Title: Towards macroscopic quantum superpositions & Theory of knots of light

Date: Feb 11, 2009 02:00 PM

URL: http://pirsa.org/09020017

Abstract: To interface photons with solid-state devices, we investigated the coupling of optically active quantum dots with optical micro- and nano-cavities. Initial experimental progress have led to the unexpected observation of ultra low threshold lasing of a photonic crystal defect mode cavity embedded with only 1 to 3 InAs self-assembled quantum dots as gain medium. Photon correlation measurements confirmed the transition from a thermal light source to a coherent light source. We also report on micro-pillar cavities with integrated oxidation apertures and electronic gates that provide an 80MHz single photon source with controllable polarization. A second set of experiments will be addressed that has as long-term aim the transfer of a superposition of a photon propagating in two directions into a superposition of two center-of-mass motions of a tiny mirror that is placed in one path of the photon. A crucial part of the proposed experiment is an optical cavity with one end mirror as small as 20 Âµm in diameter attached to a high Q mechanical cantilever. Such a system has been achieved with an optical quality factor of 2,100 and a mechanical quality factor of 100,000. This provides an excellent interferometric measurement of the thermal motion of the micro-mechanical system. The thermal motion of the center-of-mass mode can be counter acted using a feedback circuit to modulate an additional optical force. Experimental results will be shown that demonstrate the optical cooling from room temperature to 135 mK.

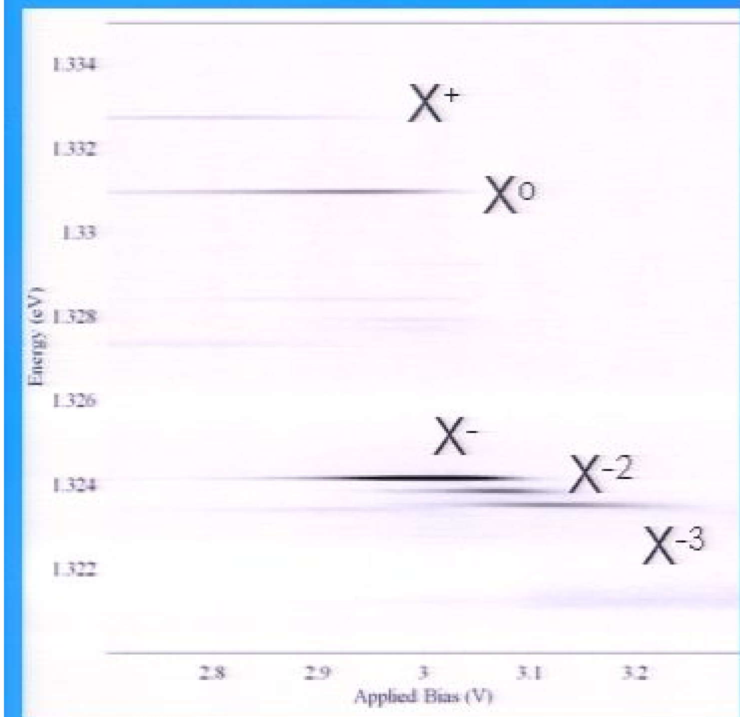
Solid State Cavity QED
Towards Macroscopic Quantum Superpositions
Knots of Light

D. Bouwmeester

PI Febr.11 2009

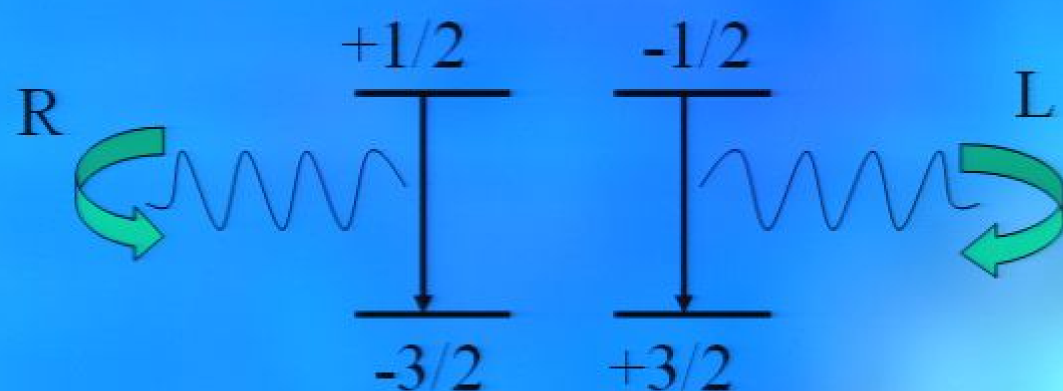
$$|\Psi\rangle = \alpha|\text{UCSB}\rangle + \beta|\text{Leiden}\rangle$$

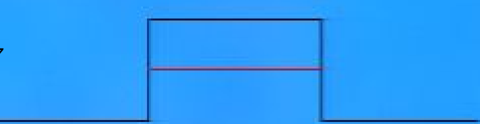
Self-assembled GaAs/InGaAs QUANTUM DOTS



add extra electron to QD

Spin of extra electron is qubit (0.1ms?) coupled to excitons (gates ns)





Hybrid QP schemes

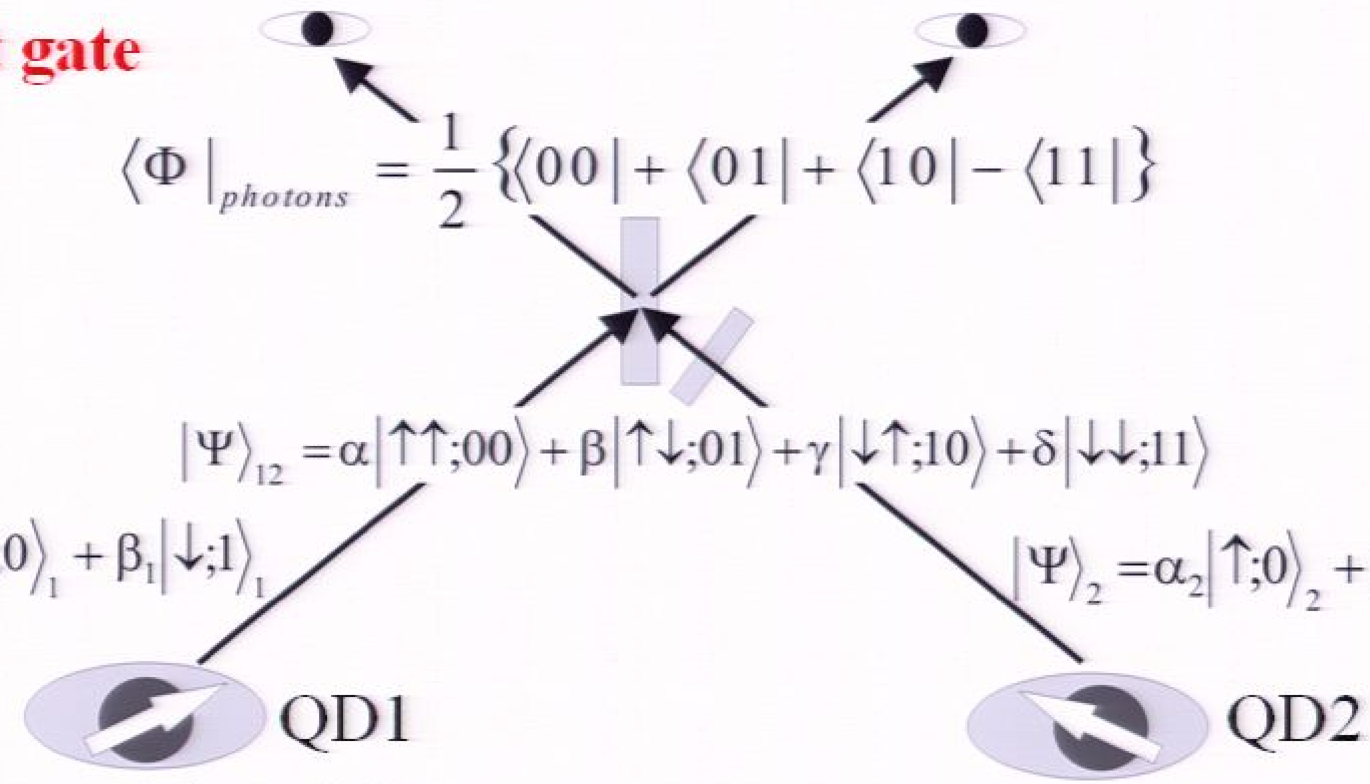
$$\langle \Phi | \Psi \rangle_{12} = \alpha |\uparrow\uparrow\rangle + \beta |\uparrow\downarrow\rangle + \gamma |\downarrow\uparrow\rangle - \delta |\downarrow\downarrow\rangle$$

2 qubit gate

$$\langle \Phi |_{photons} = \frac{1}{2} \{ \langle 00 | + \langle 01 | + \langle 10 | - \langle 11 | \}$$

$$|\Psi\rangle_{12} = \alpha |\uparrow\uparrow;00\rangle + \beta |\uparrow\downarrow;01\rangle + \gamma |\downarrow\uparrow;10\rangle + \delta |\downarrow\downarrow;11\rangle$$

$$|\Psi\rangle_1 = \alpha_1 |\uparrow;0\rangle_1 + \beta_1 |\downarrow;1\rangle_1 \qquad |\Psi\rangle_2 = \alpha_2 |\uparrow;0\rangle_2 + \beta_2 |\downarrow;1\rangle_2$$



$$|\Psi\rangle_1 = \alpha_1 |\uparrow\rangle_1 + \beta_1 |\downarrow\rangle_1 \qquad |\Psi\rangle_2 = \alpha_2 |\uparrow\rangle_2 + \beta_2 |\downarrow\rangle_2$$

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Linear optics distributed quantum computation, PRL **95**, 030505

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Local qubit, photon entanglement

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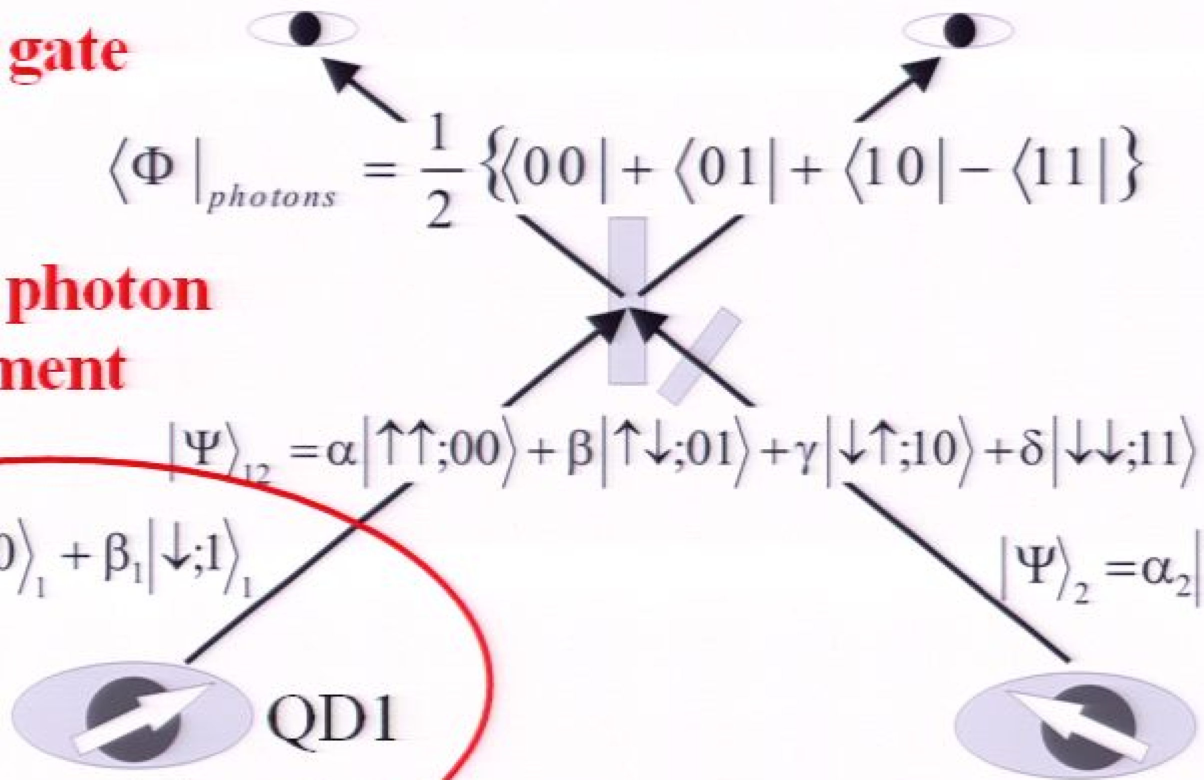

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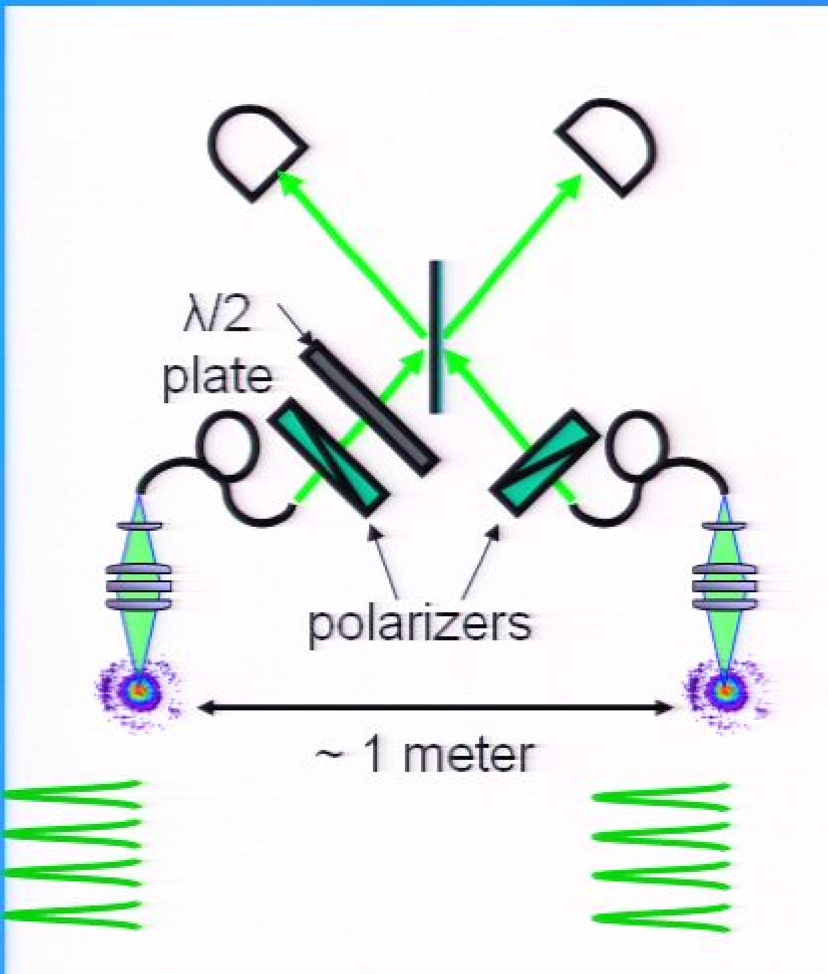
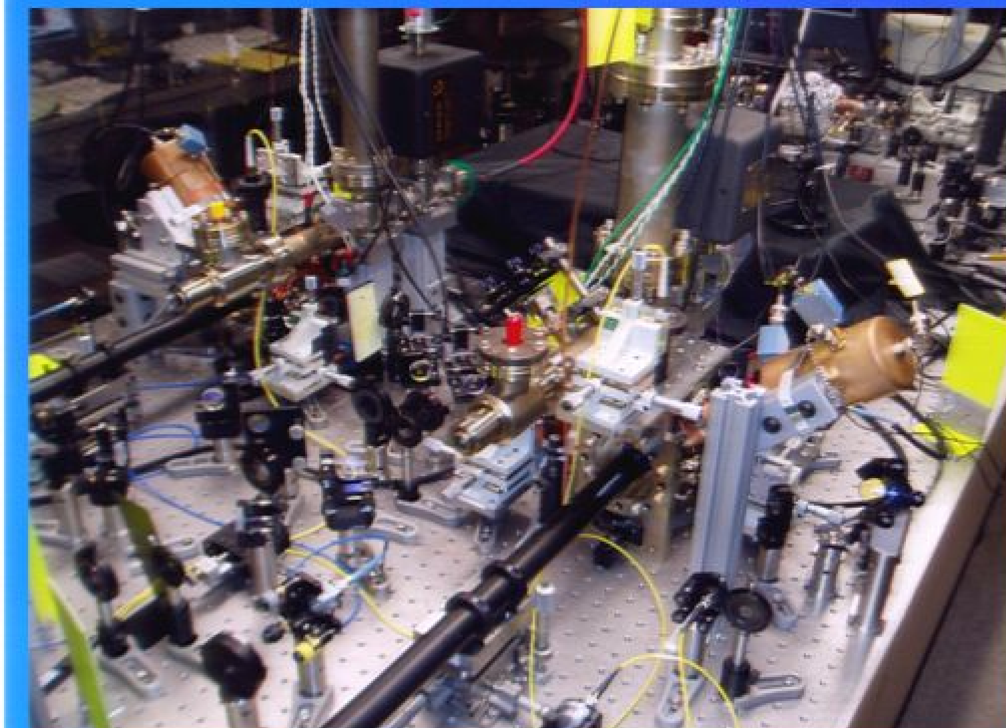


Recall Dzmitry Matsukevich talk



Fidelities good (~0.8)

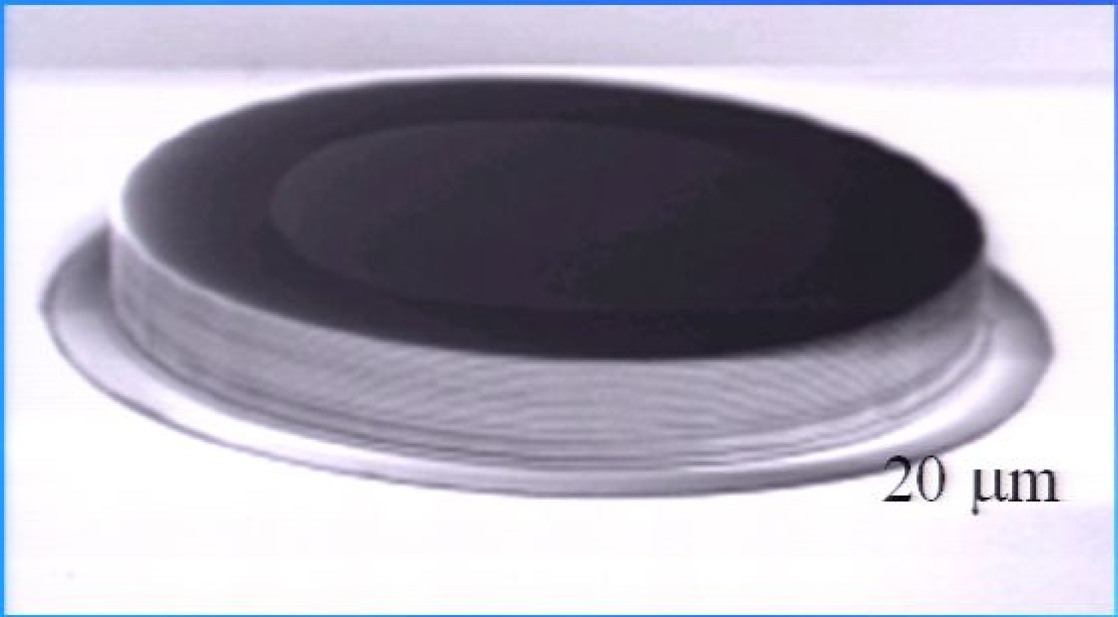
Probability of success 10^{-8}

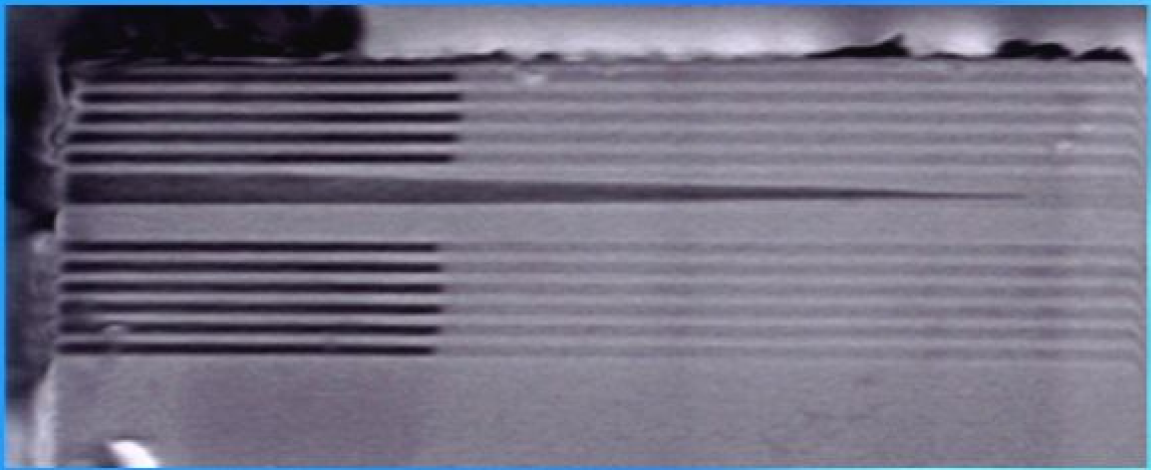
Nature **449**, 68 (2007); PRL **100**, 150404 (2008) **Monroe group**

Electron spins coupled to photons using micropillars

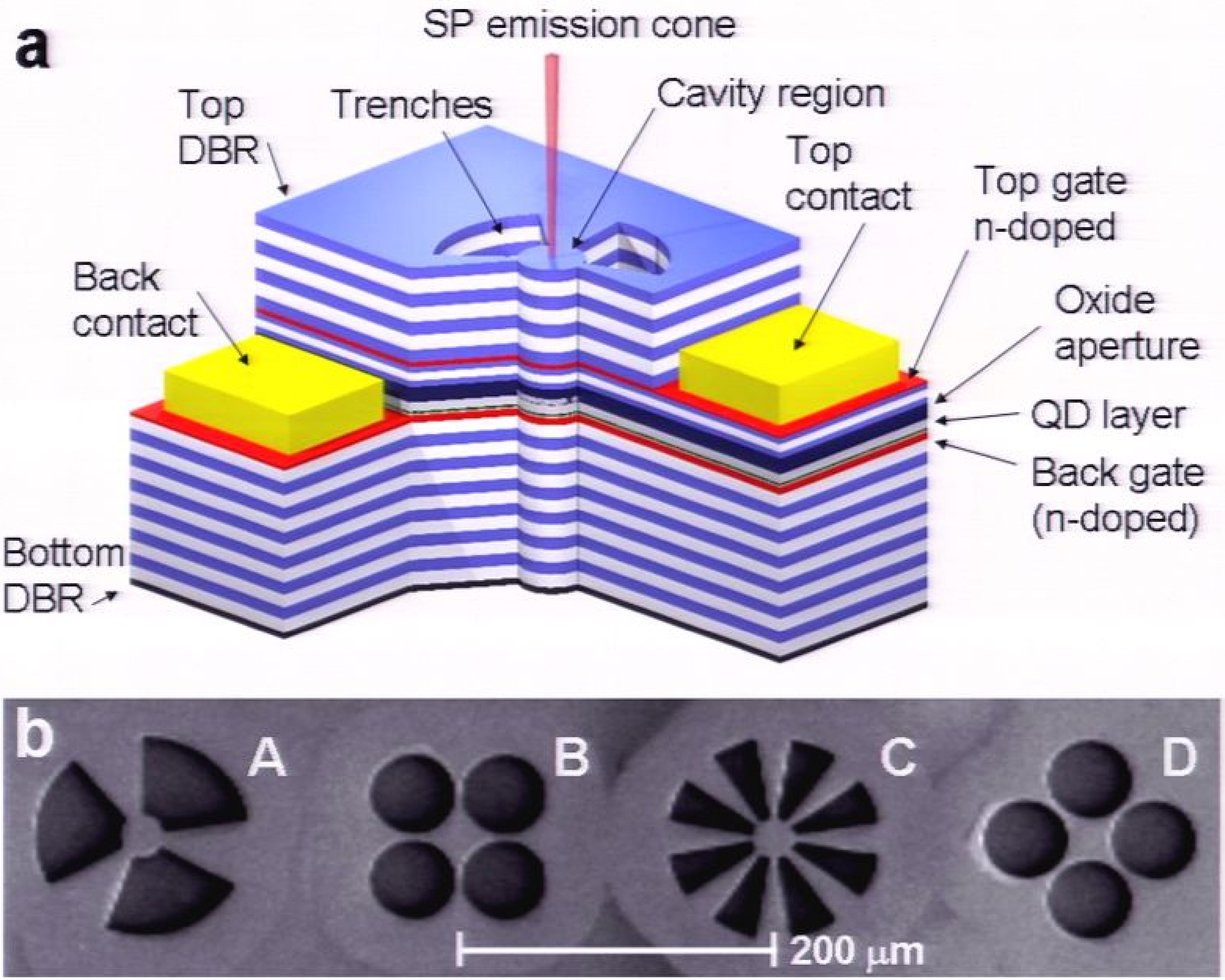
Oxide apertured micropillars

M. Pelton et al.,
PRL 2002



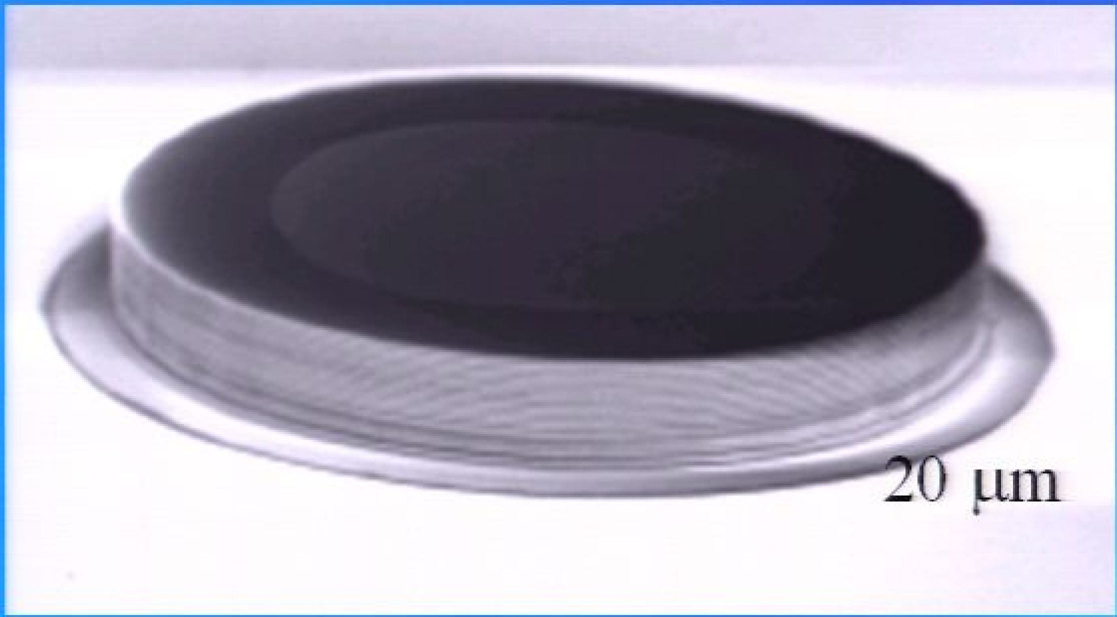


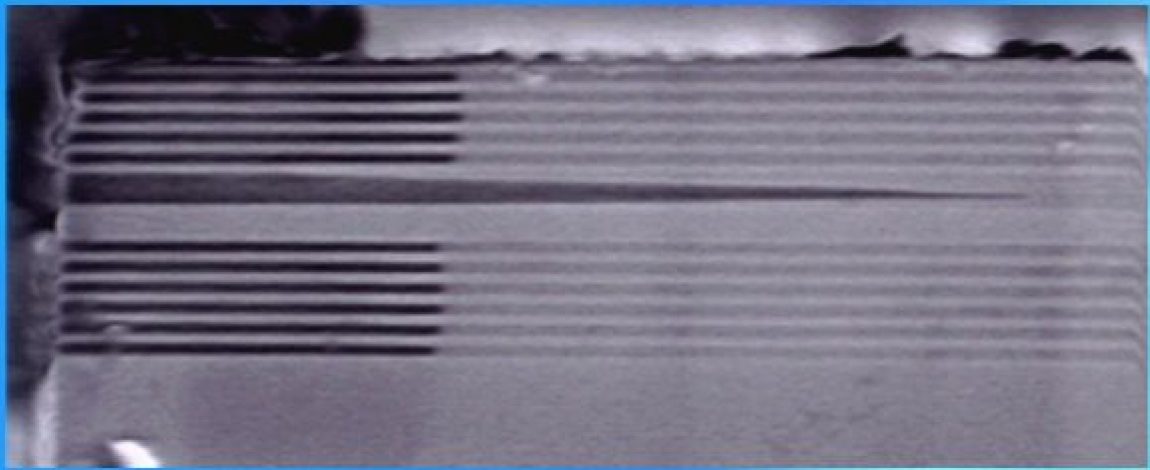
N. G. Stoltz, et al.,
Appl. Phys. Lett. **87**,
031105 (2005)


a
SP emission cone
Top
DBR
Trenches
Cavity region
Top
contact
Top gate
n-doped
Back
contact
Oxide
aperture
QD layer
Back gate
(n-doped)
Bottom
DBR
b
A
B
C
D
200 μm

Oxide apertured micropillars

M. Pelton et al.,
PRL 2002





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Appl. Phys. Lett. **87**,
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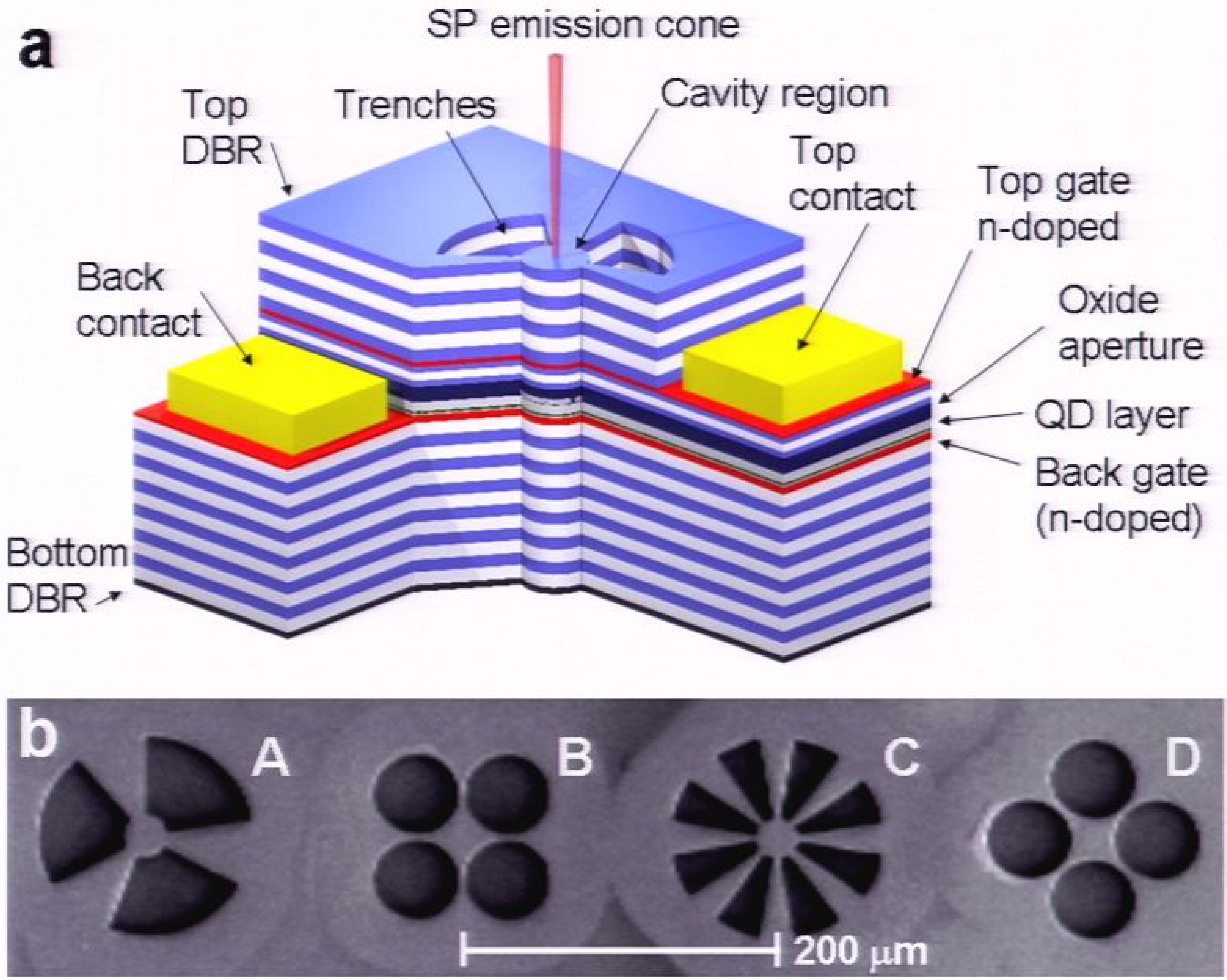
a
SP emission cone
Top DBR
Trenches
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Oxide aperture
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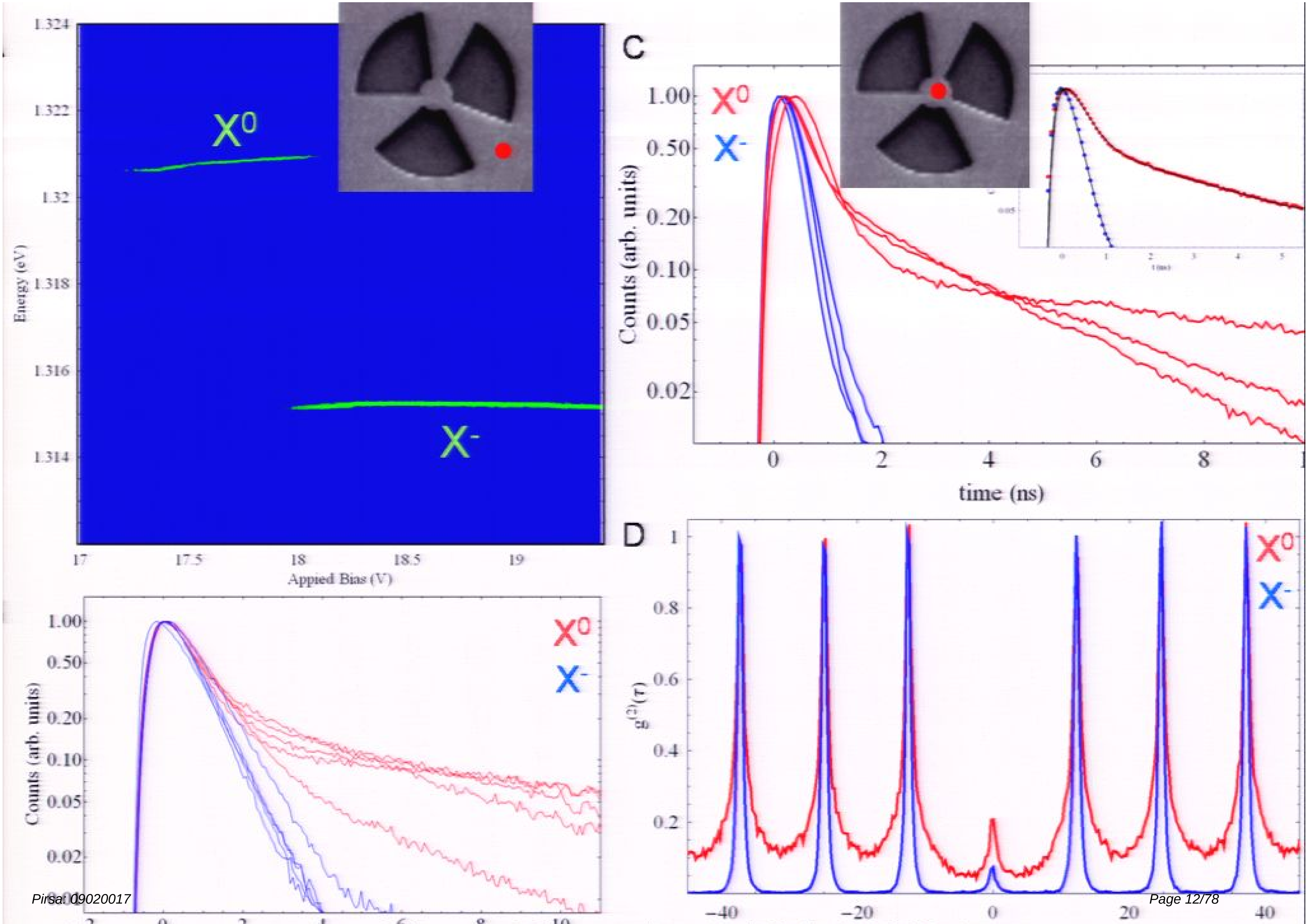
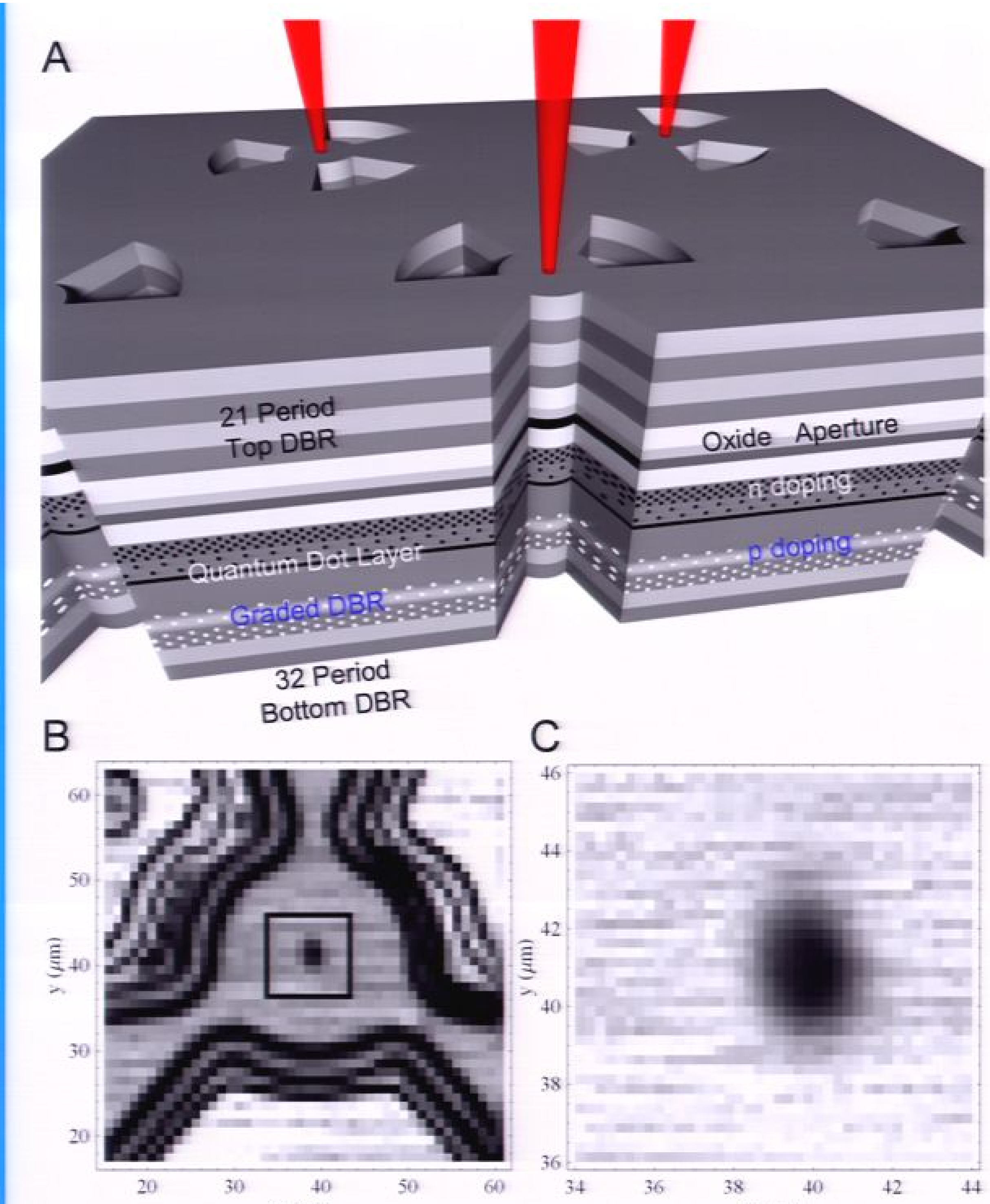
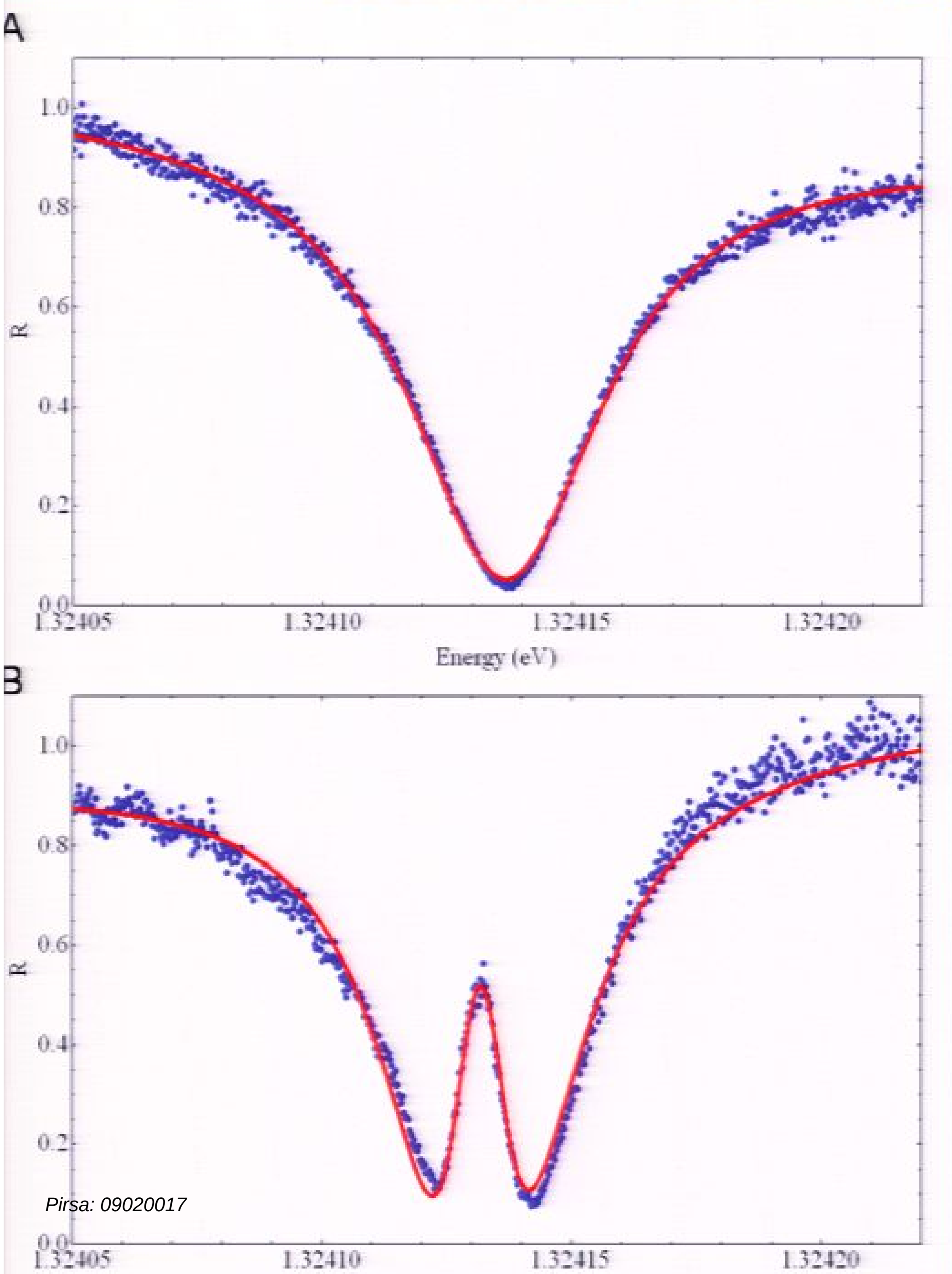


Nature Photonics 1, 704 (2007)


A
21 Period
Top DBR
Oxide Aperture
n doping
Quantum Dot Layer
p doping
Graded DBR
32 Period
Bottom DBR
B
C
y (μm)
x (μm)
20
30
40
50
60
34
36
38
40
42
44
46

Reflection Spectroscopy



Jaynes-Cummings model

$$R(\omega)=\left|1-\frac{\kappa(\gamma-i(\mu\omega-\omega_{QD}))}{(\gamma-i(\omega-\omega_{QD}))(\kappa-i(\omega-\omega_c))+g^2}\right|^2$$

κ is cavity field decay rate:

$\kappa = 24.1\mu eV$, corresponding to Q =27,000,

g is emitter-cavity coupling

$g = 9.7\,\mu eV$,

γ is emitter decay rate:

$\gamma = 1.9\mu eV$,

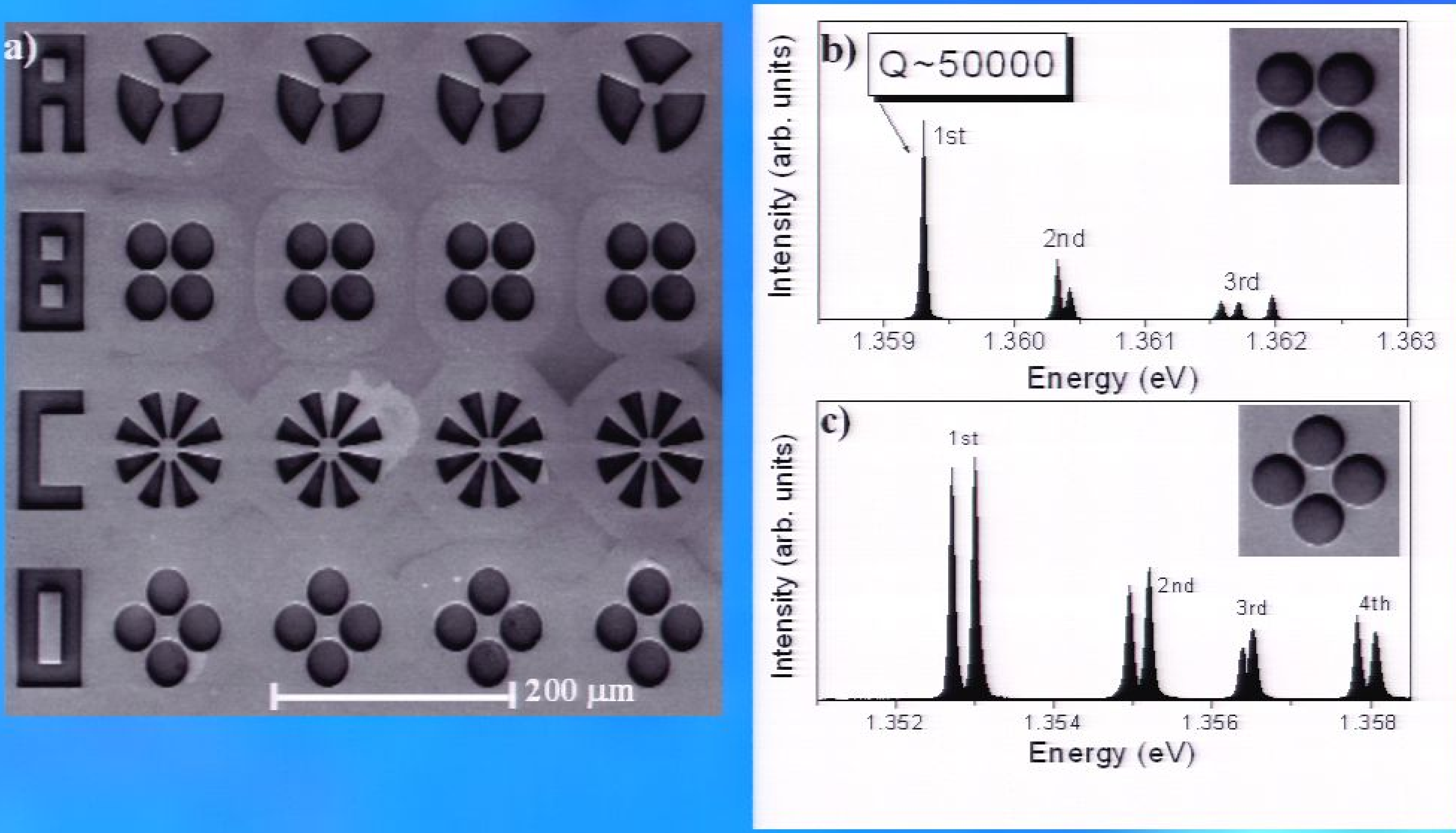
$\frac{g}{\kappa} = 0.40$, deep in Purcell (weak-coupling) regime,

$\frac{g}{\kappa} > 0.5$ is strong coupling

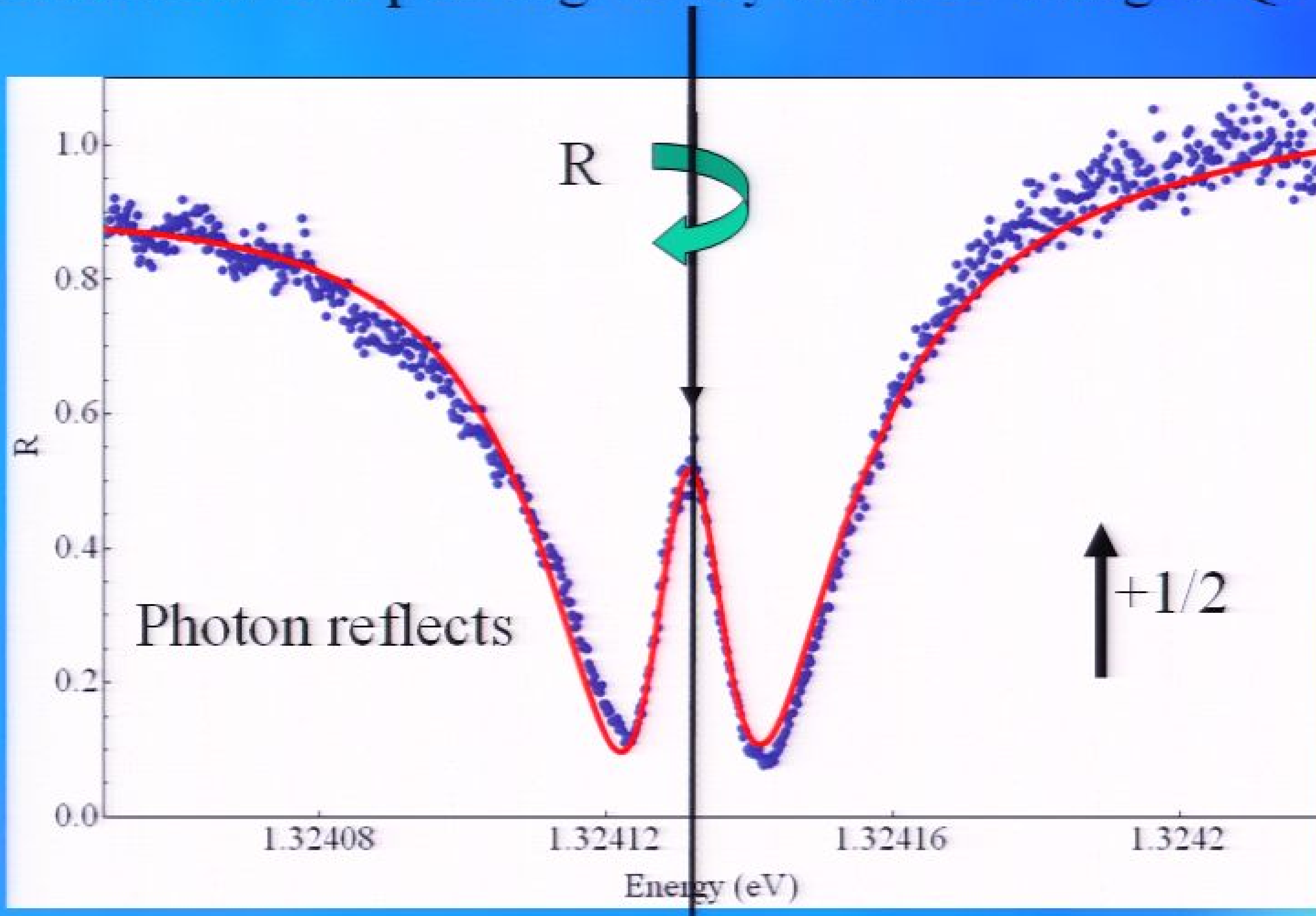
96% mode matched!!!

Ideal for hybrid QIP schemes

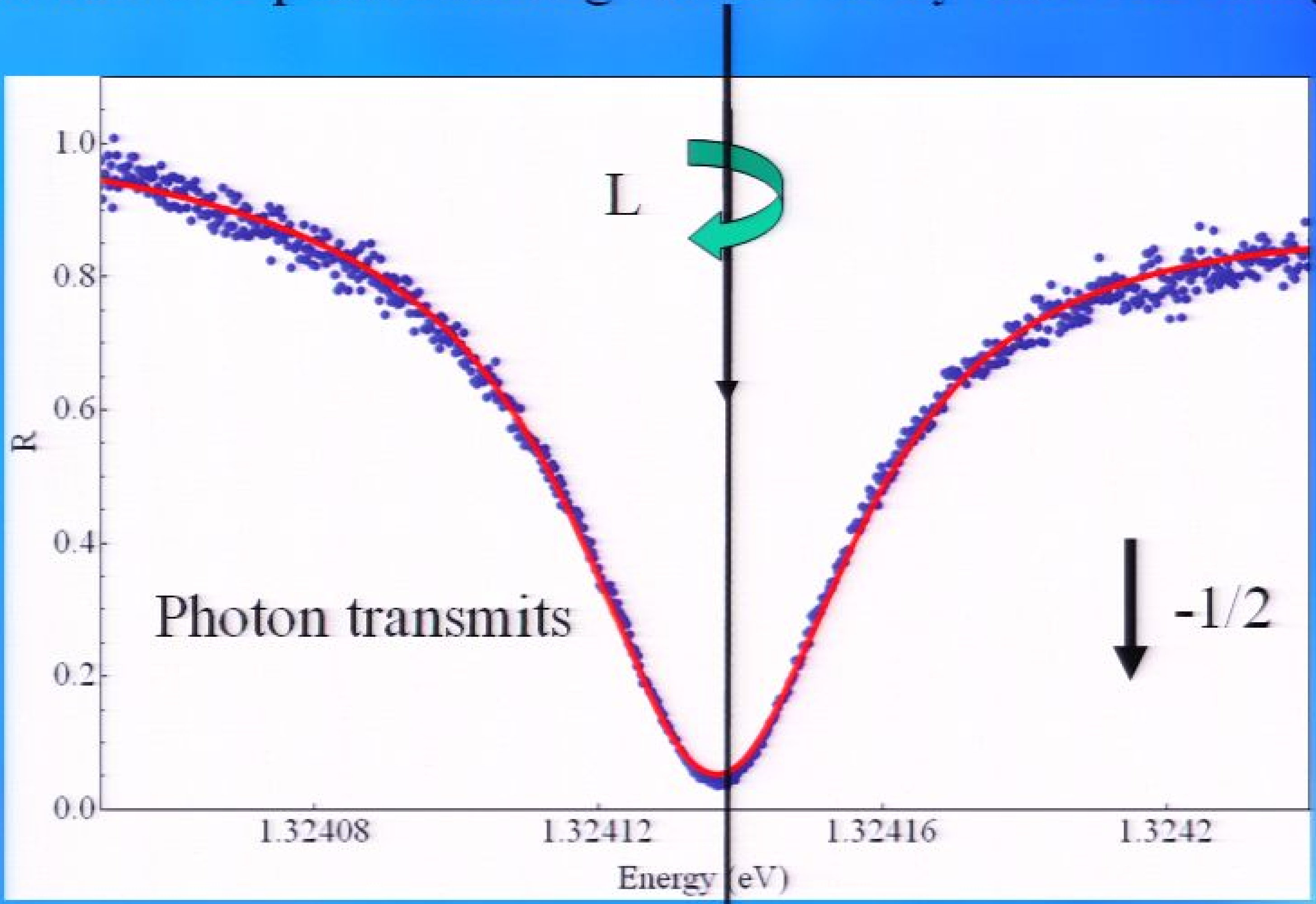
Mode polarization tuning



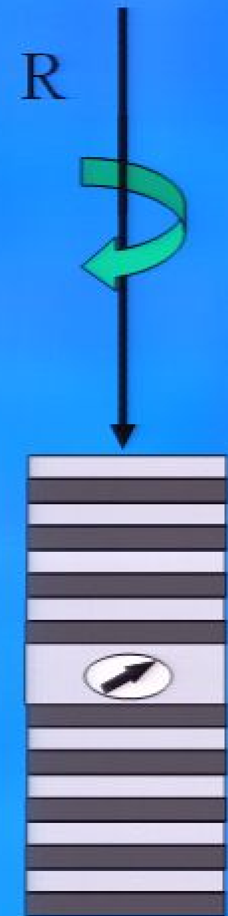
Prediction: For pol. deg. cavity and a X^- charged QD



Prediction: For polarization generate cavity and a X^- charged QD



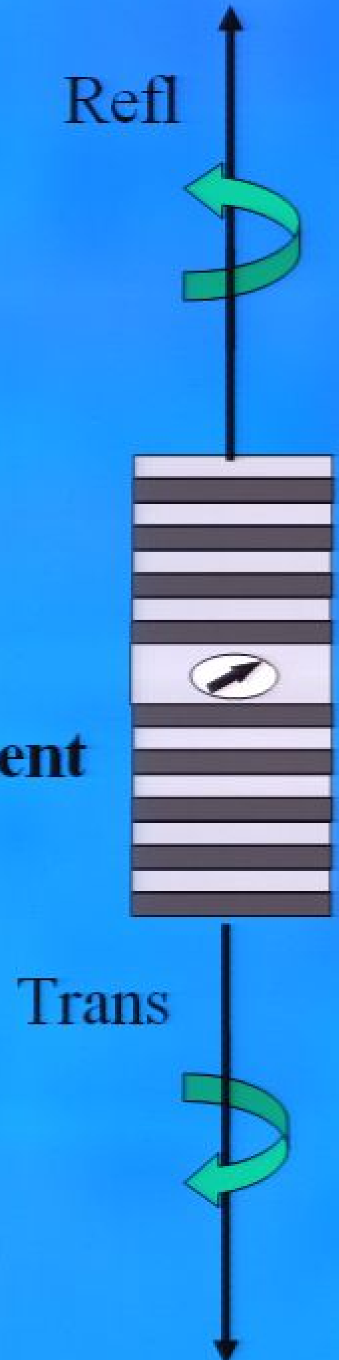
$$\left|\Psi\right\rangle = \alpha\left|\uparrow;\mathrm{Refl}\right\rangle + \beta\left|\downarrow;\mathrm{Trans}\right\rangle$$



$$|\Psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$$

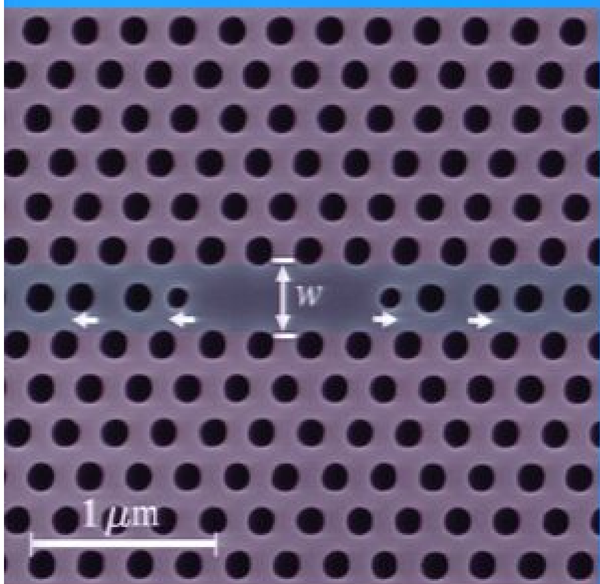
$$|\Psi\rangle = \alpha|\uparrow;\text{Refl}\rangle + \beta|\downarrow;\text{Trans}\rangle$$

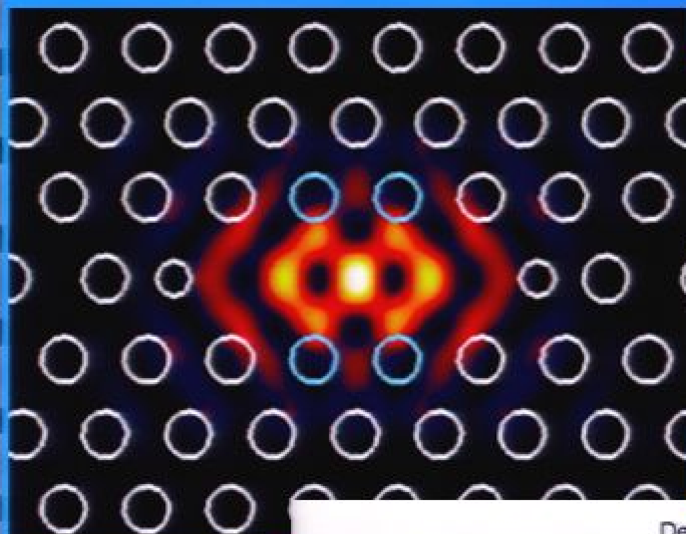
Single photon
"interaction free"
single electron spin
entanglement/measurement

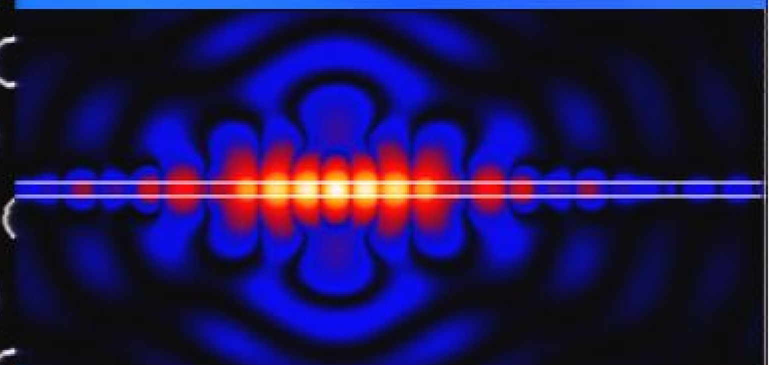


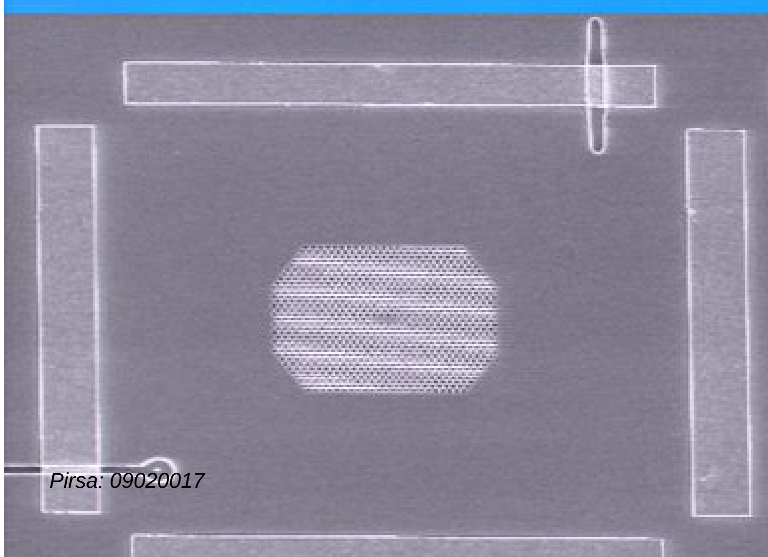
$$|\Psi\rangle = \alpha|\uparrow;\mathrm{Refl}\rangle + \beta|\downarrow;\mathrm{Trans}\rangle$$

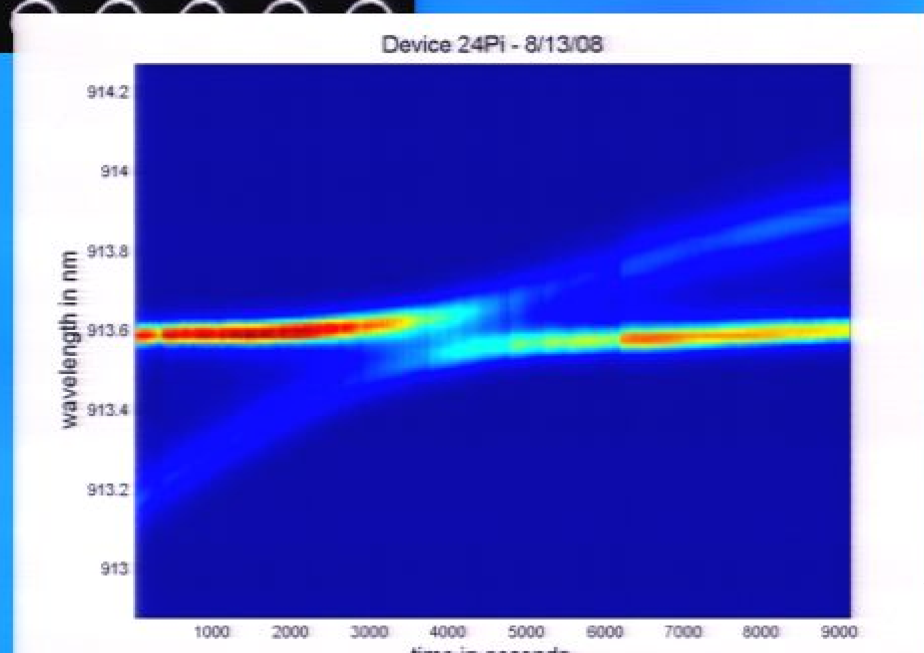
QD in Photonic Crystals

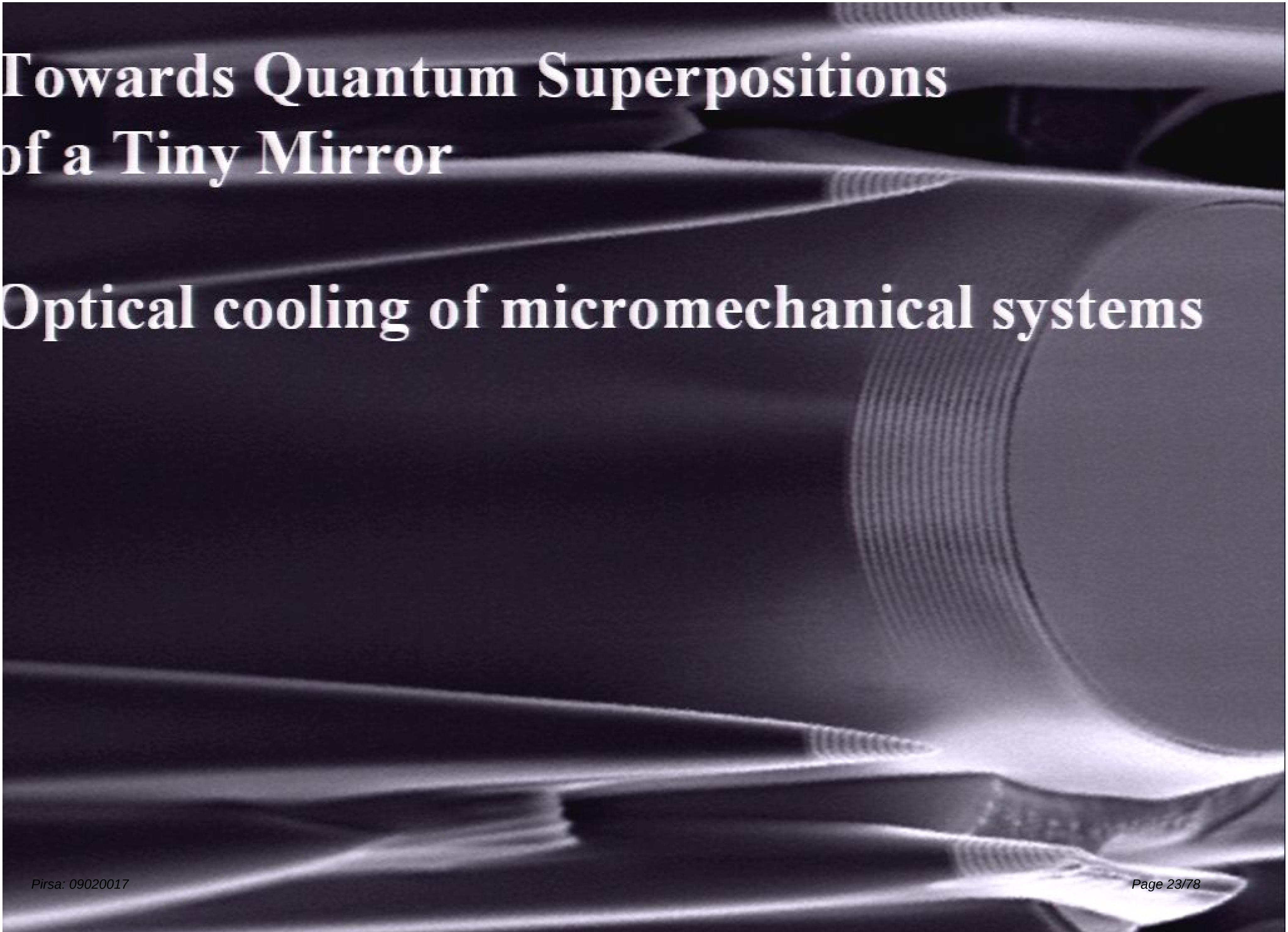












Towards Quantum Superpositions of a Tiny Mirror

Optical cooling of micromechanical systems

Niels Bohr
Copenhagen interpretation:

The wavefunction $|\Psi\rangle$ is not to be taken seriously as describing a quantum level physical reality, but is to be regarded as merely referring to our knowledge of the system.

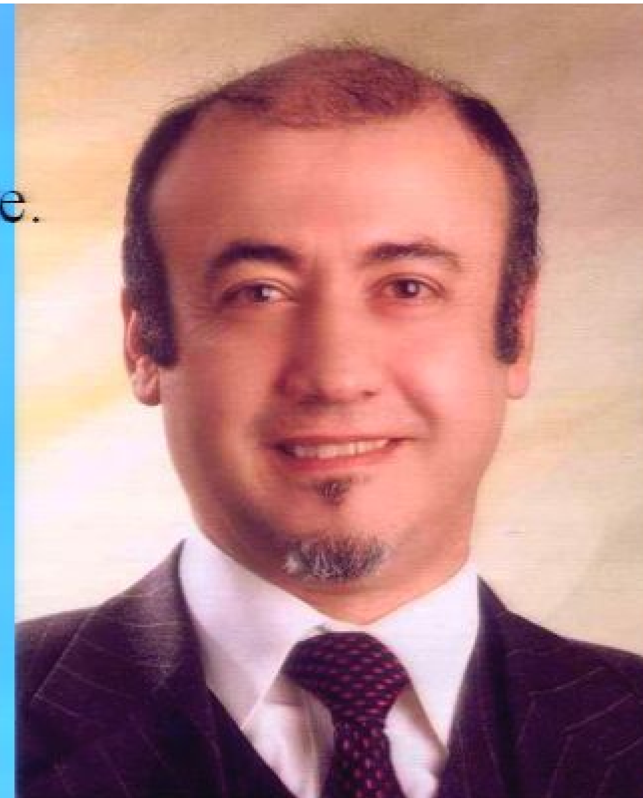
Quantum Measurements

Zurek (and others):

Environment Induced Decoherence

Caldeira-Leggett model assumes a linear coupling between the position of the system and he bath of harmonic oscillators

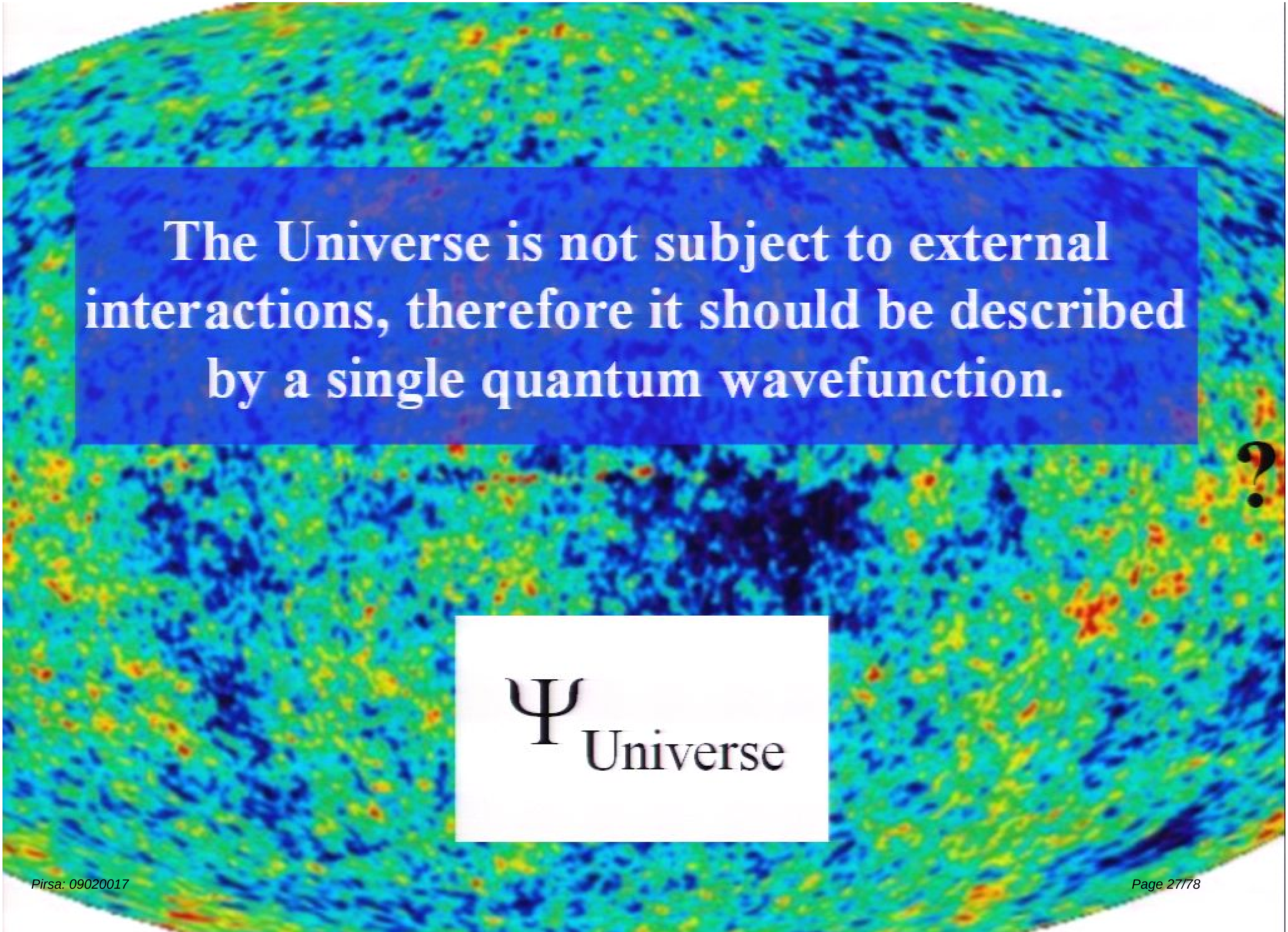
The wavefunction $|\Psi\rangle$ is a representation of a *real* physical state.



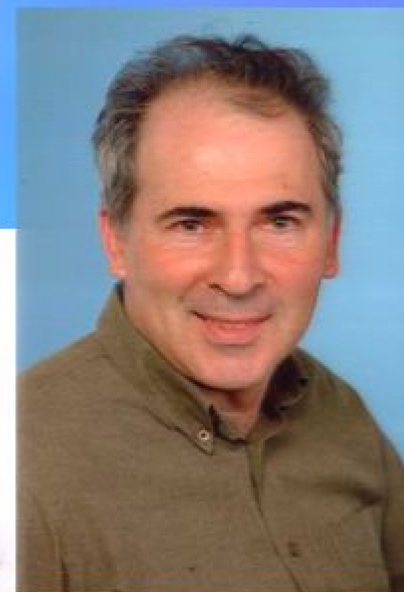
Everett

Deutsch

Many Worlds Interpretation

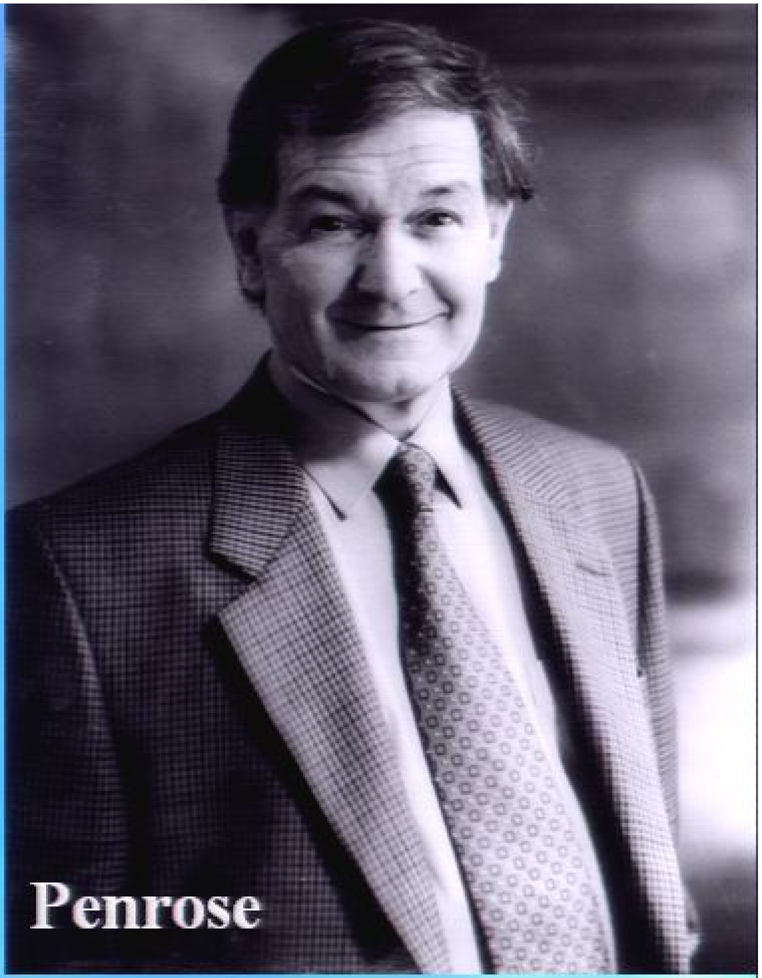

The Universe is not subject to external interactions, therefore it should be described by a single quantum wavefunction.
?
$\Psi_{Universe}$

Vaidman's watch



Penrose:

There is a conflict between Einstein's general covariance principle and the quantum superposition principle.



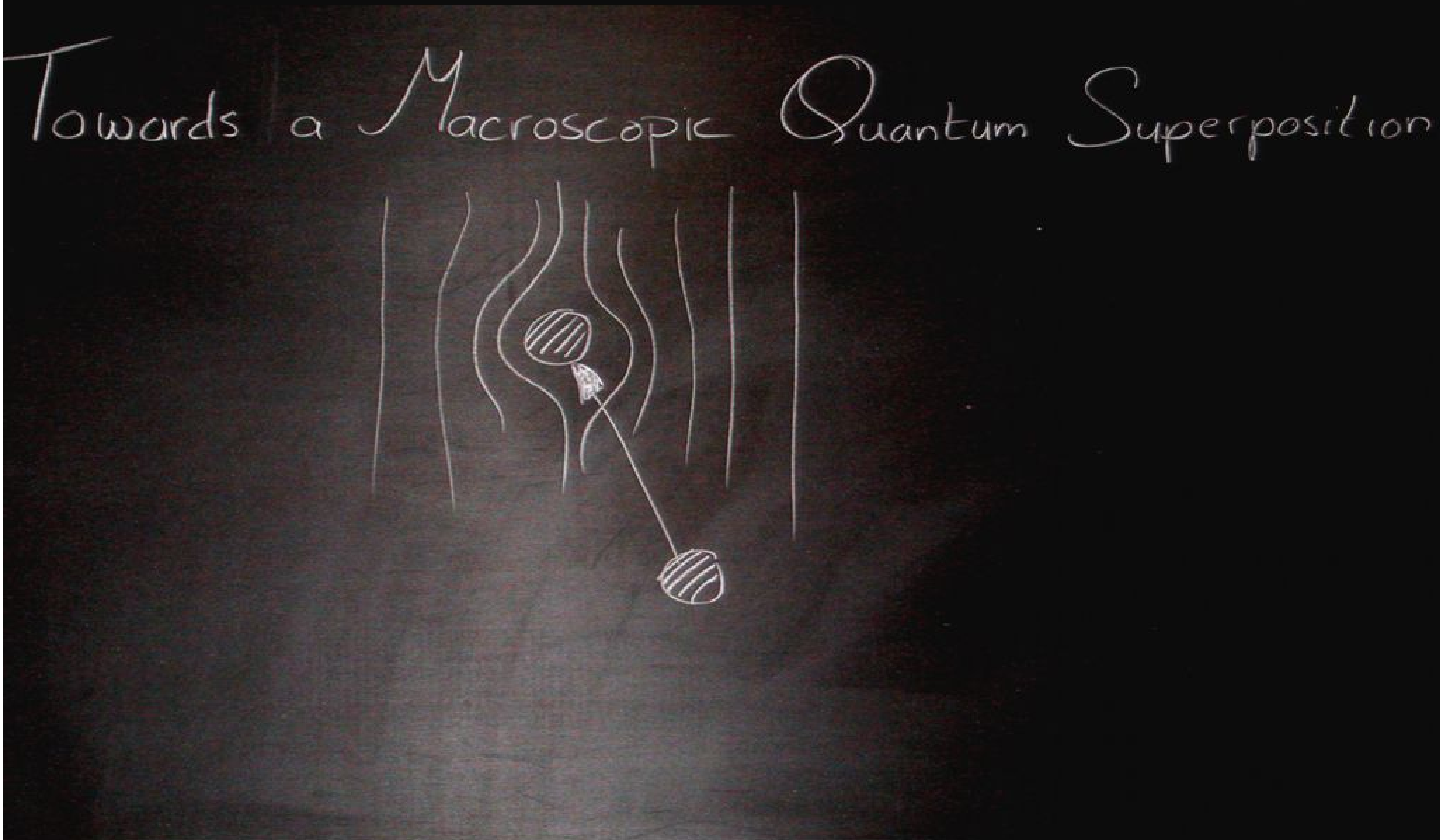
Two alternative locations of a massive object will each have stationary states, and have wavefunctions $|\Psi\rangle$ and $|\Phi\rangle$, that are eigenstates of the $\frac{\partial}{\partial t}$ operator with eigenvalues related to the energy.

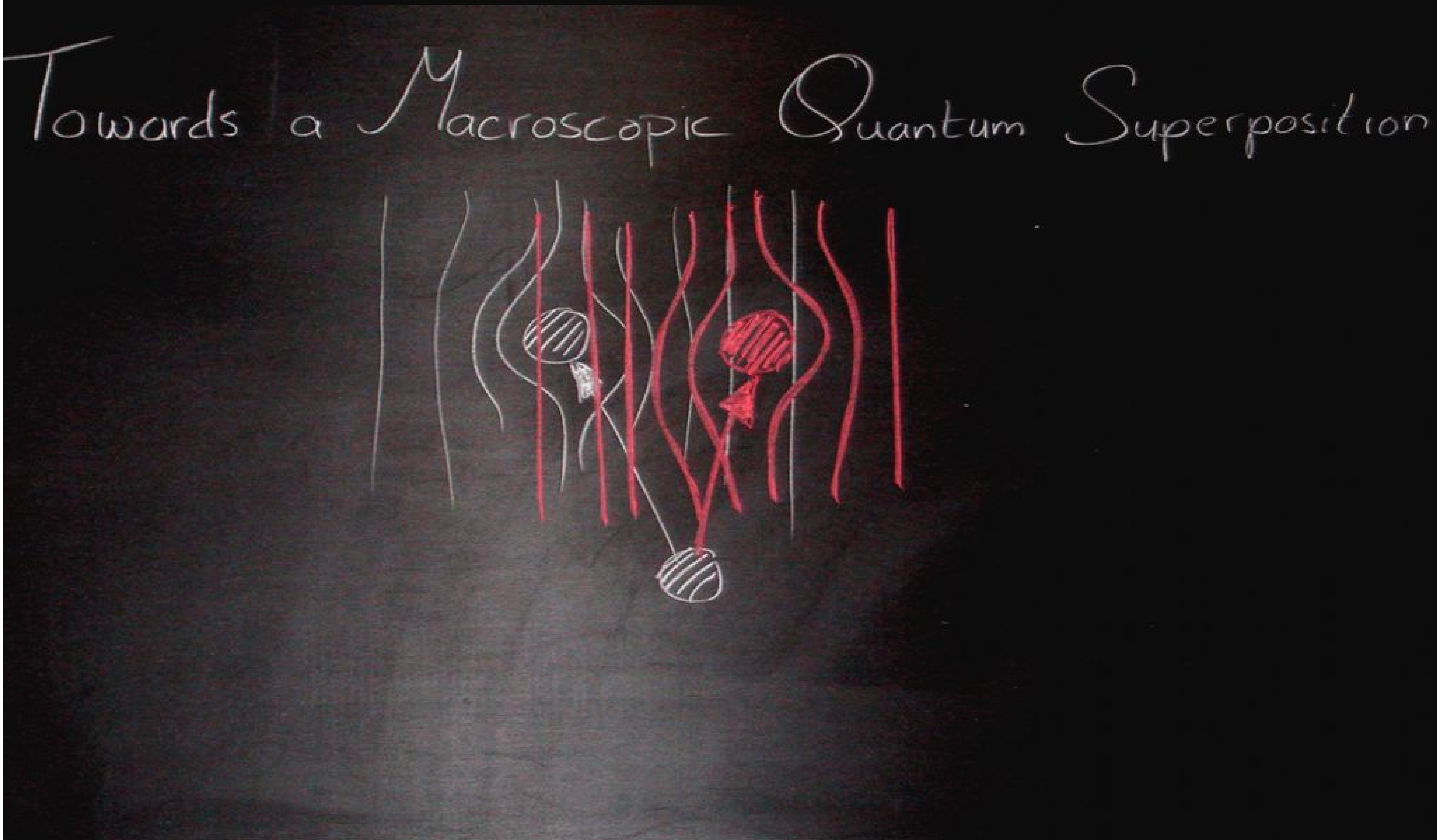
$$\frac{\partial}{\partial t}|\Psi\rangle = -i\hbar E_{\Psi}|\Psi\rangle$$

$$\frac{\partial}{\partial t'}|\Phi\rangle = -i\hbar E_{\Phi}|\Phi\rangle$$

But how to deal with superpositions

$$\left.\frac{\partial}{\partial t}\right)_{x,y,z\text{ fixed}}\left(\alpha|\Psi\rangle + \beta|\Phi\rangle\right) = ???$$


Towards a Macroscopic Quantum Superposition


Towards a Macroscopic Quantum Superposition

Consider an equal superposition $\frac{1}{\sqrt{2}}\left(|\Psi\rangle + |\Phi\rangle\right)$

f and **f'** are the acceleration 3-vectors of the free-fall motion in the two space-times (**f** and **f'** are gravitational forces per unit test mass).

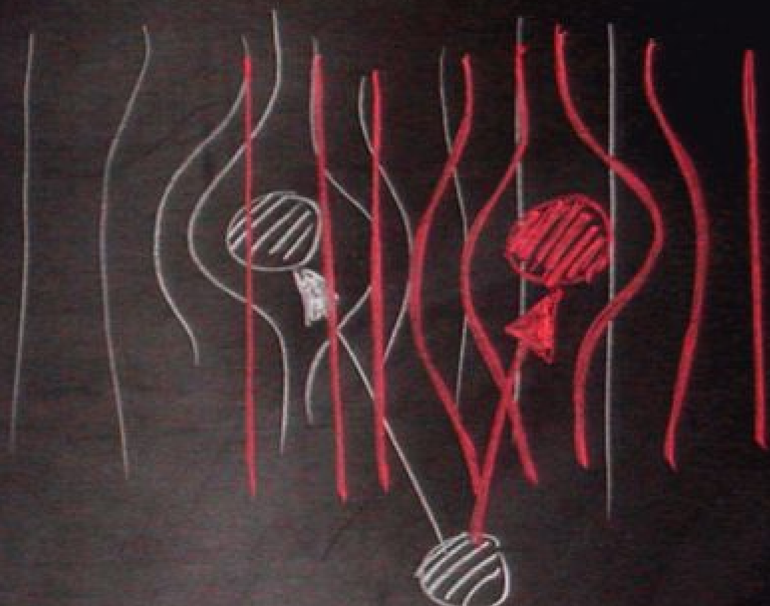
Penrose postulate: at each point the scalar $(|\mathbf{f}-\mathbf{f}'|)^2$ is a measure of incompatibility of the identification. The total measure of incompatibility (or "uncertainty) Δ at time *t* is:

$$\Delta = \int \left(\mathrm{f} - \mathrm{f}'\right)^2 \mathrm{d}^3 x$$
$$\equiv E_G$$

This is (4π) the gravitational self energy of the difference between the mass distributions of each of the two lump locations.

Prediction: The superposition state is unstable and has a lifetime of the order of $\frac{\hbar}{E_G}$

Towards a Macroscopic Quantum Superposition

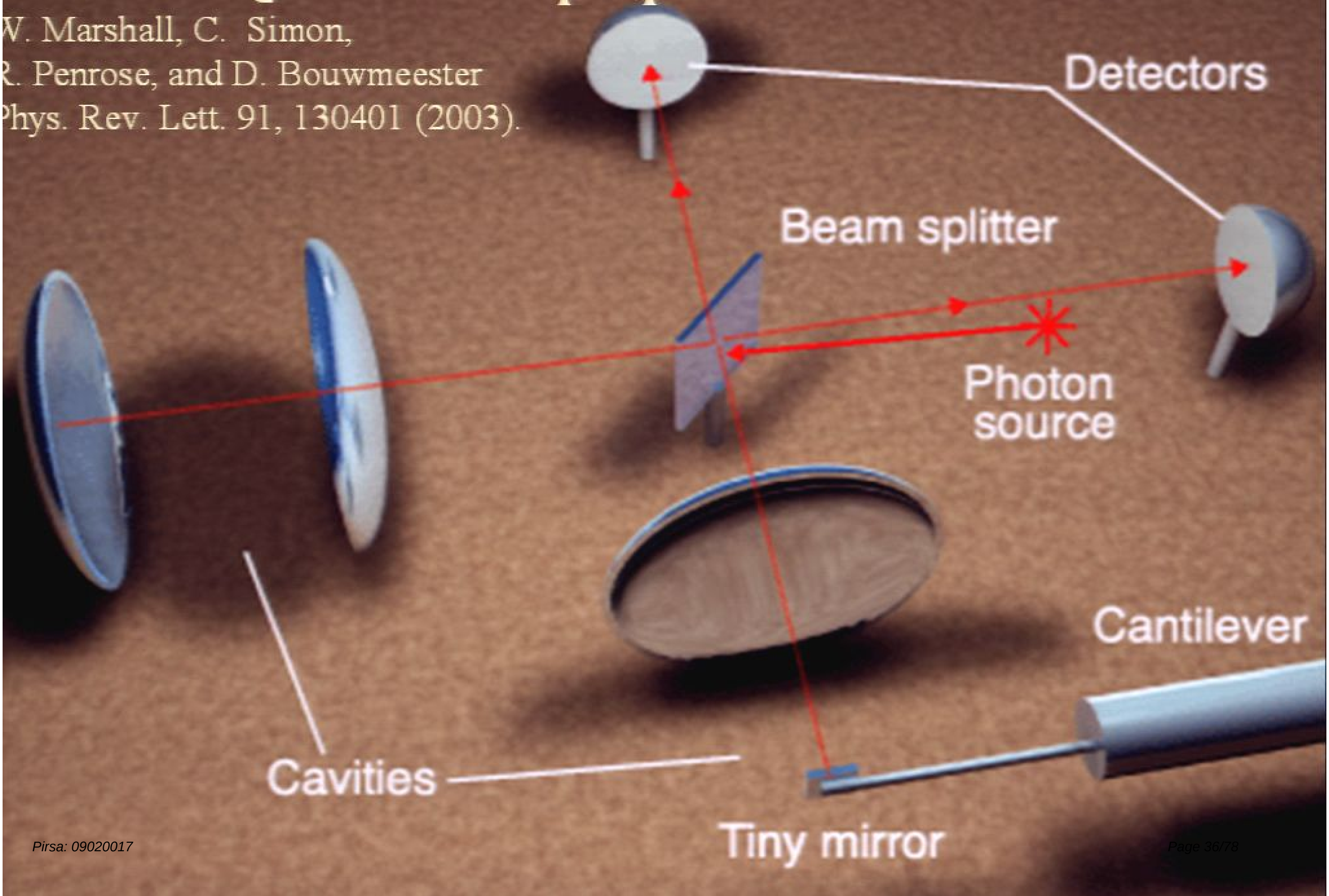


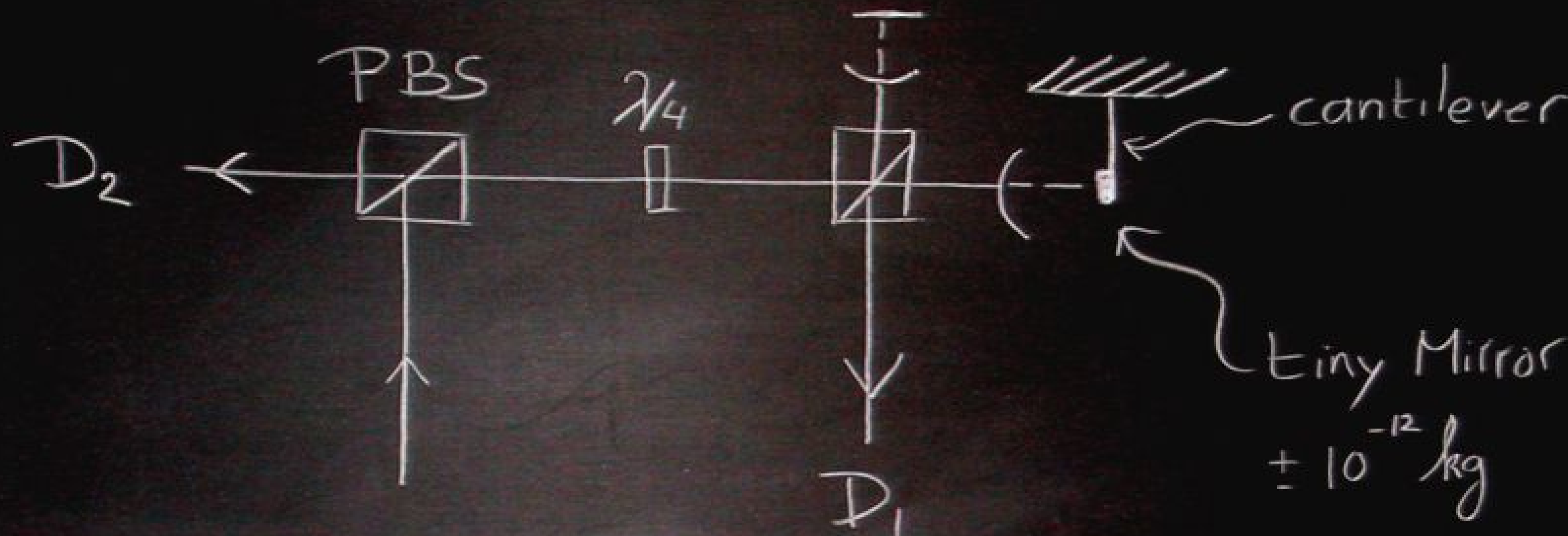
$$\Delta E_G \, \Delta t \geq \hbar$$


Oxford 2001

Towards Quantum Superpositions of a Mirror

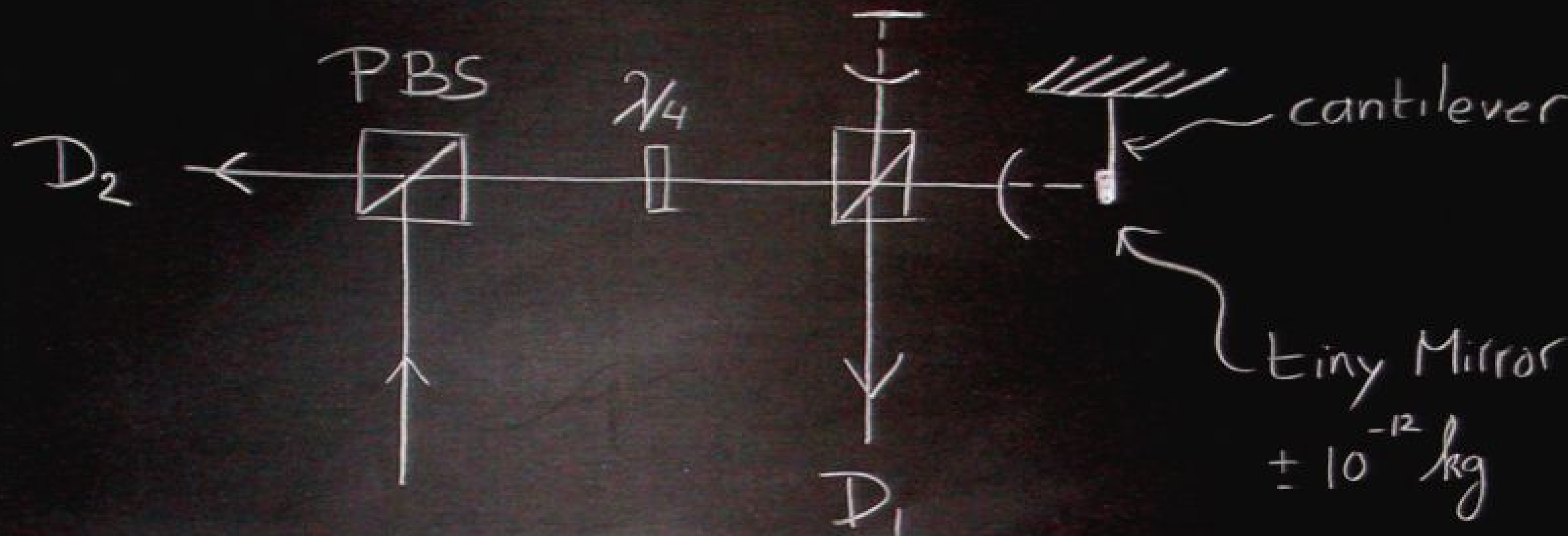
W. Marshall, C. Simon,
R. Penrose, and D. Bouwmeester
Phys. Rev. Lett. 91, 130401 (2003).





$$\mathcal{H} = \hbar\omega_c a^\dagger a + \hbar\omega_m b^\dagger b - \hbar g a^\dagger a (b + b^\dagger)$$

$$g = \frac{\omega_c}{L}\sqrt{\frac{\hbar}{2M\omega_m}}$$



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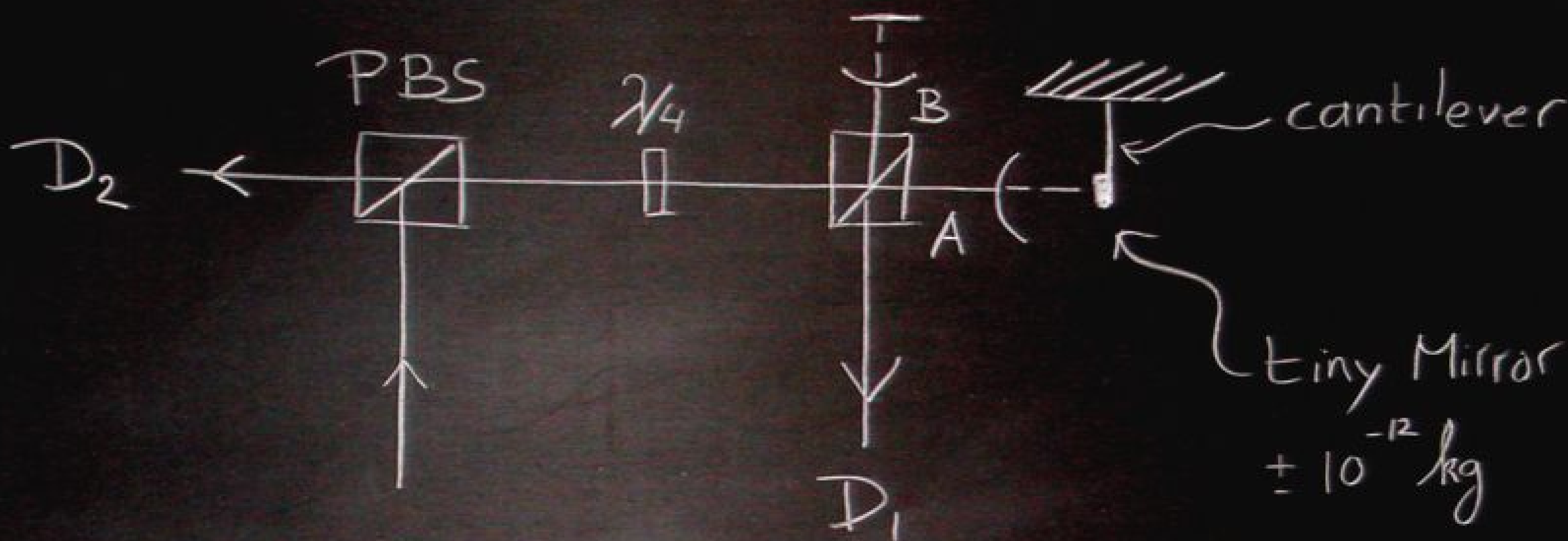
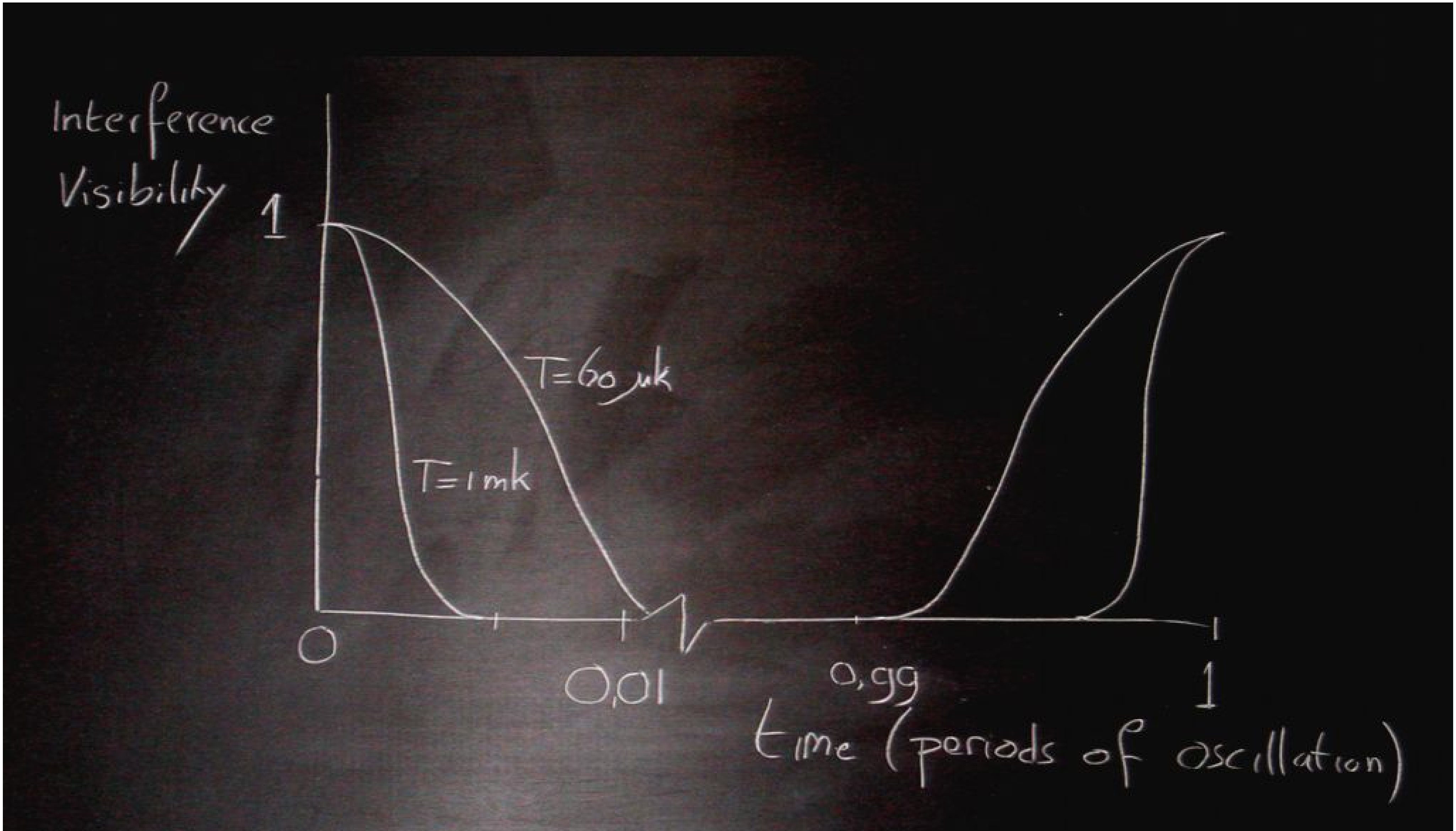
Law, PRA, **49**, 433 (1993)
Bose et al. PRA **59**, 3204 (1999)
Marshall et al. PRL **91**, 130401 (2003)



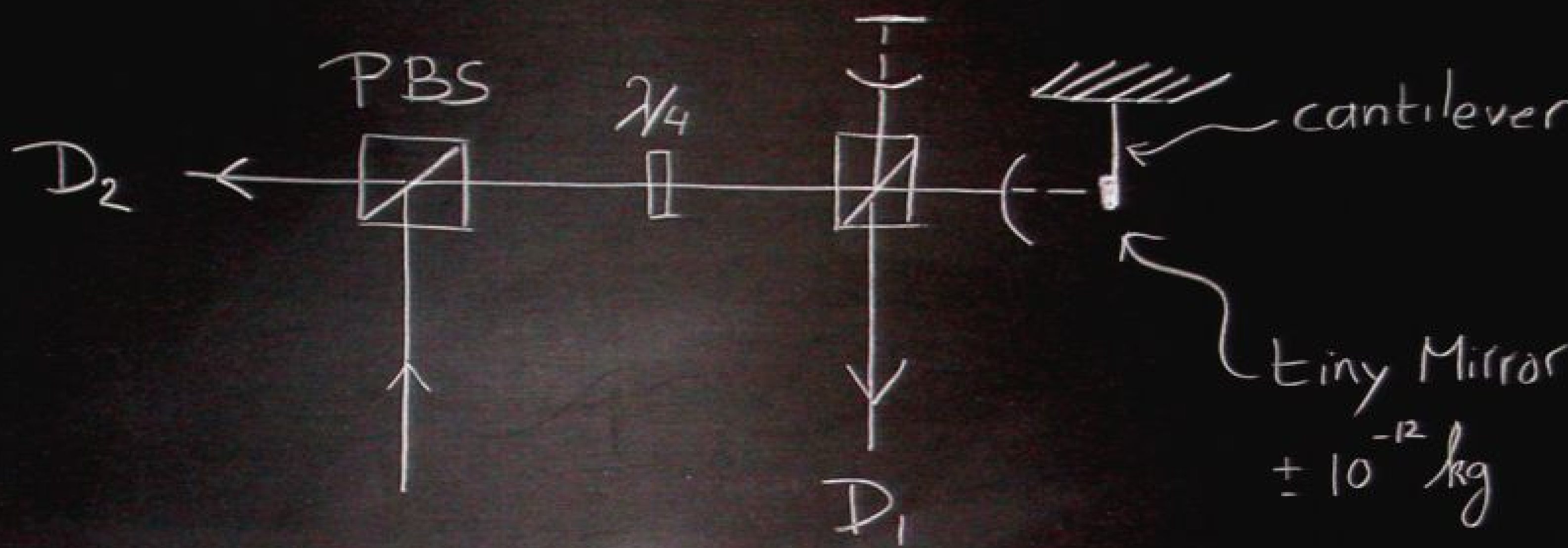
mirror in coherent state $|\beta\rangle = e^{-\frac{|\beta|^2}{2}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |n\rangle$

Initial state $|\psi(0)\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B \right) |\beta\rangle$

$|\psi(t)\rangle$ = entangled state of mirror and photon except after full period of oscillation



T is effective temperature of the fundamental resonance of cantilever



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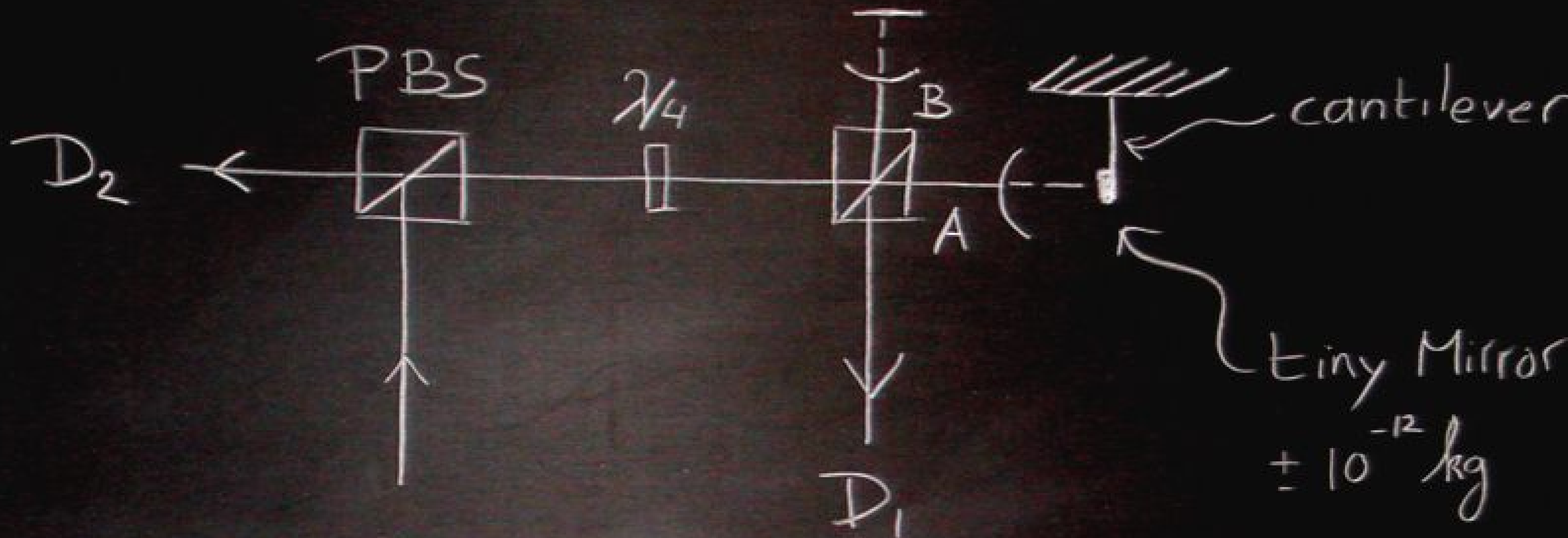
Law, PRA, **49**, 433 (1993)
Bose et al. PRA **59**, 3204 (1999)
Marshall et al. PRL **91**, 130401 (2003)



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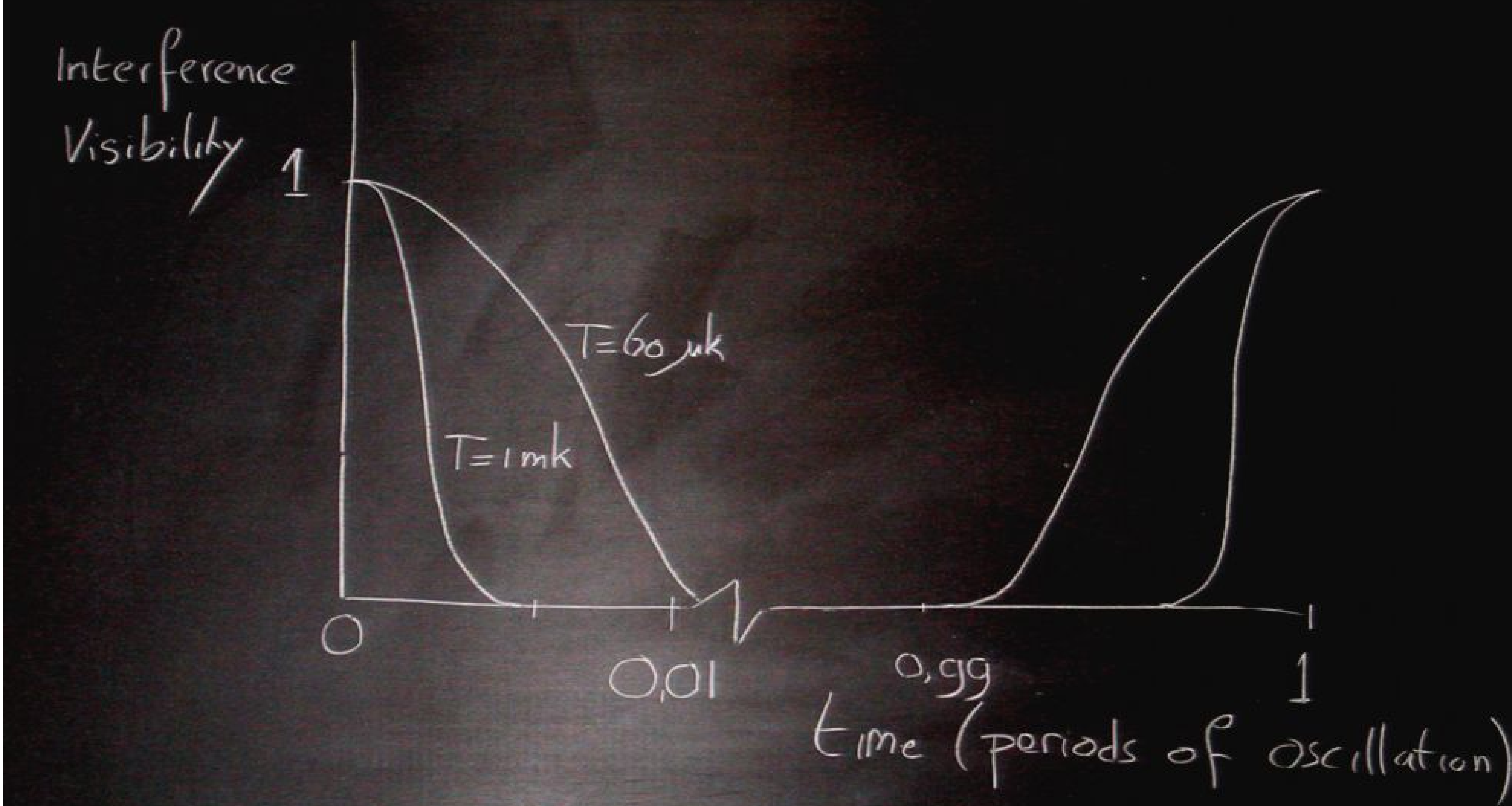


T is effective temperature of the fundamental resonance of cantilever

Experimental Requirements

1) momentum kick imparted by photon has to be larger than the initial quantum uncertainty of the mirror's momentum

$$\frac{2\hbar N^3 L}{\pi c M \lambda^2} \gtrsim 1$$

Optimum 700 nm   10x10x10 μm SiO_2/Ta_2O_5 mirror

$N \sim 10^5 - 10^6$

$L \sim 1-5\,cm$

$\omega_m \sim 2\,kHz \longrightarrow \Delta x_{mirror} = 10^{-13}\,m$

Experimental Requirements

2) environmental decoherence time ~ 1 period

$$\gamma_D = \gamma_m k_B T M (\Delta x)^2 / \hbar^2 \qquad \text{(Zurek et al)}$$

↑ damping rate cantilever

$$\rightarrow Q = \omega_m / \gamma_m \gtrsim 10^5 \qquad \text{(@3mK Rugar et al)}$$

Q=150.000 leads to required T<8mK for bulk material

Experimental Requirements

3) Stability of order $\lambda / 20N \sim 10^{-14}$ m

on timescale of experiment.

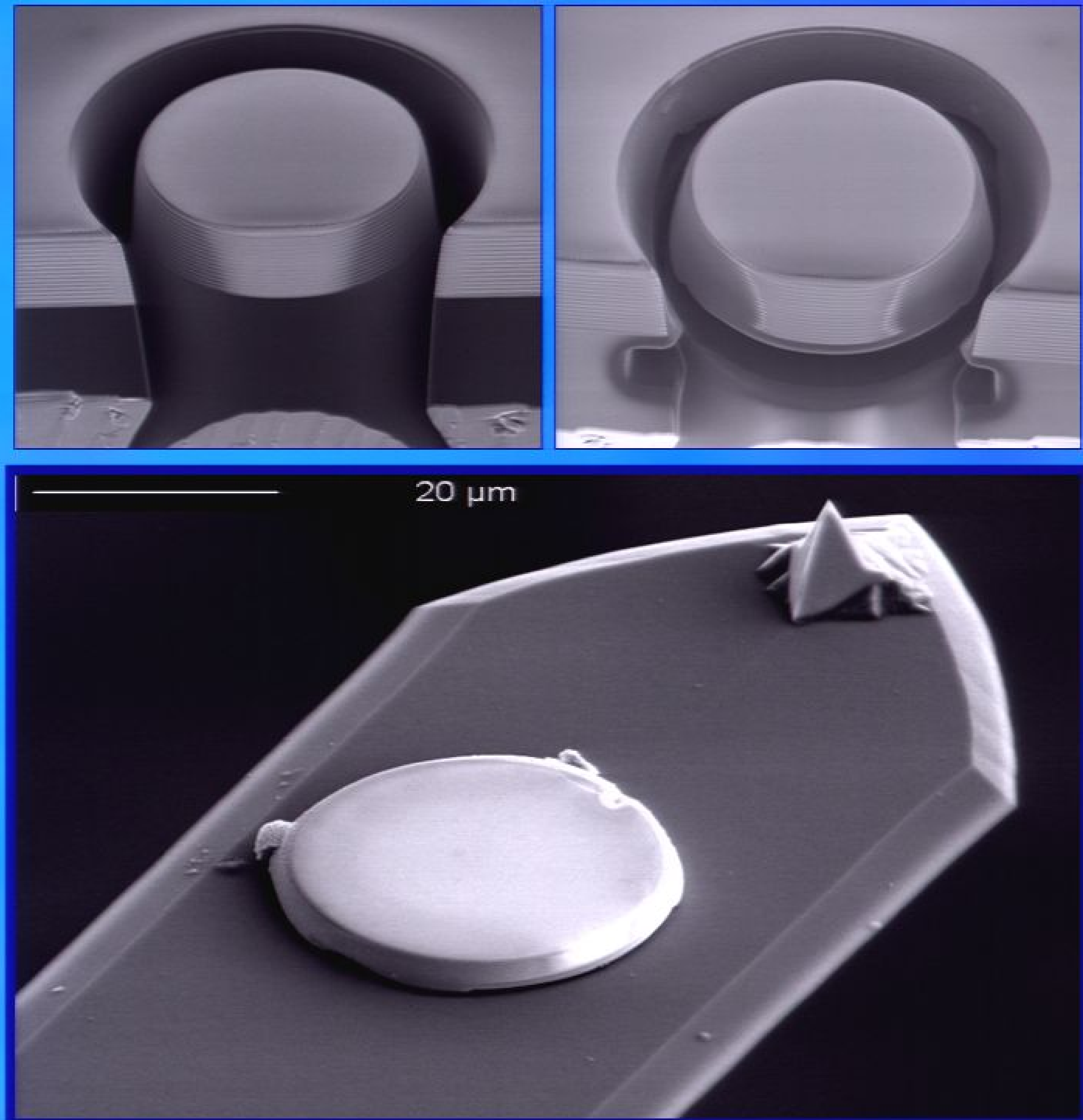
(STM 10^{-13} m/min

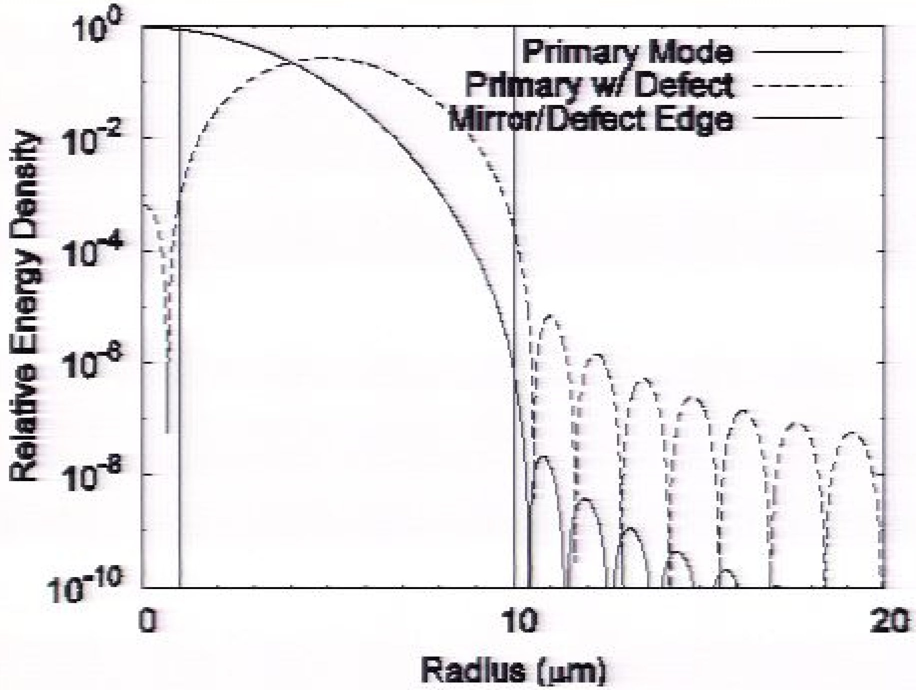
Gravitational wave detection 10^{-19} m/ms)

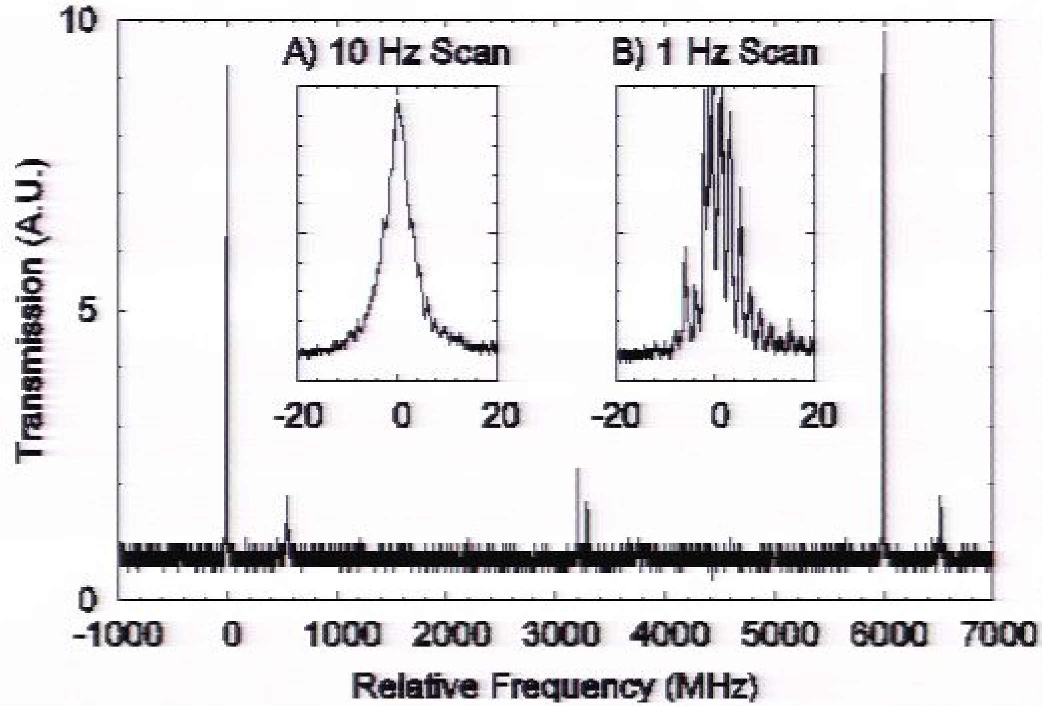
Great help Switchable mirrors

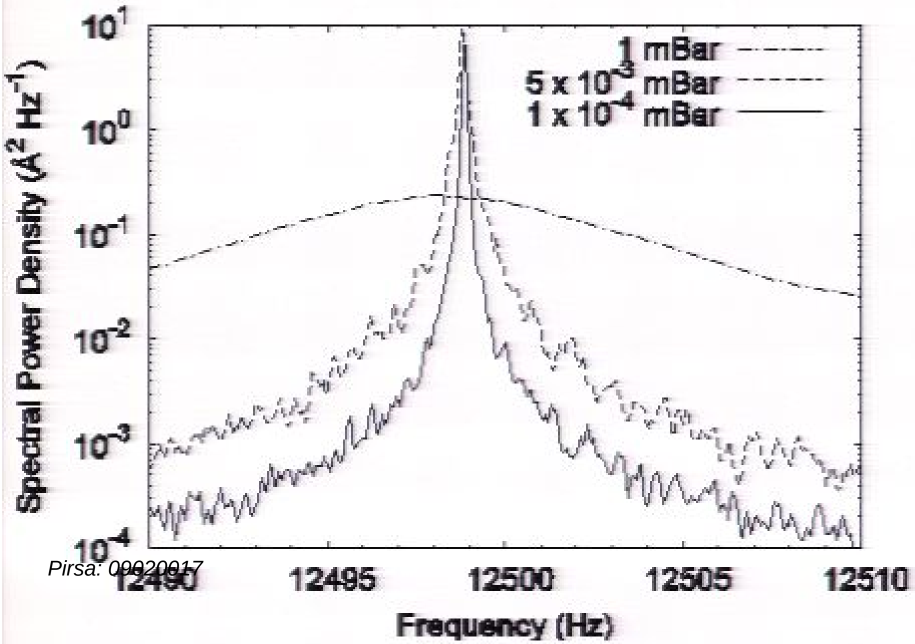
4) UUHV background density $\sim 100 \frac{\text{particles}}{\text{cm}^3}$


UCSB


20 µm



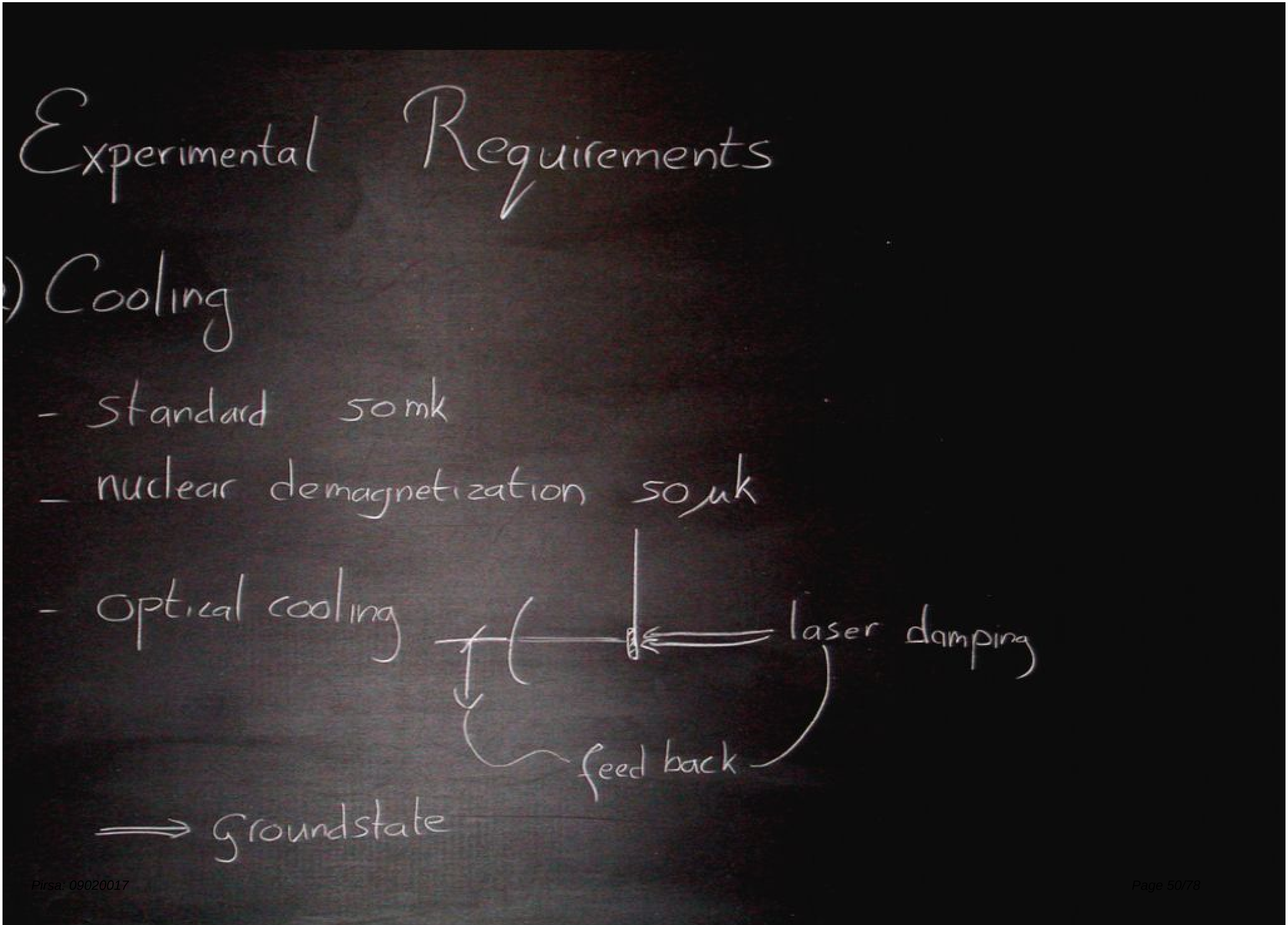




Optical Q=2100

Mechanical Q=137.000

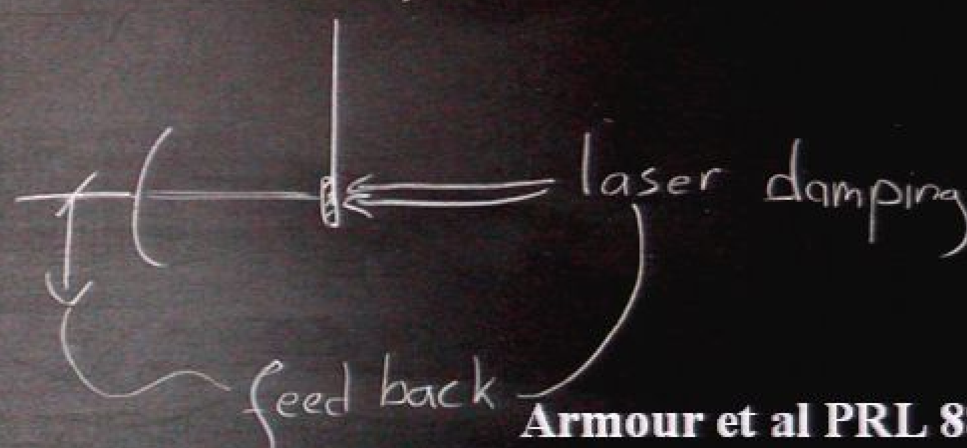
PRL **96**, 173901 (2006)


Experimental Requirements
) Cooling
- standard 50 mk
- nuclear demagnetization 50 μk
- optical cooling
laser damping
feed back
⟹ Groundstate

Experimental Requirements

2) Cooling

- standard 50 mk
- nuclear demagnetization 50 μk
- optical cooling



⟹ Groundstate

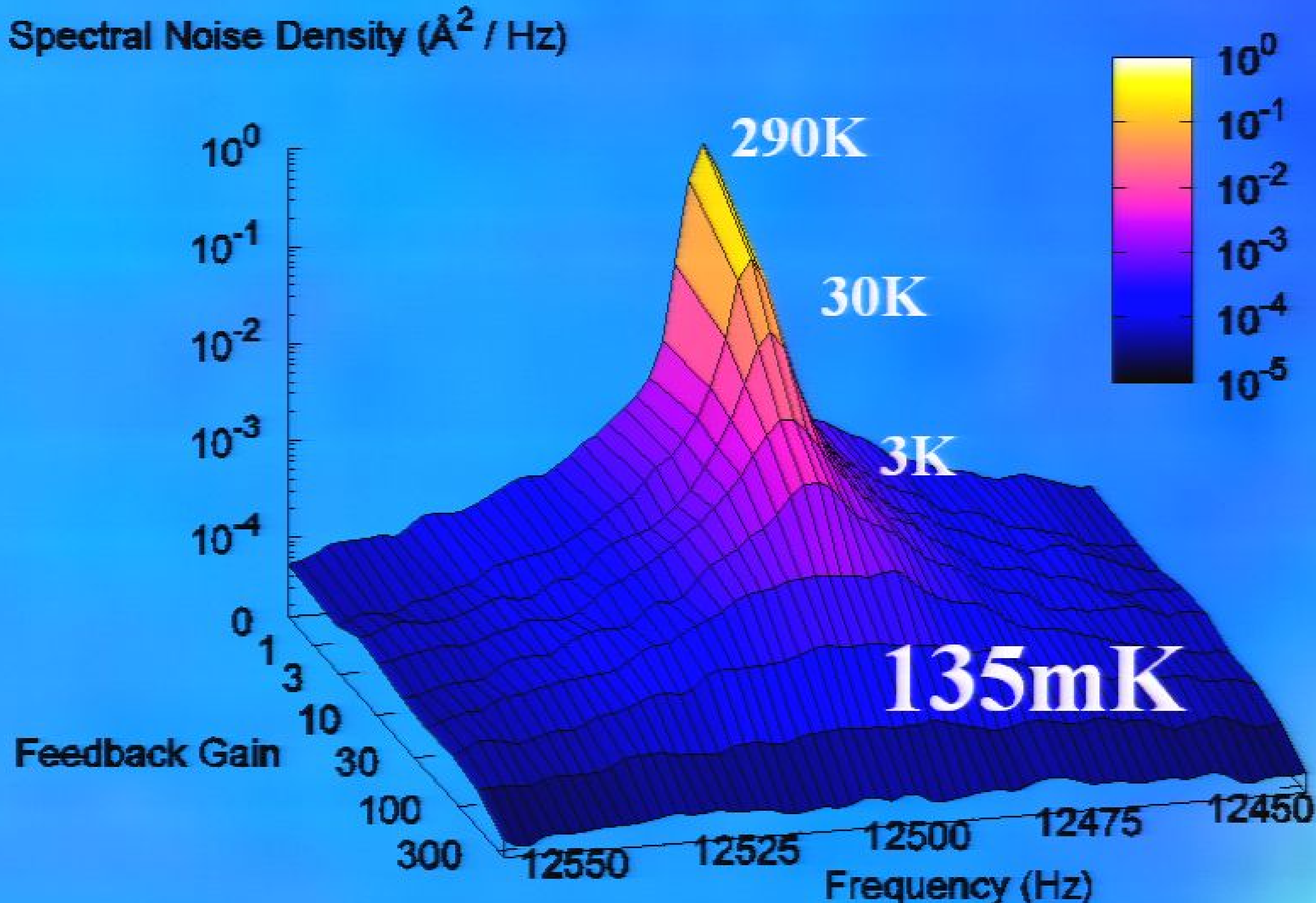
Armour et al PRL 88, 1483010 (02)

Mancini et al PRL 80, 688 (98)

Cohadon et al PRL 83, 3174 (99)

Optical Cooling

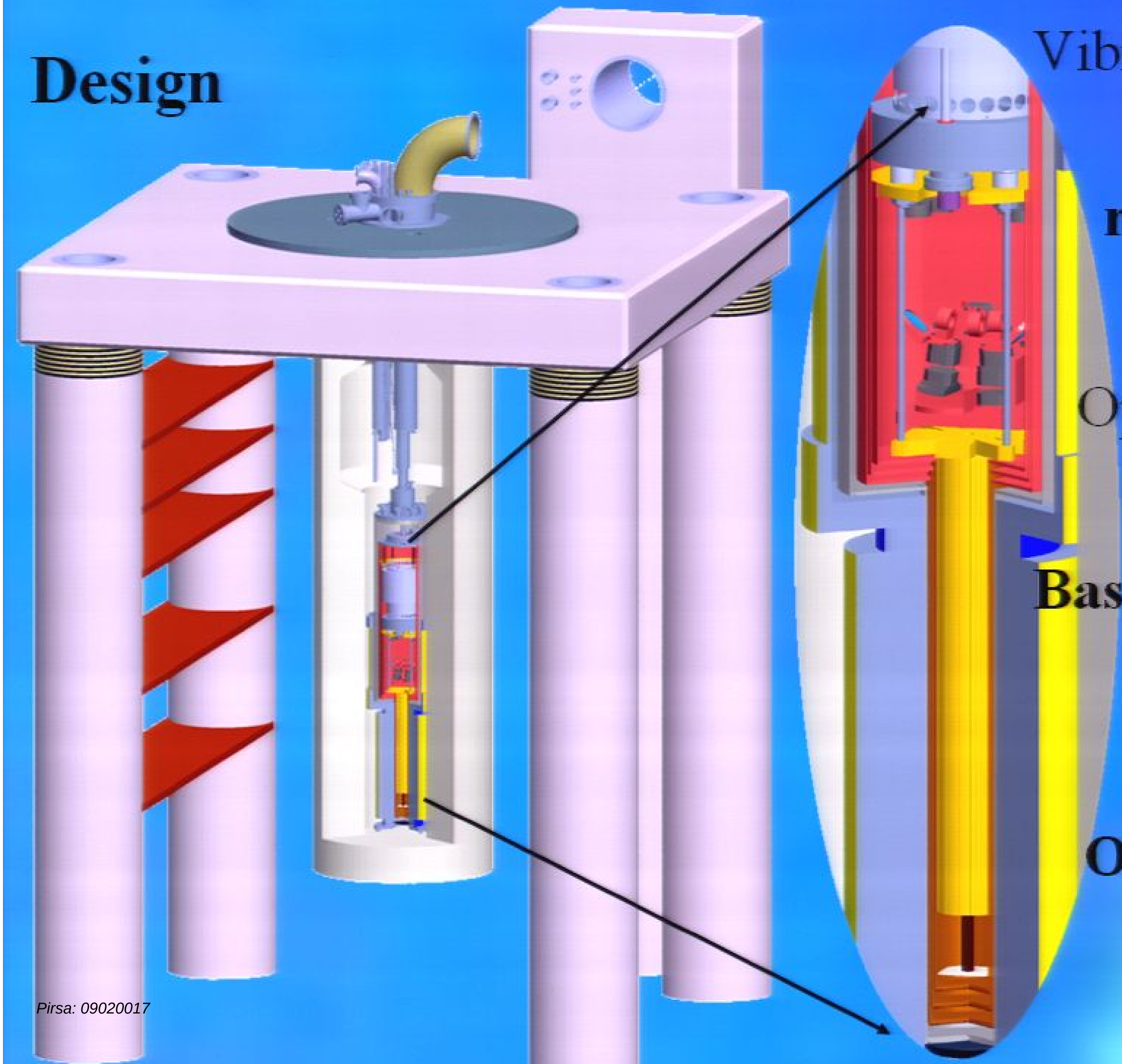
Gain factor 2500

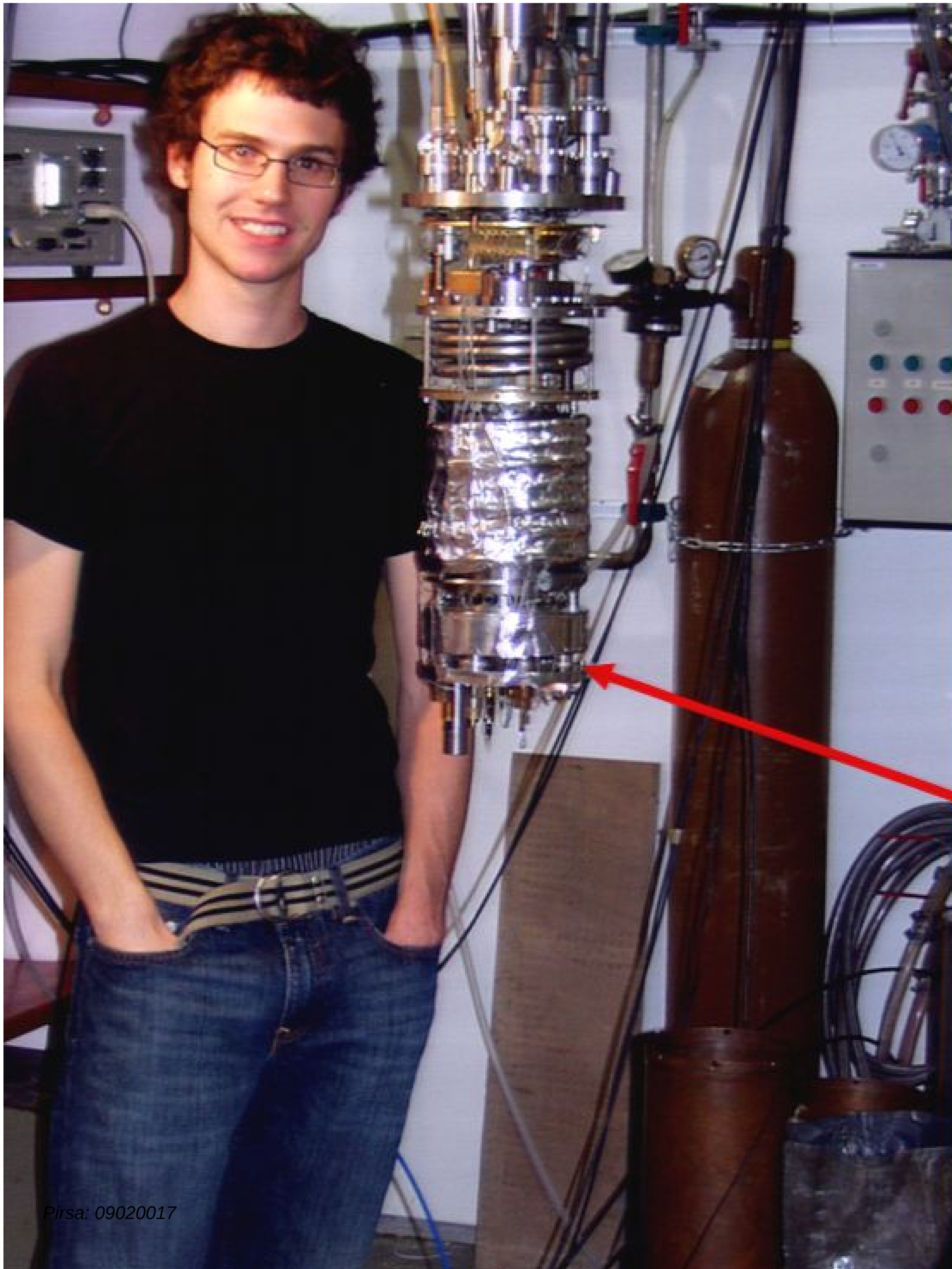


D. Kleckner and D.B. Nature **444**, 75 (2006).

Leiden, the Netherlands

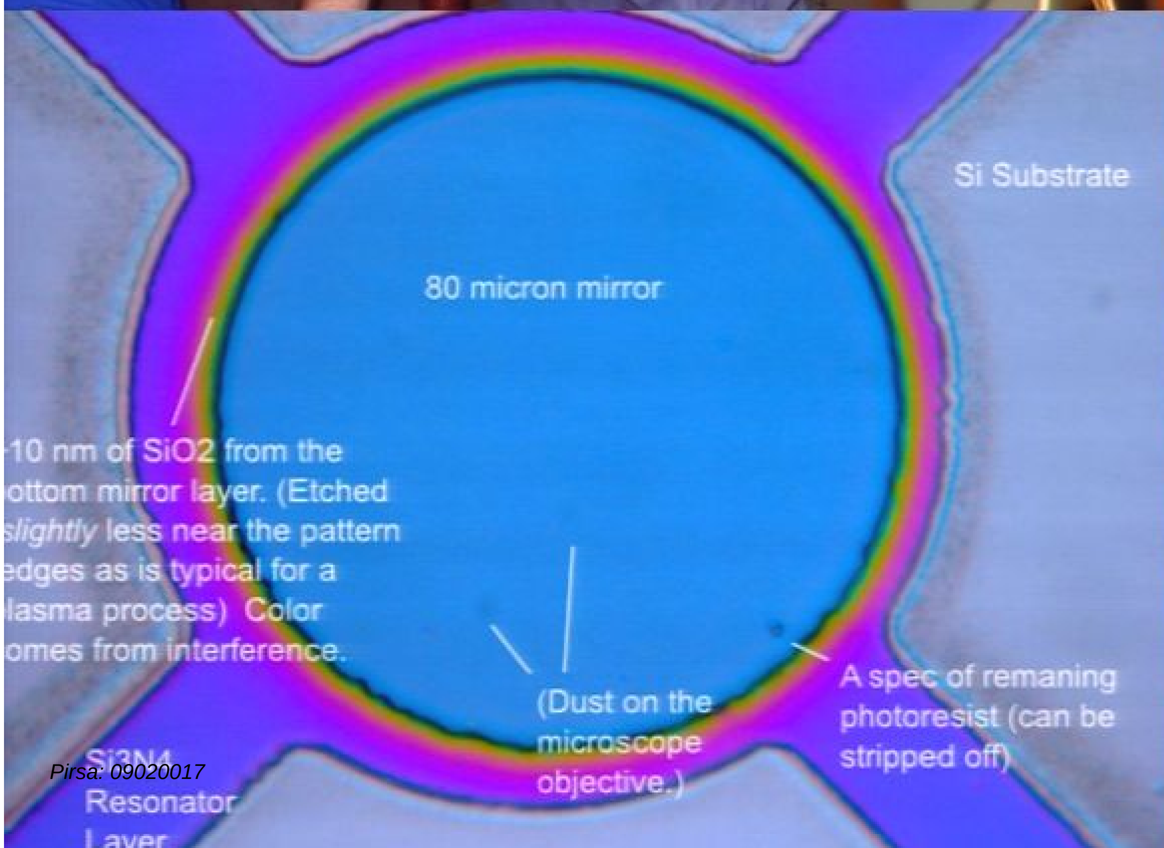
Design

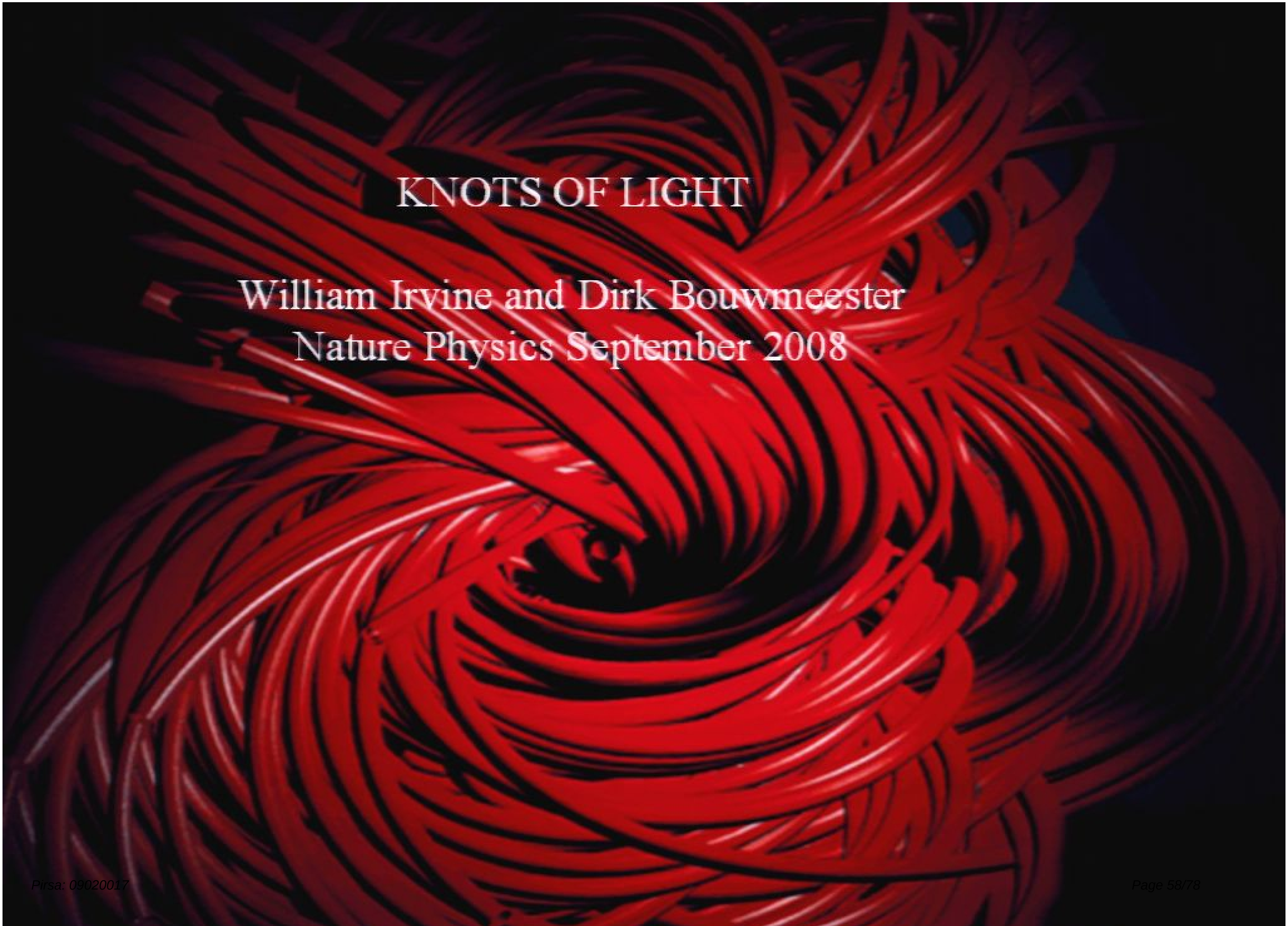

Vibration damping
Dilution refrigeration 10mK
Optics/cantilever
Base temperature 100μK
Optical cooling 100nK



Evan Jeffrey
Petro Sonin
Eric Eliel
Dustin Kleckner
Brian Pepper
Georgio Frossati
Arlette de Waard
Harmen v/d Meer

3.3mK


Si Substrate
80 micron mirror
-10 nm of SiO2 from the
ottom mirror layer. (Etched
slightly less near the pattern
edges as is typical for a
lasma process) Color
omes from interference.
(Dust on the
microscope
objective.)
A spec of remaning
photoresist (can be
stripped off)
Si3N4
Resonator


KNOTS OF LIGHT
William Irvine and Dirk Bouwmeester
Nature Physics September 2008

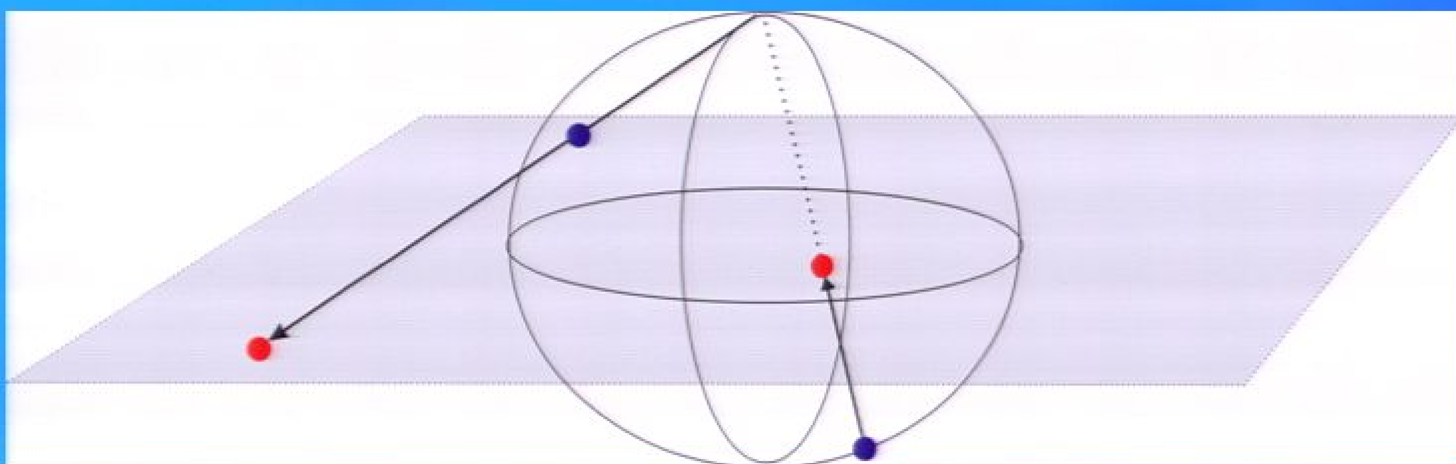
Sterographic projection

$ST^2 : S^2 \rightarrow R^2 \cup \infty$

or

$S^2 \rightarrow C \cup \infty$

Conformal transformation:
circles remain circles, and
local angles remain unchanged



similarly

$ST^3 : S^3 \rightarrow R^3 \cup \infty$

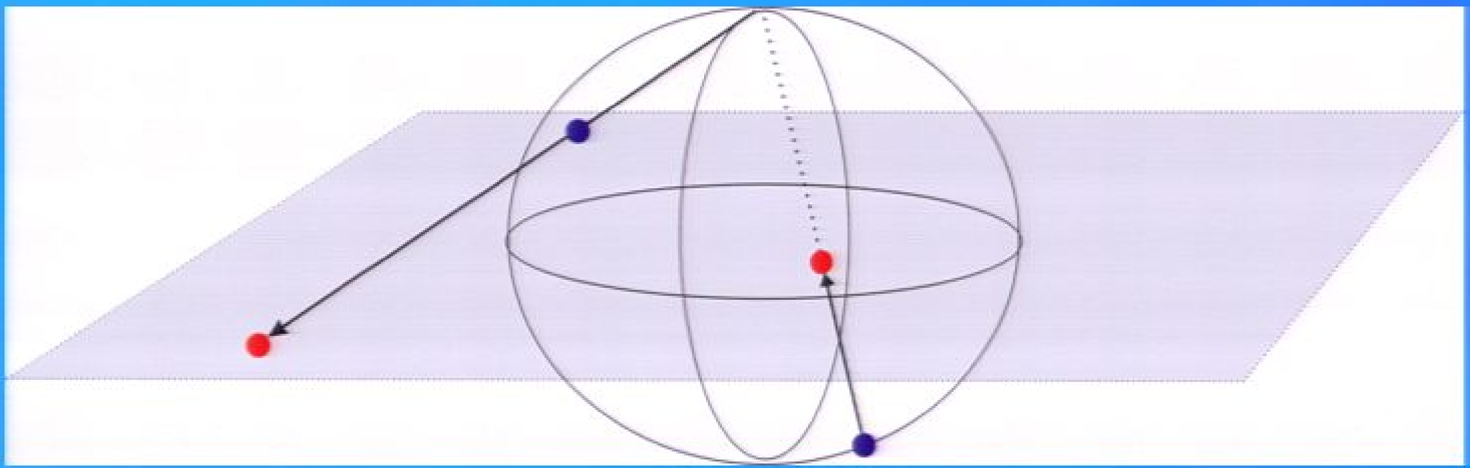
Sterographic projection

$ST^2 : S^2 \to R^2 \cup \infty$

or

$S^2 \to C \cup \infty$

Conformal transformation:
circles remain circles, and
local angles remain unchanged



similarly

$ST^3 : S^3 \to R^3 \cup \infty$

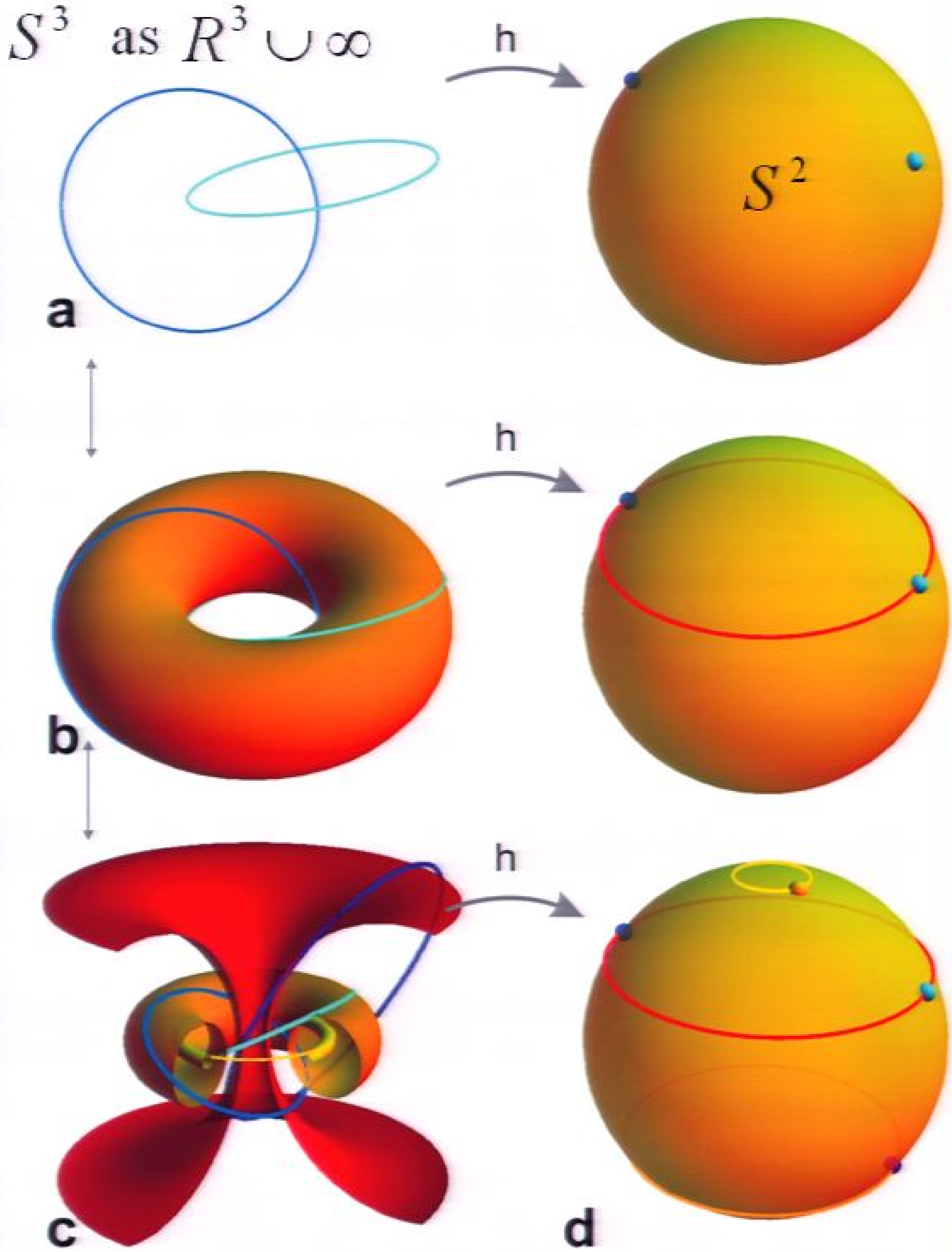
$a^2 + b^2 + c^2 + d^2 = 1,$ three − sphere in 4 − dim

$$x = \frac{a}{1-d}, \quad y = \frac{b}{1-d}, \quad z = \frac{c}{1-d}, \quad \text{and inverse}$$

$$a = \frac{2x}{r^2+1}, \quad b = \frac{2y}{r^2+1}, \quad c = \frac{2z}{r^2+1}, \quad d = \frac{r^2-1}{r^2+1}$$

Hopf map

$S^3 \to S^2$



Hopf map

$S^3 \rightarrow S^2$

S^3 can be identified with $SU(2)$, the 2 - dim unitary matrices with det $=1$

basis : $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad i\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

g element of $SU(2)$: $\quad g = \begin{pmatrix} a+id & ib-c \\ ib+c & a-id \end{pmatrix}$, with $a^2+b^2+c^2+d^2=1$

Hopf Map $\mathrm{h}_{\hat{\vec{k}}} : g \in SU(2) \cong S^3 \rightarrow \hat{\vec{n}} \in S^2$:

$$\hat{\vec{n}}^i \sigma_i = g\left(\hat{\vec{k}}^j \sigma_j\right) g^{-1}$$

note : set $g_0 e^{i\tau \hat{\vec{k}}^j \sigma_j}$, with $\tau \in [0, 2\pi)$ (a circle on S^3) map to same $\hat{\vec{n}}_0$

Hopf map

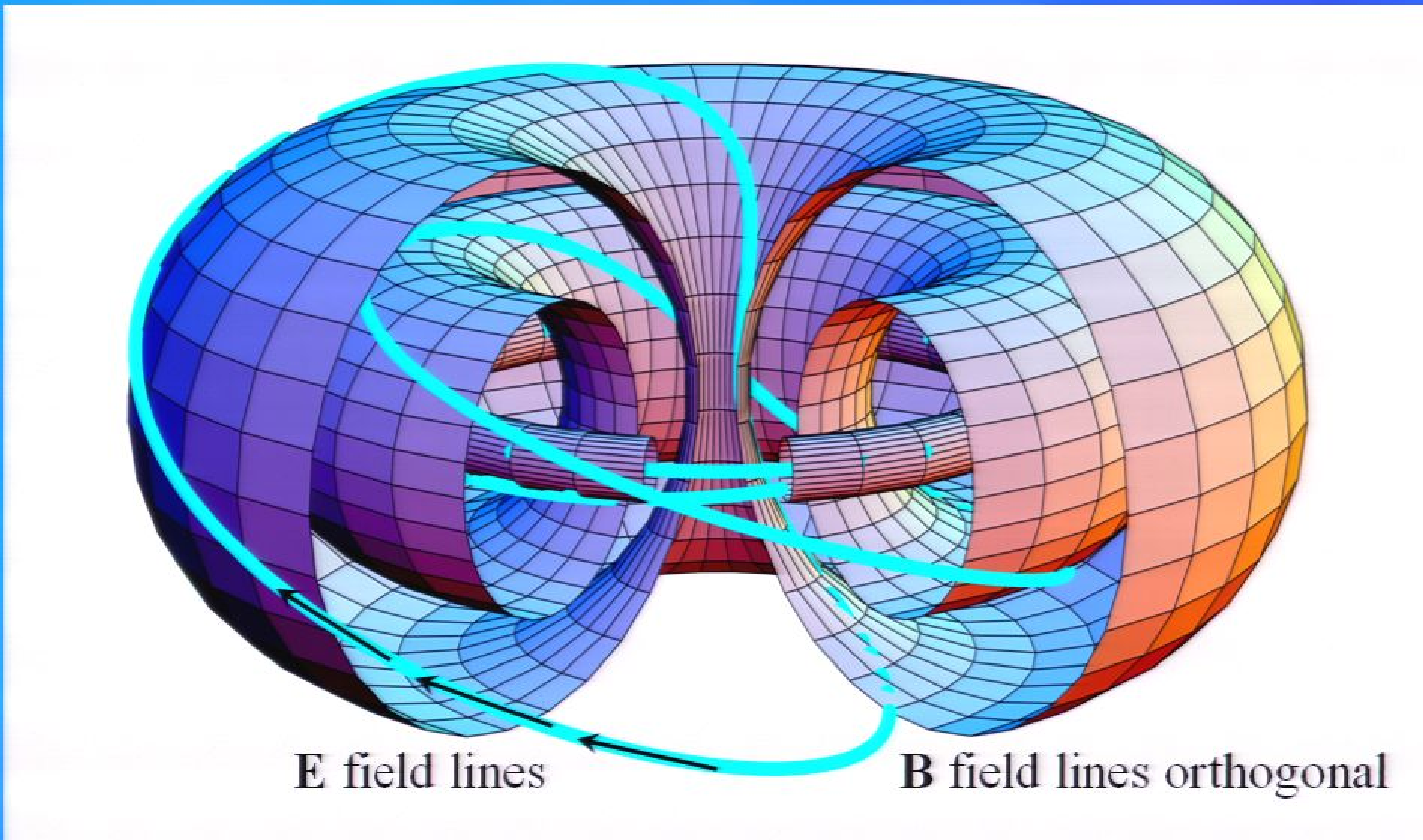
$$S^3 \to S^2$$

for specific choice of $\hat{\vec{k}}$: $ST^2 \circ h_{\hat{\vec{k}}}: \quad S^3 \to C \cup \infty$

$$\xi_{\hat{\vec{k}}=(0,0,1)} = -\frac{a+ib}{c+id},$$

and $ST^2 \circ h_{\hat{\vec{k}}} \circ ST^3 : R^3 \cup \infty \to C \cup \infty$

$$\xi_{\hat{\vec{k}}=(0,0,1)} = -\frac{x+iy}{z+i\frac{\left(r^2-1\right)}{2}}$$



A.F. Ranada and J.L. Trueba, Phys. Lett. 232 A, 25 (1997).

Source-free Maxwell's equations

$\nabla \cdot B = 0$ $\nabla \times B - \dot{E} = 0$	$\nabla \cdot E = 0$ $\nabla \times E + \dot{B} = 0$	Vector nature (Cartesian vectors)
$\partial^\mu F_{\mu\nu} = 0$	$\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$	4-Vector nature (Lorentz vectors)
$\mathrm{d}F = 0$	$\mathrm{d} \star F = 0$	Topological/geometrical features (Differential forms)

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & B^z & -B^y \\ -E^y & -B^z & 0 & B^x \\ -E^z & B^y & -B^x & 0 \end{pmatrix},$$

or $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$F = F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

F is a differential 2-form

***

Differential Forms

Needed to represent calculations, in particular integration, on manifolds (e.g. R^n, S^n) in a coordinate independent way. You can use a specific local coordinate system to do an explicit calculation.

ω denotes a 1-form. In a local coordinate system:

$$\omega = a_i \mathrm{d}x^i, \quad i = 1,\ldots,n, \text{ with } n \text{ dim of manifold.}$$

A 2-form T can be obtained from a 1-form by taking the exterior derivative d of a 1 form:

$$\mathrm{T} = \mathrm{d}\omega \equiv \frac{\partial a_i}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^i \equiv \mathrm{T}_{ij} \mathrm{d}x^j \wedge \mathrm{d}x^i \equiv \mathrm{T}_{ij} \mathrm{d}x^j \mathrm{d}x^i - \mathrm{T}_{ji} \mathrm{d}x^i \mathrm{d}x^j$$

note T is antisymmetric under exchange of i and j.

***

Differential Forms

Example with 2-dim manifold M

$$\int_M T = \int_M d\omega = \int_M \frac{\partial a_i}{\partial x^j} dx^j \wedge dx^i = \int_M \frac{\partial a_1}{\partial x^2} dx^2 dx^1 - \frac{\partial a_2}{\partial x^1} dx^1 dx^2$$

$$= \int_M \left(\frac{\partial a_1}{\partial x^2} - \frac{\partial a_2}{\partial x^1} \right) dx^1 dx^2$$

with $A \equiv \left(a_1 \hat{\vec{i}} + a_2 \hat{\vec{j}} \right)$

$$= \int_M \operatorname{curl} A \cdot dS \overset{\text{Stokes' theorem}}{=} \int_{\partial M} A \cdot dl$$

$$= \int_{\partial M} a_i dx^i = \int_{\partial M} \omega$$

conclusion for 2 - dim manifold M (possibly embedded in a higher dimensional manifold) : $\int_M d\omega = \int_{\partial M} \omega$

***

Differential Forms

Another example of use of diferential forms

Consider the specific n - form in a local basis :

$$\omega = 1 dx^1 \wedge dx^2 \wedge .... \wedge dx^n = dx^1 dx^2 .... dx^n$$

$$\int \omega = \int dx^1 dx^2 .... dx^n$$, is the volume integral in specific basis.

Consider basis transformation x_i to y_i

$$\omega = \left(\frac{\partial x^1}{\partial y^{i_1}} dy^{i_1} \right) \wedge \left(\frac{\partial x^2}{\partial y^{i_2}} dy^{i_2} \right) \wedge .... \wedge \left(\frac{\partial x^n}{\partial y^{i_n}} dy^{i_n} \right)$$

$$= \mathrm{Det}\left(\frac{\partial x^i}{\partial y^j} \right) dy^1 \wedge dy^2 \wedge .... \wedge dy^n = \mathrm{Det}\left(\frac{\partial x^i}{\partial y^j} \right) dy^1 dy^2 .... dy^n$$

By virtue of the $\wedge$ (hodge) product the specific form ω automatically transforms like a n - dimensional volume element under a local coordinate transformation!

Property 1: dF = 0 for n-form F in n-dim. manifold

example: $F = F_{ij} dx^i \wedge dx^j$, with $i, j = 1,2$

$$dF = \frac{\partial F_{ij}}{\partial x^k} dx^k \wedge dx^i \wedge dx^j, \; i, j, k = 1,2$$

dF = 0 trivially because of repeated indices.

Property 2: $d^2\omega = 0$ for any differenti al form.

Therefore if for a n-form F dF = 0, F = dA with A being a (n-1)-form

example: claim Maxwell's equations: dF = 0, with F a 2-form in 4-dim

Minkowski space F = dA, with A a 1-form in Minkowski space:

$$F = dA = \frac{\partial A_\nu}{\partial x^\mu} dx^\mu \wedge dx^\nu \rightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{with } \partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

Maxwell's equtions: dF = 0 (and d * F = 0), $dF = \frac{\partial F_{\mu\nu}}{\partial x^\eta} dx^\eta \wedge dx^\mu \wedge dx^\nu = 0$

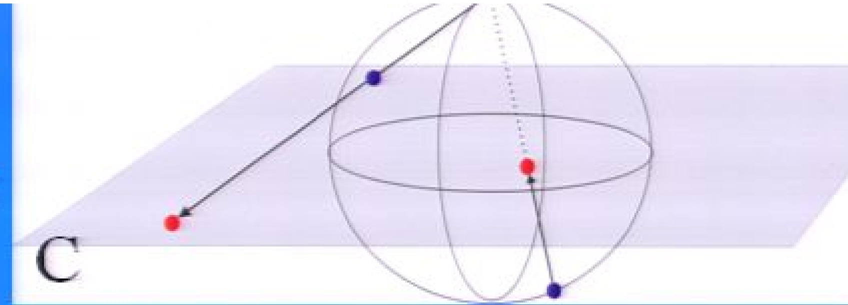
$$\Rightarrow \varepsilon^{\rho\eta\mu\nu} \partial_\eta F_{\mu\nu} = 0$$

Property 3 :

For a given map Φ between two manifold (for example the Hopf map between S^3 and S^2) properties of n - forms are preserved under pull - back. For example if the exterior drivative of a 2 - form F, dF, is zero in one of the manifolds (which is automatically the case on S^2), $dF = 0$ also on S^3

We can now construct solutions of Maxwell's equation by simply taking a 2 - form on a 2 - dimensional manifold, for example the standard local volume (area) form $dx_1 dx_2$ on S^2, and pull it back under a given Map to R^3 or Minkowski space. If the Map has non - trivial topological properties (like the Hopf map), non - trivial solutions of Maxwell's equations are obtained.

$$2\text{-form on } C: d\Theta \wedge d\overline{\Theta} = d(x-iy)\wedge d(x+iy)$$
$$= i\,dx\,dy - i\,dy\,dx + i\,dx\,dy - i\,dy\,dx$$
$$= 2i\,dx \wedge dy$$



$$2\text{-form on } S^2 \text{ projected onto } C: \frac{2\,dx\wedge dy}{\left(1+r^2\right)^2} = \frac{d\Theta \wedge d\overline{\Theta}}{i\left(1+\Theta\overline{\Theta}\right)^2}$$

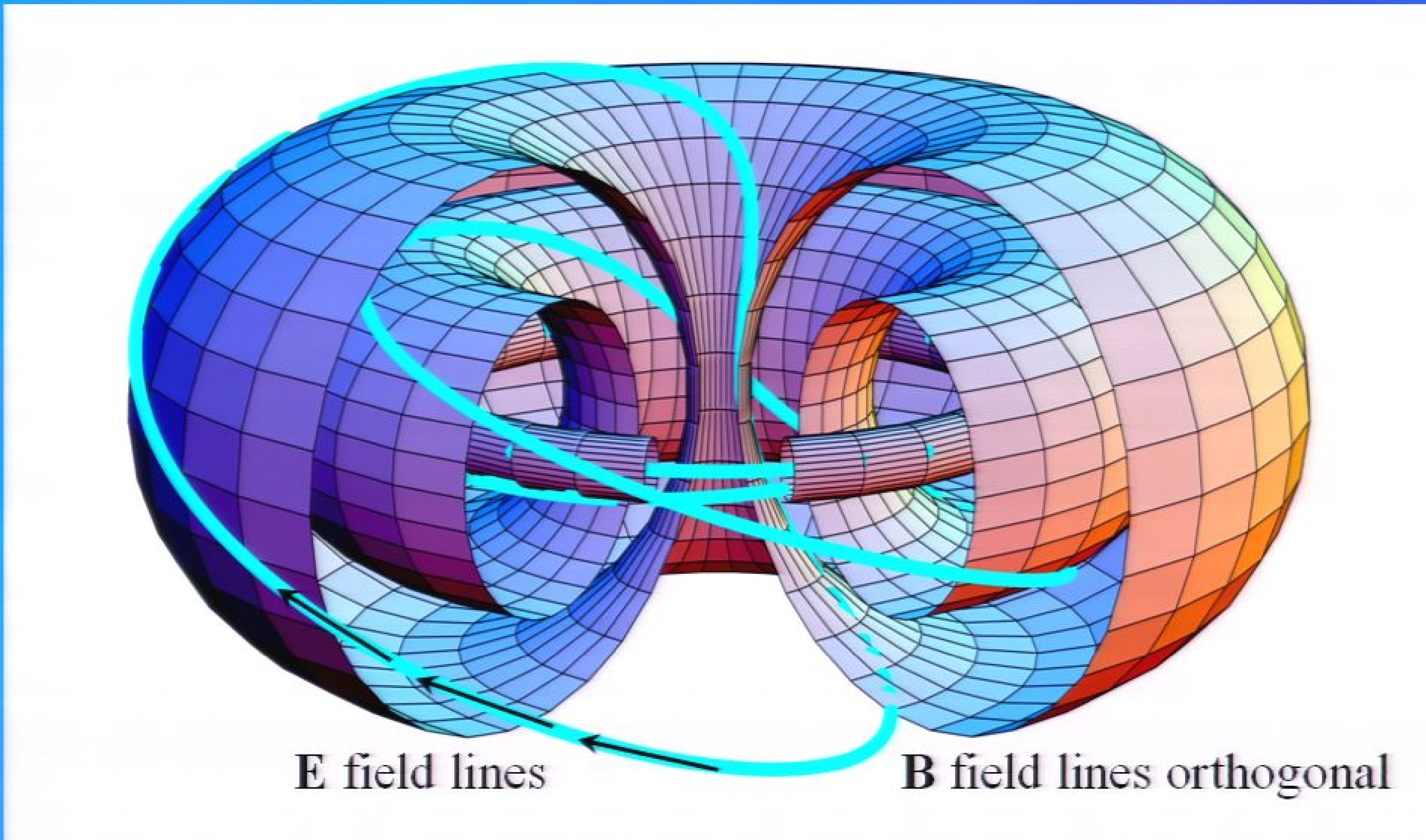
transfer 2 – form from one C to Minkowski space

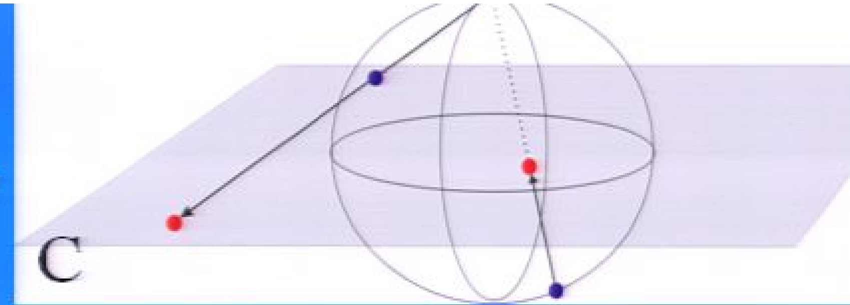
$$d\Theta \wedge d\overline{\Theta} = \frac{\partial \Theta}{\partial x^\mu} dx^\mu \wedge \frac{\partial \overline{\Theta}}{\partial x^\nu} dx^\nu = \ldots = \left(\partial_\mu \Theta \partial_\nu \overline{\Theta} - \partial_\nu \Theta \partial_\mu \overline{\Theta}\right) dx^\mu \wedge dx^\nu$$

Special solution to Mawell's equation :

$$F_{\mu\nu} = \frac{\left(\partial_\mu \Theta \partial_\nu \overline{\Theta} - \partial_\nu \Theta \partial_\mu \overline{\Theta}\right)}{i\left(1+\Theta\overline{\Theta}\right)^2}, \text{ with } \Theta(x,y,z,t=0) \overset{\text{Hopf}}{=} \frac{x+iy}{z+i\frac{\left(r^2-1\right)}{2}}$$

dF = 0 per construction (F is afteral just the area two - form of a two dimensional manifold hiding in a higher dimensional space!)


E field lines
B field lines orthogonal



$$2\text{-form on } \mathbb{C}: \mathrm{d}\Theta \wedge \mathrm{d}\overline{\Theta} = \mathrm{d}(x - iy) \wedge \mathrm{d}(x + iy)$$
$$= i\mathrm{d}x\mathrm{d}y - i\mathrm{d}y\mathrm{d}x + i\mathrm{d}x\mathrm{d}y - i\mathrm{d}y\mathrm{d}x$$
$$= 2i\mathrm{d}x \wedge \mathrm{d}y$$

$$2\text{-form on } S^2 \text{ projected onto } \mathbb{C}: \frac{2\,\mathrm{d}x \wedge \mathrm{d}y}{(1+r^2)^2} = \frac{\mathrm{d}\Theta \wedge \mathrm{d}\overline{\Theta}}{i(1+\Theta\overline{\Theta})^2}$$

transfer 2 − form from one C to Minkowski space

$$\mathrm{d}\Theta \wedge \mathrm{d}\overline{\Theta} = \frac{\partial\Theta}{\partial x^\mu}\mathrm{d}x^\mu \wedge \frac{\partial\overline{\Theta}}{\partial x^\nu}\mathrm{d}x^\nu = \ldots = \left(\partial_\mu\Theta\partial_\nu\overline{\Theta} - \partial_\nu\Theta\partial_\mu\overline{\Theta}\right)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

Special solution to Mawell's equation :

$$F_{\mu\nu} = \frac{\left(\partial_\mu\Theta\partial_\nu\overline{\Theta} - \partial_\nu\Theta\partial_\mu\overline{\Theta}\right)}{i(1+\Theta\overline{\Theta})^2}, \text{ with } \Theta(x,y,z,t=0) \overset{\text{Hopf}}{=} \frac{x+iy}{z+i\frac{(r^2-1)}{2}}$$

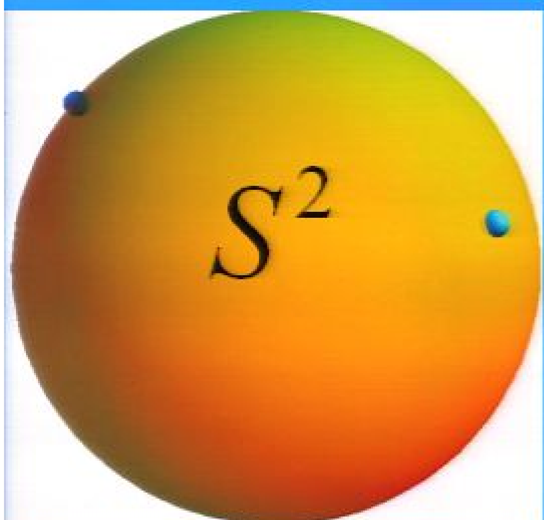
dF = 0 per construction (F is afteral just the area two - form of a two dimensional manifold hiding in a higher dimensional space!)

Source-free Maxwell's equations

dF=0, and $d * F = 0$ $\quad\quad$ $F = F_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu}$

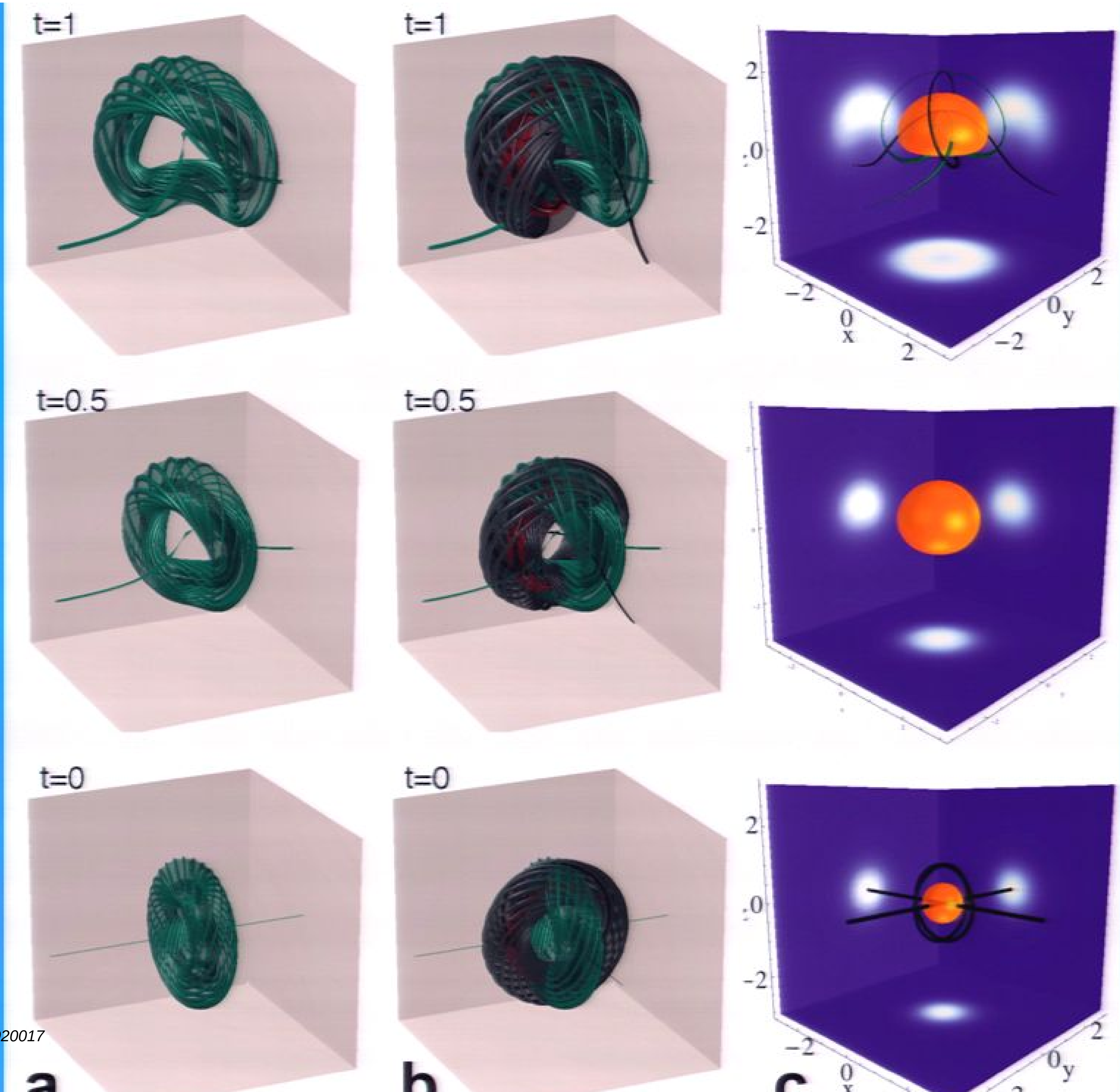
F is a differential 2-form

For differential 2-form on a 2-dim manifold such as S^2 : dF=0 always



$$S^3 \left(R^3 \cup \infty\right) \xrightarrow{\text{Hopf}} S^2 \quad (\text{with } dF = 0)$$

$$dF = 0 \quad \underset{\text{Pull back}}{\leftarrow} \quad \begin{pmatrix} \text{pull back commutes} \\ \text{with d operation} \end{pmatrix}$$


t=1
t=1
t=0.5
t=0.5
t=0
t=0
2
0
-2
-2
0
x
2
2
0
y
-2
a
b
c

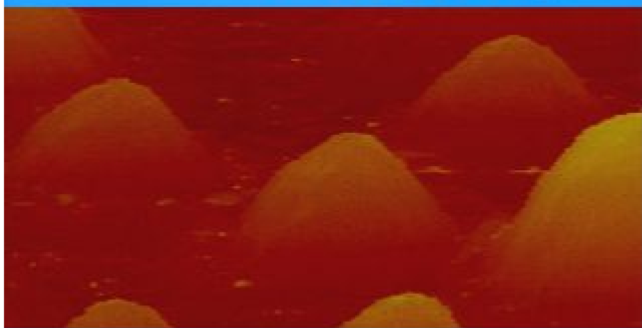
$$A_{\text{Hopf}}(\vec{r},t)=\sqrt{\frac{4}{3\pi}}\int \mathrm{d}k\,k^{3}e^{-k}\left[A_{1,1}^{\text{TE}}(k,\vec{r})-iA_{1,1}^{\text{TM}}(k,\vec{r})\right]e^{-i\omega t}+\text{c.c}$$

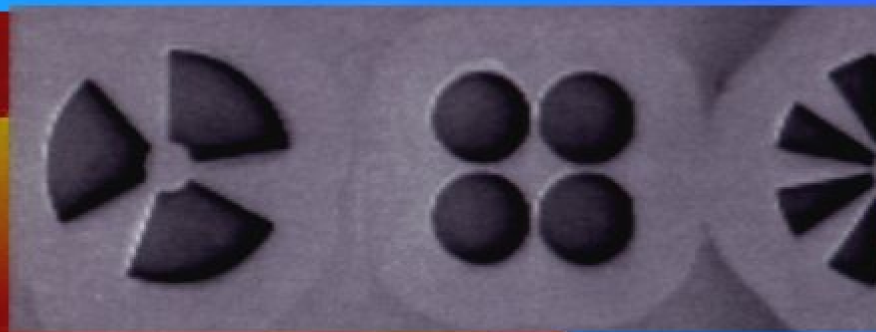
Solution is simple superposition of TE and TM vector spherical harmonics (l=m=1) and an energy spectrum

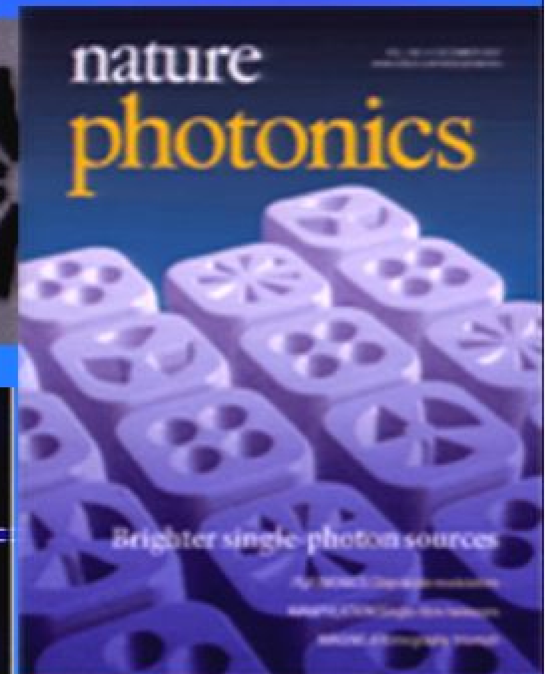
$$S(\omega)\sim\omega e^{-\omega}$$

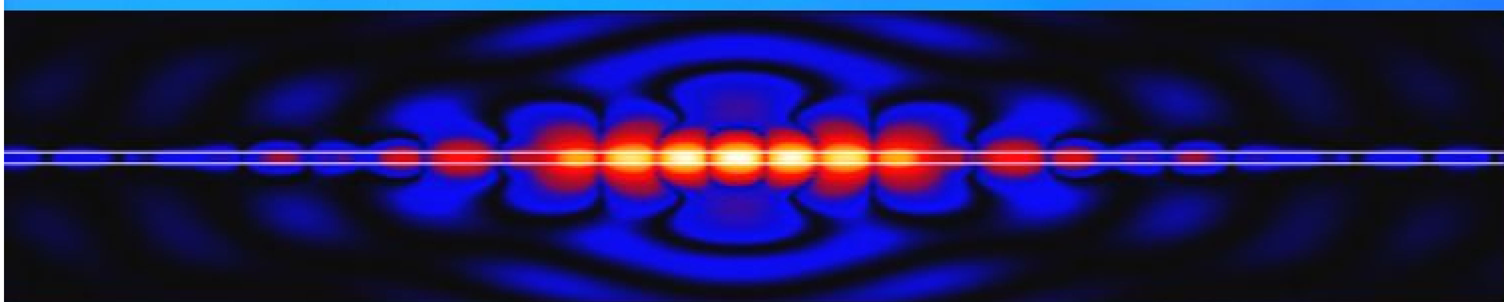
-force free fields (plasma physics, fluid dynamics)
-conserved quantities
-generalization l=m=2, 3…n
- tightly focused circular polarized Laguerre-Gaussian beams with proper pulse shape should produce such fields to good approximation
-solitons in nonlinear media
-same solutions for gravitational fields

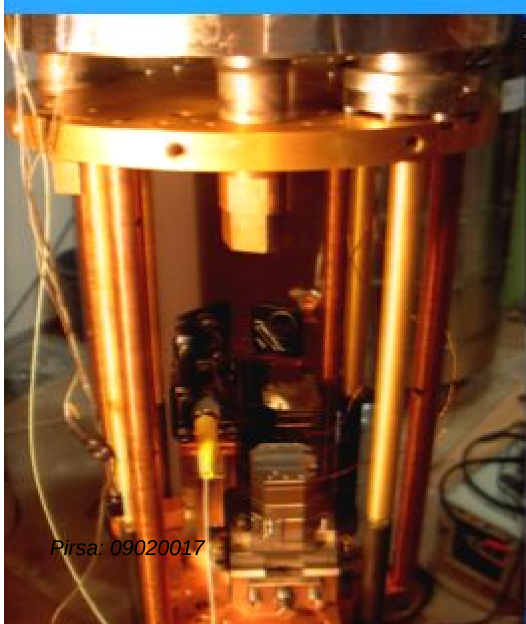
Summary

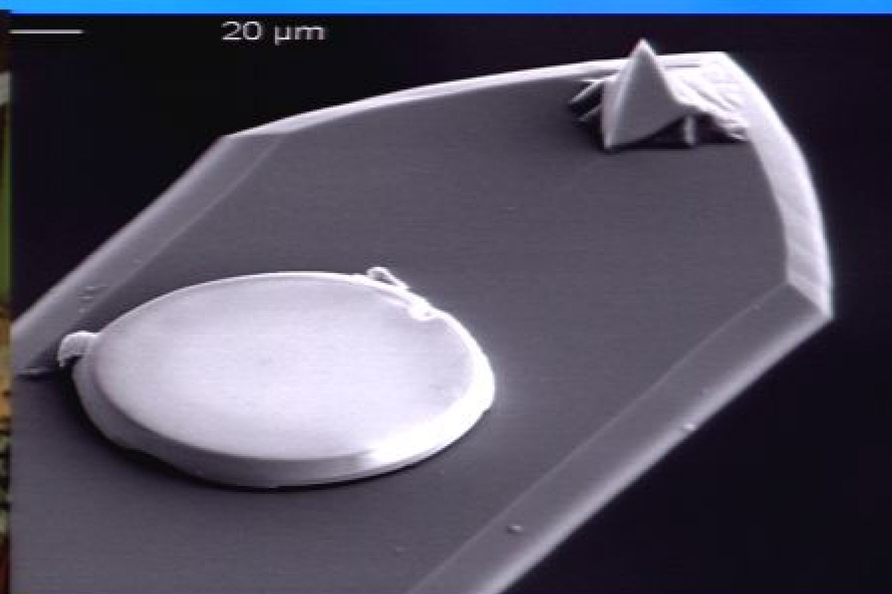





nature
photonics
Brighter single-photon sources

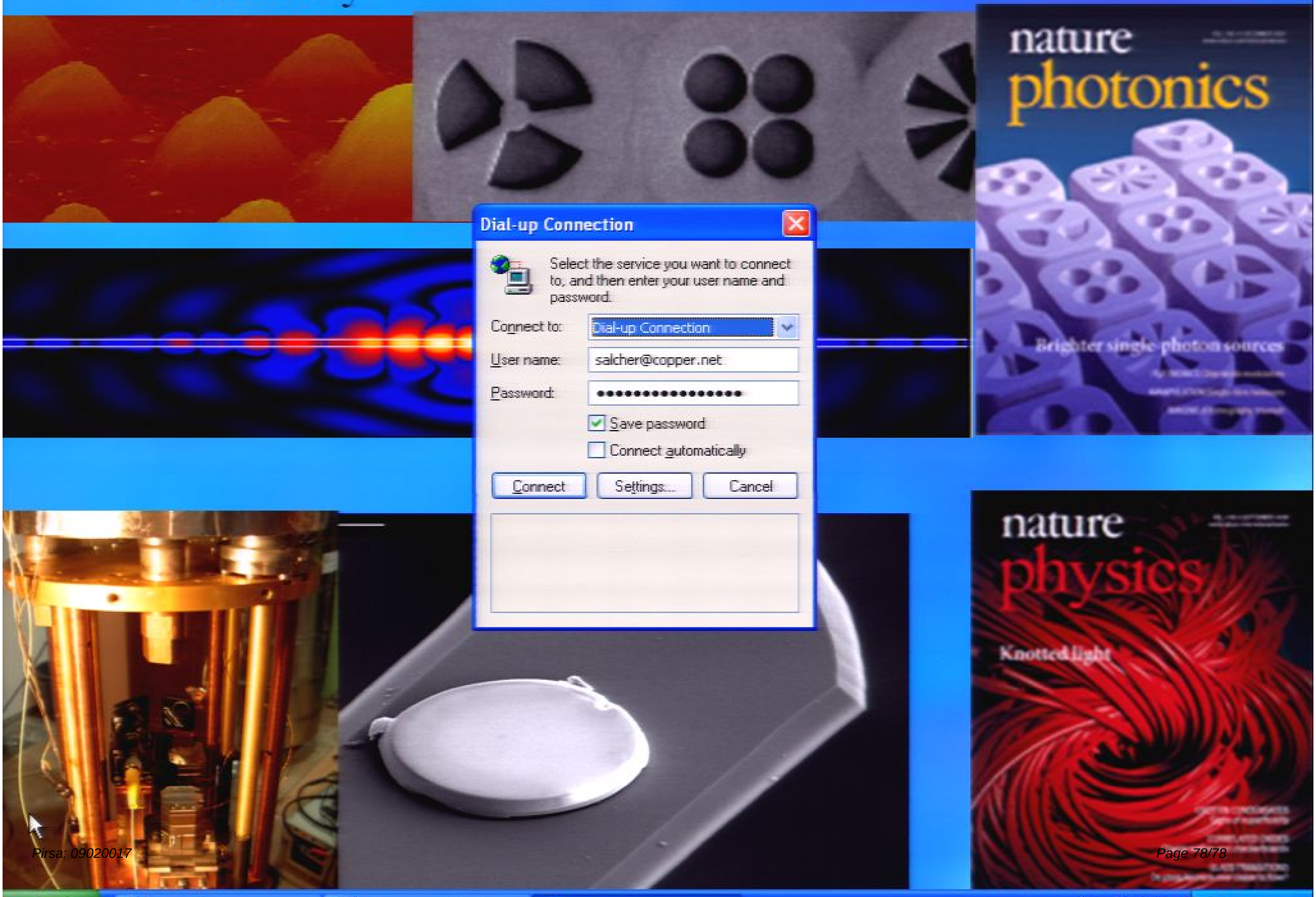





20 µm


nature
physics
Knotted light

Summary


nature
photonics
Brighter single-photon sources
nature
physics
Knotted light
Dial-up Connection
Select the service you want to connect to, and then enter your user name and password.
Connect to:
Dial-up Connection
User name:
salcher@copper.net
Password:
Save password
Connect automatically
Connect
Settings...
Cancel