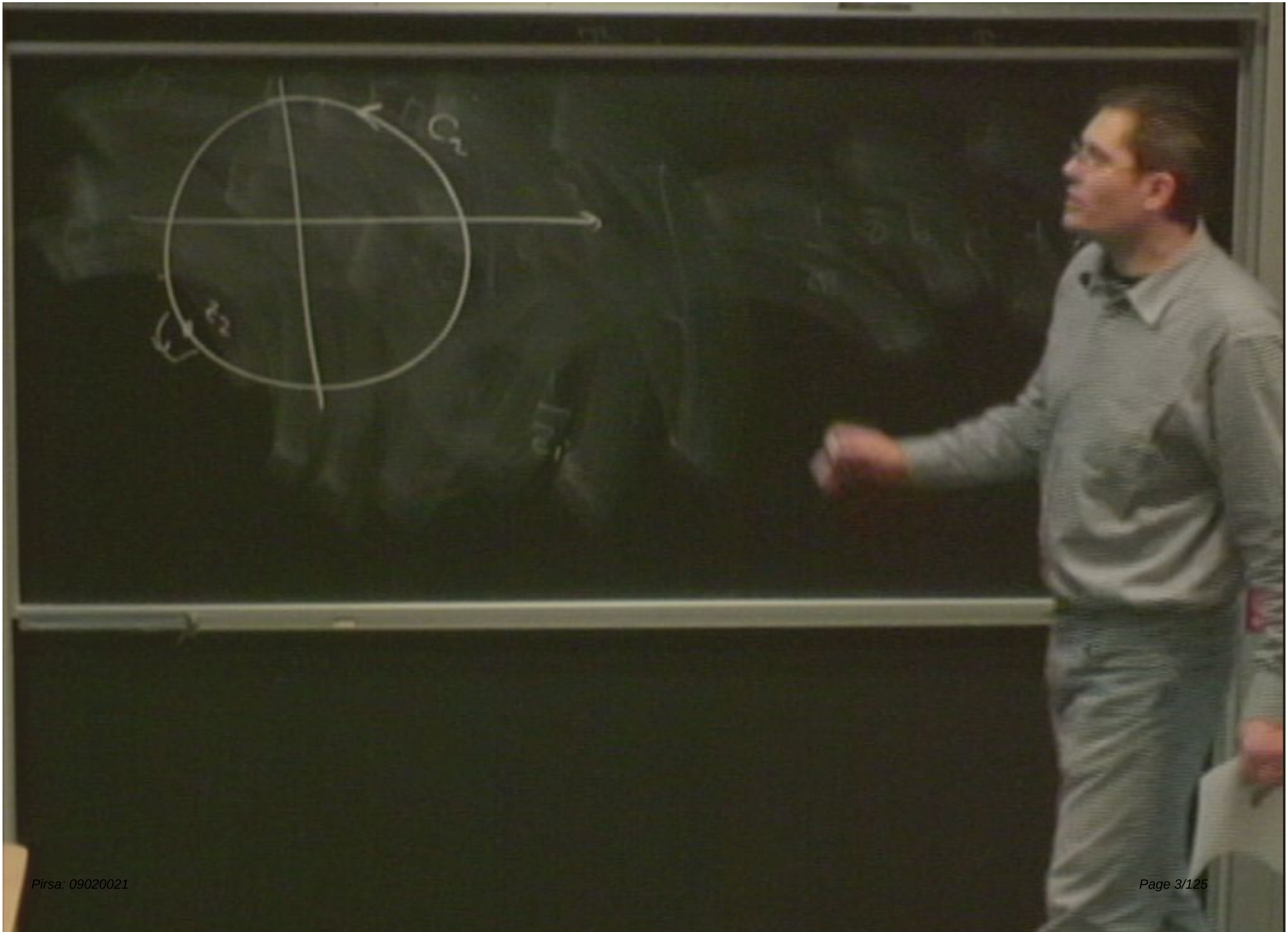
Title: Introduction to the Bosonic String Part B

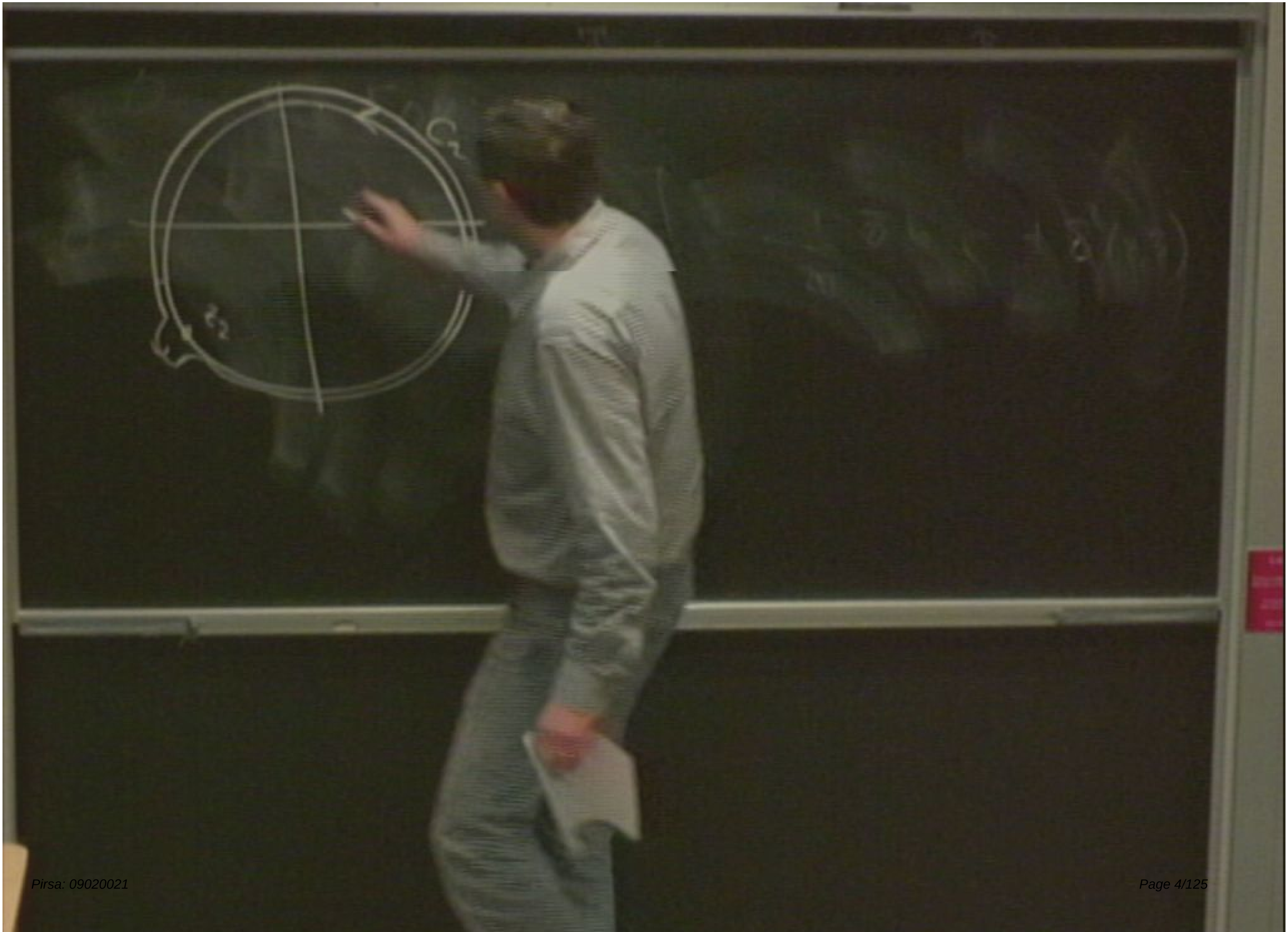
Date: Feb 27, 2009 12:00 PM

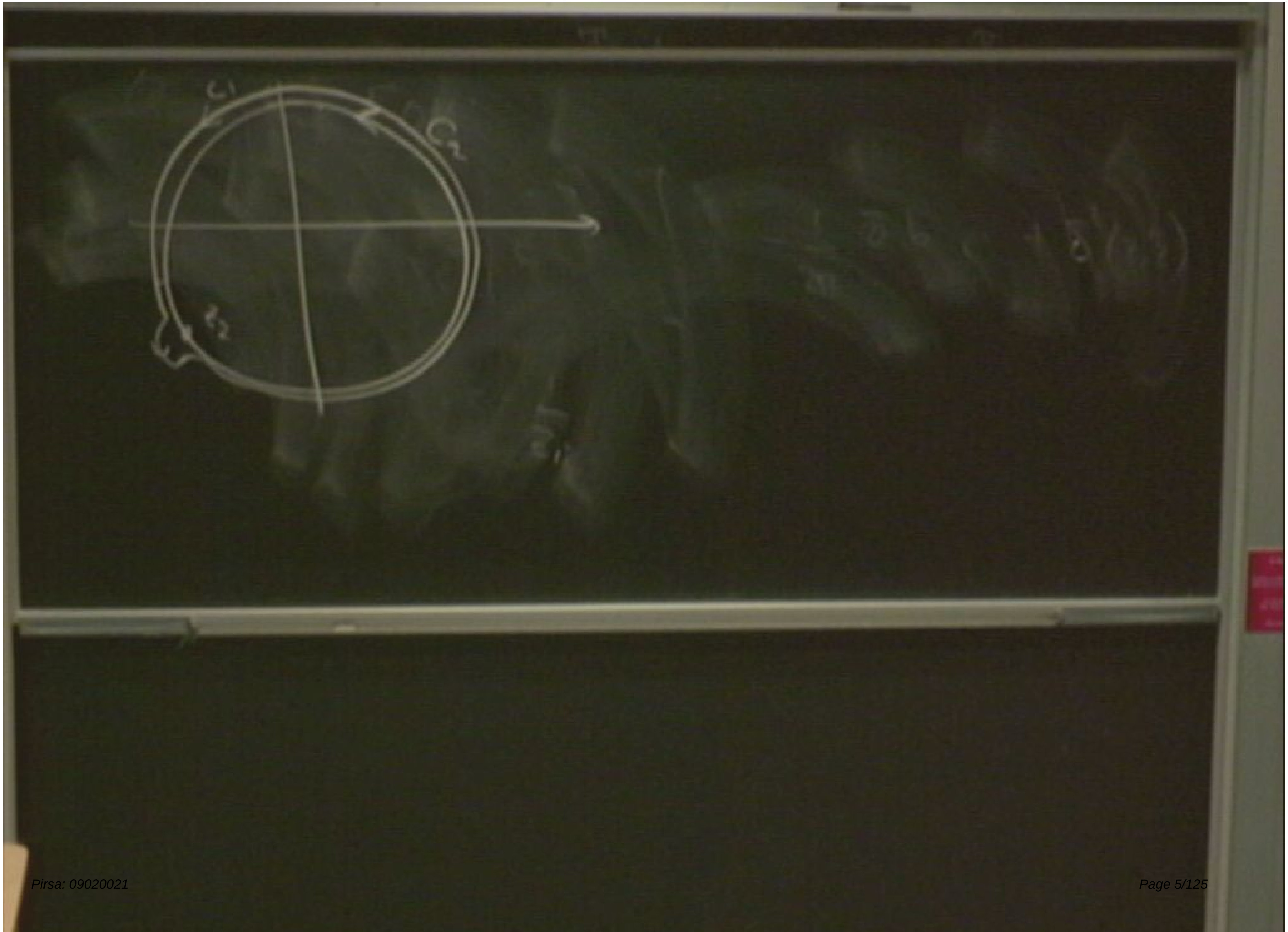
URL: http://pirsa.org/09020021

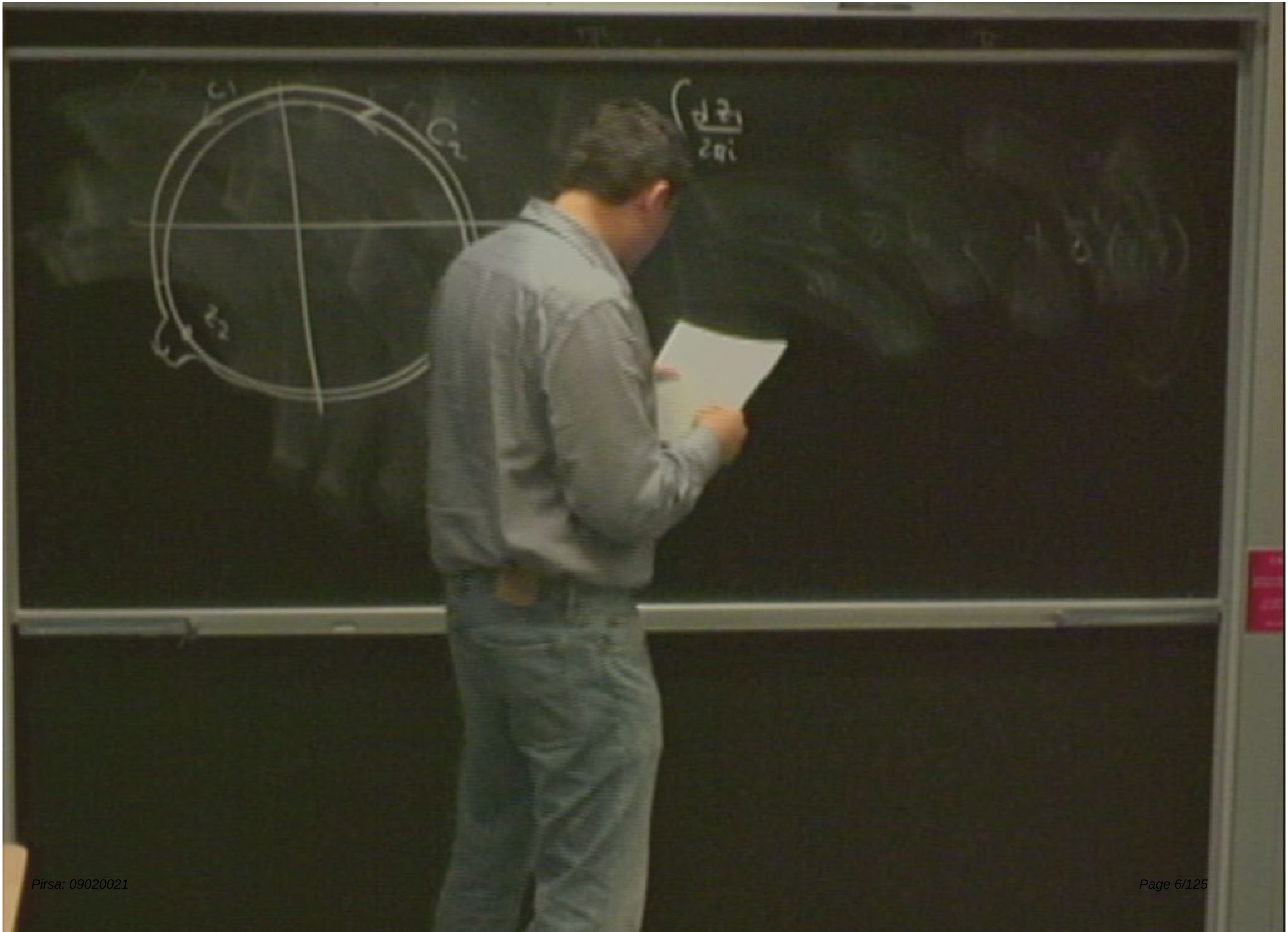
Abstract: This course provides a thorough introduction to the bosonic string based on the Polyakov path integral and conformal field theory. We introduce central ideas of string theory, the tools of conformal field theory, the Polyakov path integral, and the covariant quantization of the string. We discuss string interactions and cover the tree-level and one loop amplitudes. More advanced topics such as T-duality and D-branes will be taught as part of the course. The course is geared for M.Sc. and Ph.D. students enrolled in Collaborative Ph.D. Program in Theoretical Physics. Required previous course work: Quantum Field Theory (AM516 or equivalent). The course evaluation will be based on regular problem sets that will be handed in during the term. The primary text is the book: 'String theory. Vol. 1: An introduction to the bosonic string. J. Polchinski (Santa Barbara, KITP) . 1998. 402pp. Cambridge, UK: Univ. Pr. (1998) 402 p.' All interested students should contact Alex Buchel at abuchel@uwo.ca as soon as possible.

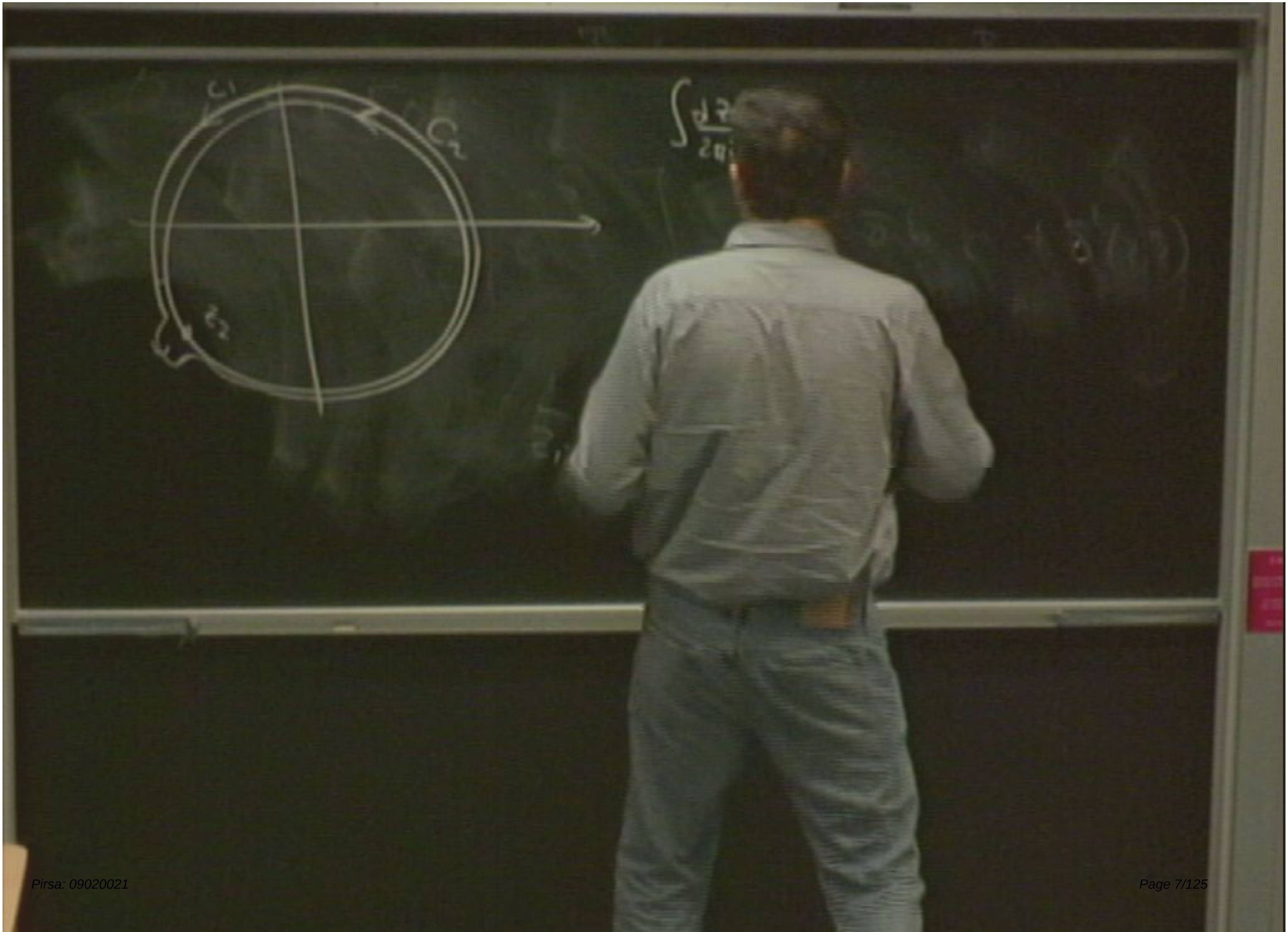


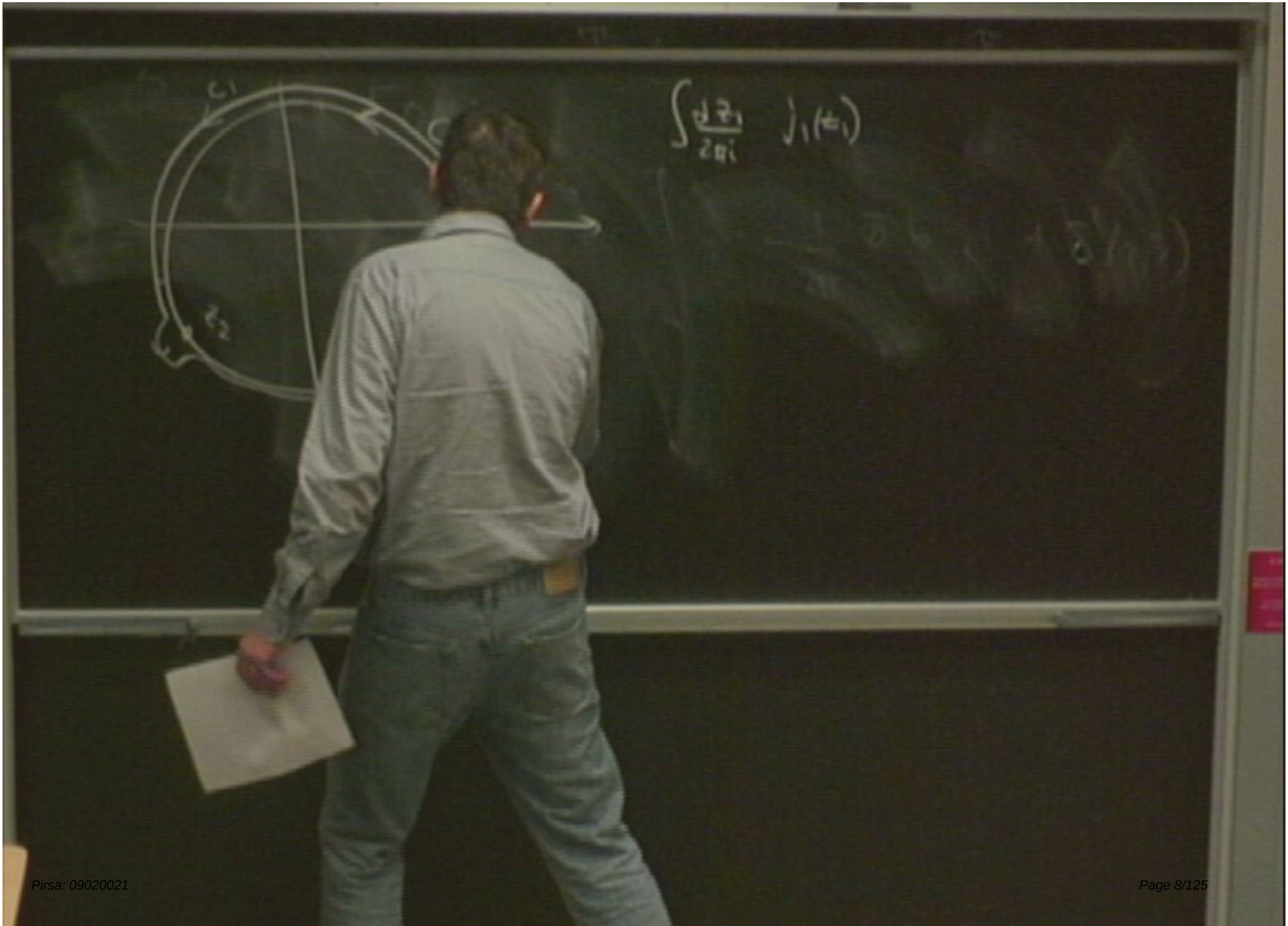
C2
ε2

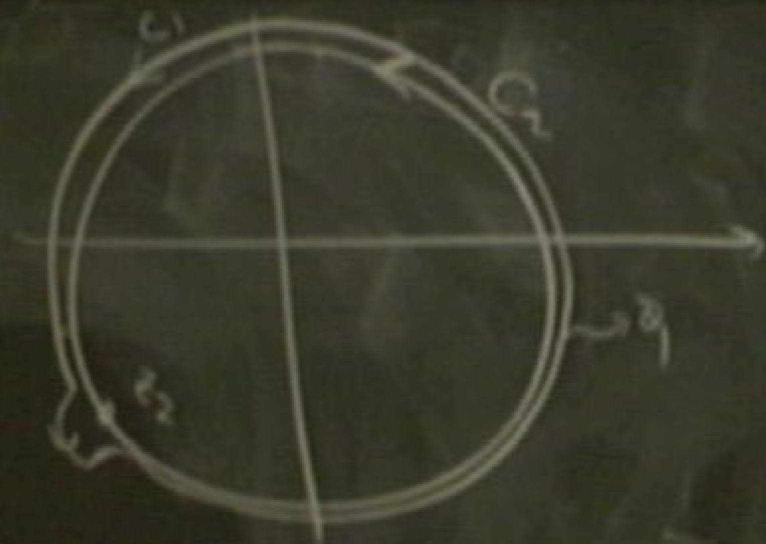

C2


C1
C2
z2

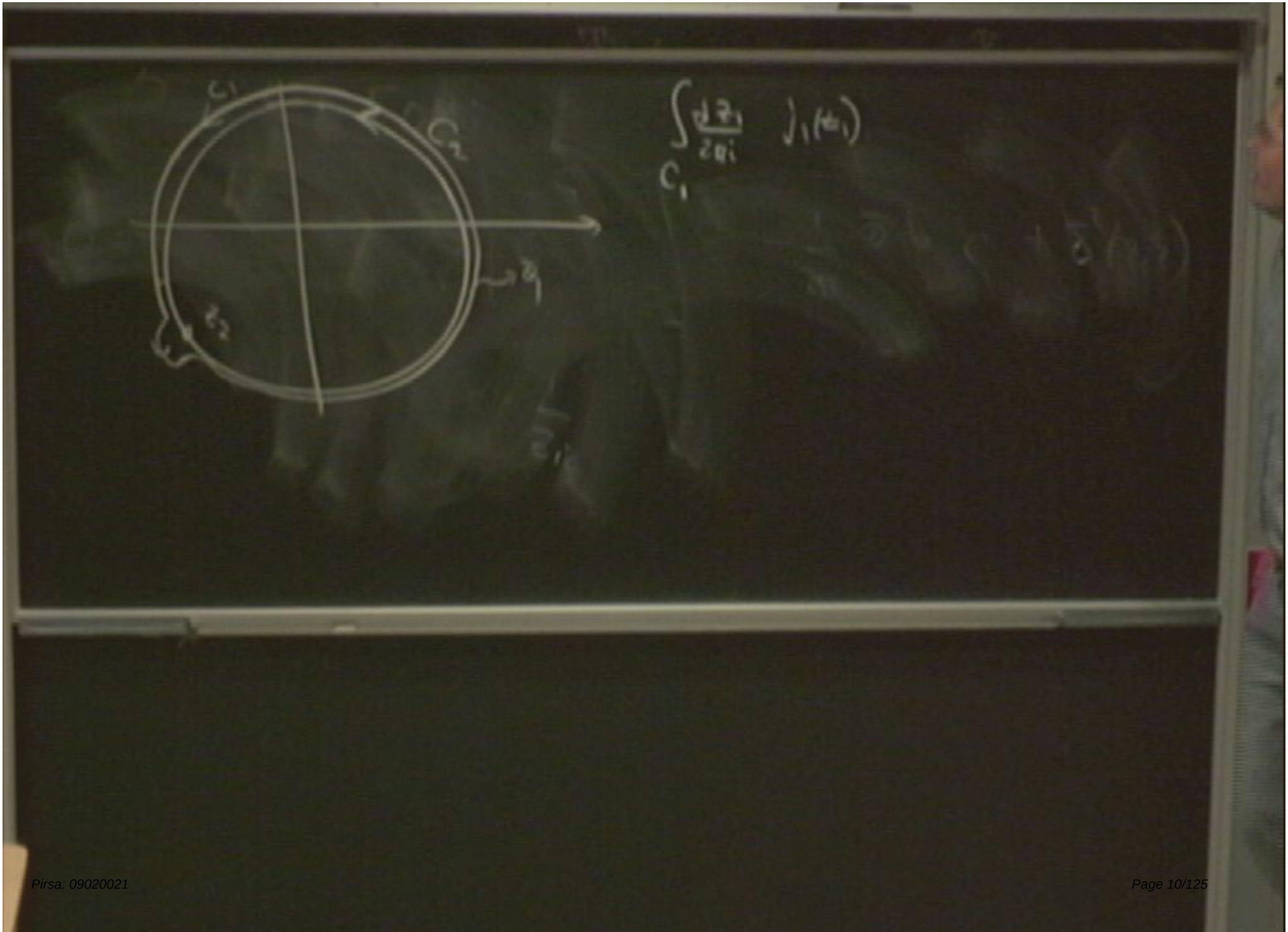


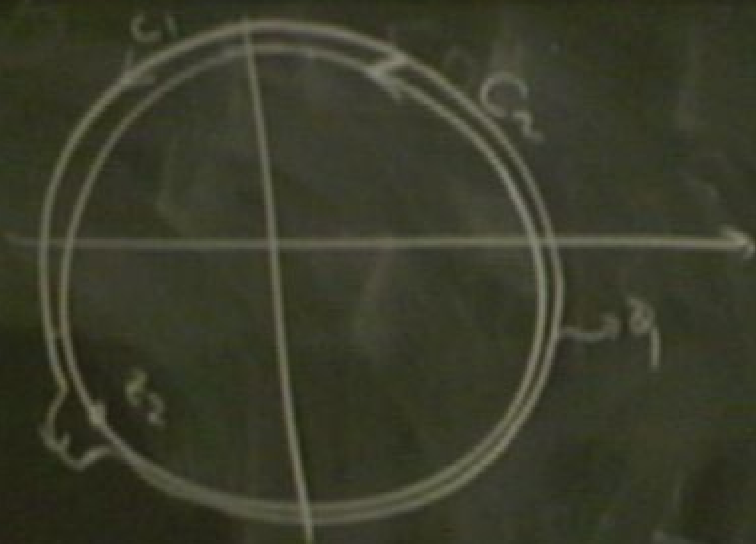

C1
C2


C_1
z_2
$\int \frac{dz_1}{2\pi i} \; j_1(z_1)$

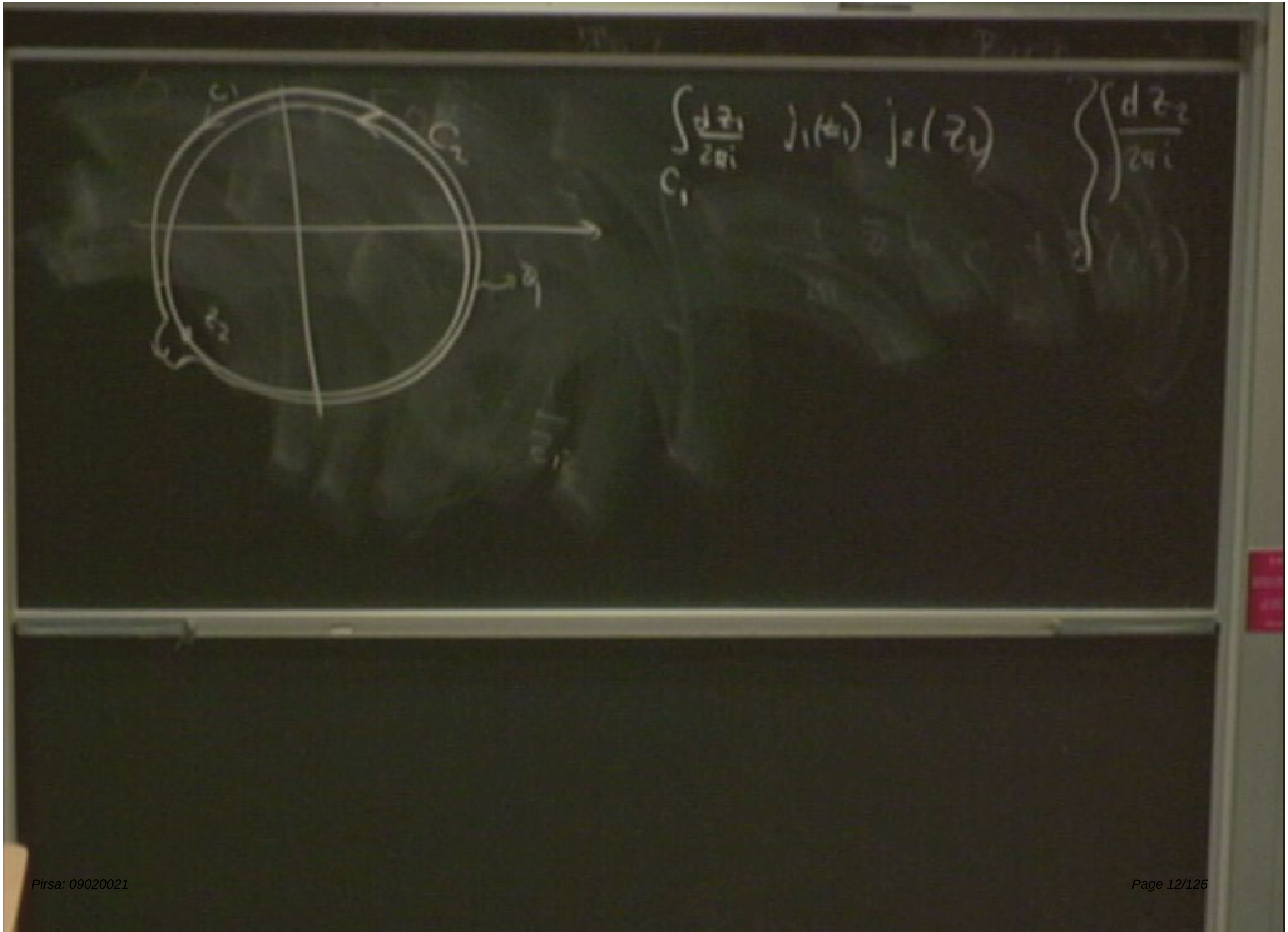


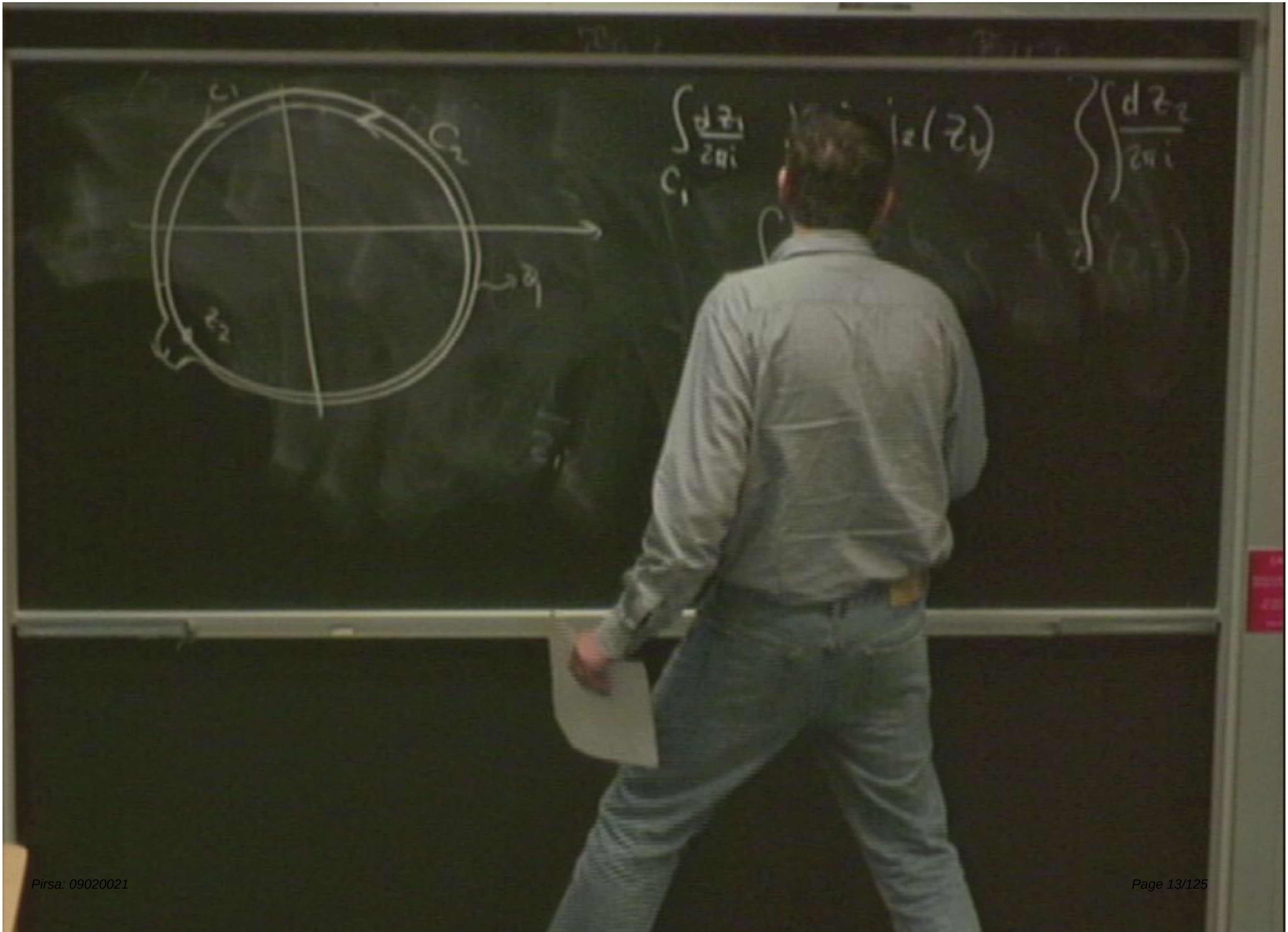
$$\int \frac{dz_1}{2\pi i} \; j_1(z_1)$$

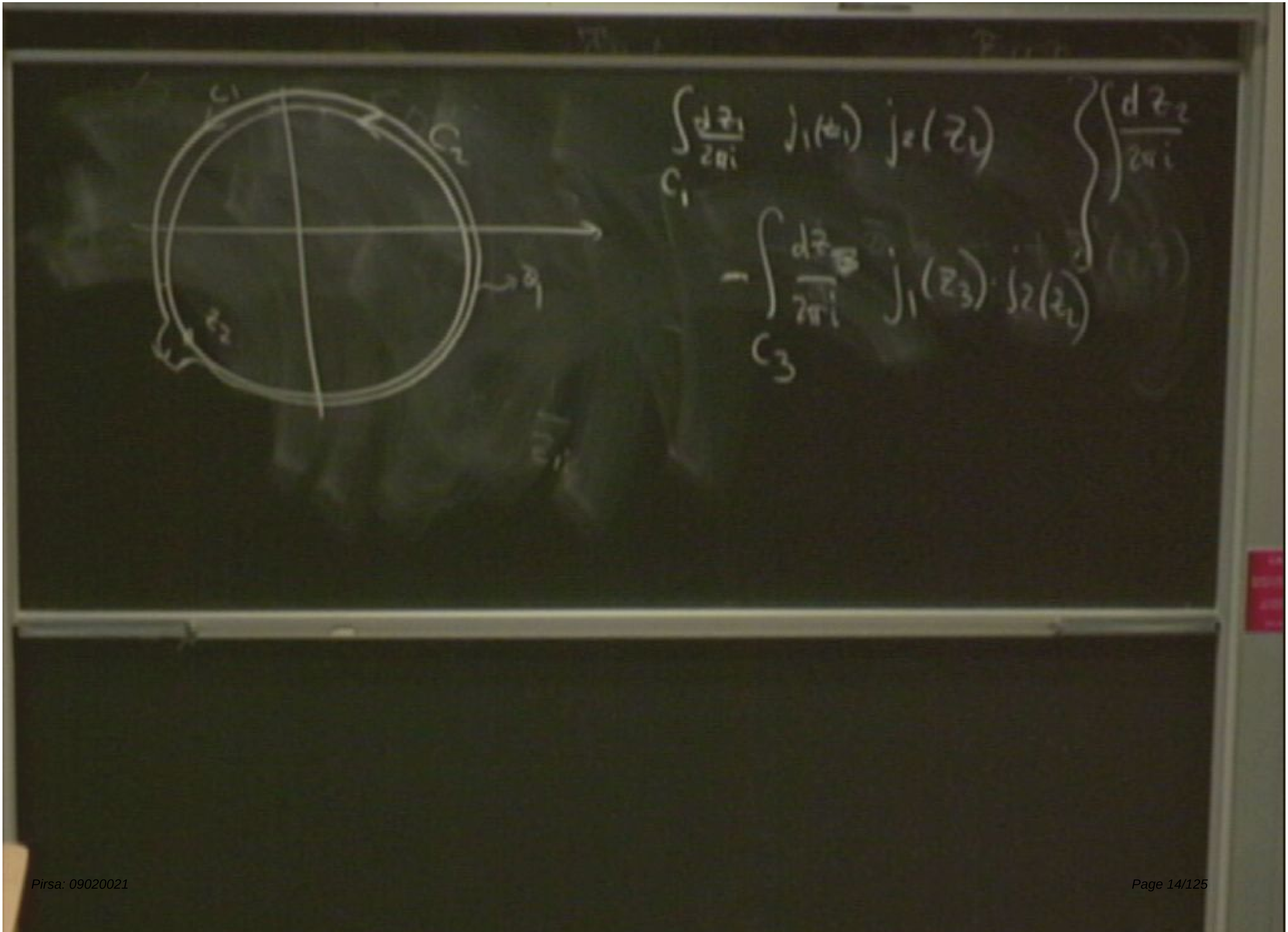

C1
C2
z1
z2
$\int_{C_1} \frac{dz_1}{2\pi i} \, j_1(z_1)$

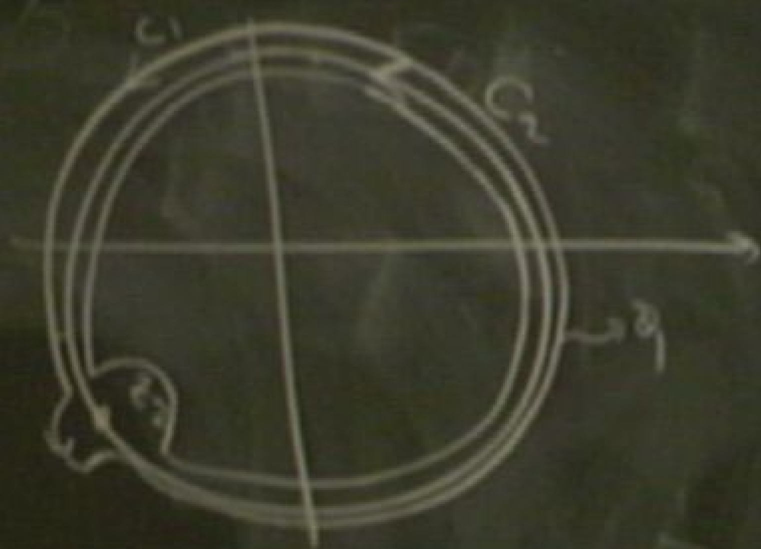


$$\int_{C_1} \frac{dz_1}{2\pi i}\, j_1(z_1)\, j_2(z_2)$$


C1
C2
z1
z2
∫ dz1/2πi j1(z1) j2(z1)
C1
∫ dz2/2πi

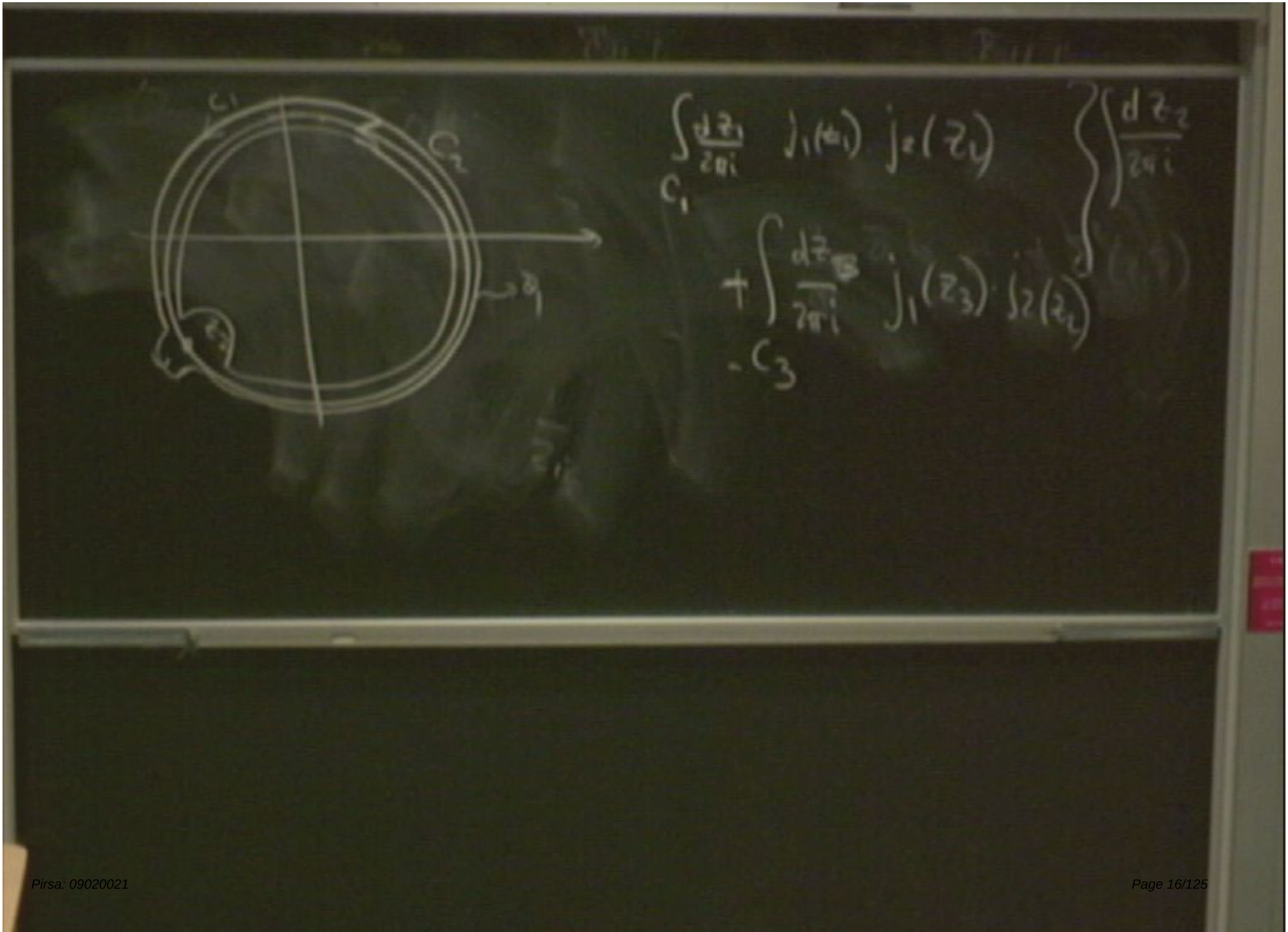

C1
C2
z2
C1


C1
C2
z1
z2
∫ dz1/2πi j1(z1) j2(z2)
C1
− ∫ dz/2πi j1(z3) · j2(z2)
C3
∫ dz2/2πi



$$\int_{C_1} \frac{dz_1}{2\pi i}\, j_1(z_1)\, j_2(z_2) \; - \int_{C_3} \frac{dz_3}{2\pi i}\, j_1(z_3)\, j_2(z_2)$$

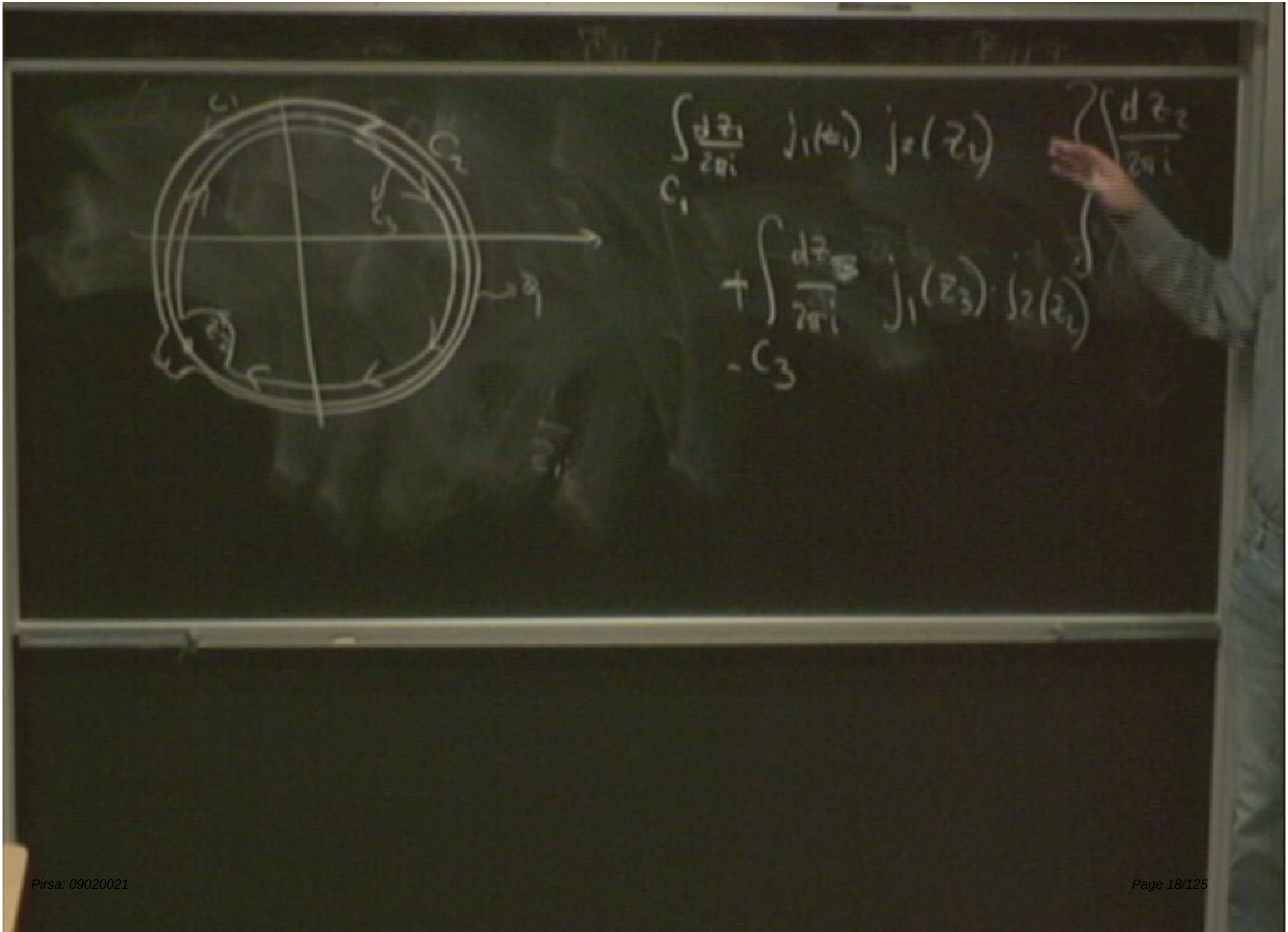
$$\int \frac{dz_2}{2\pi i}$$


c_1
C_2
z_2
$\int_{C_1} \frac{dz_1}{2\pi i} j_1(z_1)\, j_2(z_2)$
$+\int_{-C_3} \frac{dz_3}{2\pi i} j_1(z_3) \cdot j_2(z_2)$
$\int \frac{dz_2}{2\pi i}$

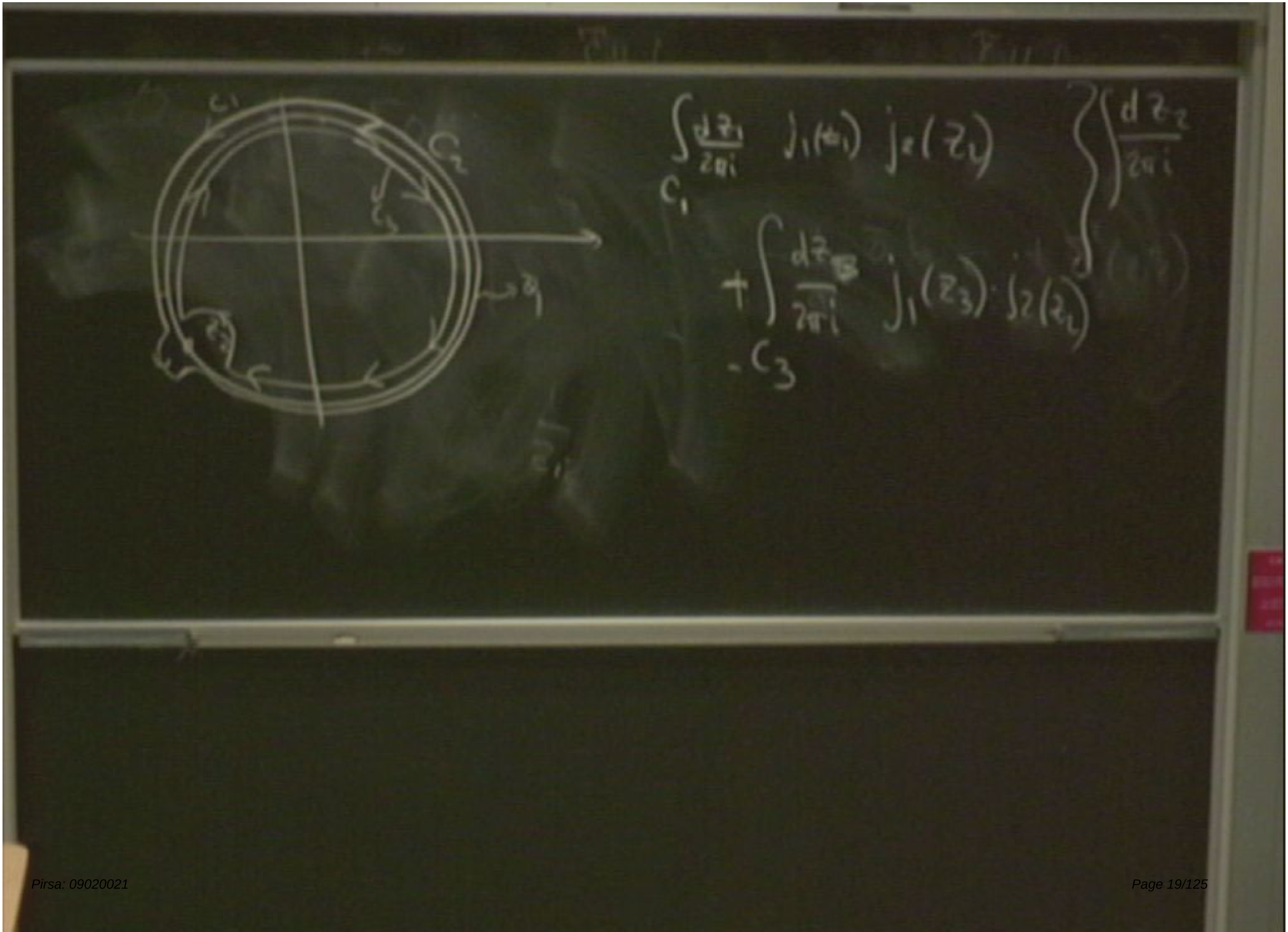
$$\int_{C_1} \frac{dz_1}{2\pi i}\; j_1(z_1)\; j_2(z_2)$$

$$+ \int_{-C_3} \frac{dz_3}{2\pi i}\; j_1(z_3) \cdot j_2(z_2)$$

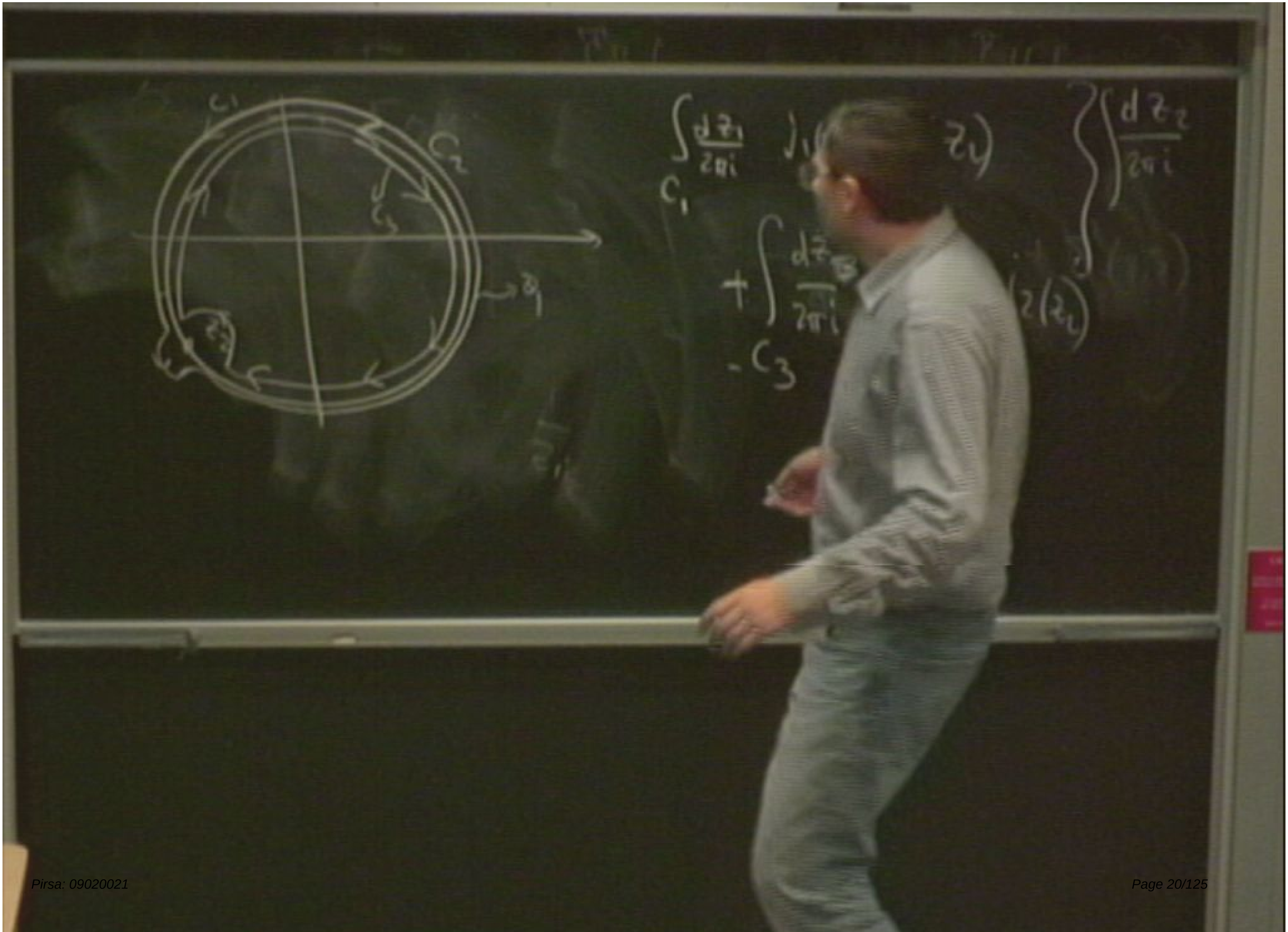
$$\int \frac{dz_2}{2\pi i}$$

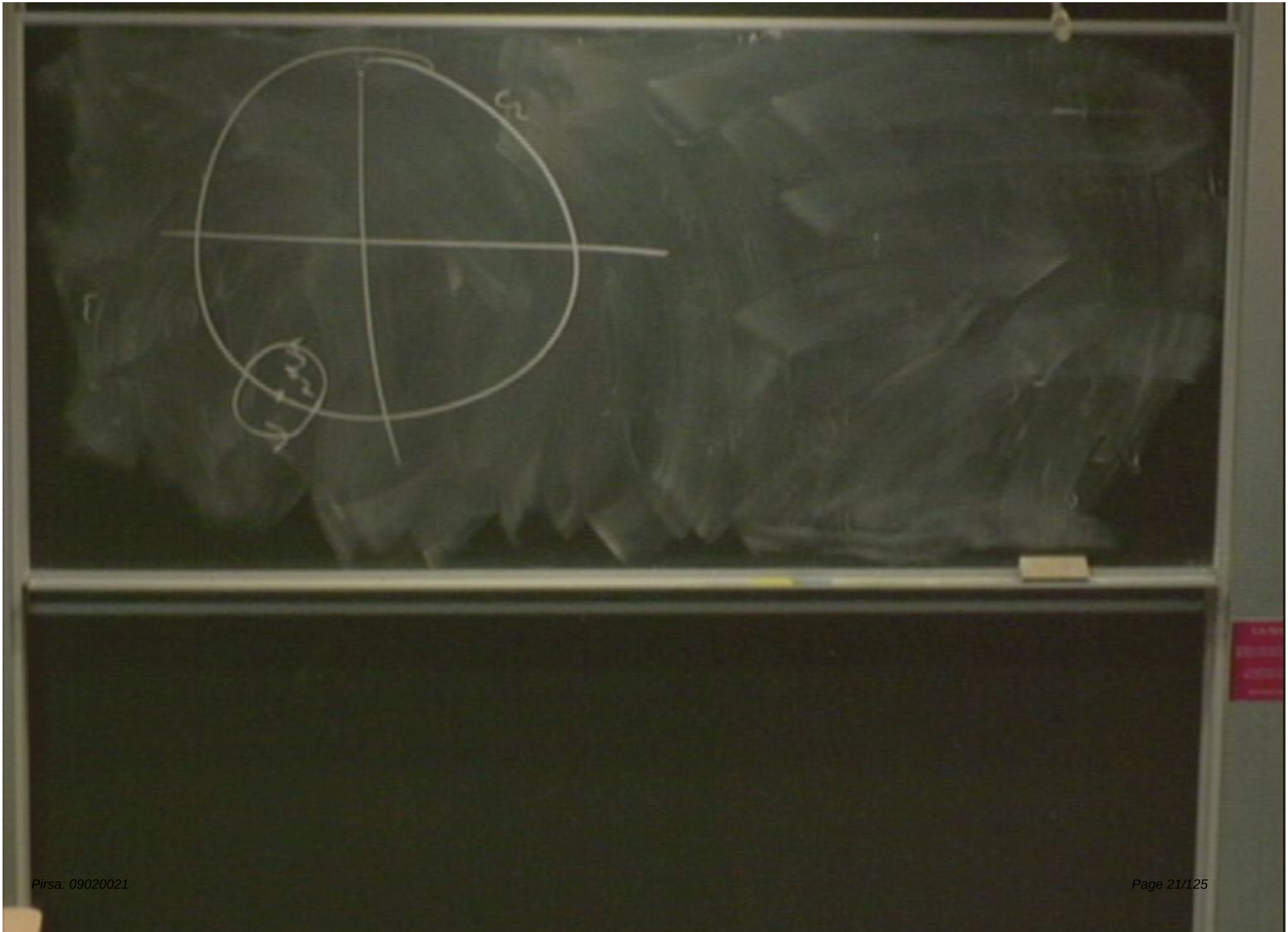


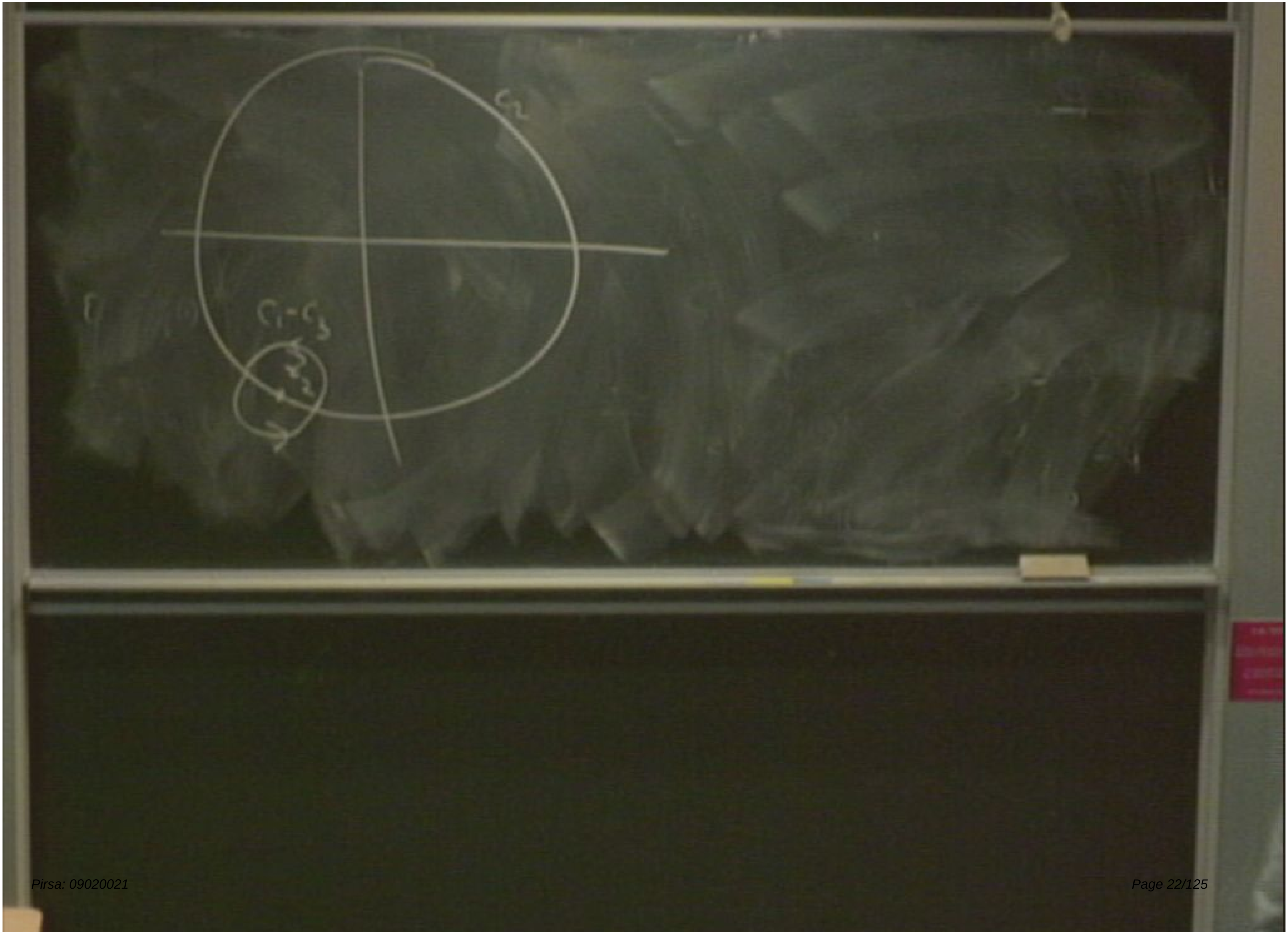
C_1
C_2
C_3
z_3
z_1
$\int_{C_1} \frac{dz_1}{2\pi i} \, j_1(z_1) \, j_2(z_1)$
$+ \int_{-C_3} \frac{dz_3}{2\pi i} \, j_1(z_3) \, j_2(z_4)$
$\int \frac{dz_2}{2\pi i}$



C_1
C_2
z_1
z_2
z_3
$\int_{C_1} \frac{dz_1}{2\pi i}\, j_1(z_1)\, j_2(z_2)$
$+ \int_{-C_3} \frac{dz_3}{2\pi i}\, j_1(z_3) \cdot j_2(z_2)$
$\int \frac{dz_2}{2\pi i}$


C1
C2
C3
2πi
- C3

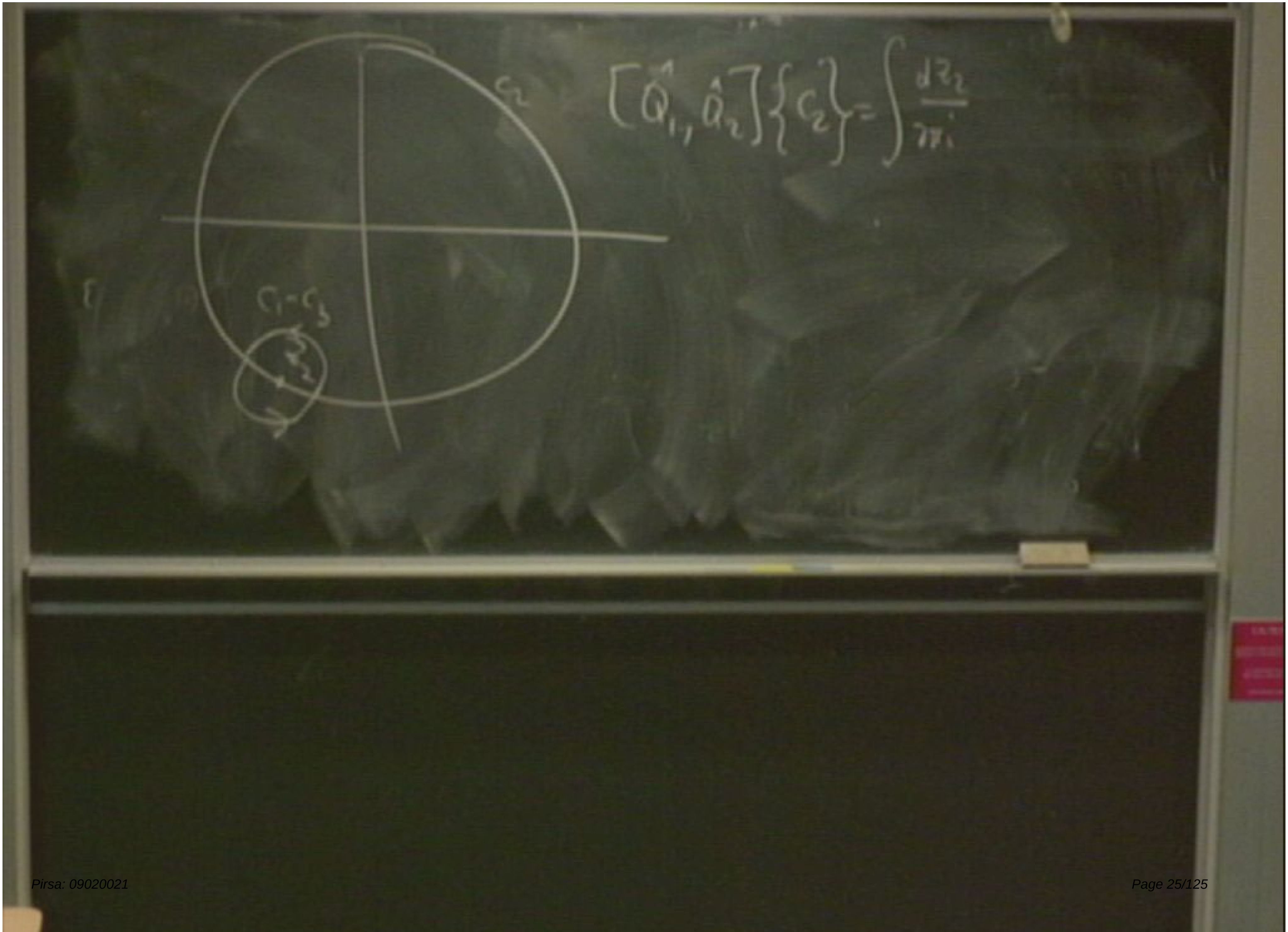


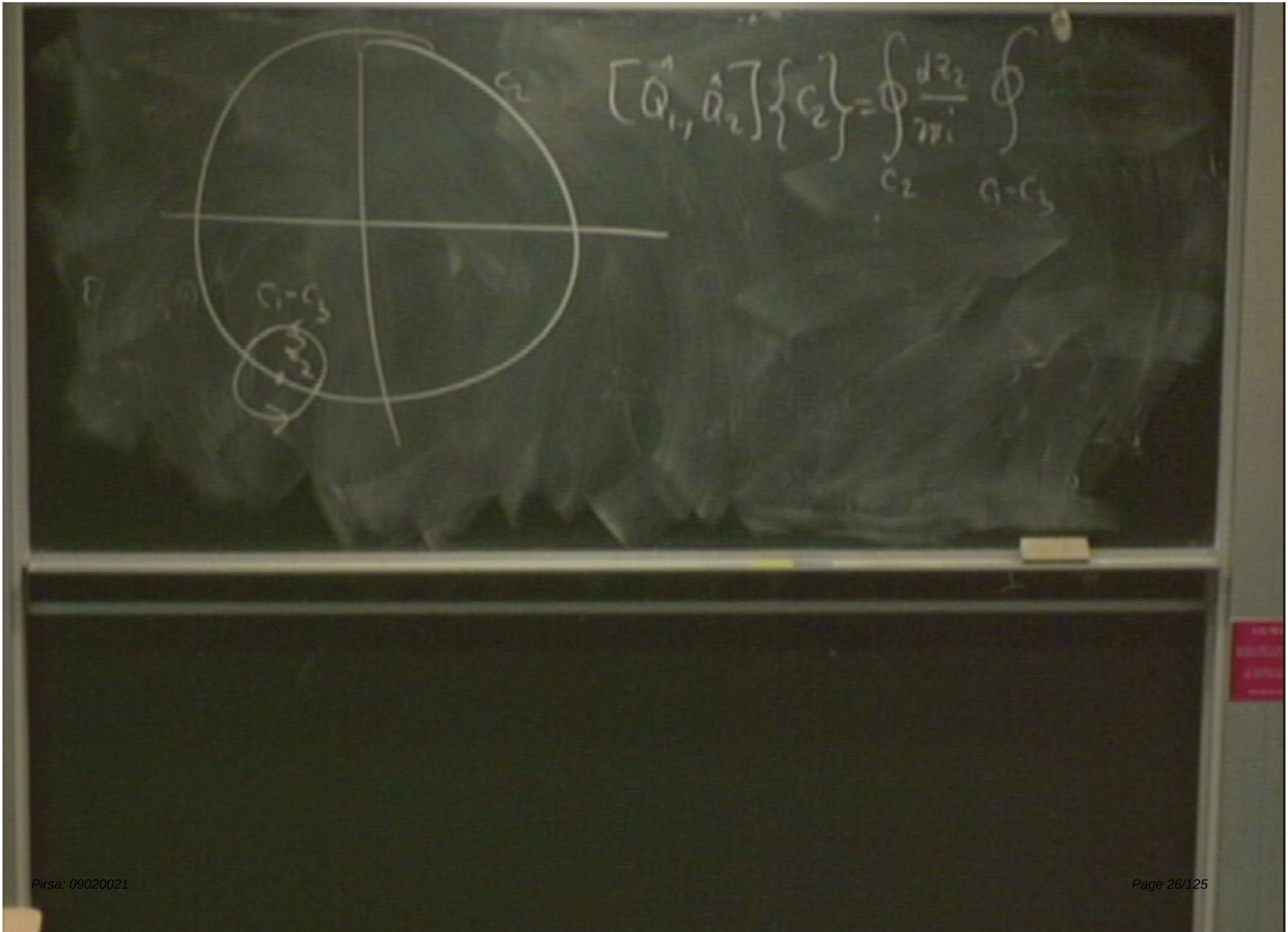


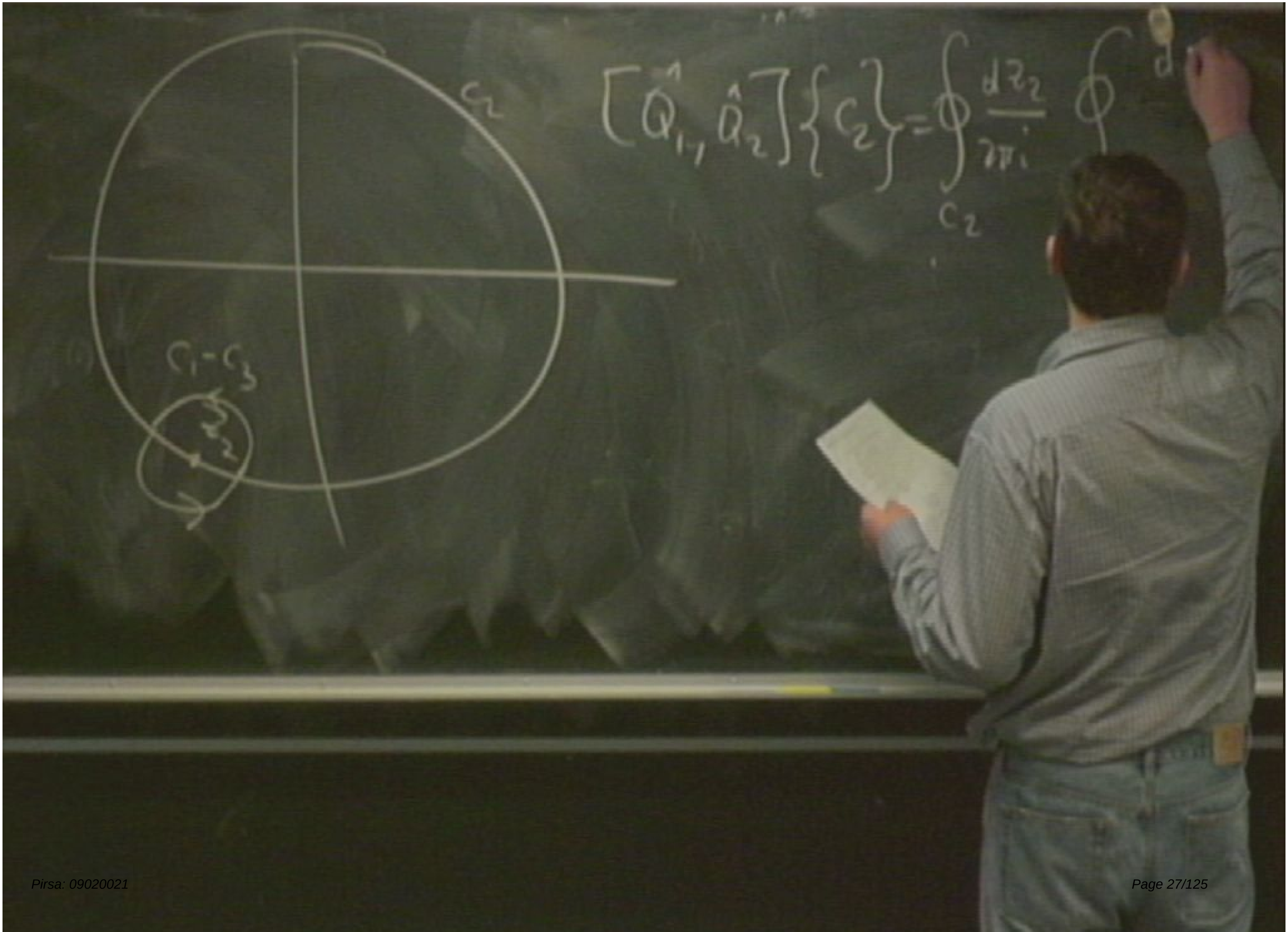
C_2
$C_1 - C_3$
ξ_2

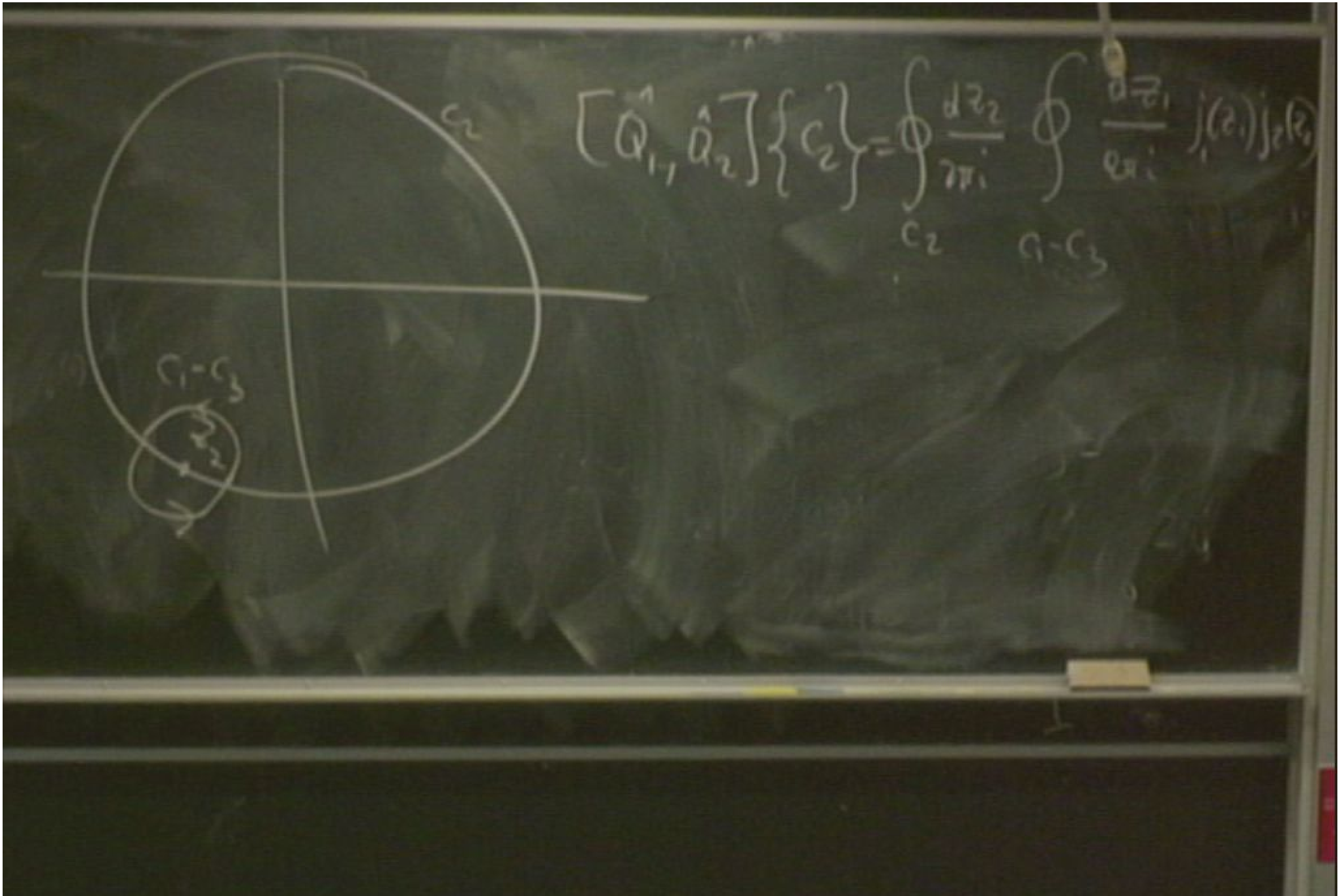

c_2
Q
$c_1 - c_3$


$[\hat{Q}_1, \hat{Q}_2]\{C_2\} =$
C_2
$C_1 - C_3$

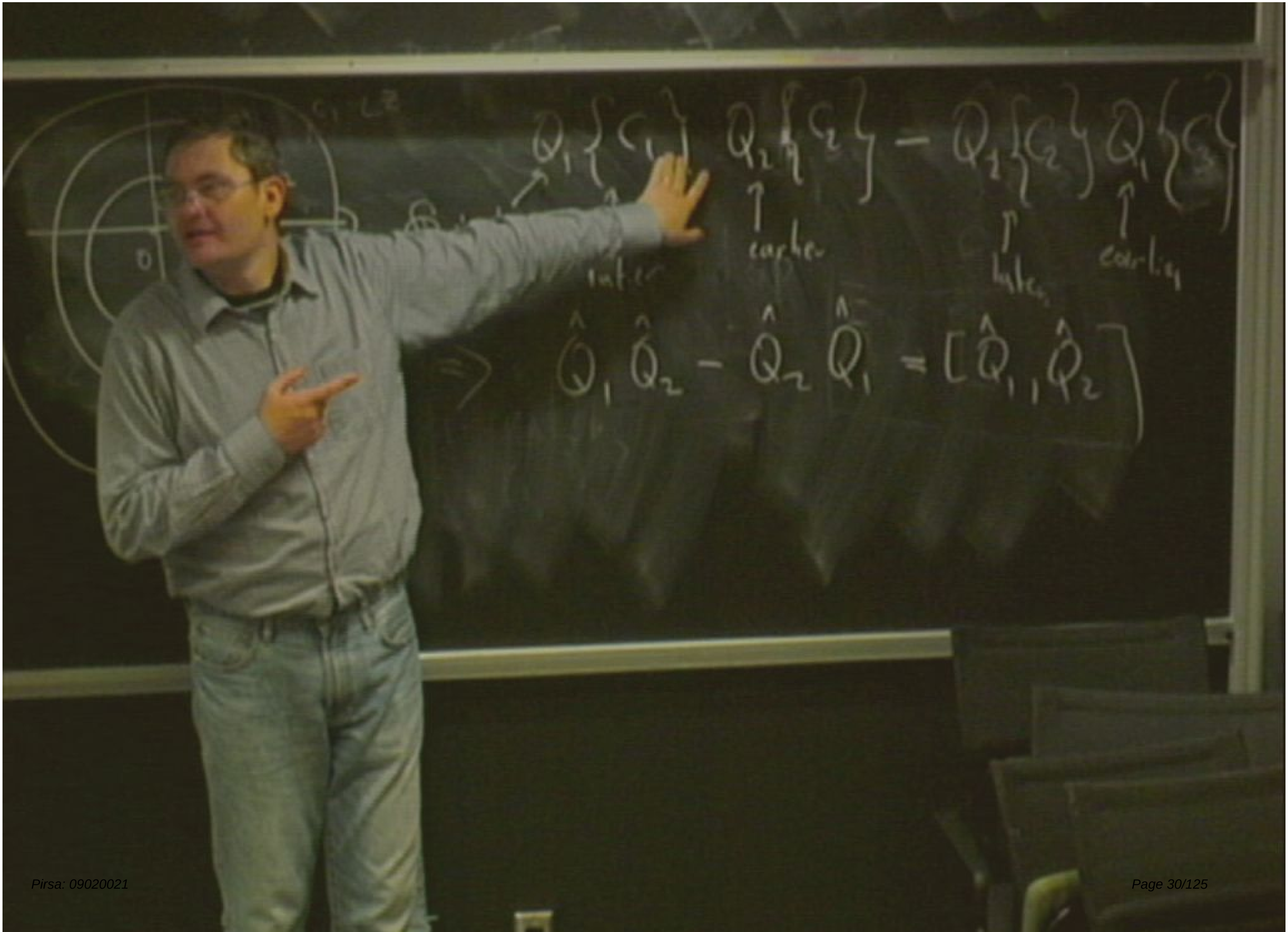

[Q̂1, Q̂2]{C2} = ∫ dz2/2πi
C2
C1 - C3
z2


$[\hat{Q}_{1,1}, \hat{Q}_2]\{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1 - C_3}$
C_2
$C_1 - C_3$
z_2

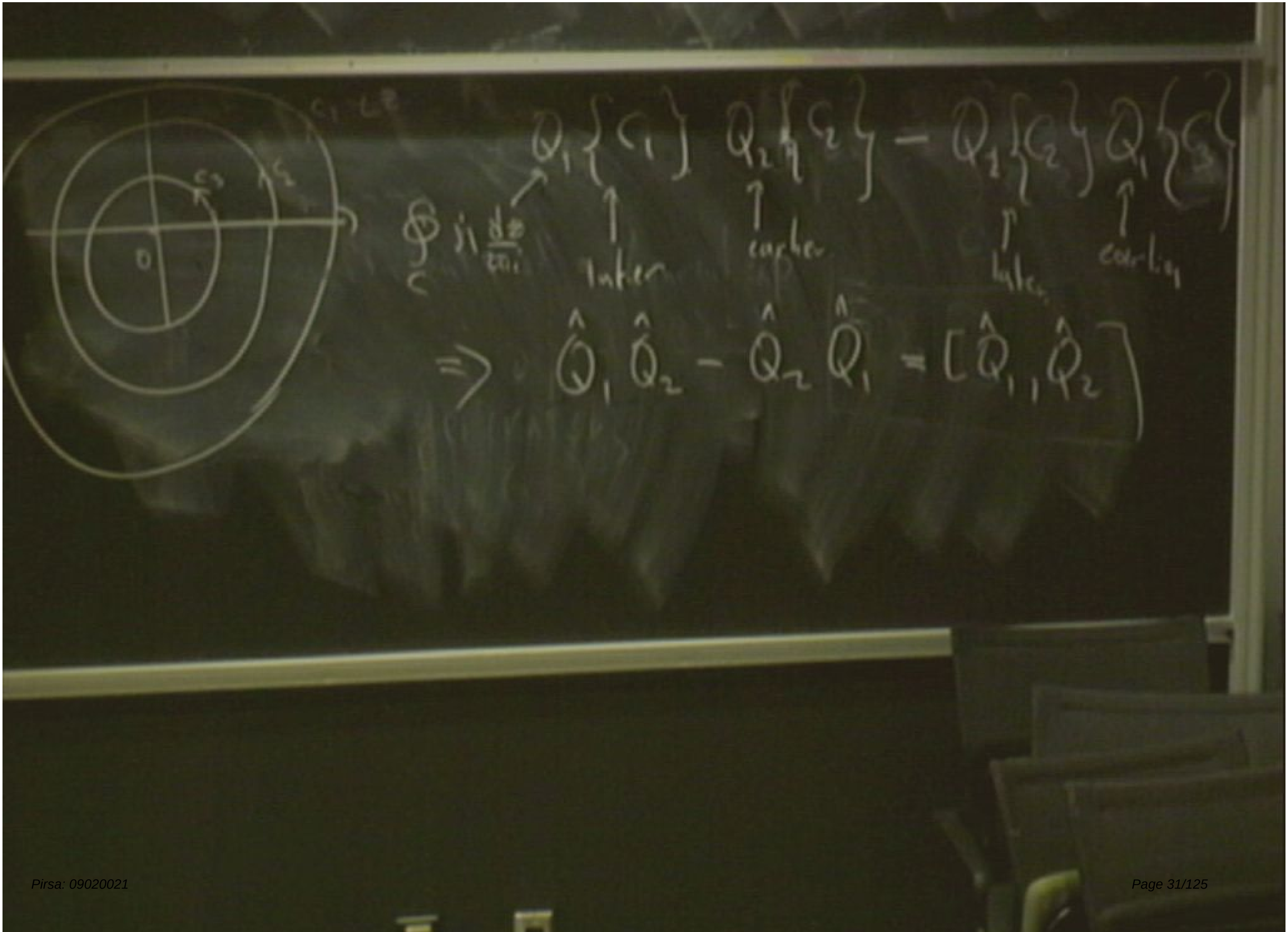

C2
C1-C3
z2
[Q̂1,1, Q̂2]{C2} = ∮ dz2/2πi ∮ d
C2


[Q̂₁, Q̂₂]{C₂} = ∮ dz₂/2πi ∮ dz₁/2πi j₁(z₁) j₂(z₂)
C₂
C₁ - C₃
C₂
C₁ - C₃
z₂

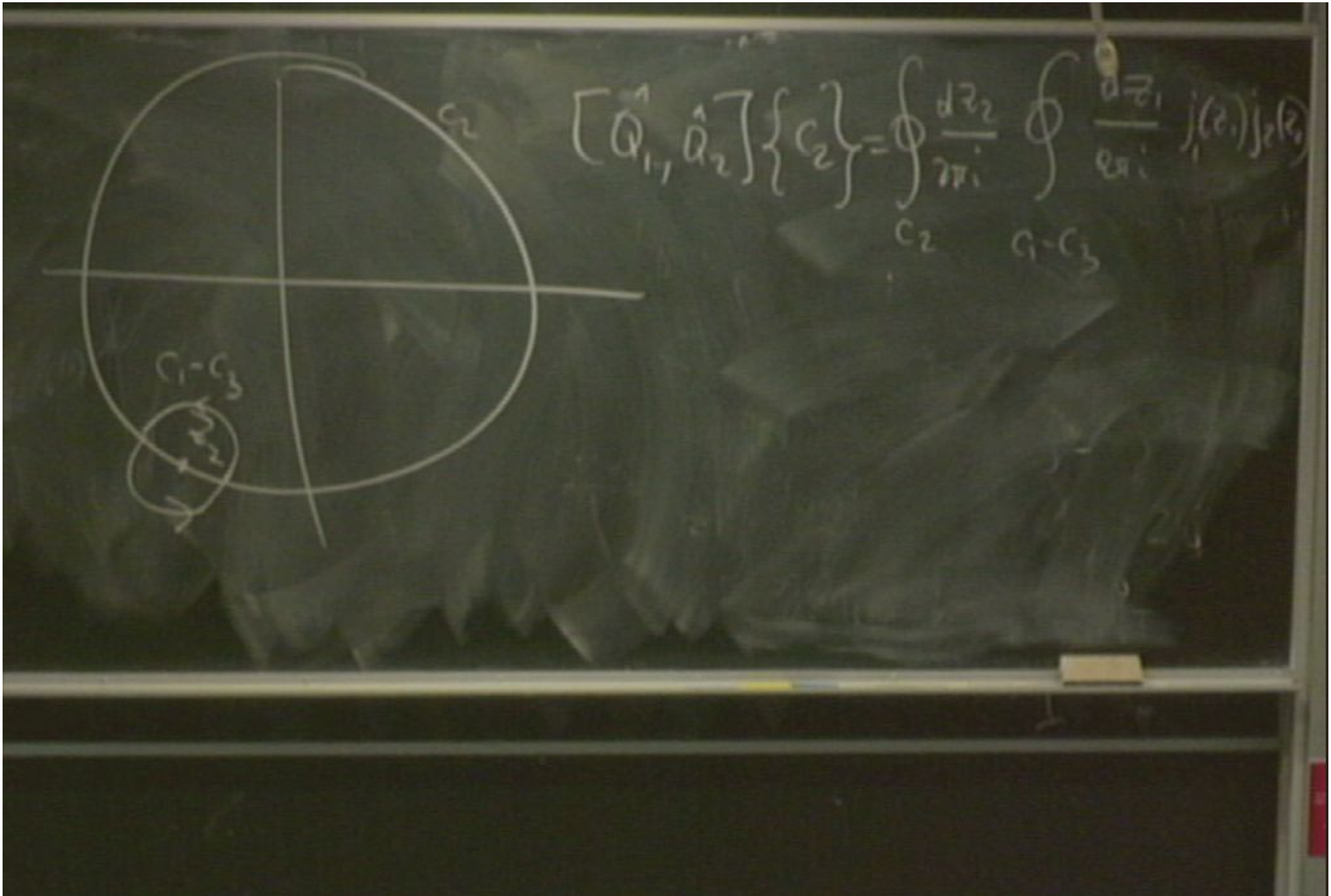
$$[\hat{Q}_1, \hat{Q}_2]\{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1 - C_3} \frac{dz_1}{2\pi i}\, j_1(z_1)\, j_2(z_2)$$



$Q_1\{C_1\}\ Q_2\{C_2\} - Q_2\{C_2\}\ Q_1\{C_3\}$
later
earlier
later
earlier
$\hat{Q}_1 \hat{Q}_2 - \hat{Q}_2 \hat{Q}_1 = [\hat{Q}_1, \hat{Q}_2]$



$Q_1\{C_1\}\ Q_2\{C_2\} - Q_1\{C_2\}\ Q_1\{C_3\}$
$\oint_C j_i \frac{d\sigma}{2\pi i}$
later
earlier
later
earlier
$\Rightarrow \hat{Q}_1 \hat{Q}_2 - \hat{Q}_2 \hat{Q}_1 = [\hat{Q}_1, \hat{Q}_2]$


$[\hat{Q}_1, \hat{Q}_2]\{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1 - C_3} \frac{dz_1}{2\pi i} j_1(z_1) j_2(z_2)$
C_2
$C_1 - C_3$
z_2

$$[\hat{Q}_{1-1}, \hat{Q}_2]\{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1 - C_3} \frac{dz_1}{2\pi i} j_1(z_1)\, j_2(z_2)$$

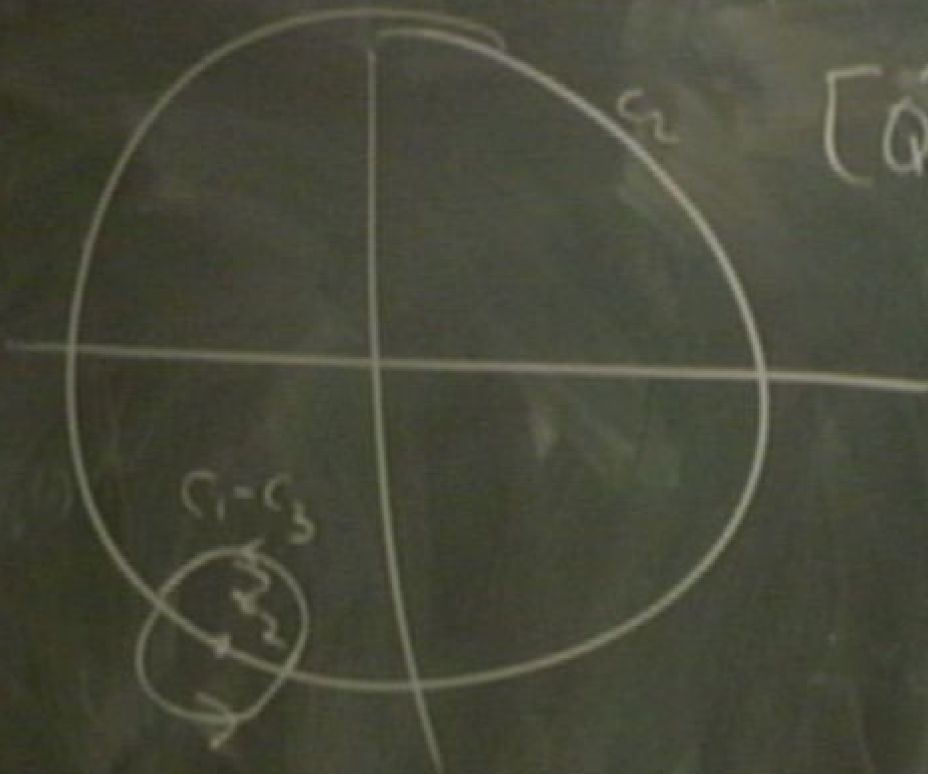
$\uparrow \bar{T}$

evaluate
path in-ky

$$[\hat{Q}_1, \hat{Q}_2]\{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1 - C_3} \frac{dz_1}{2\pi i}\, j_1(z_1)\, j_2(z_2)$$

evaluate path integ

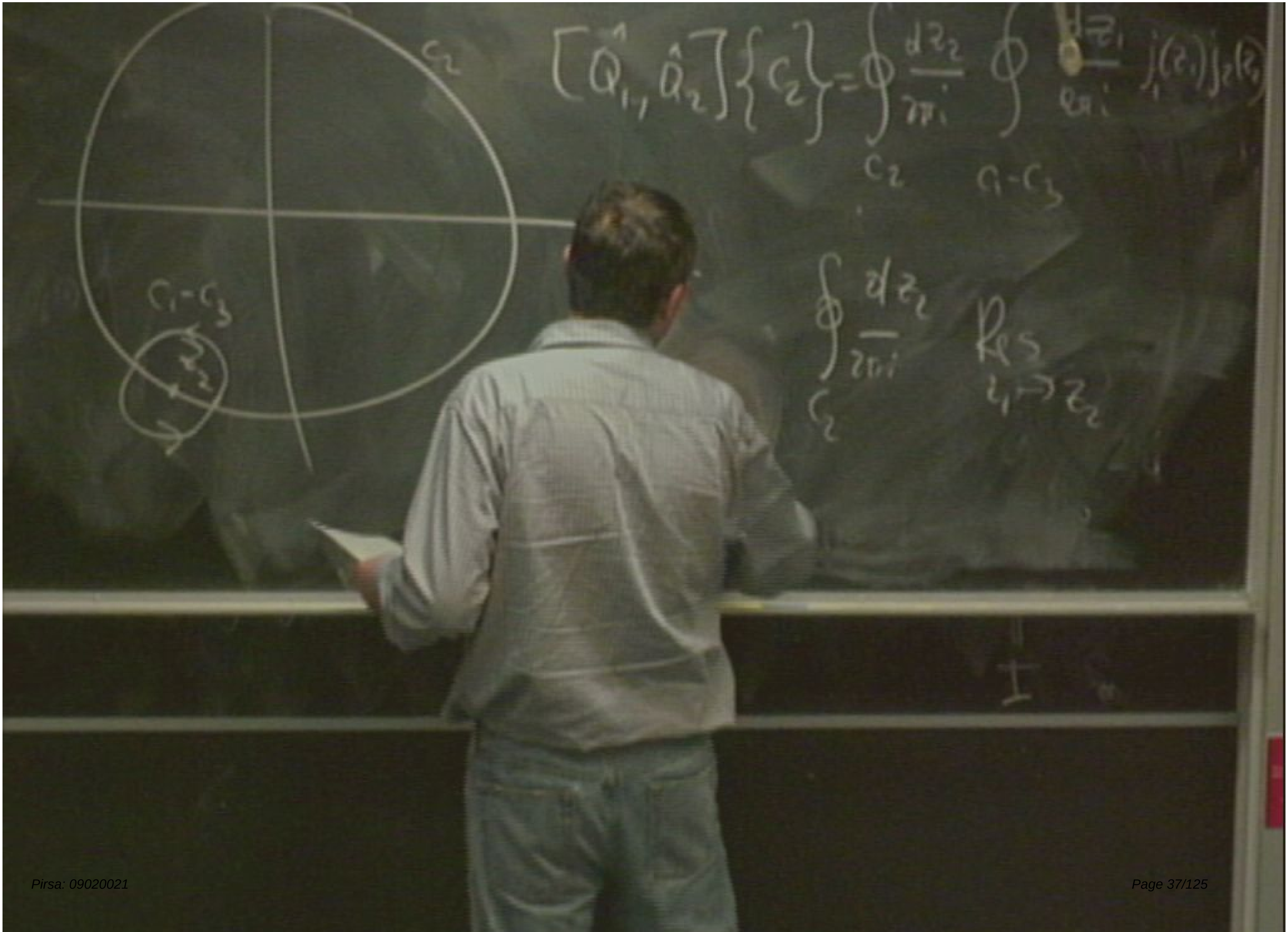
C2
C1 - C3
C2
C1 - C3
evaluate
path integ
C2

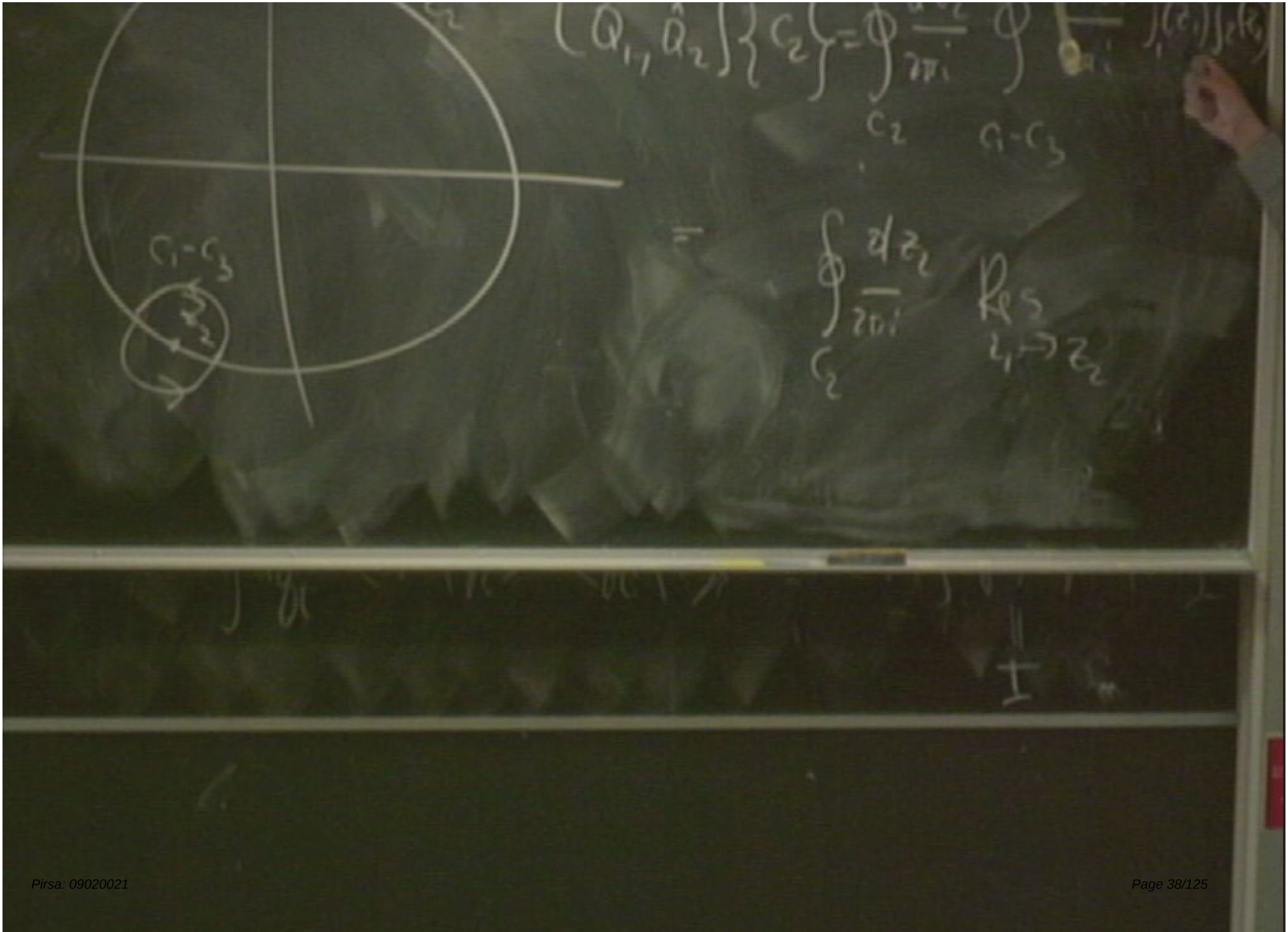


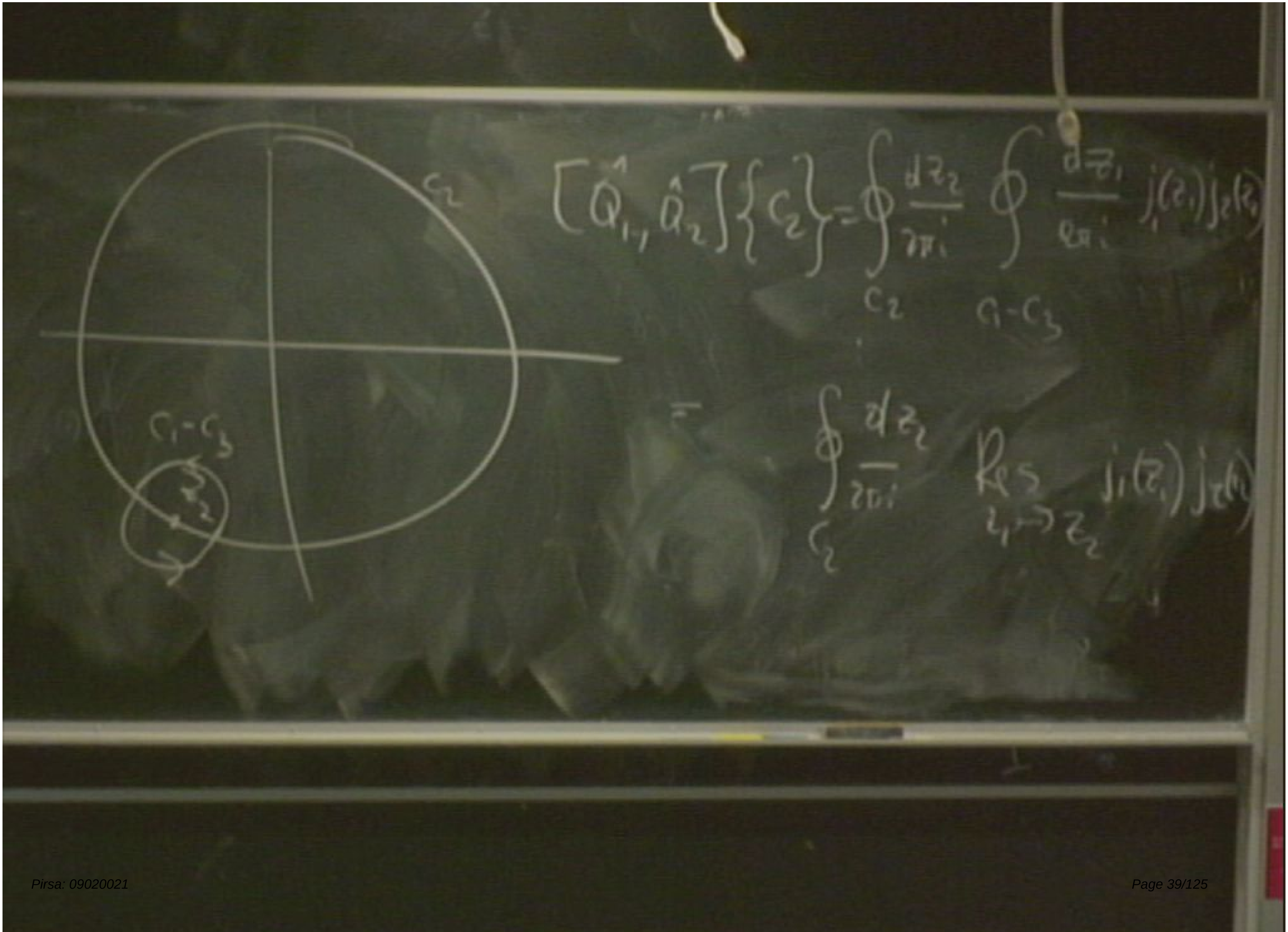
$$[\hat{Q}_1, \hat{Q}_2]\{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1 - C_3} \frac{dz_1}{2\pi i}\, j_1(z_1)\, j_2(z_2)$$

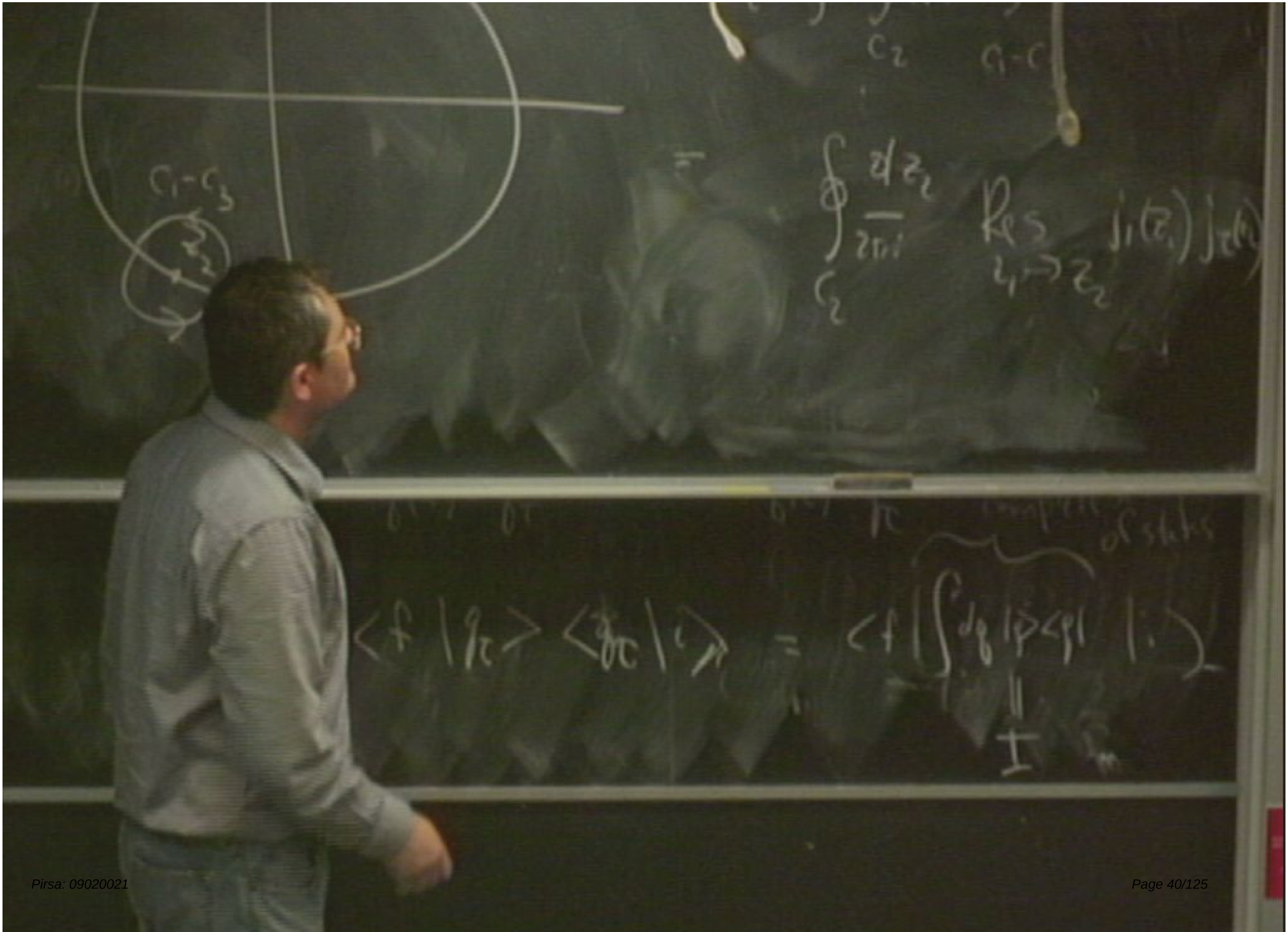
$$\uparrow \qquad \oint_{C_2} \frac{dz_2}{2\pi i}$$

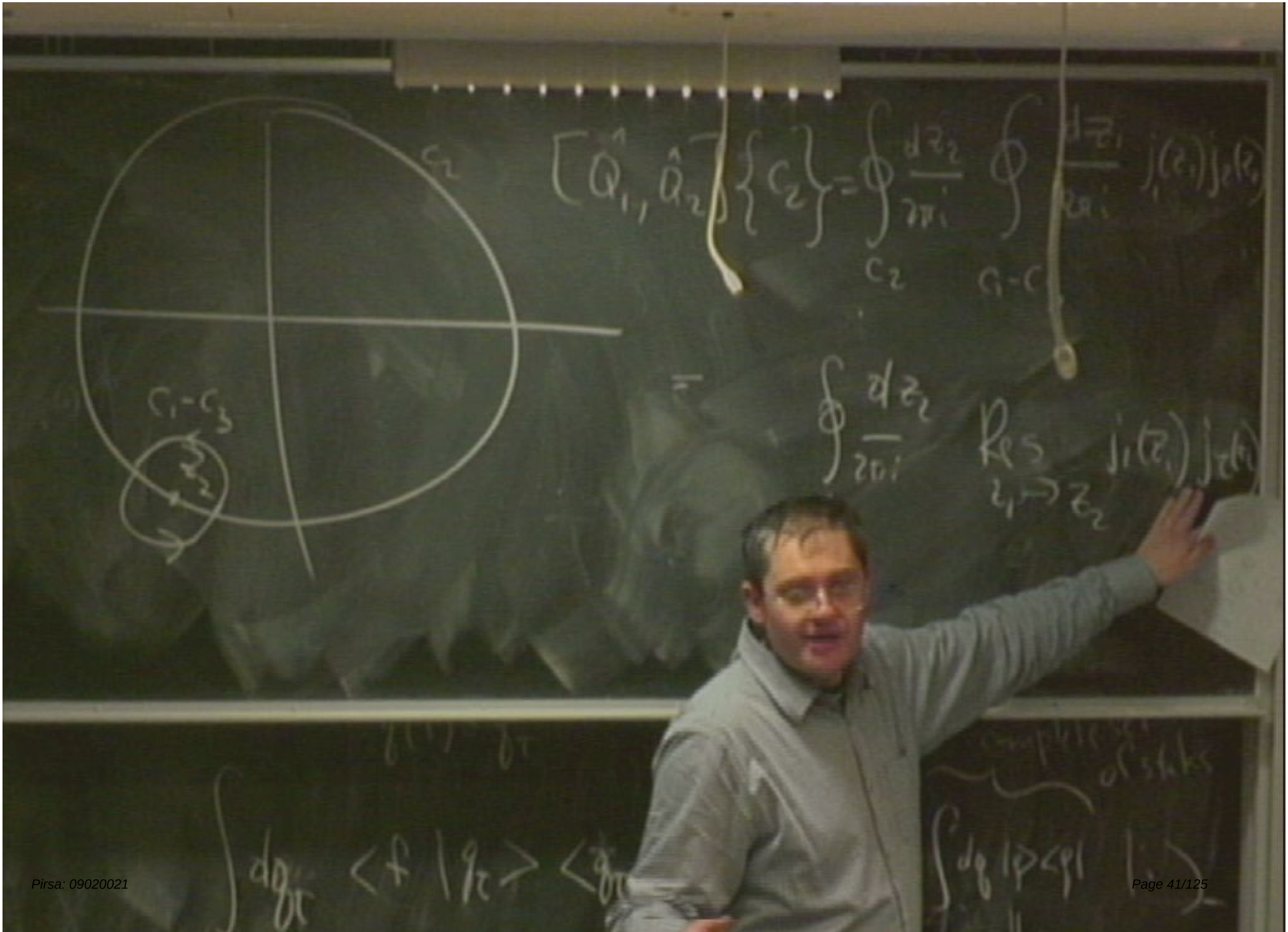
evaluate
path-ordering

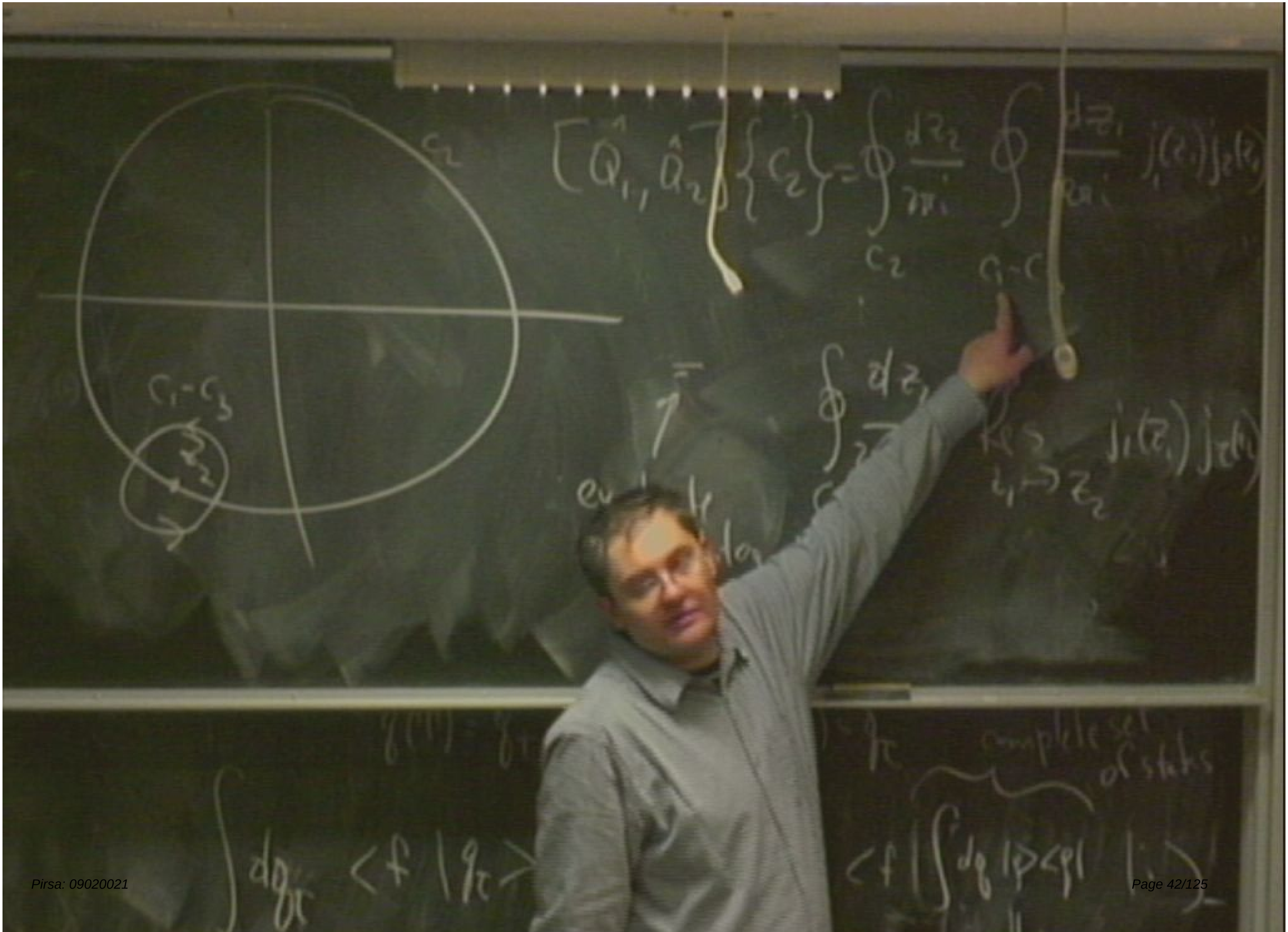

C2
C1-C3
C2
C1-C3
Res
z1→z2

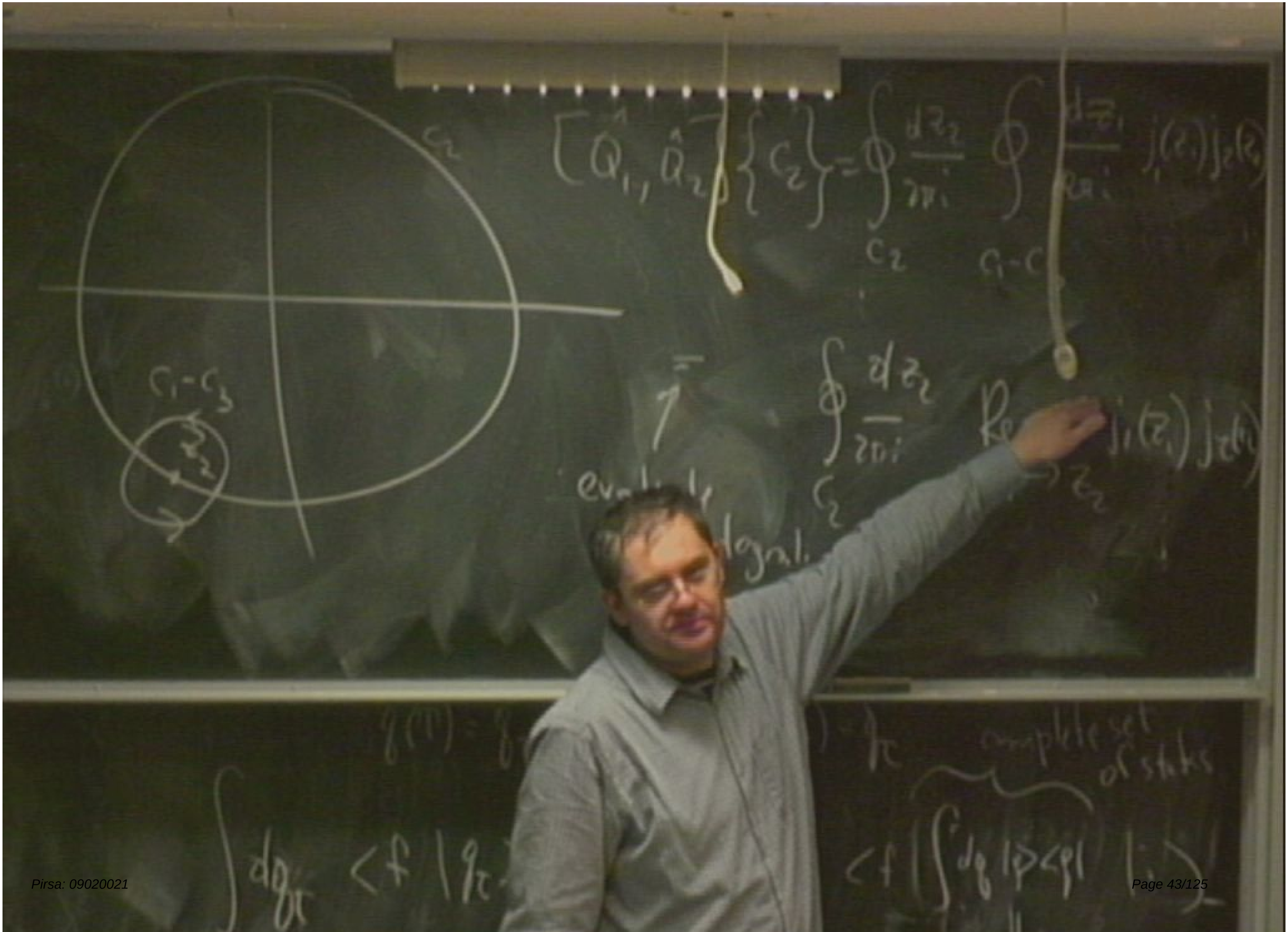

c1 - c3
c2
Res
z1 → z2

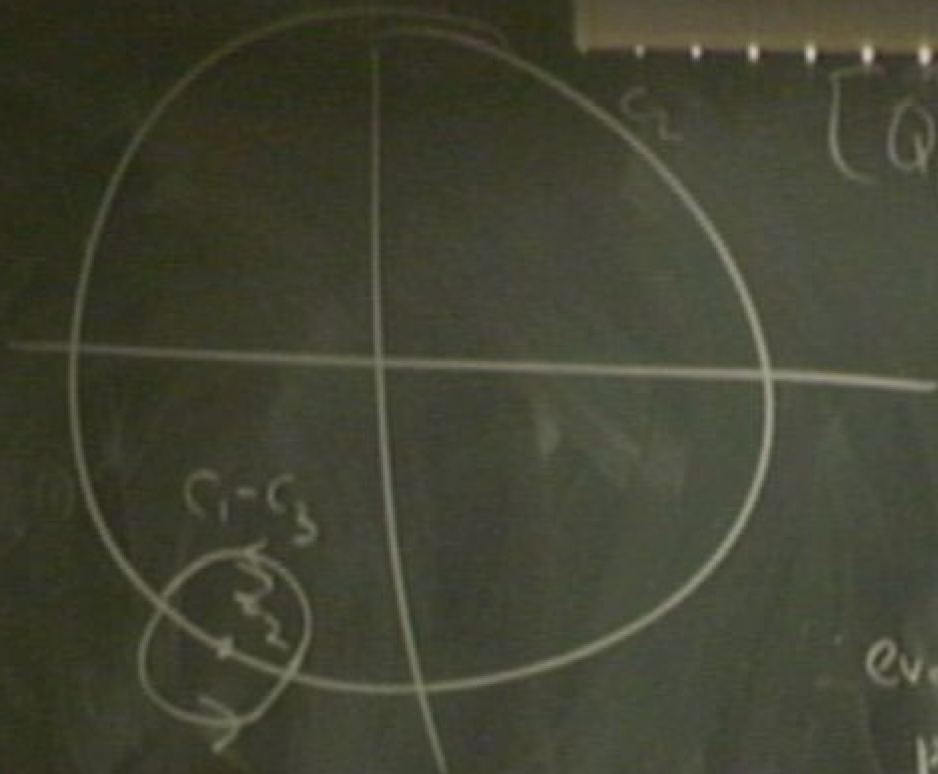

$[\hat{Q}_{1,}, \hat{Q}_2]\{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1 - C_3} \frac{dz_1}{2\pi i} j_1(z_1) j_2(z_2)$
$= \oint_{C_2} \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \to z_2} j_1(z_1) j_2(z_2)$
C_2
$C_1 - C_3$
z_2


c1 - c3
C2
Res
of states


$[\hat{Q}_1, \hat{Q}_2]\{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1 - C_3} \frac{dz_1}{2\pi i} j_1(z_1) j_2(z_2)$
$= \oint \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \to z_2} j_1(z_1) j_2(z_2)$
$C_1 - C_3$
C_2
z_2


complete set of states


complete set
of states



$$[\hat{Q}_1, \hat{Q}_2]\{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1 - C_3} \frac{dz_1}{2\pi i} \, j_1(z_1) j_2(z_2)$$

$$= \oint_{C_2} \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \to z_2} j_1(z_1) j_2(z_2)$$

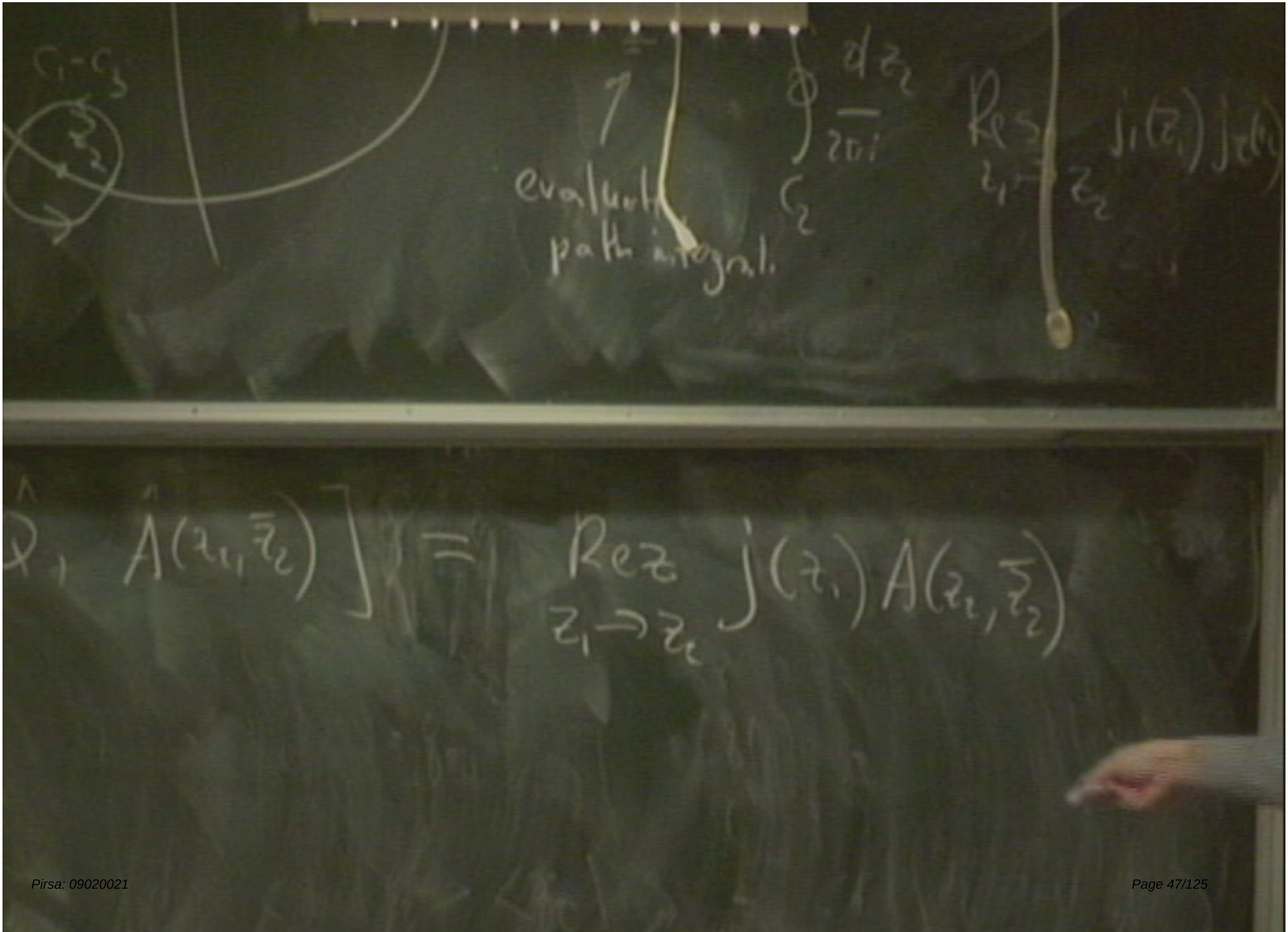
evaluate path integral.

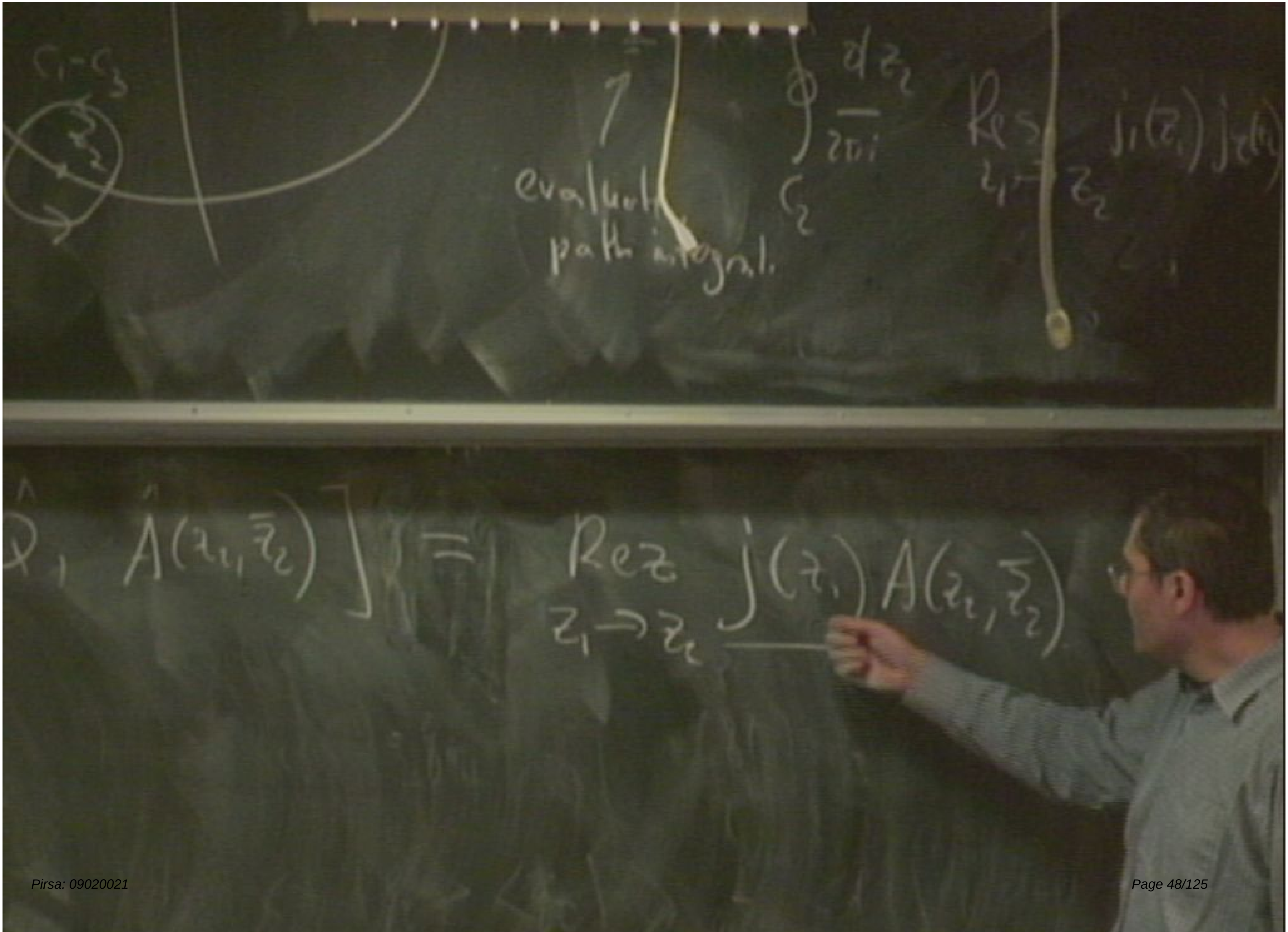
$q(T) = q_T$ $\qquad q(\tau) = q_\tau$

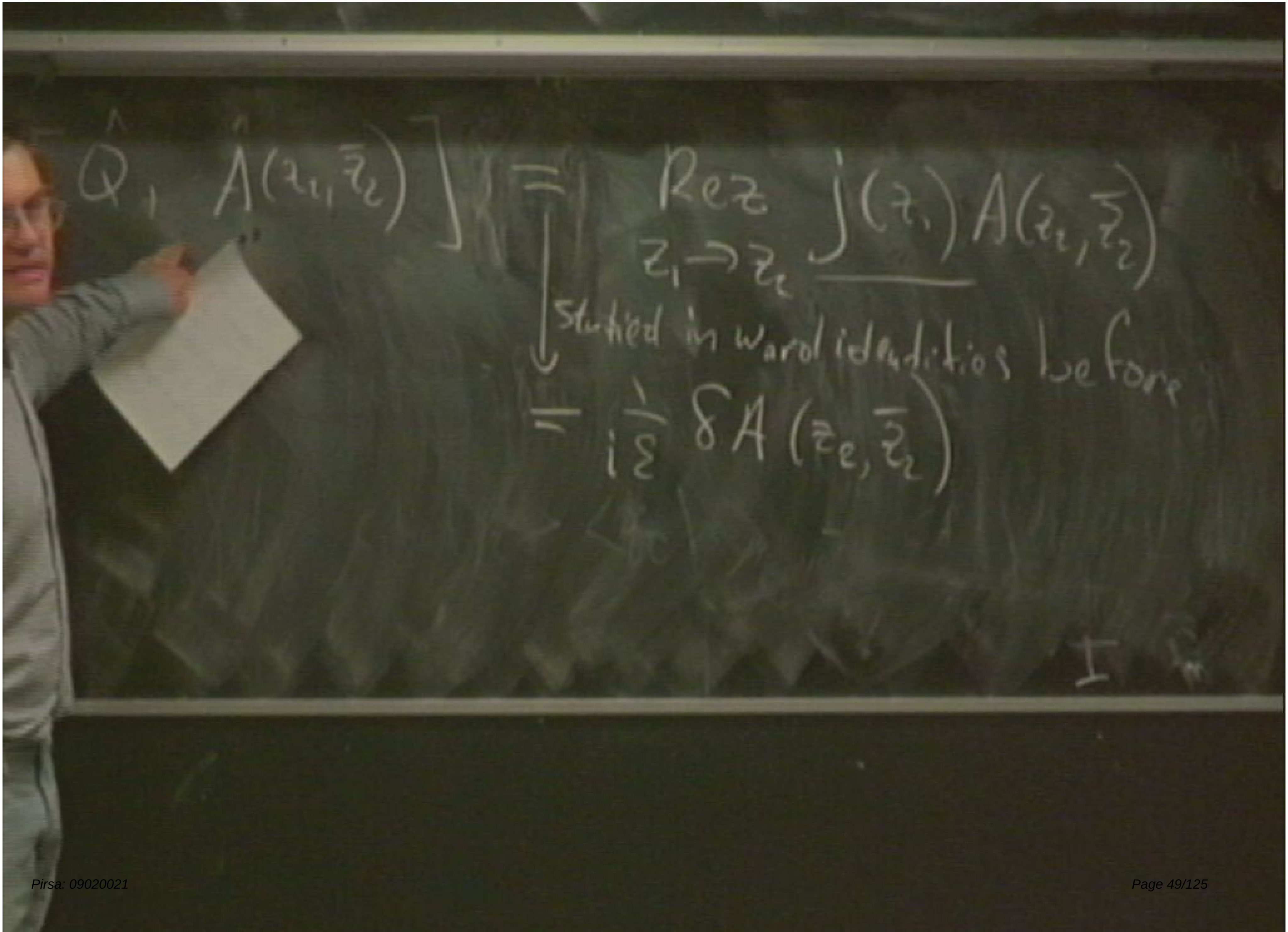
$$\int dq_\tau \langle f | q_\tau \rangle \langle q_\tau | i \rangle = \langle f | \underbrace{\Big(\int dq \, |q\rangle\langle q| \Big)}_{\text{complete set of states}} | i \rangle$$


$[\hat{Q},$

$$\left[\hat{Q}, \hat{A}(z_2,\bar{z}_2)\right] = \operatorname*{Res}_{z_1\to z_2} j(z_1)\, A(z_2,\bar{z}_2)$$


evaluate
path integral
$\oint_{C_2} \frac{dz_2}{2\pi i}$ Res$_{z_1 \to z_2}$ $j_1(z_1)\, j_2(z_2)$
$A(z_1, \bar{z}_2)\,] =$ Rez$_{z_1 \to z_2}$ $j(z_1)\, A(z_2, \bar{z}_2)$


$C_1 - C_3$
evaluate path integral.
$\oint_{C_2} \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \to z_2} j_1(z_1) j_2(z_2)$
$\operatorname{Res}_{z_1 \to z_2} j(z_1) \hat{A}(z_2, \bar{z}_2)$


$[\hat{Q}, \hat{A}(z_2, \bar{z}_2)] = \mathrm{Res}_{z_1 \to z_2} \, j(z_1) A(z_2, \bar{z}_2)$
studied in ward identities before
$= \frac{1}{i\epsilon} \delta A(z_2, \bar{z}_2)$

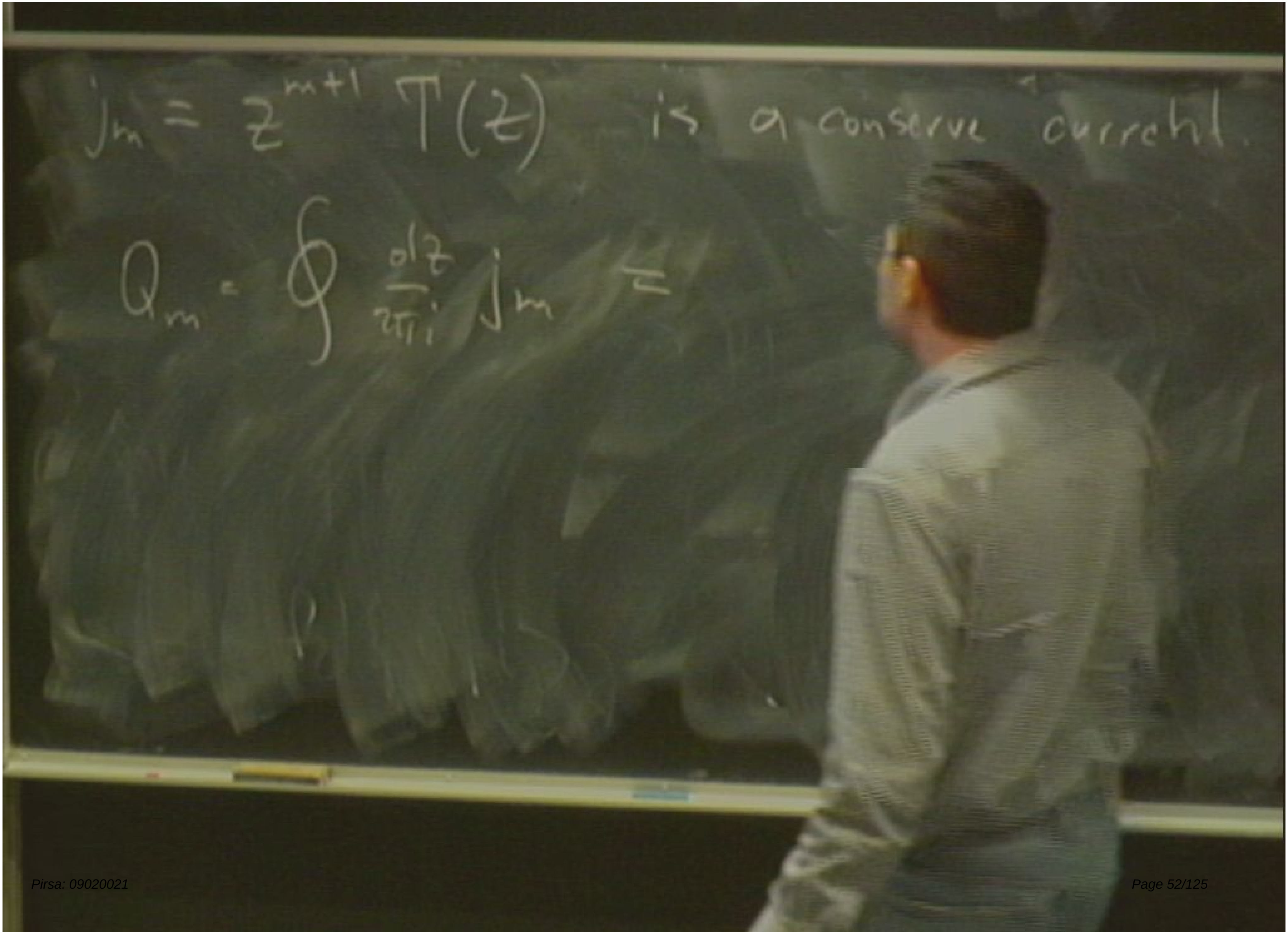
$$[\hat{Q}, \hat{A}(z_1,\bar{z}_2)] = \mathrm{Res}_{z_1 \to z_2}\, \underline{j(z_1)}\, A(z_2,\bar{z}_2)$$

studied in ward identities before

$$= \frac{1}{i\varepsilon}\,\delta A(z_2,\bar{z}_2)$$

OPE $\Rightarrow$ commutator algebra!

$$j_m = z^{m+1}\, T(z)$$


$j_m = z^{m+1}\, T(z)$ is a conserve current.
$Q_m = \oint \frac{dz}{2\pi i}\, j_m =$

$j_m = z^{m+1}\, T(z)$ is a conserve current.

$$Q_m = \oint \frac{dz}{2\pi i}\, j_m = \oint \frac{dz}{2\pi i\, z}\, z^{m+2}\, T(z$$

$j_m = z^{m+1}\, T(z)$ is a conserve current

Virasoro generators

$$Q_m = \oint \frac{dz}{2\pi i}\, j_m = \oint \frac{dz}{2\pi i\, z}\, z^{m+2}\, T(z) \equiv L_m$$

$$[L_m, L_n] = \oint_{C_2} \frac{dz_2}{2\pi i} \operatorname*{Res}_{z_1 \to z_2} j_m(z_1)\, j_n(z_2)$$

$[\hat{Q}_1, \hat{Q}_2]\{C_2\} = \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_1 - C_3} \frac{dz_1}{2\pi i} j_1(z_1) j_2(z_2)$
$\oint \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \to z_2} j_1(z_1) j_2(z_2)$
C_2
$C_1 - C_3$
z_2

$j_m = z^{m+1} T(z)$ is a conserve current

virasoro generators

$$Q_m = \oint \frac{dz}{2\pi i} j_m = \oint \frac{dz}{2\pi i\, z} z^{m+2} T(z) \equiv L_m$$

$$[L_m, L_n] = \oint_{C_2} \frac{dz_2}{2\pi i} \operatorname*{Res}_{z_1 \to z_2} j_m(z_1)\, j_n(z_2)$$

$$\mathrm{Res}\; j_m j_n = \mathrm{Res}\; z_1^{m+1} z_2^{n+1}\, T(z_1) T(z_2)$$

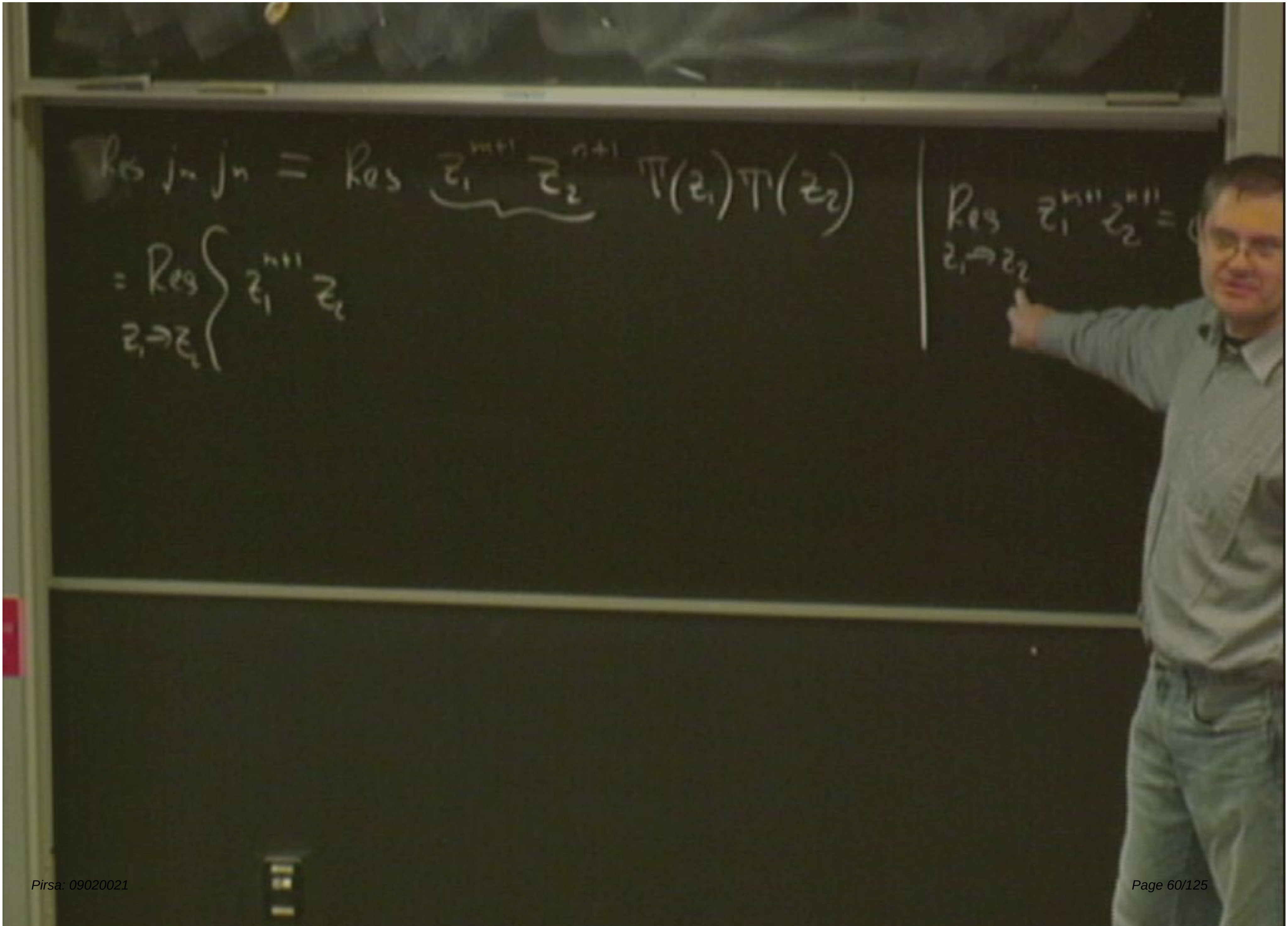
$$\mathrm{Res}\, j_m j_n = \mathrm{Res}\, \underbrace{z_1^{m+1} \; z_2^{n+1}} \, \Pi(z_1)\Pi(z_2)$$

$$= \mathrm{Res}_{z_1 \to z_2} \Big\{ z_1^{n+1} z_2$$

$$\mathrm{Res}\, j_m j_n = \mathrm{Res}\, \underbrace{z_1^{m+1} z_2^{n+1}}\, T(z_1) T(z_2)$$

$$= \operatorname*{Res}_{z_1 \to z_2} \Big\{ z_1^{m+1} z_2$$

$$\operatorname*{Res}_{z_1 \to z_2} z_1^{m+1} z_2^{n+1} =$$


$\text{Res}\, j_n j_n = \text{Res}\, z_1^{n+1} z_2^{n+1}\, T(z_1) T(z_2)$
$= \text{Reg}_{z_1 \to z_2} \{ z_1^{n+1} z_2$
$\text{Res}_{z_1 \to z_2}\, z_1^{n+1} z_2^{n+1} =$

$$\operatorname{Res} j_m j_n = \operatorname{Res} \underbrace{z_1^{m+1} z_2^{n+1}}\, T(z_1)\, T(z_2)$$

$$= \operatorname*{Res}_{z_1 \to z_2} \left\{ z_1^{m+1} z_2^{n+1} \left[\frac{c}{2 z_{12}} \right.\right.$$

$$\operatorname*{Res}_{z_1 \to z_2} z_1^{n+1} z_2^{n+1} = 0$$

$$\mathrm{Res}\, j_m j_n = \mathrm{Res}\, \underbrace{z_1^{m+1} z_2^{n+1}} T(z_1) T(z_2)$$

$$= \mathrm{Res}_{z_1 \to z_2} \left\{ z_1^{m+1} z_2^{n+1} \left[\frac{c}{2 z_{12}^4} + \frac{2}{z_{12}^2} T(z_2) + \frac{1}{z_{12}} \partial T(z_2) \right] \right\}$$

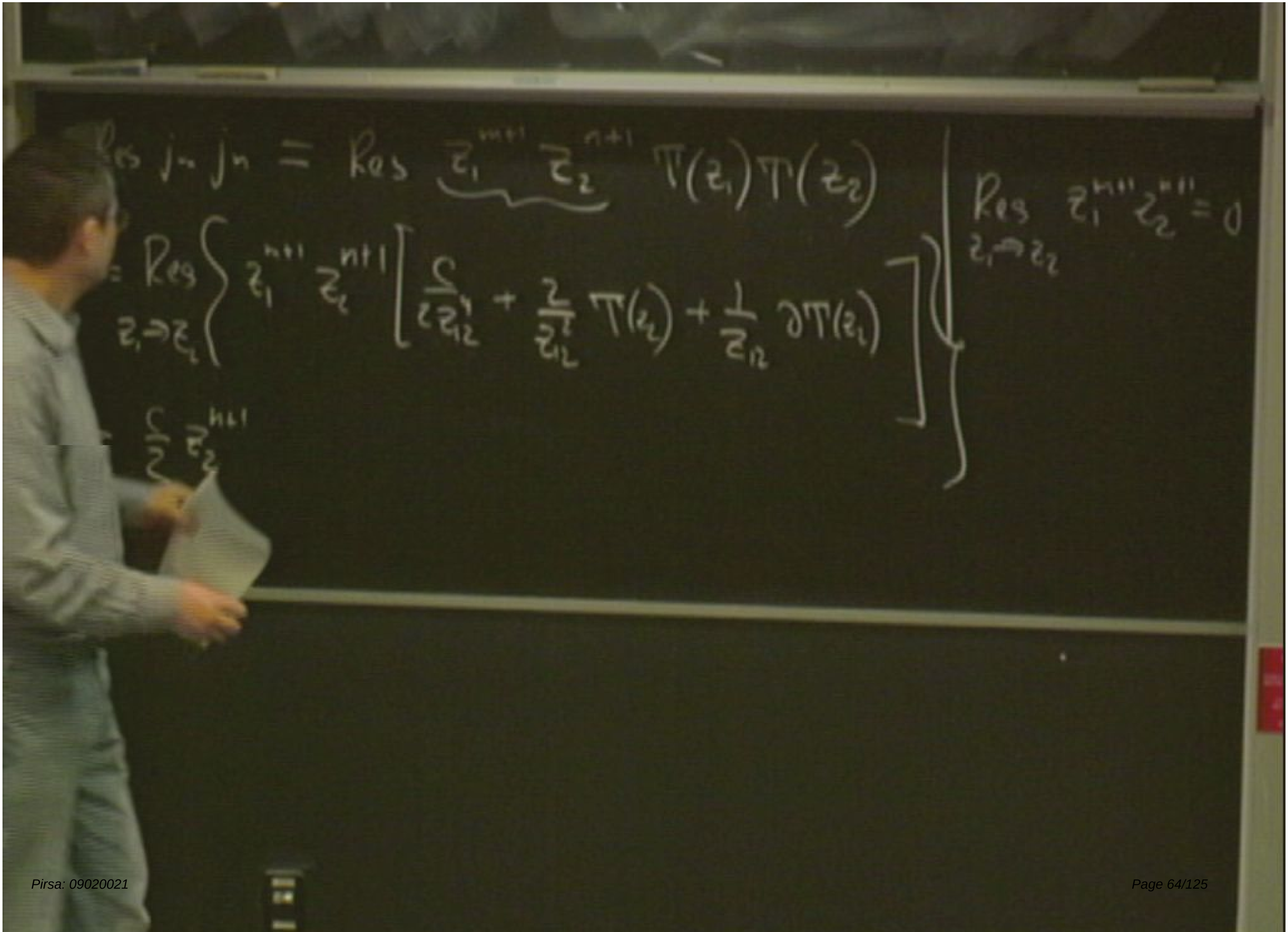
$$\mathrm{Res}_{z_1 \to z_2}\, z_1^{m+1} z_2^{n+1} = 0$$

$$\operatorname{Res} j_m j_n = \operatorname{Res} \underbrace{z_1^{m+1} \, z_2^{n+1}} \, T(z_1) T(z_2)$$

$$= \operatorname{Res}_{z_1 \to z_2} \left\{ z_1^{m+1} z_2^{n+1} \left[\frac{c}{2 z_{12}^4} + \frac{2}{z_{12}^2} T(z_2) + \frac{1}{z_{12}} \partial T(z_2) \right] \right\}$$

$$= \frac{c}{2} z_2^{n+1}$$

$$\operatorname{Res}_{z_1 \to z_2} z_1^{n+1} z_2^{n+1} = 0$$


$\mathrm{Res}\, j_n j_n = \mathrm{Res}\, z_1^{n+1} z_2^{n+1}\, T(z_1) T(z_2)$
$= \mathrm{Res}_{z_1 \to z_2} \left\{ z_1^{n+1} z_2^{n+1} \left[\frac{c}{2 z_{12}^4} + \frac{2}{z_{12}^2} T(z_2) + \frac{1}{z_{12}} \partial T(z_2) \right] \right\}$
$= \frac{c}{2} z_2^{n+1}$
$\mathrm{Res}_{z_1 \to z_2}\, z_1^{n+1} z_2^{n+1} = 0$

$$\mathrm{Res}\, j_m j_n = \mathrm{Res}\, \underbrace{z_1^{m+1}\, z_2^{n+1}}\, T(z_1)T(z_2)$$

$$= \mathrm{Res}_{z_1 \to z_2} \left\{ z_1^{m+1}\, z_2^{n+1} \left[\frac{c}{2 z_{12}^4} + \frac{2}{z_{12}^2} T(z_2) + \frac{1}{z_{12}} \partial T(z_1) \right] \right\}$$

$$= \frac{c}{2} z_2^{n+1}$$

$$\mathrm{Res}_{z_1 \to z_2}\, z_1^{m+1} z_2^{n+1} = 0$$

$$\operatorname{Res} j_m j_n = \operatorname{Res} \underbrace{z_1^{m+1} z_2^{n+1}} T(z_1) T(z_2)$$

$$= \operatorname*{Res}_{z_1 \to z_2} \left\{ z_1^{m+1} z_2^{n+1} \left[\frac{c}{2 z_{12}^4} + \frac{2}{z_{12}^2} T(z_2) + \frac{1}{z_{12}} \partial T(z_1) \right] \right\}$$

$$= \frac{c}{2} z_2^{n+1}$$

$$\operatorname*{Res}_{z_1 \to z_2} z_1^{m+1} z_2^{n+1} = 0$$

$$f(z_1) = f(z_2) + \frac{f'}{1'} z_{12} + \frac{f''}{2!} z_{12}^2 + \dots$$

$$\mathrm{Res}\ j_m j_n = \mathrm{Res}\ \underbrace{z_1^{m+1} z_2^{n+1}}\ T(z_1)T(z_2)$$

$$= \mathrm{Res}_{z_1 \to z_2} \left\{ z_1^{m+1} z_2^{n+1} \left[\frac{c}{2 z_{12}^4} + \frac{2}{z_{12}^2} T(z_2) + \frac{1}{z_{12}} \partial T(z_2) \right] \right\}$$

$$= \frac{c}{2} z_2^{n+1} \frac{1}{3!} \partial^3 \left(z_2^{m+1} \right)$$

$$\mathrm{Res}_{z_1 \to z_2}\ z_1^{m+1} z_2^{n+1} = 0$$

$$f(z_1) = f(z_2) + \frac{f'(z_2)}{1!} z_{12} + \frac{f''(z_2)}{2!} z_{12}^2 + \cdots$$

$$\mathrm{Res}\, j_m j_n = \mathrm{Res}\, \underbrace{z_1^{m+1} z_2^{n+1}} T(z_1) T(z_2)$$

$$= \mathrm{Res}_{z_1 \to z_2} \Big\{ z_1^{m+1} z_2^{n+1} \Big[\frac{c}{2 z_{12}^4} + \frac{2}{z_{12}^2} T(z_2) + \frac{1}{z_{12}} \partial T(z_2) \Big] \Big\}$$

$$= \frac{c}{2} z_2^{n+1} \frac{1}{3!} \partial^3 \left(z_2^{m+1} \right) + 2 z_2^{n+1}$$

$$\mathrm{Res}_{z_1 \to z_2} z_1^{m+1} z_2^{n+1} = 0$$

$$f(z_1) = f(z_2) + \frac{f'(z_2) z_{12}}{1!} + \frac{f''(z_2) z_{12}^2}{2!} + \cdots$$

$$\text{Res}\, j_m j_n = \text{Res}\, \underbrace{z_1^{m+1} z_2^{n+1}}\, T(z_1) T(z_2)$$

$$= \text{Res}\; z_2^{n+1} \left[\frac{c}{2 z_{12}^4} + \frac{2}{z_{12}^2} T(z_2) + \frac{1}{z_{12}} \partial T(z_1) \right]$$

$$\frac{1}{3!}\, \partial^3 \left(z_2^{n+1} \right) + 2 z_2^{n+1}$$

$$\text{Res}_{z_1 \to z_2}\, z_1^{m+1} z_2^{n+1} = 0$$

$$f(z_1) = f(z_2) + \frac{f'(z_2) z_{12}}{1!} + \frac{f''(z_2) z_{12}^2}{2!} + \cdots$$

$$\mathrm{Res}\, j_m j_n = \mathrm{Res}\, \underbrace{z_1^{m+1} \, z_2^{n+1}} \, T(z_1) T(z_2)$$

$$= \mathrm{Res}_{z_1 \to z_2} \left\{ z_1^{m+1} z_2^{n+1} \left[\frac{c}{2 z_{12}^4} + \frac{2}{z_{12}^2} T(z_2) + \frac{1}{z_{12}} \partial T(z_2) \right] \right\}$$

$$= \frac{c}{2} z_2^{n+1} \frac{1}{3!} \partial^3 \left(z_2^{m+1} \right) + 2 z_2^{n+1} \frac{1}{1!} \partial \left(z_2^{m+1} \right) T(z_2)$$

$$\mathrm{Res}_{z_1 \to z_2} z_1^{m+1} z_2^{n+1} = 0$$

$$f(z_1) = f(z_2) + \frac{f'(z_2)}{1!} z_{12} + \frac{f''(z_2)}{2!} z_{12}^2 + \cdots$$

$$\operatorname{Res} j_m j_n = \operatorname{Res} \underbrace{z_1^{m+1} z_2^{n+1}} T(z_1) T(z_2)$$

$$= \operatorname*{Res}_{z_1 \to z_2} \left\{ z_1^{m+1} z_2^{n+1} \left[\frac{c}{2 z_{12}^4} + \frac{2}{z_{12}^2} T(z_2) + \frac{1}{z_{12}} \partial T(z_2) \right] \right\}$$

$$= \frac{c}{2} z_2^{n+1} \frac{1}{3!} \partial^3 \left(z_2^{m+1} \right) + 2 z_2^{n+1} \frac{1}{1!} \partial \left(z_2^{m+1} \right) T(z_2)$$

$$+ z_2^{n+m+1} \partial T(z_2)$$

$$\operatorname*{Res}_{z_1 \to z_2} z_1^{m+1} z_2^{n+1} = 0$$

$$f(z_1) = f(z_2)$$

$$+ \frac{f'(z_2)}{1!} z_{12}$$

$$+ \frac{f''(z_2)}{2!} z_{12}^2 + \cdots$$

$$\operatorname{Res} j_m j_n = \operatorname{Res} \underbrace{z_1^{m+1} z_2^{n+1}} T(z_1) T(z_2)$$

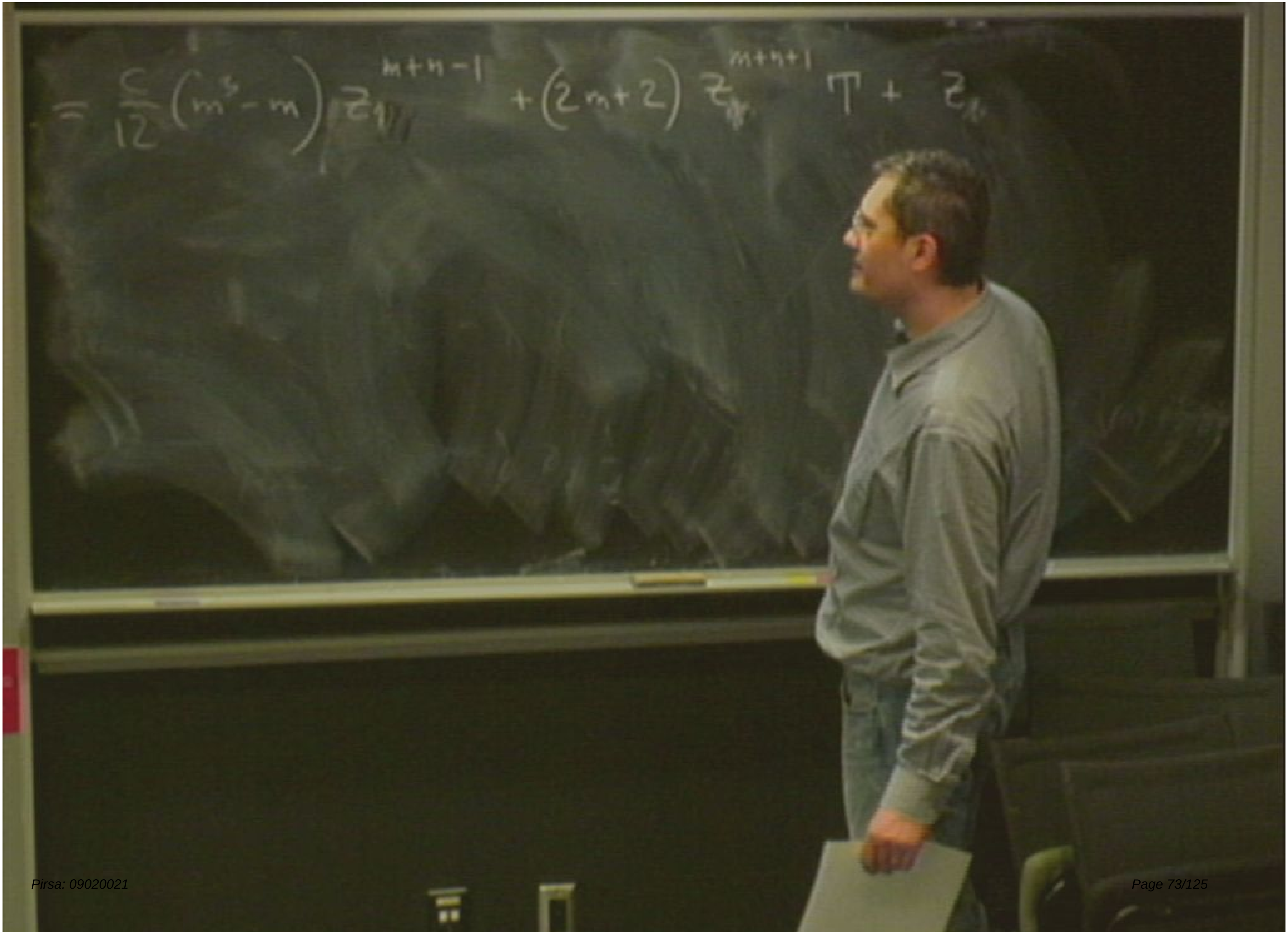
$$= \operatorname*{Res}_{z_1 \to z_2} \left\{ z_1^{m+1} z_2^{n+1} \left[\frac{c}{2 z_{12}^4} + \frac{2}{z_{12}^2} T(z_2) + \frac{1}{z_{12}} \partial T(z_2) \right] \right\}$$

$$= \frac{c}{2} z_2^{n+1} \frac{1}{3!} \partial^3 \left(z_2^{m+1} \right) + 2 z_2^{n+1} \frac{1}{1!} \partial \left(z_2^{m+1} \right) T(z_2)$$

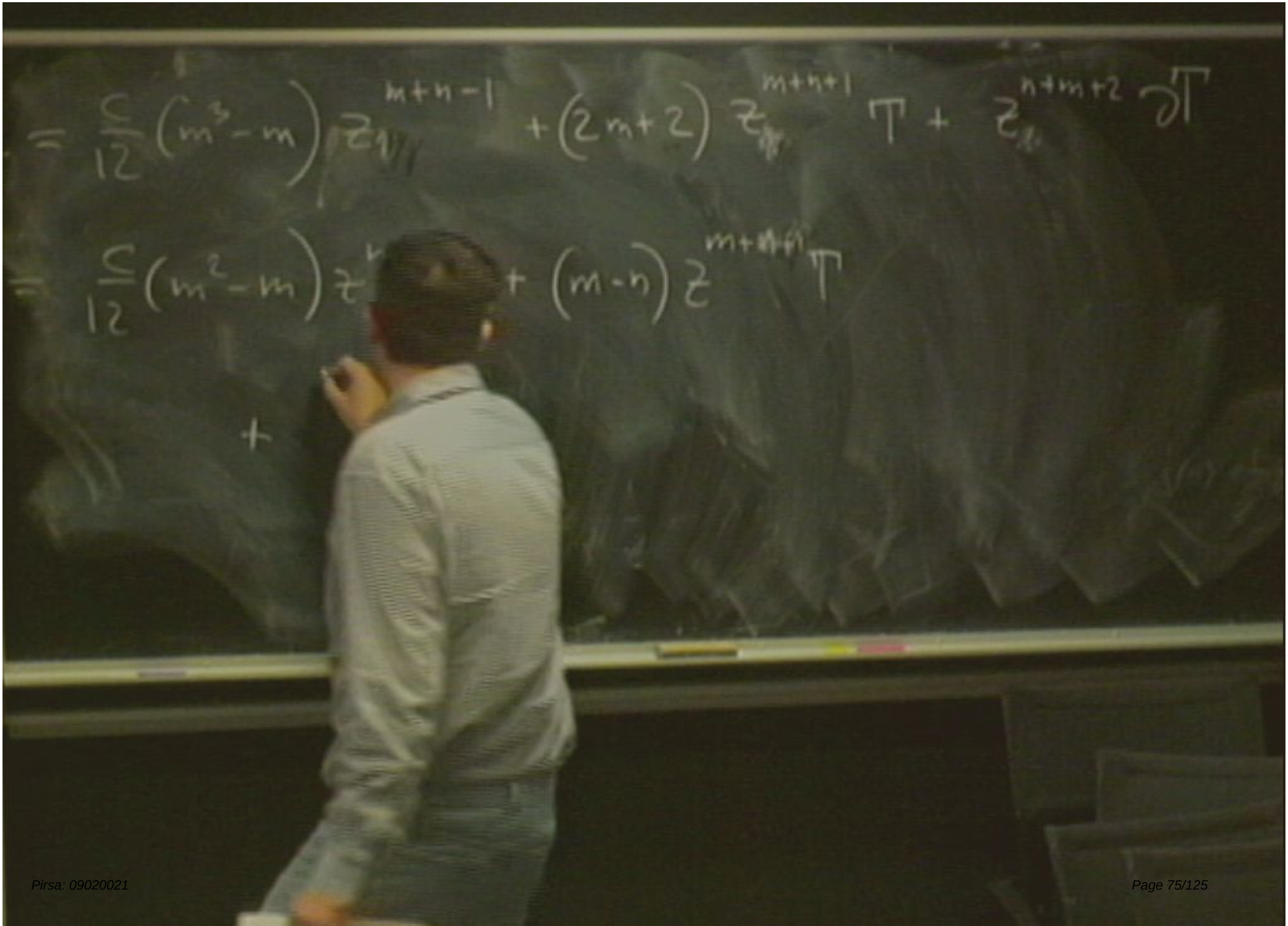
$$+ z_2^{n+m+2} \partial T(z_2)$$

$$\operatorname*{Res}_{z_1 \to z_2} z_1^{m+1} z_2^{n+1} = 0$$

$$f(z_1) = f(z_2) + \frac{f'(z_2) z_{12}}{1!} + \frac{f''(z_2) z_{12}^2}{2!} + \cdots$$


= c/12 (m^3 - m) z^{m+n-1} + (2m+2) z^{m+n+1} T + z

$$= \frac{c}{12}(m^3 - m)\, z_1^{m+n-1} + (2m+2)\, z_1^{m+n+1}\, T + z_1^{n+m+2}\, \partial T$$


$= \frac{c}{12}(m^3 - m) z^{m+n-1} + (2m+2) z^{m+n+1} T + z^{n+m+2} \partial T$
$= \frac{c}{12}(m^2 - m) z + (m-n) z^{m+n+1} T$
$+$

$$= \frac{c}{12}\left(m^3 - m\right) z^{m+n-1} + \left(2m+2\right) z^{m+n+1}\, T + z^{n+m+2}\, \partial T$$

$$= \frac{c}{12}\left(m^2 - m\right) z^{m+n-1} + \left(m-n\right) z^{m+n+1}\, T$$

$$+ \left\{ \left(m+n+2\right) z^{m+n+1}\, T + z^{m+n+2}\, \partial T \right\}$$

$$= \frac{c}{12}\left(m^3 - m\right) z^{m+n-1} + \left(2m+2\right) z^{m+n+1}\, T + z^{n+m+2}\, \partial T$$

$$= \frac{c}{12}\left(m^2 - m\right) z^{m+n-1} + \left(m-n\right) z^{m+n+1}\, T$$

$$+ \left\{ \left(m+n+2\right) z^{m+n+1}\, T + z^{m+n+2}\, \partial T \right\}$$

$$= \frac{c}{12}\left(m^3 - m\right) z^{m+n-1} + (2m+2)\, z^{m+n+1}\, T + z^{n+m+2}\, \partial T$$

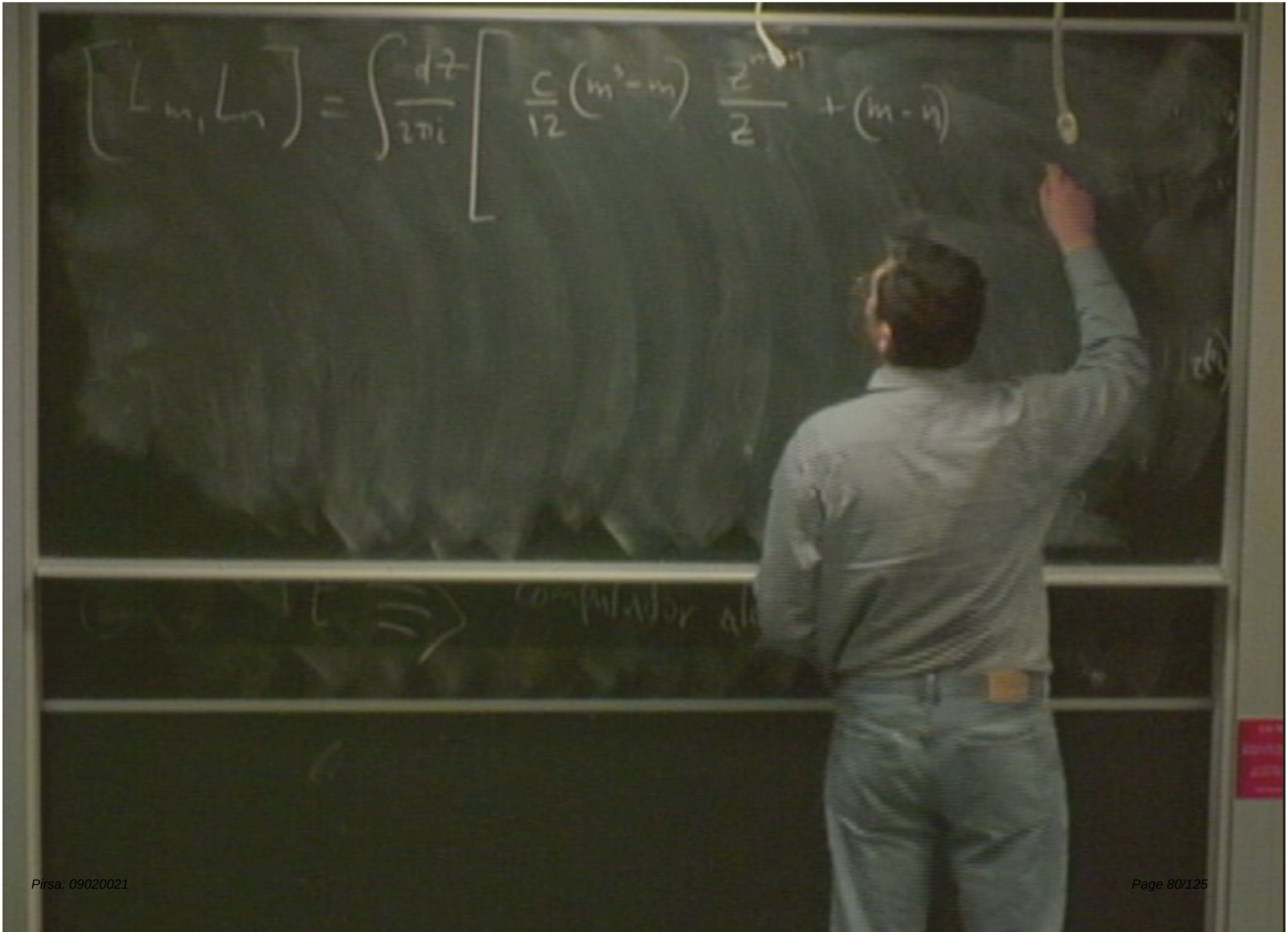
$$= \frac{c}{12}\left(m^2 - m\right) z^{m+n-1} + (m-n)\, z^{m+n+1}\, T$$

$$+ \left\{ \underbrace{(m+n+2)\, z^{m+n+1}\, T + z^{m+n+2}\, \partial T}_{\partial\left[z^{m+n+2}\, T\right]} \right\}$$

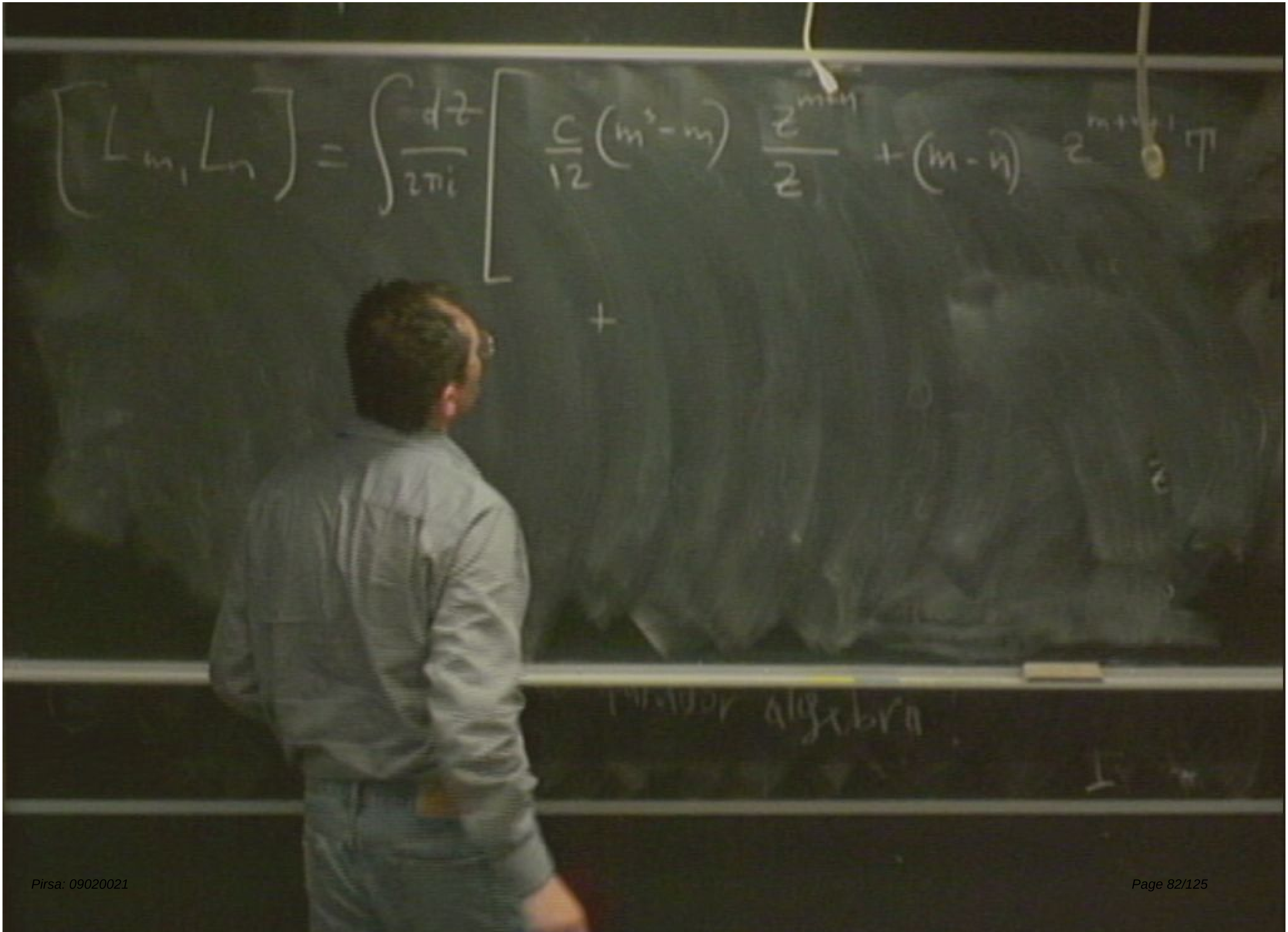
$$= \frac{c}{12}\left(m^3 - m\right) z^{m+n-1} + (2m+2)\, z^{m+n+1}\, T + z^{n+m+2}\, \partial T$$

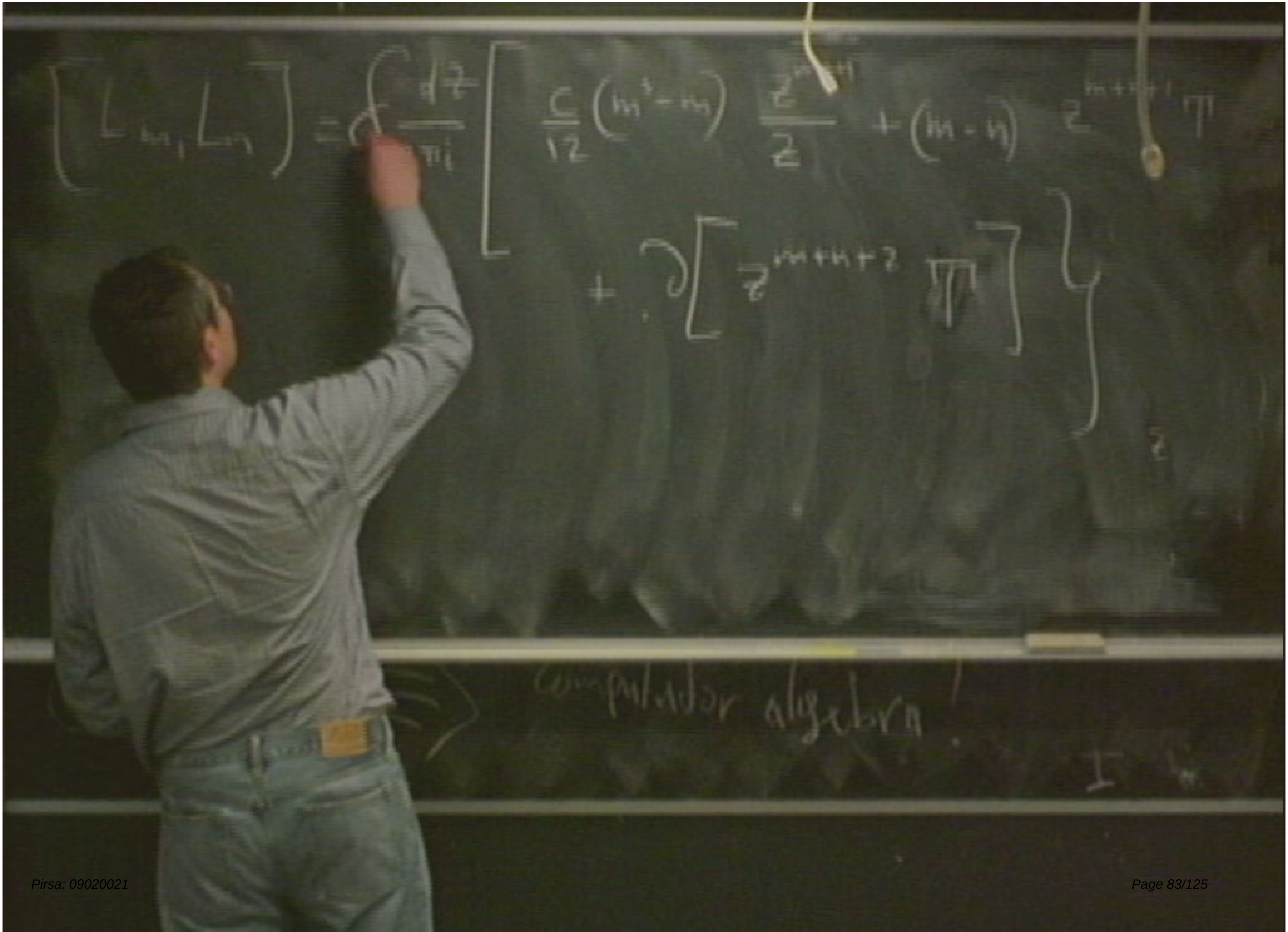
$$= \frac{c}{12}\left(m^2 - m\right) z^{m+n-1} + (m-n)\, z^{m+n+1}\, T$$

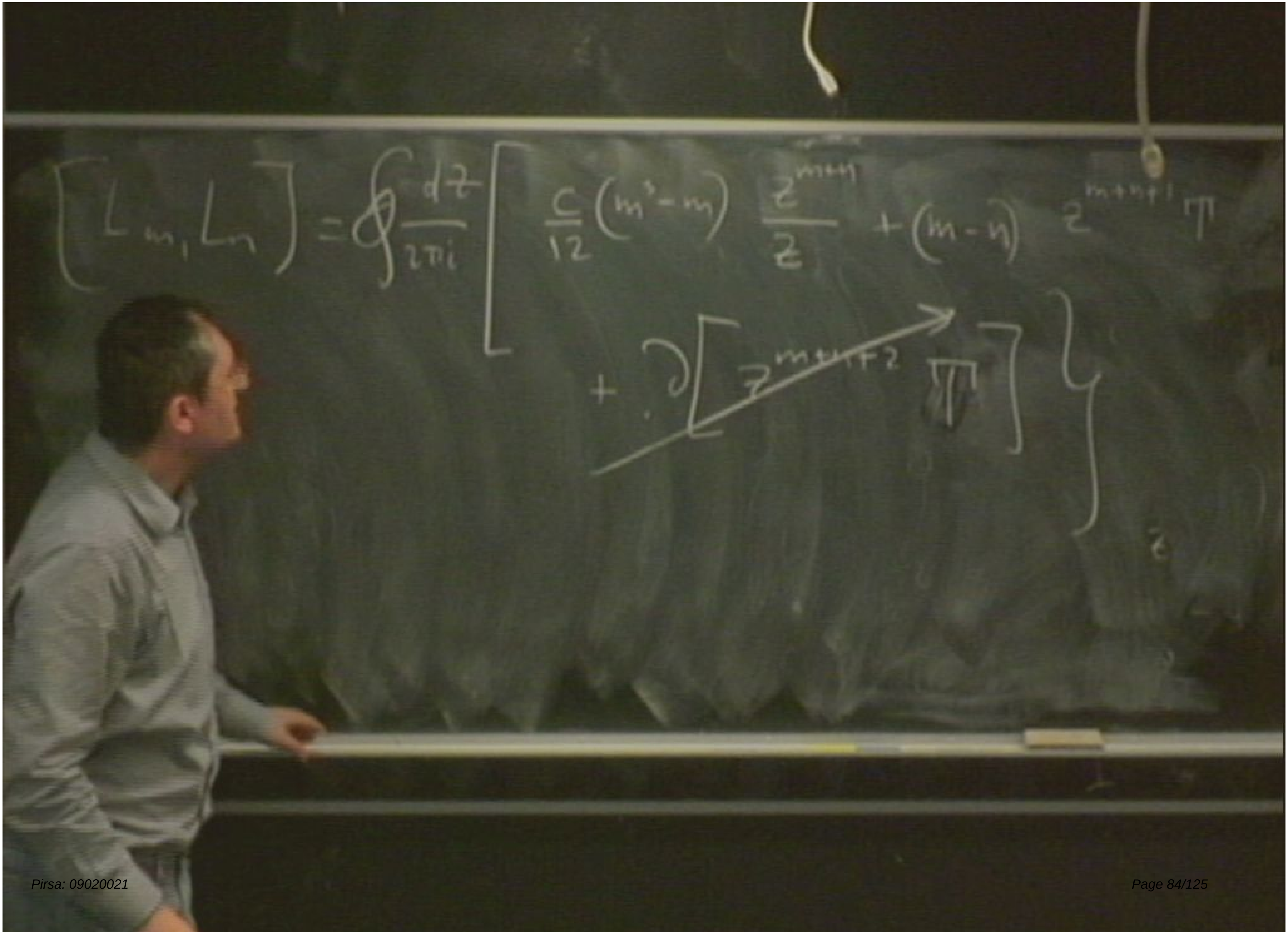
$$+ \left\{ \underbrace{(m+n+2)\, z^{m+n+1}\, T + z^{m+n+2}\, \partial T}_{\partial\left[z^{m+n+2}\, T \right]} \right\}$$

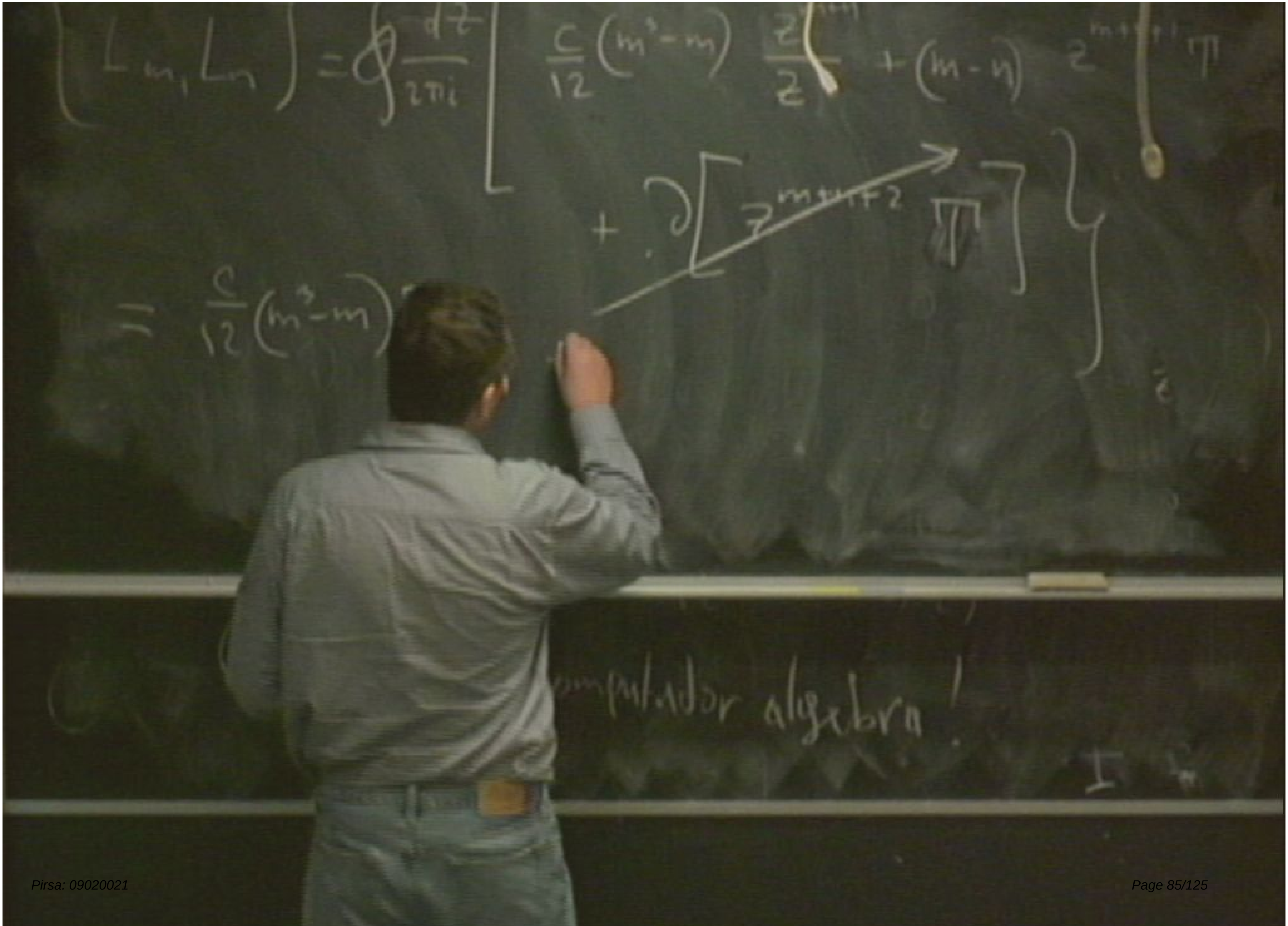

$[L_m, L_n] = \oint \frac{dz}{2\pi i} \Big[\frac{c}{12}(m^3 - m) \frac{z^{m+n}}{z} + (m-n)$
commutator al

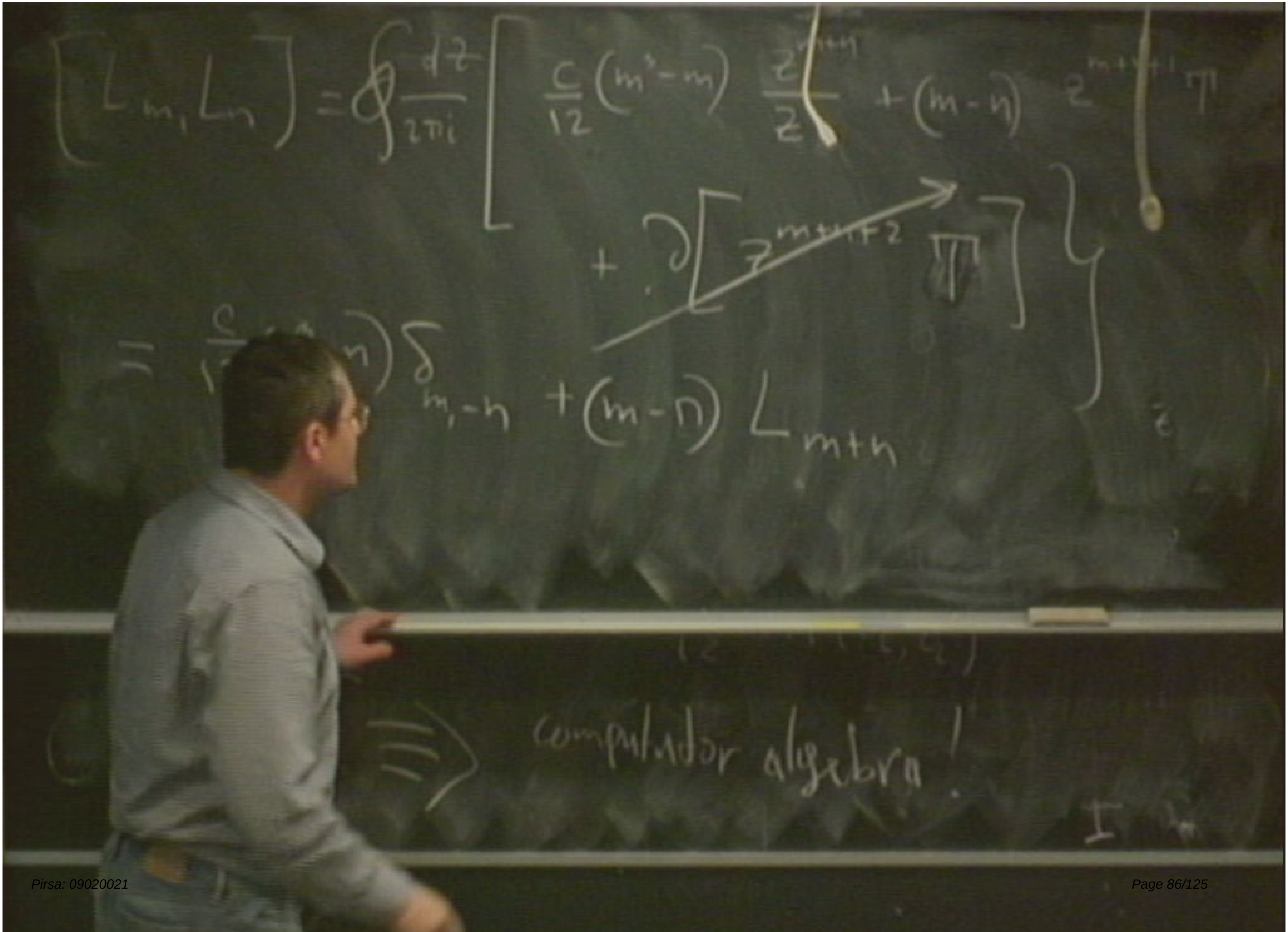
$$\left[L_m, L_n \right] = \oint \frac{dz}{2\pi i} \left[\frac{c}{12} \left(m^3 - m \right) \frac{z^{m+n}}{z} + (m-n) \; z^{m+n} \right.$$

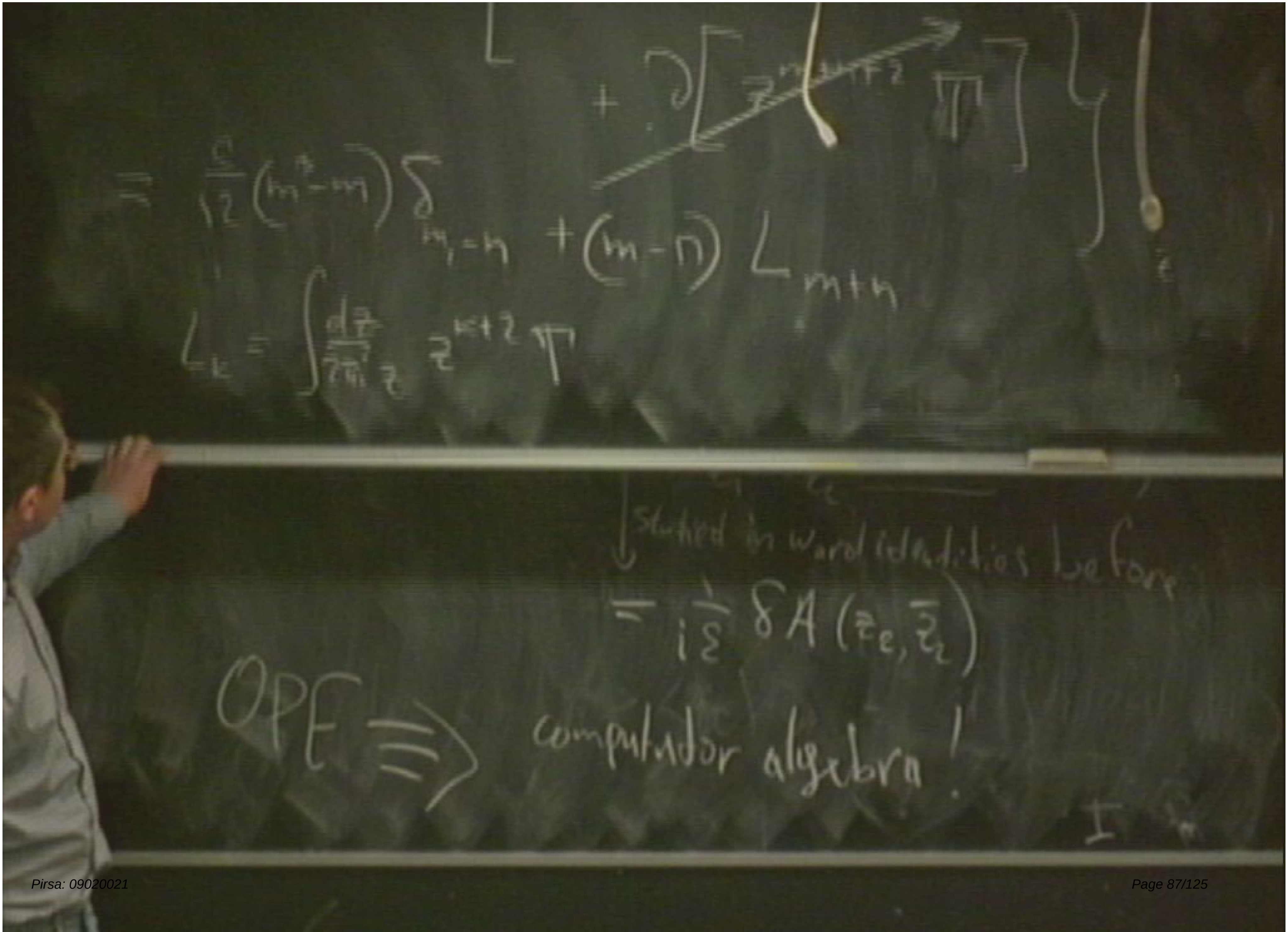

$[L_m, L_n] = \int \frac{dz}{2\pi i} \Big[\frac{c}{12}(m^3 - m) \frac{z^{m+n}}{z} + (m-n)\, z^{m+n+1}\, T$
$+$
algebra


$[L_m, L_n] = \oint \frac{dz}{2\pi i} \Big[\frac{c}{12}(m^3-m) \frac{z^{m+n}}{z} + (m-n)\, z^{m+n+1}\, T + \partial [z^{m+n+2}\, T] \Big]$
commutator algebra!


$[L_m, L_n] = \oint \frac{dz}{2\pi i} \Big[\frac{c}{12}(m^3 - m)\frac{z^{m+n}}{z} + (m-n)\, z^{m+n+1}\, T$
$+ \cancel{\partial [z^{m+n+2}\, T]} \Big] \Big\}$


ommutator algebra !


$[L_m, L_n] = \oint \frac{dz}{2\pi i} \left[\frac{c}{12}(m^3 - m) \frac{z^{m+n}}{z} + (m-n) z^{m+n+1} T \right.$
$+ \partial [z^{m+n+2} T] \}$
$= \frac{c}{12}(m^3 - m)\delta_{m,-n} + (m-n) L_{m+n}$
$\Rightarrow$ commutator algebra!


= c/12 (m^3 - m) δ_{m,-n} + (m-n) L_{m+n}
L_k = ∫ dz/2πi z^{k+2} T
= 1/iε δA(z_2, z̄_2)
OPE ⟹ commutator algebra !

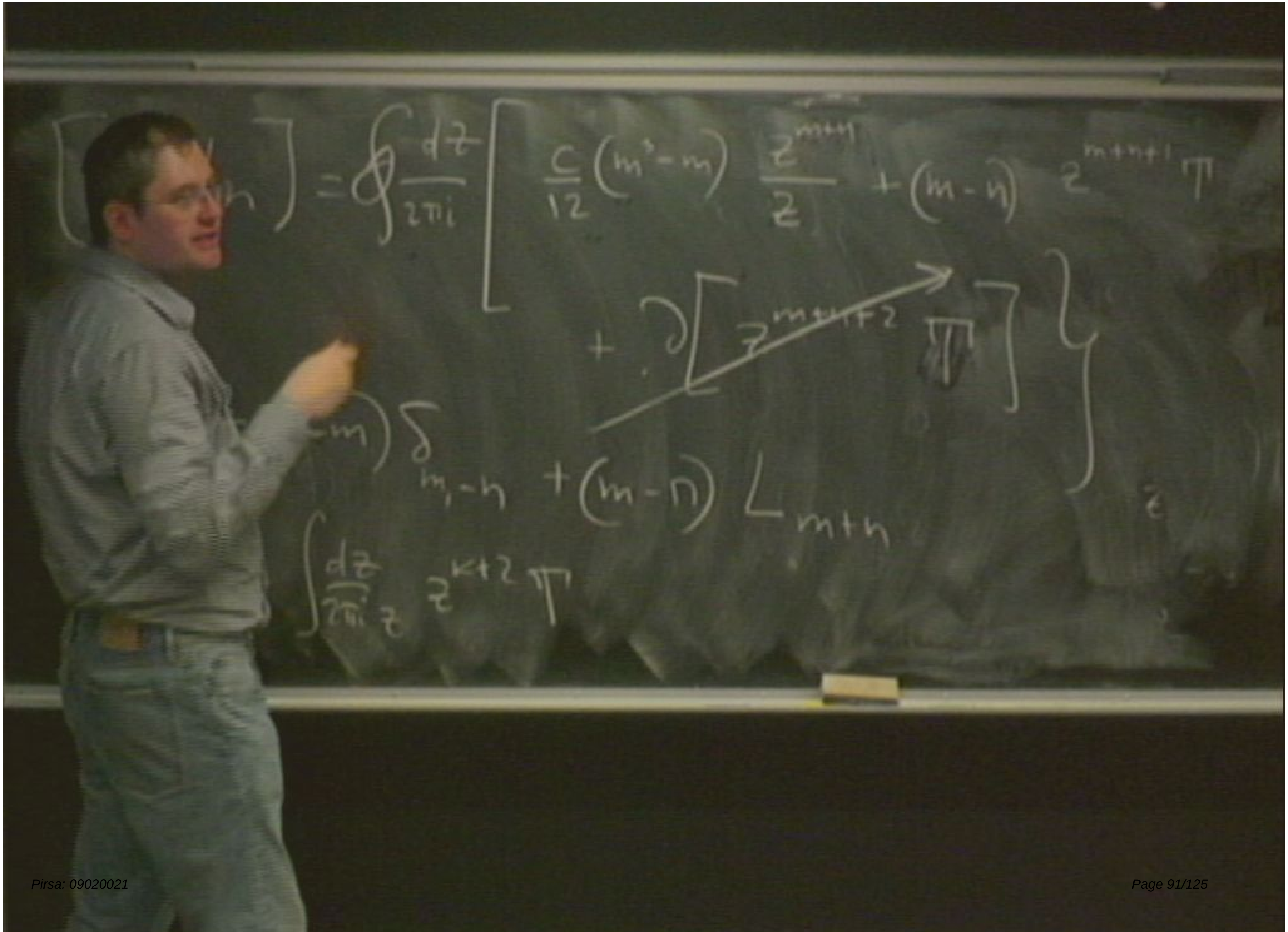
$$[L_m, L_n] =$$

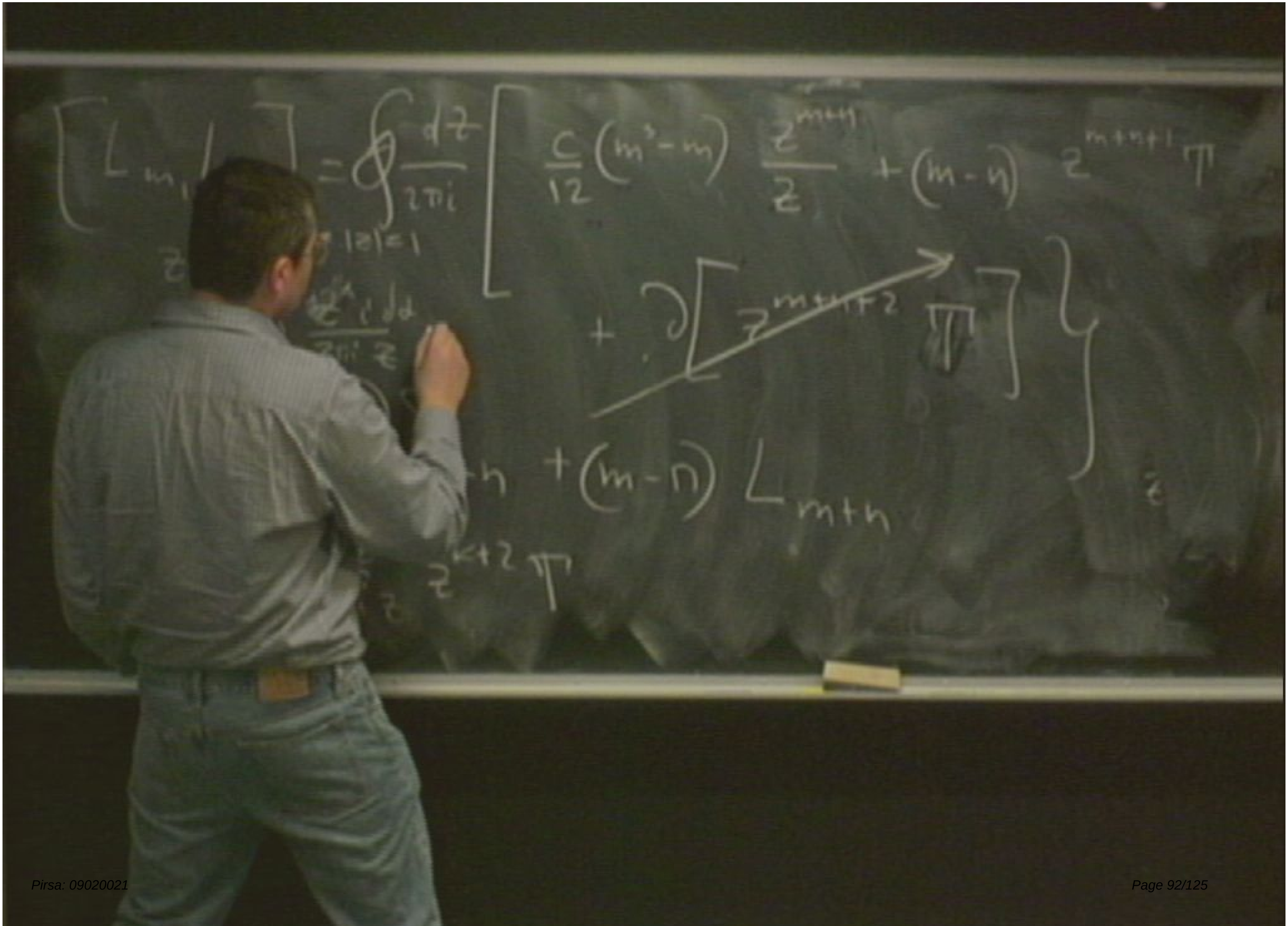
$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\,\delta_{m,-n}$$

$$\left[L_m, L_n\right] = \oint \frac{dz}{2\pi i}\left[\frac{c}{12}(m^3-m)\frac{z^{m+n}}{z} + (m-n)\, z^{m+n+1}\, T + \cancel{\partial\left[z^{m+n+2}\, T'\right]}\right]$$

$$= \frac{c}{12}(m^3-m)\,\delta_{m,-n} + (m-n)\, L_{m+n}$$

$$L_k = \int \frac{dz}{2\pi i\, z}\, z^{k+2}\, T$$


$= \oint \frac{dz}{2\pi i} \left[\frac{c}{12}(m^3-m) \frac{z^{m+n}}{z} + (m-n) z^{m+n+1} T \right.$
$+ \partial \left[z^{m+n+2} T \right] \}$
$\delta_{m,-n} + (m-n) L_{m+n}$
$\oint \frac{dz}{2\pi i \, z} z^{k+2} T$



$$[L_m, L_n] = \oint \frac{dz}{2\pi i} \Big[\frac{c}{12}(m^3 - m) \frac{z^{m+n}}{z} + (m-n)\, z^{m+n+1}\, T + \partial\big[z^{m+n+2}\, T \big] \Big\}$$

$C: |z| = 1$

$z = e^{i\alpha}$

$$\int_0^{2\pi} \frac{e^{i\alpha}\, i\, d\alpha}{2\pi i\, e^{i\alpha}}\, e^{i\alpha(m+n)}$$

$$= \frac{c}{12}(m^3 - m)\, \delta_{m,-n} + (m-n)\, L_{m+n}$$

$$L_k = \int \frac{dz}{2\pi i\, z}\, z^{k+2}\, T$$

$$[L_m, L_n] = \oint \frac{dz}{2\pi i} \Big[\frac{c}{12}(m^3-m)\frac{z^{m+n}}{z} + (m-n)\, z^{m+n+1}\, T + \partial\left[z^{m+n+2}\, T \right] \Big\}$$

$z = e^{i\alpha}$ $C: |z|=1$

$$\int_0^{2\pi} \frac{i\, d\alpha}{2\pi i}\, e^{i\alpha(m+n)}$$

$$= \frac{c}{12}(m^3-m)\,\delta_{m,-n} + (m-n)\, L_{m+n}$$

$$L_k = \int \frac{dz}{2\pi i\, z}\, z^{k+2}\, T$$

$$[L_m, L_n] = \oint \frac{dz}{2\pi i}\left[\frac{c}{12}(m^3-m)\frac{z^{m+n}}{z} + (m-n)\, z^{m+n+1}\, T + \partial\left[\cancel{z^{m+n+2}\, T}\right]\right]$$

$C: |z| = 1$

$z = e^{i\alpha}$

$$\int_0^{2\pi} \frac{e^{i\alpha}\, i\, d\alpha}{2\pi i}\, e^{i\alpha(m+n)}$$

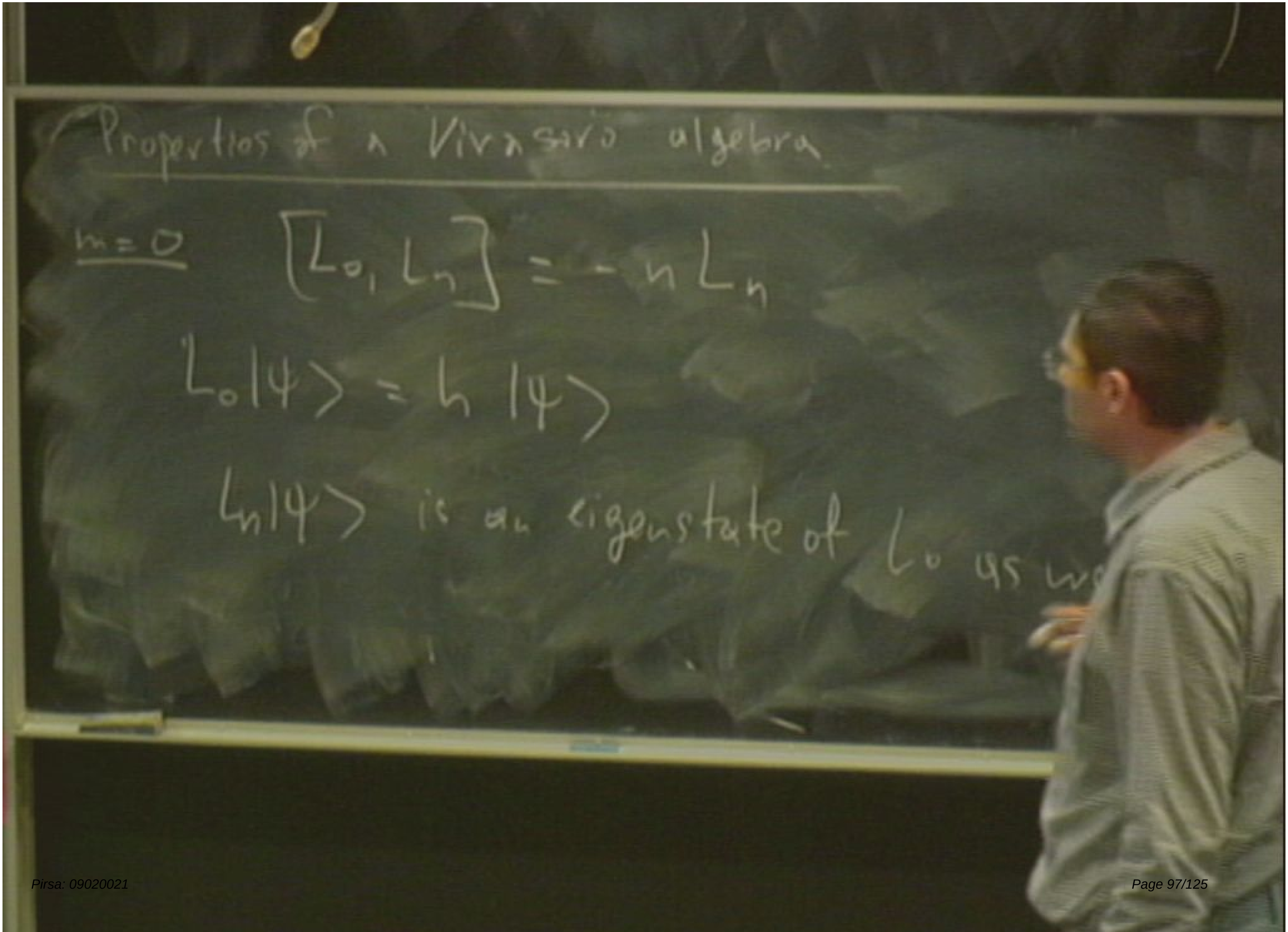
$$= \frac{c}{12}(m^3-m)\,\delta_{m,-n} + (m-n)\, L_{m+n}$$

$$L_k = \int \frac{dz}{2\pi i\, z}\, z^{k+2}\, T$$

Properties of a Virasoro algebra

$m = 0$ $\qquad [L_0, L_n] = -n L_n$

$L_0 |\psi\rangle$


Properties of a Virasoro algebra
m = 0
[L_0, L_n] = - n L_n
L_0 |ψ> = h |ψ>
L_n |ψ> is an eigenstate of L_0 as we

$L_0|\psi\rangle = h|\psi\rangle$

$L_n|\psi\rangle$ is an eigenstate of L_0 as well

$$= \frac{c}{2} z_2^{n+1} \frac{1}{3!} \partial^3\left(z_2^{n+1}\right) + 2 z_2^{n+1} \frac{1}{1!} \partial\left(z_2^{m+1}\right) T(z_2)$$
$$+ z_2^{n+m+2} \partial T(z_2)$$

$+ \frac{f'(z) z}{1!}$

$+ \frac{f''(z) z^2}{2!} z_{12}$

Properties of a Virasoro algebra

$\underline{m=0}$ $[L_0, L_n] = -n L_n$

$L_0 |\psi\rangle = h |\psi\rangle$

$L_n|\psi\rangle$ is an eigenstate of L_0 as well

$L_0 (L_n|\psi\rangle)$

$$= \frac{c}{2} z_2^{n+1} \frac{1}{3!} \partial^3 (z_2^{n+1}) + 2 z_2^{n+1} \frac{1}{1!} \partial (z_2^{m+1}) T(z_2)$$

$$+ z_2^{n+m+2} \partial T(z_2 \ldots$$

$+ \frac{f'(z_2) z_{\ldots}}{1}$

$+ \frac{f''(z) z^2_{12}}{2!}$

Properties of a Virasoro algebra

$\underline{m=0}$ $[L_0, L_n] = -n L_n \quad = \quad L_0 L_n = -n L_n + L_n L_0$

$L_0|\psi\rangle = h|\psi\rangle$

$L_n|\psi\rangle$ is an eigenstate of L_0 as well

$L_0(L_n|\psi\rangle) = L_n L_0|\psi\rangle - n L_n|\psi\rangle =$

$$= \frac{c}{2} z_2^{n+1} \frac{1}{3!} \partial^3 (z_2^{m+1}) + 2 z_2^{n+1} \frac{1}{1!} \partial (z_2^{m+1}) T(z_2) + z_2^{n+m+2} \partial T(z_2)$$

$$+ \frac{c' \, z_{12}}{1'} + \frac{c''(4) z_{12}^2}{2!} + \cdots$$

Properties of a Virasoro algebra

$\underline{m=0}$ $\quad [L_0, L_n] = -n L_n \quad = \quad L_0 L_n = -n L_n + L_n L_0$

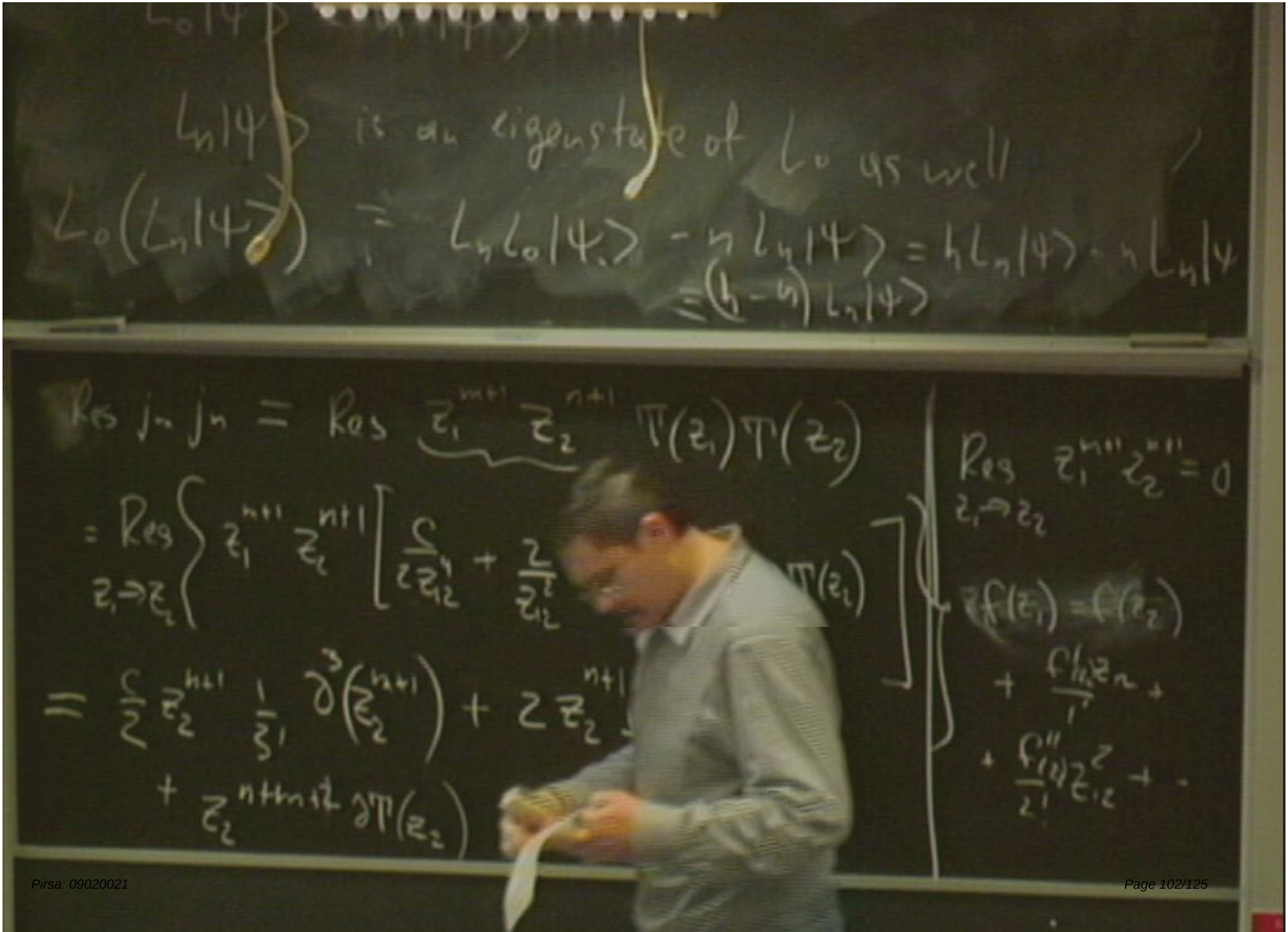
$$L_0 |\psi\rangle = h |\psi\rangle$$

$L_n|\psi\rangle$ is an eigenstate of L_0 as well

$$L_0 (L_n |\psi\rangle) = L_n L_0 |\psi\rangle - n L_n |\psi\rangle = h L_n |\psi\rangle - n L_n |\psi\rangle = (h-n)$$

$$= \frac{c}{2} z_2^{n+1} \frac{1}{3!} \partial^3 \left(z_2^{m+1}\right) + 2 z_2^{n+1} \frac{1}{1!} \partial \left(z_2^{m+1}\right) T(z_2)$$
$$+ z_2^{n+m+2} \partial T(z_2)$$

$$+ \frac{C_{12} z_{12}}{1} + \frac{C''(4) z_{12}^2}{2!} + \cdots$$


$L_n|\psi\rangle$ is an eigenstate of L_0 as well
$L_0(L_n|\psi\rangle) = L_n L_0|\psi\rangle - n L_n|\psi\rangle = h L_n|\psi\rangle - n L_n|\psi$
$= (h-n) L_n|\psi\rangle$
Res $j_m j_n$ = Res $z_1^{m+1} z_2^{n+1} T(z_1) T(z_2)$
Res $z_1^{n+1} z_2^{n+1} = 0$
$z_1 \to z_2$

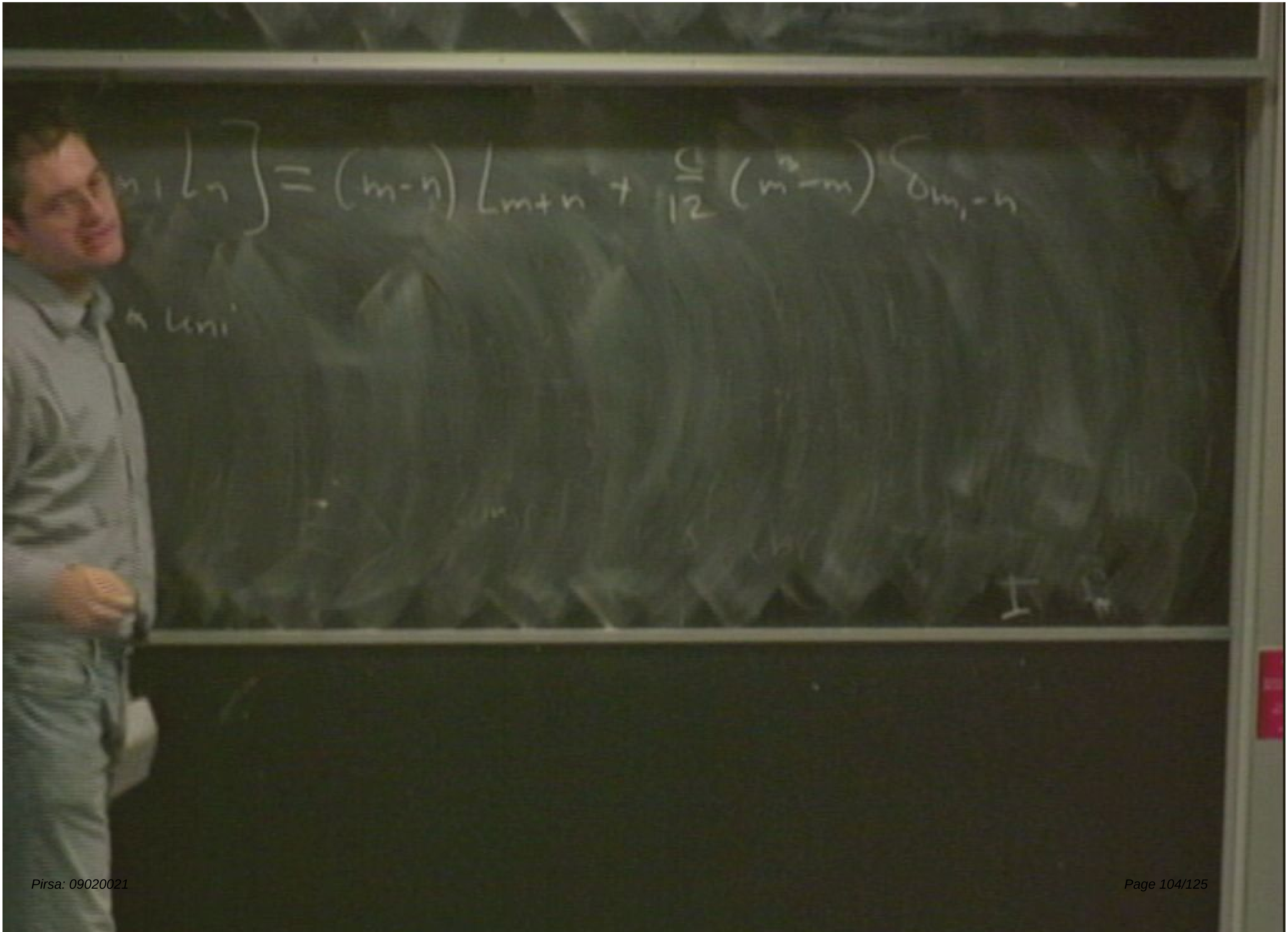
$L_0|\psi\rangle = \ldots L_n|\psi\rangle$

$L_n|\psi\rangle$ is an eigenstate of L_0 as well

$$L_0(L_n|\psi\rangle) = L_n L_0|\psi\rangle - n L_n|\psi\rangle = h L_n|\psi\rangle - n L_n|\psi\rangle$$
$$= (h-n) L_n|\psi\rangle$$

$\Rightarrow$ Generators with $n>0$ lower h

$n<0$ raise h


$m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n}$

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3-m)\,\delta_{m,-n}$$

$[\ ]$ a universal subalgebra of Virasoro algebra

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$\exists$ a universal subalgebra of Virasoro algebra

$$L_0, L_\pm \Rightarrow [L_0, L_1] = -L_1 \quad [L_0, L_{-1}] = L_{-1} \quad [L_1, L_{-1}] = 2L_0$$

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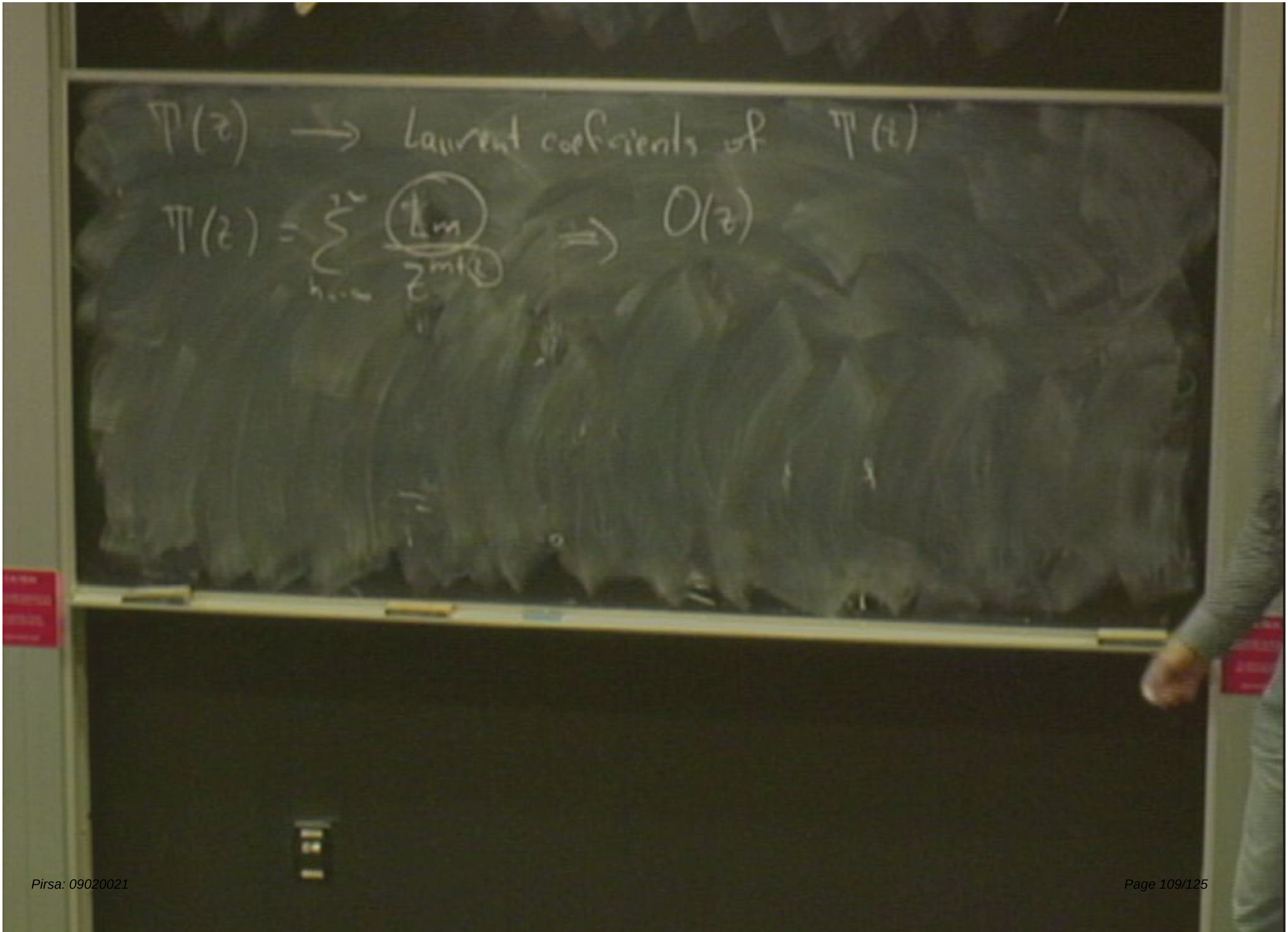
$SL(2,R)$

$\Psi(z) \longrightarrow$ Laurent coefficients of $\Psi(z)$

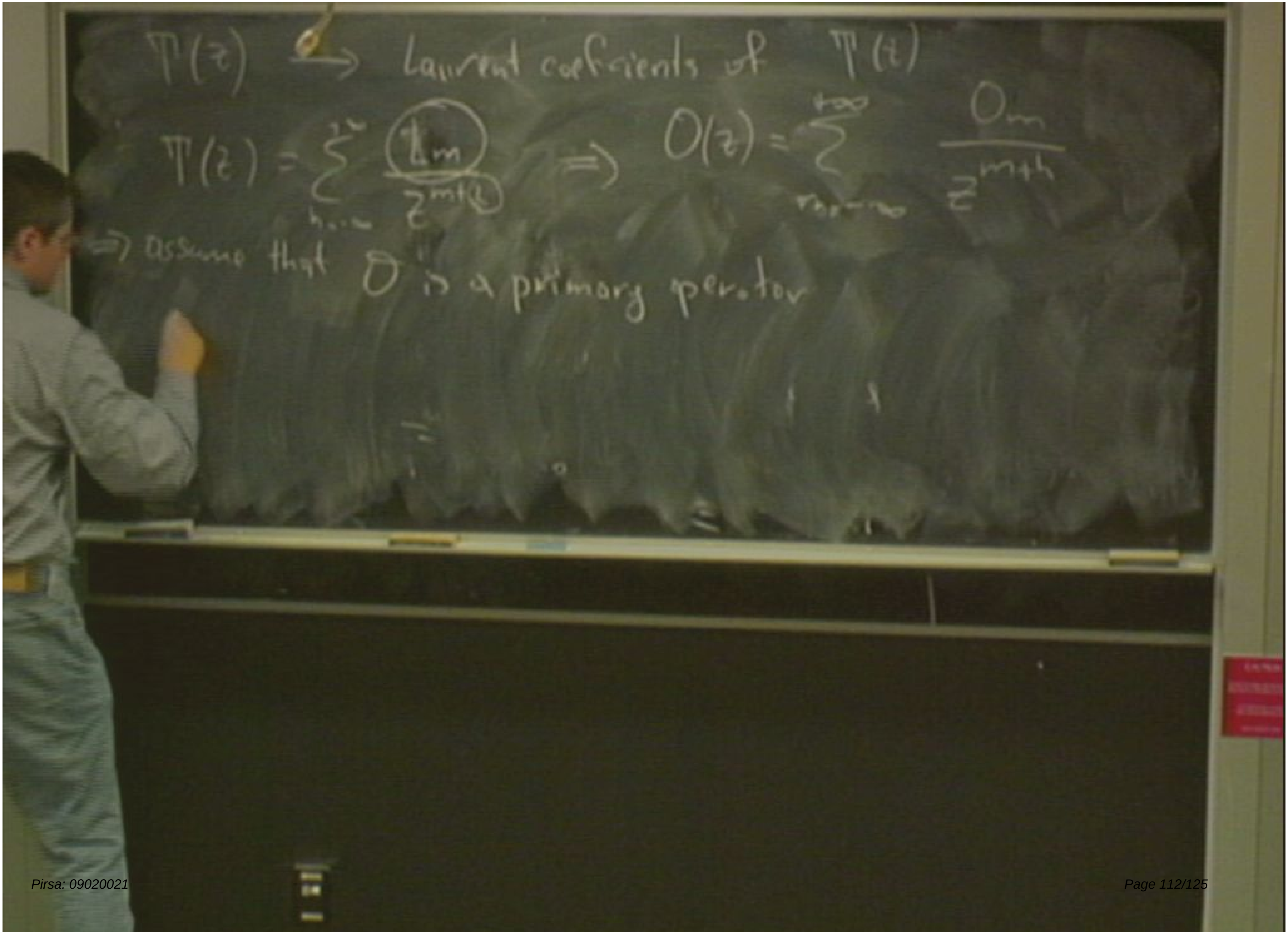
$$\Psi(z) = \sum_{n=-\infty}^{+\infty} \frac{\Psi_m}{z^{m+1/2}}$$


Ψ(z) → Laurent coefficients of Ψ(z)
Ψ(z) = Σ_{n=-∞}^{∞} ψ_m / z^{m+1/2} ⇒ O(z)

$\Psi(z) \longrightarrow$ Laurent coefficients of $\Psi(z)$

$$\Psi(z) = \sum_{m=-\infty}^{+\infty} \frac{\Psi_m}{z^{m+h}} \Longrightarrow O(z) = \sum_{m=-\infty}^{+\infty} \frac{O_m}{z^{m+h}}$$


T(z) ⟹ Laurent coefficients of T(z)
T(z) = Σ_{n=-∞}^{+∞} L_m / z^{m+2} ⟹ O(z) = Σ_{m=-∞}^{+∞} O_m / z^{m+h}

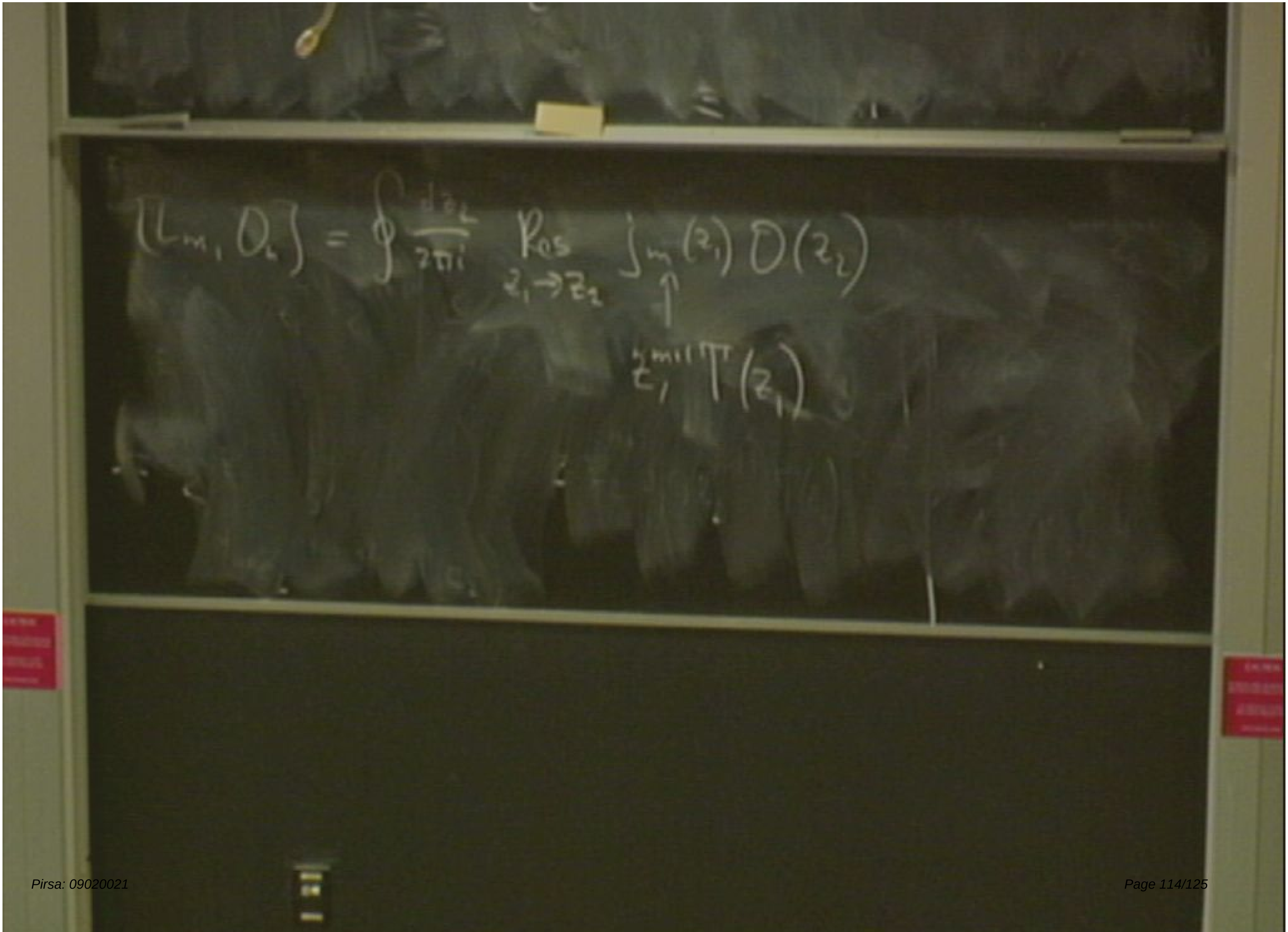

T(z) → Laurent coefficients of T(z)
$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_m}{z^{m+2}} \Longrightarrow O(z) = \sum_{m=-\infty}^{+\infty} \frac{O_m}{z^{m+h}}$
⟹ assume that O is a primary operator

$T(z) \longrightarrow$ Laurent coefficients of $T(z)$

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_m}{z^{m+2}} \Longrightarrow O(z) = \sum_{m=-\infty}^{+\infty} \frac{O_m}{z^{m+h}}$$

$\Rightarrow$ assume that O is a primary operator

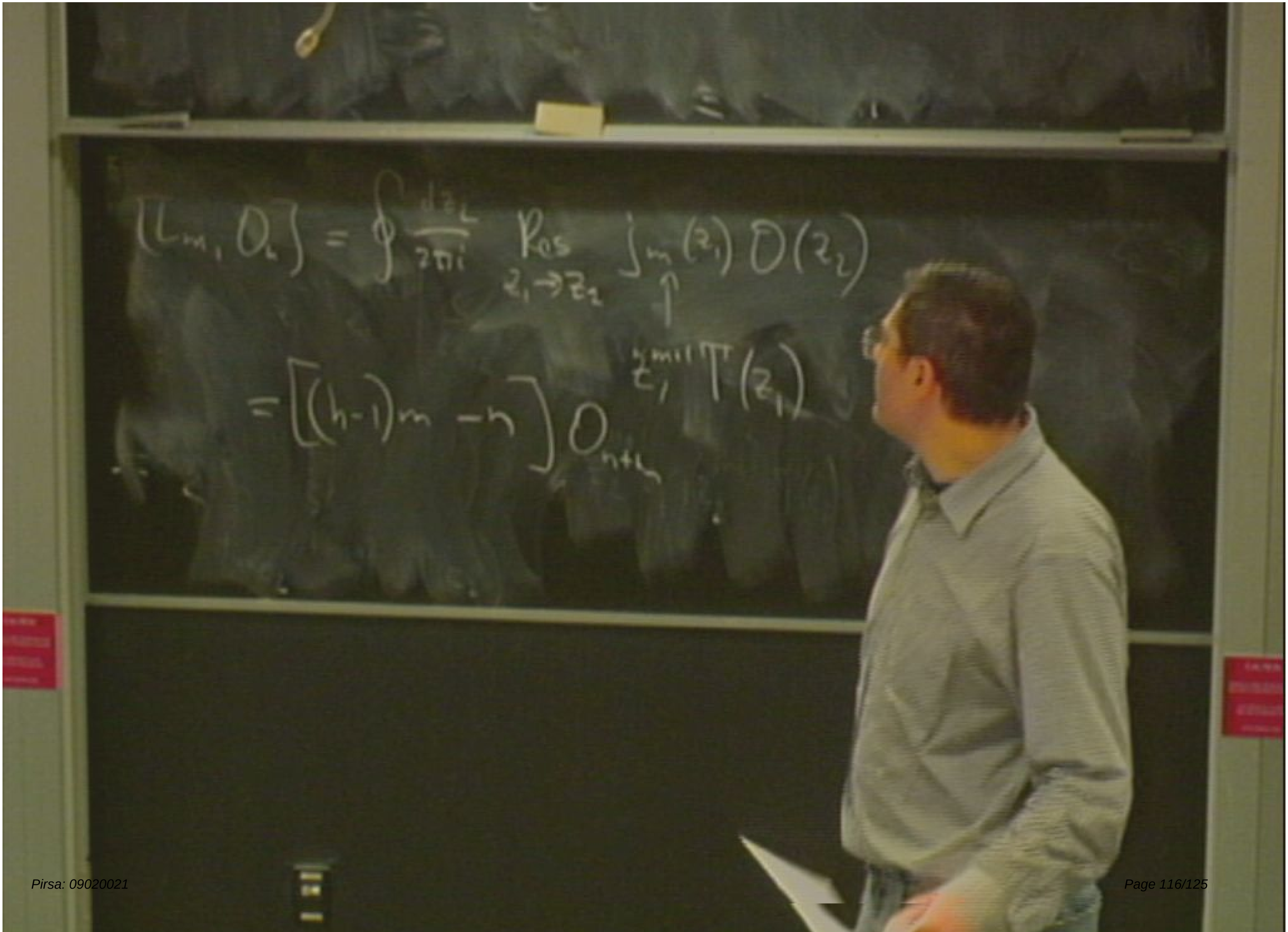
$$T(z)\, O = \frac{h}{z^2} O + \frac{1}{z} \partial O$$


$[L_m, O_2] = \oint \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \to z_2} j_m(z_1)\, O(z_2)$
$z_1^{m+1} T(z_1)$

$$[L_m, O_n] = \oint \frac{dz_2}{2\pi i} \operatorname*{Res}_{z_1 \to z_2} j_m(z_1)\, O(z_2)$$

$$j_m(z_1) = z_1^{m+1} T(z_1)$$

$$= [(h-1)m - n]\, O_{n+m}$$


$[L_m, O_n] = \oint \frac{dz_2}{2\pi i} \mathrm{Res}_{z_1 \to z_2} j_m(z_1) O(z_2)$
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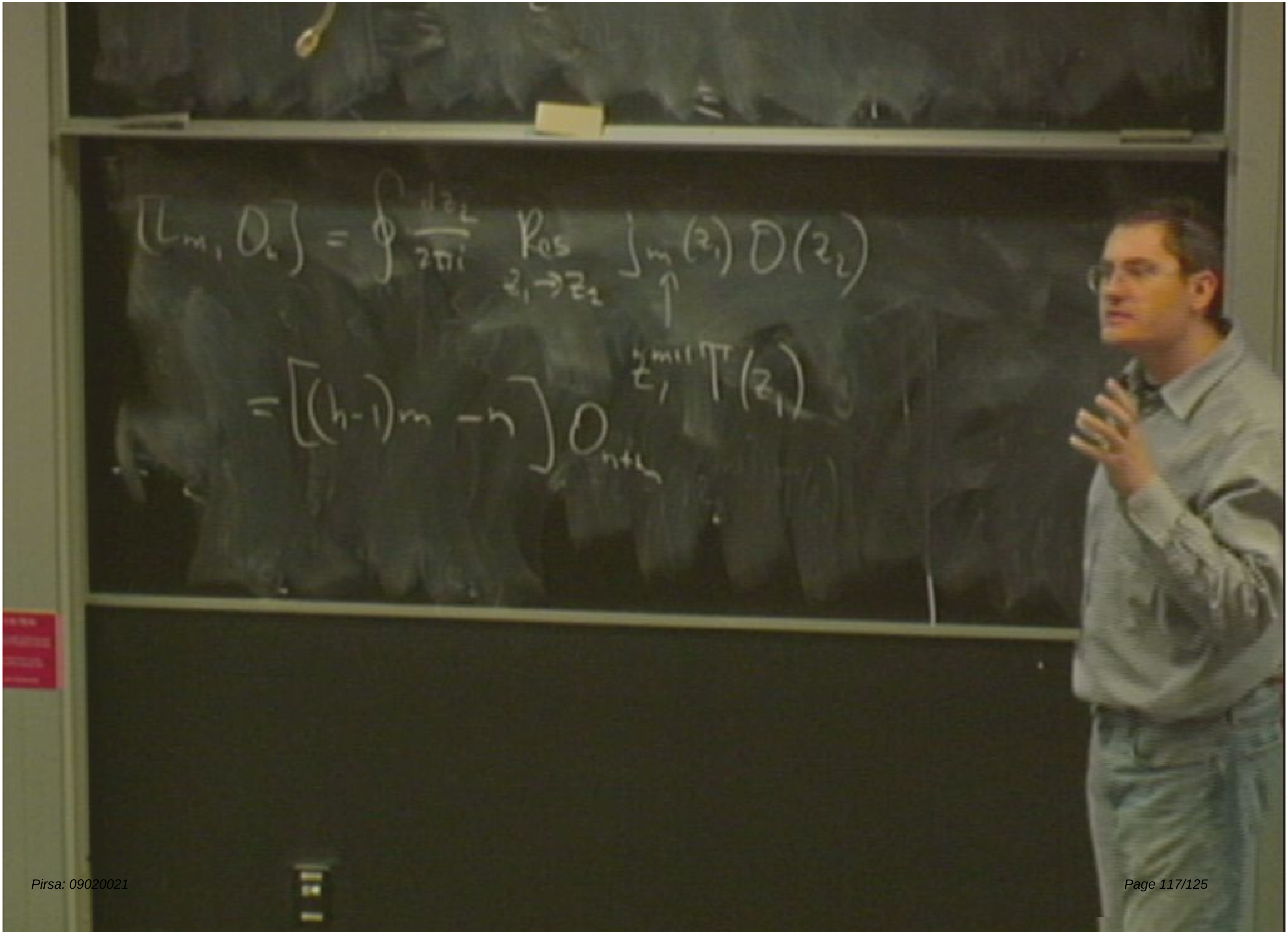
$[L_m, O_n] = \oint \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \to z_2} j_m(z_1)\, O(z_2)$
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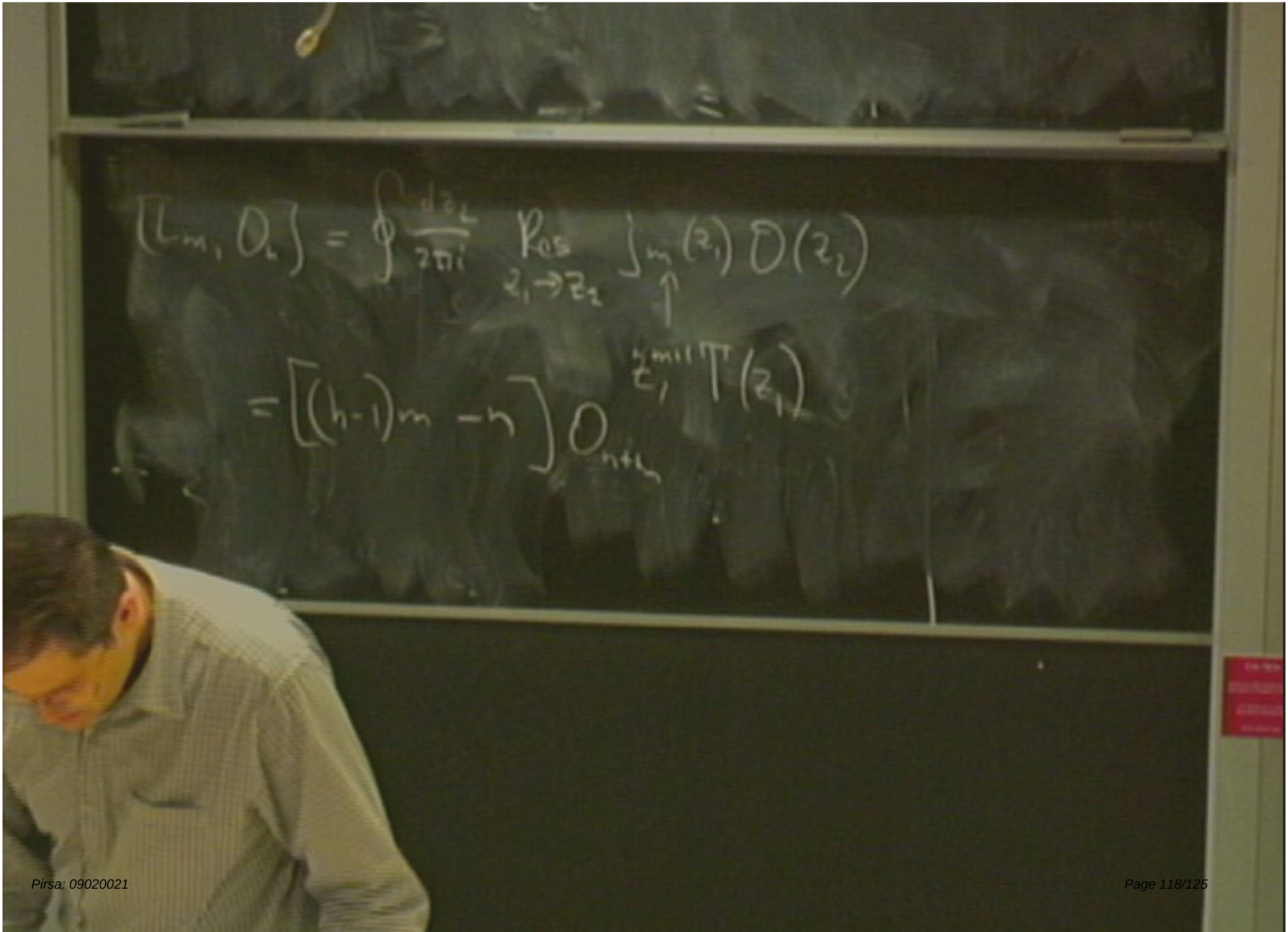
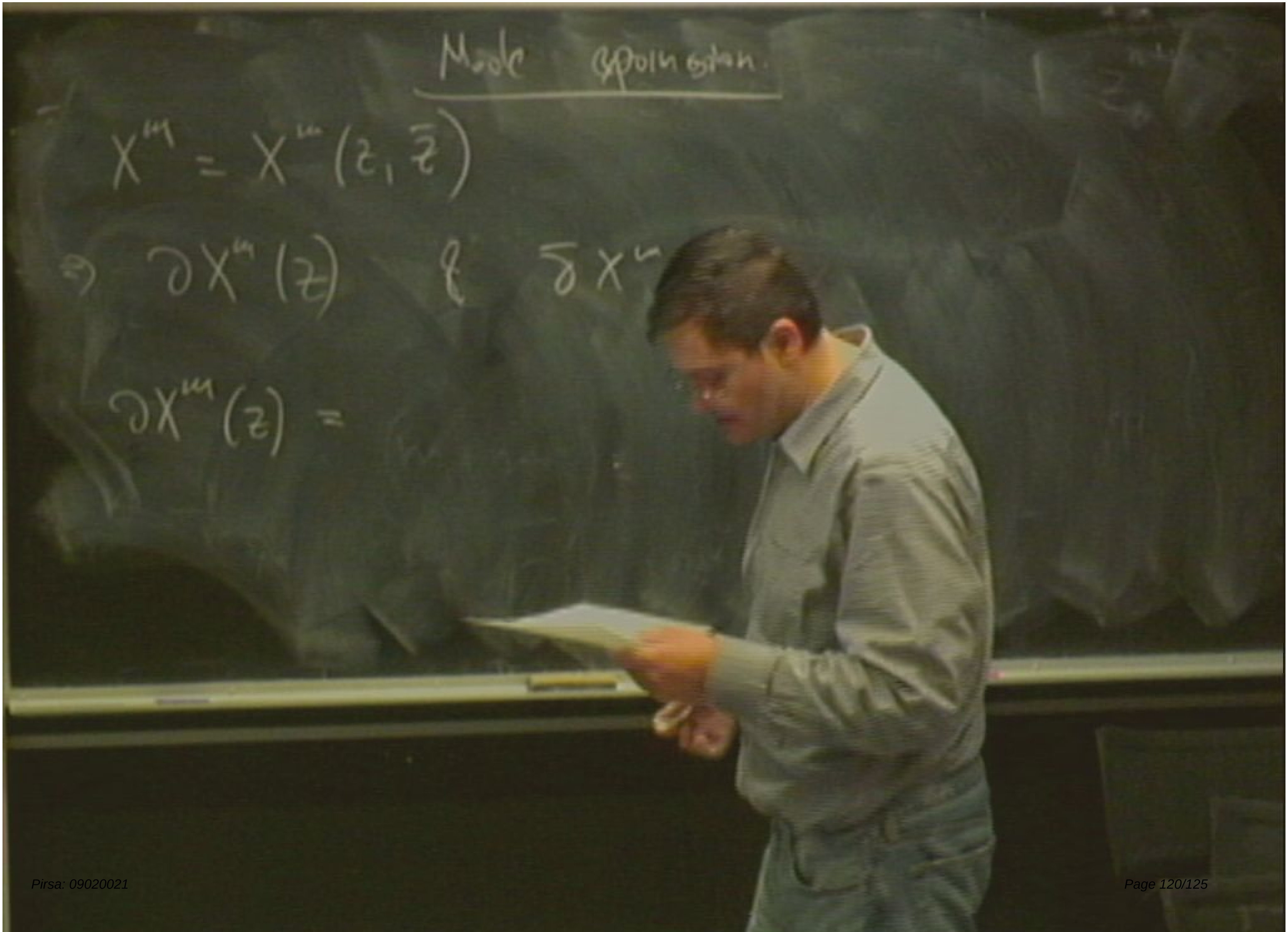

[L_m, O_n] = ∮ dz_2/2πi Res_{z_1→z_2} j_m(z_1) O(z_2)
z_1^{m+1} T(z_1)
= [(h-1)m - n] O_{n+m}


Mode expansion
$X^m = X^m(z, \bar{z})$
$\Rightarrow \partial X^m(z)$


Mode expansion.
$X^m = X^m(z, \bar{z})$
$\Rightarrow \partial X^m(z)$ & $\bar{\partial} X^m$
$\partial X^m(z) =$

Mode expansion

$$X^\mu = X^\mu(z, \bar{z})$$

$$\Rightarrow \partial X^\mu(z) \quad \& \quad \bar{\partial} X^\mu(\bar{z})$$

$$\partial X^\mu(z) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum \frac{\alpha_m^\mu}{z^{m+1}}$$

Mode expansion.

$$X^\mu = X^\mu(z, \bar{z})$$

$$\Rightarrow \quad \partial X^\mu(z) \quad \& \quad \bar{\partial} X^\mu(\bar{z})$$

$$\partial X^\mu(z) = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{-\infty}^{+\infty} \frac{\alpha_m^\mu}{z^{m+1}} \qquad \bar{\partial} X^\mu = -i\left(\frac{\alpha'}{2}\right)^{1/2} \sum \frac{\tilde{\alpha}_m^\mu}{\bar{z}^{m+1}}$$

$\Rightarrow$ We want $X(z, \bar{z})$ to be single valued on $\mathbb{C}$

$\Rightarrow$ We want $X(z,\bar{z})$ to be single valued on $\mathbb{C}$

Look at $\underline{m=0}$

$$\alpha_0^\mu \ln z + \tilde{\alpha}_0^\mu \ln \bar{z}$$

$\Rightarrow$ We want $X(z, \bar{z})$ to be single valued on $\mathbb{C}$

Look at $\underline{m=0}$

$$\alpha_0^\mu \ln z + \tilde{\alpha}_0^\mu \ln \bar{z} \Rightarrow \ln z\bar{z} \Rightarrow \boxed{\alpha_0^\mu = \tilde{\alpha}_0^\mu}$$