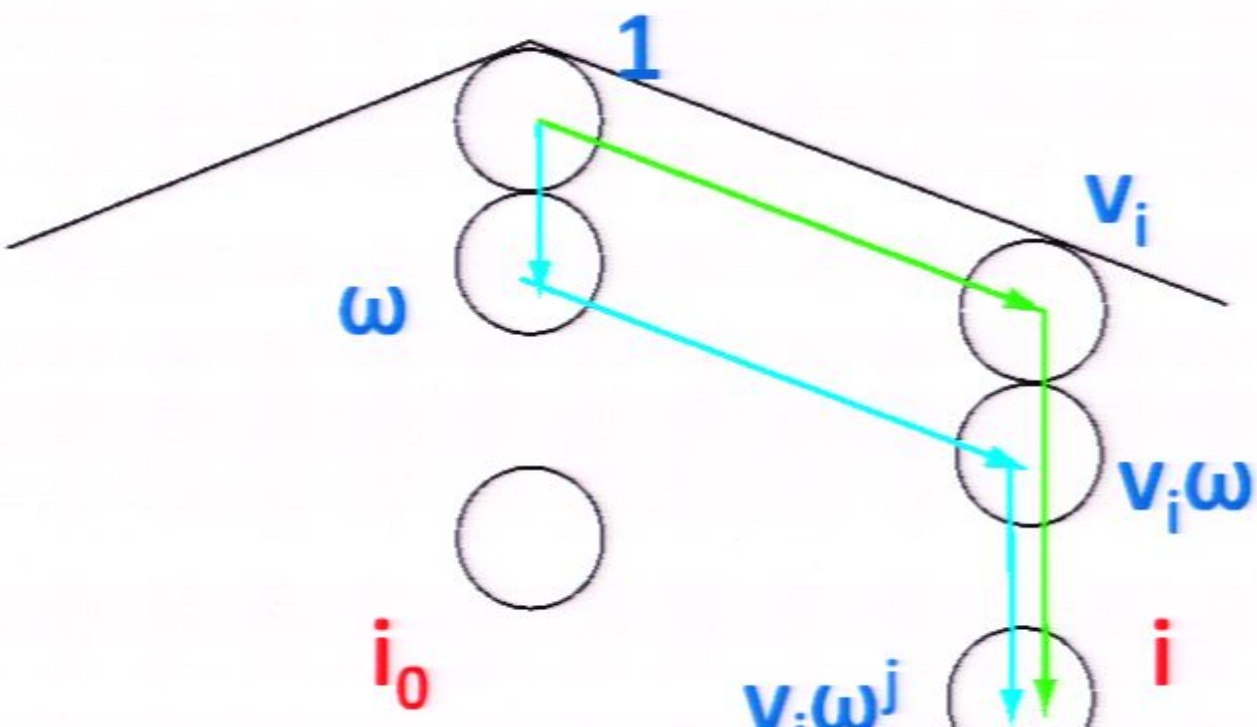

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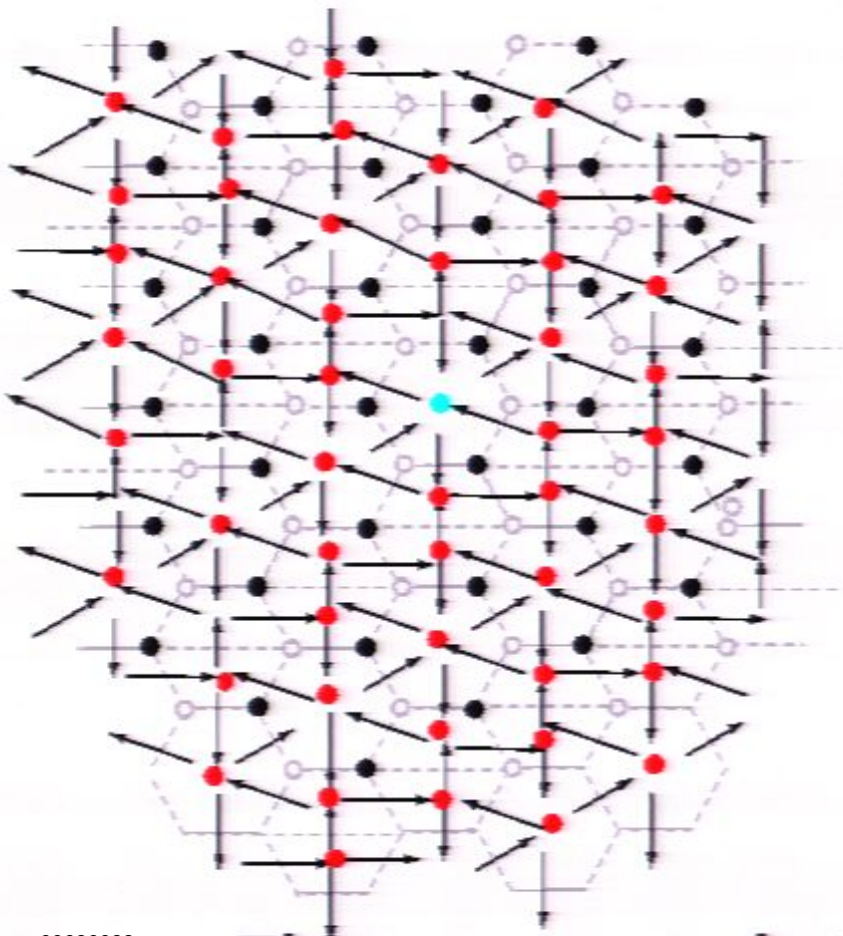

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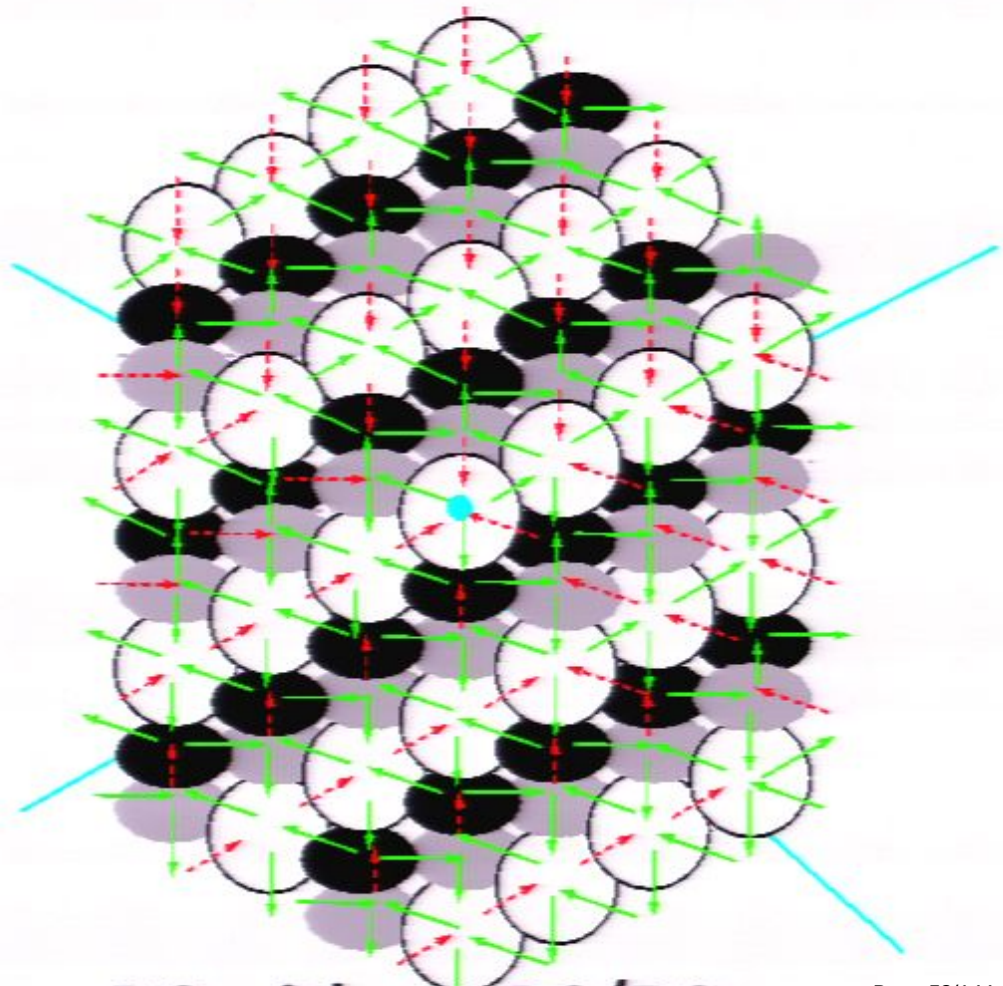

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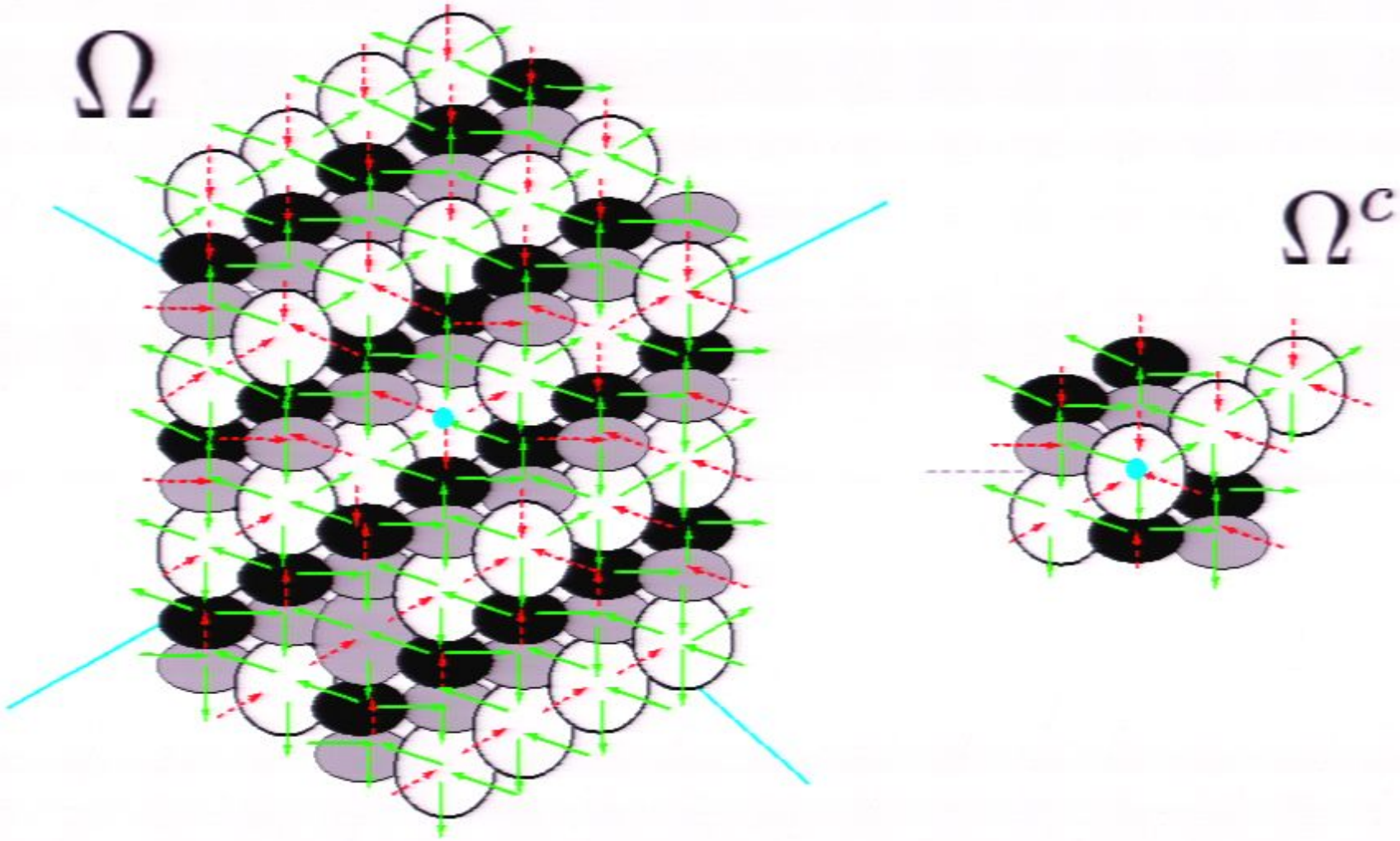
molten crystal

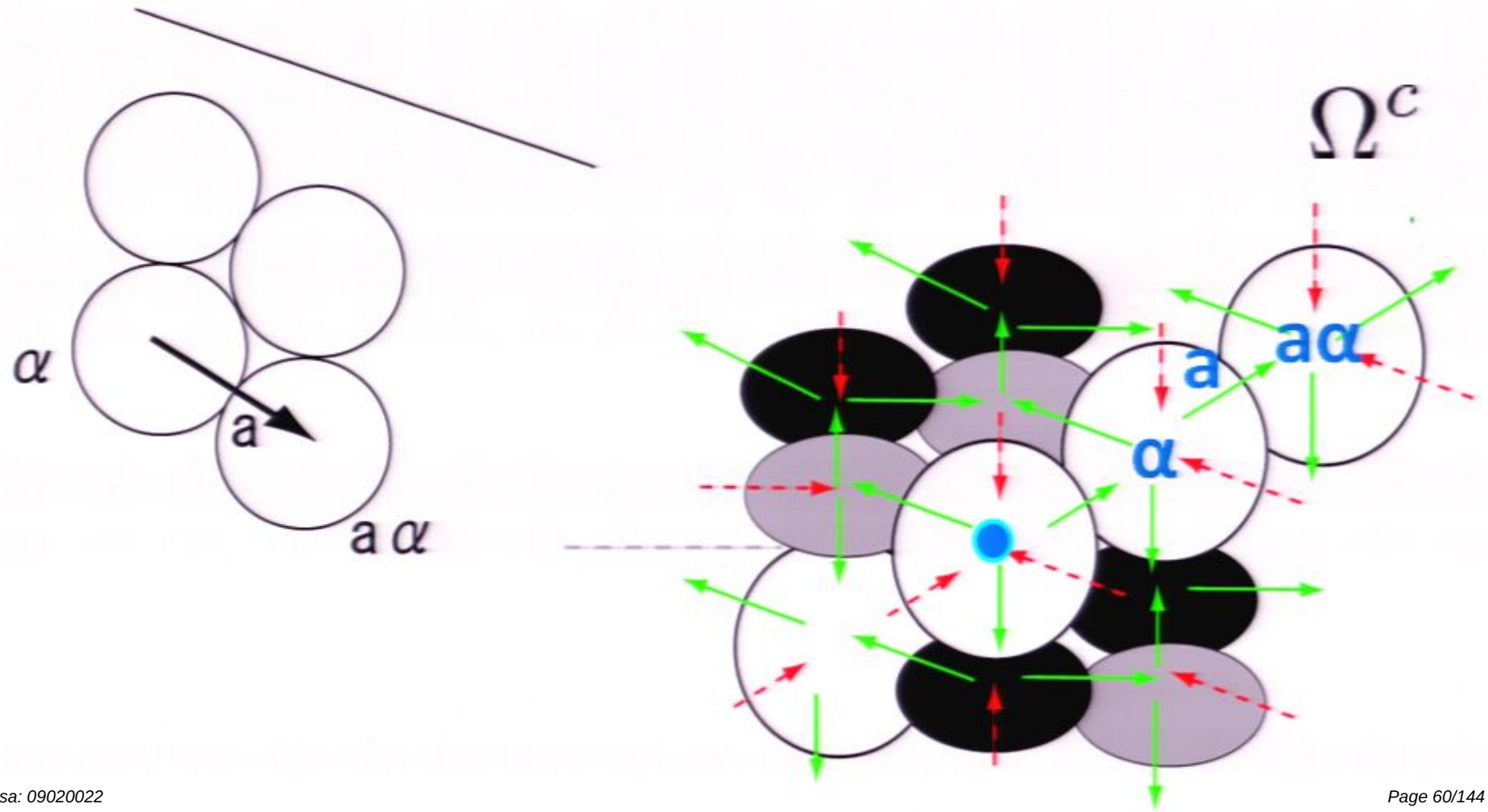


Adding D0/D2 charges corresponds to molten crystals

BPS bound state = molten crystal

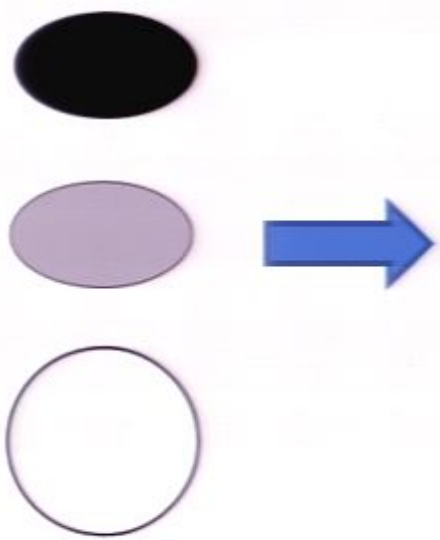
Melting Rule:

$$\text{For } a \in A, \quad a\alpha \in \Omega^c \Rightarrow \alpha \in \Omega^c$$



◆**BPS bound states are constructed out of atoms, which carries linear combination of D-brane charges**

(Similar to quark models, as opposed to Sakata model)



Arbitrary BPS bound state

(Reminiscent of bubbling AdS geometry [LLM]
Fuzzball conjecture [Mathur])

Melting Rule:

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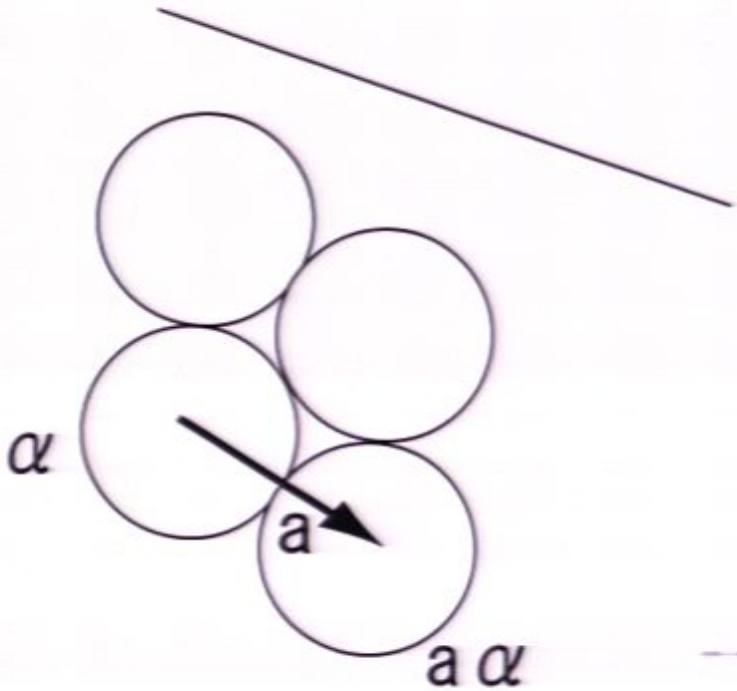
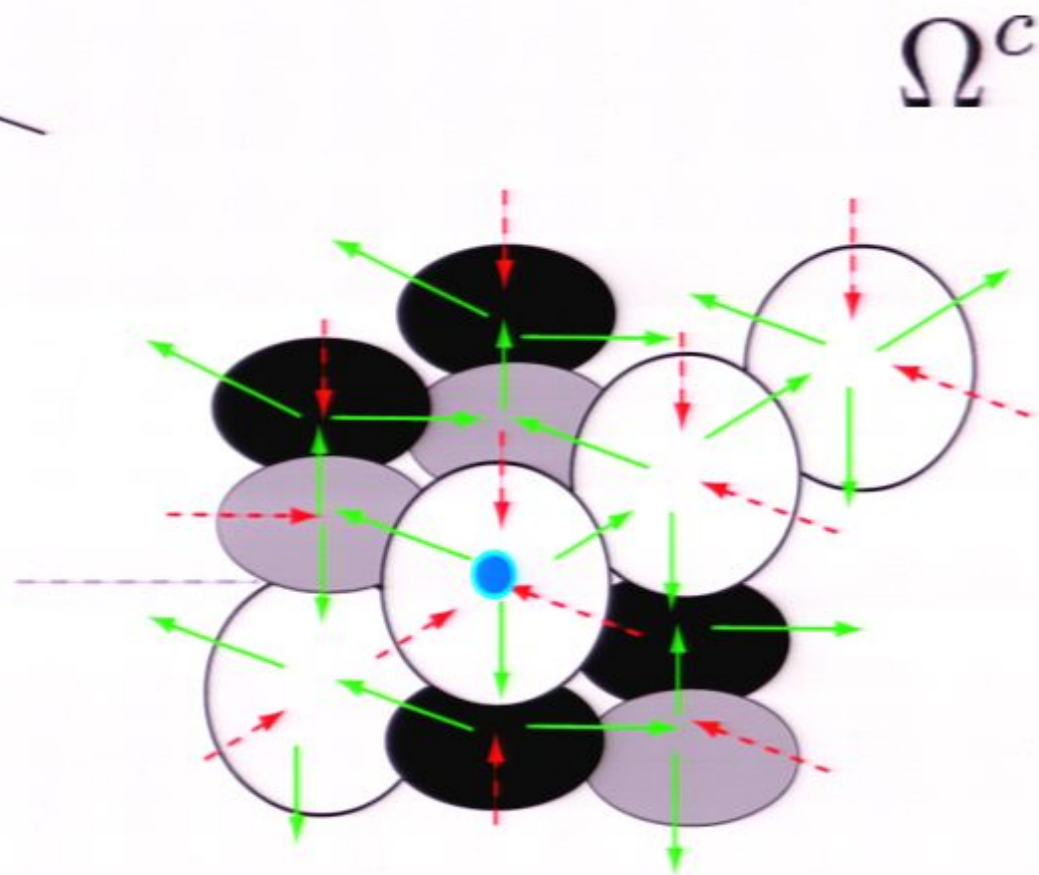
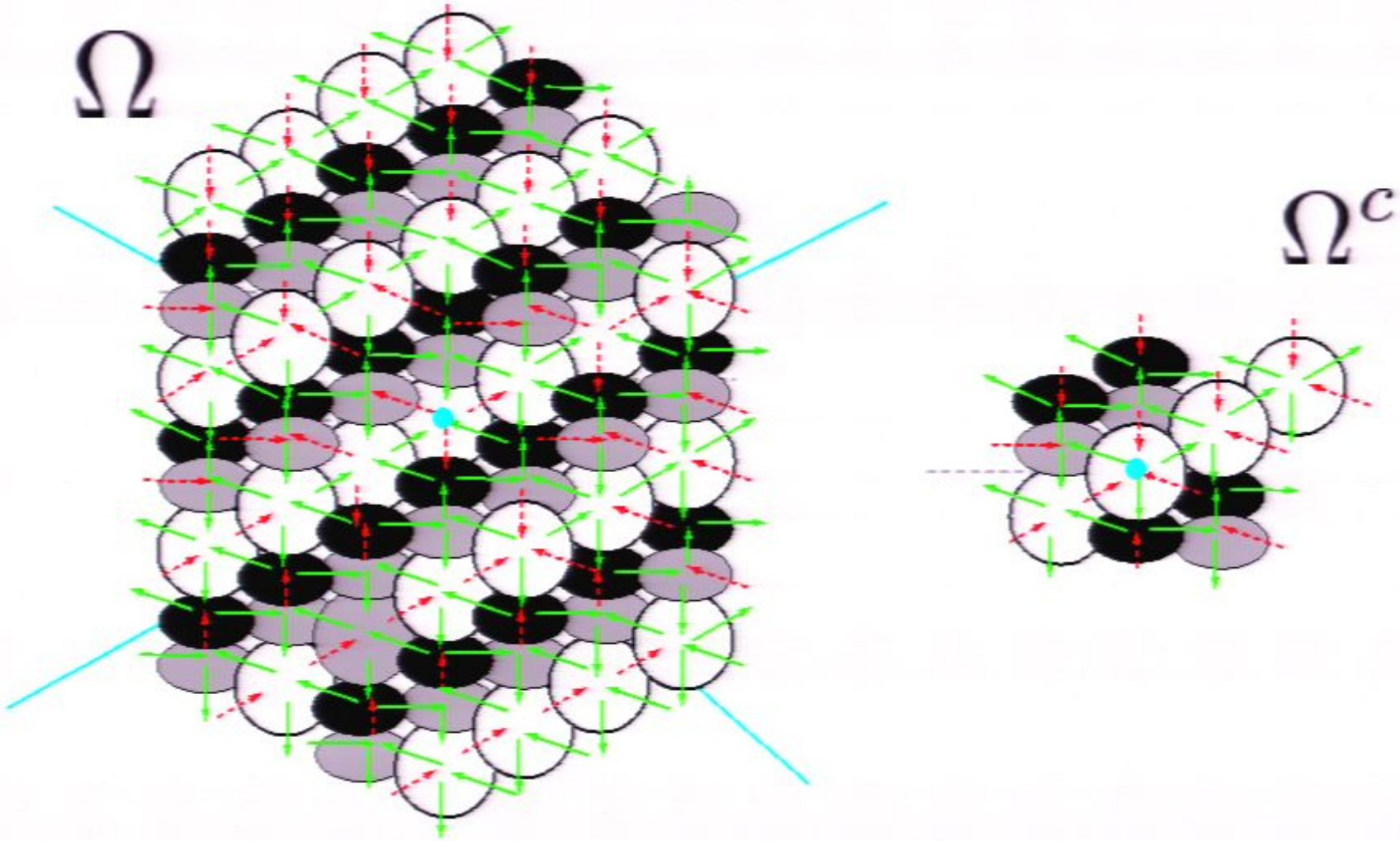




molten crystal



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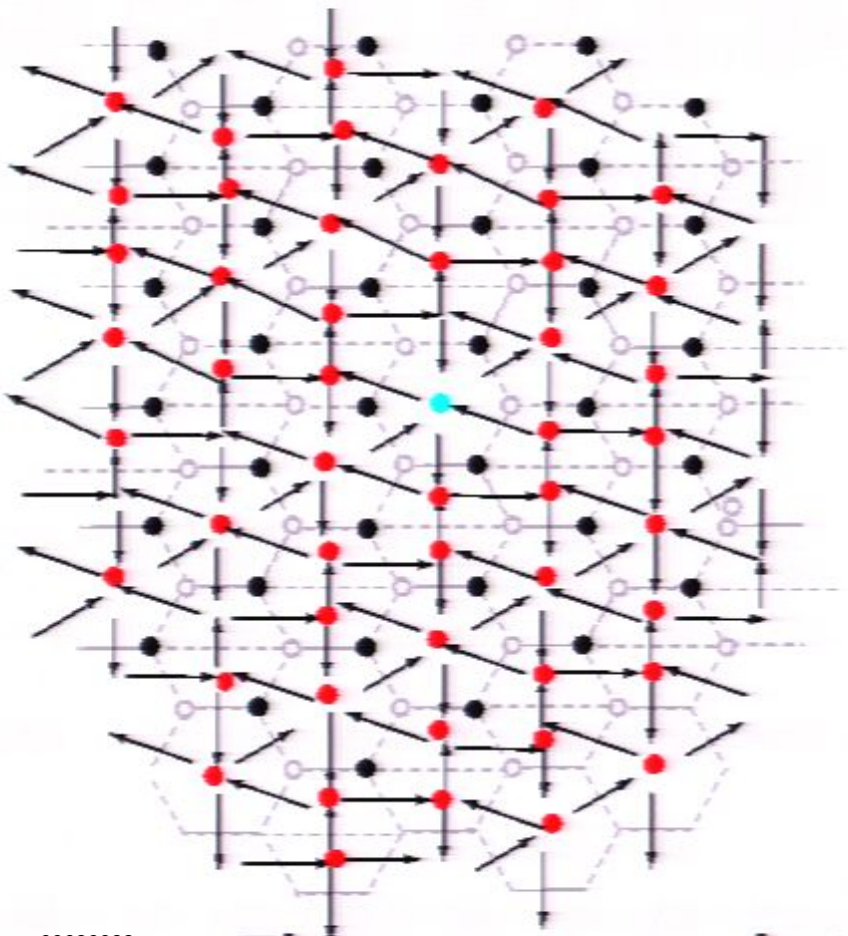

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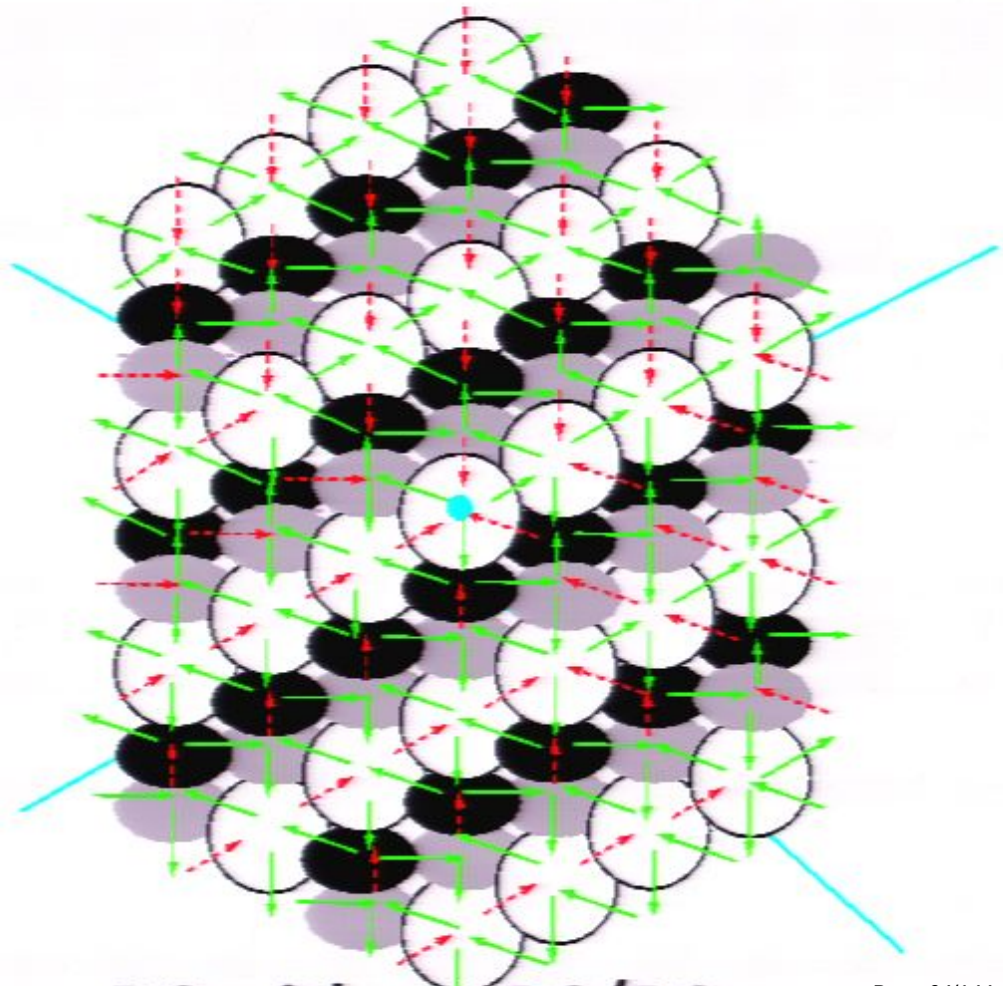

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090210Crystal.pptx - Microsoft PowerPoint

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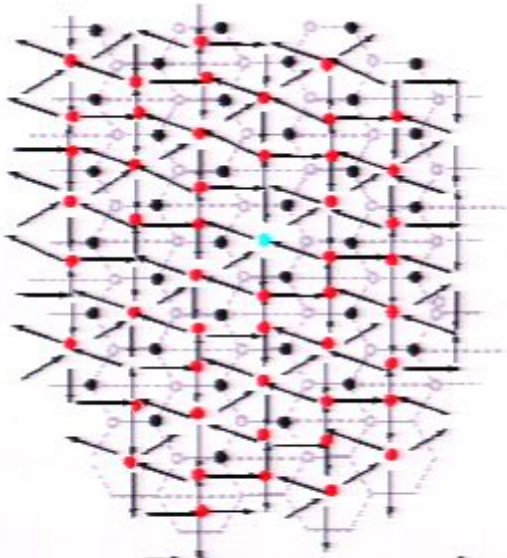
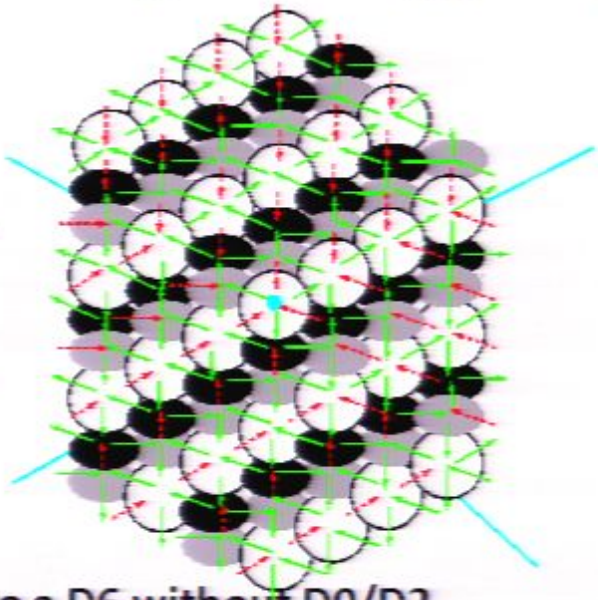
Unmolten crystal (ground state; pyramid)





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クリックしてノートを入力

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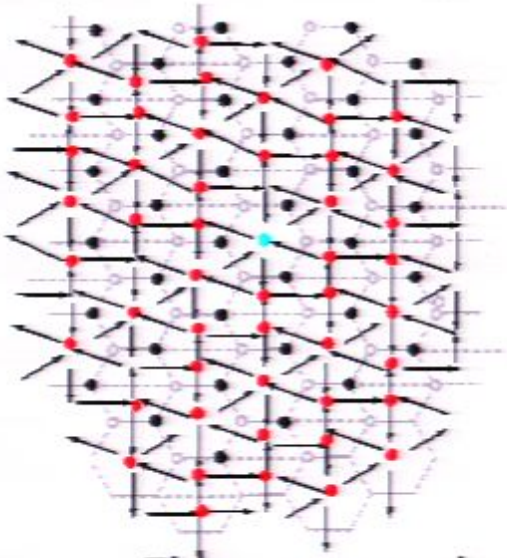

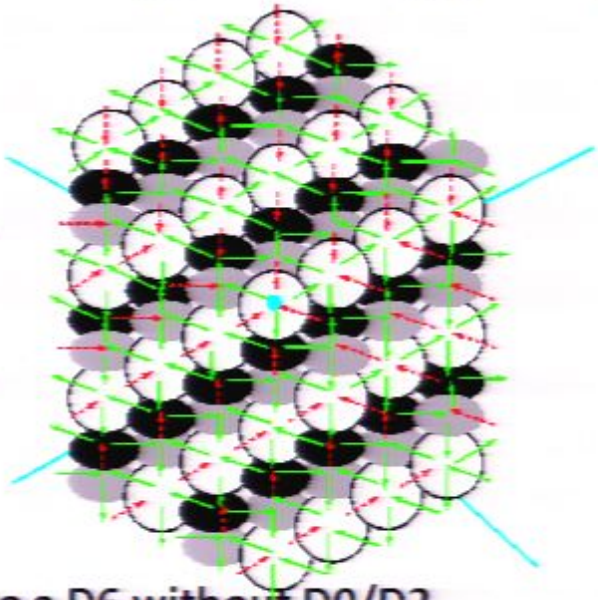

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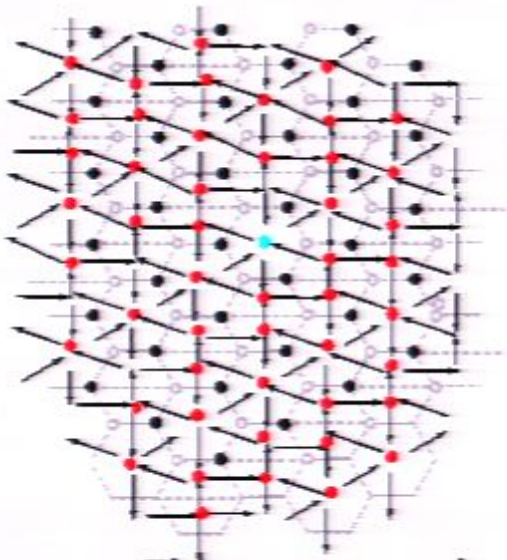

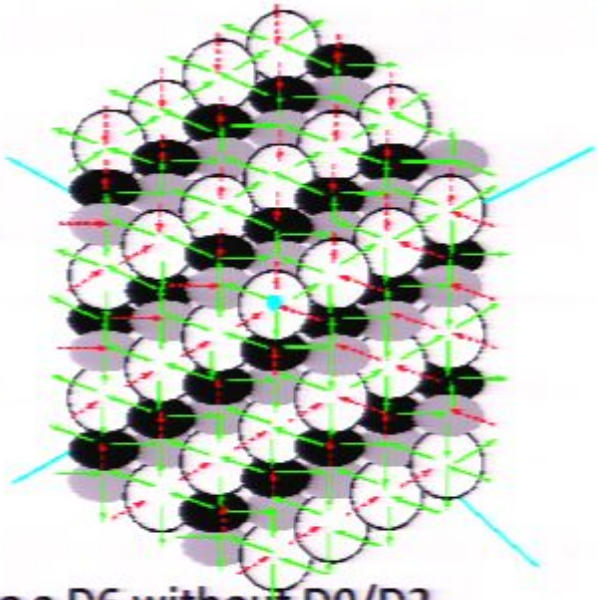

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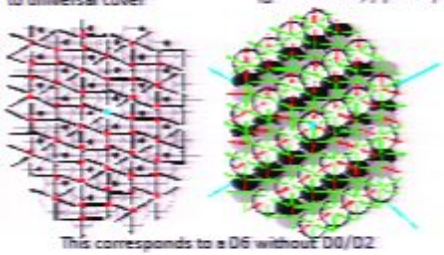
③New Crystal Melting Model

BPS state= a configuration of molten crystal

An example of a crystal



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クリックしてノートを入力

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ホーム 挿入 デザイン アニメーション スライドショー 校閲 表示 Acrobat

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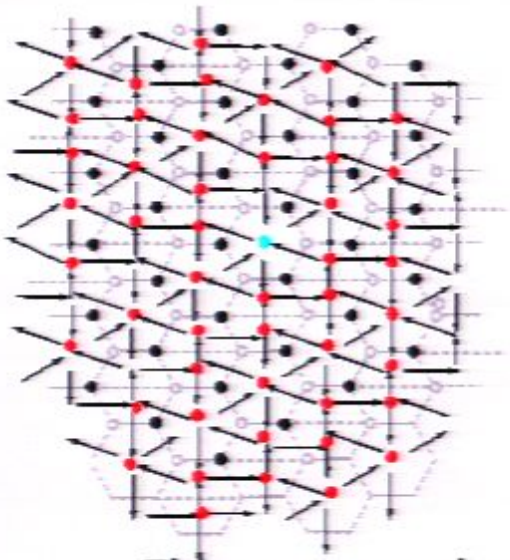

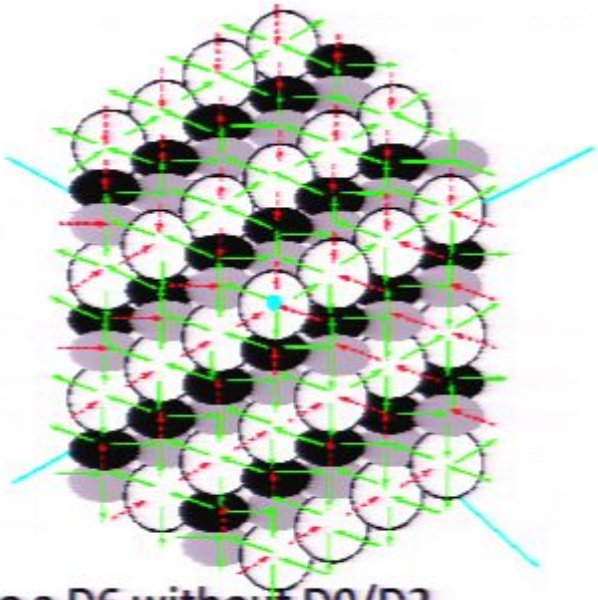

This corresponds to a D6 without D0/D2

Summary

1. New crystal melting which captures the BPS states of D0/D2/D6-branes; applicable to an *arbitrary* toric Calabi-Yau manifold

   BPS bound state ⟷ Molten Crystal

   New features: noncommutativity, "colors" of atoms...

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More to come!

- Thermodynamic limit of the statistical model (w/ H. Ooguri, to appear)

  Classical geometry reproduced from crystal: Meaning of fluctuations, implications for BH?

(Figure from Okounkov-Reshetikhin-Vafa)

- Relation with OSV?

  $Z_{BH} = |Z_{top}|^2 \qquad Z_{BH} = Z_{top}$

  $Z_{top} \sim |Z_{top}|^{2?}$ (modulo wall crossing)

  S-duality?

  Determinations of top. string partition function?

- Topological vertex for NCDT?
- Generalization of DT(/GW) to D6/D4/D2/D0 (including D4-brane charge)

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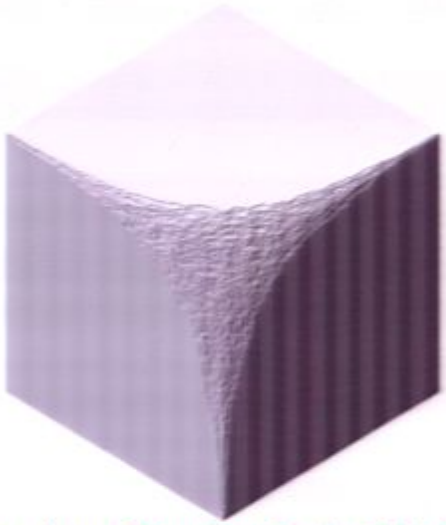

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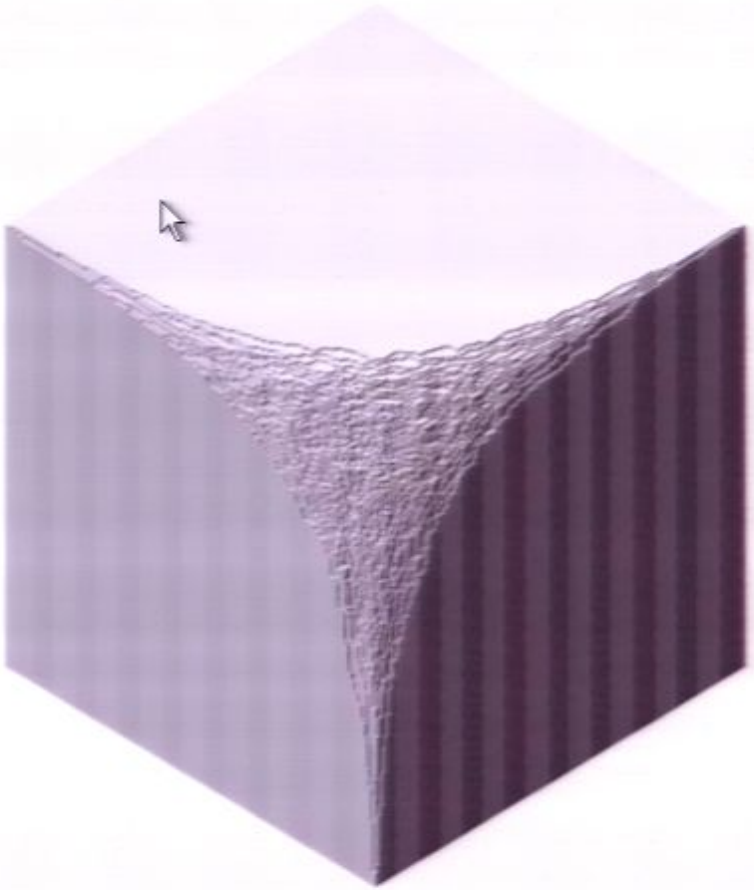

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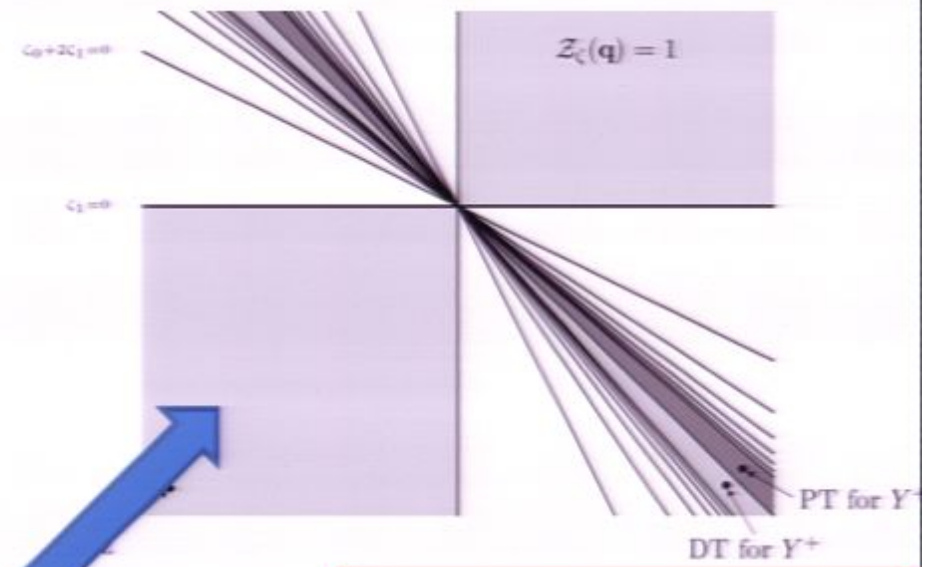
Wall Crossing Formula
[Kontsevich-Soibelman, Denef-Moore, Gaiotto-Moore-Neitzke,…]



$$Z_{\mathrm{NCDT}} = \left(\prod_k (1-q^k)^{-k}\right)^2 \prod_l (1-Qq^k)^k \boxed{\prod_k (1-Q^{-1}q^k)^k}$$

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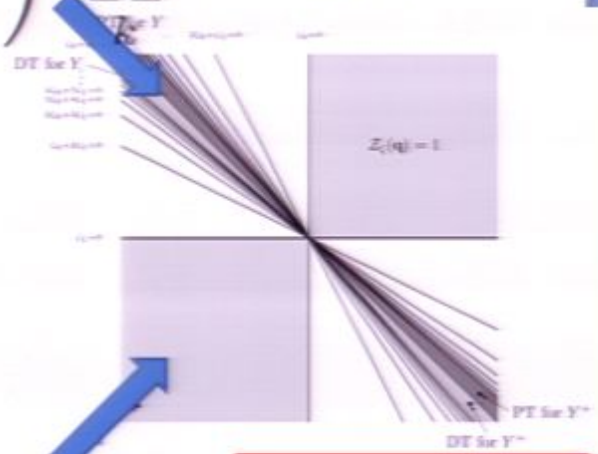
Example: conifold

$$Z_{\mathrm{DT}} = \left(\prod_k (1-q^k)^{-k}\right)^2 \prod_k (1-Qq^k)^k$$

$$q = e^{-g_s}, Q = e^{-t}$$

Wall Crossing Formula
[Kontsevich-Soibelman, Denef-Moore, Gaiotto-Moore-Neitzke,...]



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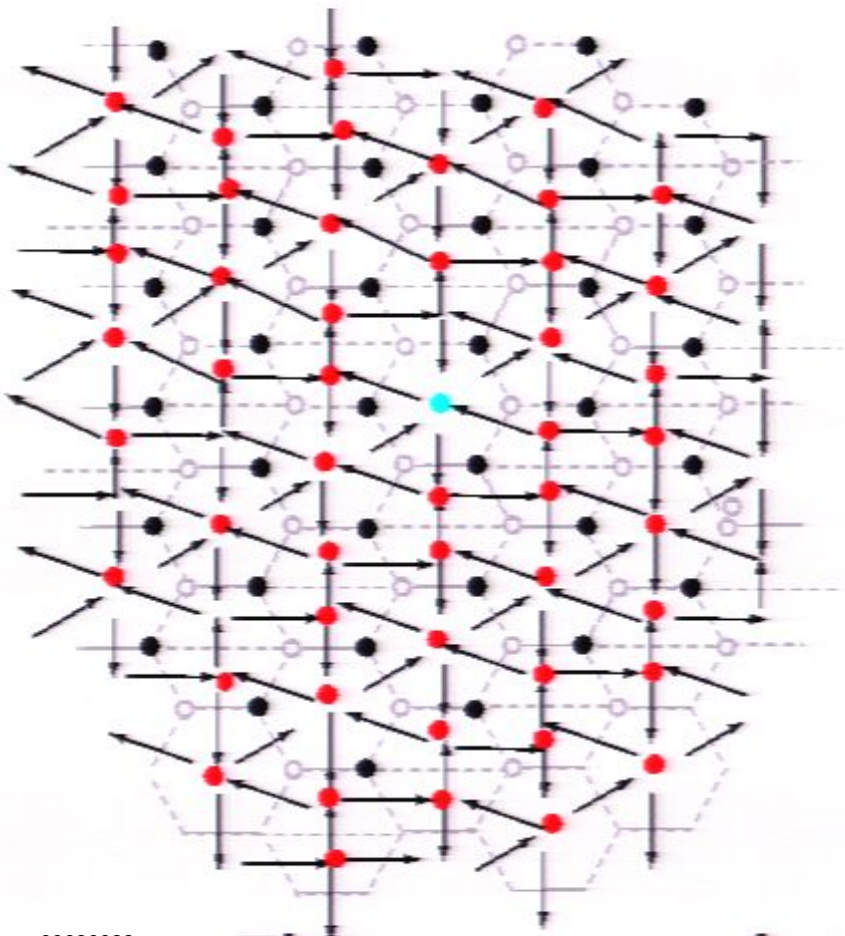
クリックしてノートを入力

An example of a crystal

Lift of quiver diagram on T^2 to universal cover



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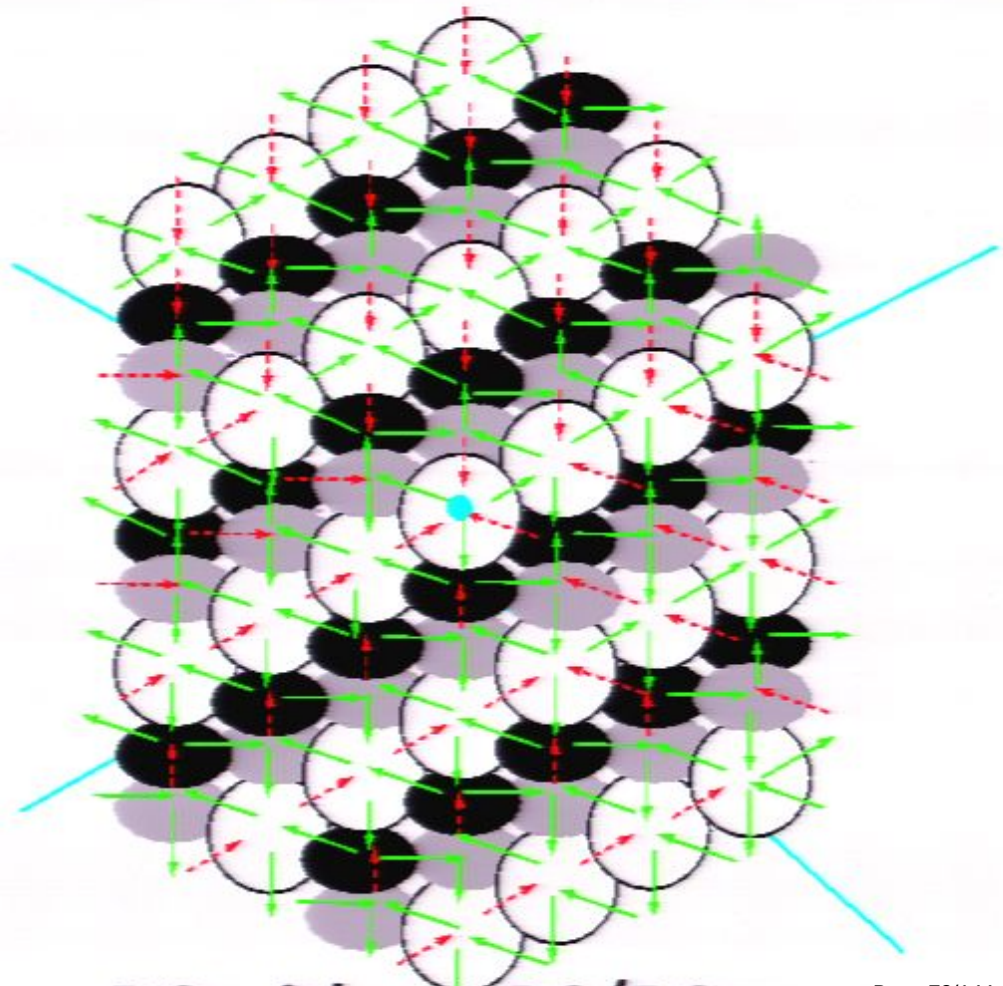

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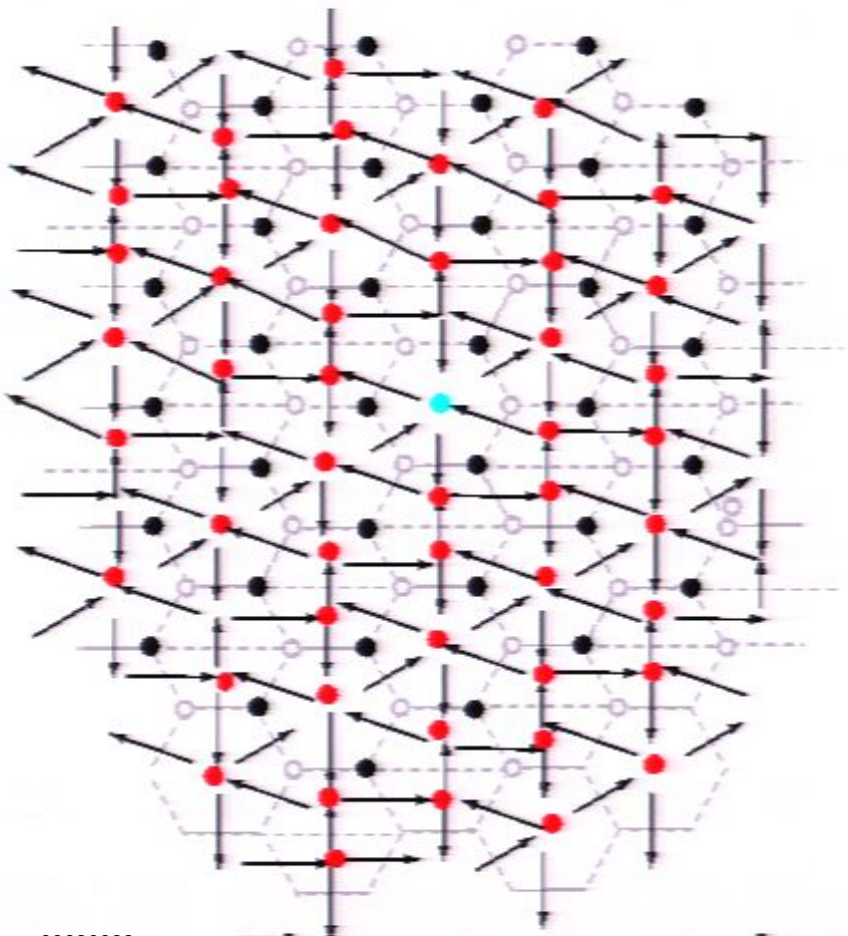

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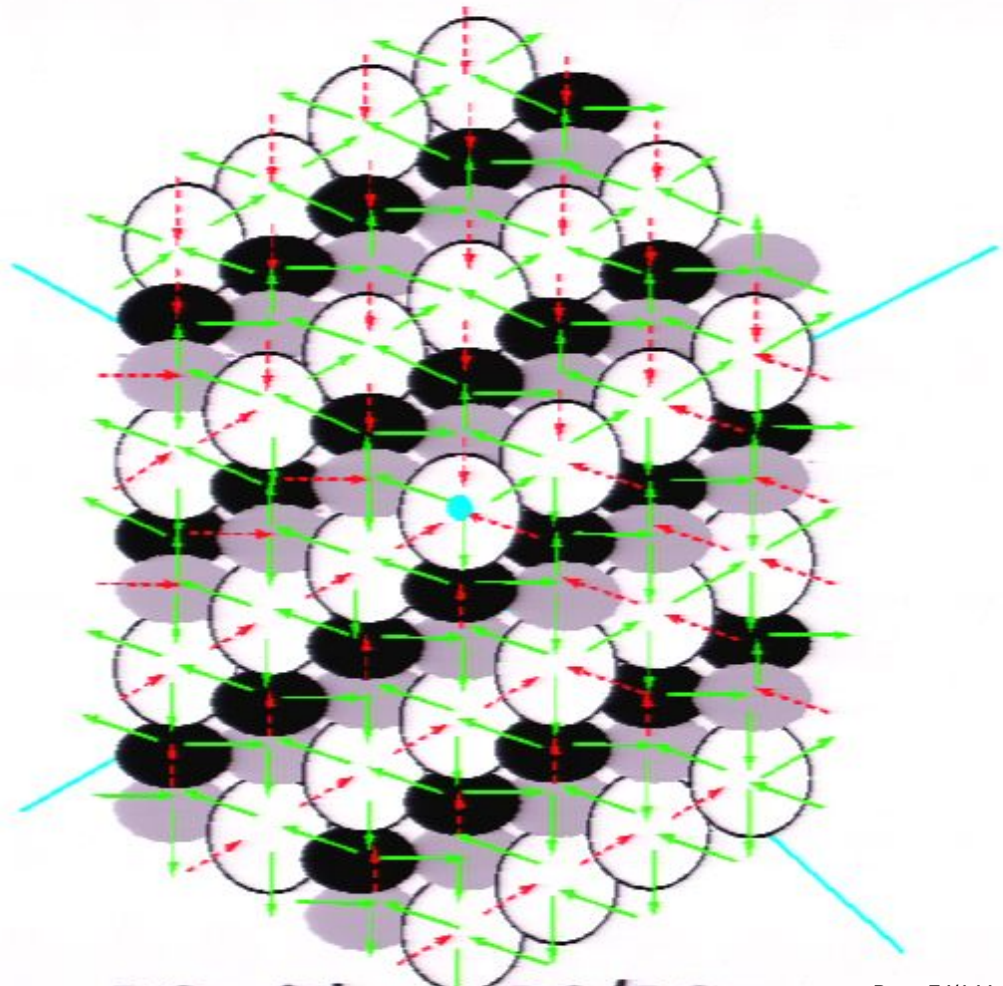

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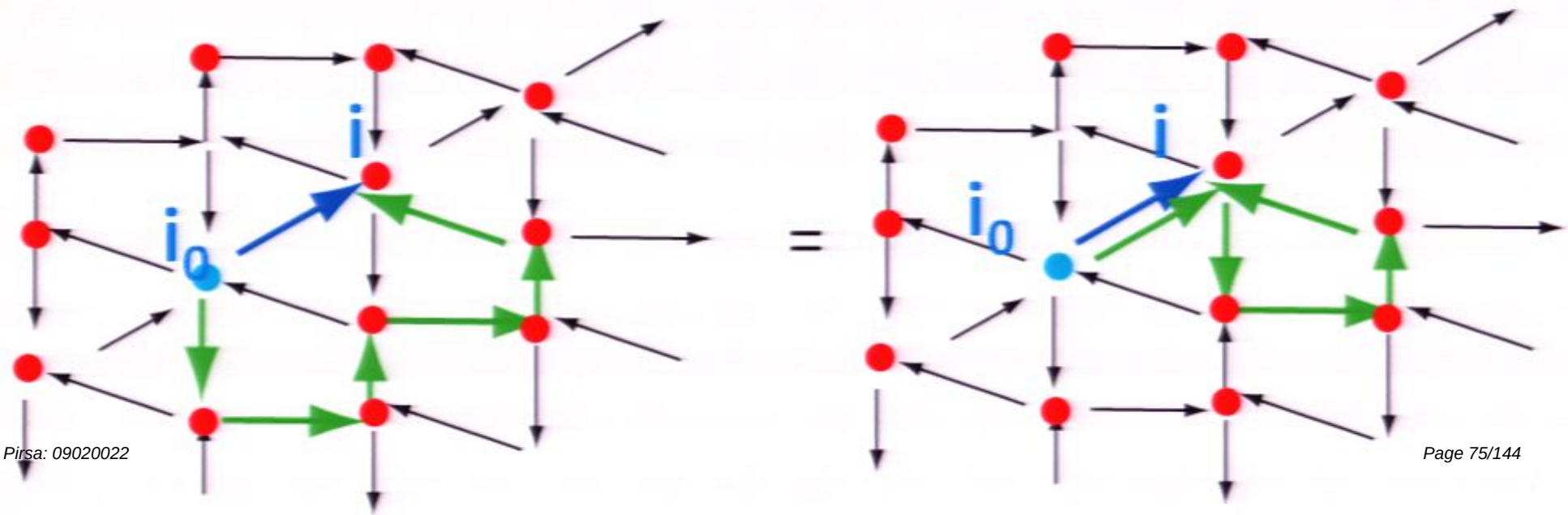

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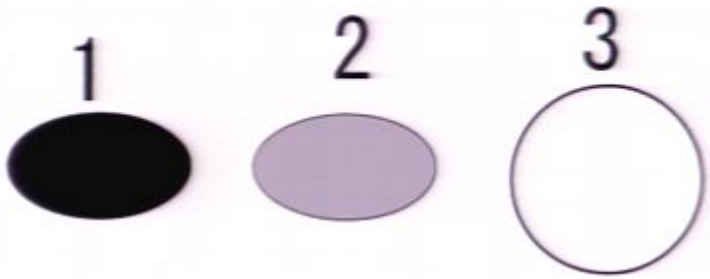

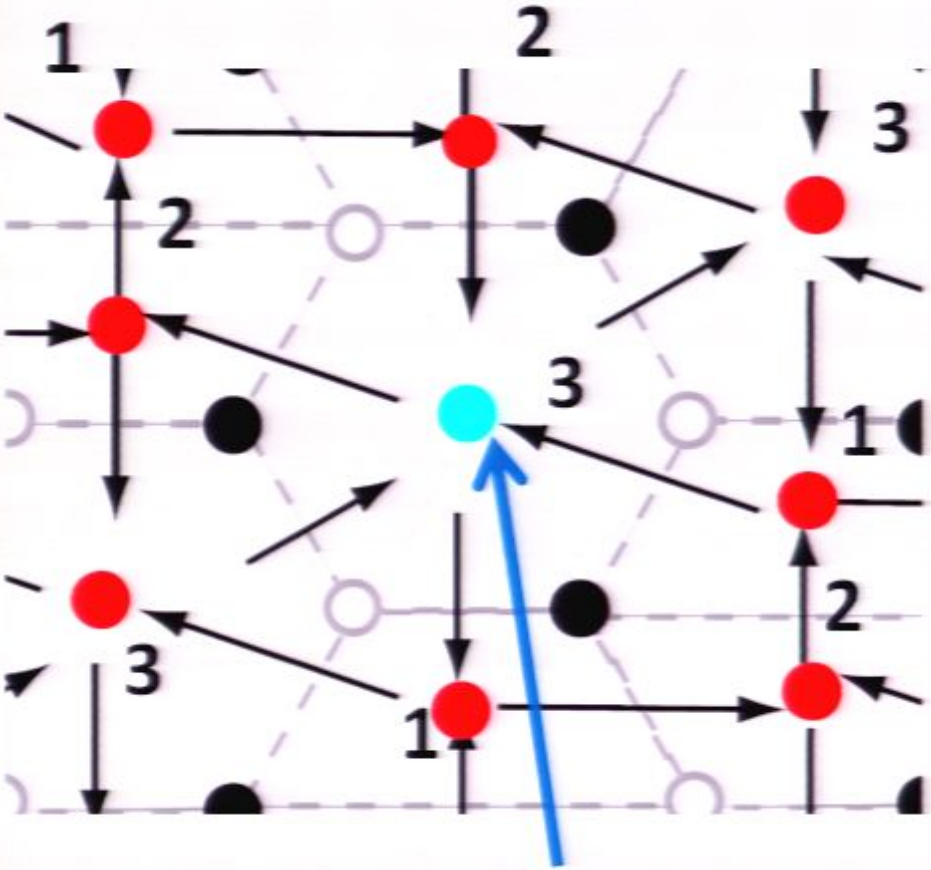

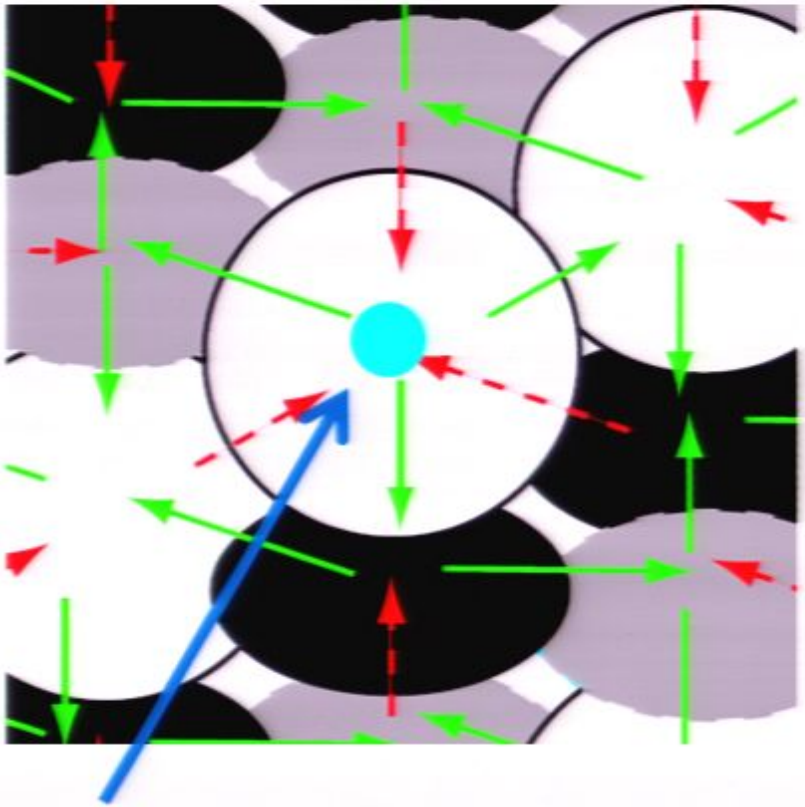

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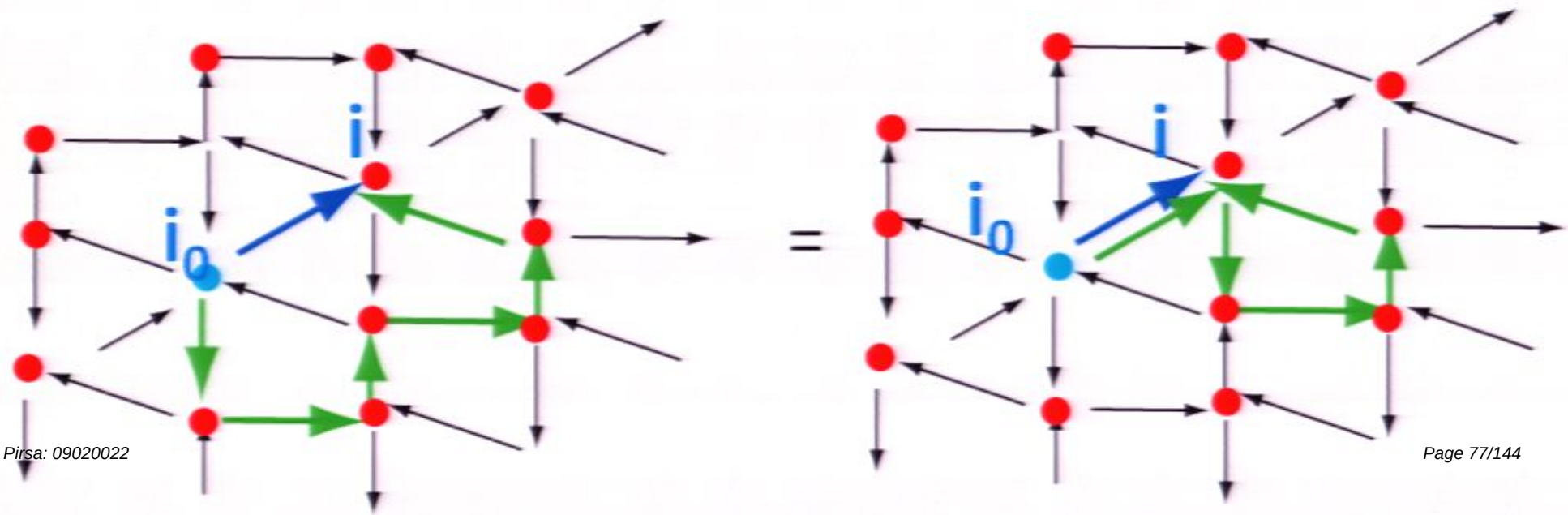

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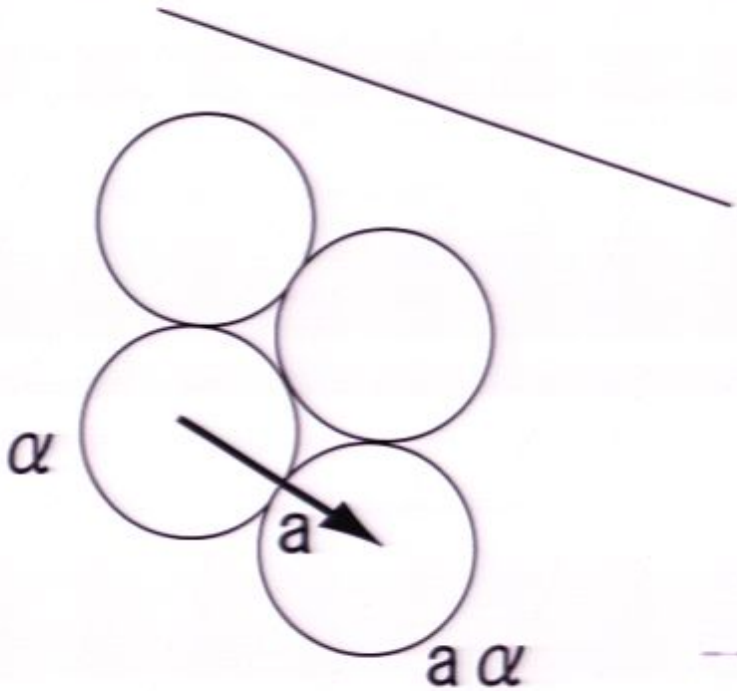
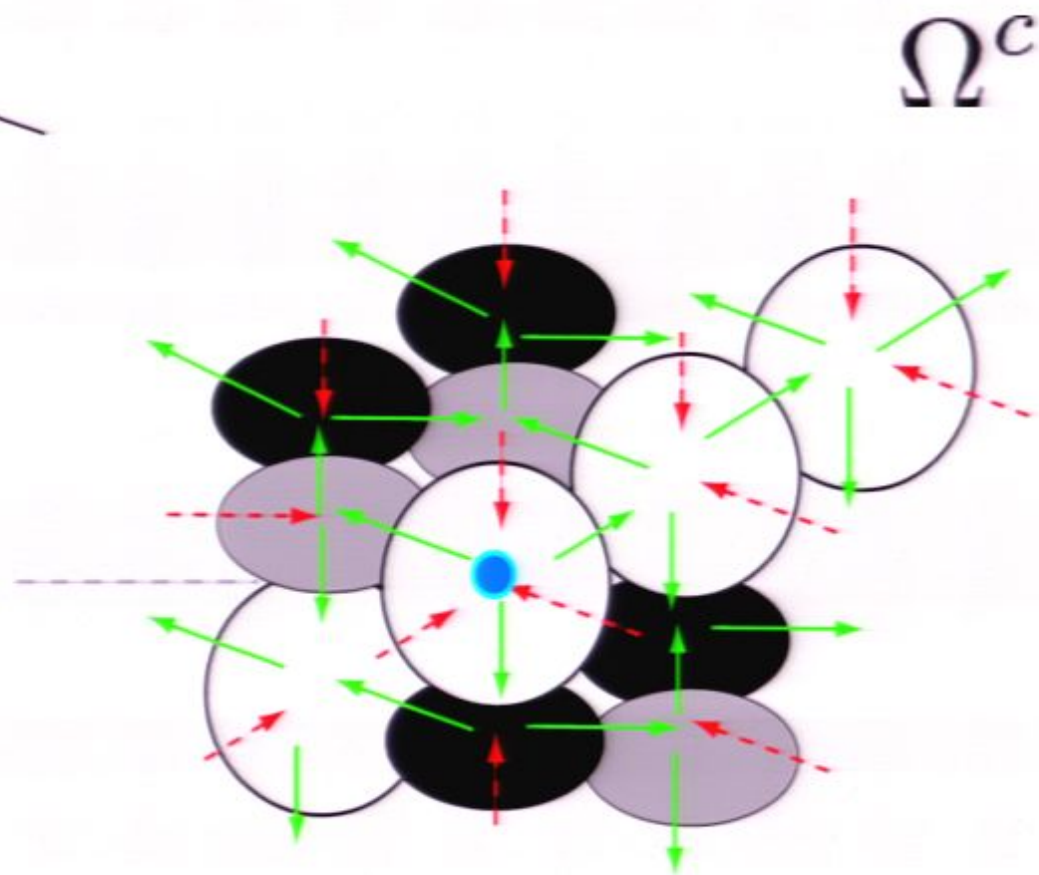
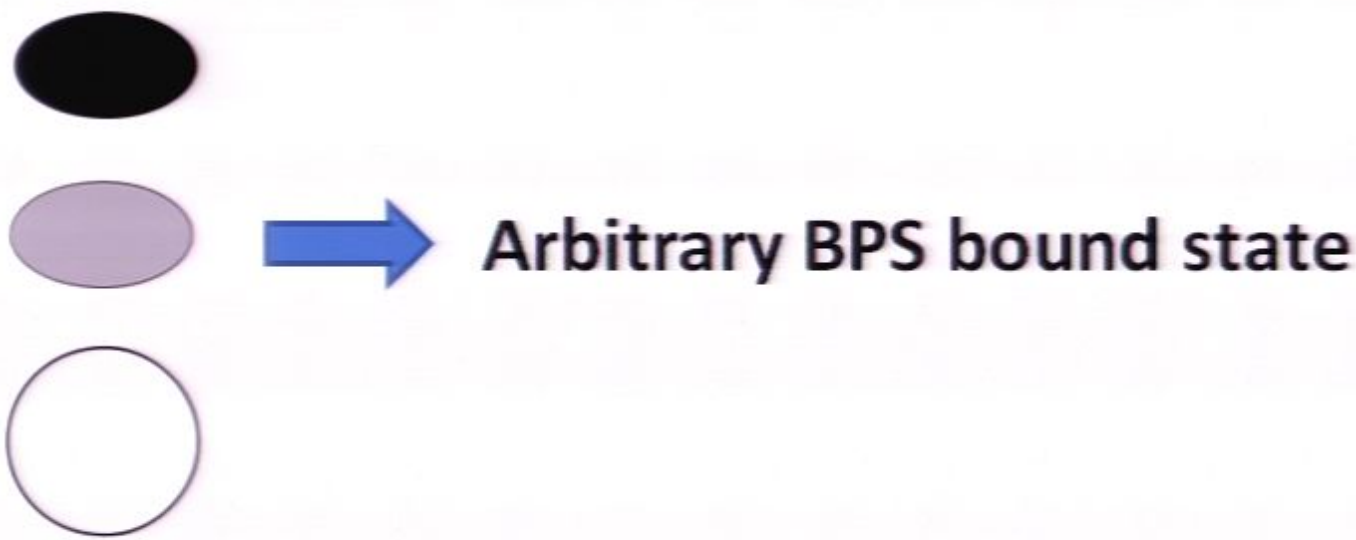




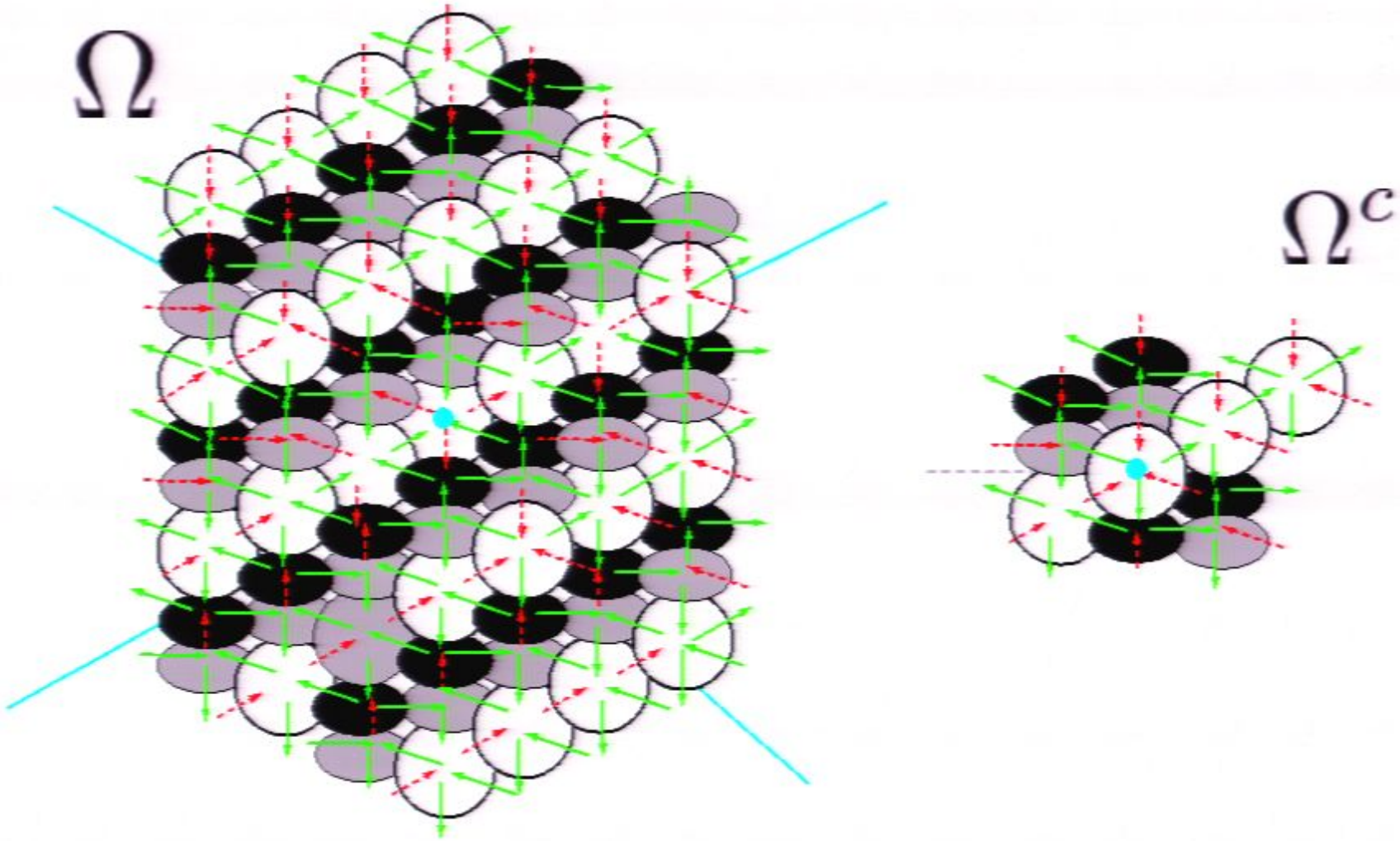
◆**BPS bound states are constructed out of atoms, which carries linear combination of D-brane charges**

(Similar to quark models, as opposed to Sakata model)



(Reminiscent of bubbling AdS geometry [LLM]
Fuzzball conjecture [Mathur])

molten crystal



Adding D0/D2 charges corresponds to molten crystals

BPS bound state = molten crystal

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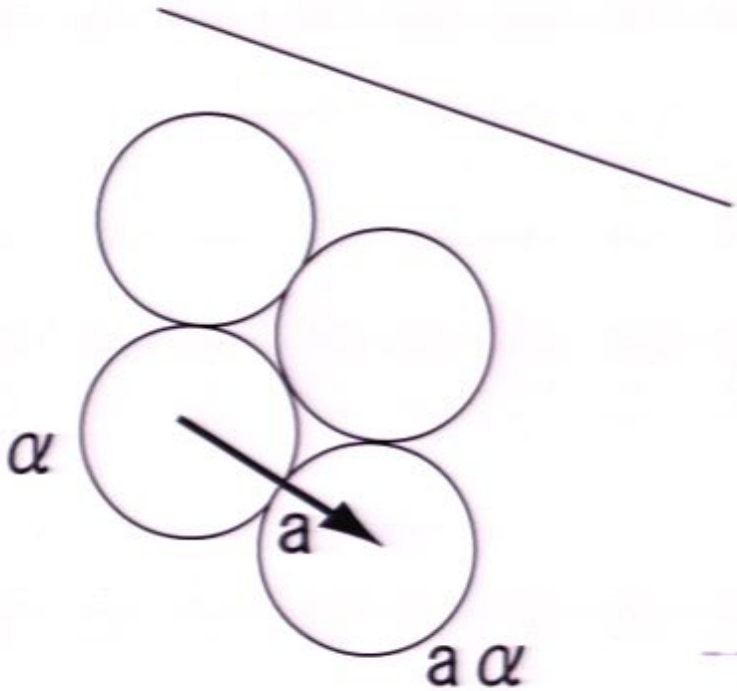

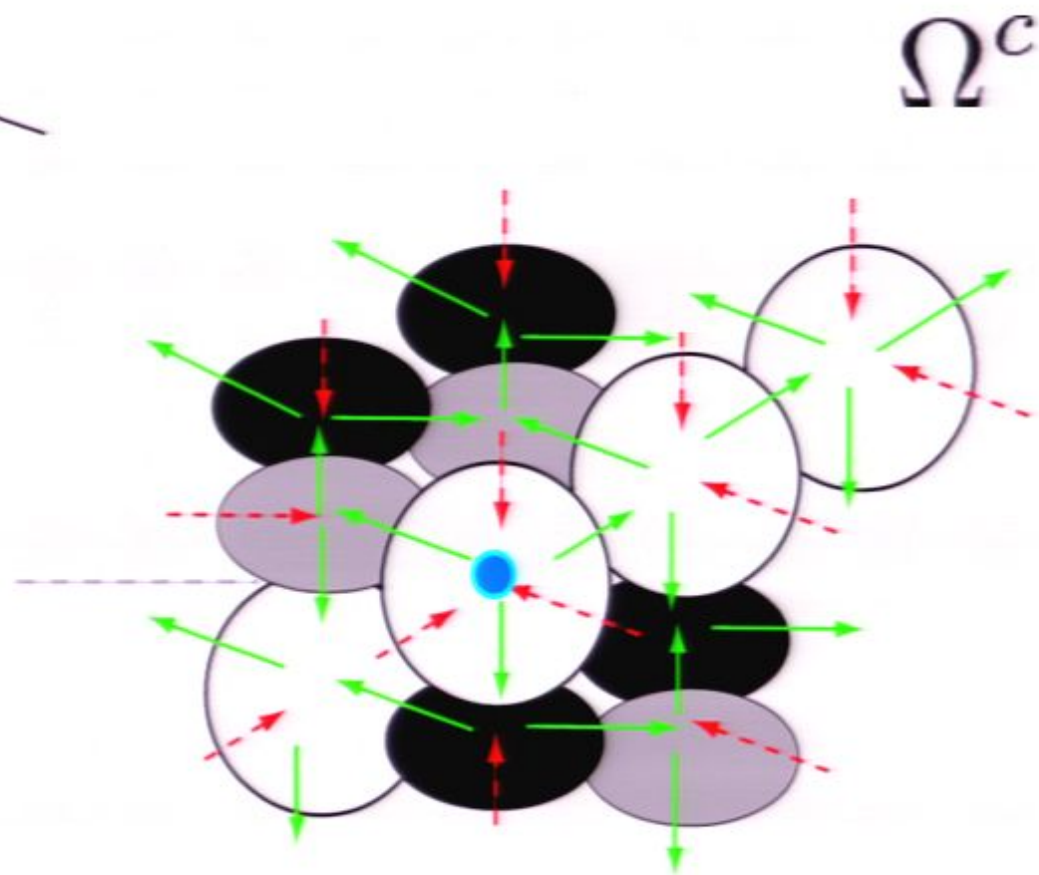

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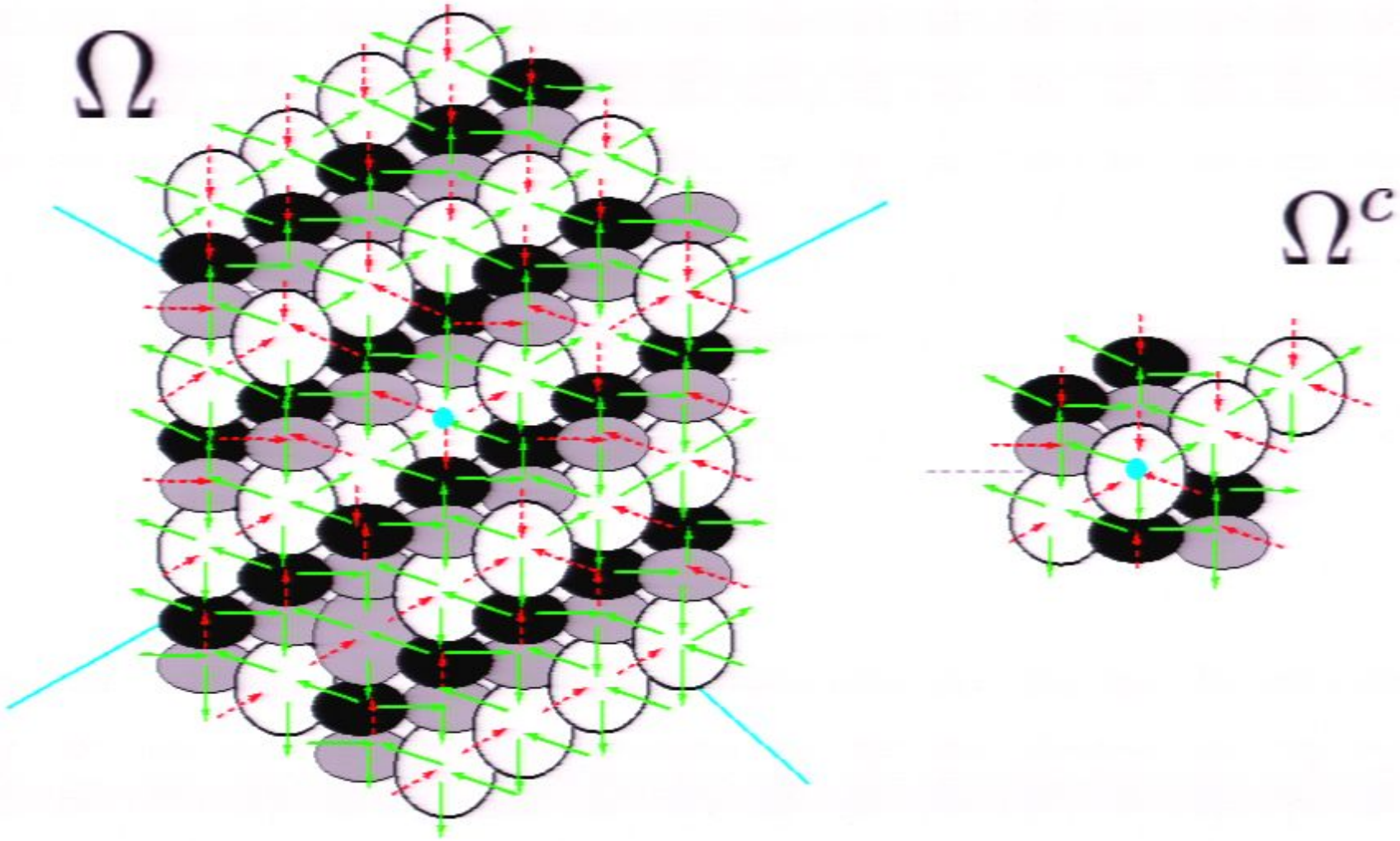

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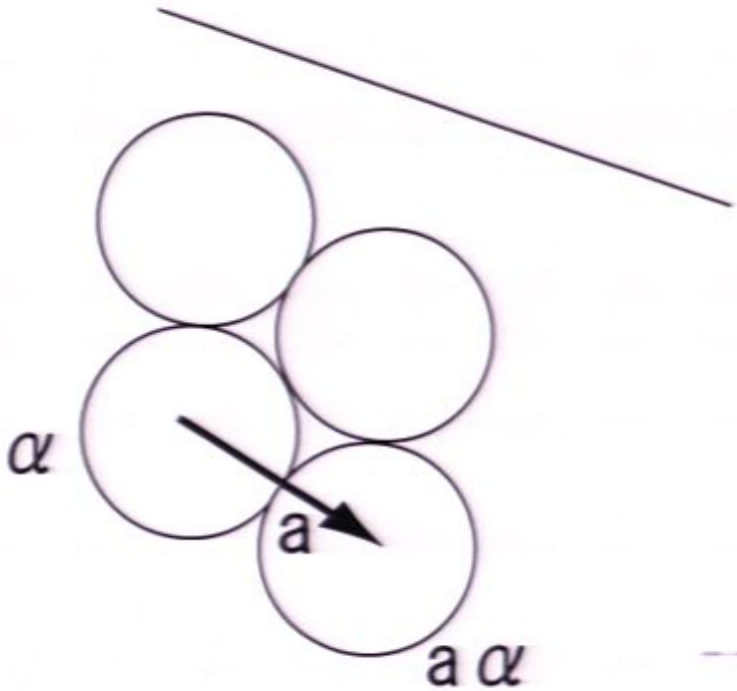

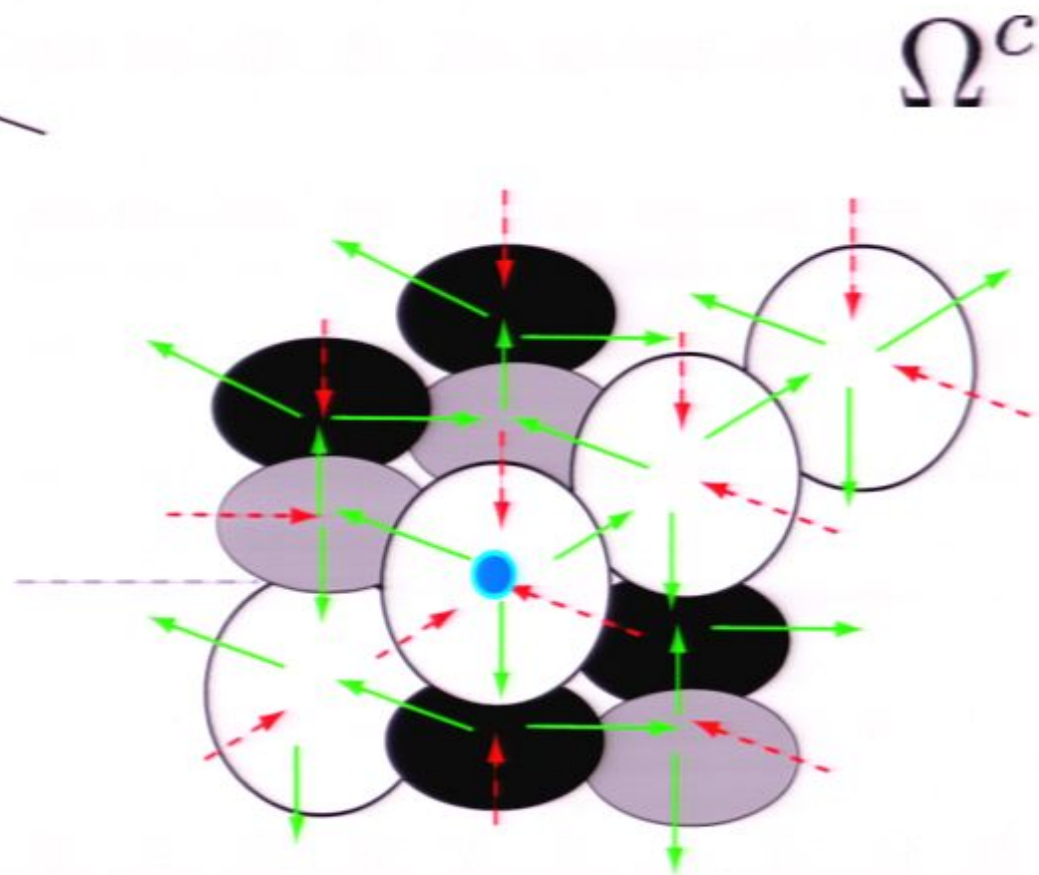

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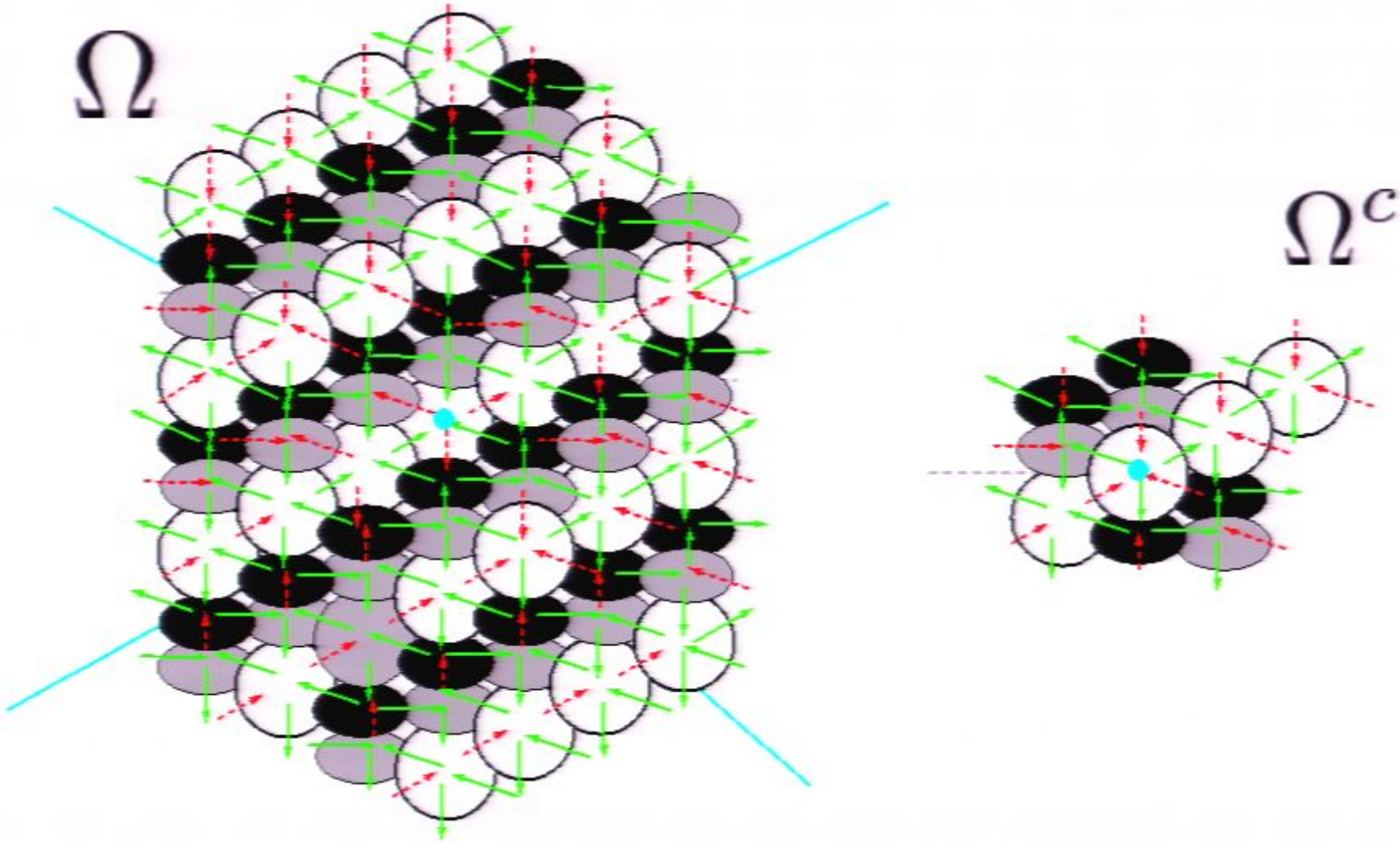

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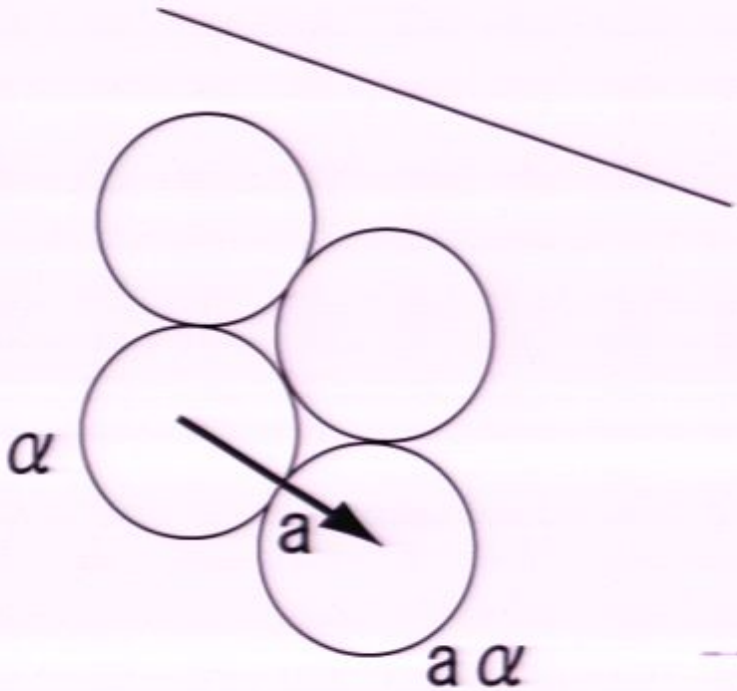

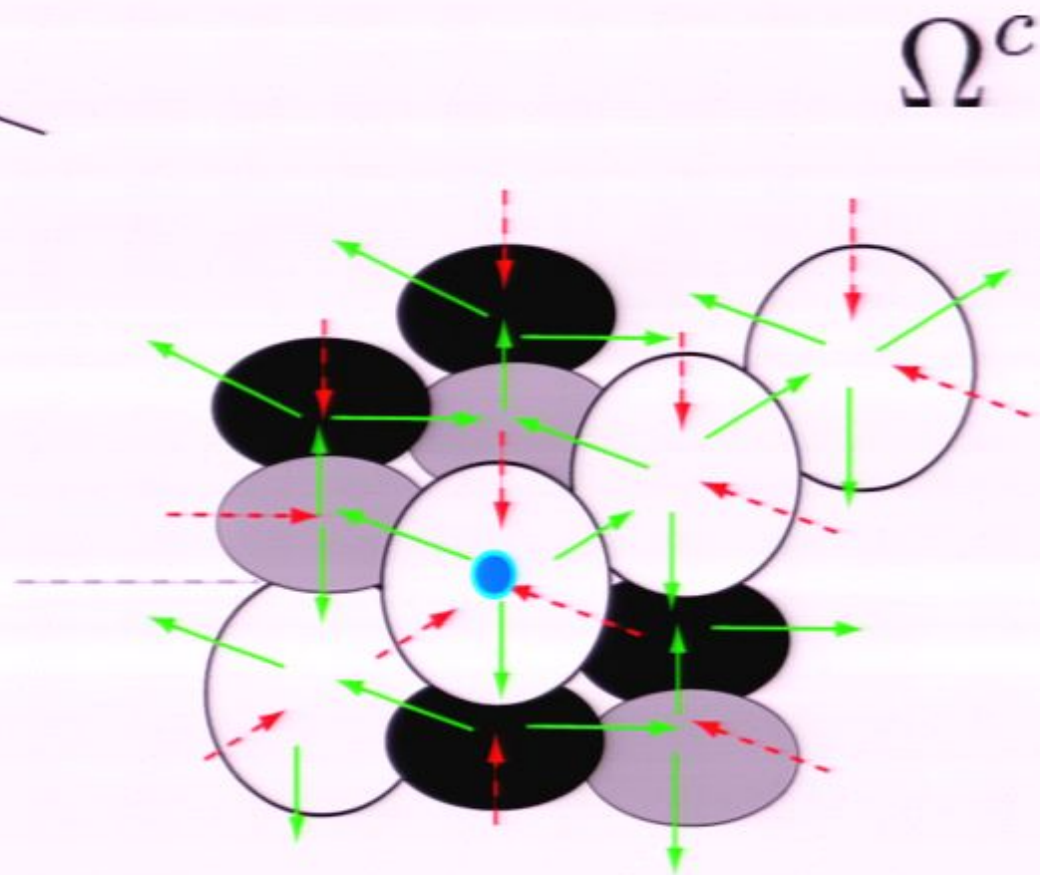

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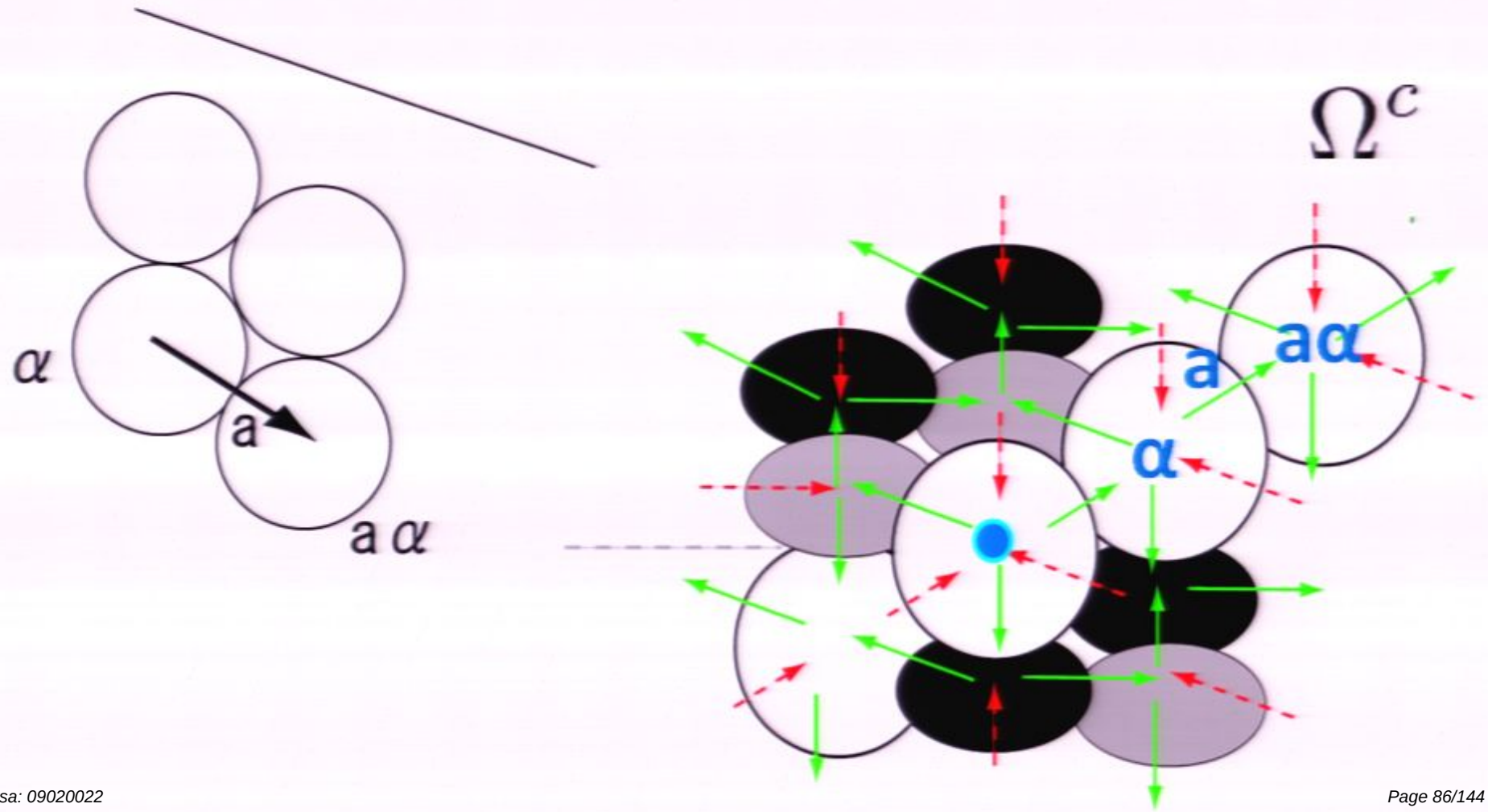
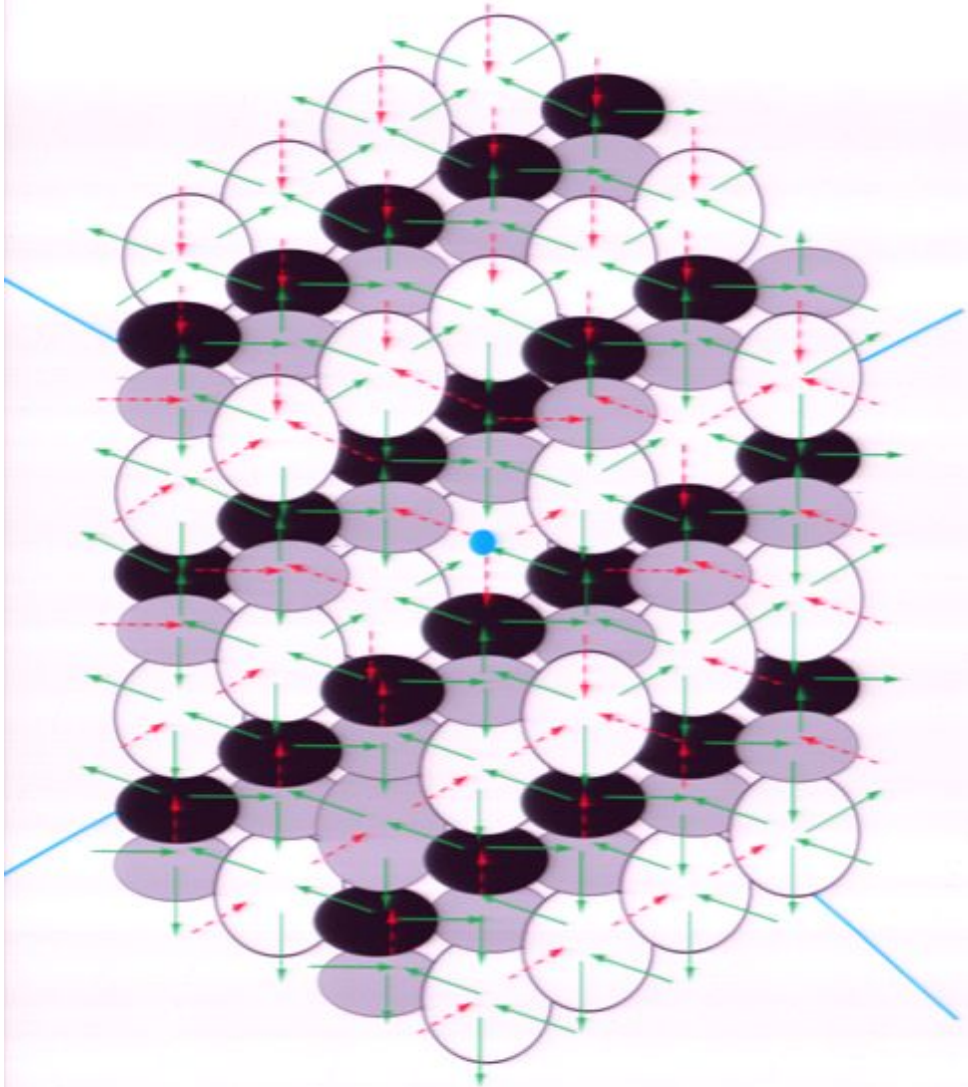
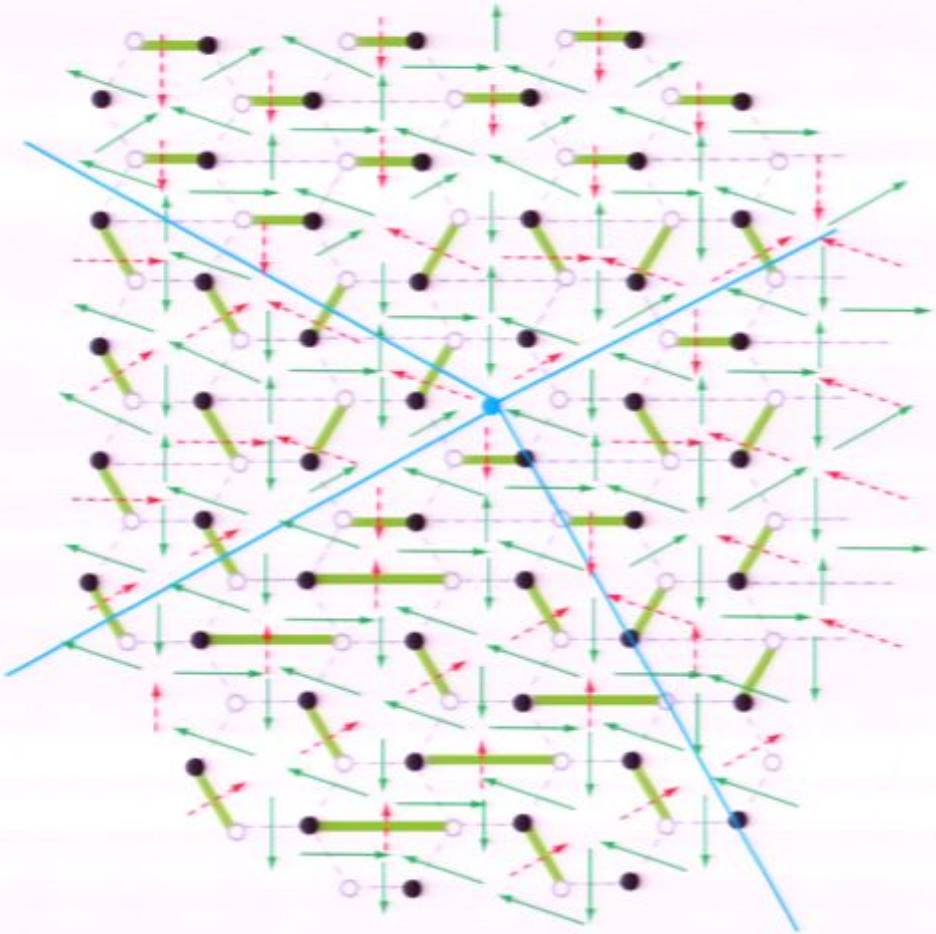


④Proof via Non-commutative Donaldson-Thomas Theory

Formulated equivalently as a dimer model





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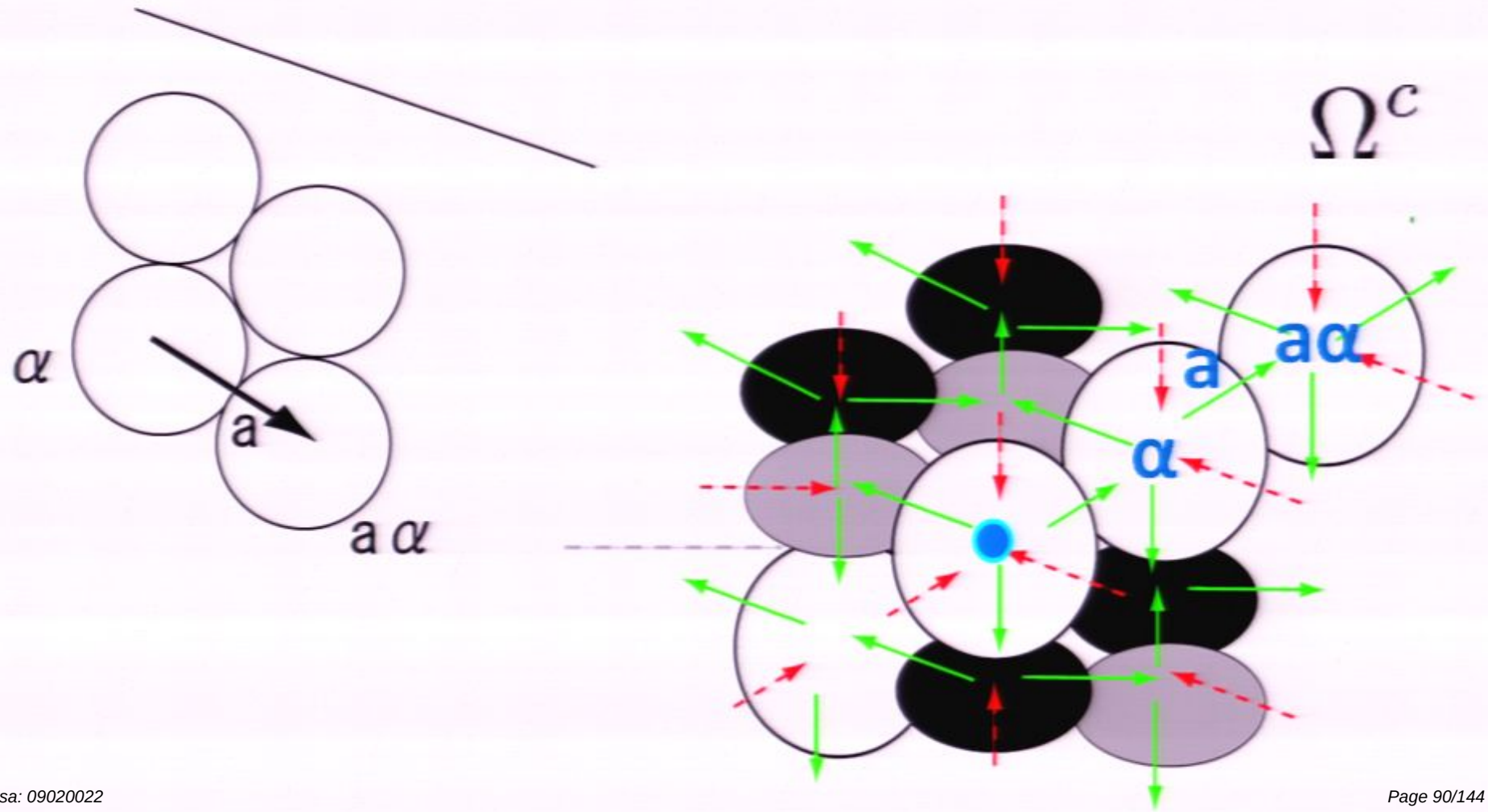

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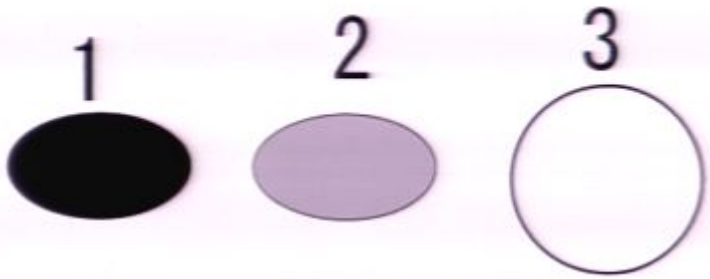

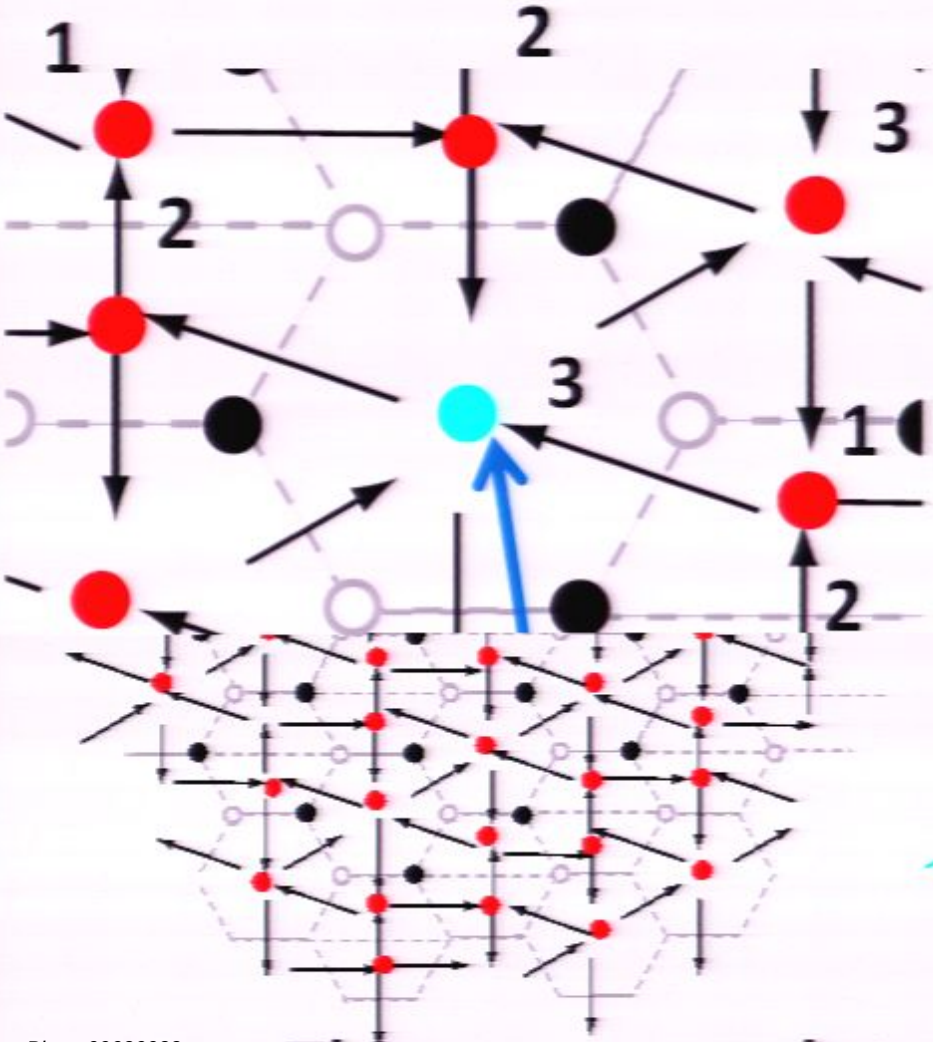

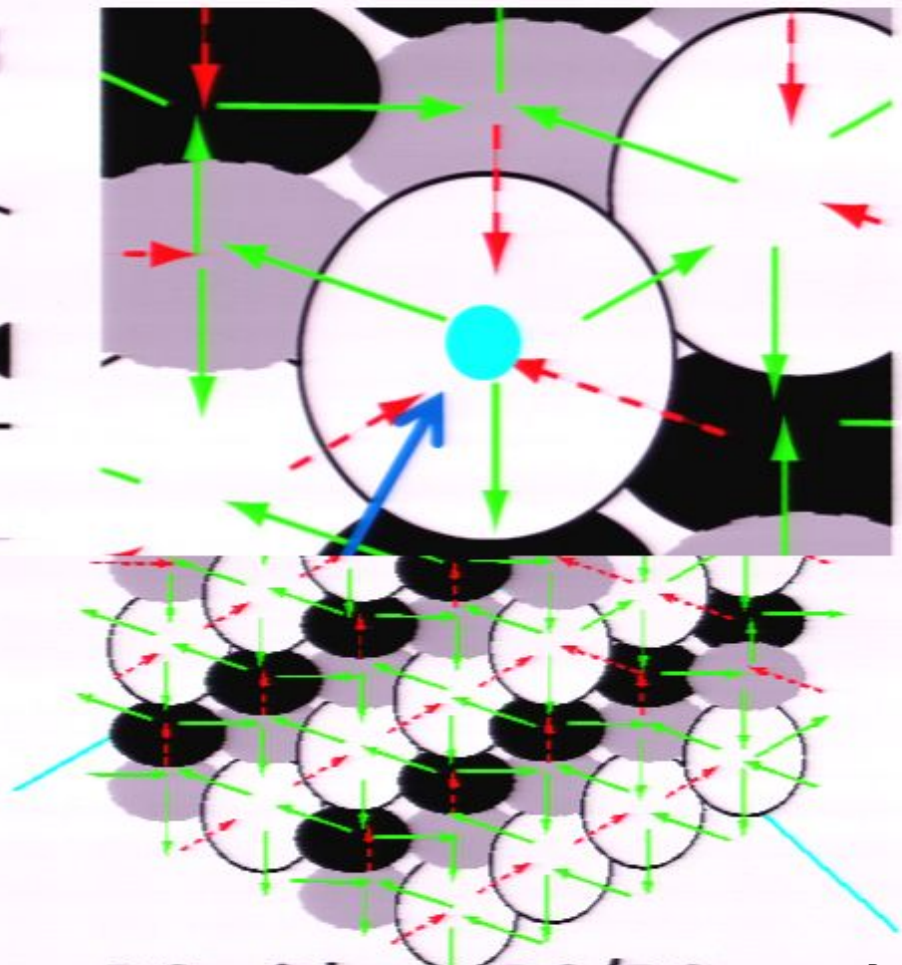

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An example of a crystal

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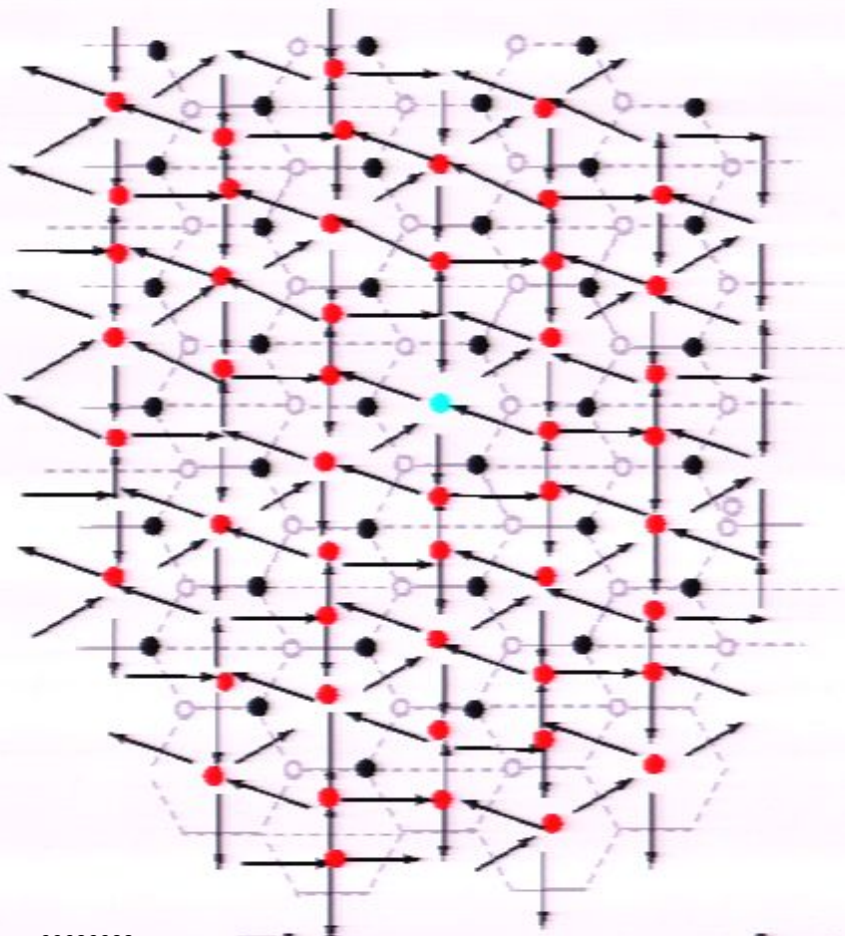

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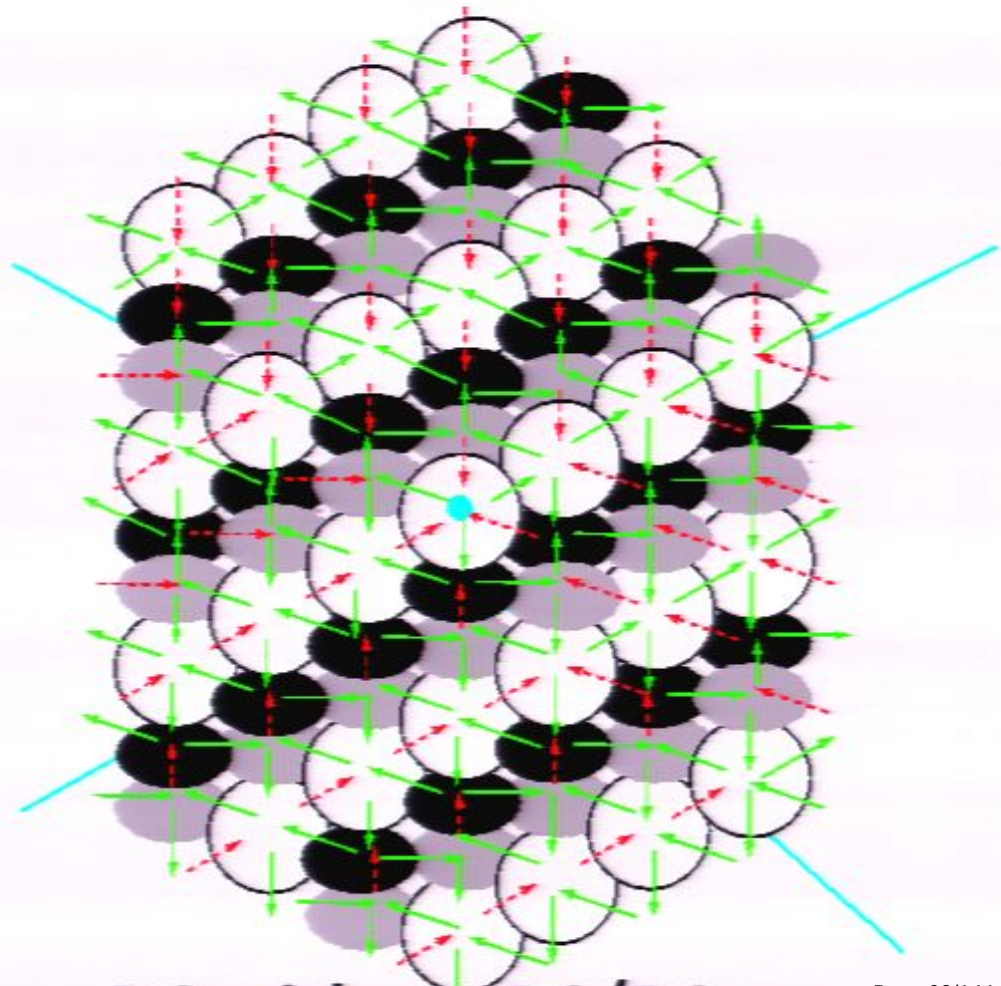

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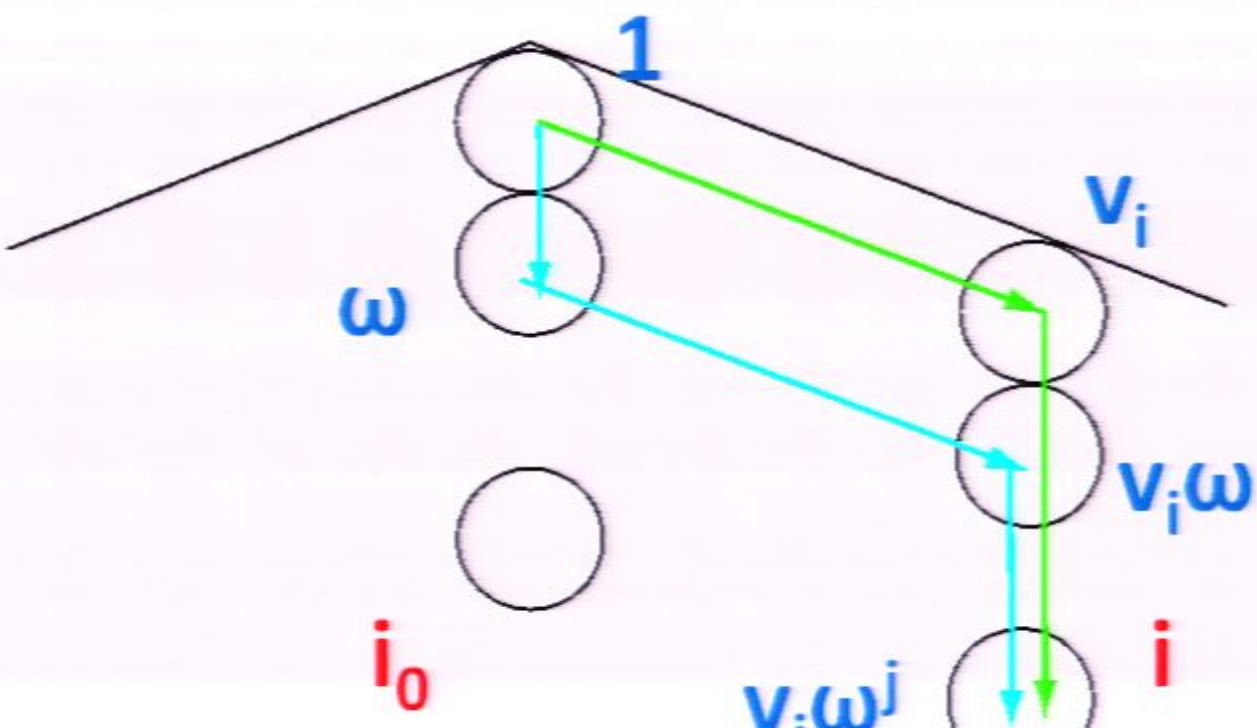


④Proof via Non-commutative Donaldson-Thomas Theory

Mathematically, "counting" of BPS states of this quiver QM (computation of Witten index) is formulated as the "non-commutative Donaldson-Thomas theory"

Don't be afraid of mathematical theory, since most of the things we physicists already know in different language.....

The Path Algebra

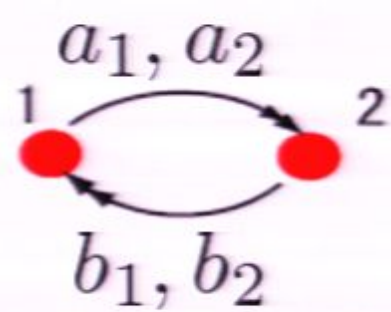
Consider the ring A of all possible paths (operators) modded out by F-term constraints

- **Path algebra $\mathbb{C}Q$: spanned by the set of paths on the quiver, multiplication defined by concatenation of paths (noncommutative)**
- **A is the factor algebra by an ideal $\mathcal{F}$ gen. by F-term constraints:**

$$A = \mathbb{C}Q/\mathcal{F}$$

Example: conifold [Klebanov-Witten '98]

$$W = \mathrm{tr}(a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1)$$



$$A = \mathbb{C}[e_1, e_2]\langle a_1, a_2, b_1, b_2\rangle / \left(a_1 b_i a_2 = a_2 b_i a_1, b_1 a_i b_2 = b_2 a_i b_1\right)_{i=1,2}$$

- Computation of Witten index
  =counting of θ-stable A-modules fixed by T^2

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A-module= representation of
bifundamental fields
with F-term constraints imposed

D-term constraints

Localization: virtue of toric geometry

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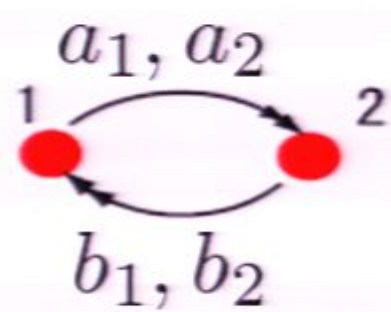

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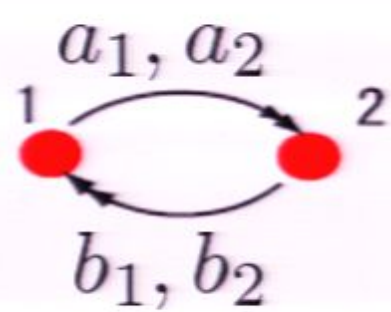

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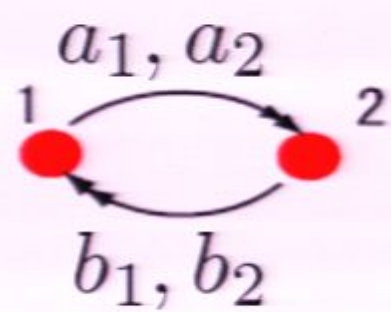

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Localization: virtue of toric geometry

◆From a molten crystal, we can explicitly construct an A-module (and thus a BPS state) by

1. a basis e_α for each atom α
2. $a(e_\alpha)=e_\beta$ for each arrow $a: \alpha \to \beta$ w/ α, β in Ω^c
   (otherwise $a(e_\alpha)=0$)

$$M = \oplus_{\alpha \in \Omega} e_\alpha$$

Mathematics results [Szendroi'07, Mozgovoy-Reineke '08] say that all BPS states can be constructed in this way

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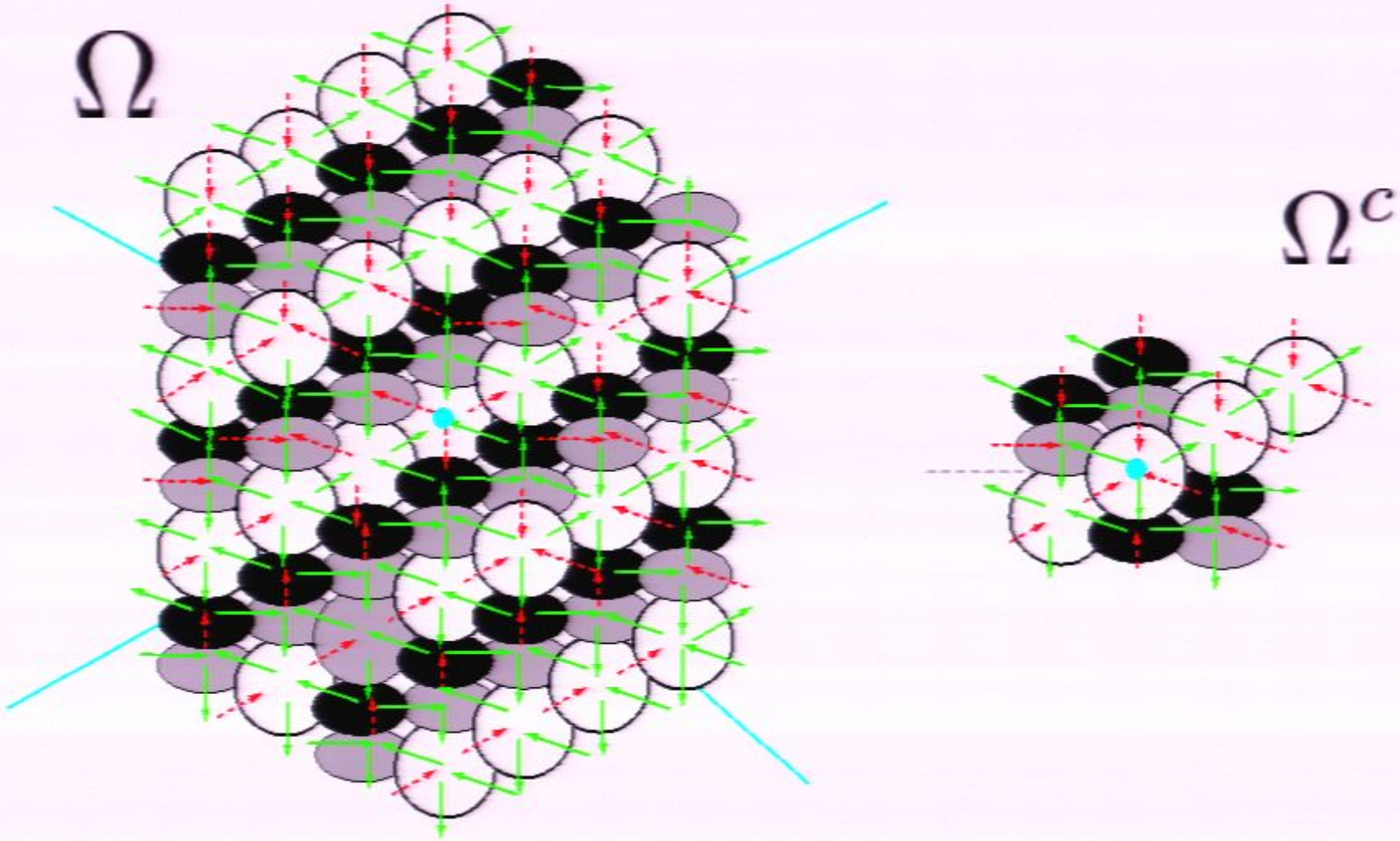
④Proof via Non-commutative Donaldson-Thomas Theory

molten crystal

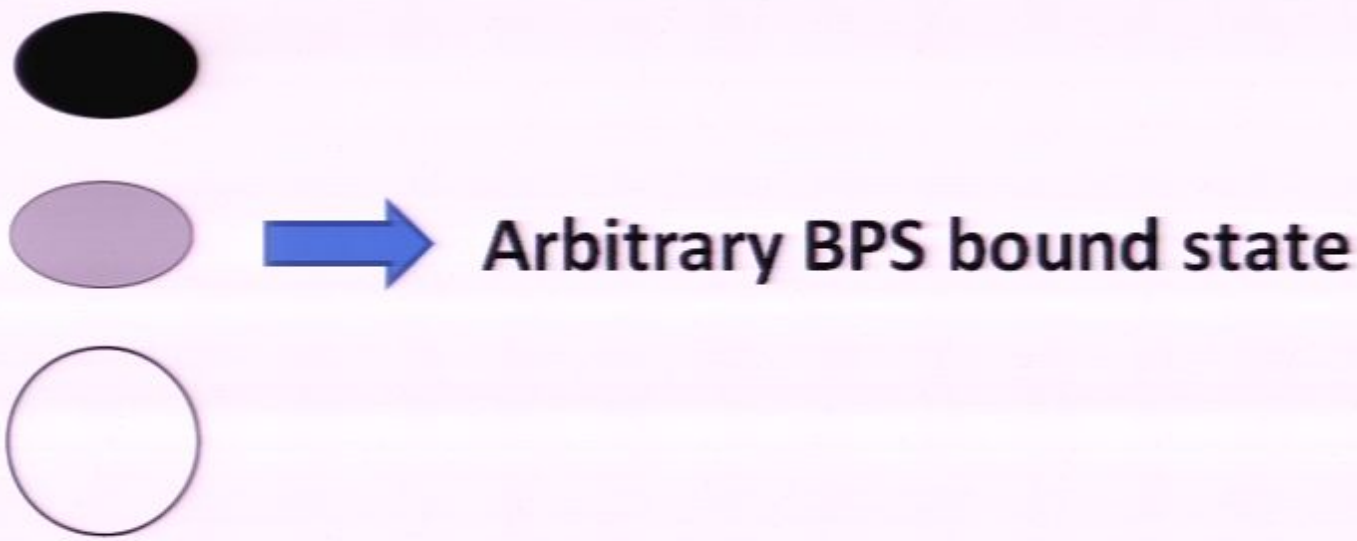


Adding D0/D2 charges corresponds to molten crystals

BPS bound state = molten crystal

◆**BPS bound states are constructed out of atoms,**
which carries linear combination of D-brane charges

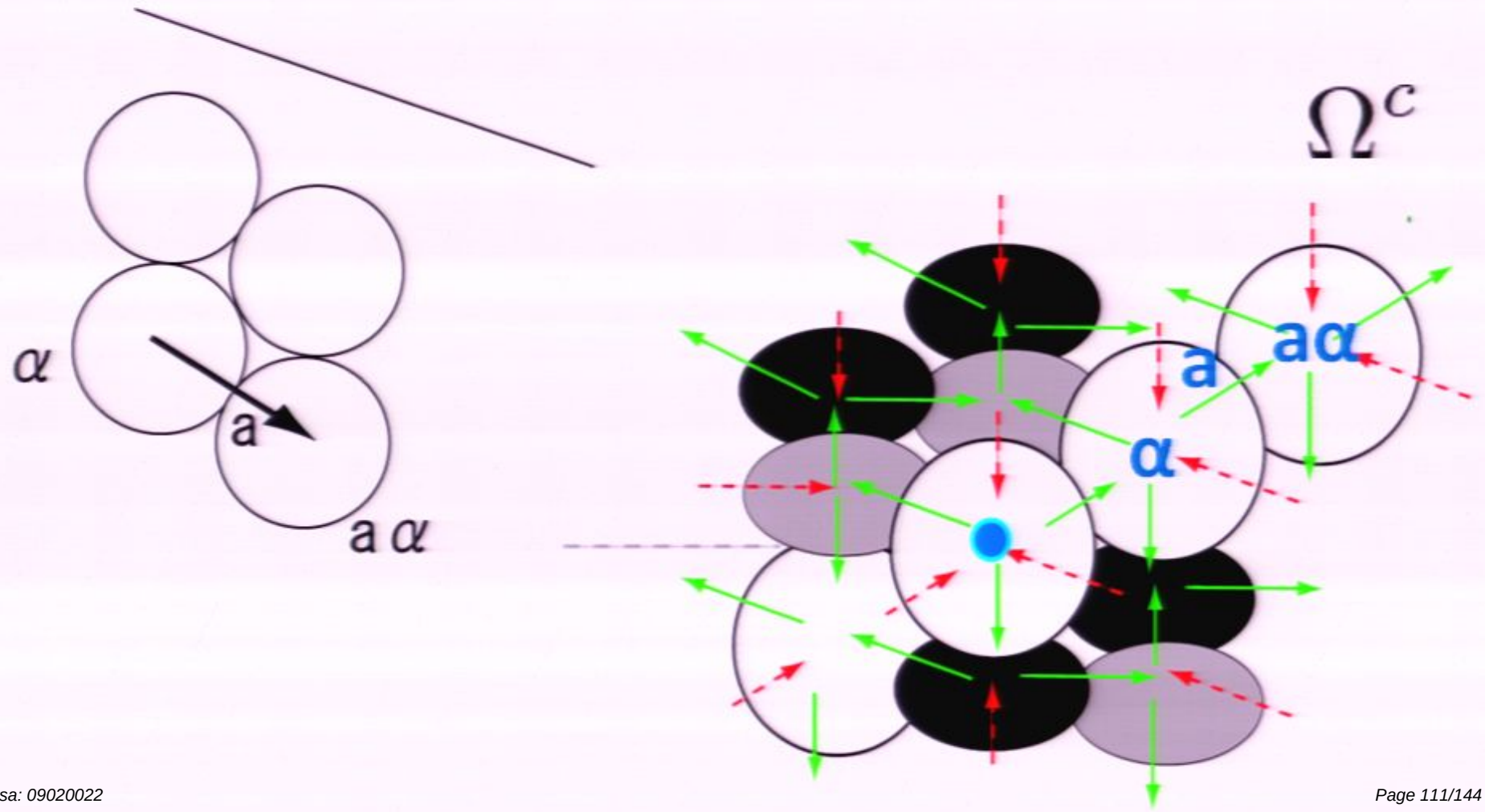
(Similar to quark models, as opposed to Sakata model)



(Reminiscent of bubbling AdS geometry [LLM]
Fuzzball conjecture [Mathur])

Melting Rule:

$$\text{For } a \in A, \quad a\alpha \in \Omega^c \Rightarrow \alpha \in \Omega^c$$



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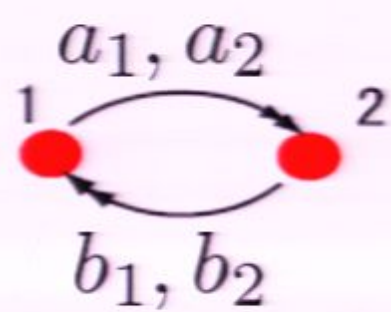

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⑤Discussions on Wall Crossing

Wall crossing

✓ **commutative DT theory**

$$Z_{\mathrm{BH}} = Z_{\mathrm{top}}$$

LHS is evaluated at different points of Kahler moduli

✓ **noncommutative DT theory**

$$Z'_{\mathrm{BH}} = Z_{\text{crystal melting}}$$

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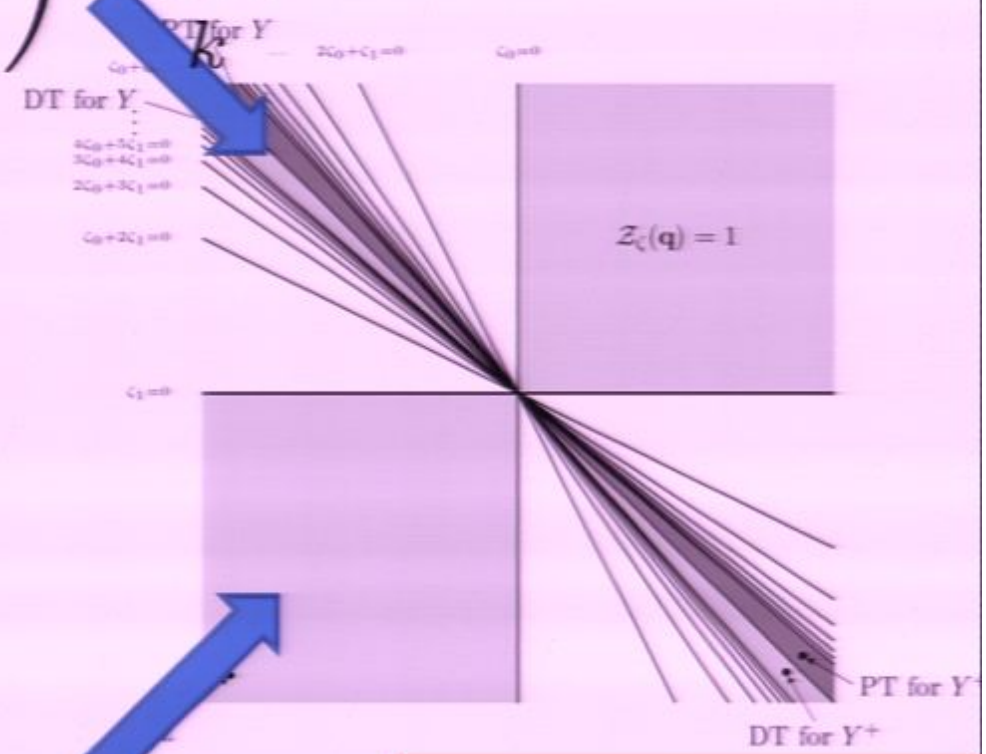

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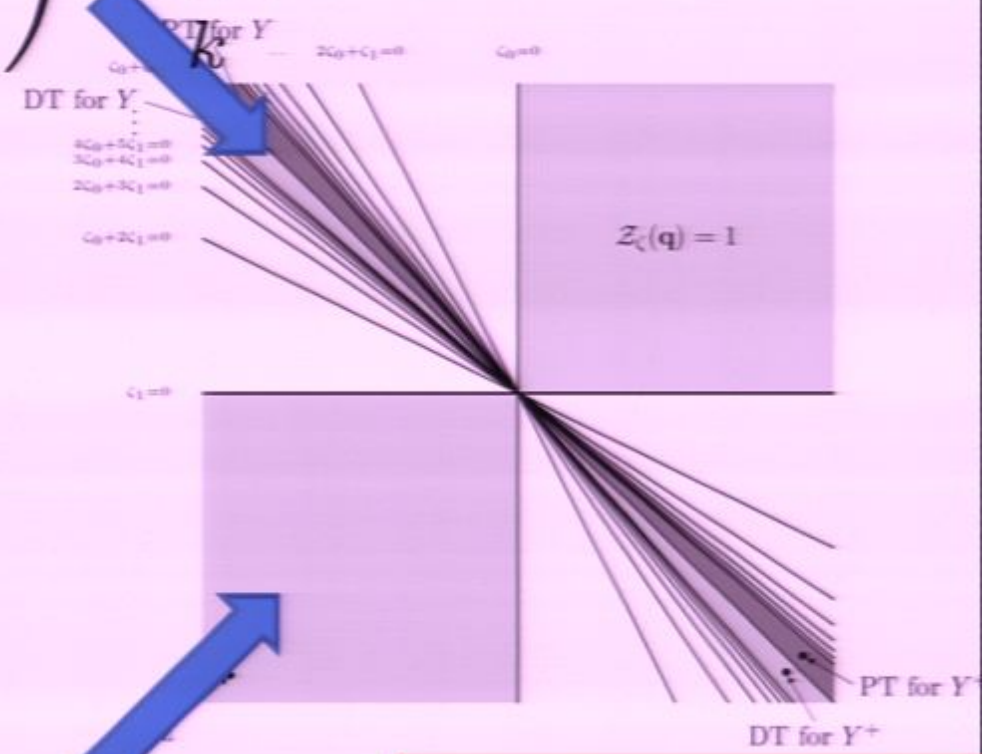

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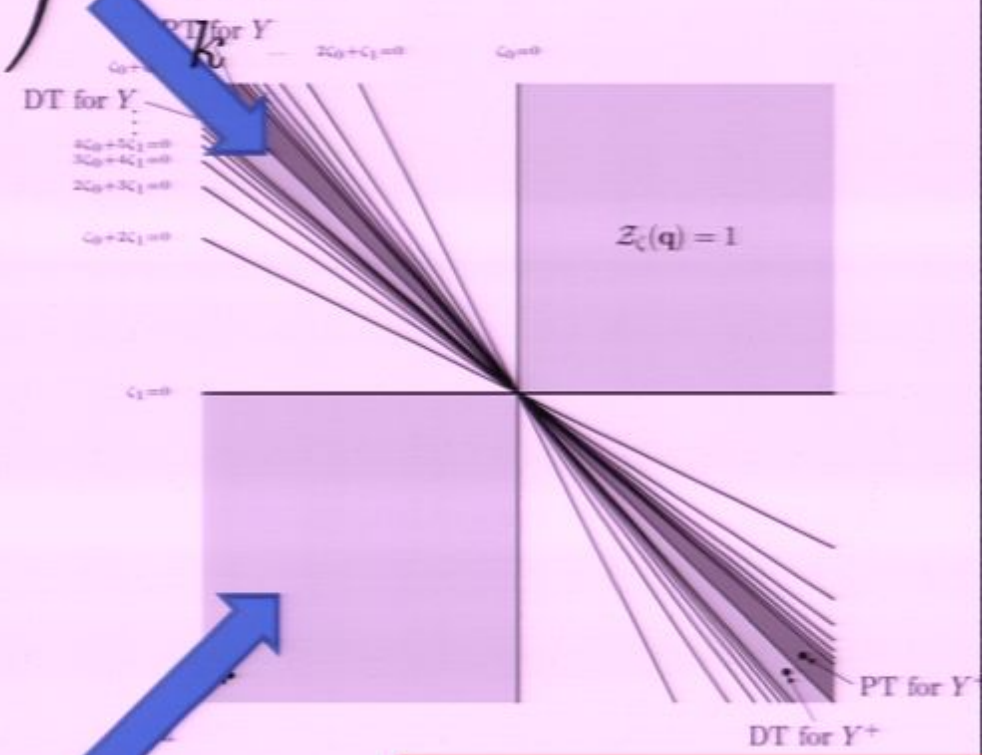

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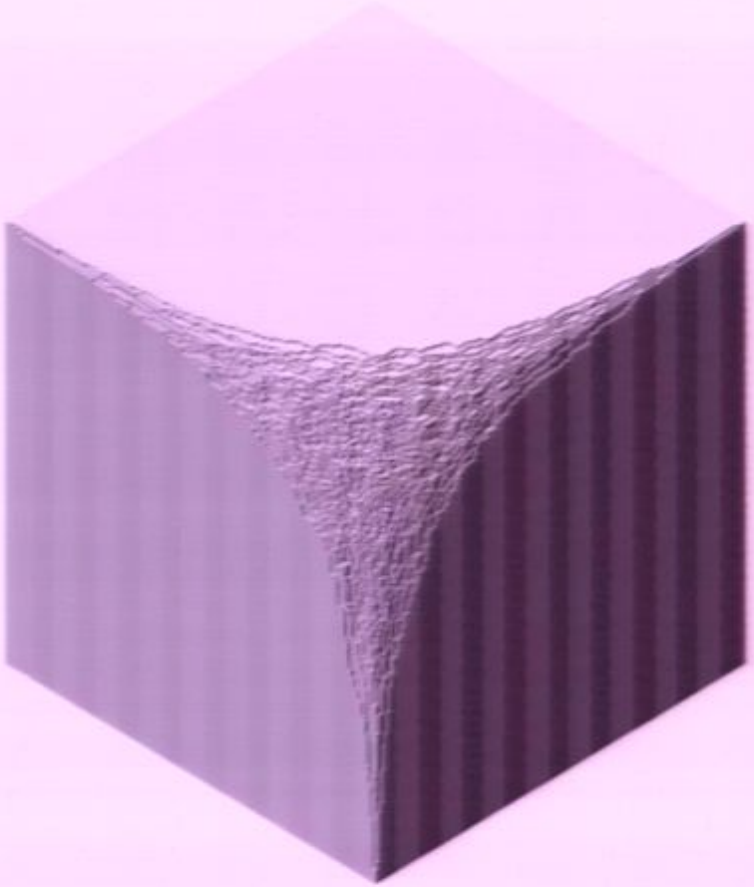

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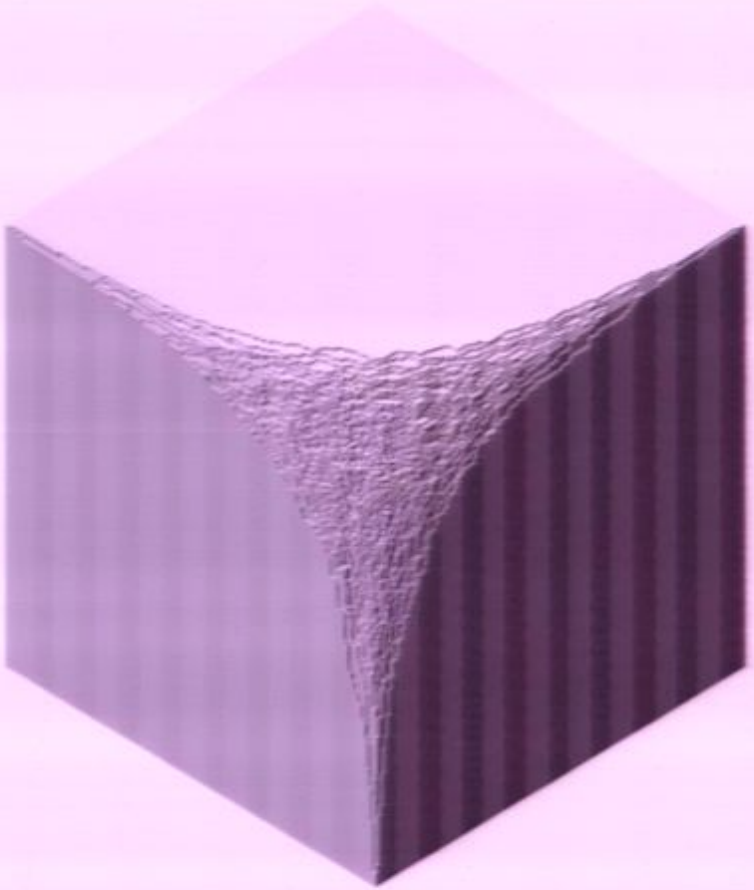

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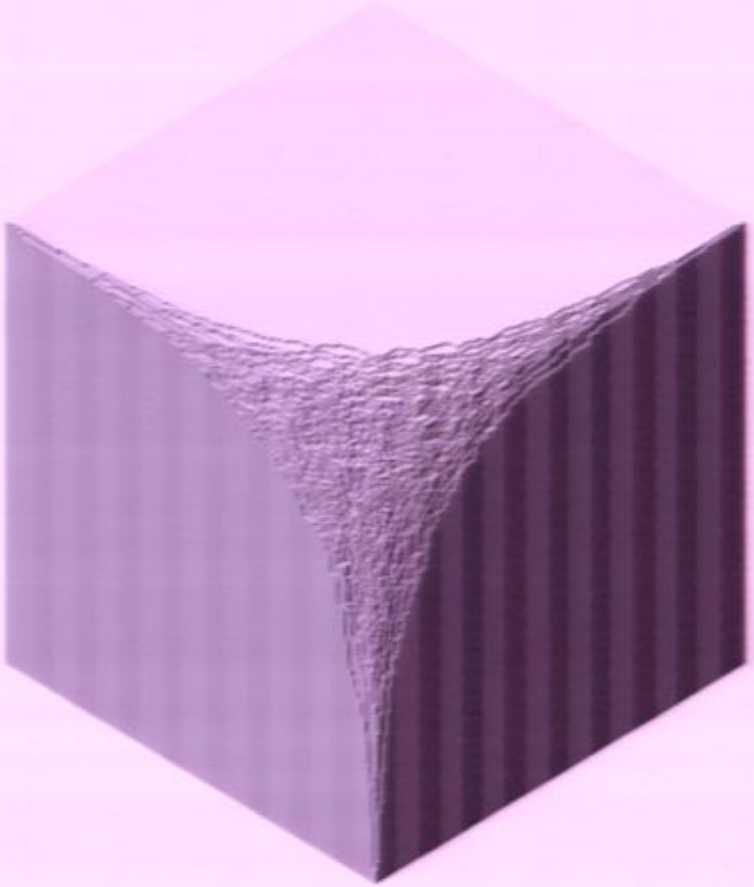

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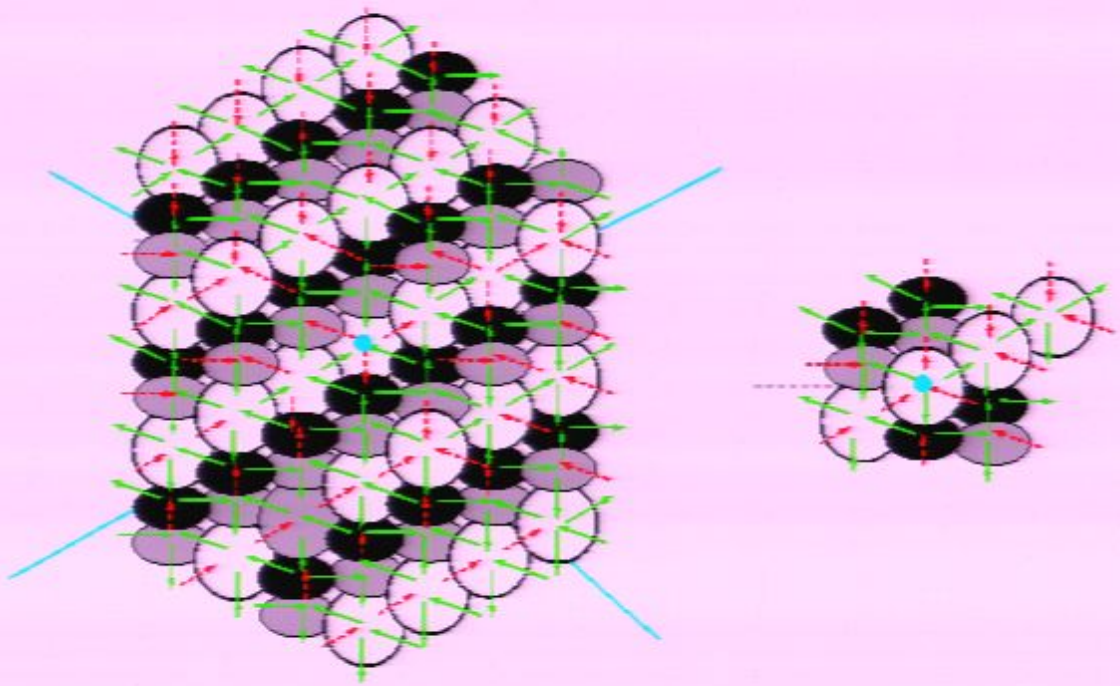

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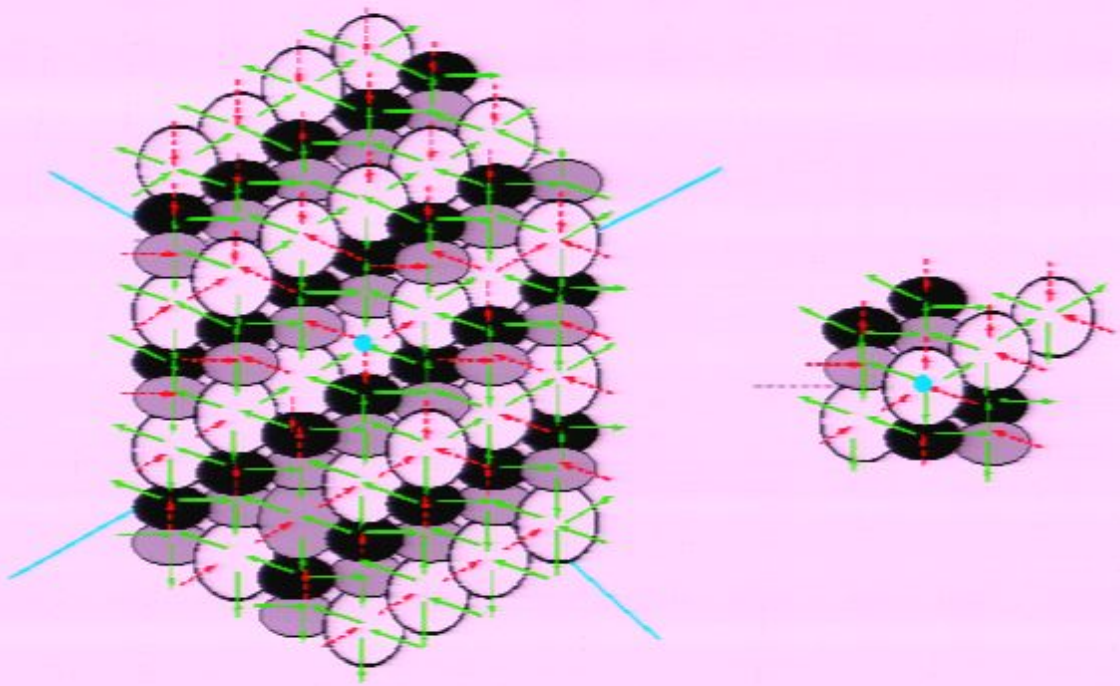

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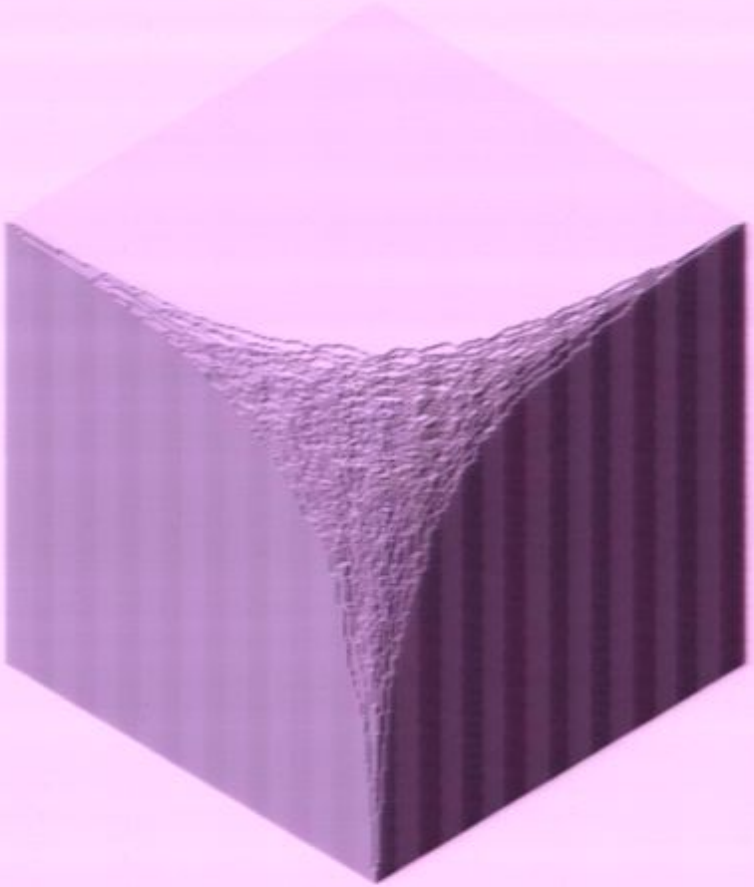

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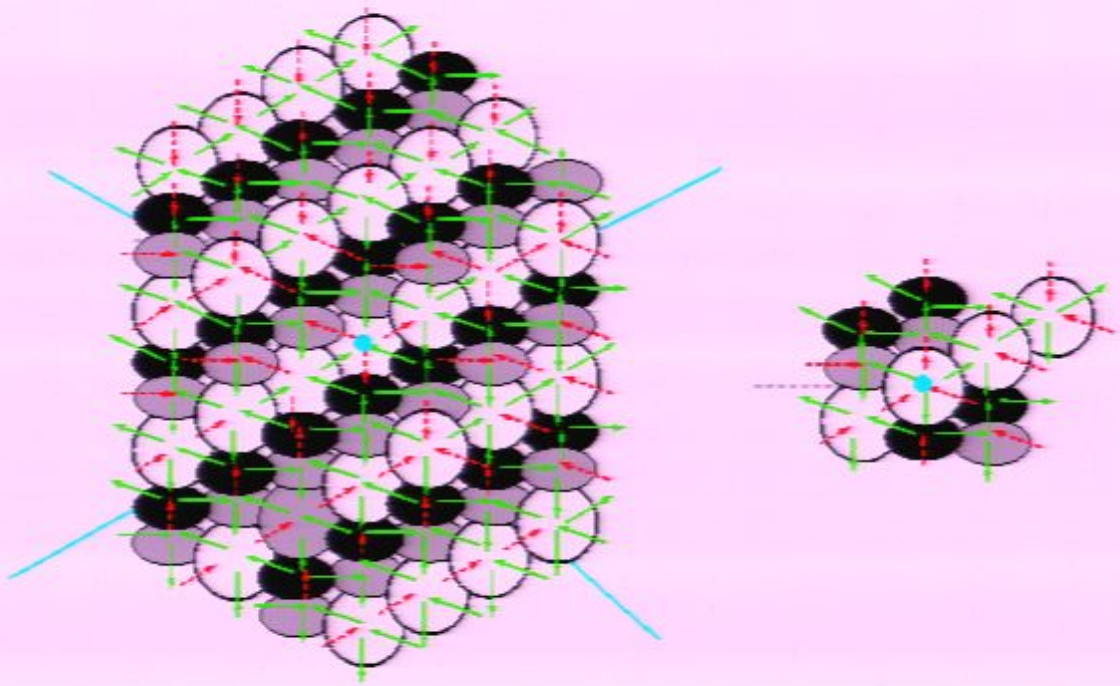

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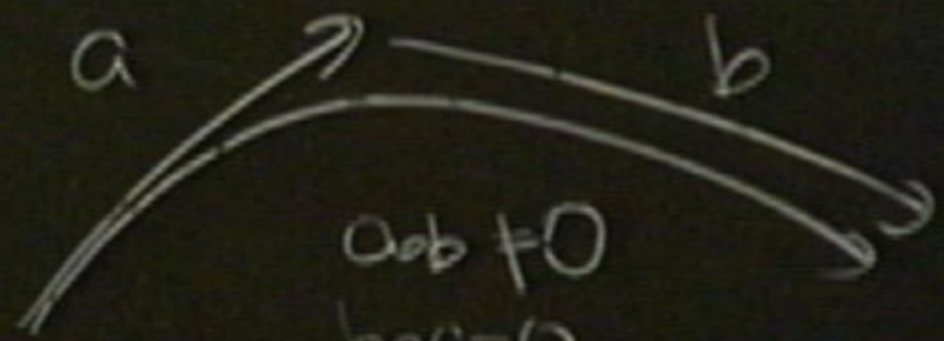

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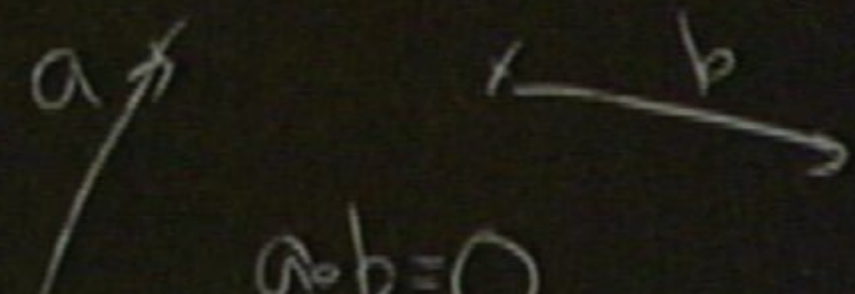

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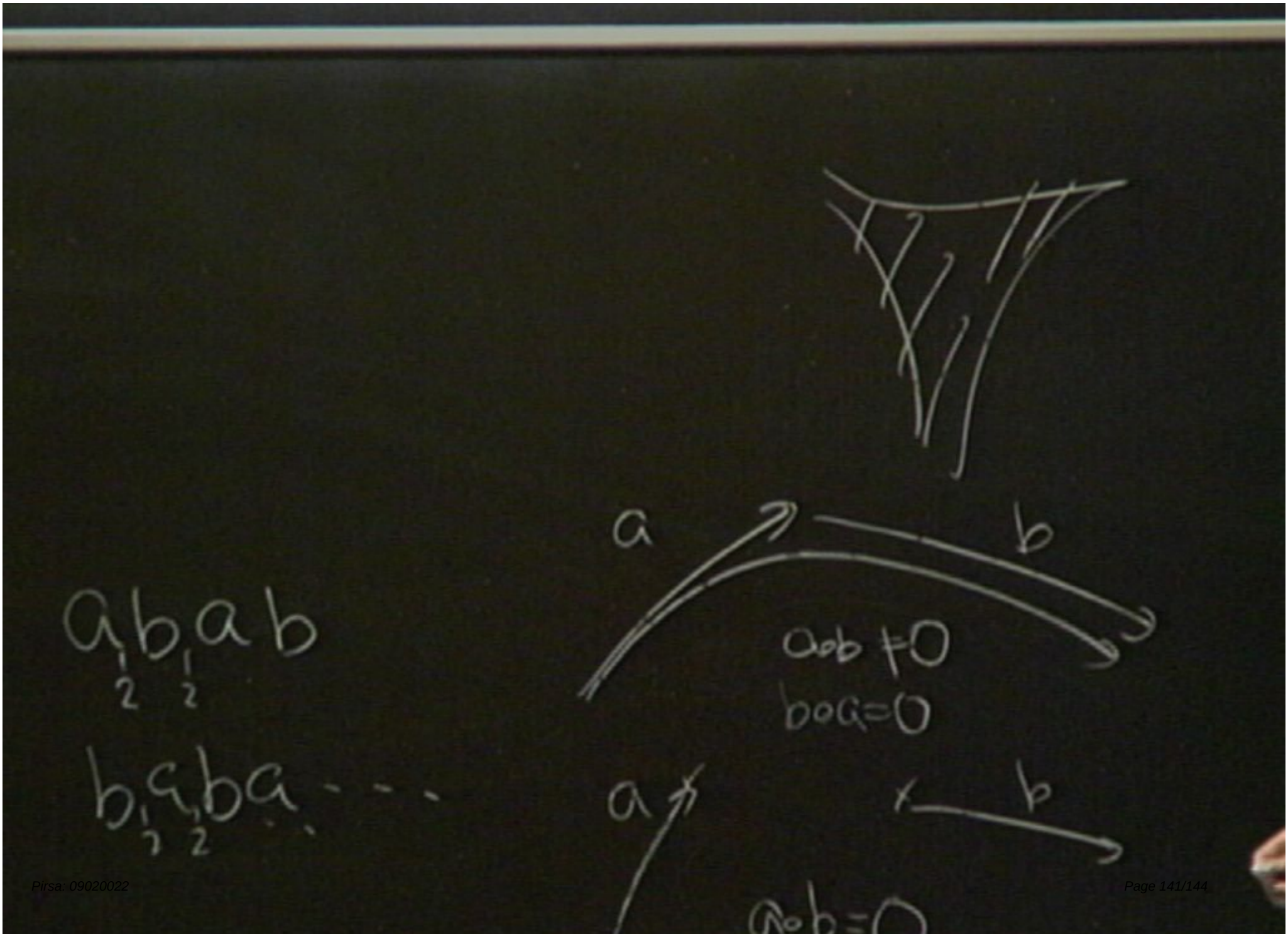
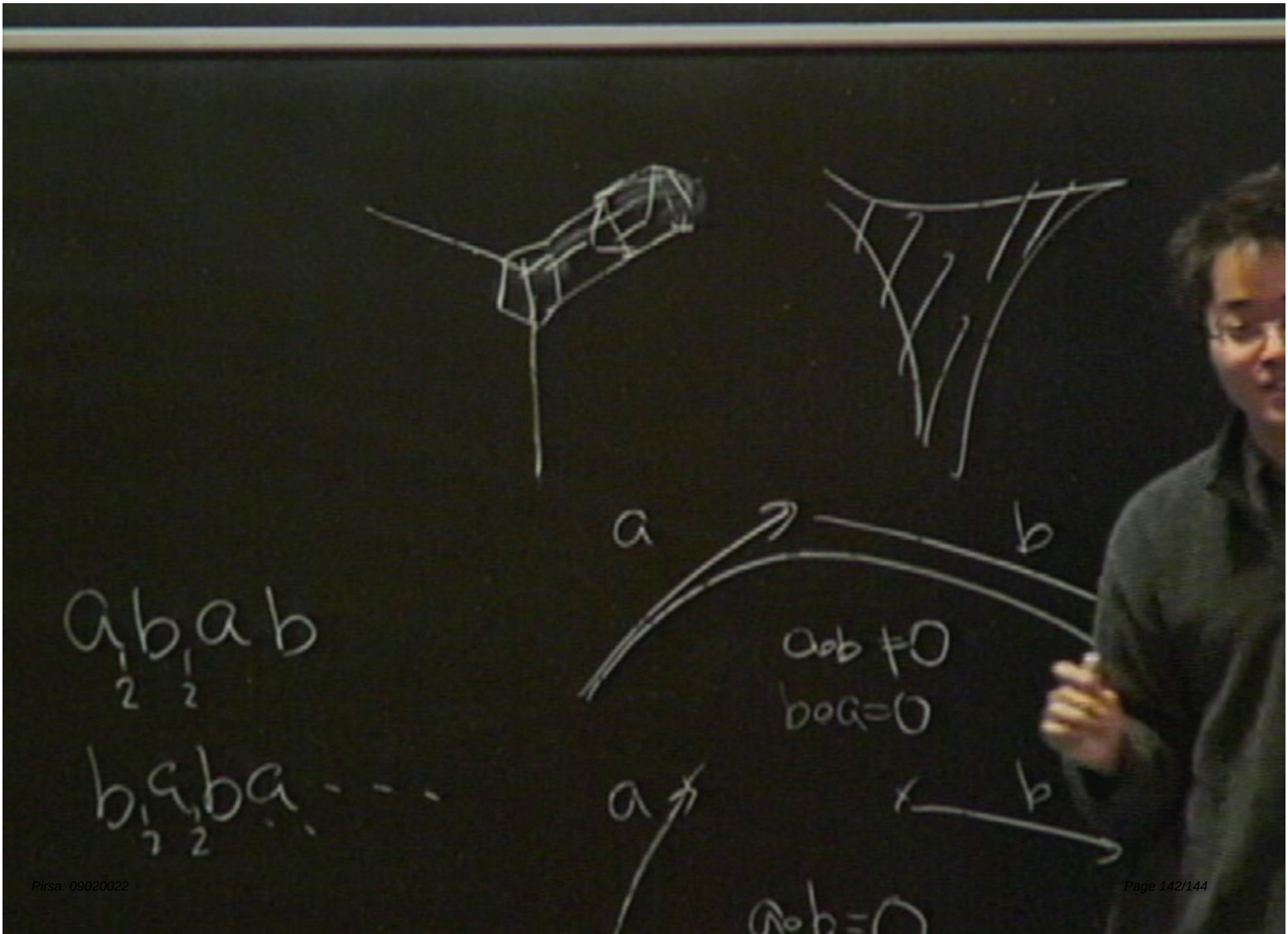

a
b
abab
aob ≠ 0
boa = 0
baba ...
a
b
aob = 0


a
b
a b a b
1 1
2 2
a∘b ≠ 0
b∘a = 0
b a b a - - -
1 1
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a
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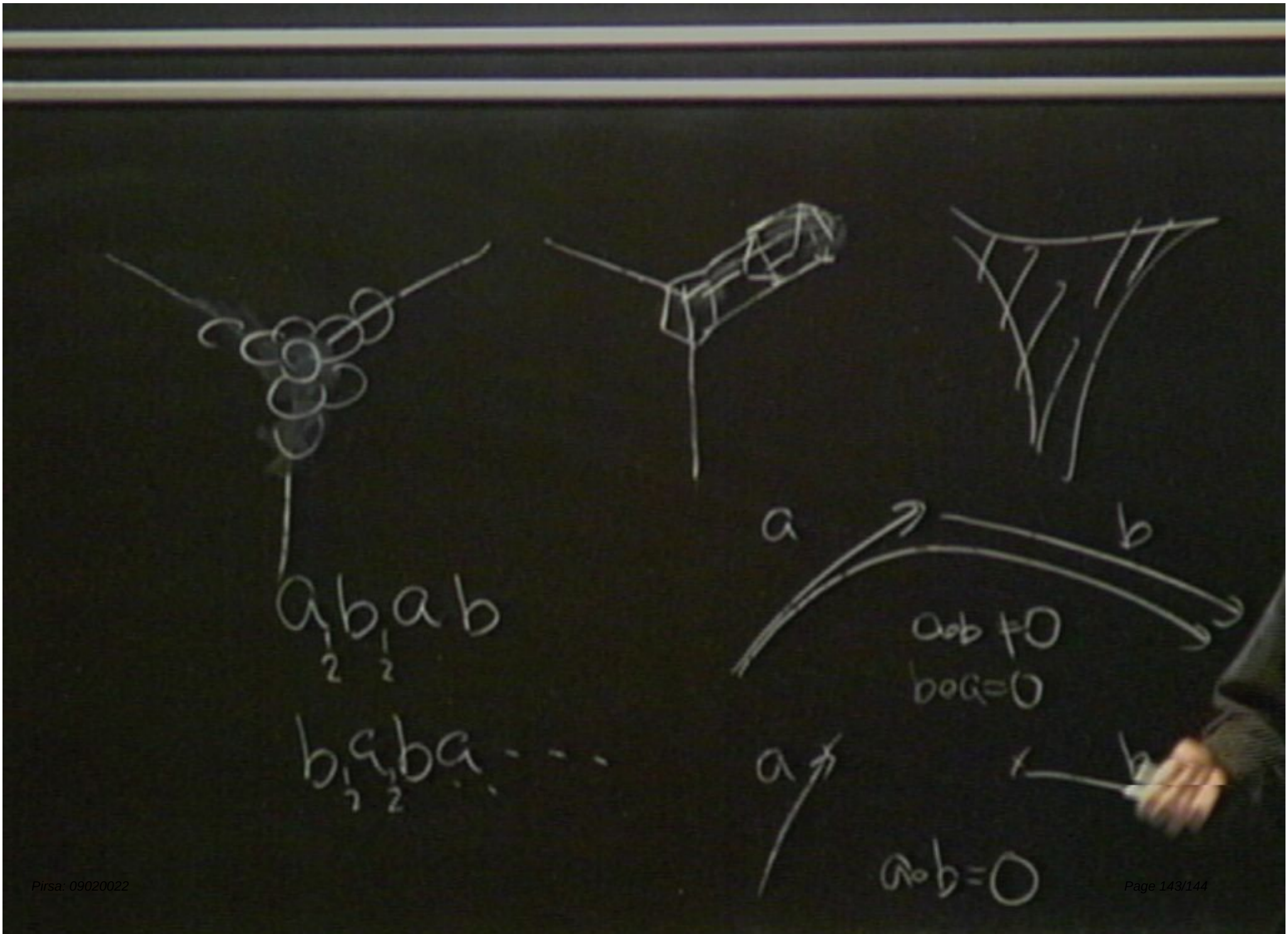
a b a b
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$a_1 b_1 a b$

$2 \quad 2$

$b_1 a_1 b a \cdots$

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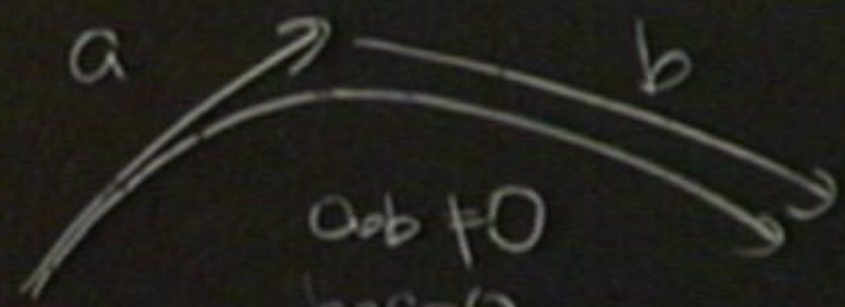

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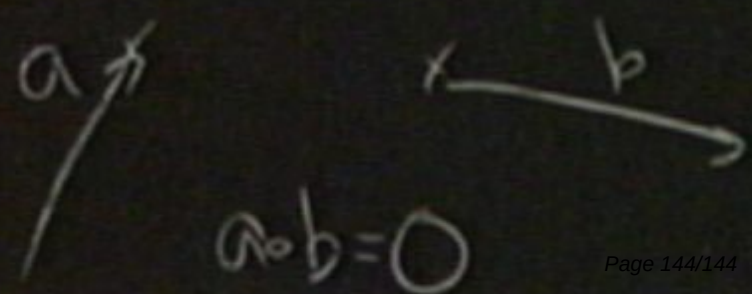

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