

Title: Introduction to the Bosonic String

Date: Feb 13, 2009 10:00 AM

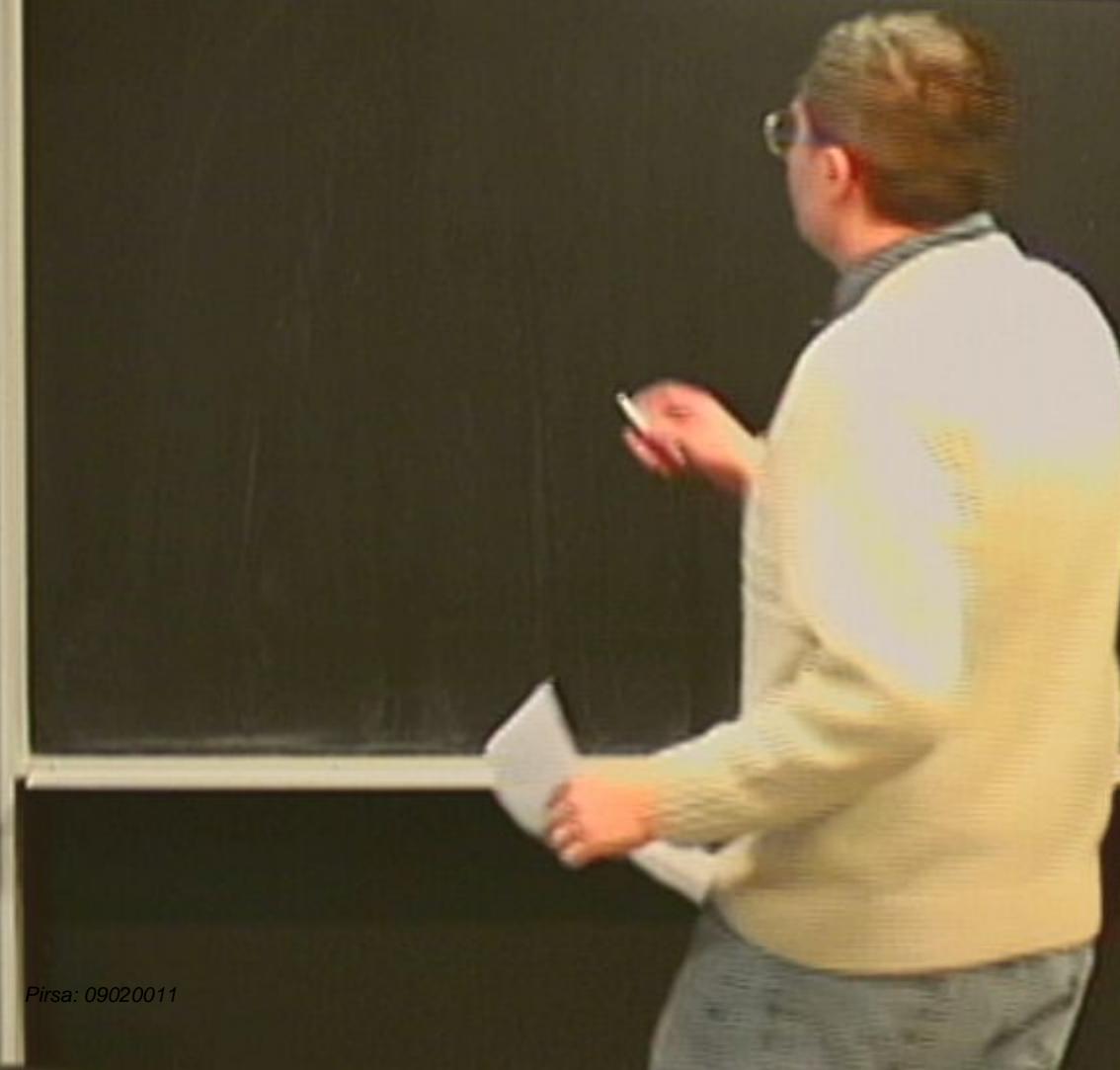
URL: <http://pirsa.org/09020011>

Abstract: This course provides a thorough introduction to the bosonic string based on the Polyakov path integral and conformal field theory. We introduce central ideas of string theory, the tools of conformal field theory, the Polyakov path integral, and the covariant quantization of the string. We discuss string interactions and cover the tree-level and one loop amplitudes. More advanced topics such as T-duality and D-branes will be taught as part of the course. The course is geared for M.Sc. and Ph.D. students enrolled in Collaborative Ph.D. Program in Theoretical Physics. Required previous course work: Quantum Field Theory (AM516 or equivalent). The course evaluation will be based on regular problem sets that will be handed in during the term. The primary text is the book: 'String theory. Vol. 1: An introduction to the bosonic string. J. Polchinski (Santa Barbara, KITP) . 1998. 402pp. Cambridge, UK: Univ. Pr. (1998) 402 p.' All interested students should contact Alex Buchel at abuchel@uwo.ca as soon as possible.

Conformal Field Theory (CFT)



Conformal Field Theory (CFT)



Conformal Field Theory (CFT)



Conformal Field Theory (CFT)

→ - string



Conformal Field Theory (CFT)

= string theory



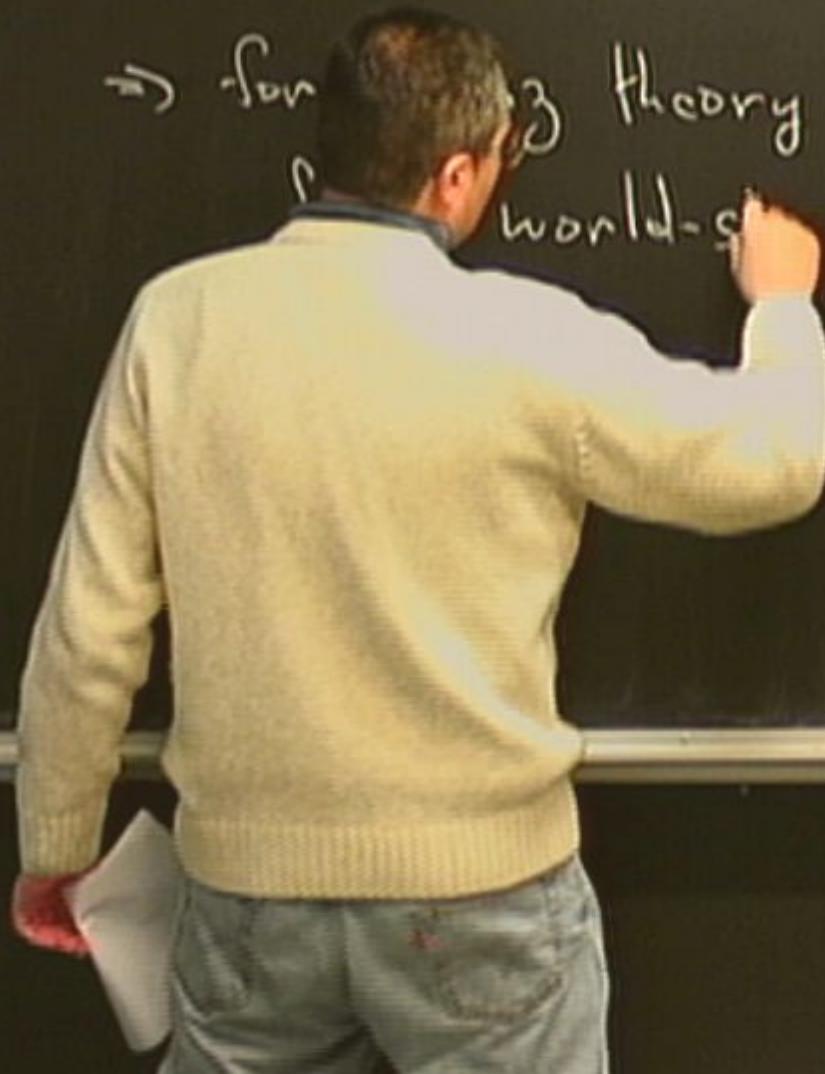
Conformal Field Theory (CFT)

- string theory



Conformal Field Theory (CFT)

→ for theory
world-s



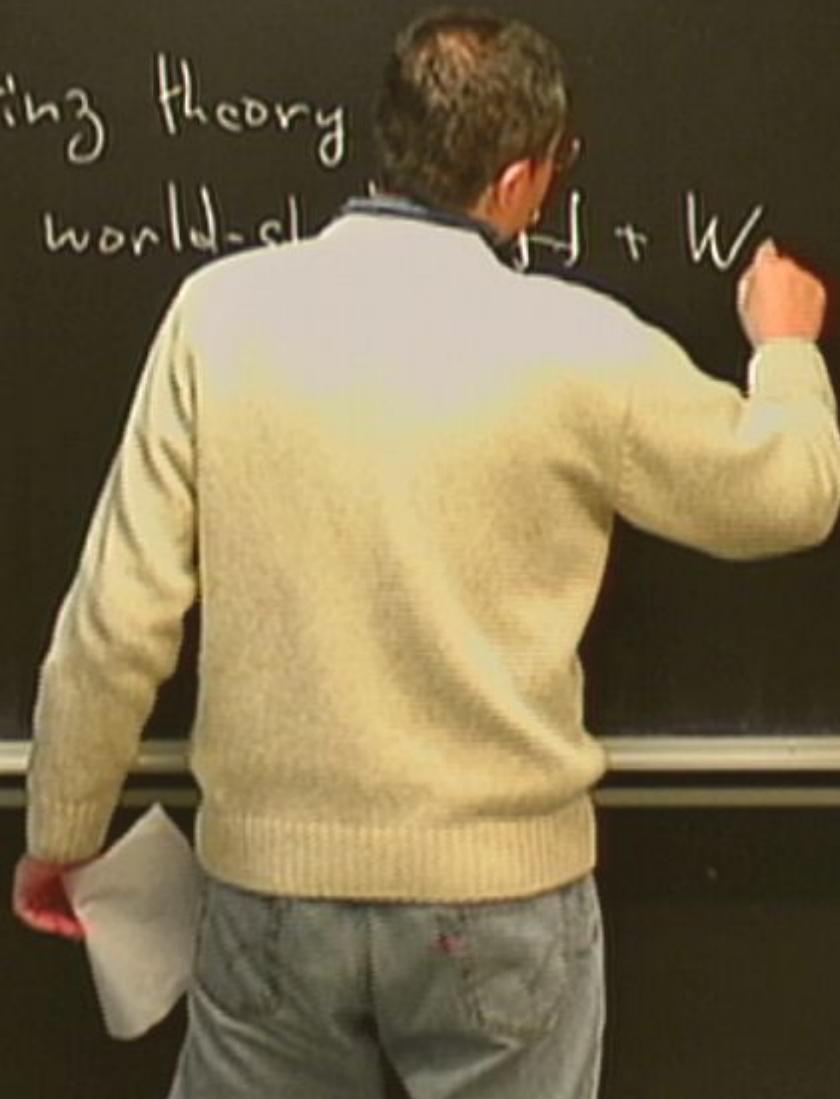
Conformal Field Theory (CFT)

→ for string
fix sheet di

Conformal Field Theory (CFT)

→ for string theory

6x world-sheets $H + W$



Conformal Field Theory (CFT)

→ for string theory
fix world-sheet diffl + we



Conformal Field Theory (CFT)

→ for string theory

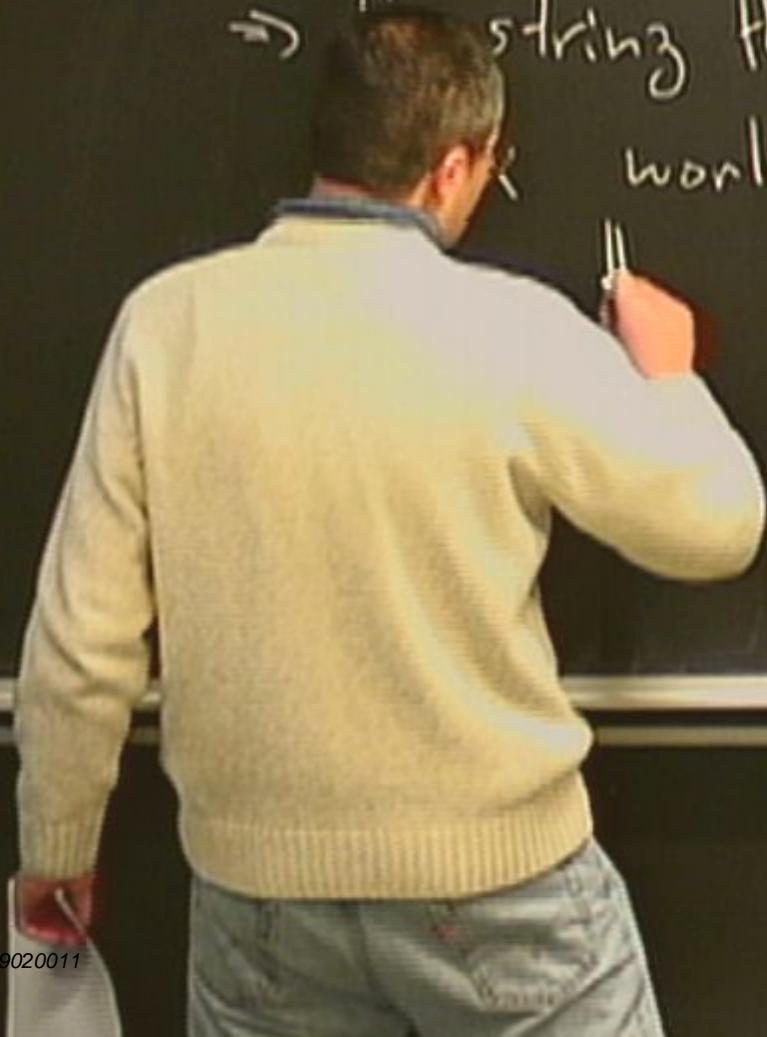
fix world-sheet diff + Weyl

Conformal Field Theory (CFT)

→ for string theory
fix world-sheet diff + Weyl

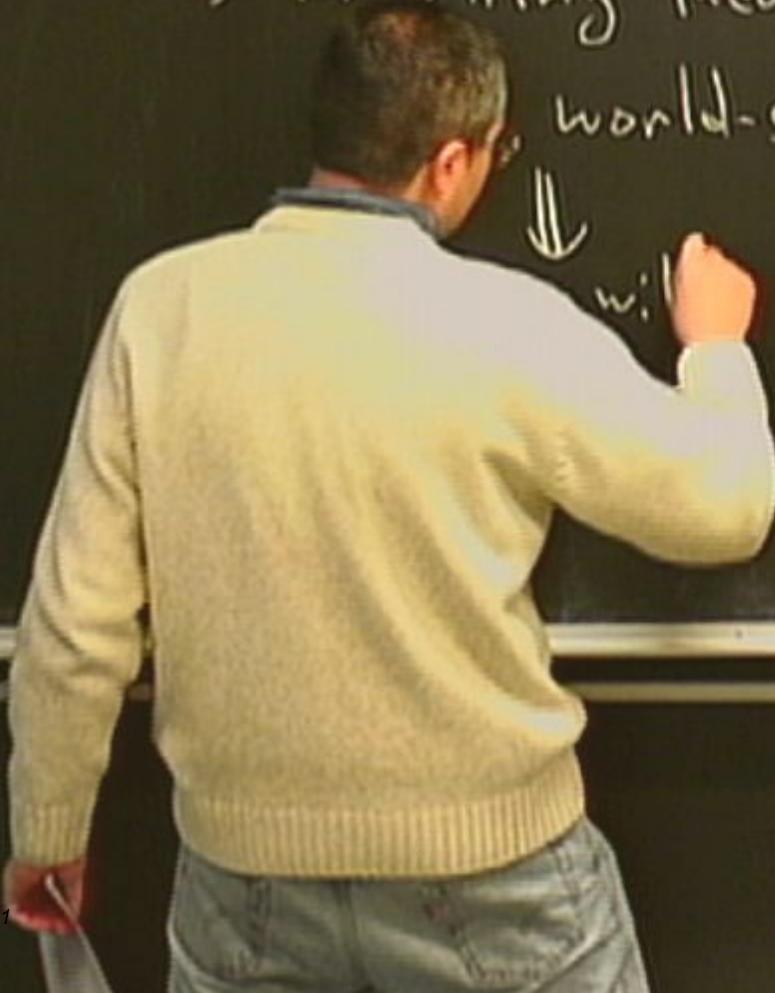
Conformal Field Theory (CFT)

→ string theory
world-sheet diff + Weyl



Conformal Field Theory (CFT)

→ for string theory
world-sheet diff + Weyl



Conformal Field Theory (CFT)

→ for string theory
fix no exact diff + Weyl

CFT

Conformal Field Theory (CFT)

⇒ for string theory

Fix [street diff + Weyl]

left with

F

Conformal Field Theory (CFT)

→ for string theory

fix world-sheet diff + Weyl

↓
we will be left with

CFT

Conformal Field Theory (CFT)

→ for string theory

fix world-sheet diff + Weyl

we will be left with

CFT

Conformal Field Theory (CFT)

→ for string theory

fix world-sheet diff + Weyl

we will be left with

CFT

Conformal Field Theory (CFT)

→ for string theory
fix world-sheet diff + Weyl
 \Downarrow
we will be left with
CFT

→ for string theory

Six world-sheet d.f. + Weyl

↓
we will be left with
CFT



→ for string theory
fix world-sheet dif + Weyl
we will be left with
CFT



\Rightarrow for string theory

Six world-sheet diff + Weyl

\Downarrow
we will be left with

\Rightarrow CFT

CFT

→ for string theory

Six world-sheet diff + Weyl

↓
we will be left with

CFT

⇒ CFT

→ for string theory

Six world-sheet diff + Weyl

↓
we will be left with

CFT

⇒ CFT

\Rightarrow for string theory

Six world-sheet diff + Weyl

\Downarrow
we will be left with

CFT

$\Rightarrow C$

$x \rightarrow \lambda x$

\Rightarrow for string theory

Six world-sheet diff + Weyl

\Downarrow
we will be left with

\Rightarrow CFT $x \rightarrow \lambda x$ CFT

\Rightarrow for string theory

δx world-sheet diff + Weyl

\Downarrow
we will be left with

\Rightarrow CFT

CFT

$\rightarrow \delta x$

\Rightarrow for string theory

$\{x\}$ world-sheet diff + Weyl

\Downarrow
we will be left with

\Rightarrow CFT $x \rightarrow g x$ CFT

\Rightarrow for string theory

$\{x\}$ world-sheet diff + Weyl

\Downarrow
we will be left with

\Rightarrow CFT $x \rightarrow g x$ CFT

\Rightarrow for string theory

Six world-sheet diff + Weyl

\Downarrow
we will be left with

$$\Rightarrow \text{CFT} \quad x \rightarrow g x \quad \text{CFT}$$

\Rightarrow for string theory

Six world-sheet diff + Weyl

\Downarrow
we will be left with

\Rightarrow CFT $x \rightarrow g x$ CFT

→ for string theory

fix world-sheet diff + Weyl



we will be left with

⇒ CFT

CFT

$x \rightarrow \xi x$ (a scaling transformation part)

\Rightarrow CFT $x \rightarrow \xi x$ CFT
(a scaling transformation is part of Confor

CFT is shown up in condensed matter @

\Rightarrow CFT $x \rightarrow \xi x$ CFT
(a scaling transformation is part of Confor

CFT is shown up in condensed matter @
2nd order phase transitions

\Rightarrow CFT $x \rightarrow \xi x$ CFT
(a scaling transformation is part of Confor

CFTs show up in condensed matter @
2nd order phase transitions

Notes

"Applied CFT" Paul Ginsparg

D-massless scalars in 2-dim. (toy CFT).

D-massless scalars in 2-dim. (toy CFT)

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left[\partial_1 X^\mu \partial_1 X^\nu + \partial_2 X^\mu \partial_2 X^\nu \right]$$

D-massless scalars in 2-dim. (toy CFT).

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left[\partial_1 X^a \partial_1 X^a - \gamma_2 X^a \partial_2 X_a \right]$$

↑ Polyakov action with $b_{ab} \Rightarrow \gamma_{ab}$

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left[\partial_a X^a \bar{\gamma}^1 X^a - \partial_b X^b \bar{\gamma}_2 X^b \right]$$

↑ Polyakov action with $\gamma_{ab} \Rightarrow \eta_{ab} \rightarrow \delta_{ab}$
 (σ_0, σ_1) Wick rotation

Polyakov action with

$$\begin{pmatrix} \sigma_0, \sigma_1 \\ \tau \end{pmatrix} \xrightarrow{\text{Wick rotation}}$$

$$\begin{array}{c} h_{ab} \mapsto \gamma_{ab} \rightarrow \delta_{ab} \\ \text{Wick rotation} \\ \left| \sigma_2 = i\sigma_0 \right. \end{array}$$

$$\begin{pmatrix} 1 & 0 \\ \tau & \sigma \end{pmatrix}$$

Wick rotation

$$(e_2, e_1)$$

$$|e_2 = i e_0 |$$

$$z = \sigma' + i e^2$$

$$\gamma_2 =$$

$$\bar{z} = \sigma' - i e^2$$

Polyakov action with
 (σ_0, σ_1) $\xrightarrow{\text{Wick rotation}}$ (σ_2, σ_1) $\xrightarrow{\gamma_{ab} \Rightarrow \gamma_{ab} \rightarrow \delta_{ab}}$
 $\tau \parallel \sigma$ Wick rotation $\boxed{\sigma_2 = i \sigma_0}$

$$\begin{aligned}
 z &= \sigma^1 + i \sigma^2 \\
 \bar{z} &= \sigma^1 - i \sigma^2
 \end{aligned}
 \quad
 \begin{aligned}
 \partial_z &= \partial_1 - i \partial_2 = \frac{1}{2} (\partial_1 - i \partial_2) \\
 \bar{\partial}_z &= \partial_{\bar{z}} = \frac{1}{2} (\partial_1 + i \partial_2).
 \end{aligned}$$

$$\partial_z \bar{z} = 1 \quad \bar{\partial}_z z = 0 \quad \dots$$

$$\begin{pmatrix} \sigma_1 & \sigma_2 \\ \tau & \sigma \end{pmatrix}$$

Wick rotation

$$(\sigma_2, \sigma_1)$$

$$\left| \sigma_2 \equiv i\sigma_0 \right\rangle$$

$$z = \sigma^1 + i\sigma^2 \quad \partial_z = \partial_x = \frac{1}{2}(\partial_1 - i\partial_2)$$

$$\bar{z} = \sigma^1 - i\sigma^2 \quad \bar{\partial}_z = \partial_{\bar{x}} = \frac{1}{2}(\partial_1 + i\partial_2)$$

$$\partial_{\bar{z}} z = 1 \quad \bar{\partial} z = 0$$

$$\begin{matrix} z \\ \uparrow \\ \text{holomorphic} \end{matrix}$$

$$\begin{matrix} \bar{z} \\ \uparrow \\ \text{anti holomorphic} \end{matrix}$$

$$\mathcal{O} = (\mathcal{O}', \mathcal{O}'')$$

$$\mathcal{O}_+^z = \mathcal{O}' + i\mathcal{O}''$$

$$\mathcal{O}_-^z = \mathcal{O}' - i\mathcal{O}''$$

$$\mathcal{O} = (O^1, O^2)$$

$$O_{\bar{z}}^z = O^1 + i O^2$$

$$O_{\bar{z}}^{\bar{z}} = O^1 - i O^2$$

$$\text{Sols} \rightarrow \begin{pmatrix} g_{zz} & g_{z\bar{z}} \\ g_{\bar{z}z} & g_{\bar{z}\bar{z}} \end{pmatrix}$$

$$U^2 = U' - iJ_L$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \end{pmatrix}$$

$$U^z = U' - i\partial_L$$

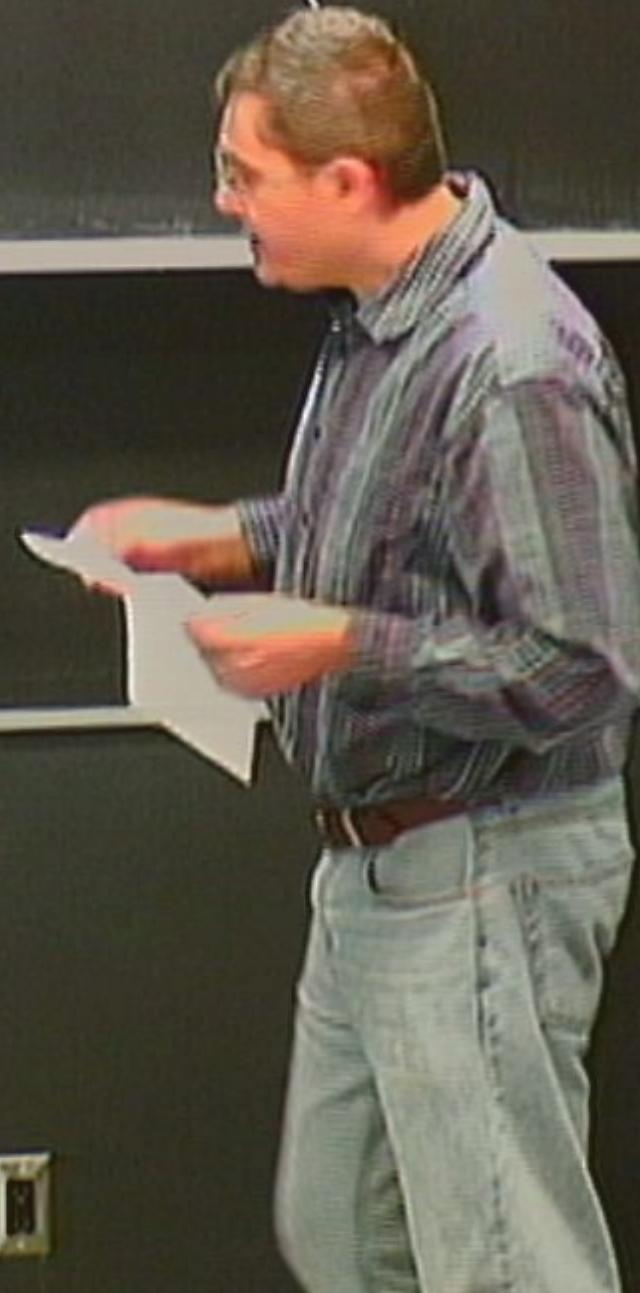
$$g^{z\bar{z}} = \bar{g}^{\bar{z}z} = 2$$

$$\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$$

$$\sqrt{g} = \frac{1}{2}$$

dz

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



$$\sqrt{g} = \frac{1}{2}$$

$$d\sigma^1 d\sigma^2 \equiv \partial^1 \sigma \wedge \partial^2 \sigma$$



$$\sqrt{g} = \frac{1}{2} d\sigma^1 d\sigma^2 \equiv \partial^2 \sigma = \underbrace{\partial \sigma^1}_{d\sigma^2} d\sigma^2$$

$$\sqrt{g} = \frac{1}{2} d\sigma^1 d\sigma^2 \equiv d\sigma^2$$

$$\text{Tr}g = \frac{1}{2} \int d\sigma^1 d\sigma^2 \det g = \frac{1}{2} \int d\sigma^1 d\sigma^2$$

$$\int d\sigma \delta(\sigma_1, \sigma_2) = \int d\sigma \delta(\sigma_1) \delta(\sigma_2) = 1$$

$$\int d^2z \delta^2(z, \bar{z}) = 1 \Rightarrow \delta^2(z, \bar{z}) = \frac{1}{2} \delta^2(\sigma_1, \sigma_2)$$

$$\int g = \frac{1}{2} \int d\sigma^1 d\sigma^2 \equiv d^2\sigma = \frac{1}{2} d\sigma^1 d\sigma^2$$

$$\int d\sigma \delta(\sigma_1, \sigma_2) = \int d^2\sigma \delta(\sigma_1) \delta(\sigma_2) = 1$$

$$\int d^2z \delta^2(z, \bar{z}) = 1 \Rightarrow \delta^2(z, \bar{z}) = \frac{1}{2} \delta^2(\sigma_1, \sigma_2)$$

Divergence theorem.

In above notation,

$$S = \frac{1}{2\pi\alpha'} \int d^2z \partial X^\mu \bar{\partial} X_\mu$$

$$S = \frac{1}{2\pi\omega} \int d^2z \sum_{m,n} \partial X^m \bar{\partial} X_n$$

\rightarrow EOM

$$0 = \frac{\delta S}{\delta X^m} = - \frac{1}{2\pi\omega} \partial \bar{\partial} X^m = 0$$

Paul Ginsparg

$$0 = \frac{\delta S}{\delta x^m} = - \frac{e}{2\pi\epsilon_0} \partial_\nu x^m = 0$$
$$\boxed{\partial_\nu x^m = 0}$$

CFT is shown to be a consistent theory

2nd order equations

Notes

"Applied

Wyl Ginsparg

$$\partial X^m - \frac{2\pi i z^m}{\partial z} = 0$$
$$\boxed{\partial \bar{\partial} X^m = 0}$$
$$X^m = X^m(z, \bar{z})$$

$$\bar{\partial} [\partial X^m] = 0$$

$$\partial X^m + \frac{2\pi i \omega}{\partial z} (z) = 0$$
$$\boxed{\bar{\partial} X^m = 0}$$
$$X^m = X^m(z, \bar{z})$$

$$\bar{\partial} [\partial X^m] = 0$$

$$\partial X^m = \partial X^m(z)$$

$$\bar{\partial} X^m = \bar{\partial} X^m(\bar{z})$$

or holomorphic
function.

$$\partial X^m - \frac{2\pi i}{\partial \bar{z}} X^m = 0$$

$\boxed{\partial \bar{\partial} X^m = 0}$

$$X^m = X^m(z, \bar{z})$$

$$\partial[\partial X^m] = 0$$

a holomorphic function.

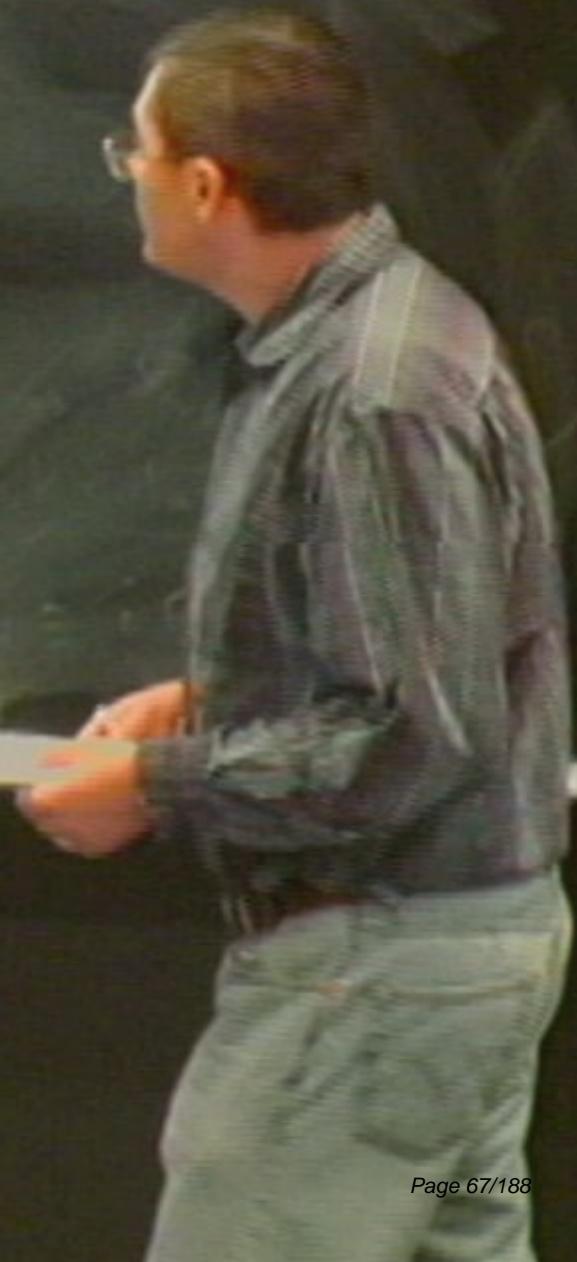
$$\partial X^m = \partial X^m(z)$$

$$\partial[\bar{\partial} X^m] = 0$$

$$\bar{\partial} X^m = \bar{\partial} X^m(\bar{z})$$

Some relations

$$\int [dx] e^{-S}$$



Some relations

$$\langle \mathcal{F}[x] \rangle = \int [dx] e^{-S}$$

Some relations

$$\langle \mathcal{F}[x] \rangle = \int [dx] e^{-S} \mathcal{F}[x]$$

Some relations

$$\langle \mathcal{F}[x] \rangle = \frac{\int [dx] e^{-S} \mathcal{F}[x]}{\int [dx] e^{-S}}$$

$$\langle \mathcal{F}[x] \rangle = \frac{\int [dx] e^{-S} \mathcal{F}[x]}{\int [dx] e^{-S}}$$

$\langle \text{ol}$

holomorphic

↑
anti holomorphic

$$\langle \mathcal{F}[x] \rangle = \frac{\int [dx] e^{-S} \mathcal{F}[x(\tau, \bar{\tau})]}{\int [dx] e^{-S}}$$

||

$$\langle \phi | \mathcal{F} | \phi \rangle$$

$\langle 0 | \hat{T} | 0 \rangle$

$$\hat{T}_1 [X(z_1, \bar{z}_1)] \quad \hat{T}_1 [X(z, \bar{z})] \quad \hat{T}_2 [X(z_2, \bar{z}_2)]$$

$\langle 0 | \hat{T} | 0 \rangle$

$$\int dX \bar{e}^{\hat{T}_1 [X(z_1, \bar{z}_1)]} \hat{T}_1 [X(z, \bar{z})] \hat{T}_2 [X(z_2, \bar{z}_2)]$$

$\langle 0 | \hat{F} | 0 \rangle$

$$\int d\vec{x} e^{\hat{F}_1[X(z_1, \bar{z}_1)] + \hat{F}_2[X(z_2, \bar{z}_2)]}$$

$$\langle 0 | \hat{F}_1 \hat{F}_2 | 0 \rangle$$

$\langle 0 | \hat{F} | 0 \rangle$

$$\int d\mathbf{x} e^{\hat{F}_1[X(z_1, \bar{z}_1)]} \hat{F}_1[X(z, \bar{z})] \int \hat{F}_2[X(z_2, \bar{z}_2)]$$

$\langle 0 | \hat{F}_1 \hat{F}_2 | 0 \rangle$

$\langle 0 | \hat{T} | 0 \rangle$

$$\int dX e^{\hat{F}_1[X(z_1, \bar{z}_1)]} \hat{F}_1[X(z, \bar{z})] \hat{F}_2[X(z_2, \bar{z}_2)]$$

$\langle 0 | \hat{F}_1 \hat{F}_2 | 0 \rangle$

$\langle 1 | \hat{T} | 2 \rangle$

$\langle 0 | \hat{T} | 0 \rangle$

$$\int d\mathbf{x} e^{\hat{F}_1[X(z_1, \bar{z}_1)]} \hat{F}_1[X(z, \bar{z})] \int \hat{F}_2[X(z_2, \bar{z}_2)]$$

$\langle 0 | \hat{F}_1 \hat{F}_2 | 0 \rangle$

$$\langle 1 | \hat{F}_1 | 2 \rangle \quad |2\rangle = \hat{F}_2 |0\rangle$$

Statement:

$$\int dx \geq [something] > 0$$

$$\int dx \delta_x [\text{something}] = 0$$

$$0 = \int [dx] \frac{\Sigma}{\delta x^m}$$



$$\left[\int dx \partial_x [something] \right] = 0$$

$$0 = \int [dx] \sum \frac{\delta}{\delta x^m} [something]$$



$$\int dx \delta_x [something] = 0$$

$$0 = \int [dx] \sum \frac{\delta}{\delta x^m} [something]$$



$$O = \int [dx] \frac{\Sigma}{\delta x^m} [\text{something}]$$

$$\int d\bar{z} (\gamma_z O^- + \partial_{\bar{z}} O^+) = i \oint (O^- d\bar{z} - O^+ dz)$$

$$O = \int [dx] \frac{\delta}{\delta x^\mu} [\bar{e}^S]$$



$$0 = \int [dx] \frac{\delta}{\delta x^\mu} [\bar{e}^S] = - \int [dx] \bar{e}^S \frac{\delta S}{\delta x^\mu}$$

$$0 = \int [dx] \frac{\delta}{\delta x^m} [e^{-S}] = - \int [dx] e^{-S} \frac{\delta S}{\delta x^m}$$
$$= - \int [dx] e^{-S}$$

$$0 = \int [dx] \frac{\delta}{\delta x^m} [\bar{e}^{-S}] = - \int [dx] \bar{e}^{-S} \frac{\delta S}{\delta x^m}$$

$$= - \int [dx] \bar{e}^{-S} \left(\frac{i}{2\pi\lambda} \partial \bar{\delta} x^m \right)$$

$$\underline{\underline{O}} = \int [dx] \frac{\delta}{\delta x^m} \left[e^{-S} \right] = - \int [dx] e^{-S} \frac{\delta S}{\delta x^m}$$

$$= - \int [dx] e^{-S} \left(\frac{\partial}{\partial x^m}, \partial \bar{x}^m \right) = \frac{1}{\pi^d} \langle \partial \bar{x}^m \rangle$$

$$\underline{\underline{O}} = \int [dx] \frac{\delta}{\delta x^m} \left[e^{-S} \right]_{z=z'}^{z=z''} = - \int [dx] e^{-S} \frac{\delta S}{\delta x^m}$$

$$= - \int [dx] e^{-S} \left(\frac{e}{2\pi i}, \partial \bar{\partial} x^m \right) = \frac{1}{\pi i} \langle e | \overline{\partial \bar{\partial} x^m} \rangle$$

$$\underline{O} = \int [dx] \frac{\delta}{\delta x^m_{(z,z')}} \left[e^{-S} \right] = - \int [dx] e^{-S} \frac{\delta S}{\delta x^m}$$

$$= - \int [dx] e^{-S} \left(\frac{1}{2\pi i} \partial \bar{\partial} x^m \right) = \frac{1}{\pi i} \langle \partial \bar{\partial} x^m \rangle$$

$$\langle \partial \bar{\partial} x^m \rangle_F = 0$$

$$\underline{\underline{O}} = \int [dx] \frac{\delta}{\delta x^m} \left[e^{-S} \right]_{z=z'} = - \int [dx] e^{-S} \frac{\delta S}{\delta x^m}$$

$$= - \int [dx] e^{-S} \left(\frac{1}{2\pi i} \partial \bar{\partial} x^m \right) = \frac{1}{\pi i} \langle \partial \bar{\partial} x^m \rangle$$

$$\langle \bar{\partial} x \rangle_T = 0$$

$$\bar{\partial} x = 0 \text{ as an operator statement}$$

$$\begin{aligned}
 \underline{\underline{O}} &= \int [dx] \frac{\delta}{\delta x^m} \left[e^{-S} \right] = - \int [dx] e^{-S} \frac{\delta S}{\delta x^m} \\
 &\quad \text{if } O = 0 \quad z \neq z' \\
 &= - \int [dx] e^{-S} \left(\frac{z}{2\pi i}, \bar{\partial} \bar{x}^m \right) = \frac{1}{\pi i} \langle \bar{\partial} \bar{x}^m \rangle \\
 \langle \bar{\partial} \bar{x}^m \rangle_F &= 0 \quad \text{as an operator statement}
 \end{aligned}$$

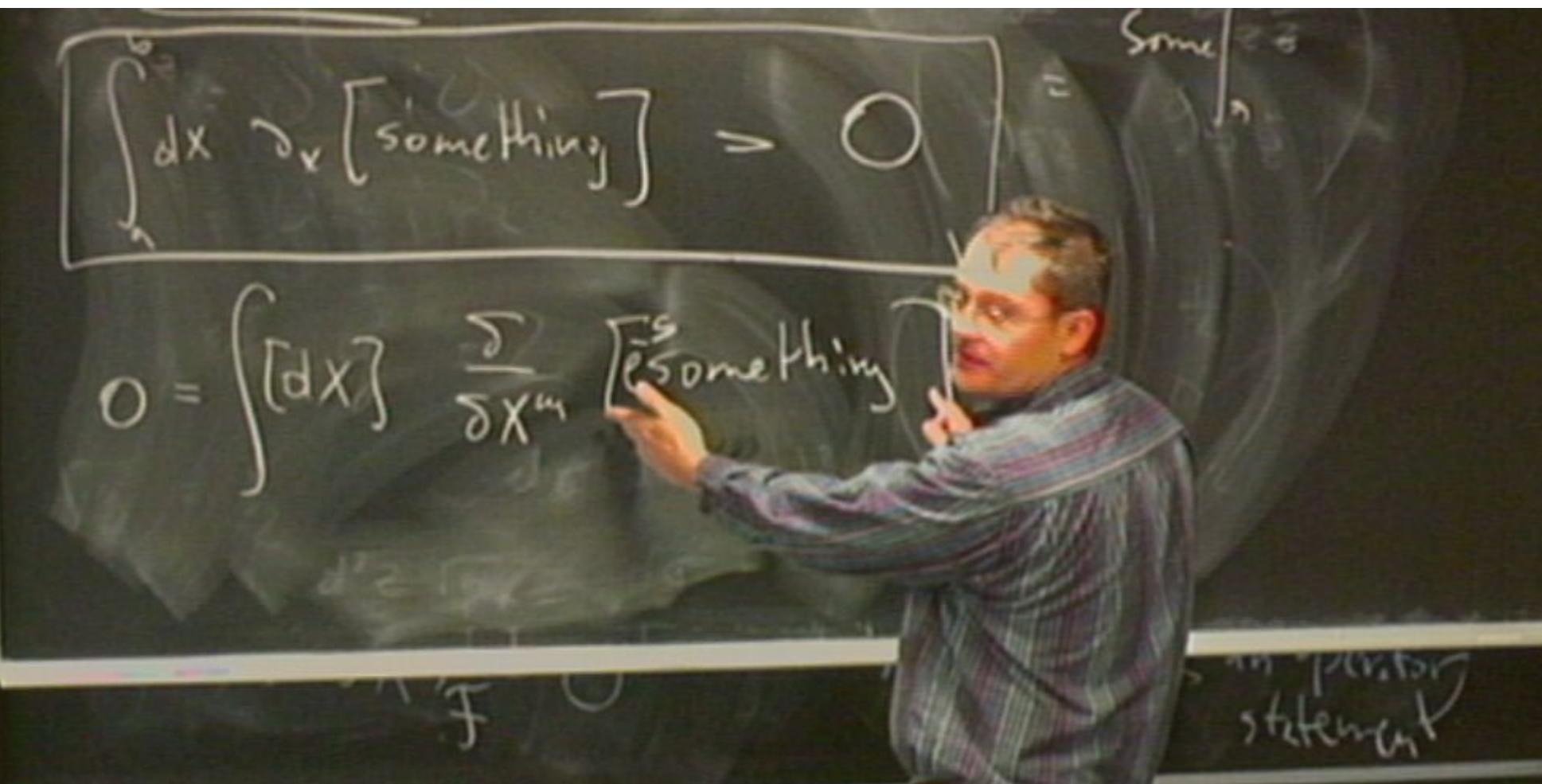
$$\int dx \rightarrow [something] = 0$$

Some

$$0 = \int [dx] \sum \frac{\delta}{\delta x^m} [something]$$

is an operator statement





$$\int dx \partial_x [\text{something}] > 0$$

$$0 = \int [dx] \sum \frac{\delta}{\delta x^m} [\text{something}]$$

$$S = \frac{1}{2m} \int dx | \partial x |^2$$

$\Gamma \vdash 0$ is an operator statement

$$D = \int [dx] \frac{\delta}{\delta X^{\mu}(z)} \left[e^{-S} \right]$$



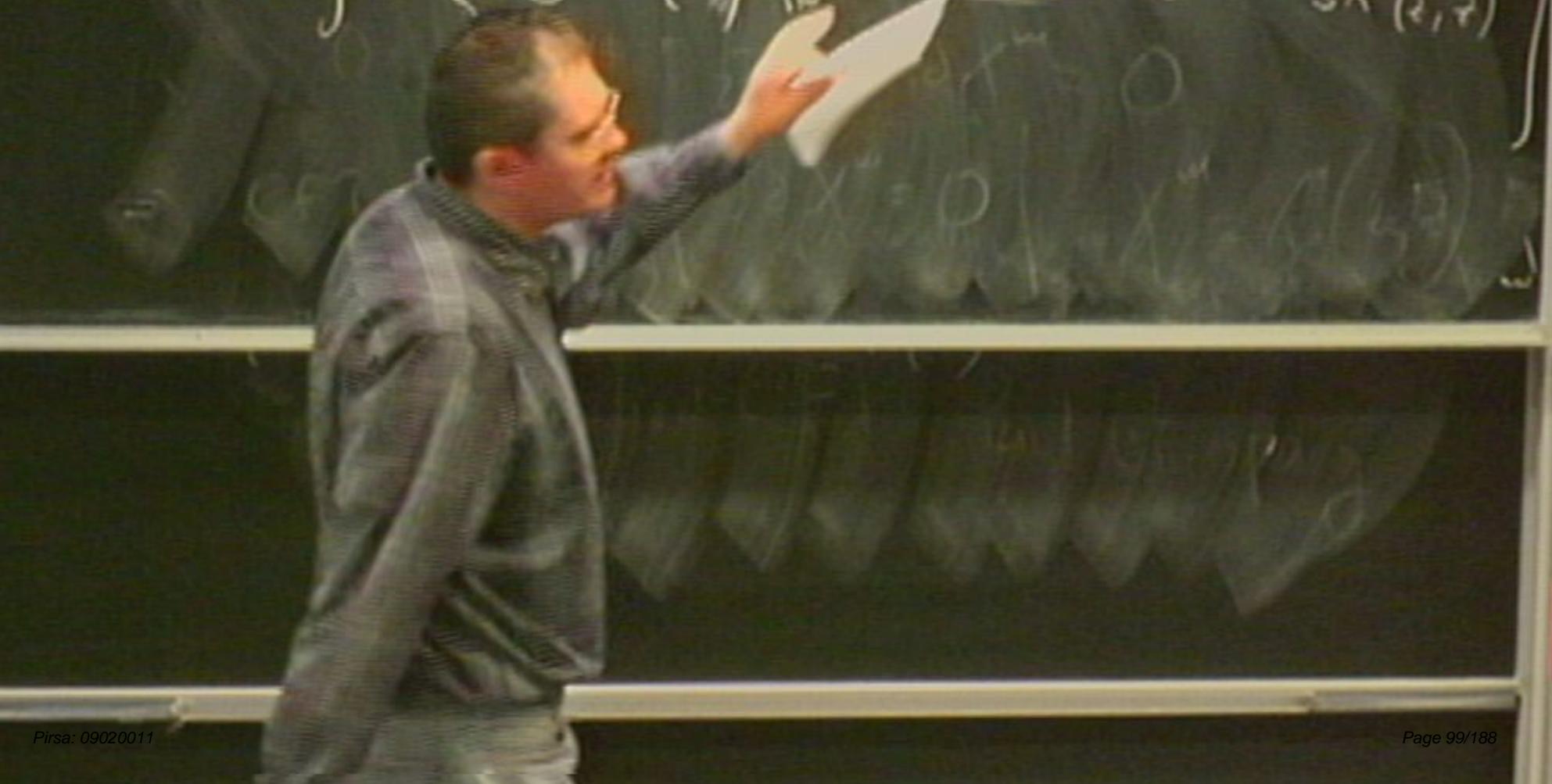
$$0 = \int [dx] \delta_{\theta} X^{\mu}(z, \bar{z}) \left[e^{-S} X^{\nu}(z', \bar{z}') \right]$$



$$D = \int [dx] \frac{\partial}{\partial x^m} (z, \bar{z}) \left[e^{-S} \mathcal{K}(z', \bar{z}') \right]$$
$$= \int dx \left\{ e^{-S} (-i)^2 \frac{1}{\pi v'} \partial z \bar{\partial} x^m \right\}$$



$$O = \int [dx] \frac{\delta}{\delta X^m(\tau, \vec{x})} \left[e^{-S} \lambda(x', \vec{x}') \right] \\ - \int dx \left\{ \bar{e}^S \cdot (-1)^2 \frac{1}{2!} \partial \bar{\partial} X^m + \bar{e}^{-S} \cdot \frac{\delta X^i(\tau', \vec{x}')}{\delta X^m(\tau, \vec{x})} \right\}$$



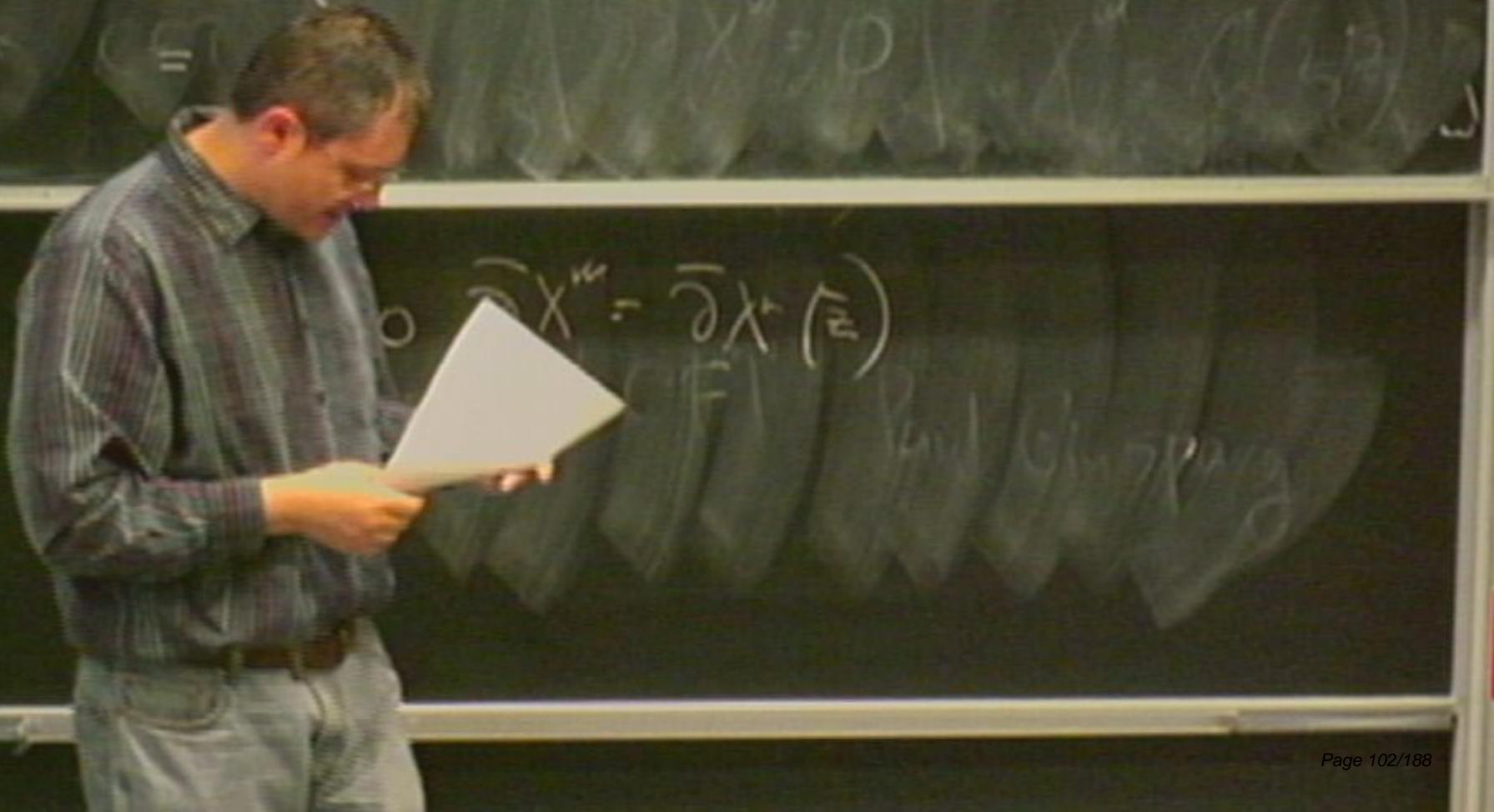
$$J^{\mu} = \delta X^{\mu}(z, \bar{z}) \left[e^{-\lambda((\sigma), t)} \right] \\ = \int d[\chi] \left\{ e^{-S} (-i)^2 \frac{1}{4!} \cdot \partial \bar{\partial} \chi^m + e^{-S} \cdot \frac{\delta X^{\nu}(z', \bar{z}')}{\delta X^{\mu}(z, \bar{z})} \right\}$$

$$\partial[\delta X^{\mu}] = \partial \lambda^{\mu}(\tilde{z})$$

$$= \int d(\bar{x}) \left\{ \bar{e}^{-S} \bar{x}^{\mu} (-i)^2 \frac{1}{\pi} \partial_{\bar{x}^{\mu}} \partial_{\bar{x}^{\nu}} \bar{x}^{\nu} + \bar{e}^{-S} \cdot \frac{\delta x'(\tau', \bar{z}')}{\delta x^{\mu}(\tau, \bar{z})} \right\}$$

$$\partial (\bar{\partial} x^{\mu}) = 0 \quad \bar{\partial} x^{\mu} = \bar{\partial} x^{\mu}(\tau, \bar{z})$$

$$= \int d[x] \left\{ \bar{e}^S \dot{x}^\nu (-i)^2 \frac{1}{\pi \delta^2} \cdot \partial \bar{\partial} x^\mu (\bar{z}, \bar{z}) + \bar{e}^S \cdot \underbrace{\dot{x}^\nu}_{x^\mu (\bar{z}, \bar{z})} \right\}$$
$$\delta_{\mu\nu} \delta^2 (z' - z, \bar{z}' - \bar{z})$$



$$0 = \int [dx] \frac{\delta}{\delta X^m(z, \bar{z})} \left[e^{-S} X(z', \bar{z}') \right] +$$

$$\int dx \left\{ \bar{e}^{-S} \left(-\frac{1}{4\pi} \cdot \partial_z \bar{\partial}_z X(z, \bar{z}) + e^{-S} \frac{\delta X^m(z, \bar{z})}{\delta X^m(z, \bar{z})} \right) \right.$$

$$\left. - \frac{1}{4\pi} \partial_z \bar{\partial}_z X(z, \bar{z}) \right\}$$



$$D = \int [dx] \frac{\delta}{\delta X^m(z, \bar{z})} \left[e^{-S} X^m(z', \bar{z}') \right] |_{z=z'} \cdot$$

$$= \int dx \left\{ \bar{e}^{-S} \dot{X}^m(-) \frac{1}{\pi \delta^2} \cdot \partial_z \bar{\partial}_{\bar{z}} X^m(z, \bar{z}) + \bar{e}^{-S} \cdot \frac{\delta X^m(z', \bar{z}')}{\delta X^m(z, \bar{z})} \right.$$

$$\left. \frac{1}{\pi \delta^2} \partial_z \bar{\partial}_{\bar{z}} z' \right\}$$



$$\begin{aligned}
 &= \int dX \left[e^{-S_X} \delta X(\bar{z}, \bar{z}) \left[(e) \cdot \lambda^{(\sigma, \epsilon)} \right] \right] \\
 &\equiv \int dX \left\{ e^{-S_X} \left(-\frac{1}{2} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \delta \bar{X}^{(i)} + e^{-S} \frac{\delta X^{(i)}(\bar{z}', \bar{z}')}{\delta X^{(i)}(\bar{z}, \bar{z})} \right) \right. \\
 &\quad \left. - \frac{1}{4!} \partial_1 \bar{\partial}_2 \langle X^{(i)}(\bar{z}, \bar{z}) X^{(j)}(\bar{z}', \bar{z}') \rangle \right\}
 \end{aligned}$$

$$\partial \bar{\partial} X = 0 \quad \partial X = \partial \lambda^*(\bar{z})$$

$$= \int d[\bar{x}] \left\{ \bar{e}^S \bar{x} \left((-1)^2 \frac{1}{\pi r'} \cdot \bar{\partial} \bar{\partial} X''(r, \bar{r}) + \bar{e}^{-S} \cdot \frac{\delta X'(r', \bar{r}')}{\delta X''(r, \bar{r})} \right) \right\}$$

$$= - \frac{1}{\pi r'} \bar{\partial}_2 \bar{\partial}_2 \langle X''(r, \bar{r}) X'(r', \bar{r}') \rangle + \Gamma'' \langle \delta^2(r' - r, \bar{r}' - \bar{r}) \rangle$$

$$\partial \bar{\partial} X'' = 0 \quad \bar{\partial} X'' = \bar{\partial} X''(\bar{r})$$

$$= \frac{1}{\pi i} \partial_z \bar{\partial}_z \langle X(z, \bar{z}) X(z', \bar{z}') \rangle + \text{Im} \langle \delta^2(z - z') \rangle$$

$$\frac{1}{\pi i} \partial_z \bar{\partial}_z \langle X(z, \bar{z}) X(z', \bar{z}') \rangle = -\eta^{(n)} \delta^2(z - z')$$

$$= \frac{1}{\pi i} \partial_2 \bar{\partial}_2 \langle X(z, \bar{z}) X(z', \bar{z}') \rangle + \text{Im} \langle \delta^2(z - z') \rangle$$

$$\frac{1}{\pi i} \partial_2 \bar{\partial}_2 \langle X(z, \bar{z}) X(z', \bar{z}') \rangle = - \text{Im} \langle \delta^2(z - z') \rangle$$

Normal ordering

$$= \frac{1}{\pi^2} \partial_2 \bar{\partial}_2 \langle X(\vec{r}, \vec{z}) X(\vec{r}', \vec{z}') \rangle + \text{Im} \langle \delta^2(\vec{r} - \vec{r}') \rangle$$

Normal ordering

$$\partial \bar{\partial} X'' \Big|_{\text{classically}} = 0$$

$$\partial \bar{\partial} X'' \Big|_{\mathcal{Q}} \neq 0$$

$$\exists \bar{z} \exists x^*(z_1, \bar{z}_1) X^*(z_1, \bar{z}_1)$$



$$\bar{\partial} \chi^m(z_1, \bar{z}_1) \chi^n(z_2, \bar{z}_2) = -\pi \delta^{mn} \delta^2(z_{12})$$
$$z_{12} = z_1 - z_2$$

$$\Im \bar{\partial} \chi^m(z_1, \bar{z}_1) \chi^m(z_2, \bar{z}_2) = - \underbrace{\pi \delta' \cdot \delta^{(n)} \delta'(z_{12})}_{\text{Quantum correction to classical EOM}} \quad z_{12} = z_1 - z_2$$

$$\Im \bar{\chi}^m(z_1, \bar{z}_1) \cdot \chi^m(z_2, \bar{z}_2) = - \underbrace{\pi \alpha' \cdot \eta^{m*} \delta'(z_{12})}_{\text{Quantum correction to classical EOM}} \quad z_{12} = z_1 - z_2$$

$$\bar{\chi}^m(z_1, \bar{z}_1) \cdot \chi^m(z_2, \bar{z}_2) = - \underbrace{\pi \alpha' \cdot \delta^{(n)} \delta'(z_{12})}_{\bar{z}_{12} = z_1 - z_2}$$

Quantum correction
to classical EOM

$$; \chi^m(z_1, \bar{z}) \chi^m(z_2, \bar{z}_2);$$

\$\downarrow f\$

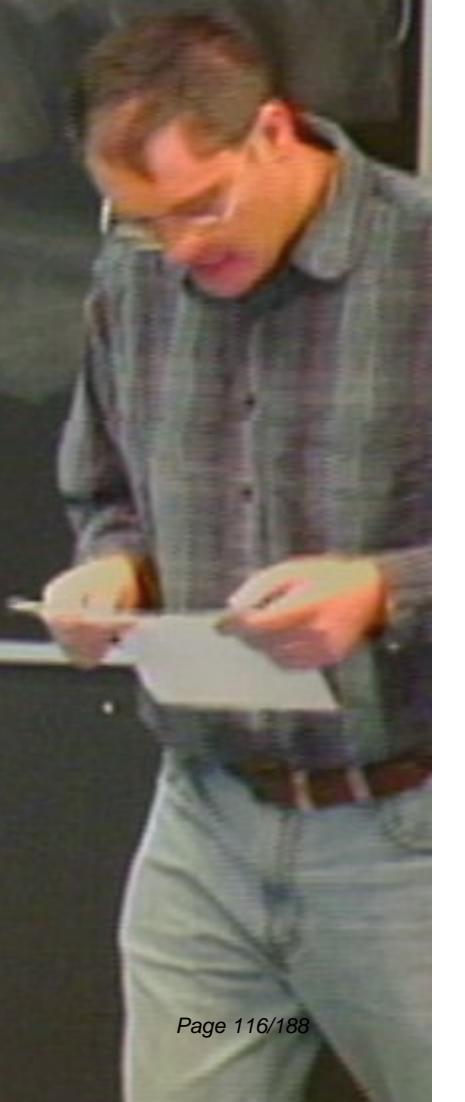
$$\bar{\chi}_1 \bar{\chi}^*(z_1, \bar{z}_1) \cdot \chi^* \chi(z_2, \bar{z}_2) = - \underbrace{i \pi \delta' \cdot \delta^{(n)} \delta^*(z_{12})}_{\text{Quantum correction}} z_{12} = z_1 - z_2$$

$$\bar{\chi}_1 ; \bar{\chi}(z_1, \bar{z}) \chi^*(z_2, \bar{z}_2) ; = 0 \quad \text{to classical EOM}$$

if

Def

$$\therefore X^m \cdot := X^m$$



Def

$$: X^m : = X^m$$

$$: X^m X^n : = X^{m+n} X^0$$

Def

$$[X^m] := X^m$$

$$[X' X] = X' X + \frac{1'}{2} [Y^{(n)}] P_n(z_2)$$

Def

$$[X^m] := X^m$$

$$[X^m X^k] = X^m X^k + \frac{1}{2} [p^{(m)}] P_n(z_2)$$

$$[X^m X^k] = -\pi \delta^{mk}$$

Def

$$[X^m] := X^m$$

$$[X^m X^0] = X^m X^0 + \frac{1}{2} \eta^{m0} P_n(z_2)$$

$$[\bar{\partial}_1 X^m X^0] = -\bar{\partial}_1 \eta^{m0} \delta'(z_{12})$$

Def

$$[X^m] := X^m$$

$$[X^m X^k] = X^m X^k + \frac{\alpha'}{2} \eta^{mn} \partial_n |z_2|^2$$

$$[\bar{\partial}_1 X^m X^k] = -\bar{\partial}_1 \eta^{mn} \delta_{(n}^{(k)} + \frac{\alpha'}{2} \eta^{mn} \bar{\partial}_1 \partial_1 |z_2|^2$$

Because: $2\pi \ln |z|^2 = 2\pi \delta^2(z)$

Because : $\partial\bar{\partial} \ln |z|^2 = 2\pi \delta^2(z)$

Proof

If $z \neq 0$

$$\partial\bar{\partial} [\ln z + \ln \bar{z}] =$$



Because: $2\pi \ln |z|^2 = 2\pi \delta^2(z)$

Proof

If $z \neq 0$

$$\Im [b_z + b_{\bar{z}}] = 0 \quad \left\{ \delta^2(z) = 0 \right.$$

Because : $2\pi \ln|z|^c = 2\pi \delta^c(z)$

Proof

If $z \neq 0$

$$\Im \left[\ln z + \ln \bar{z} \right] = 0 \quad \left\{ \delta^c(z) = 0 \right.$$

$$\int d^2z \Im \ln|z|^2 =$$

Because : $2\pi \ln |z|^2 = 2\pi \delta^c(z)$

Proof

If $z \neq 0$

$$\Im \left[\ln z + \ln \bar{z} \right] = 0 \quad \left\{ \delta^c(z) = 0 \right.$$

$$\text{LHS} \int d^2z \Im \ln |z|^2 =$$

$$\text{RHS} \int z^{\frac{1}{2}} \bar{z}^{\frac{1}{2}} d^2z = 2\pi$$

Because : $\partial_z \bar{z} \ln |z|^2 = 2\pi \delta^c(z)$

Proof

If $z \neq 0$

$$\partial_z \left[\ln z + \ln \bar{z} \right] = 0 \quad \left\{ \delta^c(z) = 0 \right.$$

$$\text{LHS} \int d^2z \partial_z \ln |z|^2 =$$

$$\text{RHS} \int z^{\alpha} \bar{z}^{\beta} d^2z = 2\pi$$

Because : $\partial\bar{\partial} \ln|z|^2 = 2\pi \delta^2(z)$

Proof

If $z \neq 0$

$$\partial\bar{\partial} [\ln z + \ln \bar{z}] = 0 \quad \left\{ \delta^2(z) = 0 \right.$$

$$\text{LHS} \int d^2z \partial\bar{\partial} \ln|z|^2 = \int d^2z \partial_z \bar{\partial}^z$$

$$= \bar{\partial} \ln|z|^2$$

$$\text{RHS} \int \bar{\partial} \ln|z|^2 d^2z = 2\pi$$

Because : $\partial_z \bar{\partial} \ln |z|^2 = 2\pi \delta^c(z)$

Proof

If $z \neq 0$

$$\partial_z \left[\ln z + \ln \bar{z} \right] = 0 \quad \left\{ \delta^c(z) = 0 \right.$$

$$\text{LHS} \int d^2z \partial_z \bar{\partial} \ln |z|^2 = \int d^2z \partial_z V^z$$

$$V^z = \bar{\partial} \ln |z|^2 = \bar{\partial} [\ln z + \ln \bar{z}] \\ = \frac{1}{z}$$

$$\text{RHS} \int z^{-1} \bar{z} - d^2z = 2\pi$$

Because : $\partial \bar{\partial} \ln |z|^2 = 2\pi \delta^2(z)$

Proof

If $z \neq 0$

$$\partial \bar{\partial} [h_z + h_{\bar{z}}] = 0 \quad \left\{ \delta^2(z) = 0 \right.$$

$$\text{LHS} \int_{|z|=1} d^2z \partial \bar{\partial} \ln |z|^2 = \int_{|z|=1} d^2z h_z \circ \bar{z} \stackrel{\text{defn}}{=} \oint_{|z|=1} 0^2 dz$$

$$\text{RHS} \int_{|z|=1} z \bar{z} \delta^2 - dz = 2\pi$$

$$= \frac{1}{2}$$

Because : $\partial_z \bar{\partial} \ln |z|^2 = 2\pi \delta^c(z)$

Proof

If $z \neq 0$

$$\partial_z \left[\ln z + \ln \bar{z} \right] = 0 \quad \left\{ \delta^c(z) = 0 \right.$$

$\int d^2z \partial_z \ln |z|^2 = \int d^2z \partial_z \ln z + \int_{|z|=1} d^2z \partial_z \ln \bar{z}$

$$\text{LHS} \int_{|z|=1} d^2z \partial_z \ln |z|^2 = \int_{|z|=1} d^2z \partial_z \ln z = \int_{|z|=1} d^2z \delta^c(z) = \frac{1}{2\pi} \int_{|z|=1} \frac{d^2z}{z} = \frac{1}{2\pi} \int_{|z|=1} \frac{d^2z}{z} = \frac{1}{2\pi} \int_{|z|=1} \frac{d^2z}{z} = \frac{1}{2\pi}$$
$$\text{RHS} \int d^2z \partial_z \ln |z|^2 = 2\pi$$

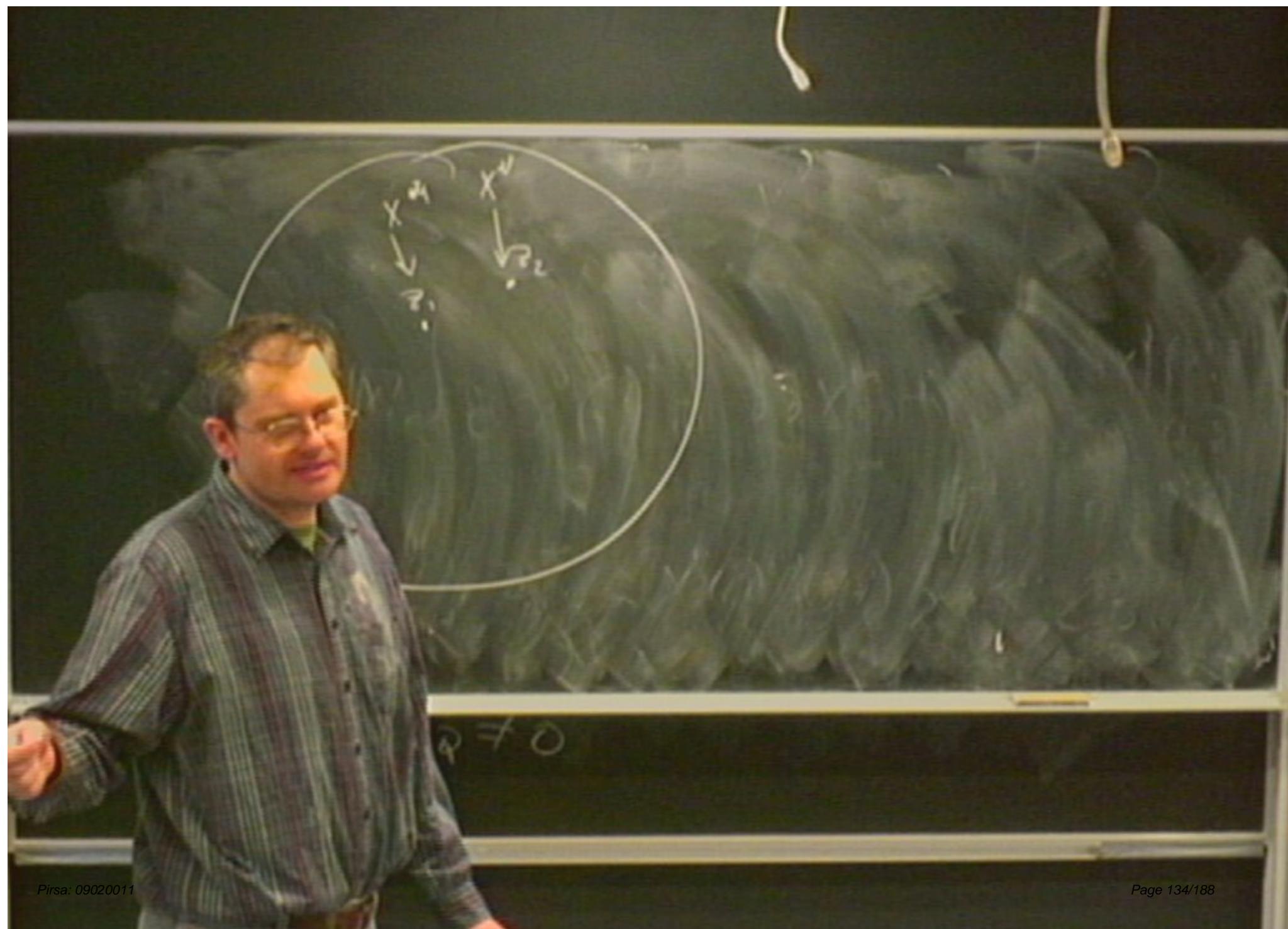
$$\begin{aligned}
 \underline{\underline{O}} &= \int [dx] \frac{\delta}{\delta x^m} \left[e^{-S} \right] = - \int [dx] e^{-S} \frac{\delta S}{\delta x^m} \\
 &\quad T = 0^\circ C \quad z \neq z' \\
 &= - \int [dx] e^{-S} \left(\frac{e}{2\pi\hbar} \partial \bar{x}^m \right) = \frac{1}{\pi\hbar} \langle \partial \bar{x}^m \rangle \\
 \langle \bar{x}^m \rangle_T &= 0 \quad \text{as an operator statement}
 \end{aligned}$$

Def

$$:X^m: = X^m$$

$$:X^m X': = X^m X' + \frac{\alpha'}{2} \eta^{(m)} h |z_2|^2$$

$$\partial_1 \bar{\partial}_1 :X^m X': = -\pi \omega' \eta^{(m)} \delta(z_2) + \frac{\alpha'}{2} \eta^{(m)} \partial_1 \bar{\partial}_1 h |z_2|^2 = 0$$





$$l_Q \neq 0$$



$\langle A:$

$A_1(z_n, \bar{z}_n) \rangle = 1$

$$\langle A_i(z_1, \bar{z}_1) \cdots A_j(z_n, \bar{z}_n) \rangle = 0$$

↑
correlation functions are observables in CFT

$$\langle A_i(z_1, \bar{z}_1) \cdots A_j(z_n, \bar{z}_n) \rangle = 0$$

\uparrow

correlation functions are observables in CFT

$z_{ij} \rightarrow 0$

$$\langle A_i(z_1, \bar{z}_1) \cdots A_j(z_n, \bar{z}_n) \rangle = 0$$

↑
correlation functions are observables in CFT

$z_{ij} \rightarrow 0$
interested in singularities

Def

$$: X^m : \rightarrow X^m$$

$$: X^m X': = X^m X' + \frac{\omega'}{2} \eta^{(m)} \bar{h} |z_{,2}|^2$$

$$i\bar{\partial}_1 : X^m X': = -i\omega' \eta^{(m)} \delta(z_{,2}) + \frac{\omega'}{2} \eta^{(m)} \bar{\partial}_1 \bar{\partial}_1 \bar{h} |z_{,2}|^2 = 0$$

DPE,

$$A_i(\mathcal{G}_1) \quad A_j(\mathcal{G}_2) =$$

DPE:

$$\langle A_i(\sigma_1) | A_j(\sigma_2) \rangle = \sum_k c_{ij}^k (\sigma_1 - \sigma_2) \langle A_k(\sigma_1) |$$



DPE:

$$\langle A_1(\sigma_1) A_2(\sigma_2) \dots \rangle \rightarrow \sum_k c_{ij}^k (\sigma_1 - \sigma_2) \langle A_k(\sigma) \dots \rangle$$

DPE:

$$\langle A_1(\sigma_1) A_2(\sigma_2) \dots \rangle \rightarrow \sum_k c_{ij}^k (\sigma_1 - \sigma_2) \langle A_k(\sigma) \dots \rangle$$

Typically:

DPE:

$$\langle A_i(\sigma_1) A_j(\sigma_2) \dots \rangle \rightarrow \sum_k c_{ij}^k (\sigma_1 - \sigma_2) \langle A_k(r) \dots \rangle$$

Typically, $c_{ij}^k \propto (r_{12})^k$

DFT:

$$\langle A_i(\sigma_1) A_j(\sigma_2) \dots \rangle \rightarrow \sum_k c_{i,j}^k (\sigma_1 - \sigma_2) \langle A_k(\sigma_1) \dots \rangle$$

Typically, $c_{i,j}^k \sim (r_{1,2})^k$

In normal QFT

DPT:

$$\langle A_i(\sigma_1) | A_j(\sigma_2) \dots \rangle \rightarrow \sum_k c_{ij}^k (\sigma_1 - \sigma_2) \langle A_k(r_1) \dots \rangle$$

Typically $c_{ij}^k \sim (r_{12})^k$

In normal QFT

radius of convergence is 0

Typically, $\phi_i \propto (r_{12})^k$

In normal QFT

In CFT's DPFs have finite radius of convergence

convergence

$$\partial_1 \bar{\partial}_1 : X^\mu X_\mu : = -\pi \alpha' \gamma^{(m)} \delta'(z_{12}) + \frac{\alpha'}{2} \gamma^{(m)} \partial_1 \bar{\partial}_1 \ln |z_{12}|^2 = 0$$

Typically, $\phi_i \propto (r_{12})^k$ | In normal QFT

In CFT's OPEs have finite radius of convergence $\neq 0$

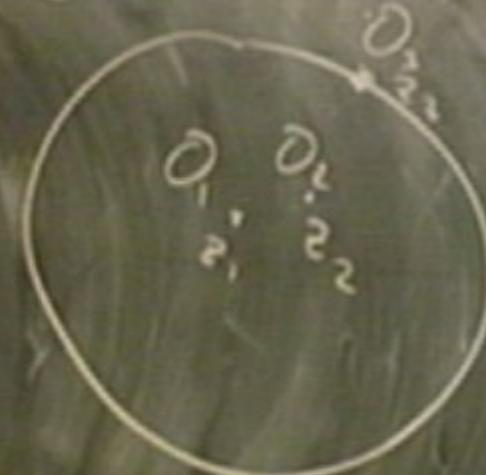
$$O_1(z_1, \bar{z}_1) O_2(z_2, \bar{z}_2) = \frac{\text{convergence!}}{z_{12}^{d-2}}$$

$$\partial_1 \bar{\partial}_1 : X^\mu X_\mu : = -\pi \alpha' \eta^{\mu\nu} \delta(z_{12}) + \frac{\alpha'}{2} \eta^{\mu\nu} \partial_1 \bar{\partial}_1 \ln |z_{12}|^2 = 0$$

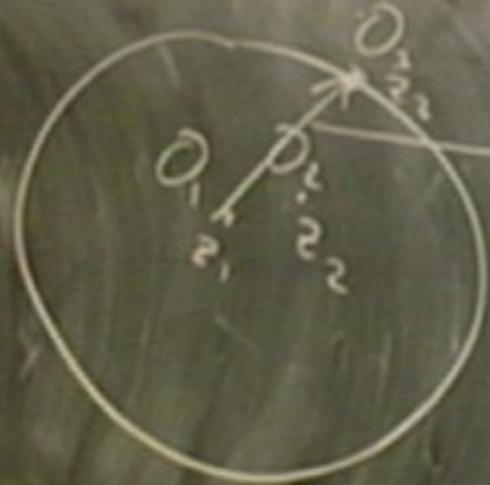
$$\sum C_{1,2}^{(k)} \mathcal{O}_{1,k}(z_2)$$

E

$$\sum_k c_{k,\ell}^{\alpha} O_{k,\ell}(z_2)$$



$$\sum_k c_k z^k$$



radius of convergence

$\oint_{\Gamma} O_1 O_2 \dots O_n$



$$\subset A_1(G_1) \subset A_2(G_2) \cdots \supset \supset C_{ij}^k(\sigma_i = \sigma_j) \subset A_r(F_r)$$

Typically, $\Omega_{ij}^k \approx (F_{ij})^k$

In CFT's OPEs have

$$O_1(z, \bar{z}) O_2(z_2, \bar{z}_2) = \frac{C_{O_1 O_2}}{|z - z_2|^{\Delta_1 + \Delta_2}}$$

In normal QFT

Radius of convergence \gg radius of



$$\langle A_1(\sigma_1) A_2(\sigma_2) \dots B \rangle \left(\sum_k \zeta^k (\sigma_1 - \sigma_2) \langle A_k(r_1) B \rangle \right)$$

Typically, $\zeta \propto (r_{12})^{-k}$ In normal QFT

In CFT's DPFs have finite radius of convergence $\neq 0$

$$O_1(z_1, \bar{z}_1) O_2(z_2, \bar{z}_2) = \frac{\text{convergence}}{z_1 - z_2}$$

$$\text{def} : X^{(1)} X^{(2)} = \circ$$

$$\text{d}.\bar{\delta}_1 : X^{(0)} X^{(2)} \dot{x}_2 = 0$$

$$5 \left[\begin{array}{l} \end{array} \right]$$



$$\bar{\partial}_1 : X^0 \times^{T_2} \cdot = 0$$

$$\bar{\partial} \left[\bar{\partial}_1 : X^m \times^0 \cdot \right] = 0$$

$$\bar{\partial}_1 \left[\cdot : \partial_1 X^m \times^0 \cdot \right] = 0$$



$$\partial_z \bar{\partial}_z : X^0(X^0) \rightarrow = 0$$

$$\textcircled{2}, \left[\partial_z : X^0(X^0) \right] = 0$$

$$\textcircled{3}, \left[: \partial_z X^0(X^0) \right] = 0$$

$\therefore \partial_z X^0(z, \bar{z}) \quad X^0(z, \bar{z})$ is a holomorphic function.

$$\textcircled{2} [\partial_1 X^a(x) \partial_1 X^b(x')] = 0$$

$\partial_1 X^a(z_1, \bar{z}_1) \partial_1 X^b(z_2, \bar{z}_2)$ is a holomorphic function.

$$[\partial_3 X^a(z_1 + \xi, \bar{z}_1 + \bar{\xi}) \partial_1 X^b(z_2, \bar{z}_2)]$$

$$\xi = z_{12}$$

correlation functions are observables in CFT
 $z_{ij} \rightarrow 0$
interested in singularities

$$\textcircled{2} [: \partial_i X^m(z) \partial_j X^n(z') :] = 0$$

$: \partial_i X^m(z_1, \bar{z}_1) \partial_j X^n(z_2, \bar{z}_2) :$ is a holomorphic function

$$: \partial_i X^m(z_1 + \xi, \bar{z}_1 + \bar{\xi}) \partial_j X^n(z_2, \bar{z}_2) : = \sum_{k=1}^{\infty} \frac{\xi^k}{k!} \cdot \delta_{ij} \partial^k X^m$$

$\tau = z_1, z_2$

correlation functions are observables in CFT

$$z_{ij} \rightarrow 0$$

interested in singularities

$$X(z_1, \bar{z}_1) X'(z_2, \bar{z}_2) : \dots : \text{is a holomorphic function}$$

$$\delta X(z_1 + \xi, \bar{z}_2 + \bar{\xi}) X'(z_2, \bar{z}_2) : = \sum_{k=1}^{\infty} \frac{\xi^k}{k!} : X' \partial^k X :$$

$\tau = z_{12}$

$$= \sum_{k=1}^{\infty} \frac{z_{12}^k}{k!} : X'(z_1, \bar{z}_1) \partial^k X(z_2, \bar{z}_2) :$$



$$\begin{aligned} & : X(z_1, \bar{z}_1) X^j(z_2, \bar{z}_2) : \text{ is a holomorphic function} \\ & : \partial_z X(z_1 + \xi, \bar{z}_2 + \bar{\xi}) X^j(z_2, \bar{z}_2) : = \sum_{k=1}^{\infty} \frac{\xi^k}{k!} : X'(z_1, \bar{z}_2) X^j(z_2, \bar{z}_2) : \\ & \quad ? = z_{12} \end{aligned}$$

$$= \sum_{k=1}^{\infty} \frac{z_{12}^k}{k!} : X'(z_1, \bar{z}_2) \partial^k X^j(z_2, \bar{z}_2) :$$

$$: X(z_1, \bar{z}_1) X^j(z_2, \bar{z}_2) : = : X(z_2, \bar{z}_2) X^j(z_1, \bar{z}_1) :$$

$$\textcircled{3} \quad [\partial_1 X^m(z) : \bar{\partial}_1 X^m(\bar{z})] = 0 \quad \left\{ \begin{array}{l} \gamma_1 [\partial_1 X^m(z) : \bar{\partial}_1 X^m(\bar{z})] = 0 \\ \gamma_2 [\partial_2 X^m(z) : \bar{\partial}_2 X^m(\bar{z})] = 0 \end{array} \right.$$

$\partial_1 X^m(z_1, \bar{z}_1) \bar{\partial}_1 X^m(\bar{z}_1, \bar{\bar{z}}_1)$ is a holomorphic function

$$[\partial_1 X^m(z_1 + \xi, \bar{z}_1 + \bar{\xi}) \bar{\partial}_1 X^m(\bar{z}_1, \bar{\bar{z}}_1)] = \sum_{k=1}^{\infty} \frac{\xi^k}{k!} \cdot X^m \partial^k X^m;$$

$\xi = z_1, \bar{z}_1$

$$= \sum_{k=1}^{\infty} \frac{\xi^k}{k!} : X^m(z_1, \bar{z}_1) \partial^k X^m(\bar{z}_1, \bar{\bar{z}}_1) :$$

$$: X^m(z_1, \bar{z}_1) \bar{\partial}^k X^m(\bar{z}_1, \bar{\bar{z}}_1) := : X^m(z_1, \bar{z}_1) X^m(\bar{z}_1, \bar{\bar{z}}_1) :$$

$$X(z_1, \bar{z}_1) X(z_2, \bar{z}_2) : = \text{if } n \text{ holomorphic functions}$$

$$\partial_z X(z_1 + \xi, \bar{z}_1 + \bar{\xi}) X(z_2, \bar{z}_2) : = \sum_{k=1}^{\infty} \frac{\xi^k}{k!} : X'(\bar{z}_1) \partial^k X :$$

$$\xi = z_{12}$$

$$= \sum_{k=1}^{\infty} \frac{z_{12}^k}{k!} : X'(z_1, \bar{z}_1) \partial^k X(z_2, \bar{z}_2) :$$

$$X(z_1, \bar{z}_1) X^j(z_2, \bar{z}_2) : = X(z_2, \bar{z}_2) X^j(z_1, \bar{z}_1) :$$

$$+ \sum_{k=1}^{\infty} \left\{ \frac{\bar{z}_{12}^k}{k!} : X^j \partial^k X : + \frac{\bar{z}_{12}^k}{k!} : X^j \bar{\partial}^k X : \right\}$$

$$O_1(z_1, \bar{z}_1) O_2(z_2, \bar{z}_2) = \sum c^k \mathcal{A}_k(z_1, \bar{z}_1)$$

A

1 - 2 - 1

$$O_1(z_1, \bar{z}_1) O_2(z_2, \bar{z}_2) = \sum c^k A_k(z_1, \bar{z}_1)$$

\downarrow
 $x^{(n)}$

\downarrow
 x'

\downarrow



$$O_1(z_1, \bar{z}_1) O_2(z_2, \bar{z}_2) = \sum c^k A_k(z_1, \bar{z}_1)$$

\Downarrow
 X^m

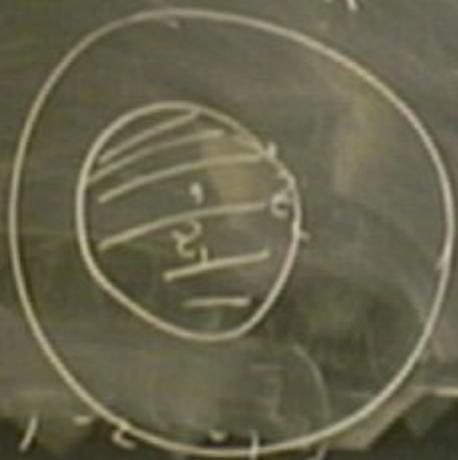
\Downarrow
 X'

\Updownarrow

$$\left\{ X^j \partial^k X'; X^j \bar{\partial}^k X' \right\}$$

$$O_1(z_1, \bar{z}_1) O_2(z_2, \bar{z}_2) = \sum c^k A_k(z_2, \bar{z}_2)$$

$$\downarrow \\ X^m$$



$$\frac{z_{12}^k}{k!}$$

$$\uparrow \\ \left\{ X' \partial^k X; X' \bar{\partial}^k X \right\}$$

$$: X^{m_1}(z_1, \bar{z}_1) \cdot X^{m_n}(z_n, \bar{z}_n) : =$$

$$X^{(m)}(z_1, \bar{z}_1) \cdots X^{(m)}(z_n, \bar{z}_n) = X^{(m)}(z, \bar{z}) + \sum$$

$$X^{(n)}(z_1, \bar{z}_1) \cdots X^{(n)}(z_n, \bar{z}_n) = X^{(1)}(z_1, \bar{z}_1) \cdots X^{(n)}(z_n, \bar{z}_n)$$

Subtraction =

+ $\sum_{\text{Subtractions}}$

$$X^{(n)}(z_1, \bar{z}_1) \cdots X^{(n)}(z_n, \bar{z}_n) = X^{(1)}(z_1, \bar{z}_1) \cdots X^{(n)}(z_n, \bar{z}_n)$$

Subtraction = \sum Subtractions

replace a pair of fields

$$\frac{1}{2} d' \eta^{ij} f_{ij} |z_j|^2$$

$$D = \int d[x] \frac{S}{\delta X_n(z_1, \bar{z}_1)} \left[e^{-S} X^{\mu_L}(z_1, \bar{z}_1) \cdots X^{\nu_n}(z_n, \bar{z}_n) \right]$$

$$D = \int d[X] \frac{\delta}{\delta X_{\alpha_1}(z_1, \bar{z}_1)} \left[e^{-S} \underbrace{X^{\alpha_1}(z_1, \bar{z}_1) \cdots X^{\alpha_n}(z_n, \bar{z}_n)}_{n-1 \text{ fields}} \right]$$

$$:X^{\alpha_1} X^{\alpha_2} \dots = X^{\alpha_1} X^{\alpha_2} + \frac{1}{2} \eta^{\alpha_1 \alpha_2} \ln |z_1|^2$$

$$X^{u_1}(z_1, \bar{z}_1) \cdot X^{u_n}(z_n, \bar{z}_n) = \underbrace{X^{u_1}(z_1, \bar{z}_1) \cdot X^{u_n}(z_n, \bar{z}_n)}_{+ \sum \text{Subtractions}}$$

Subtraction = replace a pair of fields

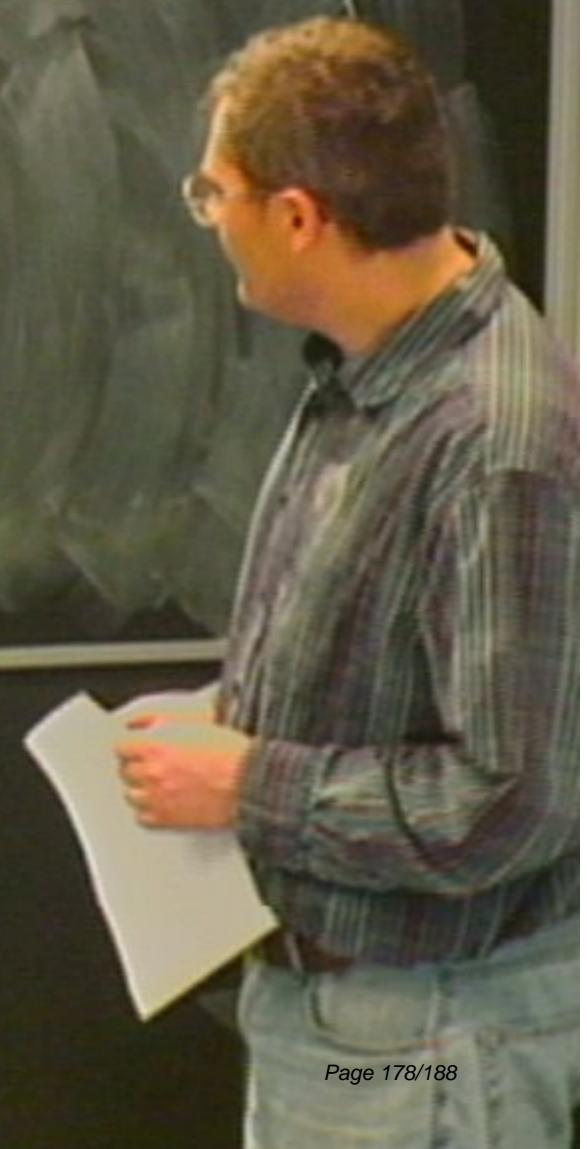
$$\frac{1}{2} d' \gamma^{u_i u_j} f_i |z_i|^2$$

$$\mathcal{T} = \mathcal{F}[x_1, \dots, x_n]$$



$$\mathcal{T} = \mathcal{F}[x_1, \dots, x_n]$$

$$:\mathcal{T}: = \exp [$$



$$\mathcal{T} = \mathcal{T}[x_1, \dots, x_n]$$

$$:\mathcal{T}: = \exp \left[\frac{g'}{4} \int d^2 z_1 d^2 z_2 \, h |z_{12}|^2 \sum_{X''(z, \bar{z}_1)} \frac{\delta}{X''(z, \bar{z}_1)} \frac{\delta}{X_u(z_1, \bar{z}_2)} \right]$$

$$\mathcal{T} = \mathcal{T}[x_1, \dots, x_n]$$

$$:\mathcal{T}: = \exp \left[\frac{d}{\hbar} \int d^2 z_1 d^2 \bar{z}_1 \, \, h(z_{12}) \frac{\sum}{z \chi''(z, \bar{z}_1)} \frac{\sum}{X_u(z, \bar{z}_1)} \right]$$

$$\hat{F} = \exp \left[-\frac{\lambda'}{4} \cdot d^2 z_1 d^2 z_2 \sum_{\delta X_{i(1)}} \sum_{\delta X_{k(2)}} \right] : \hat{f} :$$



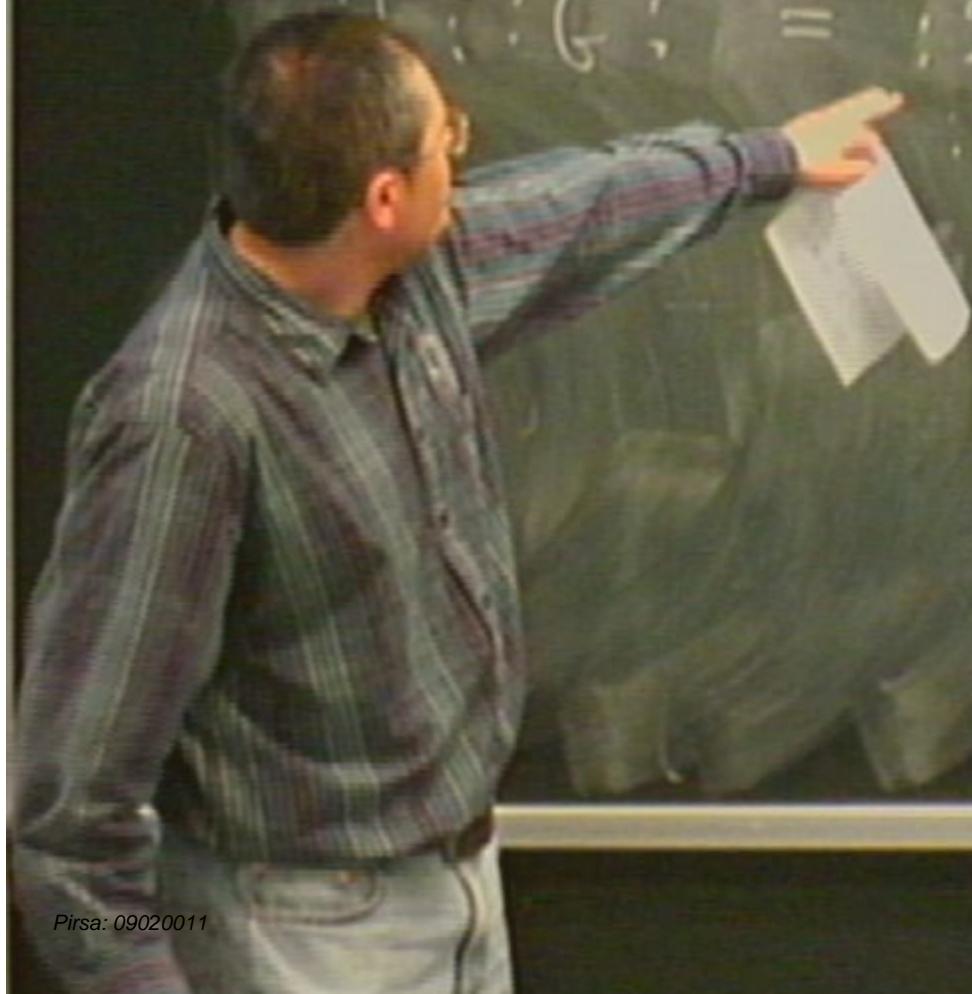
$$\mathcal{F} = \exp \left[-\frac{\lambda'}{4} |\vec{z}_1|^2 z_1 \bar{z}_2 - \sum_{\delta X_{(1)}} \sum_{\delta X_{(2)}} \right] : \mathcal{F} :$$

$$= : \tilde{f} : + \sum \text{contractions}$$

$$\text{contraction} = -\frac{1}{2} \lambda' \rho^{(m)} \ln |z_2|^2$$

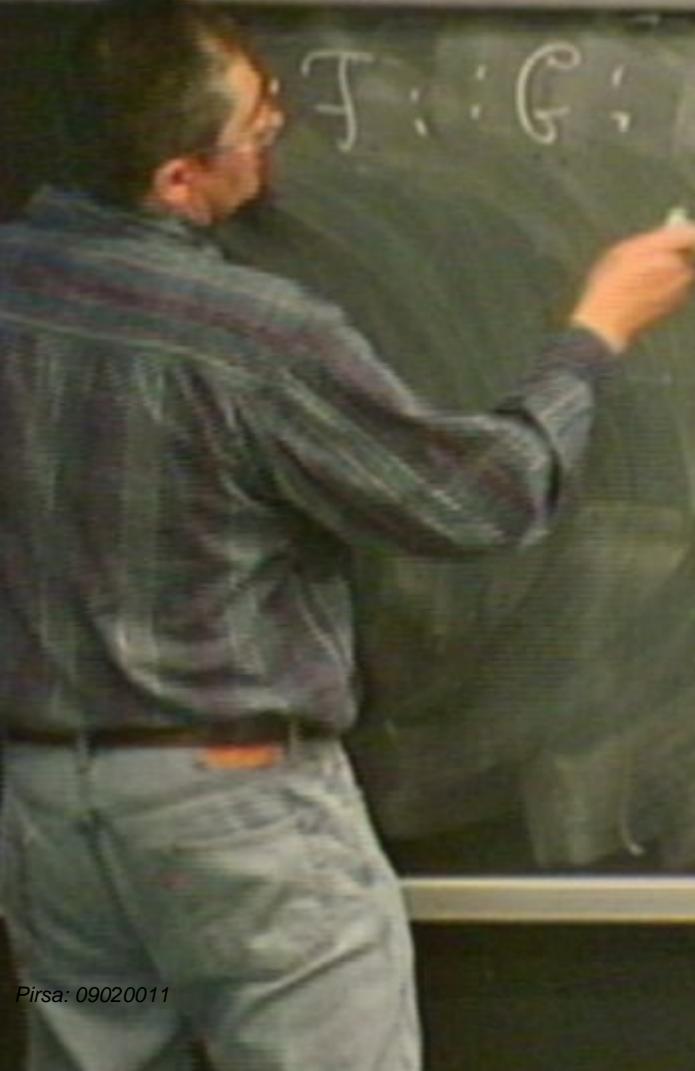
contraction = $\bar{z}^2 K \cdot k_n |\xi_2|^2$

$$T \cdot G = FG$$



$$\text{contraction} = -\bar{z}^2 \lambda \cdot k_n |z_{12}|^2$$

$$[T, [G]] = [TG] + \sum \text{cross-contractions}$$



$$\text{contraction} = -\bar{e}^2 \lambda K \cdot \ell_n |\zeta_2|^2$$

$$:\mathcal{F} : :\mathcal{G}: = :\mathcal{F}\mathcal{G}: + \sum \text{cross-contractions}$$

$$:\mathcal{F} : :\mathcal{G}: = \exp \left[-\frac{\omega'}{2} \int d^2 z_1 d^2 z_2 \ell_n |\zeta_{12}|^2 \right]$$

$$\text{contraction} = - \bar{e}^2 \lambda K \cdot k_1 |k_2|^2$$

$$:\mathcal{T} : \mathcal{G} : = :\mathcal{T}\mathcal{G}: + \sum \text{cross-contractions}$$

$$:\mathcal{T} : \mathcal{G} : = \exp \left[- \frac{d}{2} \int d^2 z_1 d^2 z_2 \ell_1 |k_2| \left(\frac{\delta}{\delta X_F(z_1, \bar{z}_1)} \right) \delta \right]$$

$$\text{contraction} = -\bar{e}^2 \lambda \cdot b_n |\zeta_2|^2$$

$$G = FG + \sum \text{cross-contractions}$$

$$G = \exp \left[-\frac{i}{2} \int d^2 z_1 d^2 z_2 b_n |\zeta_2|^2 \sum_{F} \frac{\delta}{\delta X_F(z_1, \bar{z}_1)} \frac{\delta}{\delta X_F(z_2, \bar{z}_2)} \right] \times FG$$



$$\text{contraction} = -\bar{e}^2 K \cdot k_n |z_{12}|^2$$

$$:\mathcal{T} : :\mathcal{G}: = :\mathcal{T}\mathcal{G}: + \sum \text{cross-contractions}$$

$$:\mathcal{T} : :\mathcal{G}: = \exp \left[- \frac{d'}{2} \int d^2 z_1 d^2 z_2 \ell_n |z_{12}|^2 \sum_F \frac{\delta}{\delta X_F^{(1, \bar{z}_1)}} \frac{\delta}{\delta X_G^{(2, \bar{z}_2)}} \right] \times :\mathcal{T}\mathcal{G}:$$