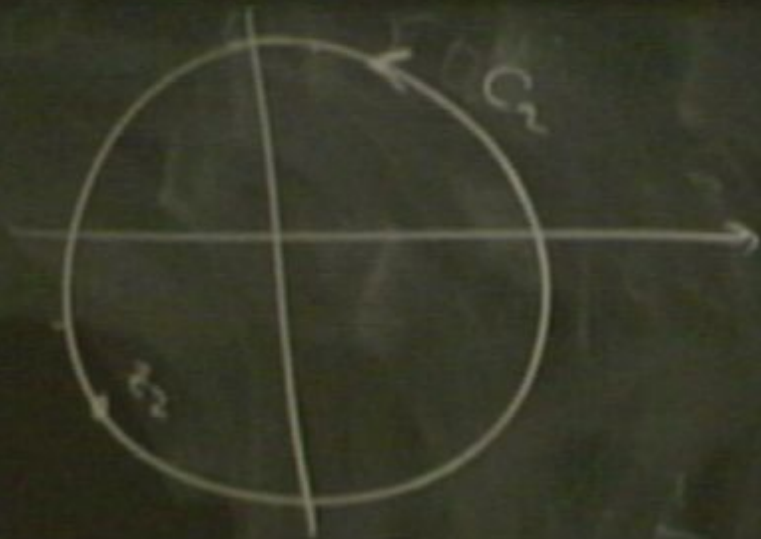


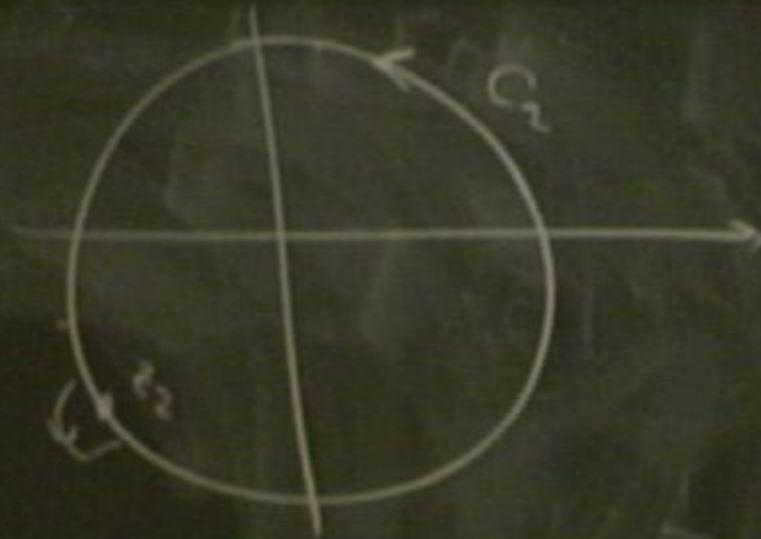
Title: Introduction to the Bosonic String Part B

Date: Feb 27, 2009 12:00 PM

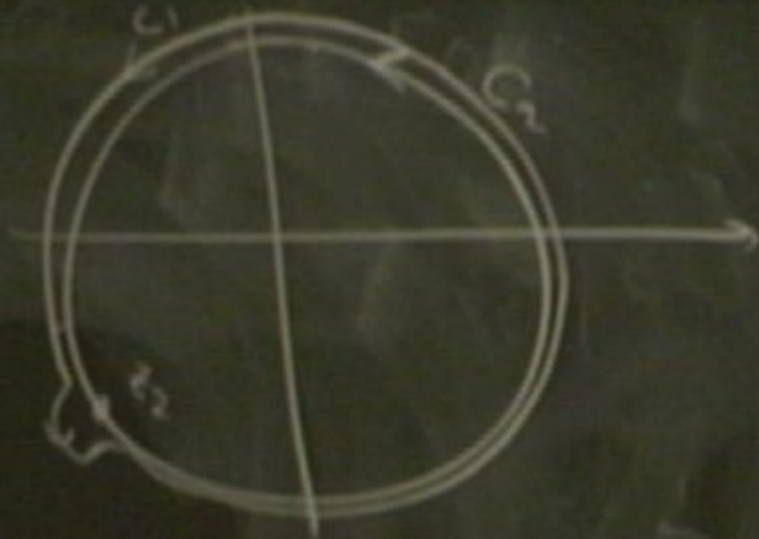
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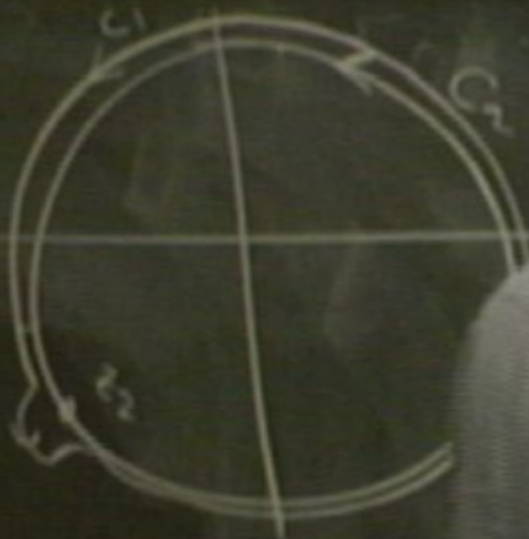
Abstract: This course provides a thorough introduction to the bosonic string based on the Polyakov path integral and conformal field theory. We introduce central ideas of string theory, the tools of conformal field theory, the Polyakov path integral, and the covariant quantization of the string. We discuss string interactions and cover the tree-level and one loop amplitudes. More advanced topics such as T-duality and D-branes will be taught as part of the course. The course is geared for M.Sc. and Ph.D. students enrolled in Collaborative Ph.D. Program in Theoretical Physics. Required previous course work: Quantum Field Theory (AM516 or equivalent). The course evaluation will be based on regular problem sets that will be handed in during the term. The primary text is the book: 'String theory. Vol. 1: An introduction to the bosonic string. J. Polchinski (Santa Barbara, KITP) . 1998. 402pp. Cambridge, UK: Univ. Pr. (1998) 402 p.' All interested students should contact Alex Buchel at abuchel@uwo.ca as soon as possible.





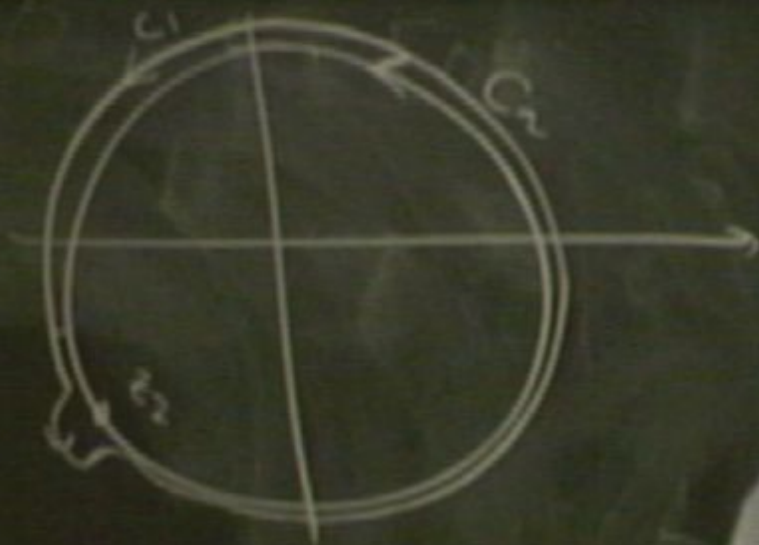






$$\frac{dz_1}{2\pi i}$$





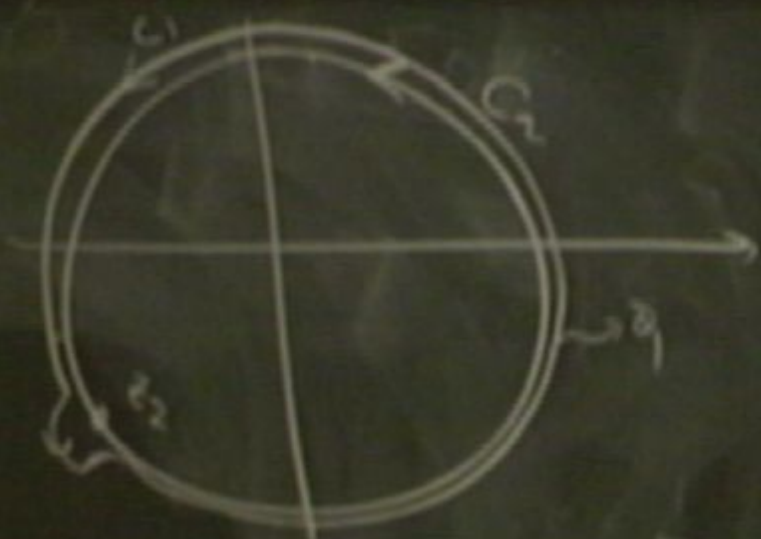
$$\int \frac{dz}{2\pi i}$$

$$f(z)$$



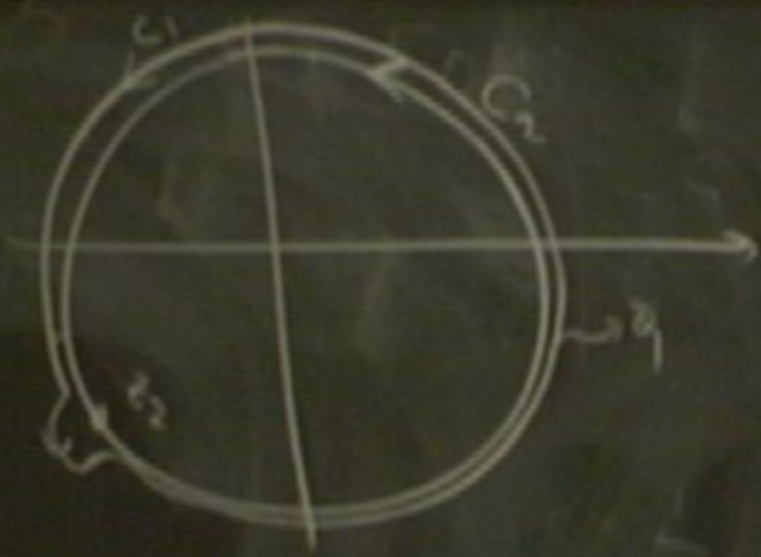
$$\int \frac{dz}{z - z_0} f_1(z)$$

$$= 2\pi i \sum \text{Res}(f_1(z), z_k)$$



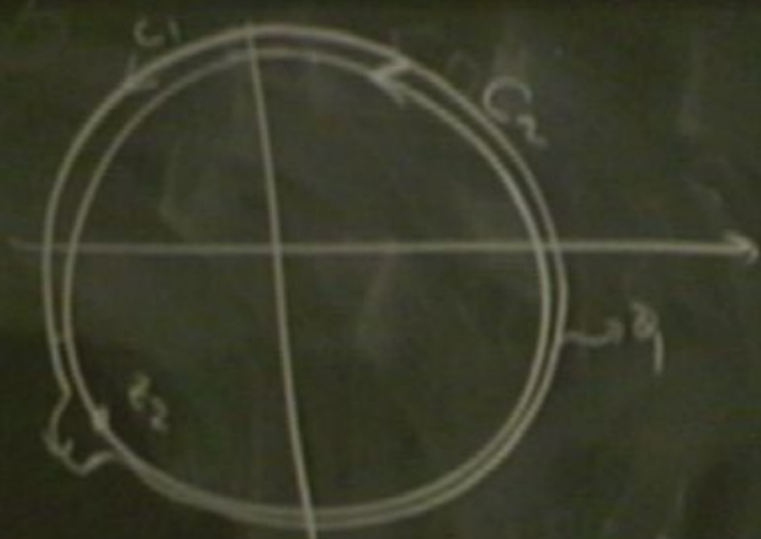
$$\int \frac{dz}{2\pi i} f(z)$$

$$= 2\pi i \sum \text{Res}(f(z))$$

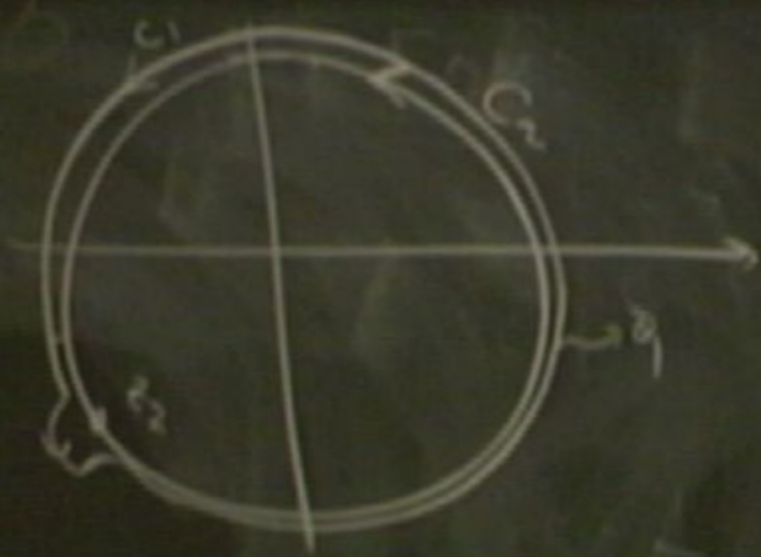


$$\int_{C_1} \frac{dz}{z-i} f_1(z)$$

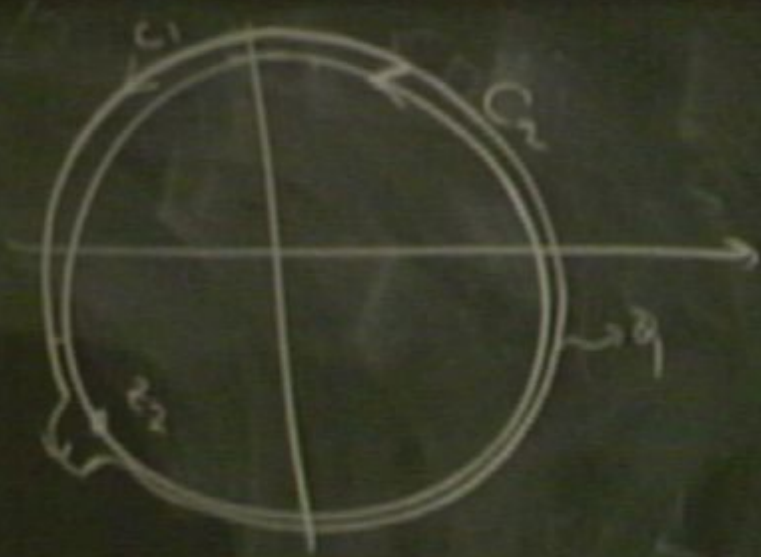
$$= 2\pi i f_1(i)$$



$$\int_{C_1} \frac{dz}{z-i} f_1(z) f_2(z)$$



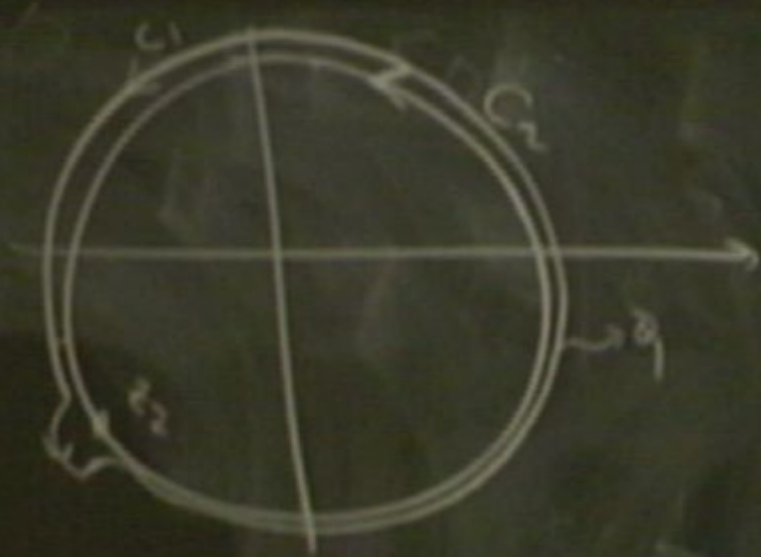
$$\int_{C_1} \frac{dz_1}{z_1} f_1(z_1) f_2(z_1) \quad \left\{ \int_{C_2} \frac{dz_2}{z_2} f_1(z_2) f_2(z_2) \right.$$



$$\int_{C_1} \frac{dz_1}{z_1 - i}$$

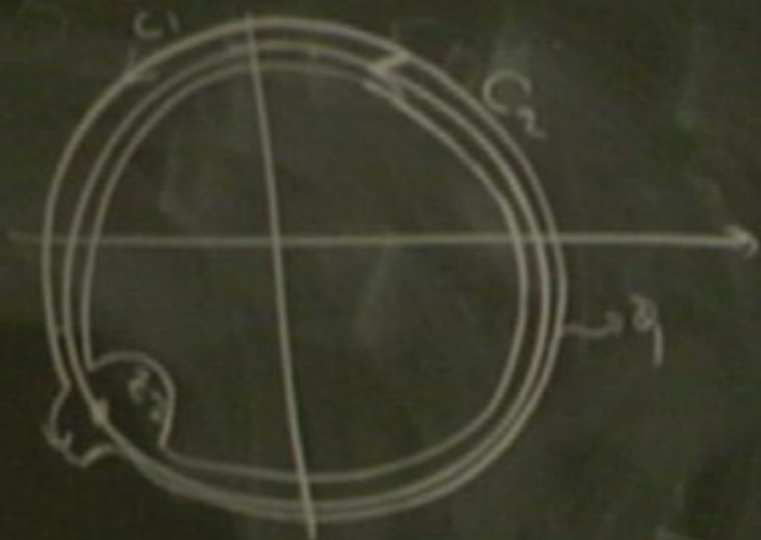
$$f(z)$$

$$\int_{C_2} \frac{dz_2}{z_2 - i}$$

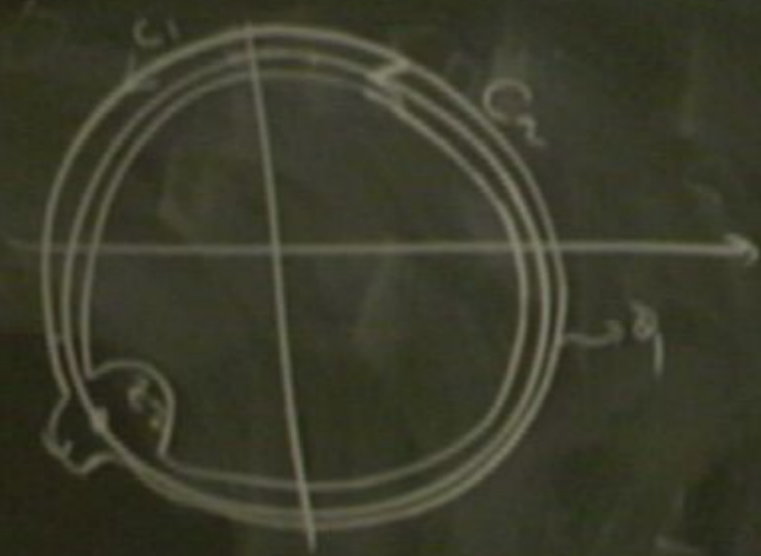


$$\int_{C_1} \frac{dz_1}{2\pi i} j_1(z_1) j_2(z_2) \left\{ \int_{C_2} \frac{dz_2}{2\pi i} \right.$$

$$\left. - \int_{C_3} \frac{dz_3}{2\pi i} j_1(z_3) j_2(z_4) \right\}$$



$$\left. \begin{aligned}
 & \int_{C_1} \frac{dz_1}{2\pi i} j_1(z_1) j_2(z_1) \\
 & - \int_{C_3} \frac{dz_3}{2\pi i} j_1(z_3) j_2(z_3)
 \end{aligned} \right\} \int \frac{dz_2}{2\pi i}$$



$$\int_{C_1} \frac{dz_1}{2\pi i} j_1(z_1) j_2(z_1) \left. \right\} \int \frac{dz_2}{2\pi i}$$

$$+ \int_{-C_3} \frac{dz_3}{2\pi i} j_1(z_3) j_2(z_3)$$



$$\int_{C_1} \frac{dz_1}{z_1} j_1(z_1) j_2(z_2) \left. \vphantom{\int_{C_1}} \right\} \int \frac{dz_2}{z_2}$$

$$+ \int_{-C_3} \frac{dz_3}{z_3} j_1(z_3) j_2(z_2)$$



$$\int_{C_1} \frac{dz_1}{2\pi i} j_1(z_1) j_2(z_1) \quad \left. \vphantom{\int_{C_1}} \right\} \int_{C_2} \frac{dz_2}{2\pi i}$$

$$+ \int_{-C_3} \frac{dz_3}{2\pi i} j_1(z_3) j_2(z_3)$$



$$\left. \int_{C_1} \frac{dz_1}{2\pi i} j_1(z_1) j_2(z_1) \right\} \frac{dz_2}{2\pi i}$$

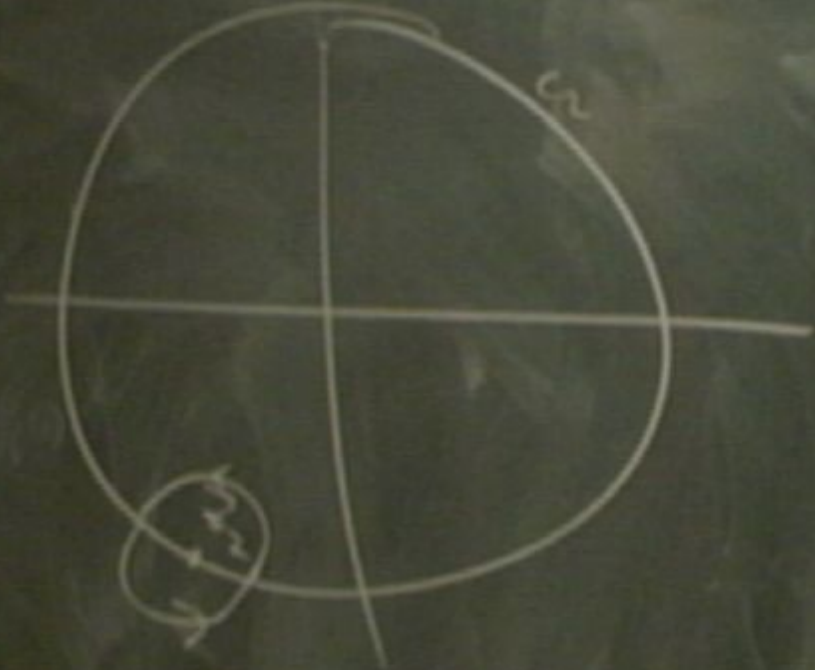
$$+ \int_{-C_3} \frac{dz_3}{2\pi i} j_1(z_3) j_2(z_2)$$



$$\int_{C_1} \frac{dz_1}{z_1} - \int_{C_2} \frac{dz_2}{z_2}$$

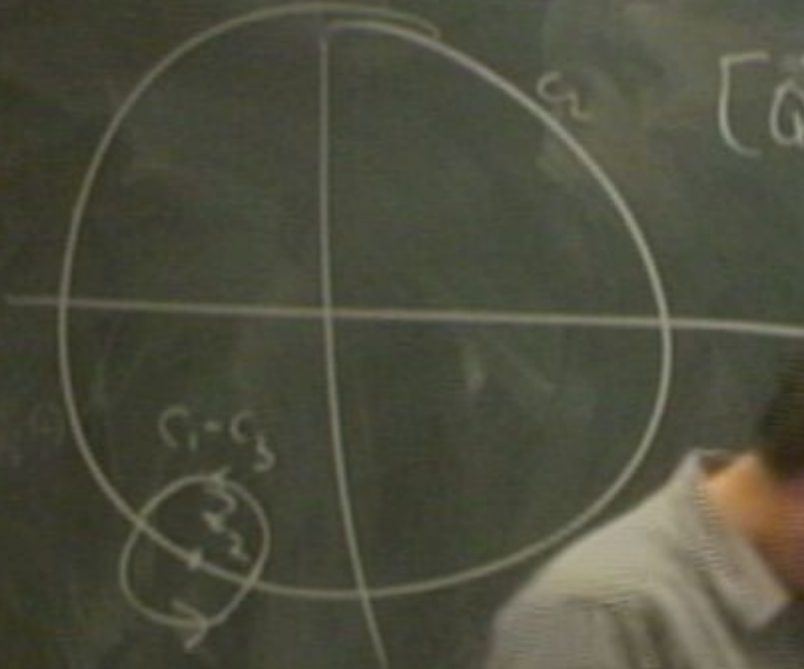
$$= \int_{C_1} \frac{dz_1}{z_1} + \int_{-C_2} \frac{dz_2}{z_2}$$

$$= \int_{C_1 \cup (-C_2)} \frac{dz}{z}$$









c_2

$$[\hat{a}_1, \hat{a}_2] \{c_2\} =$$





$$[\hat{Q}_1, \hat{Q}_2] \{c_2\} = \int \frac{d^2 z_2}{2\pi i}$$



$$[\hat{Q}_1, \hat{Q}_2] \{C_2\} = \oint_{C_2} \frac{d^2z}{m_i} \oint_{C_3} \phi$$



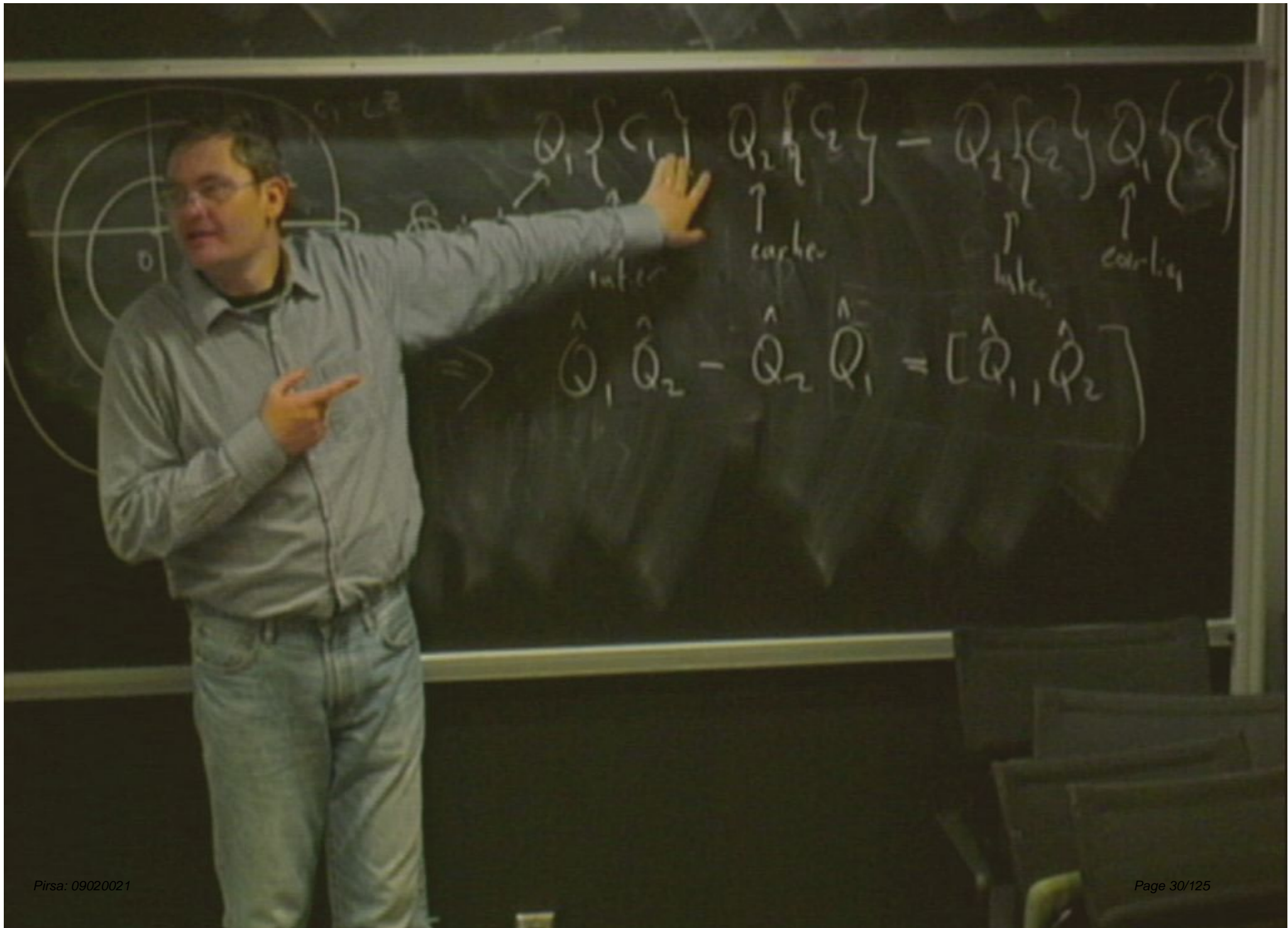
$$[\hat{Q}_1, \hat{Q}_2] \{c_2\} = \oint_{c_2} \frac{dz_2}{2\pi i} \phi$$



$$[\hat{Q}_1, \hat{Q}_2] \{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1=C_3} \frac{d\vec{z}_1}{2\pi i} j_1(z_1) j_2(z_2)$$



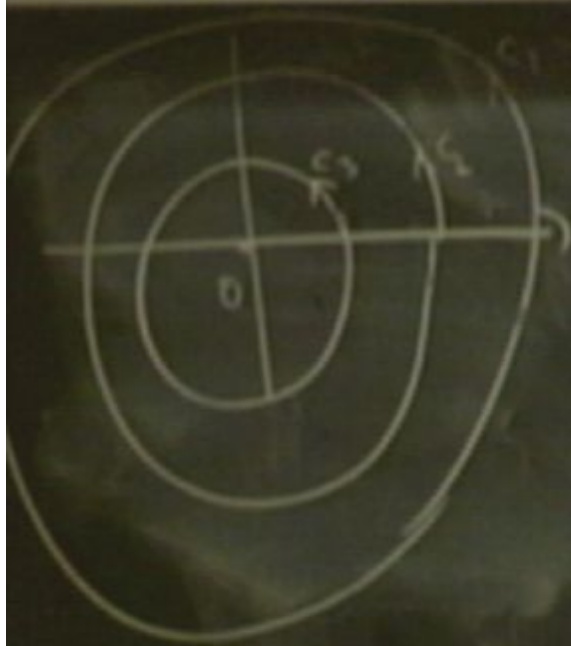
$$[\hat{Q}_1, \hat{Q}_2] \{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1-C_3} \frac{dz_1}{2\pi i} j_1(z_1) j_2(z_2)$$



$$Q_1 \{ C_1 \} - Q_2 \{ C_2 \} - Q_2 \{ C_2 \} Q_1 \{ C_1 \}$$

↑ later
↑ earlier
↑ later
↑ earlier

$$\Rightarrow \hat{Q}_1 \hat{Q}_2 - \hat{Q}_2 \hat{Q}_1 = [\hat{Q}_1, \hat{Q}_2]$$



$$\begin{aligned}
 & \hat{Q}_1 \hat{Q}_2 - \hat{Q}_2 \hat{Q}_1 = [\hat{Q}_1, \hat{Q}_2] \\
 & \begin{array}{ccccccc}
 & \hat{Q}_1 \{C_1\} & \hat{Q}_2 \{C_2\} & - & \hat{Q}_2 \{C_2\} & \hat{Q}_1 \{C_1\} & \\
 & \uparrow & \uparrow & & \uparrow & \uparrow & \\
 & \text{later} & \text{earlier} & & \text{later} & \text{earlier} & \\
 \Rightarrow & & & & & &
 \end{array}
 \end{aligned}$$

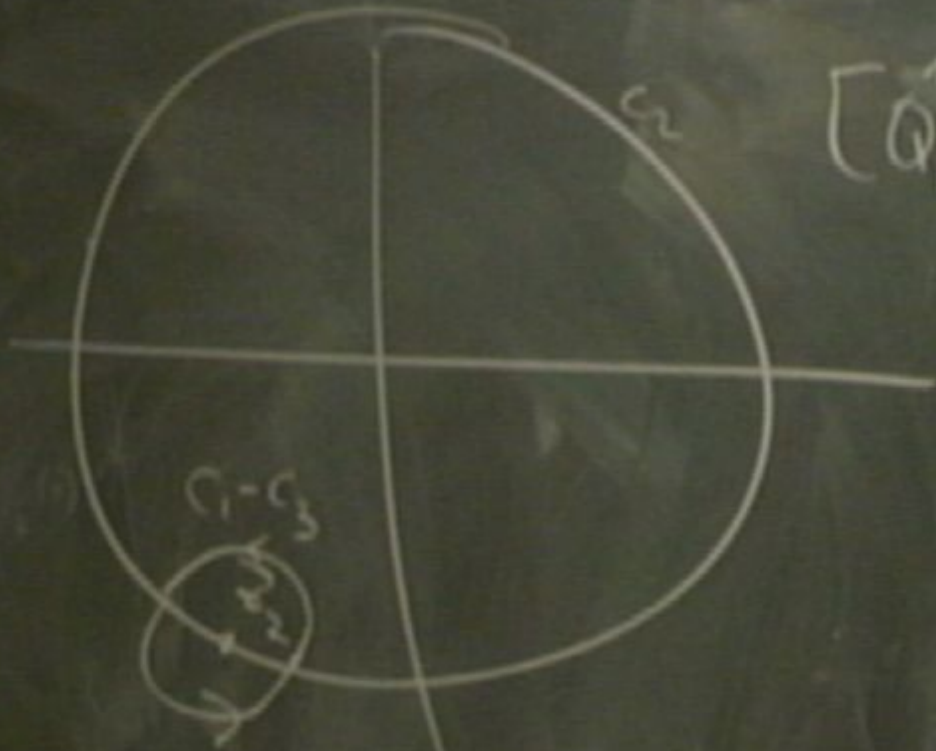


$$[\hat{Q}_1, \hat{Q}_2] \{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1 - C_3} \frac{dz_1}{2\pi i} j_1(z_1) j_2(z_2)$$



$$[\hat{Q}_1, \hat{Q}_2] \{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1-C_3} \frac{dz_1}{2\pi i} j(z_1) j(z_2)$$

↑
evaluate path by



$$[\hat{Q}_1, \hat{Q}_2] \{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1-C_3} \frac{dz_1}{2\pi i} j(z_1) j(z_2)$$

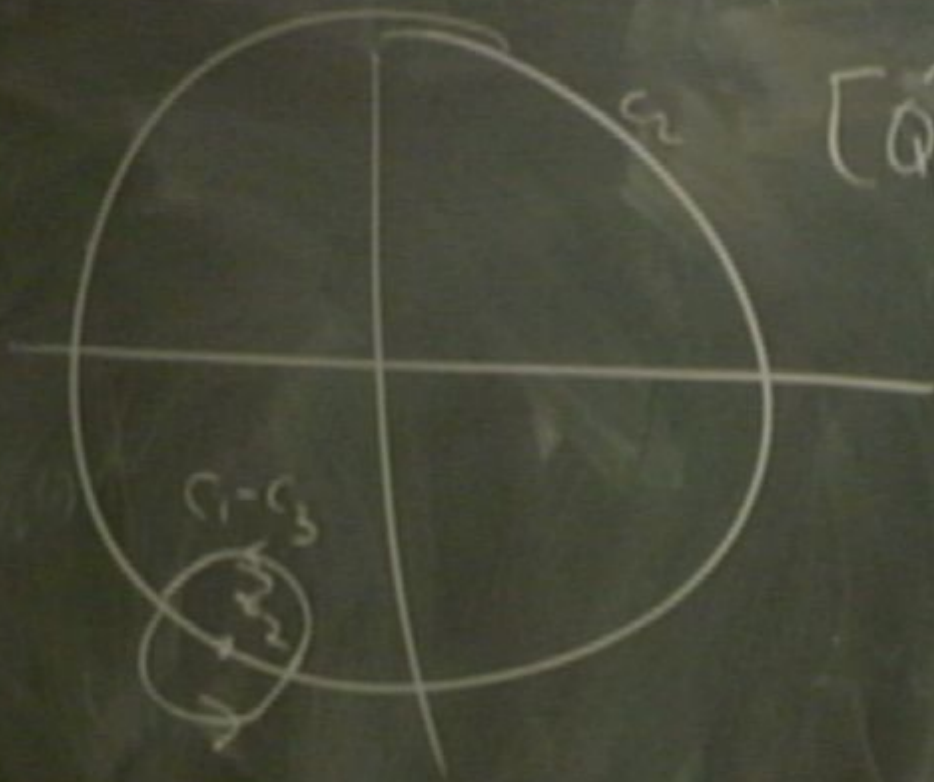
↑
evaluate path by



$$[\hat{Q}_1, \hat{Q}_2] \{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1-C_3} \frac{dz_1}{2\pi i} j(z_1) j(z_2)$$

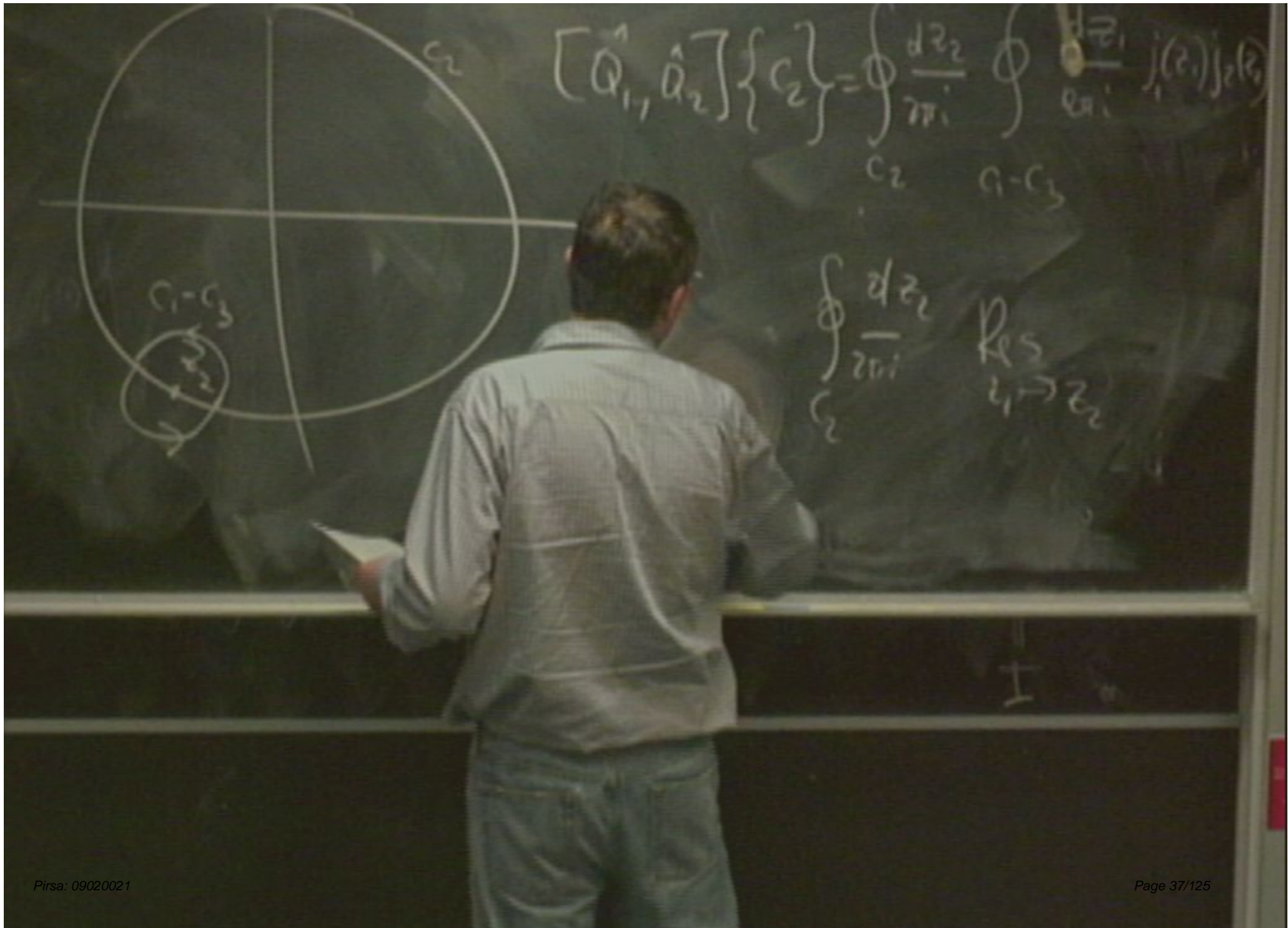
↑
Evaluate
path by C_2

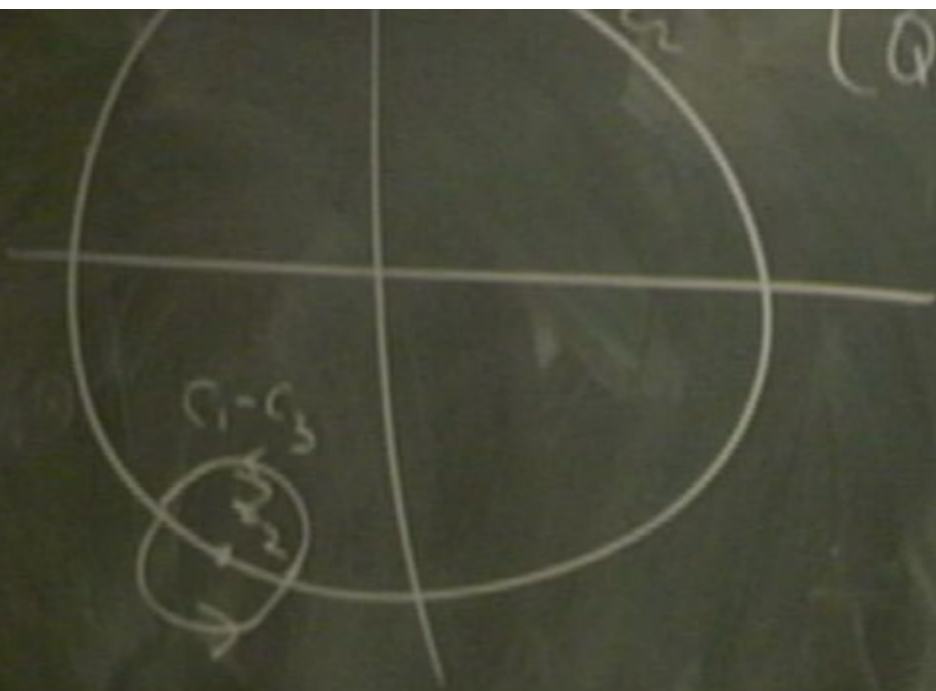
$$\oint \frac{dz_2}{2\pi i}$$



$$[\hat{Q}_1, \hat{Q}_2] \{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1=C_3} \frac{dz_1}{2\pi i} j(z_1) j(z_2)$$

$\int \frac{dz_2}{2\pi i}$
 Evaluate path by C_2





$$(Q_1, Q_2) \left\{ C_2 \right\} = \oint_{C_2} \frac{f(z)}{z - c_3} dz$$

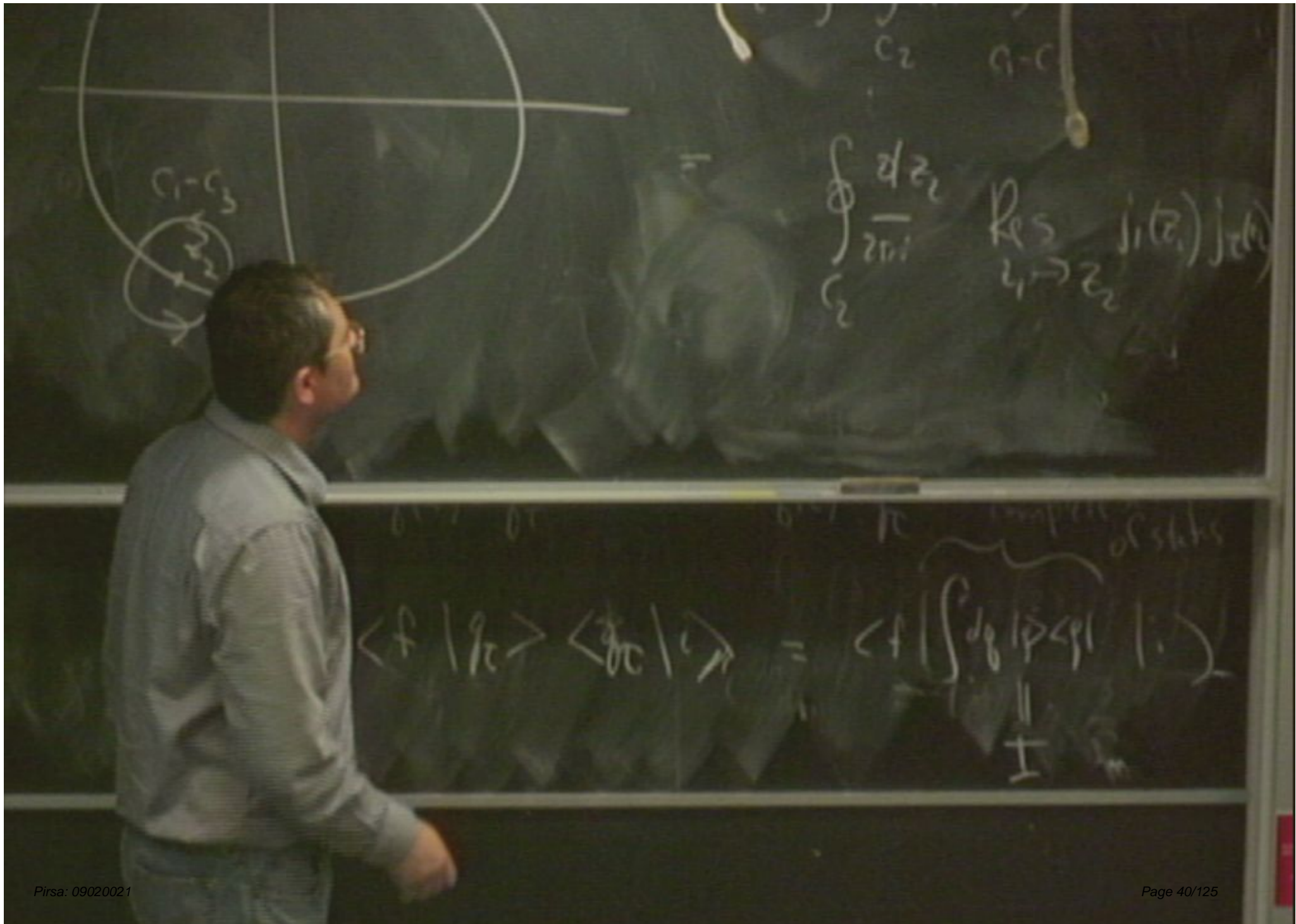
$$= \oint_{C_2} \frac{dz}{z - c_3}$$

Res
 $z_1 \rightarrow z_2$



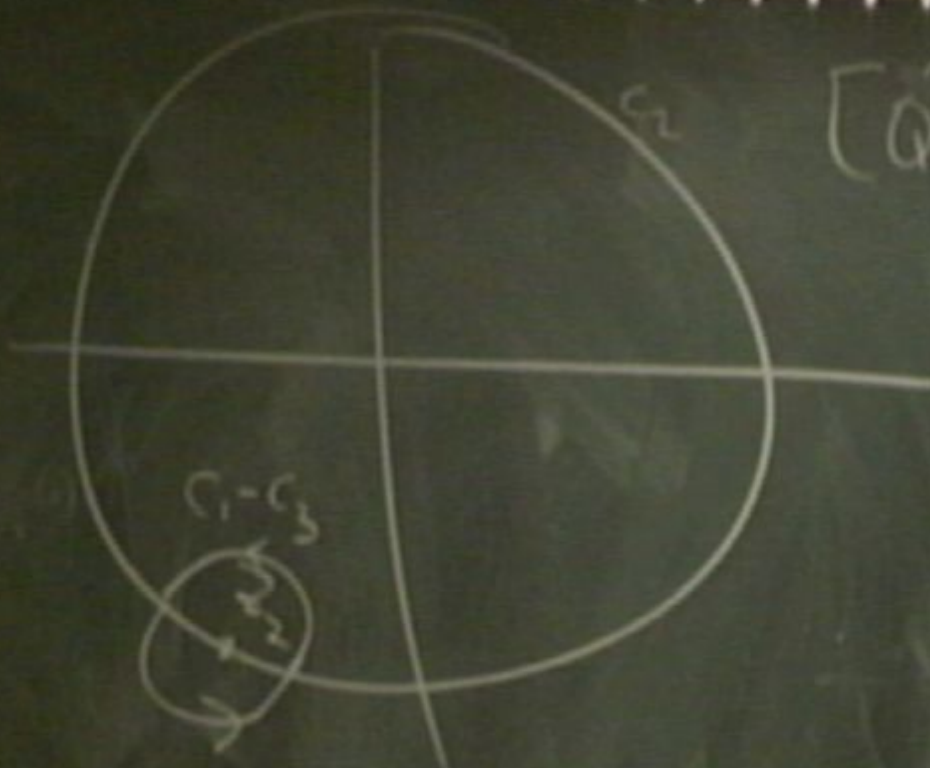
$$[\hat{Q}_1, \hat{Q}_2] \{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1-C_3} \frac{d\vec{z}_1}{2\pi i} j_1(z_1) j_2(z_2)$$

$$= \oint_{C_2} \frac{dz_2}{2\pi i} \text{Res}_{z_1 \rightarrow z_2} j_1(z_1) j_2(z_2)$$



$$\int_{\gamma} \frac{dz}{z} = 2\pi i \operatorname{Res}_{z_1 \rightarrow z_2} j_1(z_1) j_2(z_2)$$

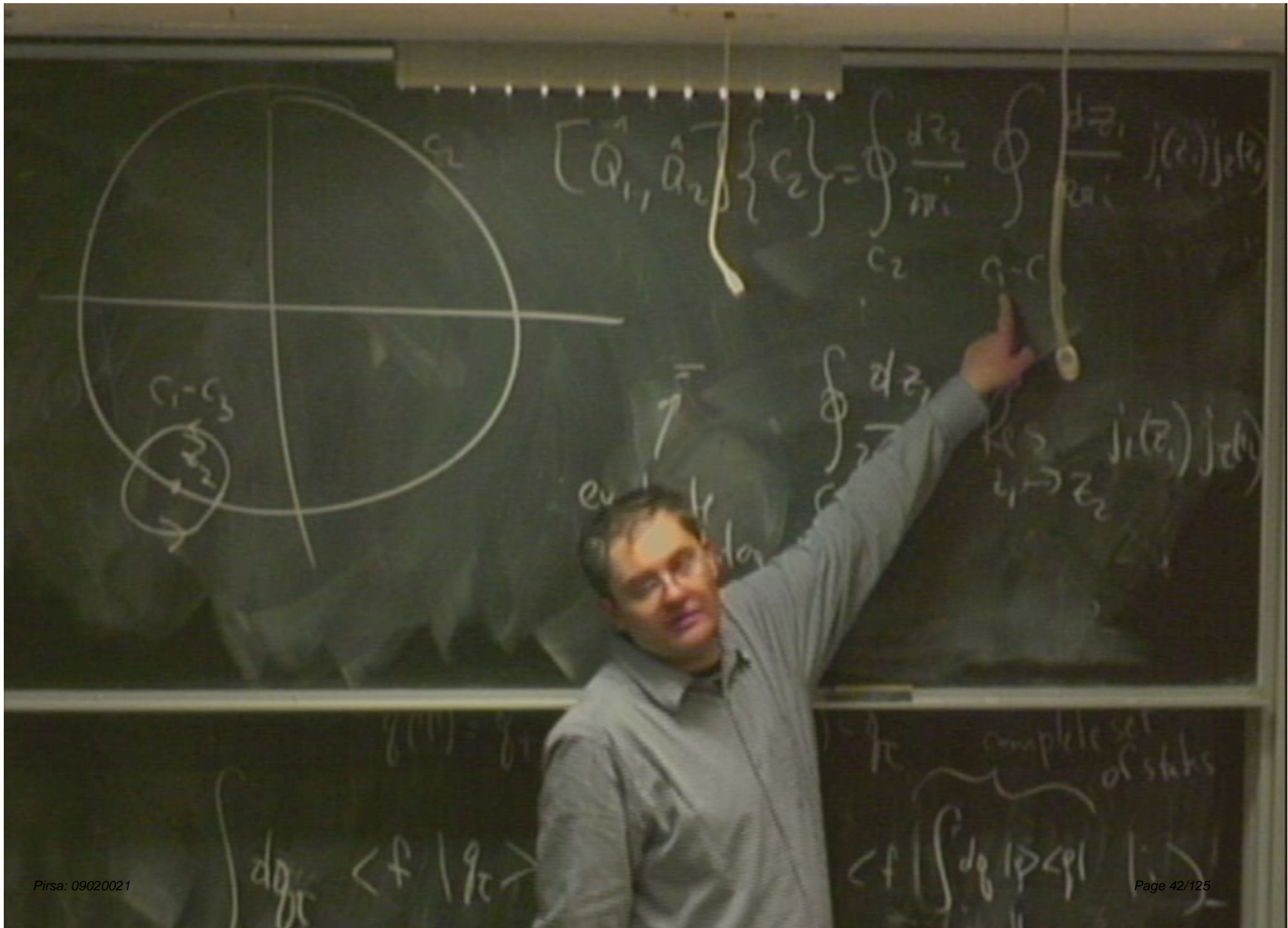
$$\langle f | g \rangle \langle g | f \rangle = \langle f | \int_{\gamma} dg |g\rangle \langle g| \cdot 1 \rangle$$



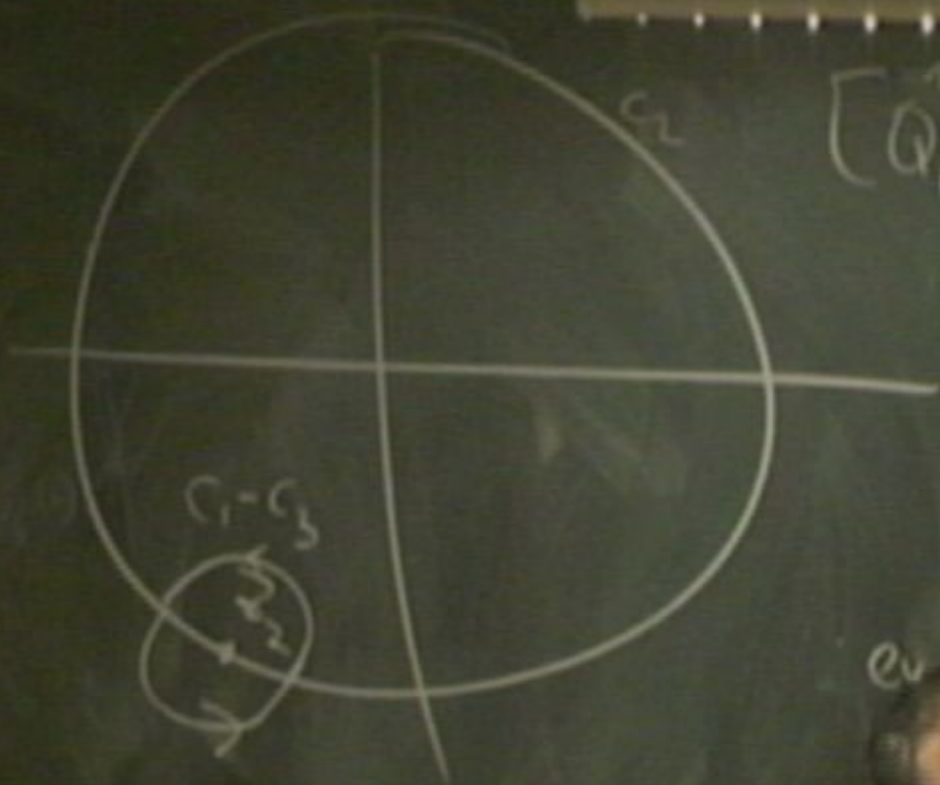
$$[Q_1, Q_2] \{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1-C} \frac{dz_1}{2\pi i} j(z_1) j(z_2)$$

$$= \oint \frac{dz_2}{2\pi i} \text{Res}_{z_1 \rightarrow z_2} j(z_1) j(z_2)$$





$$[Q_1, Q_2] \{c_2\} = \oint_{c_2} \frac{dz_2}{m_1} \oint_{c_1} \frac{dz_1}{m_1} j_1(z_1)j_2(z_2)$$



$$\oint_{c_2} \frac{dz_2}{m_1} \oint_{c_1} \frac{dz_1}{m_1} j_1(z_1)j_2(z_2)$$

$\text{Res}_{z_1} j_1(z_1)j_2(z_2)$
 $z_1 \rightarrow z_2$



$$\int dg_{\mu\nu} \langle f | \rho \rangle$$

complete set of states

$$\langle f | \int dg | \rho \rangle \langle \rho |$$



$$C_2 \quad \left[\hat{Q}_1, \hat{Q}_2 \right] \left\{ C_2 \right\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1-C_3} \frac{dz_1}{2\pi i} j_1(z_1) j_2(z_2)$$

$$\int_{C_2} \frac{dz_2}{2\pi i} \text{Res}_{z_2} j_1(z_1) j_2(z_2)$$

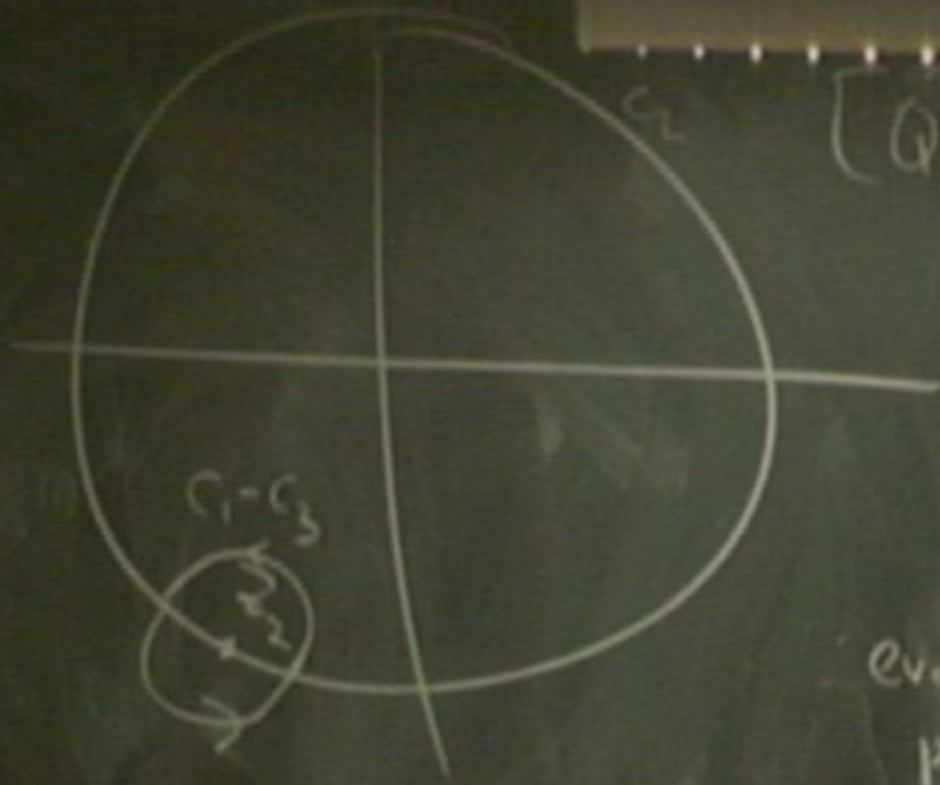
evaluate integral.



$$\int_{\mathcal{H}_g} dg \langle f | \rho \rangle$$

complete set of states

$$\langle f | \int dg | \rho \rangle \langle g | \dots \rangle$$



$$[Q_1, Q_2] \{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1-C_2} \frac{dz_1}{2\pi i} j(z_1) j(z_2)$$

evaluate path integral.

$$\oint \frac{dz_2}{2\pi i} \text{Res}_{z_1 \rightarrow z_2} j(z_1) j(z_2)$$

$$g(\tau) = g_\tau$$

$$g(\tau) = g_\tau$$

complete set of states

$$\int dg_\tau \langle f | g_\tau \rangle \langle g_\tau | i \rangle = \langle f | \int dg | p \rangle \langle p | i \rangle$$

\hat{Q}

H

$$[\hat{Q}, \hat{A}(z_1, \bar{z}_2)] = \operatorname{Res}_{z_1 \rightarrow z_2} j(z_1) A(z_1, \bar{z}_2)$$

evaluate path integral.

$$\oint \frac{dz}{z}$$

Res z_1 z_2 $j_1(z_1) j_2(z_2)$

$$\hat{\mathcal{A}}(z_1, \bar{z}_2) = \text{Res}_{z \rightarrow z_1} j(z) A(z, \bar{z}_2)$$

$\oint_{\gamma} \frac{dz}{z}$
 evaluate path integral.
 $\text{Res}_{z_1} j_1(z_1) j_2(z_2)$

$\hat{A}(z_1, \bar{z}_2) = \text{Res}_{z_1 \rightarrow z_2} j(z_1) A(z_1, \bar{z}_2)$

$\hat{Q}, \hat{A}(z_1, \bar{z}_2) \Big] = \text{Re} z \int_{z_1 \rightarrow \bar{z}_2} j(z) A(z_1, \bar{z}_2)$
 Studied in Ward identities before
 $= \frac{1}{i\epsilon} \delta A(z_1, \bar{z}_2)$

$$[\hat{Q}, \hat{A}(z_1, \bar{z}_2)] = \text{Res}_{z_1 \rightarrow z_2} j(z_1) A(z_2, \bar{z}_2)$$

↓ studied in Ward identities before

$$= i \frac{1}{\epsilon} \delta A(\bar{z}_2, \bar{z}_2)$$

OPE \Rightarrow computer algebra!

$$j_m = z^{m+1} T(z)$$

$j_m = z^{m+1} T(z)$ is a conserved current.

$$Q_m = \oint \frac{dz}{2\pi i} j_m = \mathbb{H}$$

$j_m = z^{m+1} T(z)$ is a conserved current.

$$Q_m = \oint \frac{dz}{2\pi i} j_m = \oint \frac{dz}{2\pi i} z^{m+2} T(z)$$

$J_m = z^{m+1} T(z)$ is a conserved current.

$$Q_m = \oint \frac{dz}{2\pi i} J_m = \oint \frac{dz}{2\pi i} z^{m+2} T(z) \equiv L_m$$

Virasoro
generators
↓

$$[L_m, L_n] = \oint \frac{dz_2}{2\pi i} \text{Res}_{z_1 \rightarrow z_2} J_m(z_1) J_n(z_2)$$



$J_m = z^{m+1} \pi(z)$ is a conserved current.

$$Q_m = \oint \frac{dz}{2\pi i} J_m = \oint \frac{dz}{2\pi i} z^{m+2} \pi(z) \equiv L_m$$

Virasoro
generators
↓

$$[L_m, L_n] = \oint \frac{dz_2}{2\pi i} \text{Res}_{z_1 \rightarrow z_2} J_m(z_1) J_n(z_2)$$

$$\text{Res } j_n = \text{Res } z_1^{m+1} z_2^{n+1} \pi(z_1) \pi(z_2)$$

$$\text{Res } j_m j_n = \text{Res}_{\underbrace{z_1^{m+1} z_2^{n+1}}_{z_1 \rightarrow z_2}} \pi(z_1) \pi(z_2)$$

$$= \text{Res}_{z_1 \rightarrow z_2} \left. \begin{matrix} z_1^{m+1} z_2 \\ z_1 \end{matrix} \right\}$$

$$\text{Res } j_n = \text{Res}_{\underbrace{z_1^{m+1} z_2^{n+1}}_{z_1, z_2}} \pi(z_1) \pi'(z_2)$$

$$= \text{Res}_{z_1 \rightarrow z_1} \left\{ z_1^{m+1} z_2 \right.$$

$$\left. \text{Res}_{z_1 \rightarrow z_2} z_1^{m+1} z_2^{n+1} \right.$$

$$\text{Res } j = j_n = \text{Res } \underbrace{z_1^{m+1} z_2^{n+1}} \pi(z_1) \pi'(z_2)$$

$$= \text{Res}_{z_1 \rightarrow z_2} \left\{ z_1^{m+1} z_2 \right.$$

$$\left. \text{Res } z_1^{m+1} z_2^{n+1} = 0 \right.$$



$$\text{Res } j_n = \text{Res}_{z_1^{m+1} z_2^{n+1}} \Pi(z_1) \Pi(z_2)$$

$$= \text{Res}_{z_1 \rightarrow z_2} \left[z_1^{m+1} z_2^{n+1} \right] \frac{S}{z_1 z_2}$$

$$\text{Res}_{z_1 \rightarrow z_2} z_1^{m+1} z_2^{n+1} = 0$$

$$\text{Res } j^{-n} = \text{Res } \underbrace{z_1^{m+1} z_2^{n+1}} \Pi(z_1) \Pi(z_2)$$

$$= \text{Res}_{z_1 \rightarrow z_2} \left\{ z_1^{m+1} z_2^{n+1} \left[\frac{1}{z_1 z_2} + \frac{z_1}{z_2} \Pi(z_2) + \frac{1}{z_2} \partial \Pi(z_1) \right] \right\}$$

$$\text{Res}_{z_1 \rightarrow z_2} z_1^{m+1} z_2^{n+1} = 0$$

$$\text{Res } j_n = j_n = \text{Res } \underbrace{z_1^{m+1} z_2^{n+1}} \Pi(z_1) \Pi(z_2)$$

$$= \text{Res } \left\{ z_1^{m+1} z_2^{n+1} \left[\frac{\Pi(z_1)}{z_2} + \frac{z_1 \Pi(z_1)}{z_2^2} + \frac{1}{z_2} \partial \Pi(z_1) \right] \right\}$$

$$= \sum z_2^{m+1}$$

$$\text{Res } z_1^{m+1} z_2^{n+1} = 0$$

$z_1 \leftrightarrow z_2$

$$\text{Res } j = j_n = \text{Res}_{z_1^{m+1} z_2^{n+1}} \Pi(z_1) \Pi(z_2)$$

$$= \text{Res}_{z_1 \rightarrow z_2} \left\{ z_1^{m+1} z_2^{n+1} \left[\frac{\Sigma}{z_1^2} + \frac{z_1}{z_1^2} \Pi(z_1) + \frac{1}{z_1} \partial \Pi(z_1) \right] \right\}$$

$$\text{Res}_{z_1 \rightarrow z_2} z_1^{m+1} z_2^{n+1} = 0$$

$$\Sigma z_2^{m+1}$$

$$\begin{aligned}
 \text{Res } j_n - j_n &= \text{Res } \underbrace{z_1^{m+1} z_2^{n+1}} \Pi(z_1) \Pi(z_2) \\
 &= \text{Res}_{z_1 \rightarrow z_2} \left\{ z_1^{m+1} z_2^{n+1} \left[\frac{\Sigma}{z_2^{m+1}} + \frac{z}{z_2} \Pi(z) + \frac{1}{z_2} \partial \Pi(z) \right] \right\} \\
 &= \frac{\Sigma}{z_2^{m+1}}
 \end{aligned}$$

$\text{Res}_{z_1 \rightarrow z_2} z_1^{m+1} z_2^{n+1} = 0$

$$\begin{aligned}
 \text{Res } j_n - j_n &= \text{Res } \underbrace{z_1^{m+1} z_2^{n+1}} \Pi(z_1) \Pi(z_2) \\
 &= \text{Res}_{z_1 \rightarrow z_2} \left\{ z_1^{m+1} z_2^{n+1} \left[\frac{f}{z_2} + \frac{z}{z_2} \Pi(z) + \frac{1}{z_2} \partial \Pi(z) \right] \right. \\
 &= \sum_{z_2} z_2^{m+1}
 \end{aligned}$$

$$\text{Res}_{z_1 \rightarrow z_2} z_1^{m+1} z_2^{n+1} = 0$$

$$\begin{aligned}
 f(z_1) &= f(z_2) \\
 &+ \frac{f'(z_2)}{z_2} \\
 &+ \frac{f''(z_2)}{2!} z_2^2 + \dots
 \end{aligned}$$

$$\begin{aligned}
 \text{Res } j_n &= \text{Res } \underbrace{z_1^{m+1} z_2^{n+1}} \Pi(z_1) \Pi(z_2) \\
 &= \text{Res}_{z_1 \rightarrow z_2} \left\{ z_1^{m+1} z_2^{n+1} \left[\frac{f(z)}{z_2} + \frac{z}{z_2} \Pi(z) + \frac{1}{z_2} \partial \Pi(z) \right] \right\} \\
 &= \sum_{z_2} z_2^{m+1} \frac{1}{z_2} \partial^3 \left(\frac{f(z)}{z_2} \right)
 \end{aligned}$$

$$\text{Res}_{z_1 \rightarrow z_2} z_1^{m+1} z_2^{n+1} = 0$$

$$\begin{aligned}
 f(z_1) &= f(z_2) \\
 &+ \frac{f'(z_2)}{1} z_1 - z_2 + \dots \\
 &+ \frac{f''(z_2)}{2!} z_1^2 - z_2^2 + \dots
 \end{aligned}$$

$$\text{Res } j_{-j_n} = \text{Res } \underbrace{z_1^{m+1} z_2^{n+1}} \Pi(z_1) \Pi(z_2)$$

$$= \text{Res}_{z_1 \rightarrow z_2} \left\{ z_1^{m+1} z_2^{n+1} \left[\frac{1}{z_2} + \frac{z_2}{z_2^2} \Pi(z_2) + \frac{1}{z_2} \partial \Pi(z_2) \right] \right\}$$

$$= \sum_{z_2} z_2^{m+1} \frac{1}{z_2} \partial^3 \left(\frac{z_2^{n+1}}{z_2} \right) + z_2^{n+1}$$

$$\text{Res}_{z_1 \rightarrow z_2} z_1^{m+1} z_2^{n+1} = 0$$

$$f(z_1) = f(z_2)$$

$$+ \frac{f'(z_2)}{z_2} + \frac{f''(z_2)}{2!} z_2^2 + \dots$$

$$\text{Res } j = j_n = \text{Res } \underbrace{z_1^{m+1} z_2^{n+1}} \Pi(z_1) \Pi(z_2)$$

$$= \text{Res } z_1^{n+1} \left[\frac{z_1}{z_2^{n+1}} + \frac{z_1}{z_2} \Pi(z_2) + \frac{1}{z_2} \partial \Pi(z_1) \right]$$

$$\frac{1}{z_2^{n+1}} \partial^3 \left(\frac{z_1^{n+1}}{z_2} \right) + z_2^{n+1}$$

$$\text{Res } z_1^{m+1} z_2^{n+1} = 0$$

$$z_1 \rightarrow z_2$$

$$z f(z_1) = f(z_2)$$

$$+ \frac{f'(z_2)}{z_2} +$$

$$+ \frac{f''(z_2)}{2!} z_2^2 +$$

$$\text{Res } j_{-j_n} = \text{Res } \underbrace{z_1^{m+1} z_2^{n+1}} \Pi(z_1) \Pi(z_2)$$

$$= \text{Res}_{z_1 \rightarrow z_2} \left\{ z_1^{m+1} z_2^{n+1} \left[\frac{1}{z_2} + \frac{z_2}{z_2^2} \Pi(z_2) + \frac{1}{z_2} \partial \Pi(z_2) \right] \right.$$

$$\left. = \frac{1}{z_2} z_2^{m+1} \frac{1}{z_2} \partial^3 \left(z_2^{n+1} \right) + z_2^{n+1} \frac{1}{z_2} \partial \left(z_2^{m+1} \right) \Pi(z_2) \right.$$

$$\text{Res}_{z_1 \rightarrow z_2} z_1^{m+1} z_2^{n+1} = 0$$

$$f(z_1) = f(z_2)$$

$$+ \frac{f'(z_2)}{z_2} + \frac{f''(z_2)}{2!} z_2^2 + \dots$$

$$\text{Res } j_n = \text{Res } \underbrace{z_1^{m+1} z_2^{n+1}} \Pi(z_1) \Pi(z_2)$$

$$= \text{Res}_{z_1 \rightarrow z_2} \left\{ z_1^{m+1} z_2^{n+1} \left[\frac{1}{z_2} + \frac{z_2}{z_2^2} \Pi(z_2) + \frac{1}{z_2} \partial \Pi(z_2) \right] \right.$$

$$= \frac{1}{z_2^{m+1}} \frac{1}{z_2} \partial \left(\frac{z_2^{n+1}}{z_2} \right) + z_2^{n+1} \frac{1}{z_2} \Pi(z_2) + \frac{1}{z_2^{m+1}} \frac{1}{z_2} \partial \Pi(z_2)$$

$$\text{Res}_{z_1 \rightarrow z_2} z_1^{m+1} z_2^{n+1} = 0$$

$$\Pi(z_1) = \Pi(z_2)$$

$$+ \frac{1}{z_2} \partial \Pi(z_2) + \frac{1}{z_2} \Pi(z_2)$$

$$\begin{aligned}
 \text{Res } j_n - j_n &= \text{Res } \underbrace{z_1^{m+1} z_2^{n+1}} \Pi(z_1) \Pi(z_2) \\
 &= \text{Res}_{z_1 \rightarrow z_2} \left\{ z_1^{m+1} z_2^{n+1} \left[\frac{1}{z_2} + \frac{z_2}{z_2^2} \Pi(z_2) + \frac{1}{z_2} \partial \Pi(z_2) \right] \right. \\
 &= \frac{1}{z_2} z_2^{m+1} \frac{1}{z_2} \partial^3 \left(z_2^{n+1} \right) + z_2^{n+1} \frac{1}{z_2} \partial \left(z_2^{m+1} \right) \Pi(z_2) \\
 &\quad + z_2^{n+m+2} \partial \Pi(z_2)
 \end{aligned}$$

$$\text{Res}_{z_1 \rightarrow z_2} z_1^{m+1} z_2^{n+1} = 0$$

$$\Pi(z_1) = \Pi(z_2)$$

$$\begin{aligned}
 &+ \frac{1}{z_2} \partial^2 z_2^{n+1} + \\
 &+ \frac{1}{z_2} \partial^3 z_2^{n+1} +
 \end{aligned}$$

$$= \frac{c}{12} (m^3 - m) z_1^{m+n-1} + (2m+2) z_1^{m+n+1} \pi + z_1$$



$$= \frac{c}{12} (m^3 - m) z_1^{m+n-1} + (2m+2) z_1^{m+n+1} + z_1^{n+m+2}$$

$$= \frac{c}{12} (m^3 - m) z_1^{m+n-1} + (2m+2) z_1^{m+n+1} \pi + z_1^{n+m+2} \pi^2$$

$$= \frac{c}{12} (m^2 - m) z_1^{m+n} + (m-n) z_1^{m+n+1} \pi$$



$$= \frac{c}{12} (m^3 - m) z_1^{m+n-1} + (2m+2) z_1^{m+n+1} \mathbb{T} + z_1^{n+m+2} \mathbb{T}^2$$

$$= \frac{c}{12} (m^2 - m) z_1^{m+n-1} + (m-n) z_1^{m+n+1} \mathbb{T}$$

$$+ \left\{ z_1^{m+n+2} \mathbb{T}^2 + (2+n+m) z_1^{m+n+1} \mathbb{T} \right\}$$

$$= \frac{c}{12} (m^3 - m) z^{m+n-1} + (2m+2) z^{m+n+1} \pi + z^{n+m+2} \pi$$

$$= \frac{c}{12} (m^2 - m) z^{m+n-1} + (m-n) z^{m+n+1} \pi$$

$$+ \left\{ \pi z^{2m+n+2} + z^{m+n+2} \pi \right\}$$

$$= \frac{c}{12} (m^3 - m) z^{m+n-1} + (2m+2) z^{m+n+1} + z^{n+m+2}$$

$$= \frac{c}{12} (m^2 - m) z^{m+n-1} + (m-n) z^{m+n+1}$$

$$+ \left\{ (m+n+2) z^{m+n+1} + z^{m+n+2} \right\}$$

$$= \frac{c}{12} (m^3 - m) z^{m+n-1} + (2m+2) z^{m+n+1} \pi + z^{n+m+2} \pi$$

$$= \frac{c}{12} (m^2 - m) z^{m+n-1} + (m-n) z^{m+n+1} \pi$$

$$+ \left\{ (m+n+2) z^{m+n+1} \pi + z^{m+n+2} \pi \right\}$$

$$\left[z^{m+n+2} \pi \right]$$

$$[L_m, L_n] = \int \frac{dz}{2\pi i} \left[\frac{c}{12} (m^2 - m) \frac{z^{m+n}}{z} + (m-n) \right]$$

$$[L_m, L_n] = \int \frac{dz}{2\pi i} \left[\frac{c}{12} (m^2 - m) \frac{z^{m+n}}{z} + (m-n) z^{m+n} \right]$$

$$[L_m, L_n] = \int \frac{dz}{2\pi i} \left[\frac{c}{12} (m^2 - m) \frac{z^{m+n}}{z} + (m-n) z^{m+n-1} \right]$$

+

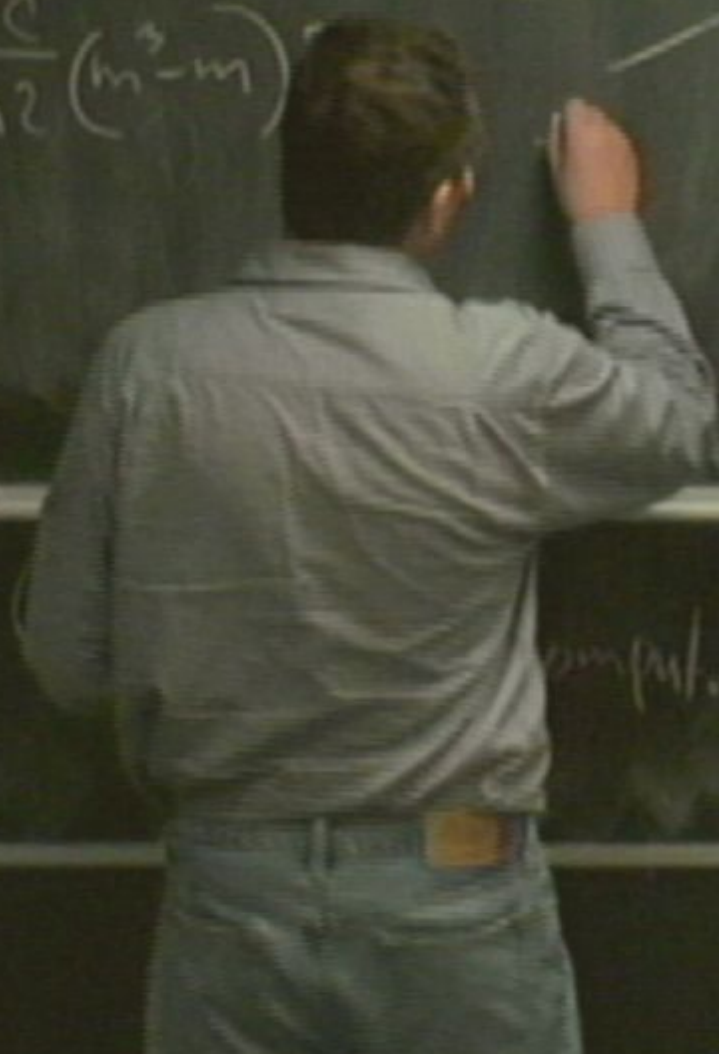
$$\begin{aligned}
 [L_{m+1}, L_{n-1}] &= \int \frac{dz}{2\pi i} \left[\frac{c}{12} (m^3 - m) \frac{z^{m+n+1}}{z} + (m-n) z^{m+n+1} \right] \\
 &+ \left[z^{m+n+2} \right]
 \end{aligned}$$

computer algebra!

$$[L_m, L_n] = \oint \frac{dz}{2\pi i} \left[\frac{c}{12} (m^2 - m) \frac{z^{m+n}}{z} + (m-n) z^{m+n+1} \right] + \dots$$

$\left[z^{m+n+2} \right]$

$$\begin{aligned}
 (L_m, L_n) &= \oint \frac{dz}{2\pi i} \left[\frac{c}{12} (m^2 - m) \frac{z^{m+n}}{z} + (m-n) z^{m+n-1} \right] \\
 &= \frac{c}{12} (m^2 - m)
 \end{aligned}$$



computer algebra!

$$[L_m, L_n] = \oint \frac{dz}{2\pi i} \left[\frac{c(m^2-m)}{12} \frac{z^{m+n}}{z} + (m-n) z^{m+n-1} \right]$$

$$= \delta_{m,-n} + (m-n) L_{m+n}$$

computer algebra!

$$\begin{aligned}
 & \frac{1}{i2\pi} \binom{m+n}{m} \int_{|z|=r} \left[\frac{dz}{z} z^{m+n} + (m-n) \frac{dz}{z} z^{m+n} \right] \\
 & L_k = \int_{\frac{1}{2\pi i}} \frac{dz}{z} z^{k+2} \pi
 \end{aligned}$$

\downarrow stated in Ward identities before
 $= \frac{1}{i2\pi} \delta A(z_e, \bar{z}_e)$

OPE \Rightarrow computer algebra!

$$[L_m, L_n] =$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}$$

$$[L_m, L_n] = \oint \frac{dz}{2\pi i} \left[\frac{c}{12} (m^2 - m) \frac{z^{m+n}}{z} + (m-n) z^{m+n-1} \right]$$

$$= \frac{c}{12} (m^2 - m) \delta_{m, -n} + (m-n) L_{m+n}$$

$$L_k = \oint \frac{dz}{2\pi i} z^{k+2} \pi$$

$$\left[\dots \right] = \oint \frac{dz}{2\pi i} \left[\frac{c(m^2-m)}{12} \frac{z^{m+1}}{z} + (m-n) z^{m+n+1} \right]$$

$$+ \left[\dots \right]$$

$$\delta_{m,-n} + (m-n) \left[\dots \right]$$

$$\oint \frac{dz}{2\pi i} z^{k+2}$$

$$\begin{aligned}
 \left[L_{m,1} \right] &= \oint_{|z|=1} \frac{dz}{2\pi i} \left[\frac{c(m^2-m)}{12} \frac{z^{m+h}}{z} + (m-h) z^{m+h-1} \right] \\
 &+ \left[z^{m+h+2} \right] \left\{ \dots \right\} \\
 &+ (m-h) \left[L_{m+h} \right] \\
 &z^{k+2\pi}
 \end{aligned}$$

$$\begin{aligned}
 [L_m, L_n] &= \oint \frac{dz}{2\pi i} \left[\frac{c}{12} (m^2 - m) \frac{z^{m+n}}{z} + (m-n) z^{m+n+1} \right] \\
 &= \frac{c}{12} (m^2 - m) \oint \frac{dz}{2\pi i} z^{m+n-1} + (m-n) \oint \frac{dz}{2\pi i} z^{m+n+1} \\
 &= \frac{c}{12} (m^2 - m) \delta_{m, -n} + (m-n) L_{m+n}
 \end{aligned}$$

$z = e^{i\theta}$, $dz = i e^{i\theta} d\theta$, $|z|=1$
 $\oint \frac{dz}{2\pi i} z^k = \int_0^{2\pi} \frac{i e^{i\theta} d\theta}{2\pi i} z^k = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i(k+1)\theta}$

$$[L_m, L_n] = \oint \frac{dz}{2\pi i} \left[\frac{c(m^2-m)}{12} \frac{z^{m+n}}{z} + (m-n) z^{m+n+1} \right]$$

$$z = e^{i\theta}, \quad c = |z|=1$$

$$\int_0^{2\pi} \frac{d\theta}{2\pi i} (i d\theta) (im)$$

$$+ \int \frac{z^{m+n+2}}{z} \left[\right]$$

$$= \frac{c}{12} (m^2-m) \oint_{m, -n} + (m-n) L_{m+n}$$

$$L_k = \oint \frac{dz}{2\pi i} z^{k+2} \pi$$

$$\begin{aligned}
 [L_m, L_n] &= \oint \frac{dz}{2\pi i} \left[\frac{c}{12} (m^2 - m) \frac{z^{m+n}}{z} + (m-n) z^{m+n+1} \pi \right] \\
 &= \frac{c}{12} (m^2 - m) \oint \frac{dz}{2\pi i} z^{m+n-1} + (m-n) \oint \frac{dz}{2\pi i} z^{m+n+1} \pi \\
 &= \frac{c}{12} (m^2 - m) \delta_{m, -n} + (m-n) L_{m+n}
 \end{aligned}$$

A hand is pointing to the term $(m-n) L_{m+n}$ on the right side of the board.

Properties of a Virasoro algebra

$$\underline{m=0} \quad [L_0, L_n] = -n L_n$$

$$L_0 |4\rangle$$

Properties of a Virasoro algebra

$$\underline{m=0} \quad [L_0, L_n] = -n L_n$$

$$L_0 |4\rangle = h |4\rangle$$

$L_n |4\rangle$ is an eigenstate of L_0 with

$$L_0 |4\rangle = \hbar |4\rangle$$

$|n\rangle$ is an eigenstate of L_0 as well

$$= \sum_{m_2} \frac{\hbar m_2}{2} \left(\frac{m_2+1}{2} \right) \psi(m_2) + 2 \sum_{m_2} \frac{\hbar m_2}{2} \left(\frac{m_2}{2} \right) \psi(m_2) + \sum_{m_2} \frac{\hbar m_2}{2} \psi(m_2)$$

$$+ \frac{\hbar m_2}{2} \psi(m_2) + \frac{\hbar m_2}{2} \psi(m_2)$$

Properties of a Virasoro algebra

$$\underline{m=0} \quad [L_0, L_n] = -n L_n$$

$$L_0 |4\rangle = 4 |4\rangle$$

$L_n |4\rangle$ is an eigenstate of L_0 as well

$$L_0(L_n |4\rangle)$$

$$= \sum_{m=2}^{\infty} \alpha_{-m} \alpha_m \left(\frac{2}{3} \right) + 2 \alpha_{-n} \alpha_n \left(\frac{2}{3} \right) + \dots$$

$$+ \dots$$

Properties of a Virasoro algebra

$$\underline{m=0} \quad [L_0, L_n] = -n L_n = L_0 L_n - L_n L_0 = m L_n + L_n L_0$$

$$L_0 |4\rangle = 4 |4\rangle$$

$L_n |4\rangle$ is an eigenstate of L_0 as well

$$L_0(L_n |4\rangle) = L_n(L_0 |4\rangle) - n L_n |4\rangle =$$

$$= \sum_{m=0}^{n-1} \frac{1}{2} \alpha_{-m} \cdot \alpha_{m+1} \cdot \alpha_{-2}^3 \binom{2n-1}{m} + 2 \alpha_{-2}^{n+1} \alpha_{-1} \binom{2n-1}{m} \alpha_{-2} + \alpha_{-2}^{n+1} \alpha_{-1} \alpha_{-2} \binom{2n-1}{m} + \alpha_{-2}^{n+1} \alpha_{-1} \alpha_{-2} \binom{2n-1}{m}$$

$$+ \frac{1}{2} \alpha_{-2}^{n+1} \alpha_{-1} \alpha_{-2} + \frac{1}{2} \alpha_{-2}^{n+1} \alpha_{-1} \alpha_{-2}$$

Properties of a Virasoro algebra

$$\underline{n \neq 0} \quad [L_0, L_n] = -n L_n = L_0 L_n - L_n L_0$$

$$L_0 |4\rangle = h |4\rangle$$

$L_n |4\rangle$ is an eigenstate of L_0 as well

$$L_0(L_n |4\rangle) = L_n(L_0 |4\rangle) - n L_n |4\rangle = h L_n |4\rangle - n L_n |4\rangle = (h - n) L_n |4\rangle$$

$$= \sum_{m=0}^{\infty} \alpha_{-m} \alpha_{m+1} \frac{1}{3} \alpha_{-2}^3 \binom{2n+1}{2} \alpha_2 + 2 \alpha_2 \alpha_{-1} \alpha_{-1} \binom{2n+1}{2} \alpha_2 + \alpha_2 \alpha_{-1} \alpha_{-1} \alpha_2 \binom{2n+1}{2} + \alpha_2 \alpha_{-1} \alpha_{-1} \alpha_2 \binom{2n+1}{2}$$

$$+ \frac{1}{2} \alpha_{-2} \alpha_2 + \frac{1}{2} \alpha_{-2} \alpha_2 + \frac{1}{2} \alpha_{-2} \alpha_2 + \frac{1}{2} \alpha_{-2} \alpha_2$$

$L_0 |4\rangle$ is an eigenstate of L_0 as well

$L_n |4\rangle$ is an eigenstate of L_0 as well

$$L_0(L_n |4\rangle) = L_n(L_0 |4\rangle) - n L_n |4\rangle = L_n |4\rangle - n L_n |4\rangle = (1-n) L_n |4\rangle$$

$$\text{Res}_{z_1} - \text{Res}_{z_2} = \text{Res}_{z_1} z_1^{n+1} z_2^{n+1} \pi(z_1) \pi(z_2)$$

$$= \text{Res}_{z_1} z_1^{n+1} z_2^{n+1} \left[\frac{1}{z_2} + \frac{z_2}{z_2^2} \right] \pi(z_1)$$

$$= \frac{1}{z_2} z_2^{n+1} \pi(z_1) + z_2^n \pi(z_1) + \dots$$

$$\text{Res}_{z_1} z_1^{n+1} z_2^{n+1} = 0$$

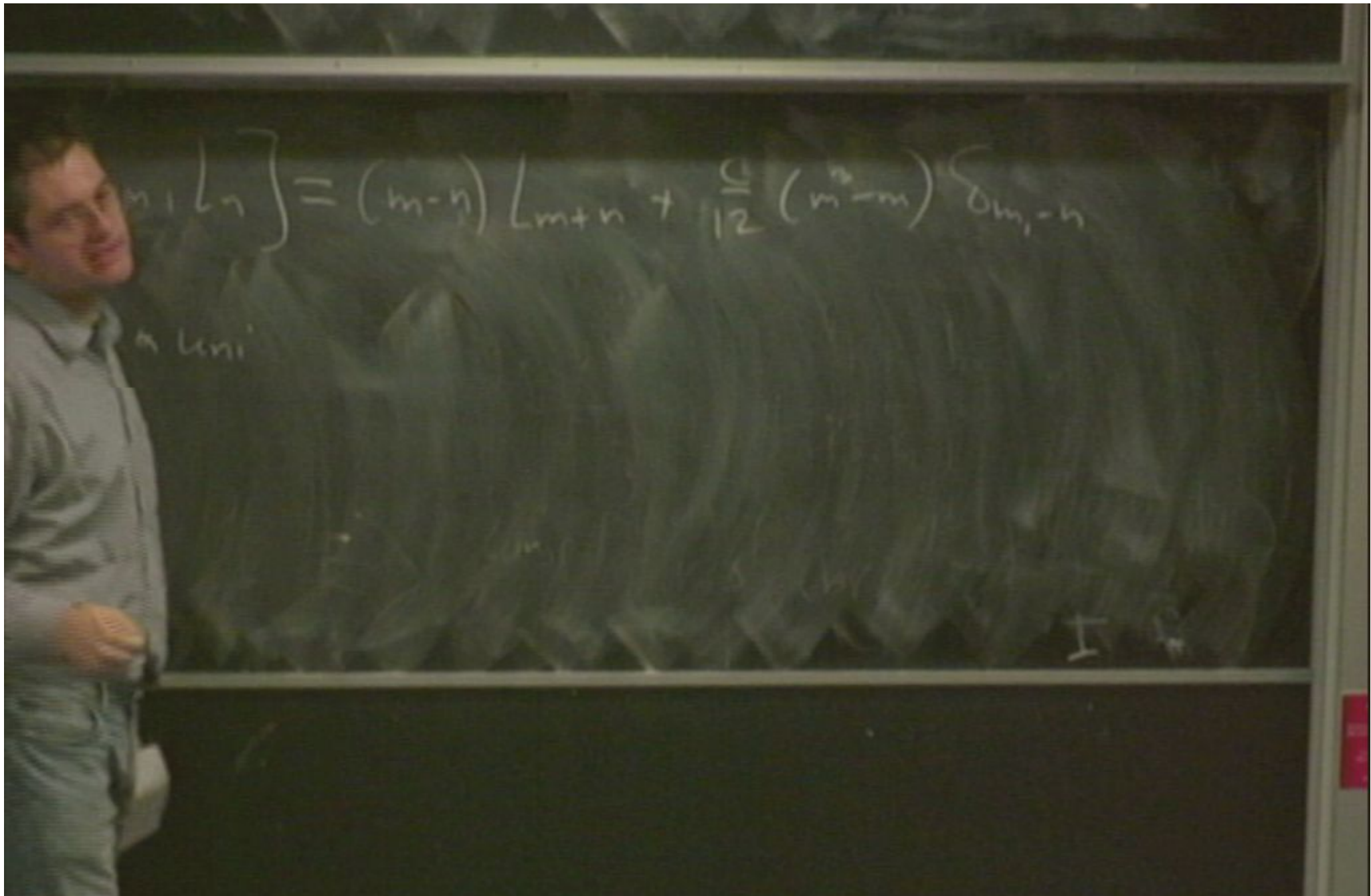
$$f(z_1) = f(z_2) + \frac{f'(z_2)}{z_2} + \frac{f''(z_2)}{2!} z_2^2 + \dots$$

$L_0 | \psi \rangle$

$L_n | \psi \rangle$ is an eigenstate of L_0 as well

$$L_0(L_n | \psi \rangle) = L_n L_0 | \psi \rangle - n L_n | \psi \rangle = L_n L_0 | \psi \rangle - n L_n | \psi \rangle = (L_0 - n) L_n | \psi \rangle$$

Generators with $n > 0$ lower L_0
 $n < 0$ raise L_0



$$\left. \begin{matrix} L_m \\ L_n \end{matrix} \right\} = (m-n) L_{m+n} + \frac{1}{12} (m^3 - n^3) \delta_{m,-n}$$

L_m

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n}$$

→ a universal subalgebra of Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n}$$

→ a universal subalgebra of Virasoro algebra

$$L_0, L_{\pm} \Rightarrow [L_0, L_1] = -L_1, [L_0, L_{-1}] = L_{-1}, [L_1, L_{-1}] = 2L_0$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n}$$

→ a universal subalgebra of Virasoro algebra

$$L_0, L_{\pm} \Rightarrow [L_0, L_1] = -L_1, [L_0, L_{-1}] = L_{-1}, [L_1, L_{-1}] = 2L_0$$

$SL(2, \mathbb{R})$

② Generation with \dots

$\pi(z) \rightarrow$ Laurent coefficients of $\pi(z)$

$$\pi(z) = \sum_{m=-\infty}^{\infty} \frac{a_m}{z^{m+1}}$$

$\pi(z) \rightarrow$ Laurent coefficients of $\pi(z)$

$$\pi(z) = \sum_{m=-\infty}^{\infty} \frac{a_m}{z^{m+1}} \Rightarrow O(z)$$

$\pi(z) \rightarrow$ Laurent coefficients of $\pi(z)$

$$\pi(z) = \sum_{m=-\infty}^{+\infty} \frac{a_m}{z^{m+1}} \Rightarrow O(z) = \sum_{m=-\infty}^{+\infty} \frac{O_m}{z^{m+1}}$$

$f(z)$ \rightarrow Laurent coefficients of $f(z)$

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{a_n}{z^{n+1}} \Rightarrow g(z) = \sum_{n=-\infty}^{\infty} \frac{a_n}{z^{n+1}}$$

$\mathbb{T}(z) \rightarrow$ Laurent coefficients of $\mathbb{T}(z)$

$$\mathbb{T}(z) = \sum_{m=-\infty}^{\infty} \frac{a_m}{z^{m+1}} \Rightarrow O(z) = \sum_{m=-\infty}^{\infty} \frac{O_m}{z^{m+h}}$$

\Rightarrow assume that \mathcal{O} is a primary operator

$\mathbb{T}(z) \rightarrow$ Laurent coefficients of $\mathbb{T}(z)$

$$\mathbb{T}(z) = \sum_{m=-\infty}^{\infty} \frac{a_m}{z^{m+2}} \Rightarrow O(z) = \sum_{m=-\infty}^{+\infty} \frac{O_m}{z^{m+h}}$$

\Rightarrow assume that \mathcal{O} is a primary operator

$$\mathbb{T}(z)\mathcal{O} = \frac{h}{2z^2}\mathcal{O} + \mathcal{O}\frac{1}{z^2}$$

$$\{L_m, O_n\} = \oint \frac{dz_2}{2\pi i} \text{Res}_{z_1 \rightarrow z_2} \int_m(z_1) O(z_2)$$

\uparrow
 $z_1^{m+1} T(z_1)$

$$\begin{aligned}
 [L_m, O_n] &= \oint \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \rightarrow z_2} \int_m(z_1) O(z_2) \\
 &= [(h-1)m - n] O_{n+h}
 \end{aligned}$$

\uparrow
 $z_1^{m+1} T(z_1)$

$$\begin{aligned}
 (L_m, O_n) &= \oint \frac{dz_1}{2\pi i} \operatorname{Res}_{z_1 \rightarrow z_2} \int_m(z_1) O(z_2) \\
 &= \left[(h-1)m - n \right] O_{n+h}
 \end{aligned}$$

\uparrow
 $z_1^{m+1} T(z_1)$

$$\begin{aligned}
 [L_m, O_n] &= \oint \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \rightarrow z_2} \int_m(z_1) O(z_2) \\
 &= \left[(h-1)m - n \right] O_{n+h}
 \end{aligned}$$

\uparrow
 $z_1^{m+1} T(z_1)$



$$\begin{aligned}
 [L_m, O_n] &= \oint \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \rightarrow z_2} \int_m(z_1) O(z_2) \\
 &= [(\hbar-1)m - n] O_{n+m}
 \end{aligned}$$

Mode Expansion

$$X^{\mu} = X^{\mu}(z, \bar{z})$$

$$\Rightarrow \partial X^{\mu}(z)$$

Modul 3: Polynomien

$$X^m = X^m(z, \bar{z})$$

$$\Rightarrow \partial X^m(z) \quad \& \quad \bar{\partial} X^m$$

$$\partial X^m(z) =$$



Mode Expansion

$$X^m = X^m(z, \bar{z})$$

$$\Rightarrow \partial X^m(z) \quad \& \quad \bar{\partial} X^m(\bar{z})$$

$$\partial X^m(z) = -i \left(\frac{\alpha'}{2} \right)^{1/2} \sum_n \frac{\alpha_{-n}^m}{z^{n+1}}$$

Made Spolnstan.

$$X^m = X^m(z, \bar{z})$$

$$\Rightarrow \partial X^m(z) \quad \delta X^m(\bar{z})$$

$$\partial X^m(z) = -i \sum_{n \neq m} \alpha_n \left(\frac{z}{2} \right)^{n-m} \frac{1}{2} \alpha_{-n}$$

$$\bar{\partial} X^m = -i \sum_{n \neq m} \alpha_{-n} \left(\frac{\bar{z}}{2} \right)^{m-n} \frac{1}{2} \alpha_n$$

\Rightarrow We want $X(z, \bar{z})$ to be single valued on \mathbb{C}

\Rightarrow We want $X(z, \bar{z})$ to be single valued on \mathbb{H}
Look at $m=0$

$$\alpha_0^m \ln z + \tilde{\alpha}_0^m \ln \bar{z}$$

\Rightarrow We want $X(z, \bar{z})$ to be single valued on \mathbb{C}
Look at $m \geq 0$

$$a_0^m \ln z + \tilde{a}_0^m \ln \bar{z} \Rightarrow \ln z \bar{z}$$

$$a_0^m = \tilde{a}_0^m$$