Title: Topos formulation of Consistent Histories

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Abstract: In this talk I will describe a topos formulation of consistent histories obtained using the topos reformulation of standard quantum mechanics put forward by Doering and Isham. Such a reformulation leads to a novel type of logic with which to represent propositions. In the first part of the talk I will introduce the topos reformulation of quantum mechanics. I will then explain how such a reformulation can be extended so as to include temporally-ordered collection of propositions as opposed to single time propositions. Finally I will show how such an extension will lead to the possibility of assigning truth values to temporal propositions.

1. INTRODUCTION

1.1 Why a History Theory?

The standard Copenhagen interpretation cannot describe closed systems, since the existence of an external observer is required.

This causes problems in any theory of quantum cosmology.

One significant recent attempt to deal with closed systems in quantum mechanics is consistent-history theory:

– *Omnes, Griffiths, Gell'mann & Hartle*: Histories as products of projection operators (therefore *not* projectors).

–*HPO formalism of consistent histories*: History propositions are identified with projection operators in bigger Hilbert space.

In this talk I will describe a new type of history quantum theory: a topos version of the temporal-logic part of the HPO formalism.

1.2 Why a topos version of the temporal logic part of the HPO formalism?

- A topos reformulation of quantum mechanics was put forward by Isham and Döring.

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  An essential ingredient is a mapping of projection operators to certain objects in a topos.

- My aim is to extend this formulation to a *history* version of quantum theory, therefore my starting point is the HPO formalism.

What is Topos Theory?

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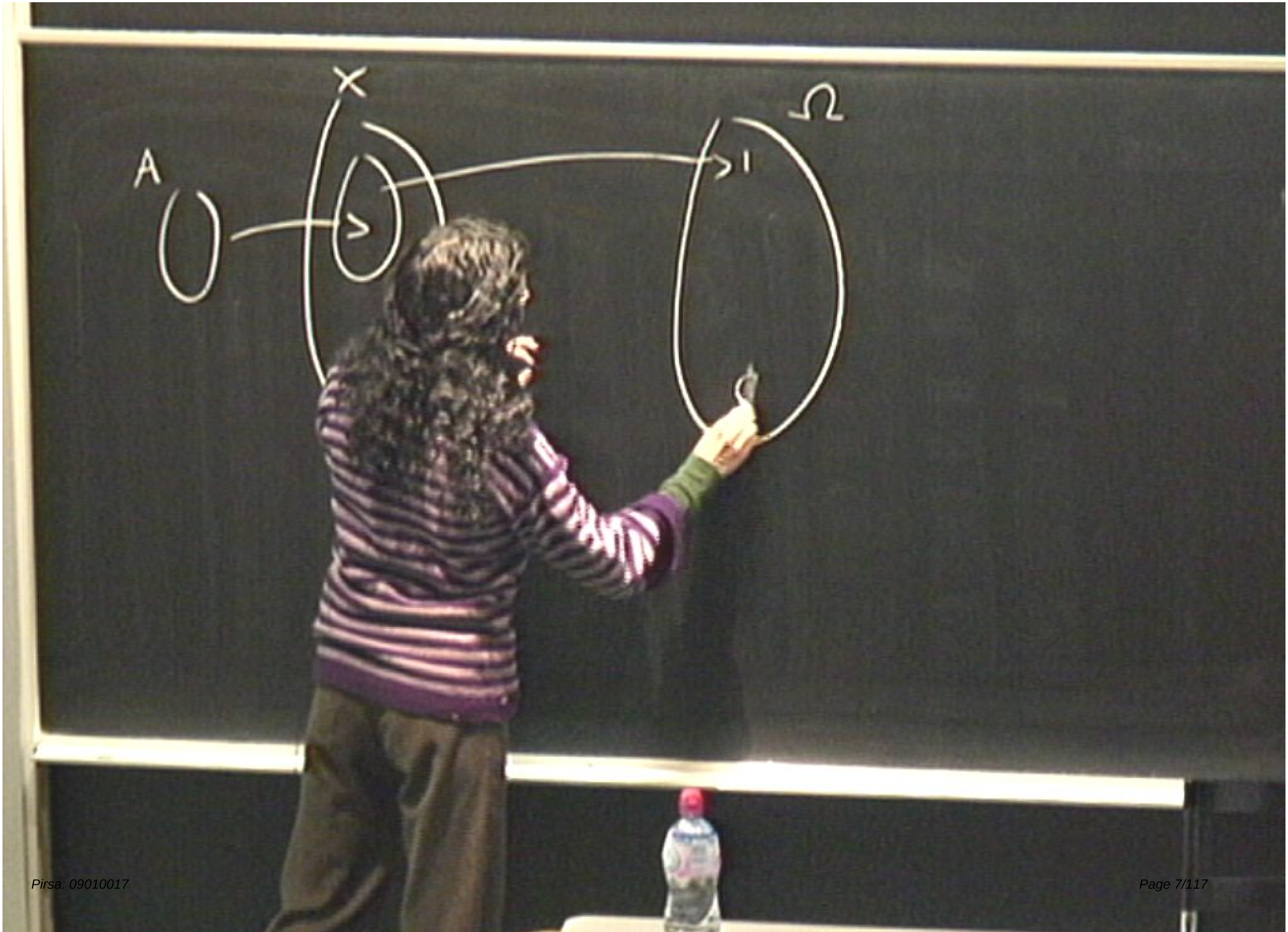
- A topos is a category which is similar to *Sets*: fundamental mathematical properties (disjoint union, Cartesian product, etc) have a topos analogue. In particular

  – **Sub-object classifier** Ω: Generalises the set $\{0, 1\}$ of truth-values in the category *Sets*.

  – Collection of all sub-objects of any object forms a **Heyting algebra**:

  A distributive algebra for which $S \vee \neg S \leq 1$. An internal logic, analogue to Boolean algebra in *Sets*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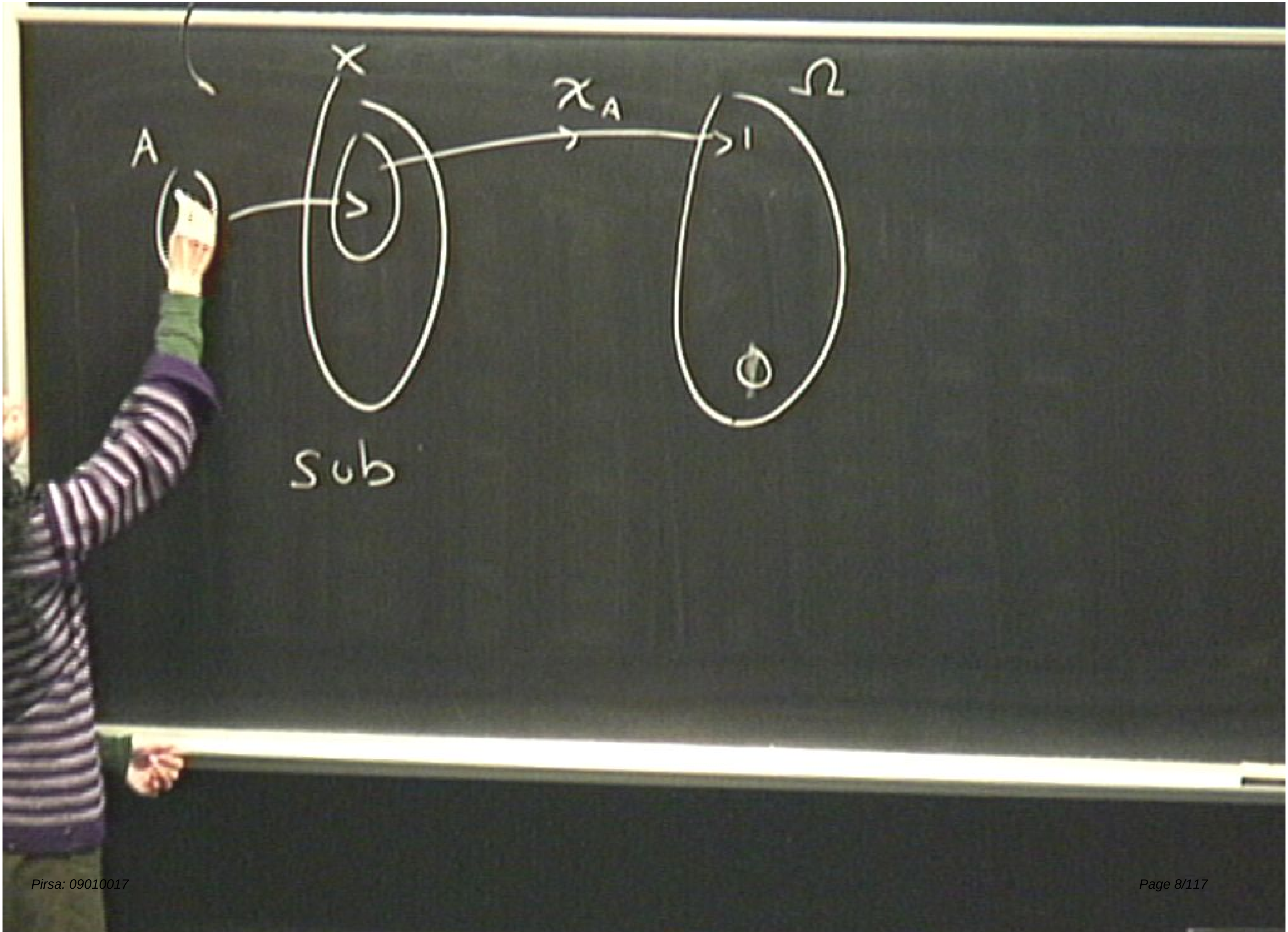
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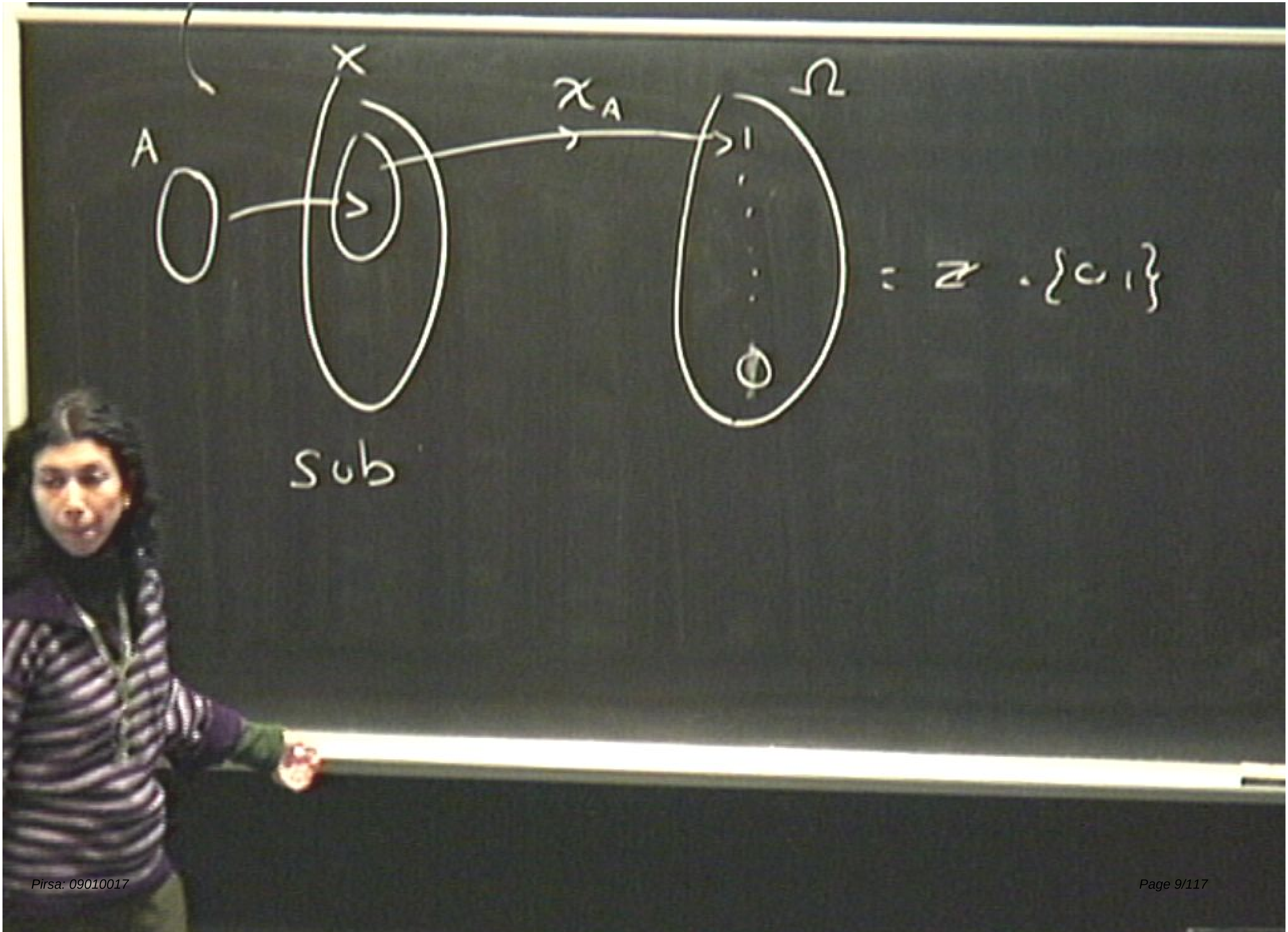
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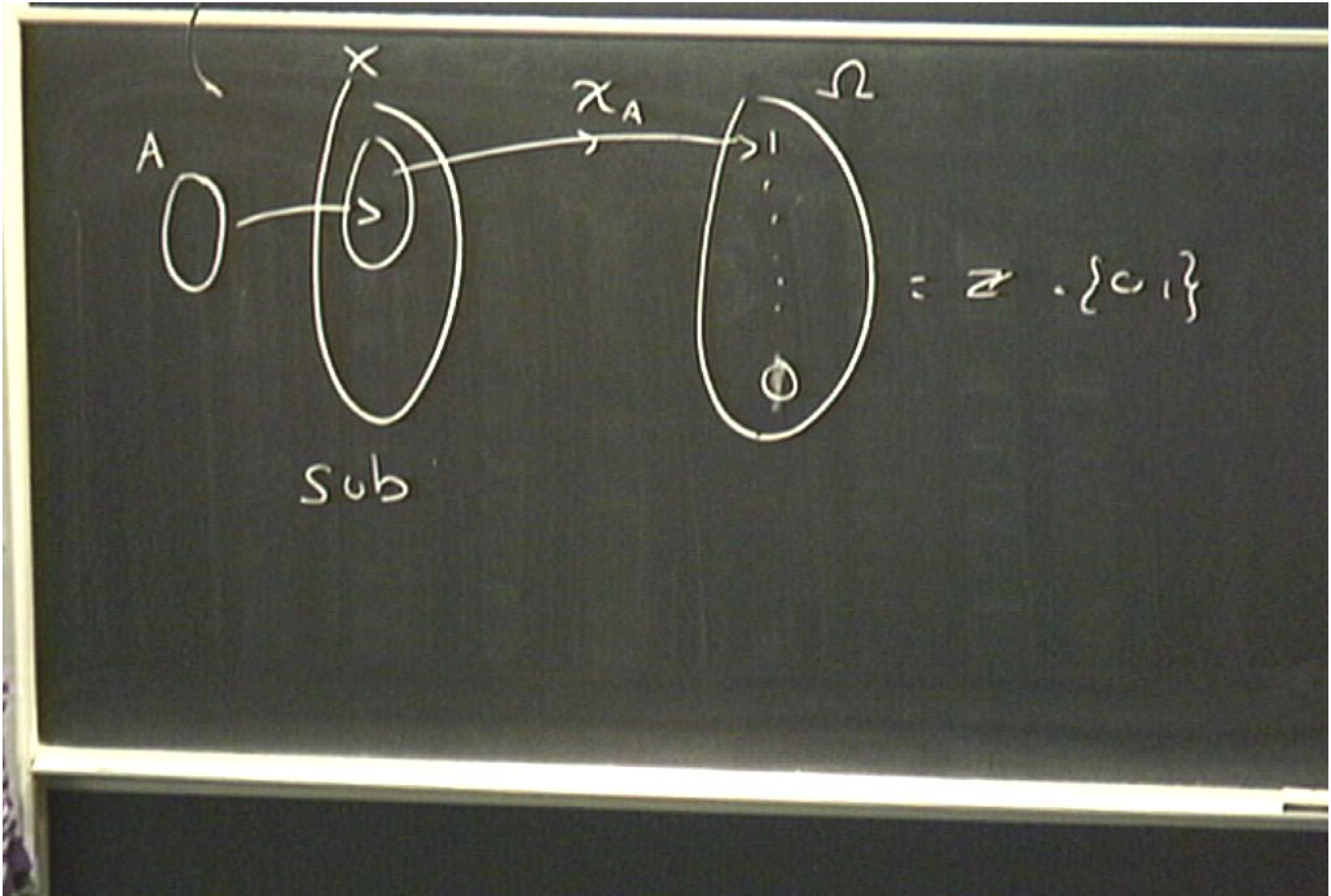
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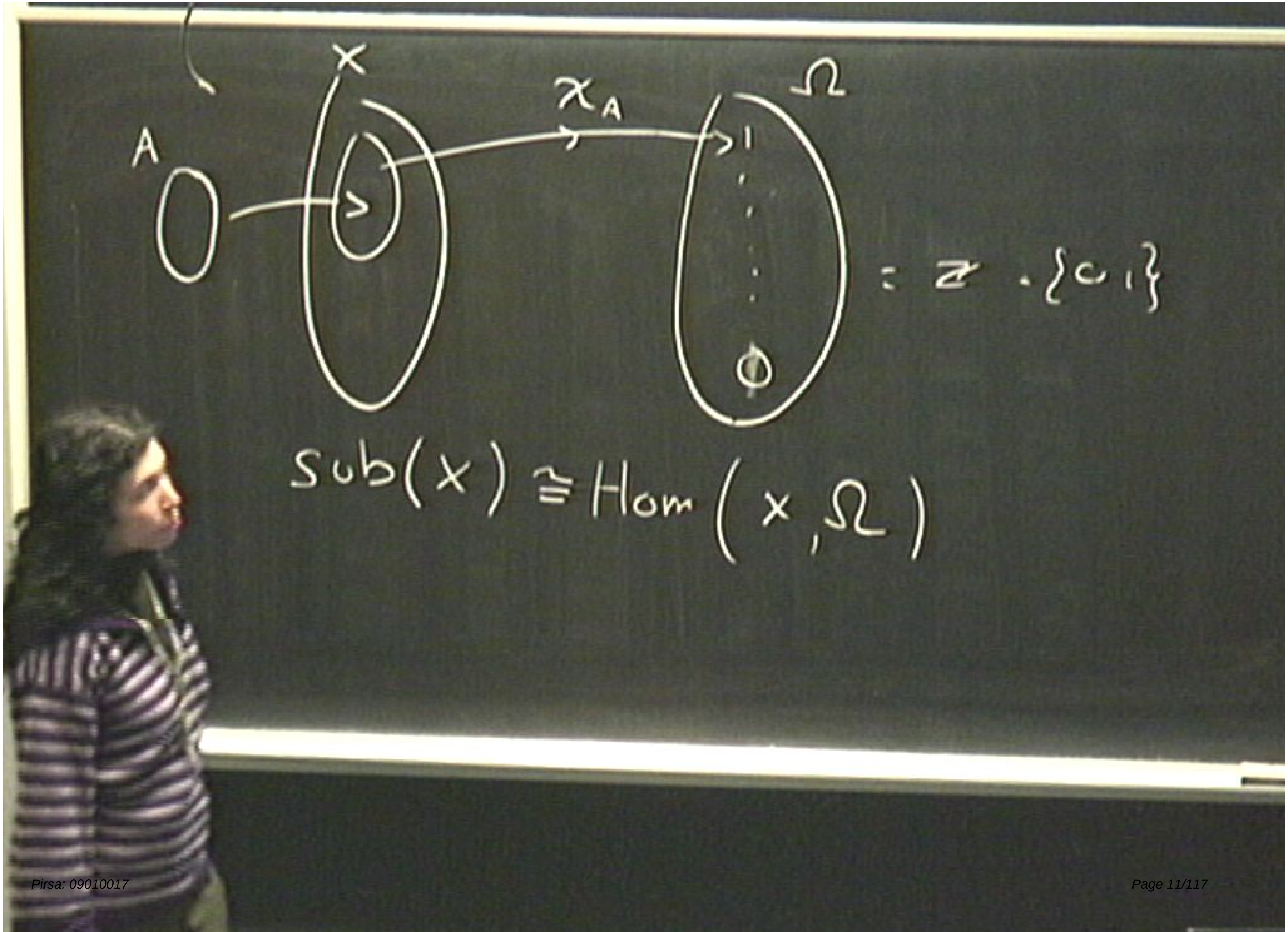
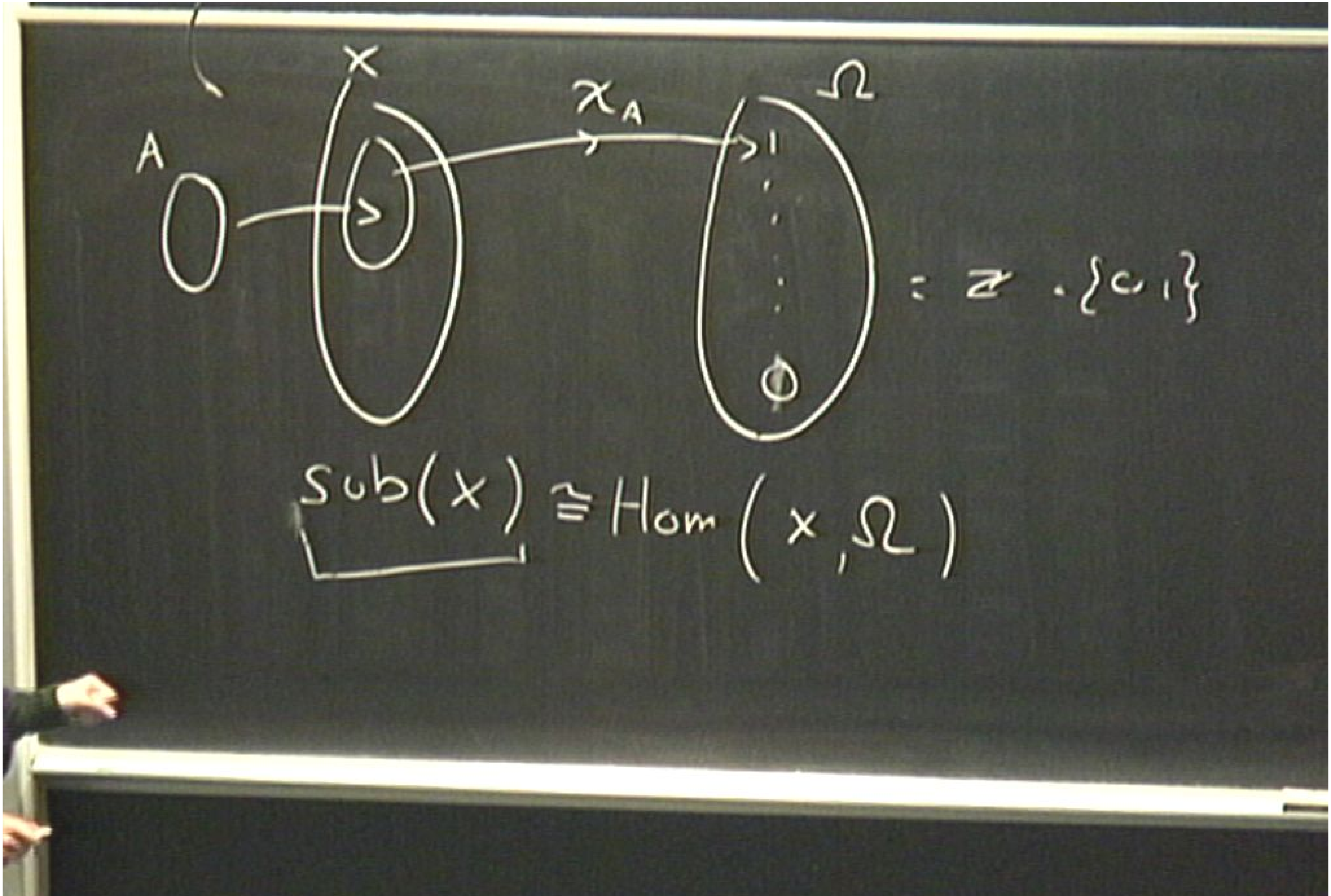
= 2 = {0, 1}
Sub


A
X
χA
Ω
1
0
{0,1}
Sub(X) ≅ Hom(X, Ω)


X
A
χ_A
Ω
1
0
$= 2 = \{0, 1\}$
$\mathrm{Sub}(X) \cong \mathrm{Hom}(X, \Omega)$

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  - **Sub-object classifier** Ω: Generalises the set $\{0, 1\}$ of truth-values in the category *Sets*.
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2.2 Why Topos Theory?

- Kocken-Specher theorem $\Longrightarrow$ non-realist interpretation of quantum theory

  For quantum cosmology, need a reformulation of quantum theory which is 'more realist'. Isham, Butterfield & Döring: can be done through *topos theory*.

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- Reformulate quantum theory to make it 'look like' classical physics:
  - Classical physics uses *Sets* as its mathematical structure.

    A topos is a category which 'looks like' *Sets*.
  - Logic of subsets in *Sets* is Boolean logic.

    Logic of subsets in a topos is a distributive logic

2.3 Which Topos?

- Need for *contexts* comes from K-S theorem: only within *abelian subalgebras* of $\mathcal{B}(\mathcal{H})$ can quantum theory 'look like' classical theory. *Contexts* form 'classical snapshots'.
  - The set of abelian subalgebras, $\mathcal{V}(\mathcal{H})$, forms a category under subset inclusion: $i_{V'V} : V' \subseteq V$

    i.e. consider all contexts at the same time!
  - **Example**: $V'' = V \cap V' \neq \emptyset$ then $\exists$ the inclusion maps $i_{V''V}$ and $i_{V''V'}$, therefore it is possible to 'relate' V and V'.

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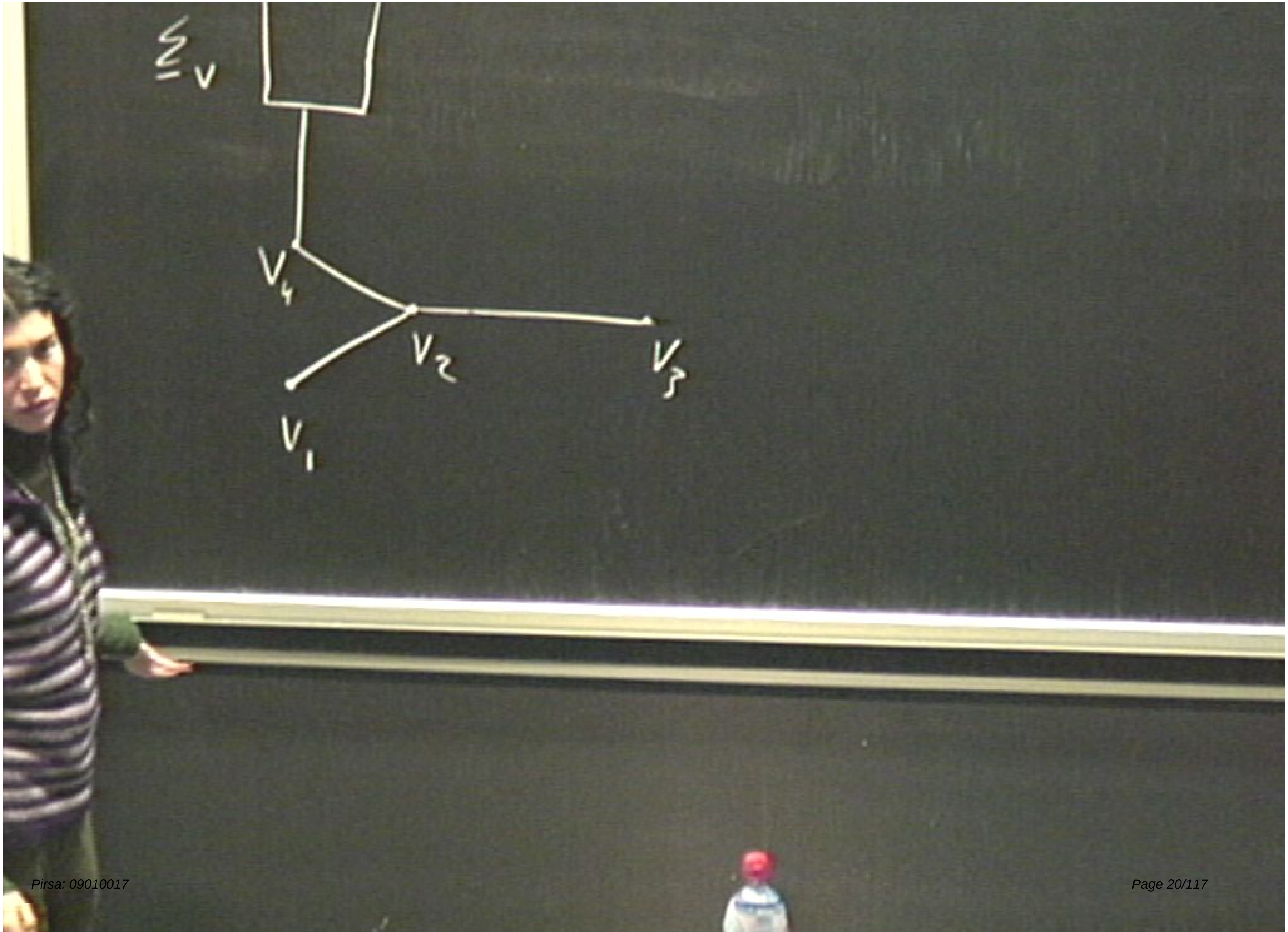
- Topos of presheaves over the category of abelian subalgebras : $Sets^{\mathcal{V}(\mathcal{H})^{op}}$.

2.4 Topos of Presheaves

Let $\mathcal{C}, \mathcal{D}$ be categories. Then a presheaf is an assignment to each $\mathcal{D}$-object A of a $\mathcal{C}$-object $X(A)$, and to each $\mathcal{D}$-arrow $f : A \to B$ a $\mathcal{C}$-arrow $X(f) : X(B) \to X(A)$ such that:

- $X(1_A) = 1_{X(A)}$

- $X(f \circ g) = X(g) \circ X(f)$ for any $g : C \to A$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V_4
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V_3
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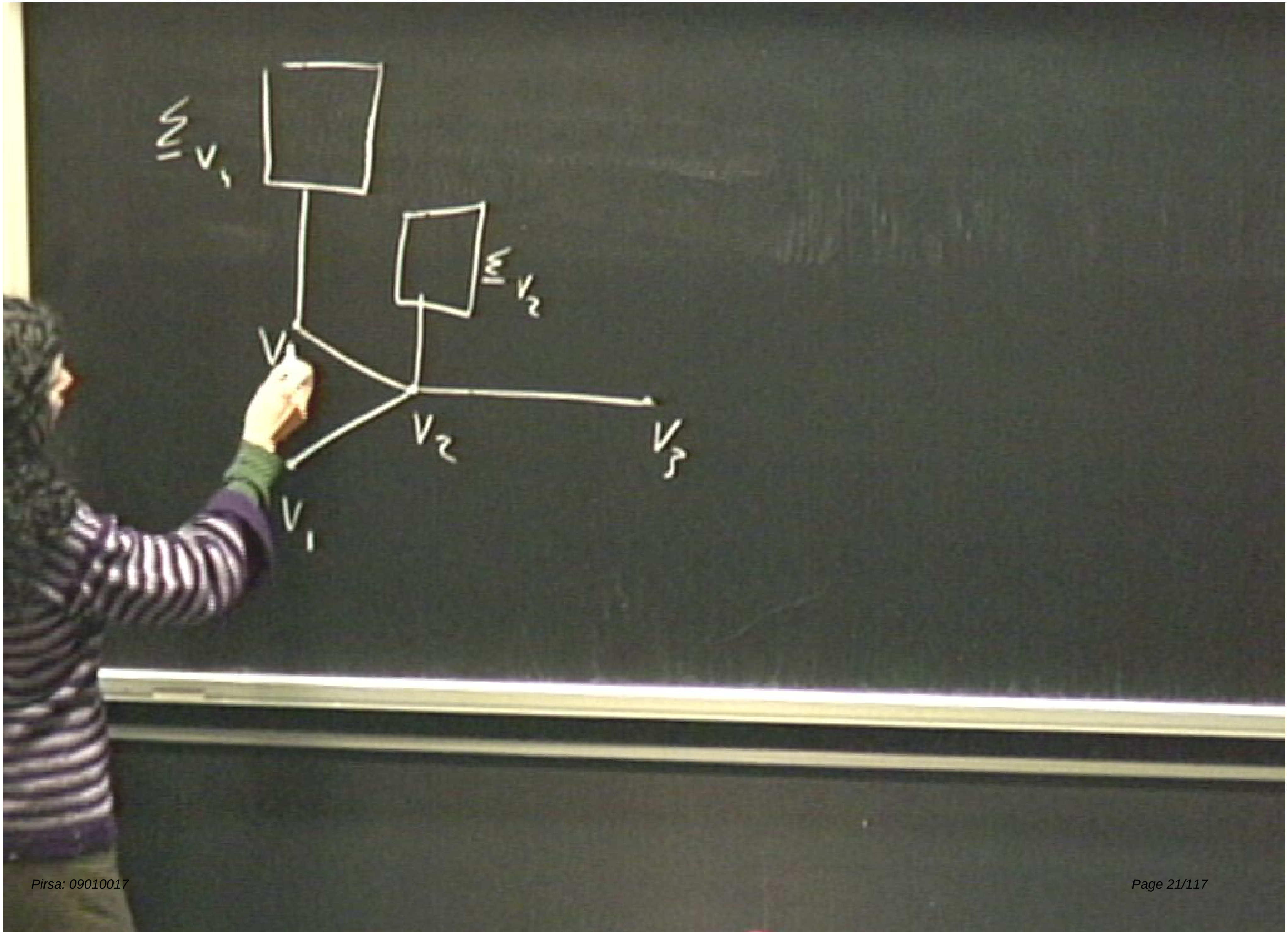
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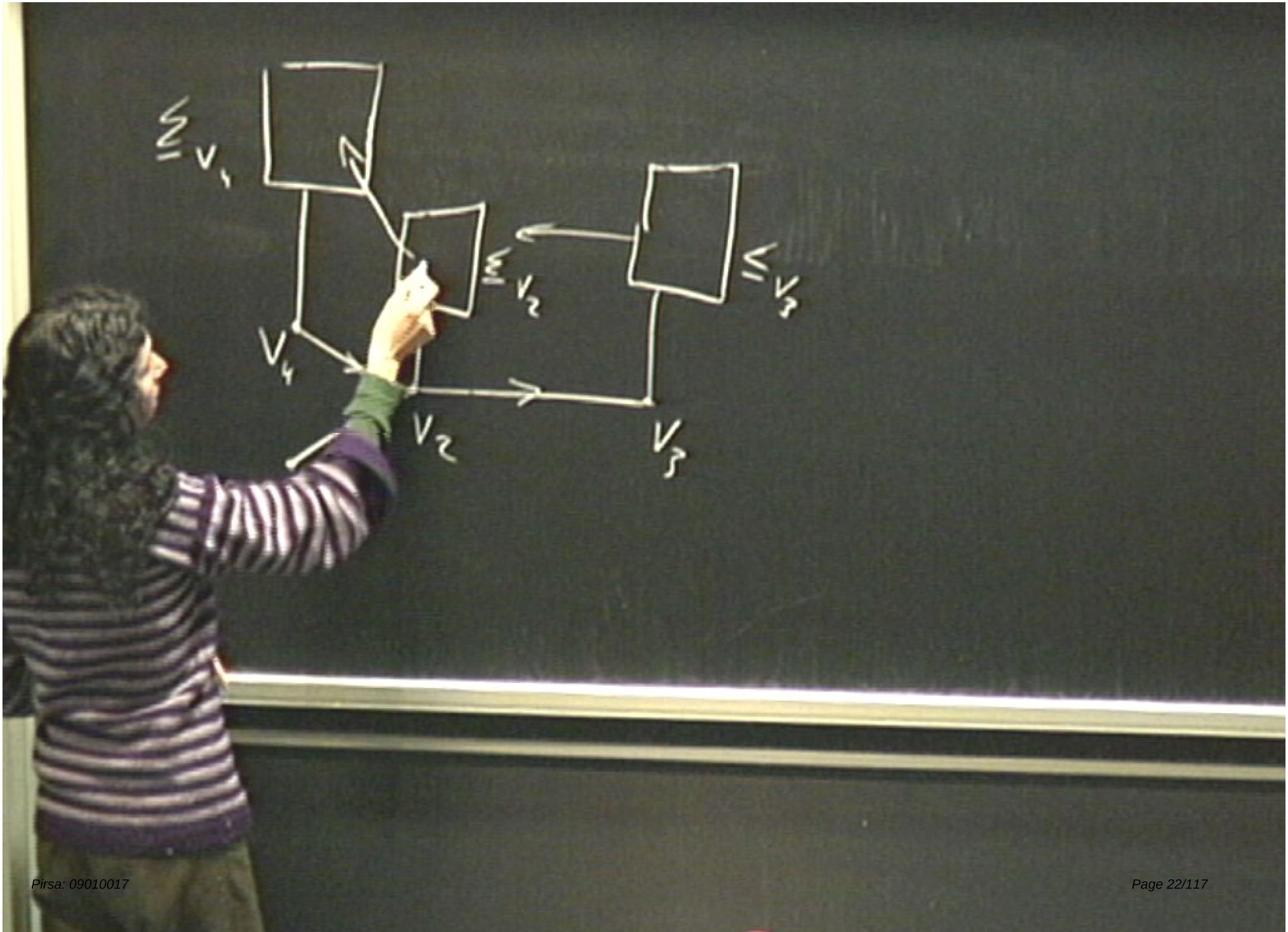
V1
V2
V3
V1


≤ v1
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V4
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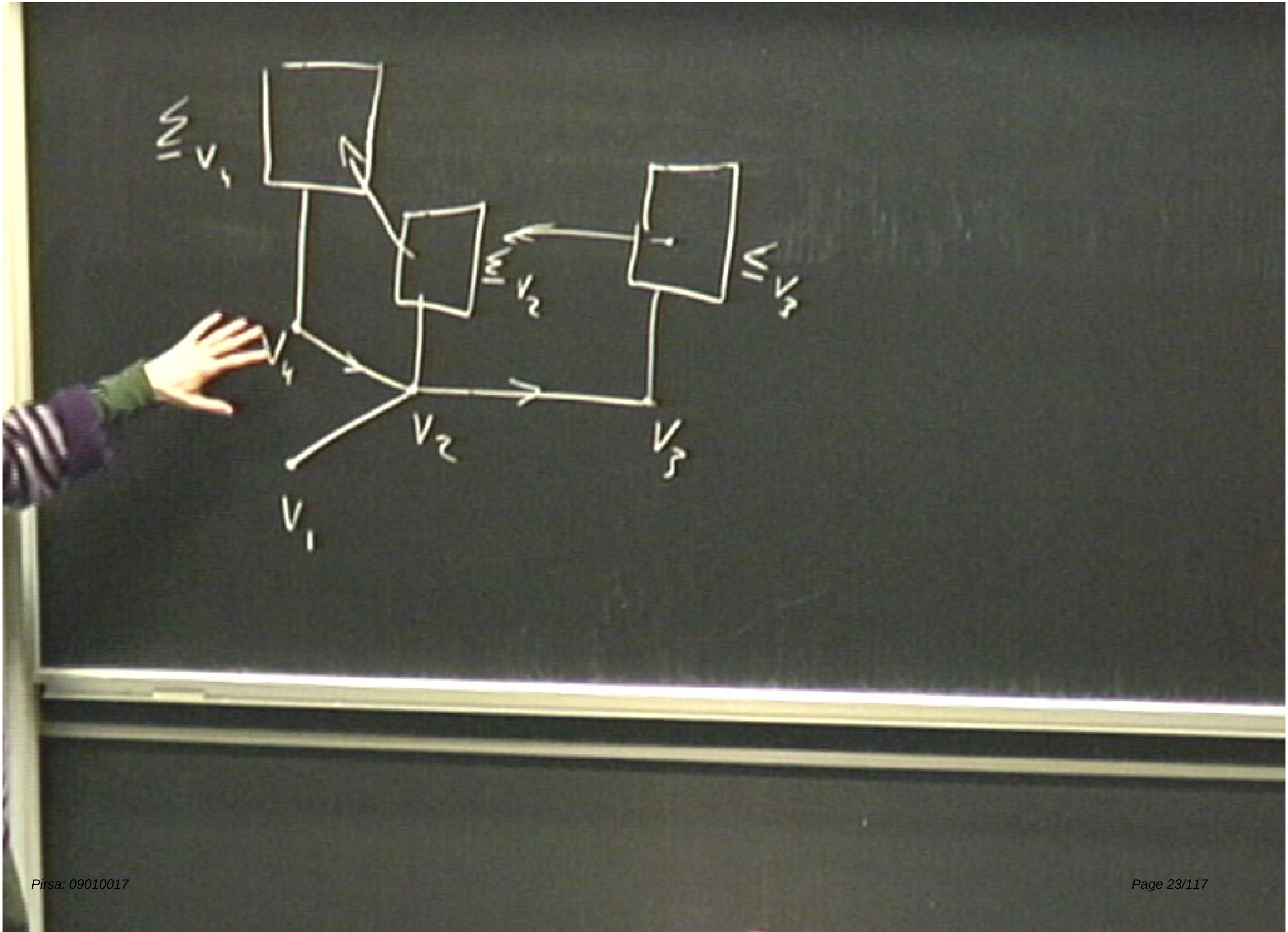

$\leq_{V_1}$
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$\leq_{V_3}$
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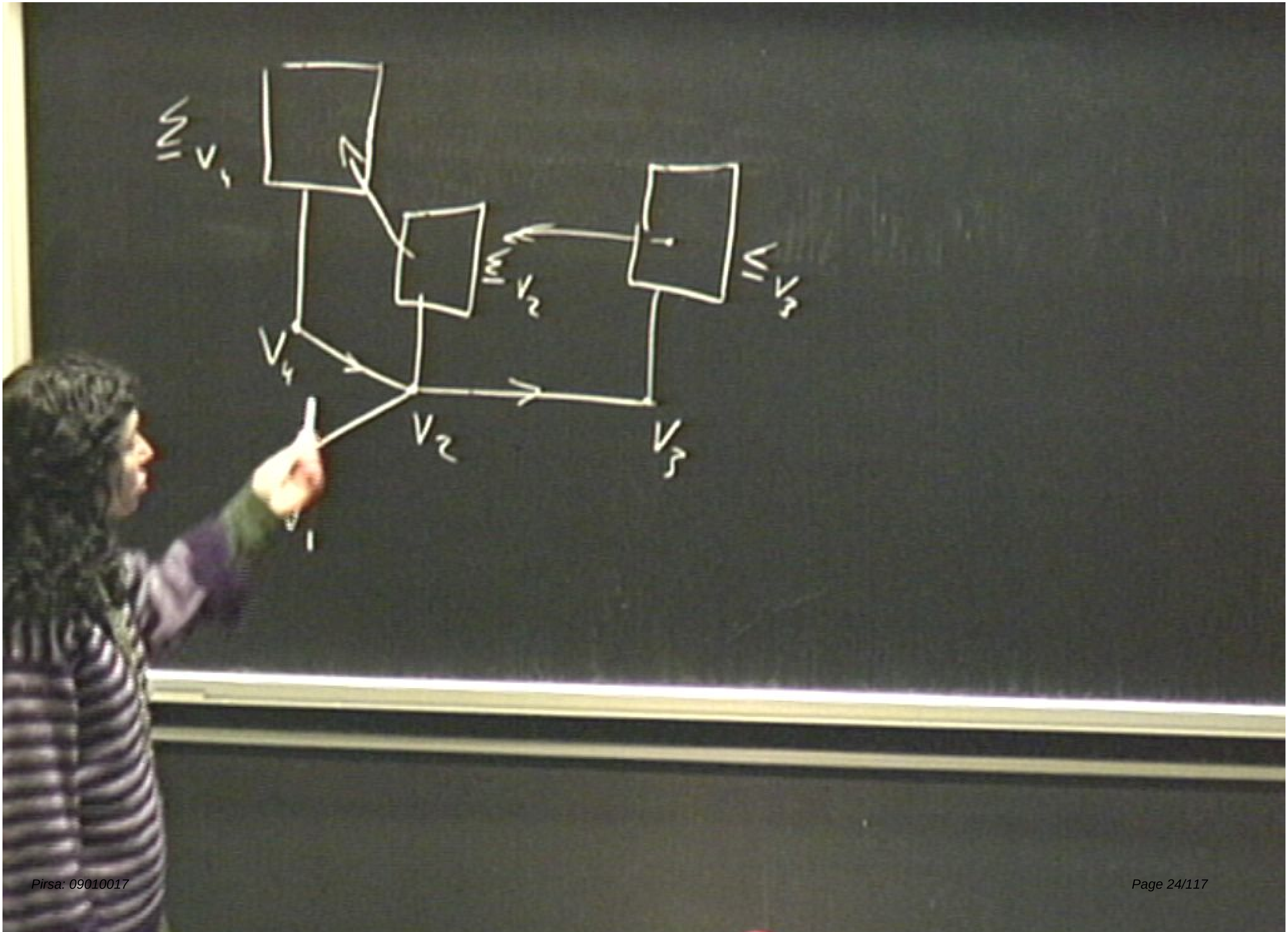

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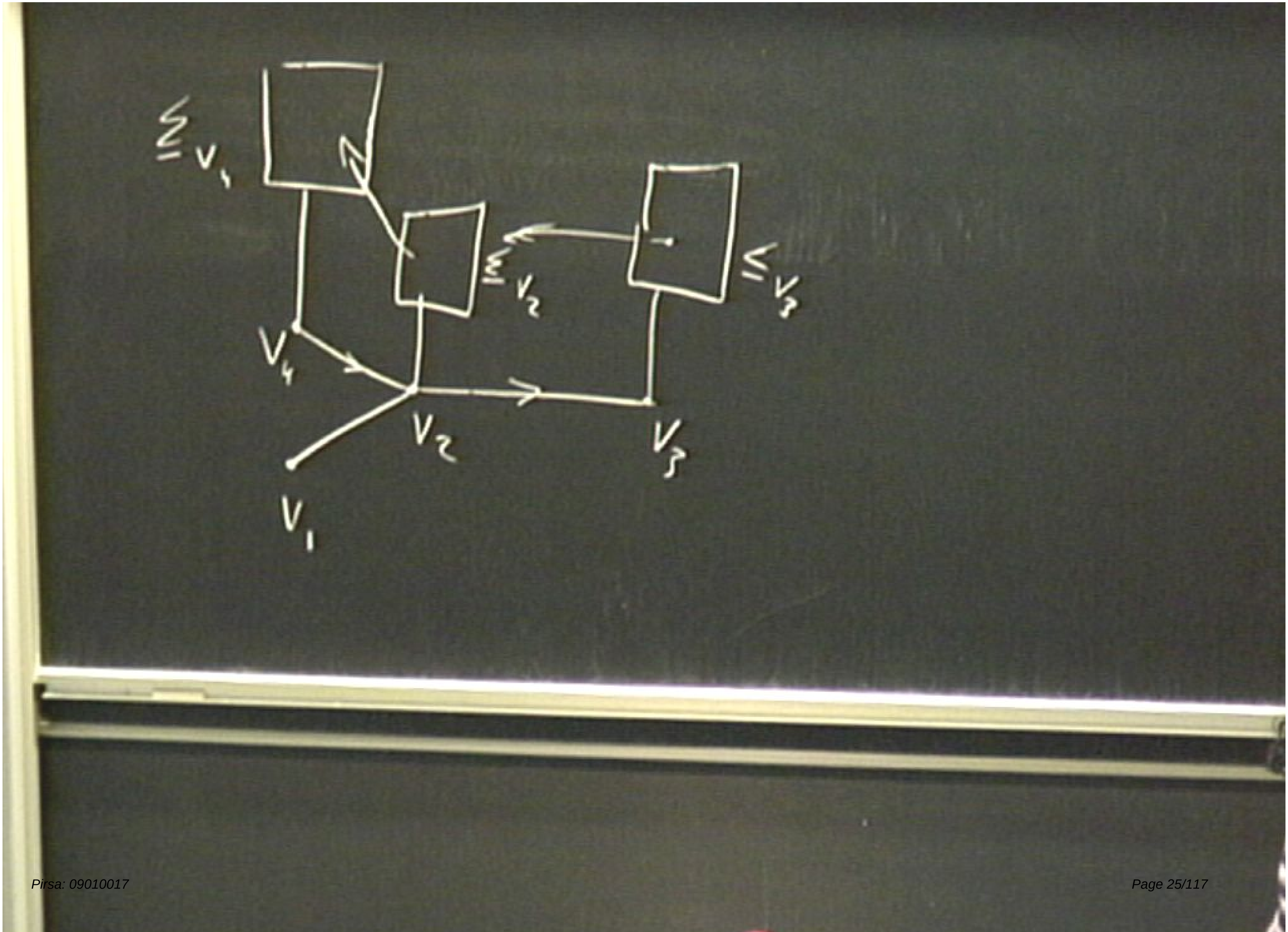

Σ_{V_1}
Σ_{V_2}
Σ_{V_3}
V_4
V_2
V_3
V_1

3. The Isham-Doering scheme

3.1 The State Object

State spaces in physics

1. *Classical physics*: Physical quantity A represented $f_A : \mathcal{S} \to \mathbb{R}$.
2. *Quantum physics*: Physical quantity A represented $\hat{A} : \mathcal{H} \to \mathcal{H}$.
3. *Topos physics*: *Spectral presheaf* $\underline{\Sigma} : \mathcal{V}(\mathcal{H}) \to$ *Set* such that

   $V \mapsto \underline{\Sigma}_V :=$ {simultaneous eigenvalues of V}; i.e., the possible values of the physical quantities in V.

   If $\hat{A} \in V$, then $f_{\hat{A}} : \underline{\Sigma}_V \to \mathbb{R}$!

$\underline{\Sigma}$ is the '**state object**' of the theory.

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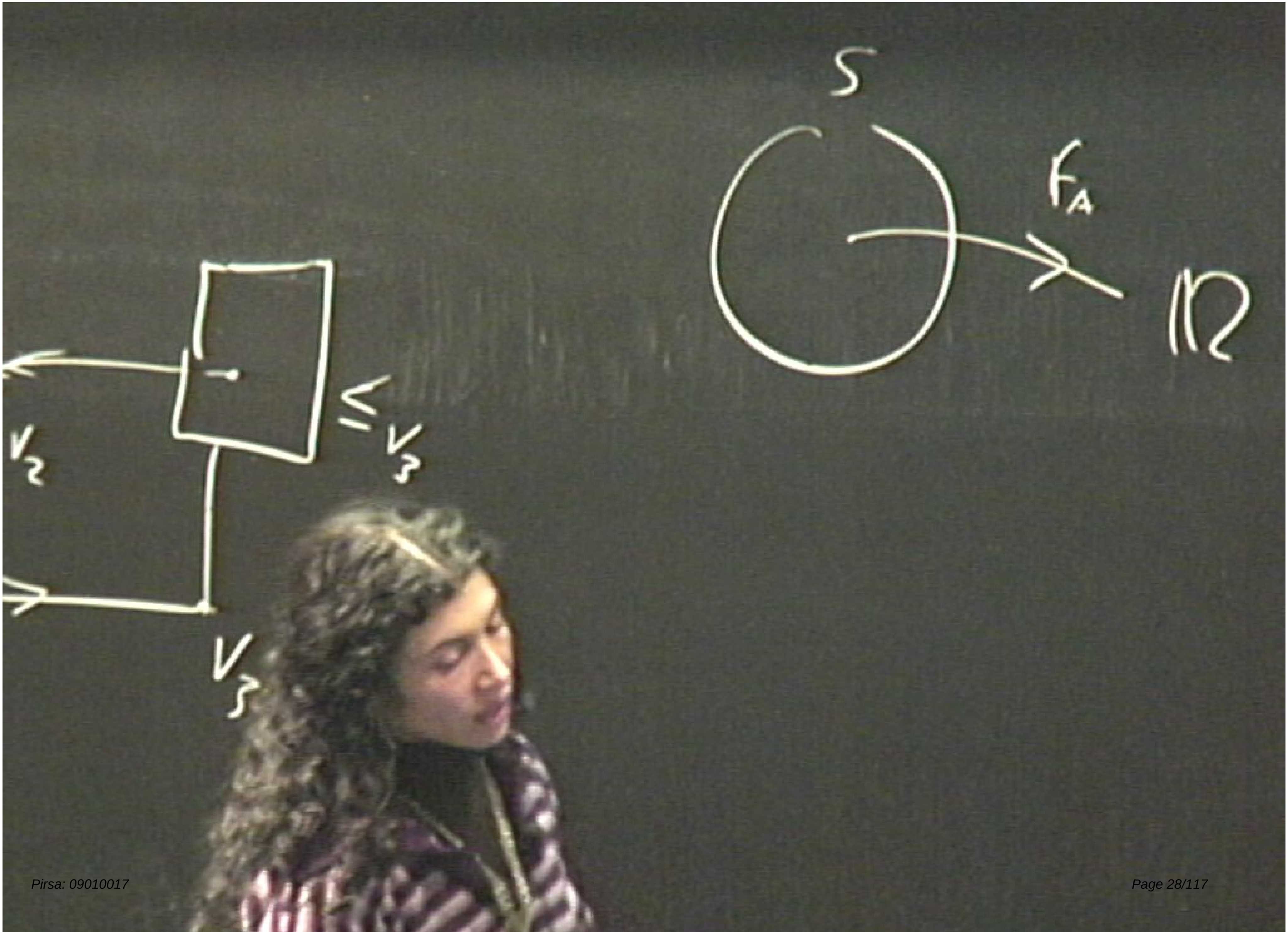
S
f_A
$\mathbb{R}$
V_2
$\leq V_3$
V_3

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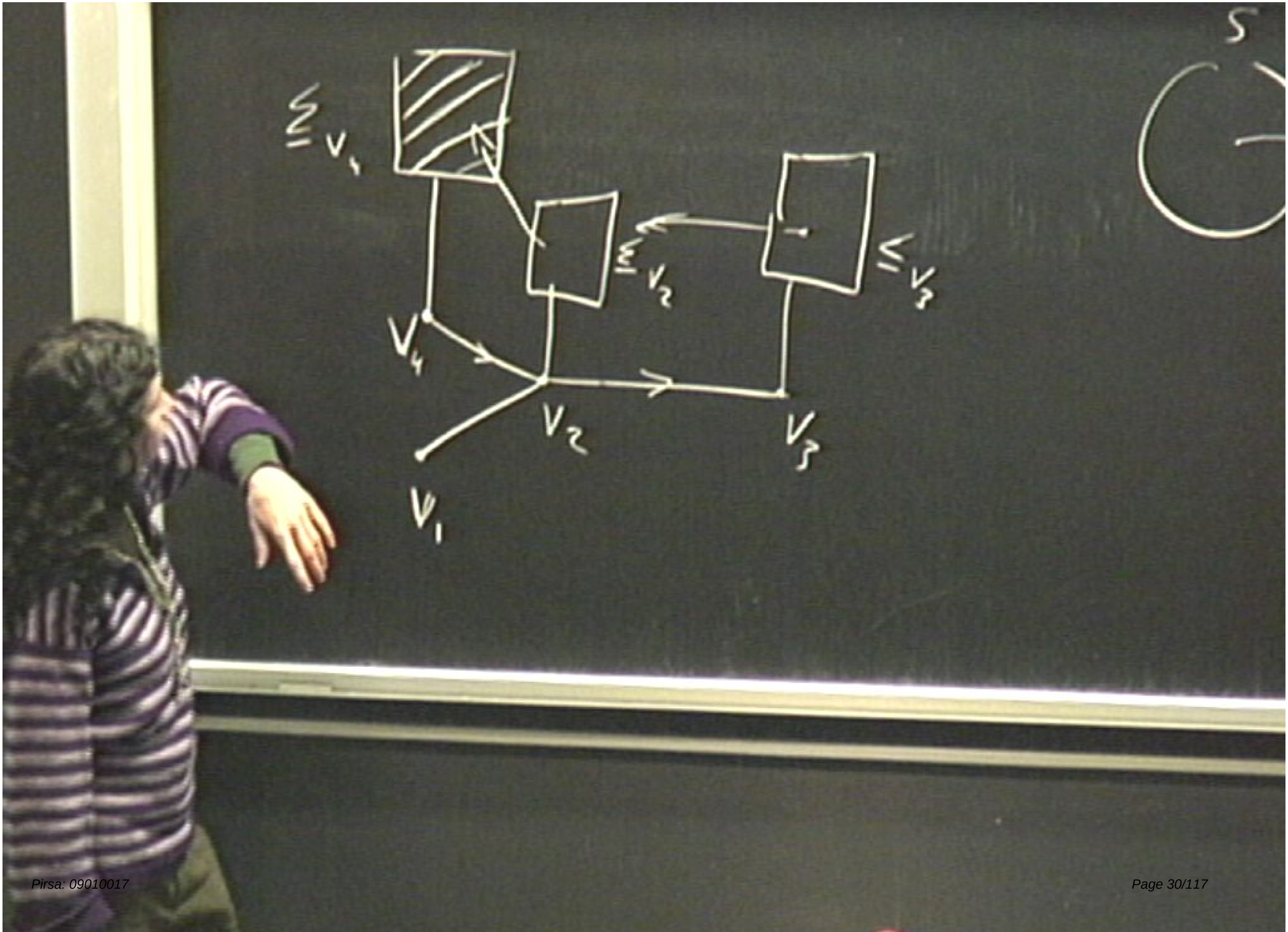
Σ_{v_4}
Σ_{v_2}
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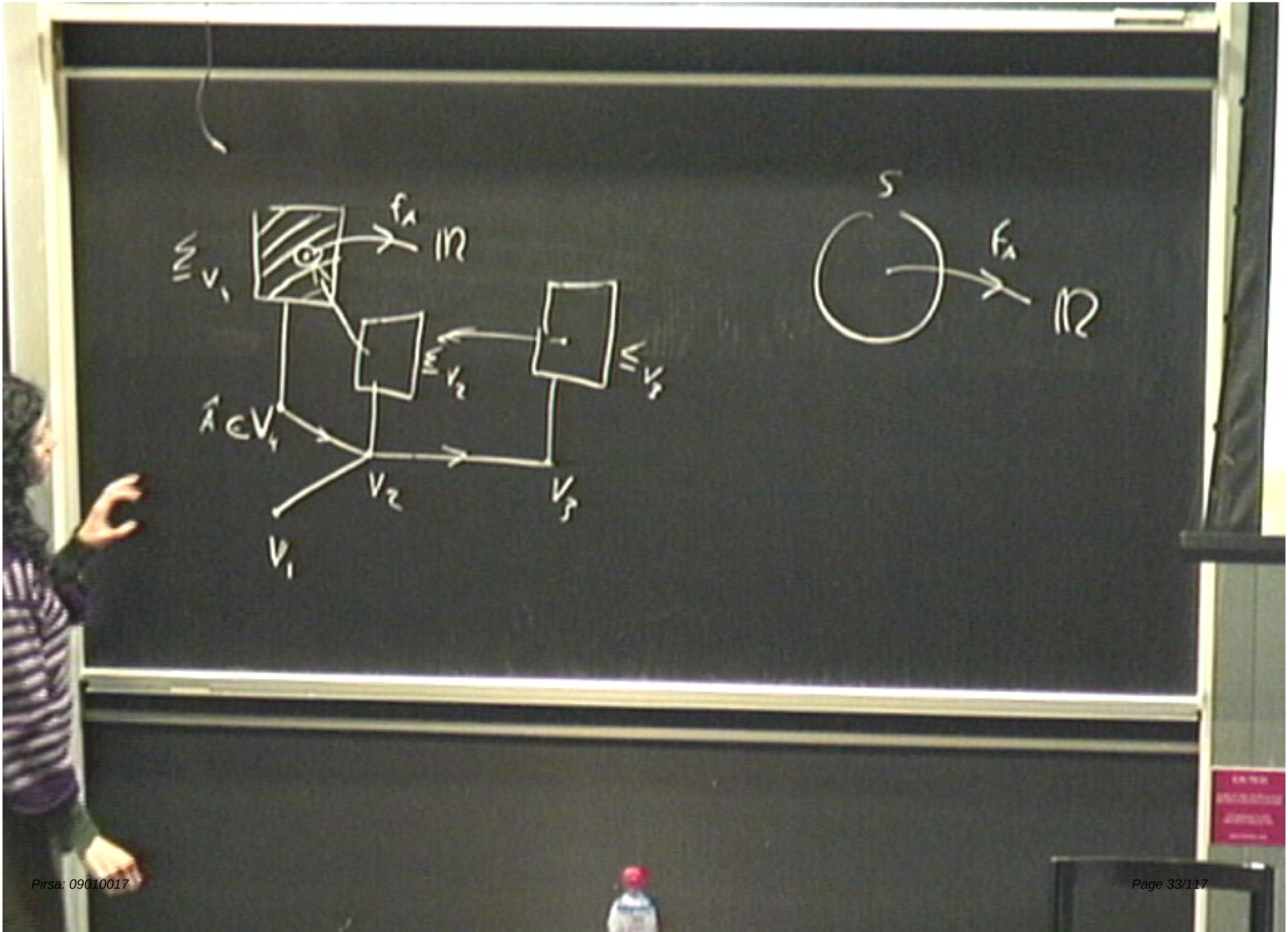
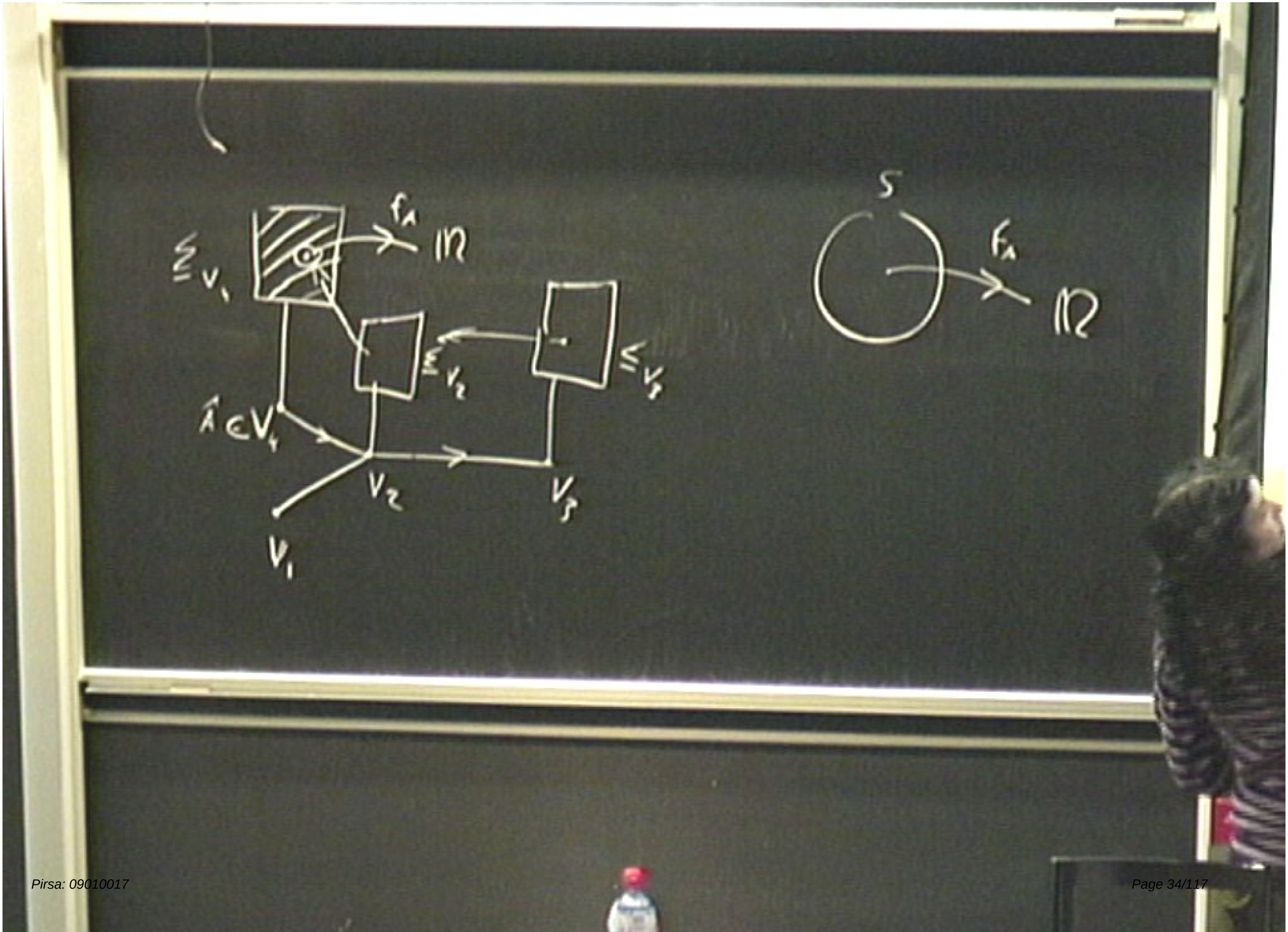


f_λ
$\mathbb{R}$
Σ_{V_1}
Σ_{V_2}
Σ_{V_3}
$\hat{\lambda} \in V_1$
V_2
V_3
V_1
S
f_λ
$\mathbb{R}$



f_A
$\mathbb{R}$
Σ_{V_1}
Σ_{V_2}
Σ_{V_3}
$\hat{A} \in V_1$
V_1
V_2
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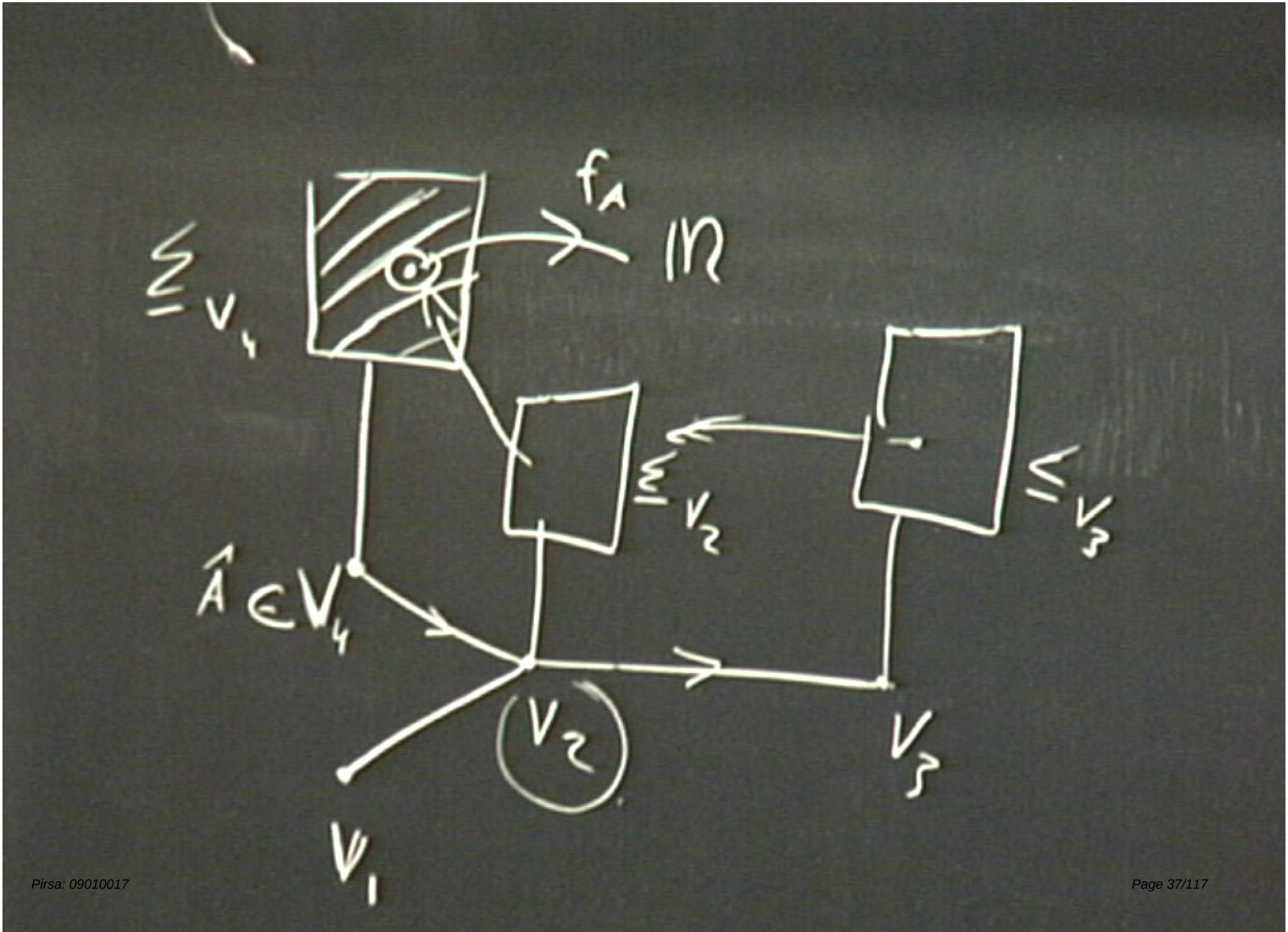

f_A
$\mathbb{R}$
Σ_{V_4}
Σ_{V_2}
Σ_{V_3}
$\hat{A} \in V_4$
V_2
V_3
V_1

3.2 Propositions

Propositions

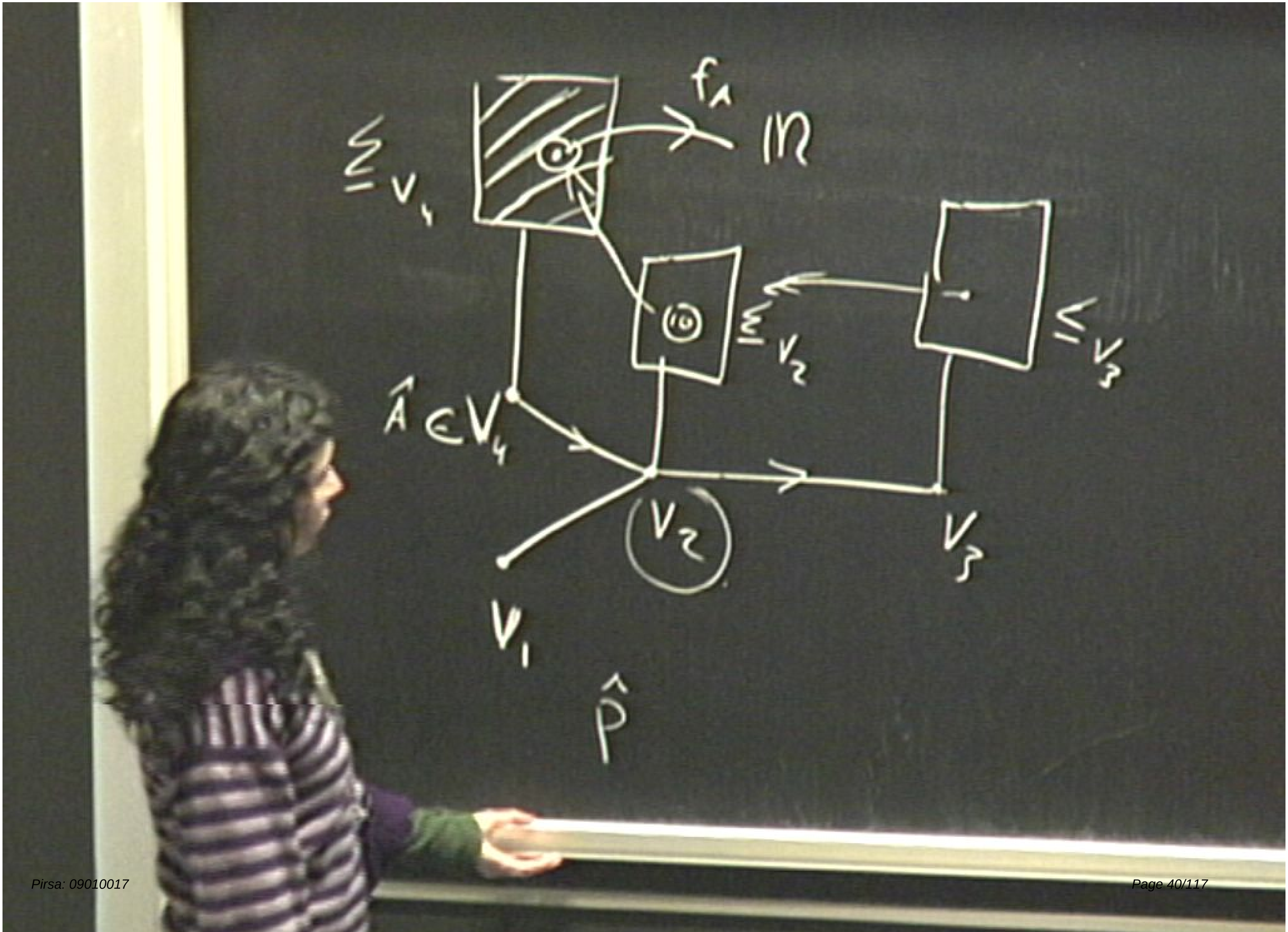
1. *Classical physics*:

$$\text{“}A \in \Delta\text{”} \rightarrow f_A^{-1}(\Delta) = \{s \in \mathcal{S} | f_A(s) \in \Delta\} \subseteq \mathcal{S}$$

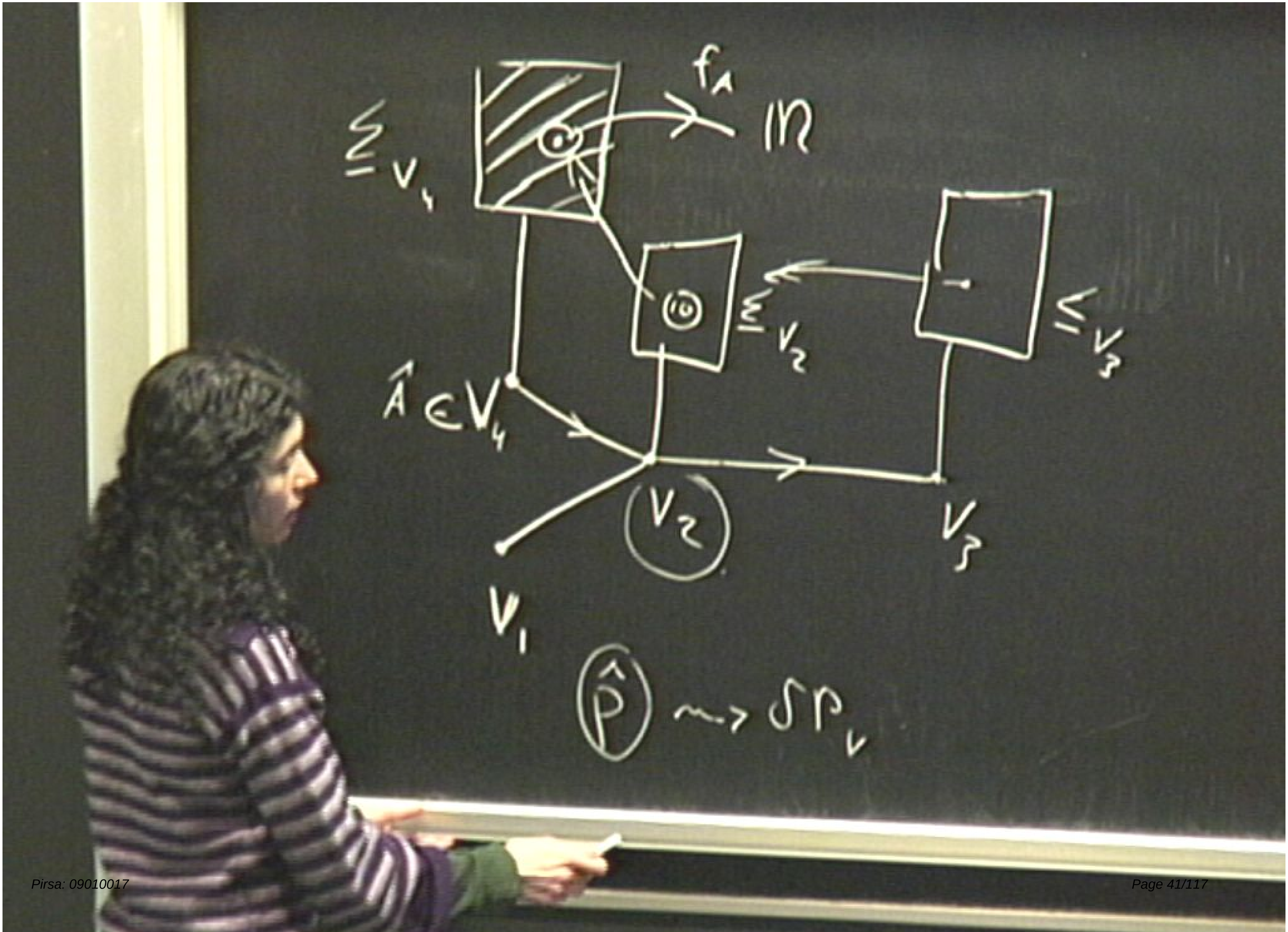
2. *Quantum theory*:

$$\hat{P} = \hat{E}[A \in \Delta] \in P(\mathcal{H})$$

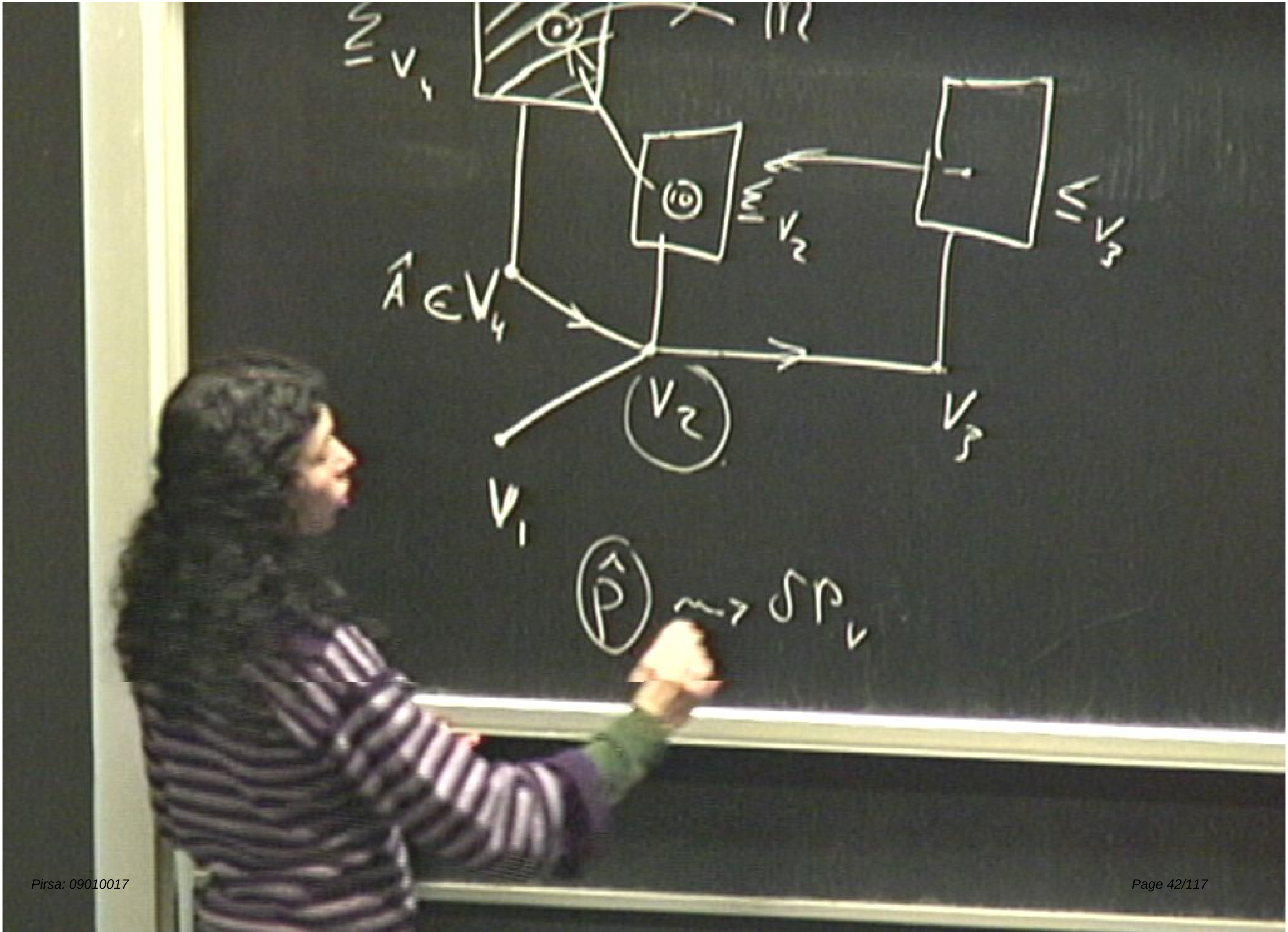
3. *Topos physics*: Need to identify $\hat{P}$ with sub-object of $\underline{\Sigma}$; i.e., for each V need subset of $\underline{\Sigma}_V$; i.e., a *projection operator* in V.



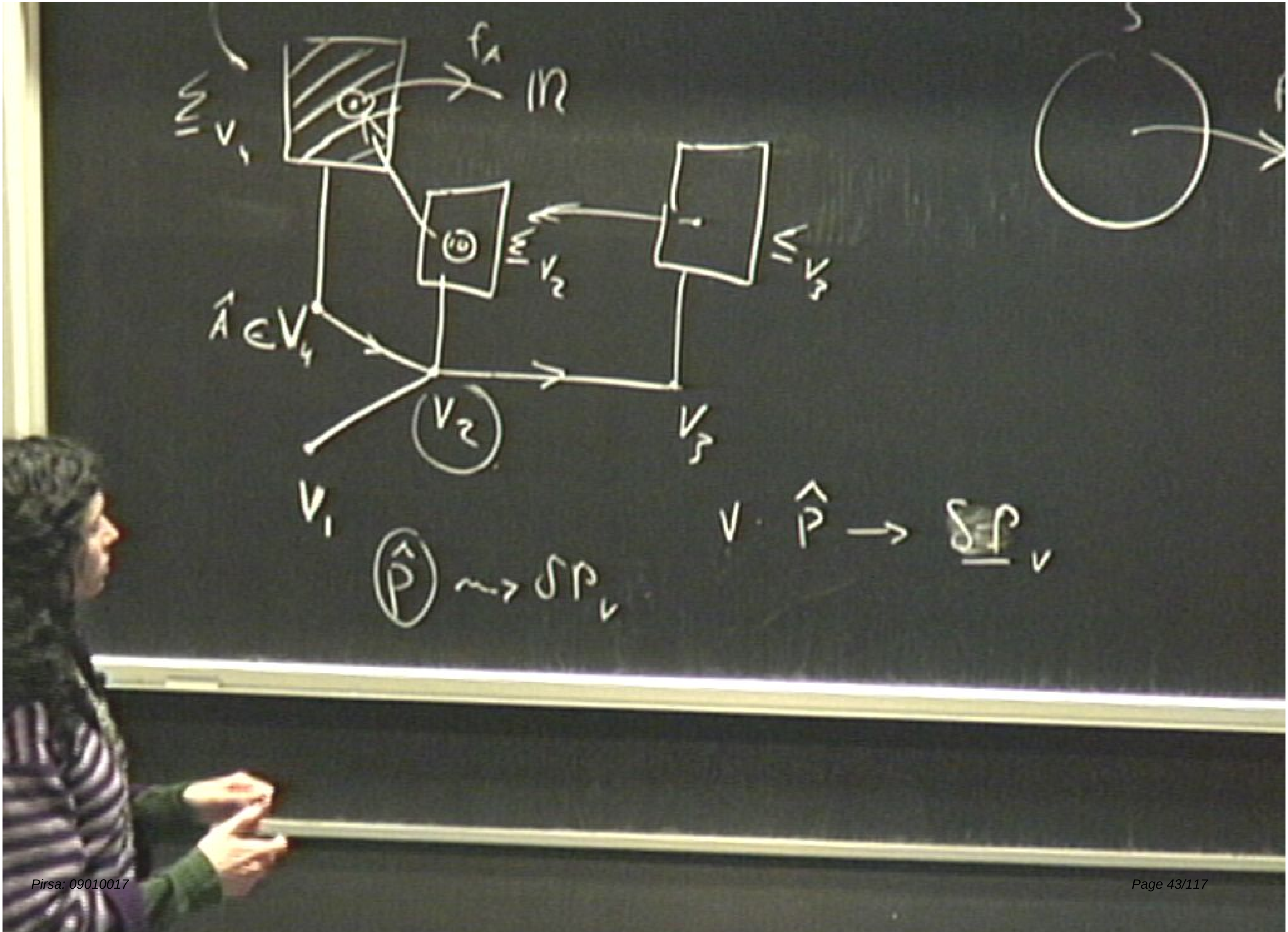
f_A
$\mathbb{R}$
Σ_{V_4}
Σ_{V_2}
Σ_{V_3}
$\hat{A} \in V_4$
V_2
V_3
V_1
$\hat{P}$



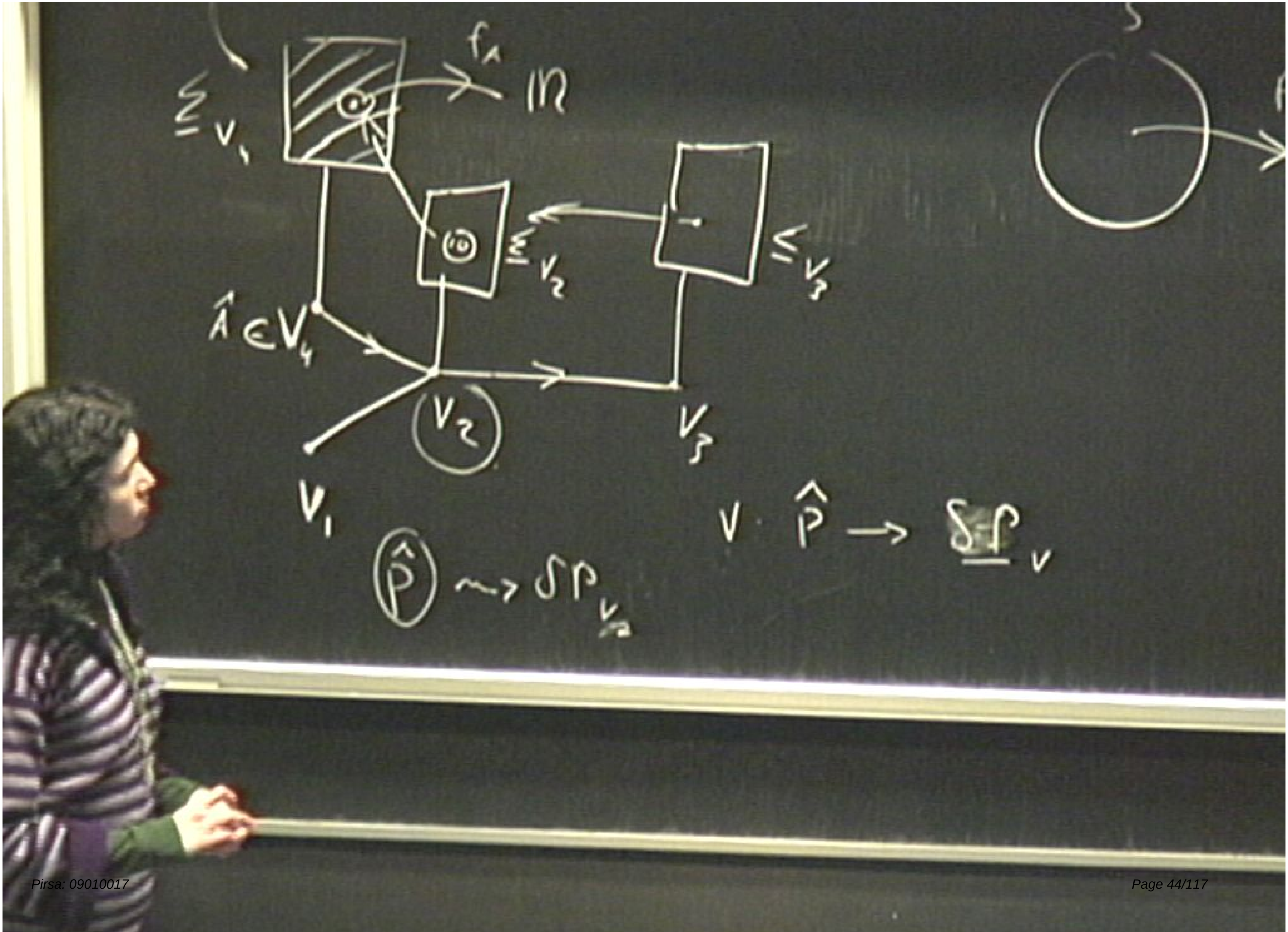
f_A
$\mathbb{R}$
Σ_{V_4}
Σ_{V_2}
Σ_{V_3}
$\vec{A} \in V_4$
V_2
V_3
V_1
$\hat{P} \rightsquigarrow \delta P_v$



Σ_{V_4}
Σ_{V_2}
Σ_{V_3}
$\vec{A} \in V_4$
V_2
V_3
V_1
$\hat{P} \rightsquigarrow \delta\rho_V$



f_A
$\mathbb{R}$
Σ_{v_1}
Σ_{v_2}
Σ_{v_3}
$\hat{A} \in V_4$
V_2
V_3
V_1
$\hat{P} \rightsquigarrow \delta P_v$
$V \cdot \hat{P} \to \delta P_v$



f_A
ℝ
Σ_{V_1}
Σ_{V_2}
Σ_{V_3}
Â ∈ V_4
V_2
V_3
V_1
P̂ ⇝ δP_{V_2}
V · P̂ → δP_V

3.3 Daseinisation

- **'Daseinisation':**

$$\delta : P(\mathcal{H}) \to P(V)$$
$$P \mapsto \delta(\hat{P})_V$$

where $\delta(\hat{P})_V := \bigwedge\{\hat{\alpha} \in P(V) | \hat{\alpha} \geq \hat{P}\}$: the 'best' approximation of $\hat{P}$ (from above) by projectors in V.

- **Relation to** $Sub(\underline{\Sigma})$:

Any projector in V gives a subset of $\underline{\Sigma}_V$. Therefore get map, for each V, $\hat{P} \mapsto \underline{\delta(\hat{P})}_V$.

Can show that this corresponds to $\delta : P(\mathcal{H}) \to Sub(\underline{\Sigma})$!

Thus δ maps propositions about a quantum system to a *distributive* lattice in a *contextual* manner.

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  In classical physics a microstate is a point in the state space.

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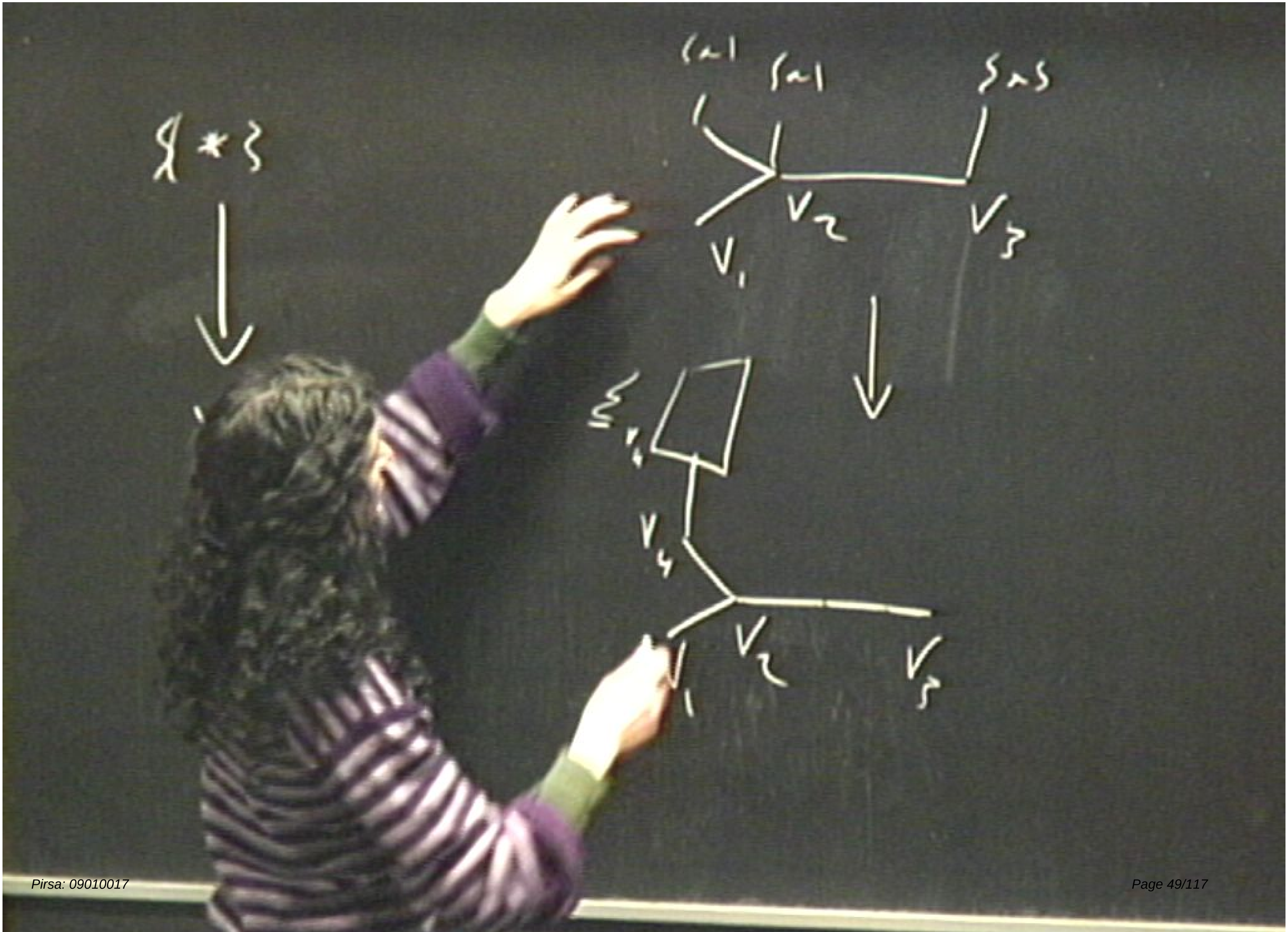
- **States in the topos formulation of quantum theory**
  - $\underline{\Sigma}$ has no 'points' $\Longrightarrow$ Equivalent to the K-S theorem
  - Topos analogues of a state is a (non-point!) sub-object of the state object $\underline{\Sigma}$:

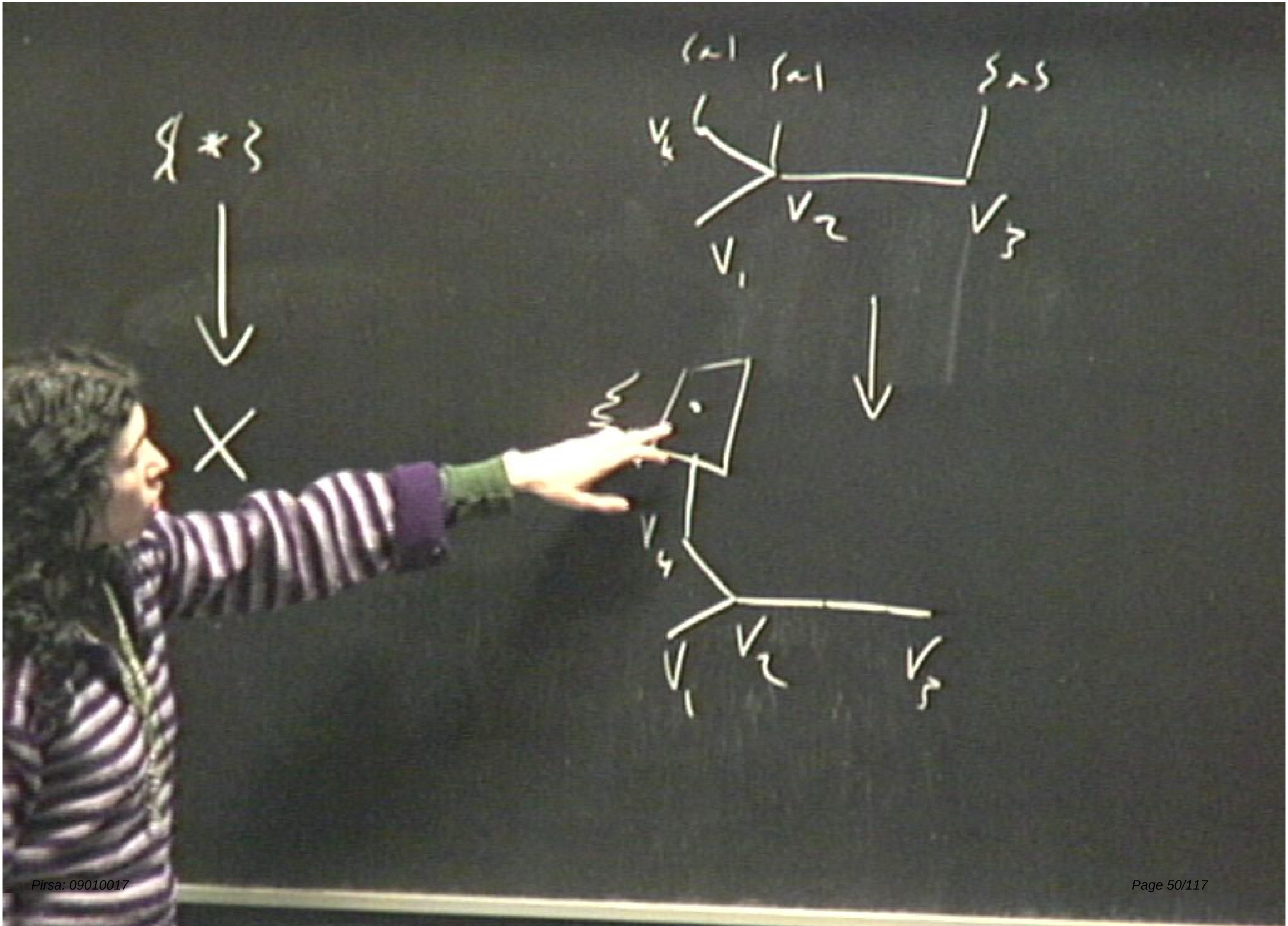
  Pseudo-state: $\underline{\mathfrak{w}}^{|\psi\rangle} := \underline{\delta(|\psi\rangle\langle\psi|)} \subseteq \underline{\Sigma}$,

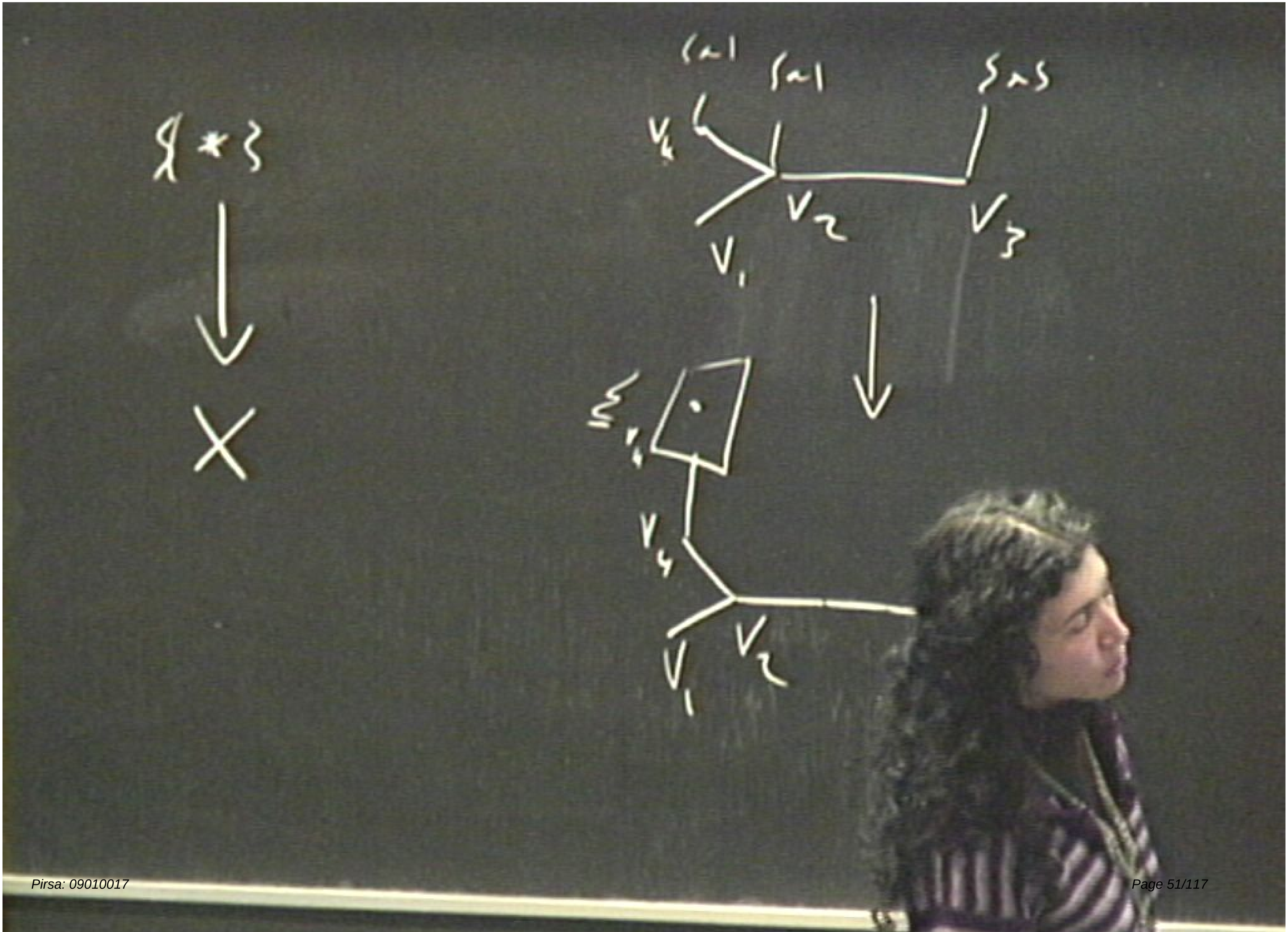
$$\delta(\,|\psi\rangle\langle\psi|)_V := \bigwedge\{\hat{\alpha} \in P(V) \mid \, |\psi\rangle\langle\psi| \leq \hat{\alpha}\}$$

$\underline{\mathfrak{w}}^{|\psi\rangle}$ is the 'closest' one can get to defining a point in $\underline{\Sigma}$.



V_1
V_2
V_3
V_4


V4
V2
V1
V3
V4
V1
V2
V3


V4
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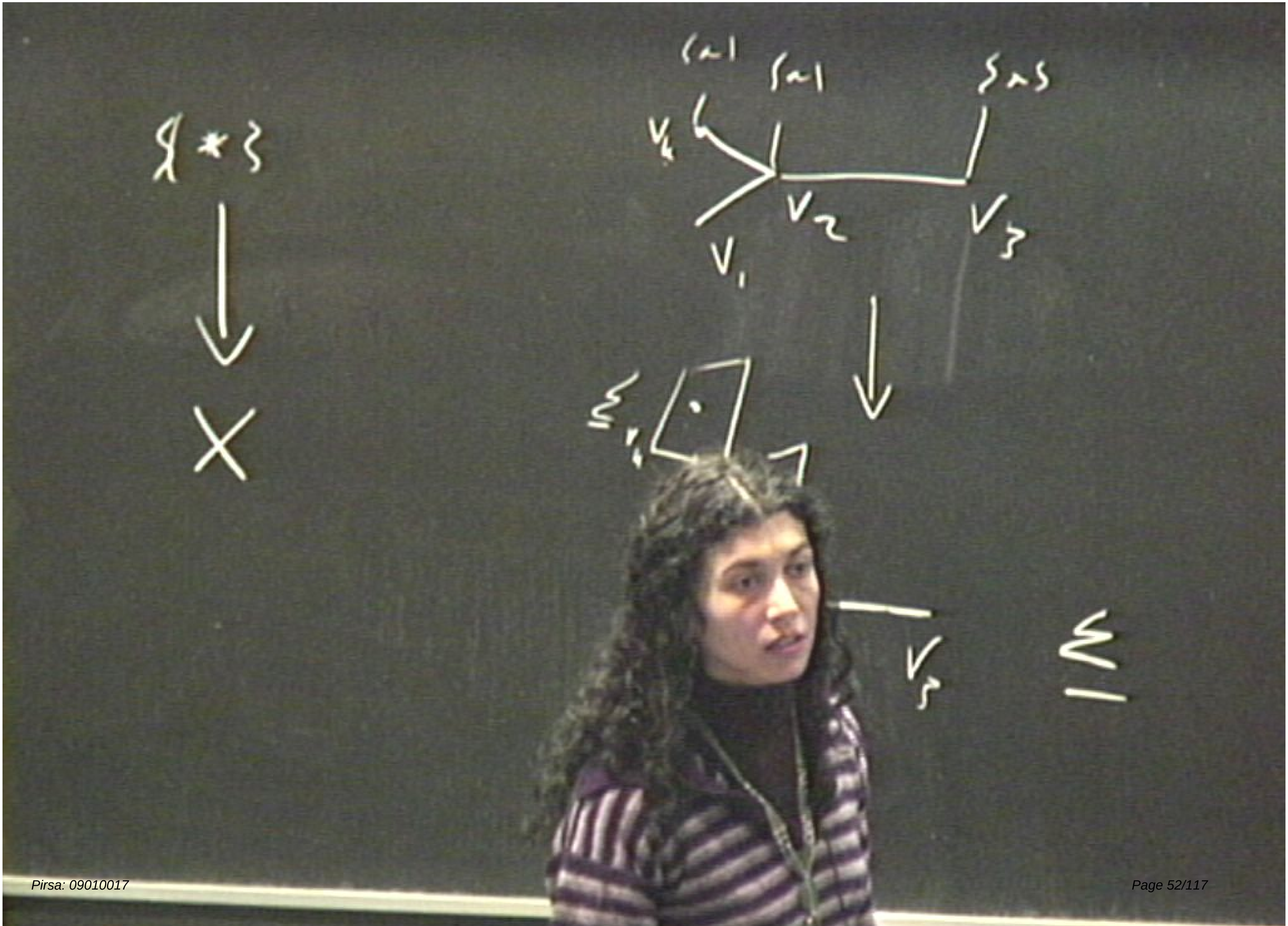
V1
V4
V1
V2



V1
V2
V3
V4
V3

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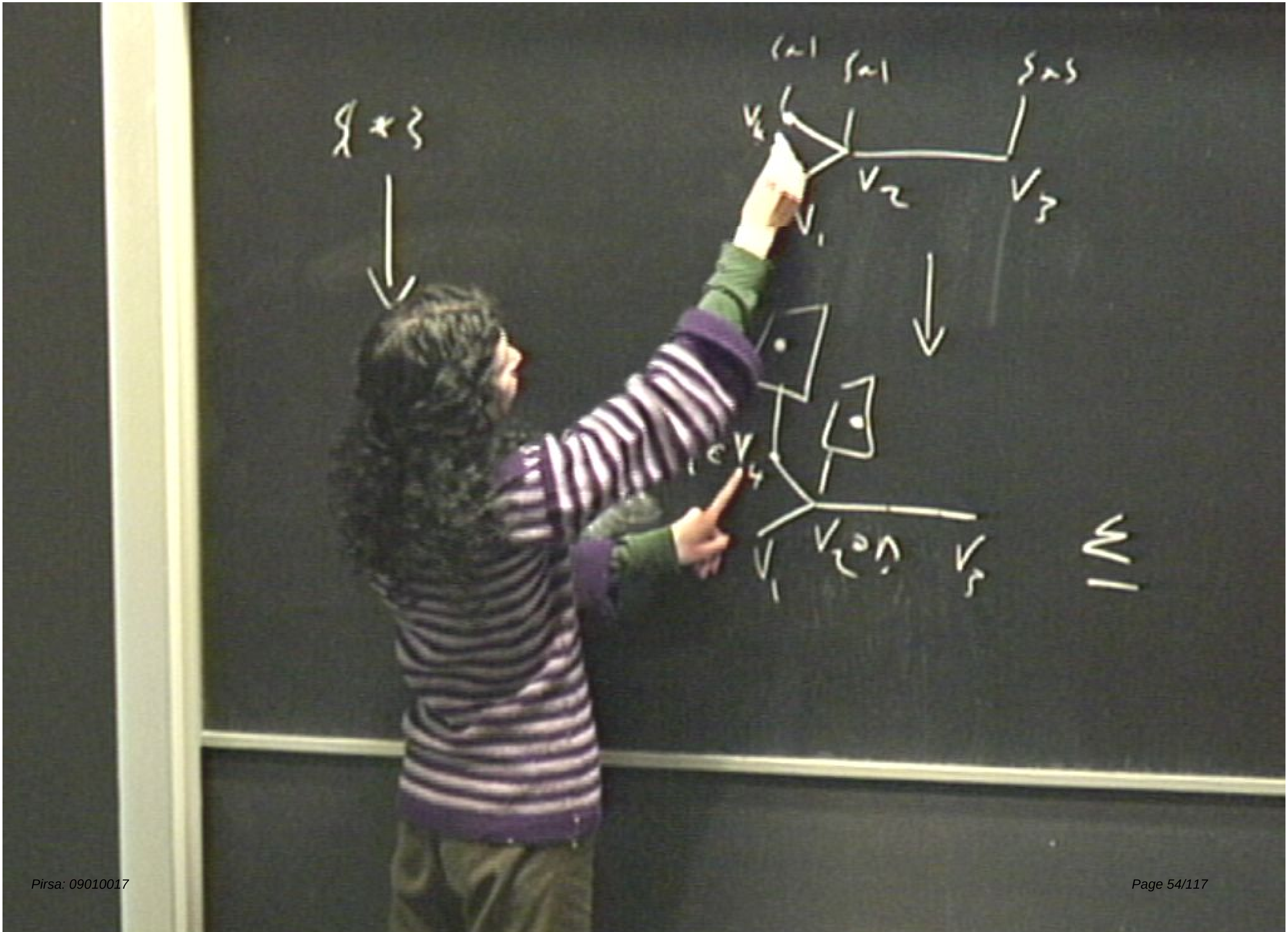

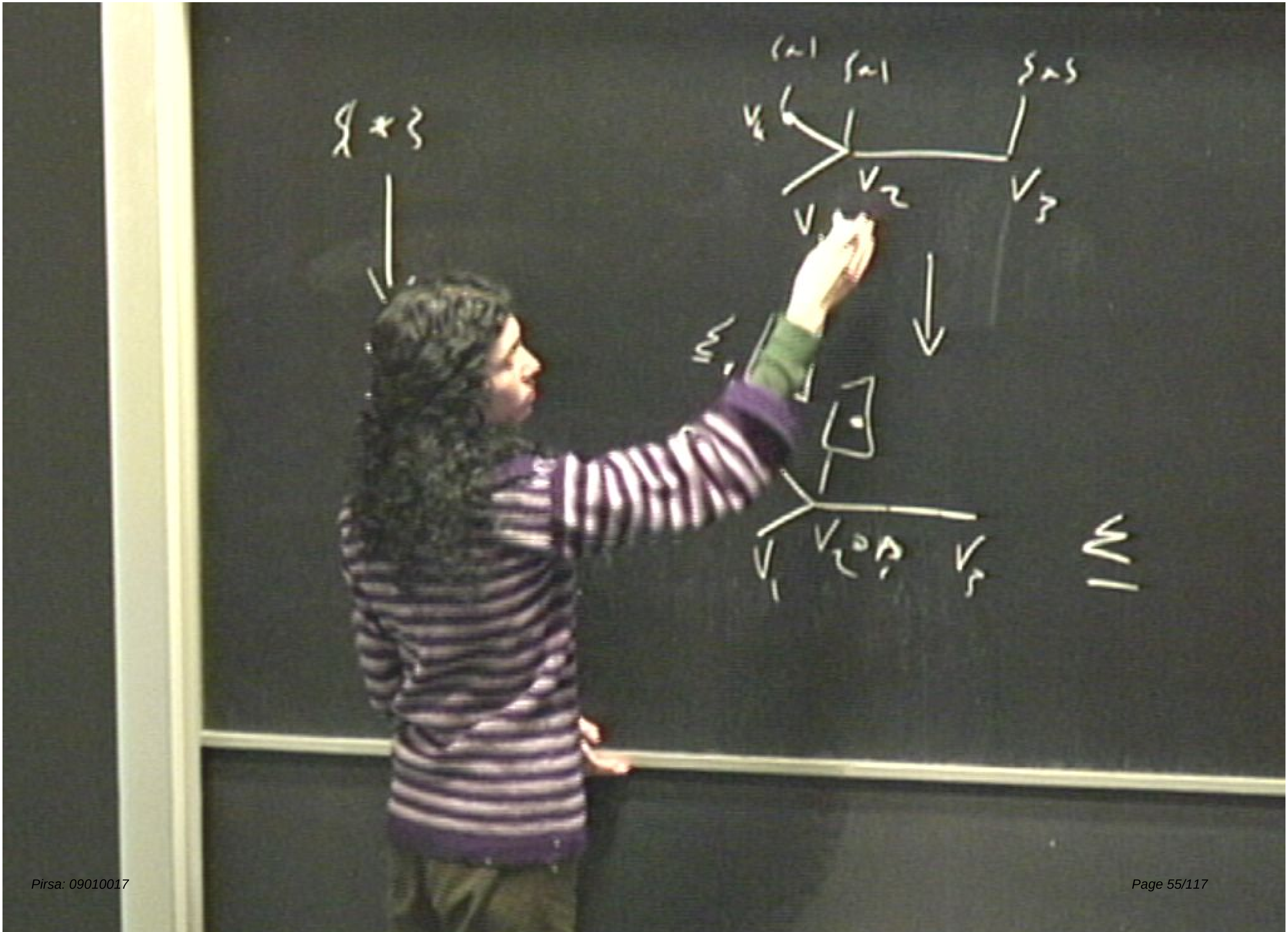

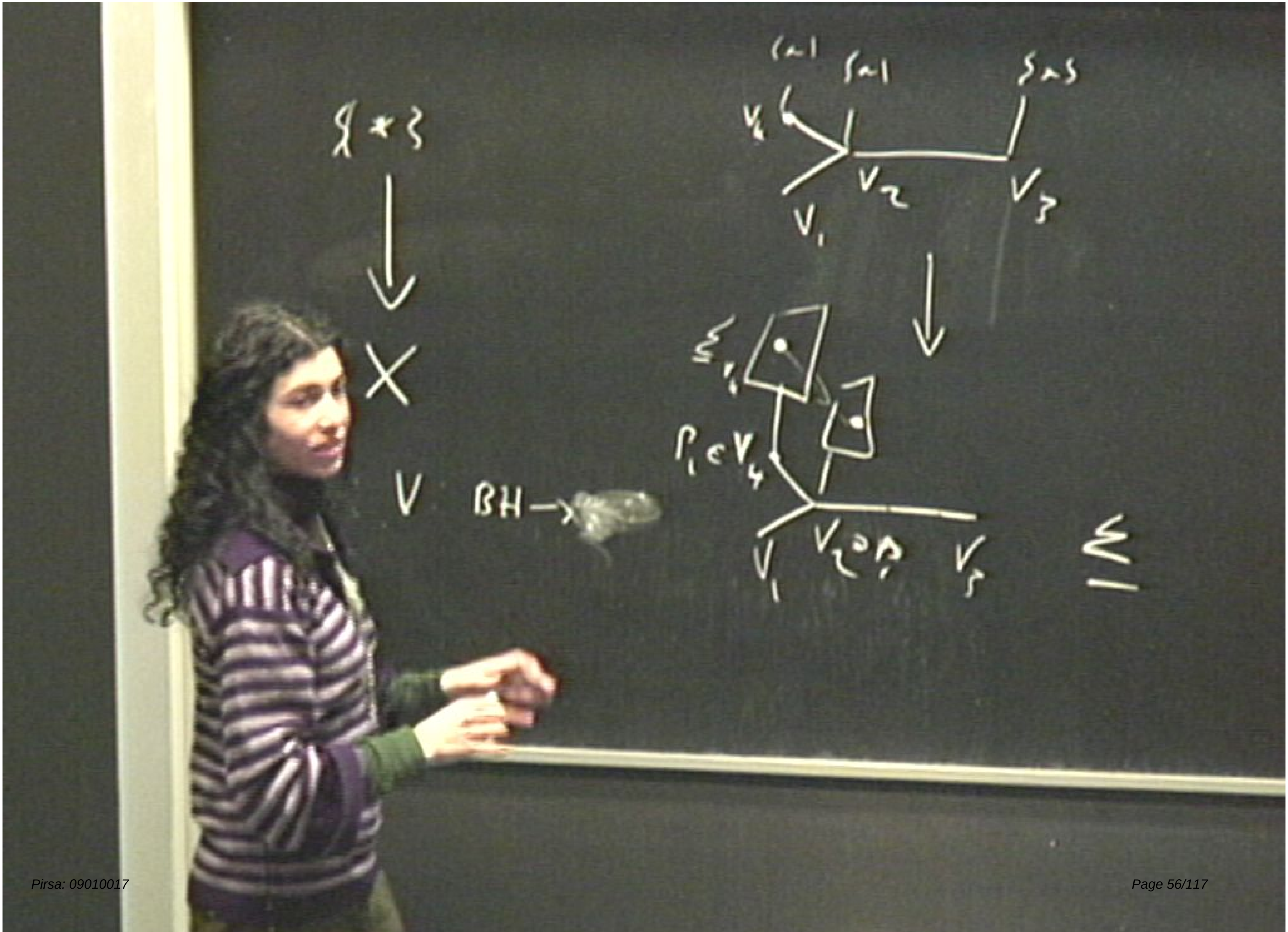
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3.4 States

- **States in classical physics**

  In classical physics a microstate is a point in the state space.

- **States in the topos formulation of quantum theory**
  - $\underline{\Sigma}$ has no 'points' $\Longrightarrow$ Equivalent to the K-S theorem
  - Topos analogues of a state is a (non-point!) sub-object of the state object $\underline{\Sigma}$:

    Pseudo-state: $\underline{\mathfrak{w}}^{|\psi\rangle} := \underline{\delta(|\psi\rangle\langle\psi|)} \subseteq \underline{\Sigma}$,

$$\delta(\,|\psi\rangle\langle\psi|)_V := \bigwedge\{\hat{\alpha} \in P(V) \mid \, |\psi\rangle\langle\psi| \leq \hat{\alpha}\}$$

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- **Truth values in the topos formulation of quantum theory**
  - In topos quantum theory, we define the proposition $\underline{\delta(\hat{P})}$ to be 'totally true' given the pseudo state $\underline{\mathfrak{w}}^{|\psi\rangle}$ iff $\underline{\mathfrak{w}}^{|\psi\rangle} \subseteq \underline{\delta(\hat{P})}$. This means that, for all V, $\mathfrak{w}_V^{|\psi\rangle} \leq \delta(\hat{P})_V$.
  - *However*, in a topos a proposition can be 'partially true' using 'contextual truth values'. At stage V we define

$$v(\underline{\mathfrak{w}}^{|\psi\rangle} \subseteq \underline{\delta(\hat{P})})_V := \{V' \subseteq V \mid \underline{\mathfrak{w}}_{V'}^{|\psi\rangle} \subseteq \underline{\delta(\hat{P})}_{V'}\}$$
$$= \{V' \subseteq V \mid \langle\psi|\delta(\hat{P})_{V'}|\psi\rangle = 1\}$$

3.5 Truth values

– This is called a *sieve* on V.

The collection, $\underline{\Omega}_V$, of all sieves on V forms a **Heyting algebra**!

– For varying V such truth values form a global section $\Gamma\underline{\Omega}$ of the sub-object classifier $\underline{\Omega}$.

The set $\Gamma\underline{\Omega}$ is itself a Heyting algebra!

– Thus we have a Heyting algebra of propositions *and* a Heyting algebra of ‘generalised’ truth values!

4. TOPOS THEORY AND THE HPO FORMALISM

4.1 HPO Formulation of Quantum Temporal Logic

- Identify the set of all history propositions with projection operators in a new Hilbert space $\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \cdots \otimes \mathcal{H}_{t_n}$
  - 'Homogeneous histories': Tensor products of projection operators

$$\alpha := \hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2} \otimes \cdots \otimes \hat{\alpha}_{t_n}$$

  - 'Inhomogeneous histories':

$$\neg(\hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2}) := \hat{1}_{\mathcal{H}_{t_1}} \otimes \hat{1}_{\mathcal{H}_{t_2}} - \hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2}$$
$$= (\neg\hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2}) \vee (\hat{\alpha}_{t_1} \otimes \neg\hat{\alpha}_{t_2}) \vee (\neg\hat{\alpha}_{t_1} \otimes \neg\hat{\alpha}_{t_2})$$

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  - 'Type III' propositions
- I.e., the tensor product implements the 'and then' temporal connective in *quantum temporal logic*.

4.2 Temporal Structure in a Heyting algebra

- **Aim**: Find a *topos* representation of the homogeneous history $\alpha = (A_1 \in \Delta_1)_{t_1} \sqcap (A_2 \in \Delta_2)_{t_2} \cdots \sqcap (A_n \in \Delta_n)_{t_n}$
  - Individual-time propositions are identified with sub-objects of the .spectral presheaf $\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_i})}$. Collection of all such sub-objects, $Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_i})})$, forms a Heyting algebra.
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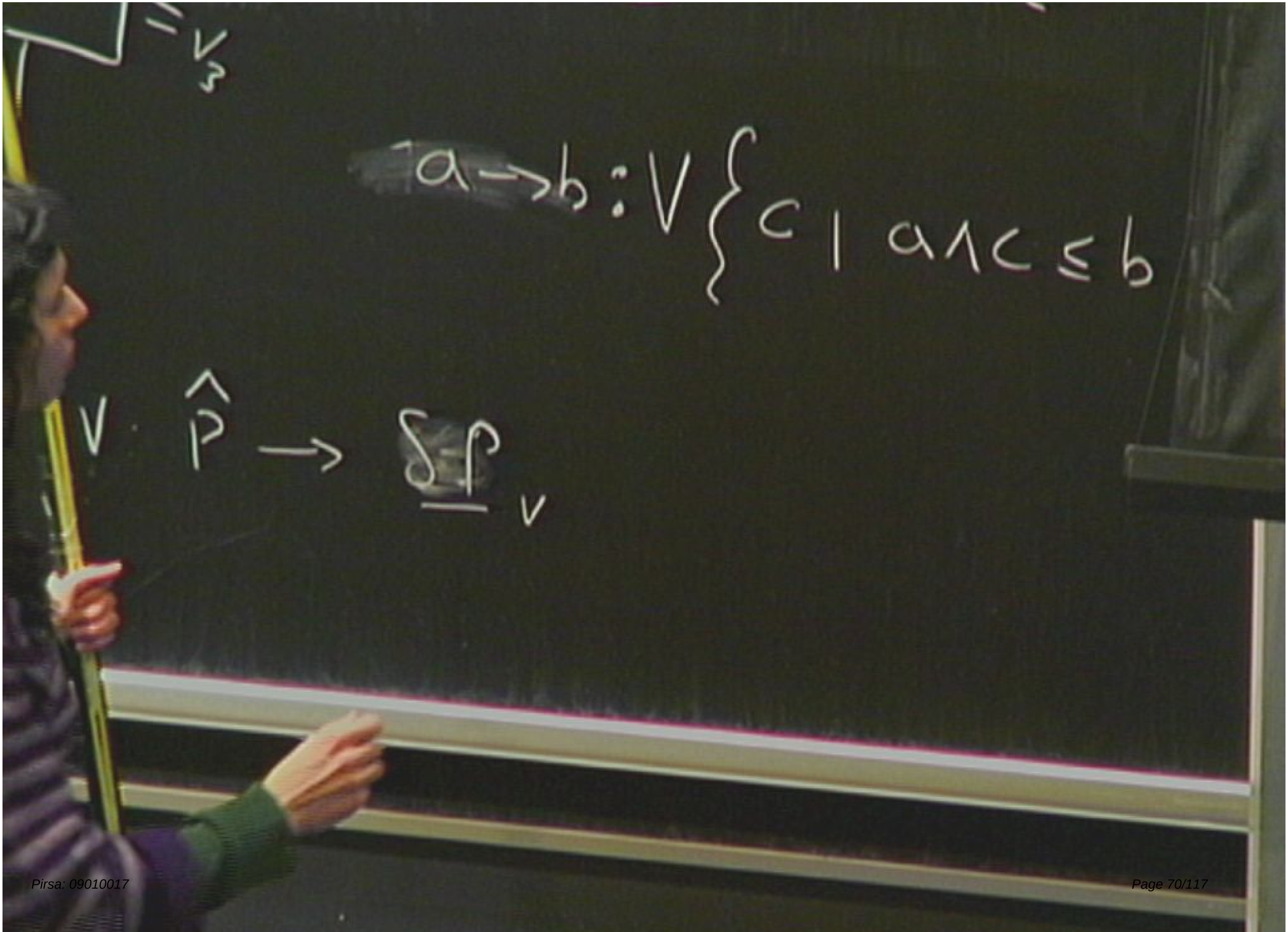
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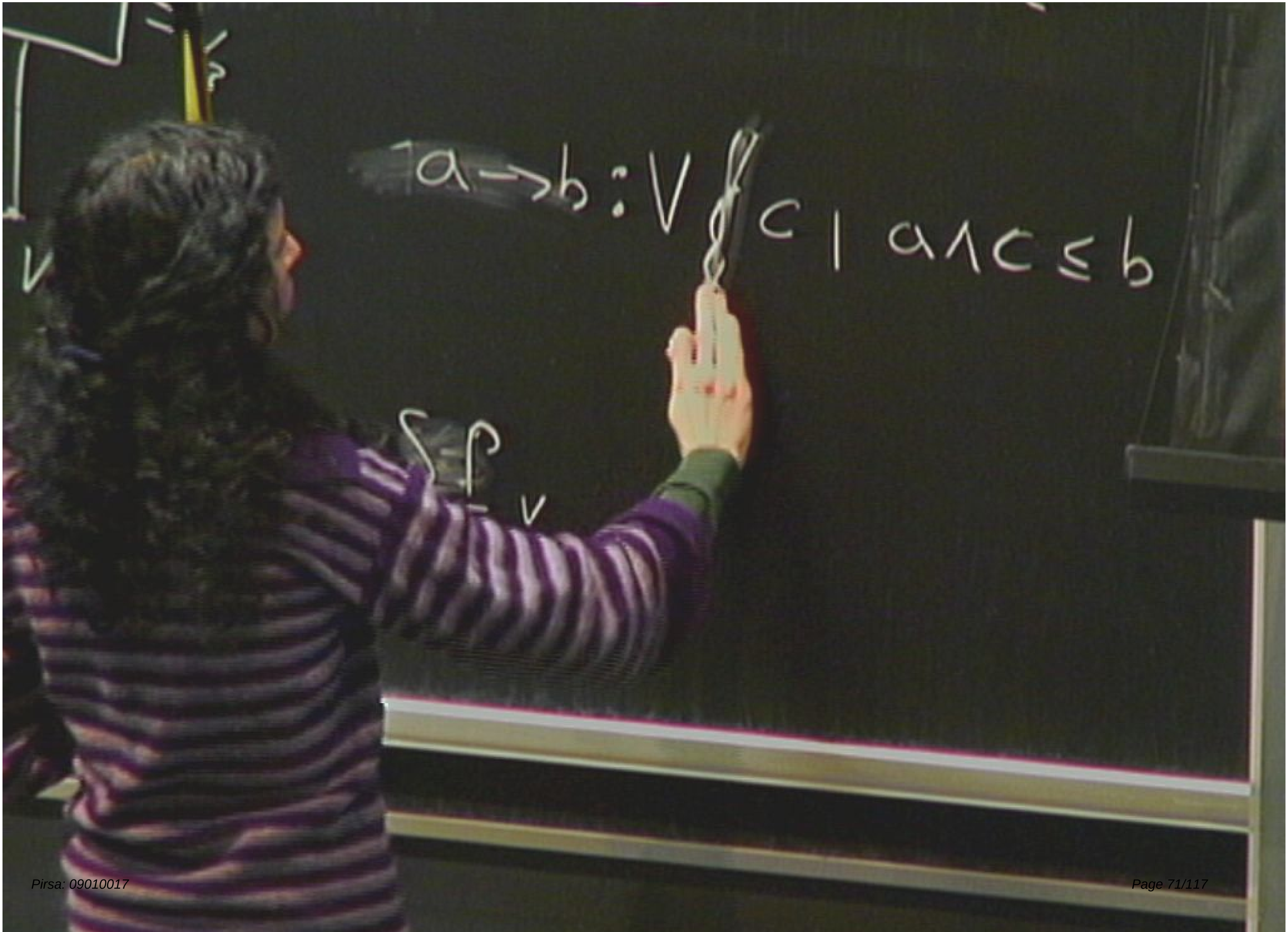
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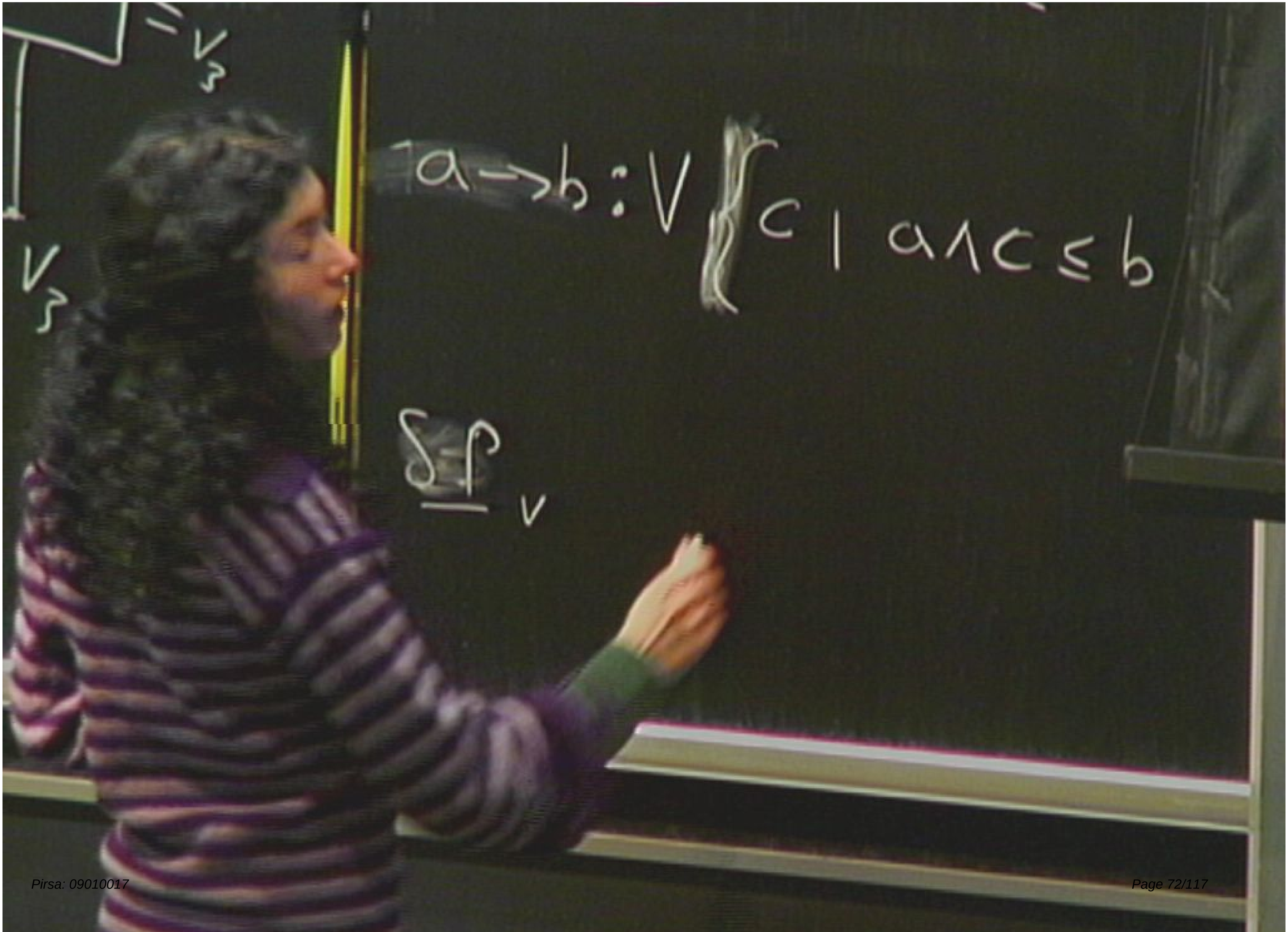
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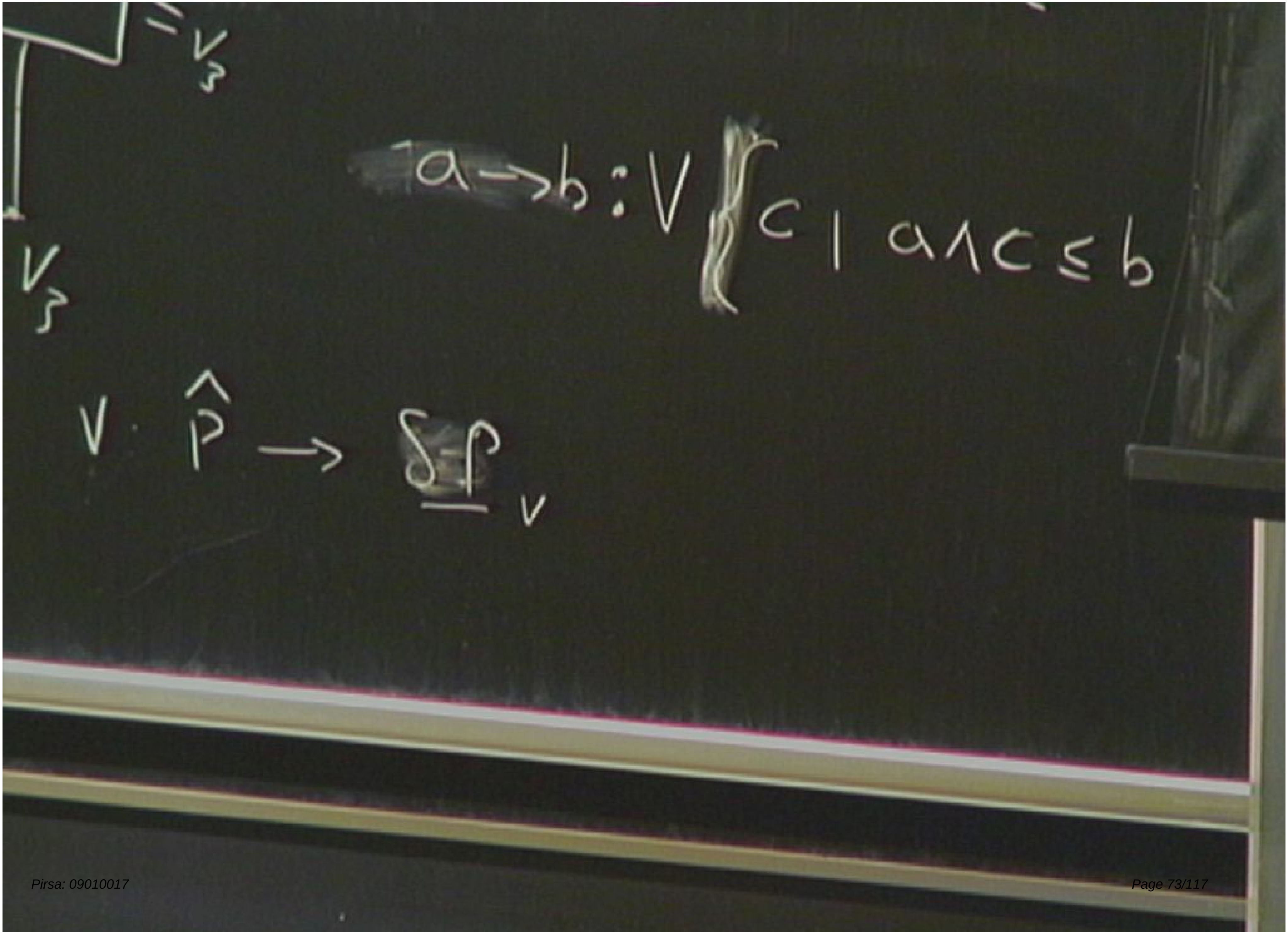
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4.3 Tensor Product in a Topos

- We need to relate the Heyting algebra $Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}) \otimes Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})})$ to sub-objects of some 'state object' in some topos related to quantum theory.

$$\hat{\alpha}_{t_1} \rightarrow \underline{\delta(\hat{\alpha}_{t_1})} \in Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}) \in Sets^{\mathcal{V}(\mathcal{H}_{t_1})^{op}}$$

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  Need to find a common topos to which both the topoi $Sets^{\mathcal{V}(\mathcal{H}_{t_1})^{op}}$ and $Sets^{\mathcal{V}(\mathcal{H}_{t_2})^{op}}$ can be related.

4.3 Tensor Product in a Topos

Intermediate topos: $Sets^{\mathcal{V}(\mathcal{H}_{t_1})^{op} \times \mathcal{V}(\mathcal{H}_{t_2})^{op}}$

$$p_1 : \mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2}) \rightarrow \mathcal{V}(\mathcal{H}_{t_1})$$
$$\langle V_1, V_2 \rangle \mapsto V_1$$
$$p_2 : \mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2}) \rightarrow \mathcal{V}(\mathcal{H}_{t_2})$$
$$\langle V_1, V_2 \rangle \mapsto V_2$$

from which we can obtain

$$p_1^* : Sets^{\mathcal{V}(\mathcal{H}_{t_1})^{op}} \rightarrow Sets^{\mathcal{V}(\mathcal{H}_{t_1})^{op} \times \mathcal{V}(\mathcal{H}_{t_2})^{op}}$$
$$p_2^* : Sets^{\mathcal{V}(\mathcal{H}_{t_2})^{op}} \rightarrow Sets^{\mathcal{V}(\mathcal{H}_{t_1})^{op} \times \mathcal{V}(\mathcal{H}_{t_2})^{op}}$$

which gives the well-defined product in $Sets^{\mathcal{V}(\mathcal{H}_{t_1})^{op} \times \mathcal{V}(\mathcal{H}_{t_2})^{op}}$

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$\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})$ contains *entangled contexts* $W = V_1 \otimes V_2 + V_3 \otimes V_4$ and so $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})^{op}} \neq \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{op} \times \mathcal{V}(\mathcal{H}_{t_2})^{op}}$

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Must find a relation between $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\mathrm{op}} \times \mathcal{V}(\mathcal{H}_{t_2})^{\mathrm{op}}}$ and $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})^{\mathrm{op}}}$

- Relation between context categories:

$$\begin{aligned} \theta : \mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2}) &\rightarrow \mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2}) \\ \langle V_1, V_2 \rangle &\mapsto V_1 \otimes V_2 \end{aligned}$$

- Induced relation between topoi:

$$\theta^* : \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})^{\mathrm{op}}} \rightarrow \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\mathrm{op}} \times \mathcal{V}(\mathcal{H}_{t_2})^{\mathrm{op}}}$$

$$\begin{aligned} (\theta^* \underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})})_{\langle V_1, V_2 \rangle} &:= (\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})})_{\theta(\langle V_1, V_2 \rangle)} = \underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})}_{V_1 \otimes V_2} \\ &\cong \underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}_{V_1} \times \underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})}_{V_2} \end{aligned}$$

using the fact that, for contexts of the form $V_1 \otimes V_2$,

$$\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})}_{V_1 \otimes V_2} \cong \underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}_{V_1} \times \underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})}_{V_2}$$

4.4 Topos Formulation of HPO

Conclusion: To account for both homogeneous and inhomogeneous ('logically entangled') propositions the *intermediate* topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\mathrm{op}} \times \mathcal{V}(\mathcal{H}_{t_2})^{\mathrm{op}}}$ suffices.

But, in full topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})^{\mathrm{op}}}$ there also arise (i) entangled *contexts* $W = V_1 \otimes V_2 + V_3 \otimes V_4$; and (ii) a third type that cannot be expressed in this way. The physical significance of these needs to be explored further.

5. Truth Values

Claim: because of the absence of state-vector reduction in the topos approach, it is meaningful to define the truth value of a (homogeneous) history proposition in terms of the truth values of the individual time components:

$$v\big((A_1 \in \Delta_1)_{t_1} \sqcap (A_2 \in \Delta_2)_{t_2};\, |\psi\rangle_{t_1}\big) := v\big(A_1 \in \Delta_1;\, |\psi\rangle_{t_1}\big) \otimes v\big(A_2 \in \Delta_2;\, |\psi\rangle_{t_2}\big)$$

where $|\psi\rangle_{t_2} = \hat{U}(t_2, t_1)\,|\psi\rangle_{t_1}$.

Want to find a 'topos interpretation' of the above equation.

Truth values

Problem: truth values belong to different topoi

$$v(A_1 \in \Delta_1; |\psi\rangle_{t_1}) := v(\underline{\mathfrak{w}}^{|\psi\rangle_{t_1}} \subseteq \underline{\delta(\hat{P}_1)}) \in \Gamma\underline{\Omega}^{\mathcal{V}(\mathcal{H}_{t_1})}$$

$$v(A_2 \in \Delta_2; |\psi\rangle_{t_2}) := v(\underline{\mathfrak{w}}^{|\psi\rangle_{t_2}} \subseteq \underline{\delta(\hat{P}_2)}) \in \Gamma\underline{\Omega}^{\mathcal{V}(\mathcal{H}_{t_2})}$$

Solution: pull back to the 'intermediate topos' $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\mathrm{op}} \times \mathcal{V}(\mathcal{H}_{t_2})^{\mathrm{op}}}$.
In $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{\mathrm{op}} \times \mathcal{V}(\mathcal{H}_{t_2})^{\mathrm{op}}}$ we have:

$$v(\underline{\mathfrak{w}}^{|\psi\rangle_{t_1}} \subseteq \underline{\delta(\hat{P}_{t_1})}) \otimes v(\underline{\mathfrak{w}}^{|\psi\rangle_{t_2}} \subseteq \underline{\delta(\hat{P}_{t_2})}) \cong v(\underline{\mathfrak{w}}^{|\psi\rangle_{t_1}} \otimes \underline{\mathfrak{w}}^{|\psi\rangle_{t_2}} \subseteq \underline{\delta(\hat{P}_{t_1})} \otimes \underline{\delta(\hat{P}_{t_2})})$$

using the fact that (theorem)

$$\boxed{\Gamma\underline{\Omega}^{\mathcal{V}(\mathcal{H}_{t_1})} \otimes \Gamma\underline{\Omega}^{\mathcal{V}(\mathcal{H}_{t_2})} \simeq \Gamma\underline{\Omega}^{\mathcal{V}(\mathcal{H}_{t_1}) \times \mathcal{V}(\mathcal{H}_{t_2})}}$$

Tensor products of Truth values

Therefore, so long as entangled contexts (plus type III) are not considered it is possible to define truth values of history propositions as tensor products of truth values of individual-time propositions.

In particular, to obtain a topos formulation of quantum history theories the *intermediate* topos suffices.

5. Summary

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  The strategy adopted was to extend the topos formulation of quantum mechanics put forward by Isham and Döring so as to include temporal propositions.

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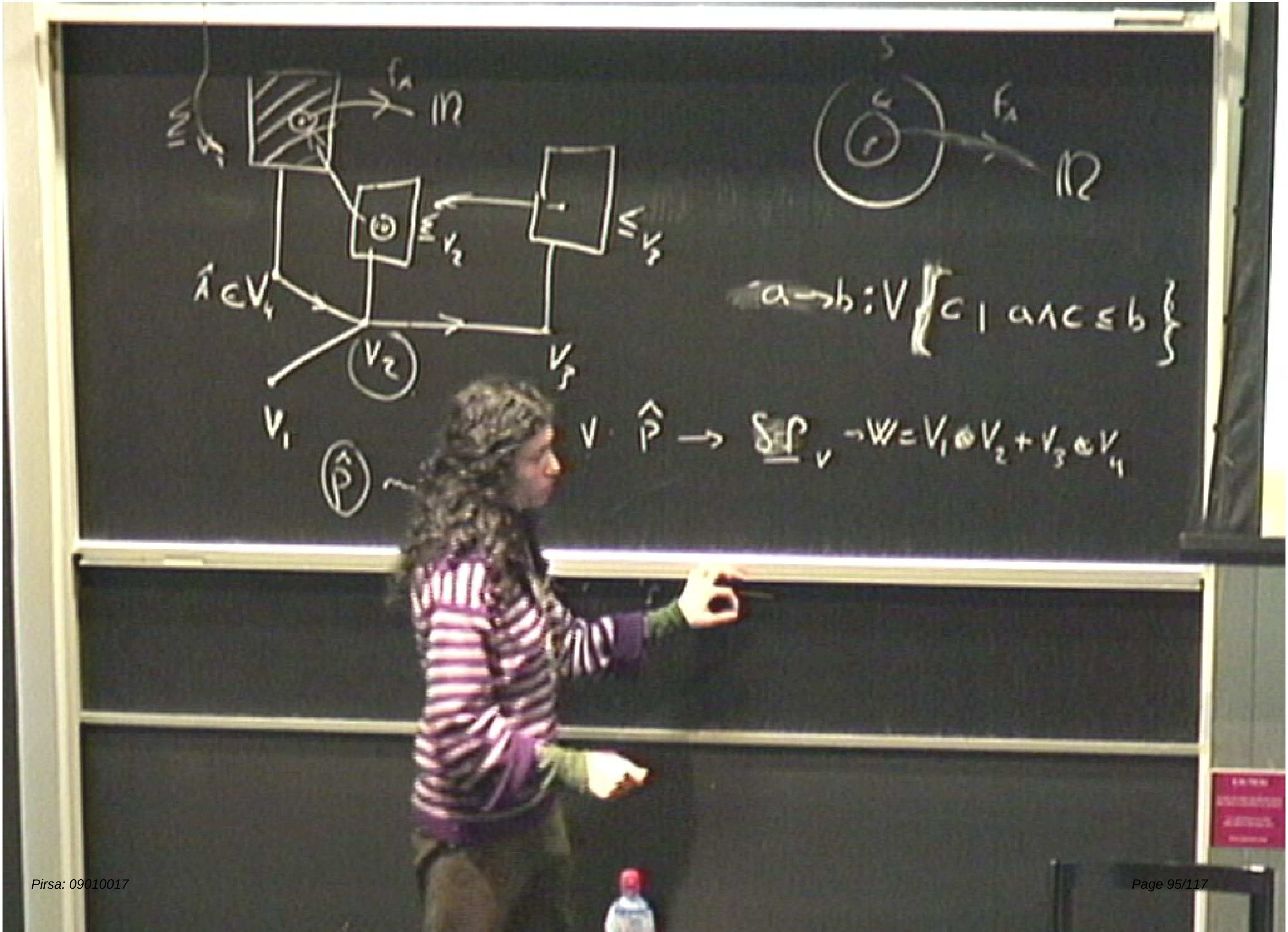
- For non-entangled contexts the two topoi coincide.

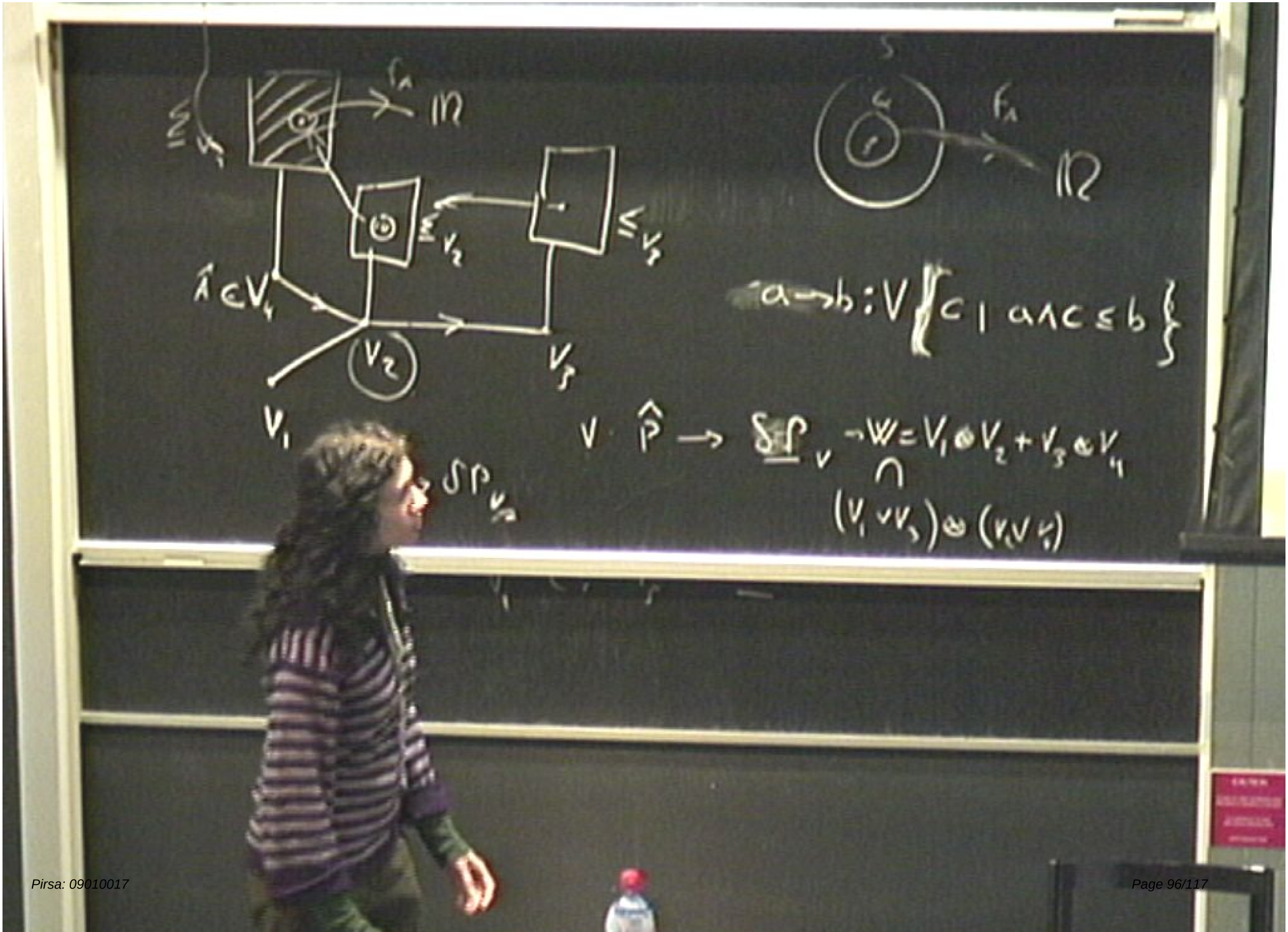
Future Work

1. Is it possible to represent, with a novel mathematical structure, *type III propositions* in topos-theoretical terms?

   – Logical entanglement is captured by the notion of tensor product of Heyting algebra.

   – Quantum entanglement *might* be captured by the notion (yet to be defined) of 'quantum' tensor product.


fλ
IR
≤V2
≤V3
λ̂ ∈ V4
V1
V2
V3
a→b : V {c | a∧c ≤ b}
ρ̂ → δρ_V ¬W = V1⊗V2 + V3⊗V4


a → b : V {c | a∧c ≤ b}
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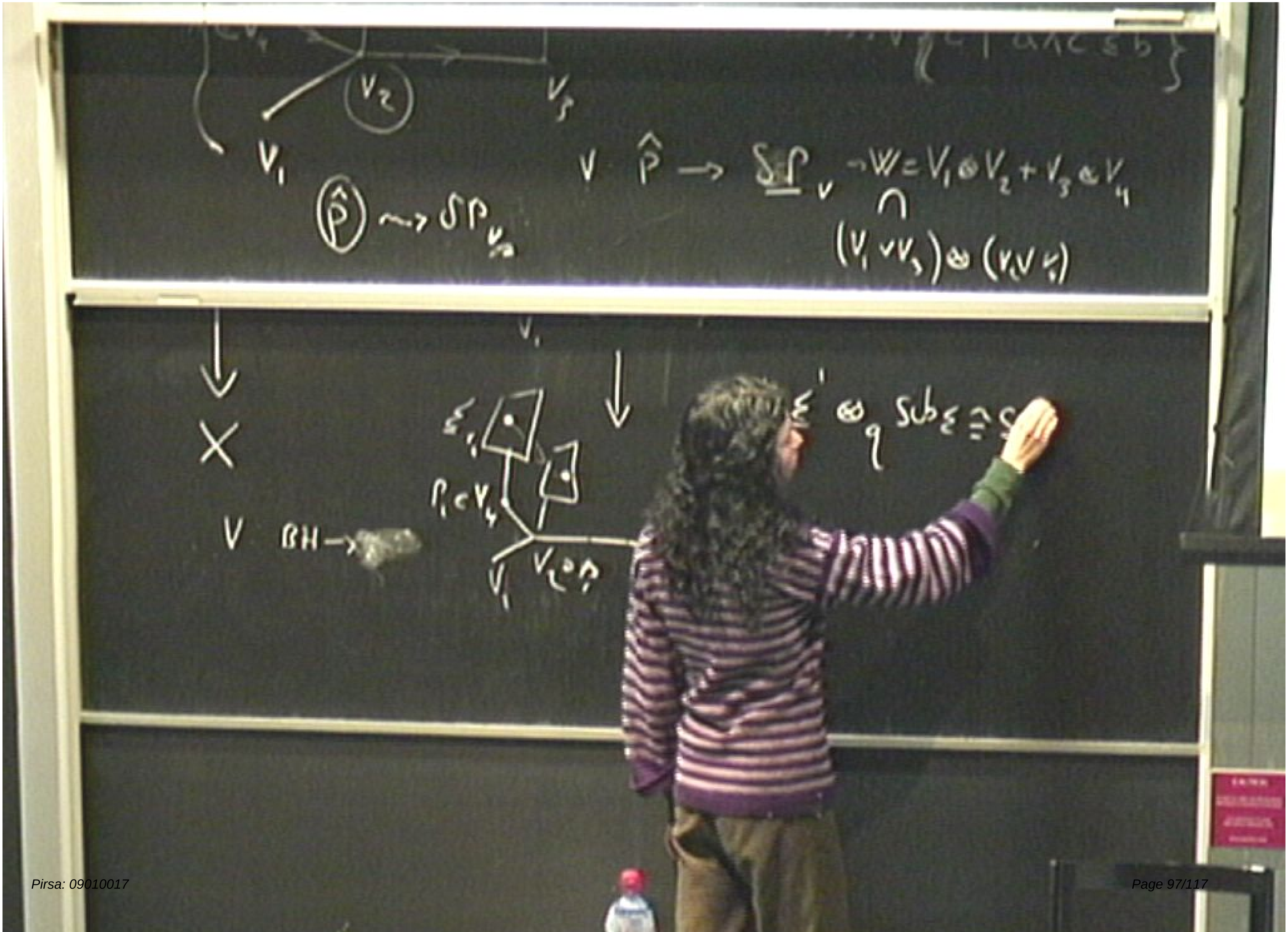
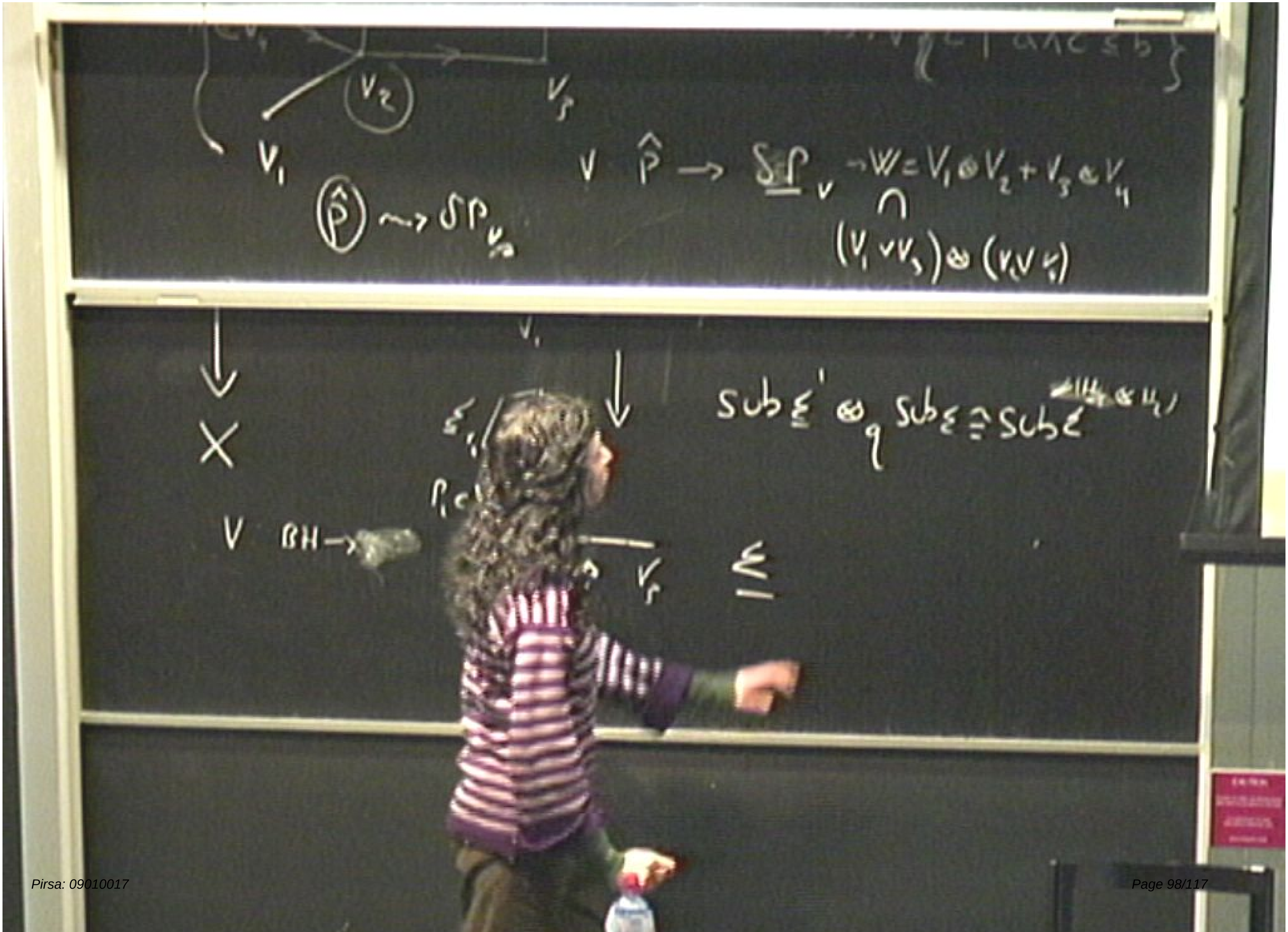
(V1 ∨ V3) ⊗ (V2 ∨ V4)





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2. Extend the topos reformulation of history theory to the case of *continuous* time.
3. Construct a history version of *quantum field theory* using the topos reformulation of history theory.

4.4 Topos Formulation of HPO

Theorem: $Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}) \otimes Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})}) \simeq Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})} \times \underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})})$

- **Conjecture:** Proposition $\alpha_1 \sqcap \alpha_2$ should be represented by
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$\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})$ contains *entangled contexts* $W = V_1 \otimes V_2 + V_3 \otimes V_4$ and so $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})^{op}} \neq \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_{t_1})^{op} \times \mathcal{V}(\mathcal{H}_{t_2})^{op}}$
- Want to find a relation between
$\underline{\delta(\hat{\alpha}_1)} \otimes \underline{\delta(\hat{\alpha}_2)} \in p_1^*(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}) \times p_2^*(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})})$ and
$\underline{\delta(\hat{\alpha}_1 \otimes \hat{\alpha}_2)} \in \underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2})}$.

4.3 Tensor Product in a Topos

- We need to relate the Heyting algebra $Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}) \otimes Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})})$ to sub-objects of some 'state object' in some topos related to quantum theory.

$$\hat{\alpha}_{t_1} \to \underline{\delta(\hat{\alpha}_{t_1})} \in Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}) \in Sets^{\mathcal{V}(\mathcal{H}_{t_1})^{op}}$$
$$\hat{\alpha}_{t_2} \to \underline{\delta(\hat{\alpha}_{t_2})} \in Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})}) \in Sets^{\mathcal{V}(\mathcal{H}_{t_2})^{op}}$$

- To what topos does the history proposition $\underline{\delta(\hat{\alpha}_{t_1})} \otimes \underline{\delta(\hat{\alpha}_{t_2})} \in Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_1})}) \otimes Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_2})})$ belong?

  Need to find a common topos to which both the topoi $Sets^{\mathcal{V}(\mathcal{H}_{t_1})^{op}}$ and $Sets^{\mathcal{V}(\mathcal{H}_{t_2})^{op}}$ can be related.

3.5 Truth values

- This is called a *sieve* on V.

  The collection, $\underline{\Omega}_V$, of all sieves on V forms a **Heyting algebra**!

- For varying V such truth values form a global section $\Gamma\underline{\Omega}$ of the sub-object classifier $\underline{\Omega}$.

> The set $\Gamma\underline{\Omega}$ is itself a Heyting algebra!

- Thus we have a Heyting algebra of propositions *and* a Heyting algebra of 'generalised' truth values!

3.5 Truth Values

- **Truth values in classical physics**

  A proposition $Q \subseteq \mathcal{S}$ is true in a state s iff $s \in Q$.

  This is equivalent to $\{s\} \subseteq Q$.

3.4 States

- **States in classical physics**

  In classical physics a microstate is a point in the state space.

- **States in the topos formulation of quantum theory**

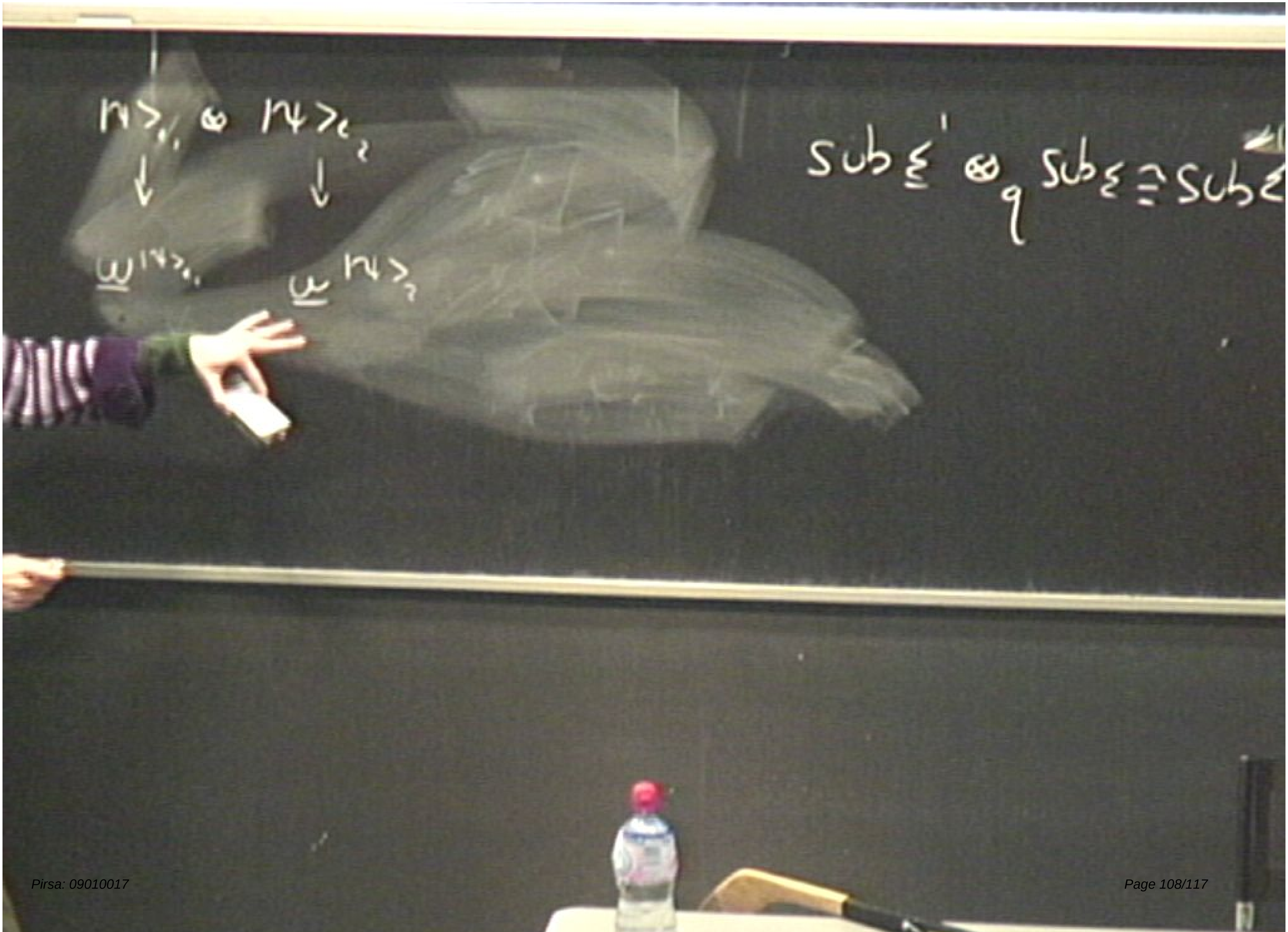
  – $\underline{\Sigma}$ has no 'points' $\Longrightarrow$ Equivalent to the K-S theorem

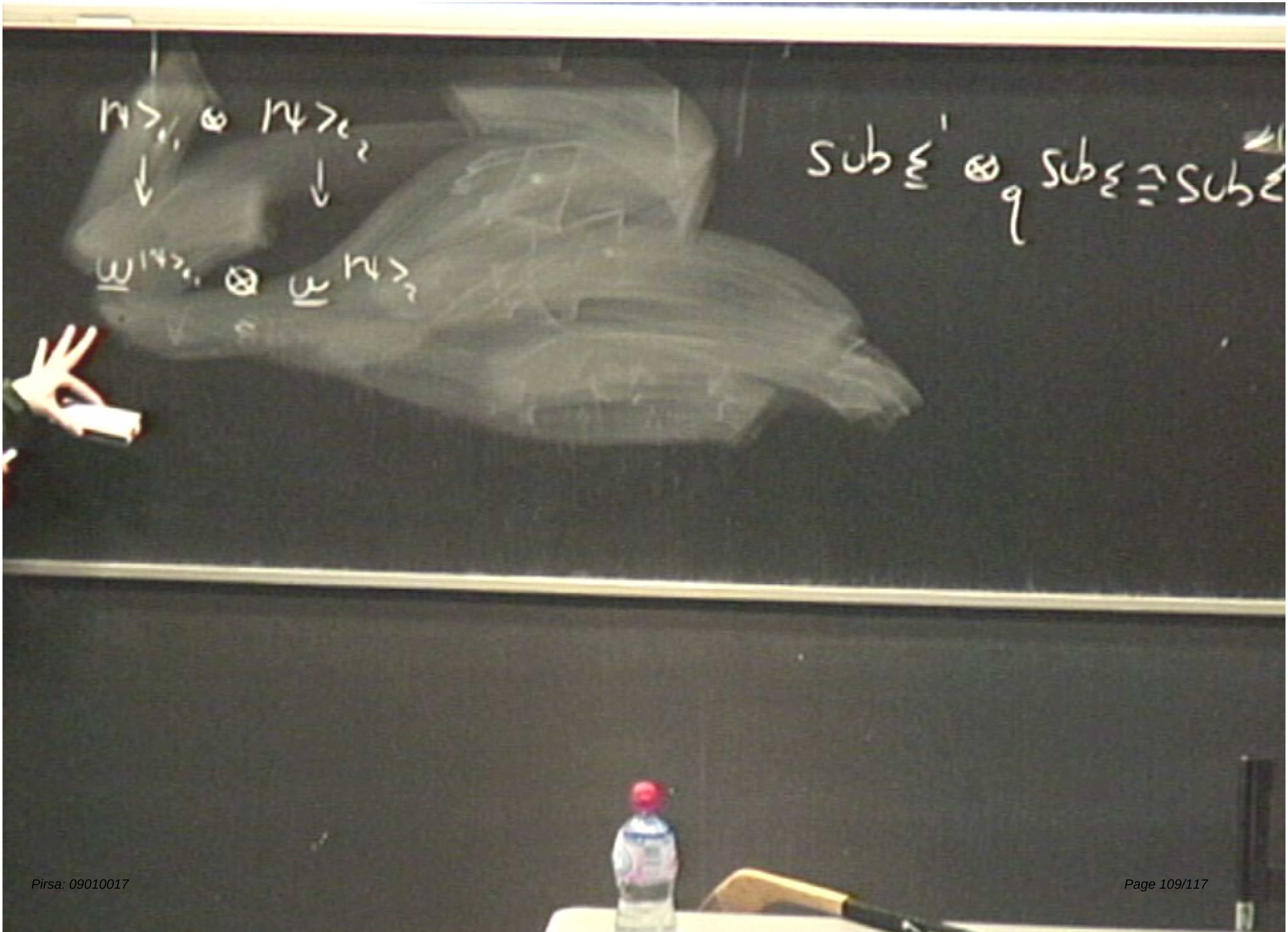
  – Topos analogues of a state is a (non-point!) sub-object of the state object $\underline{\Sigma}$:

  Pseudo-state: $\underline{\mathfrak{w}}^{|\psi\rangle} := \underline{\delta(|\psi\rangle\langle\psi|)} \subseteq \underline{\Sigma}$,

$$\delta(|\psi\rangle\langle\psi|)_V := \bigwedge\{\hat{\alpha} \in P(V) \mid |\psi\rangle\langle\psi| \leq \hat{\alpha}\}$$

$\underline{\mathfrak{w}}^{|\psi\rangle}$ is the 'closest' one can get to defining a point in $\underline{\Sigma}$.


|ψ>c1 ⊗ |ψ>c2
ω|ψ>c1
ω|ψ>2
sub ε' ⊗q sub ε ≅ sub ε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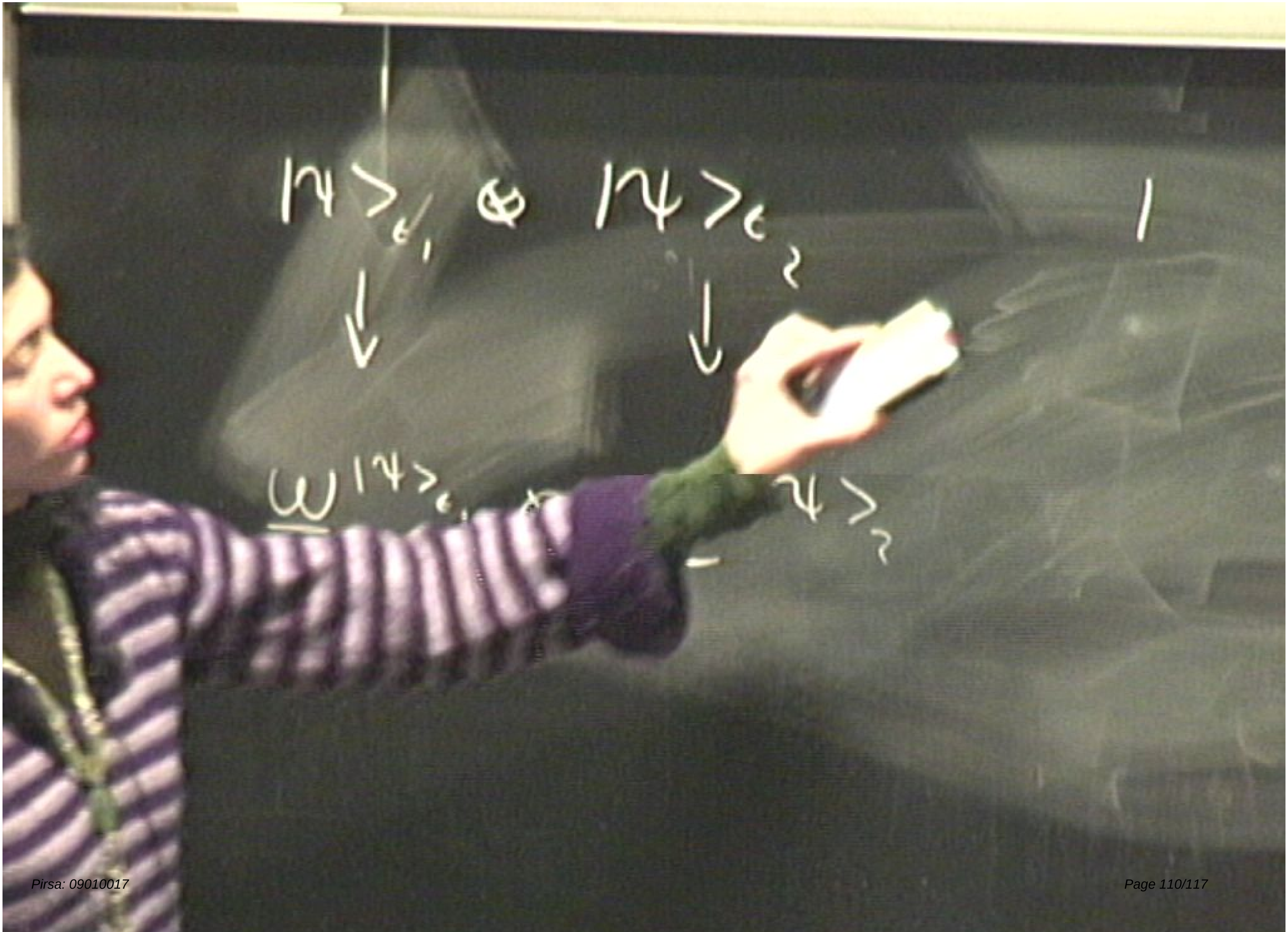
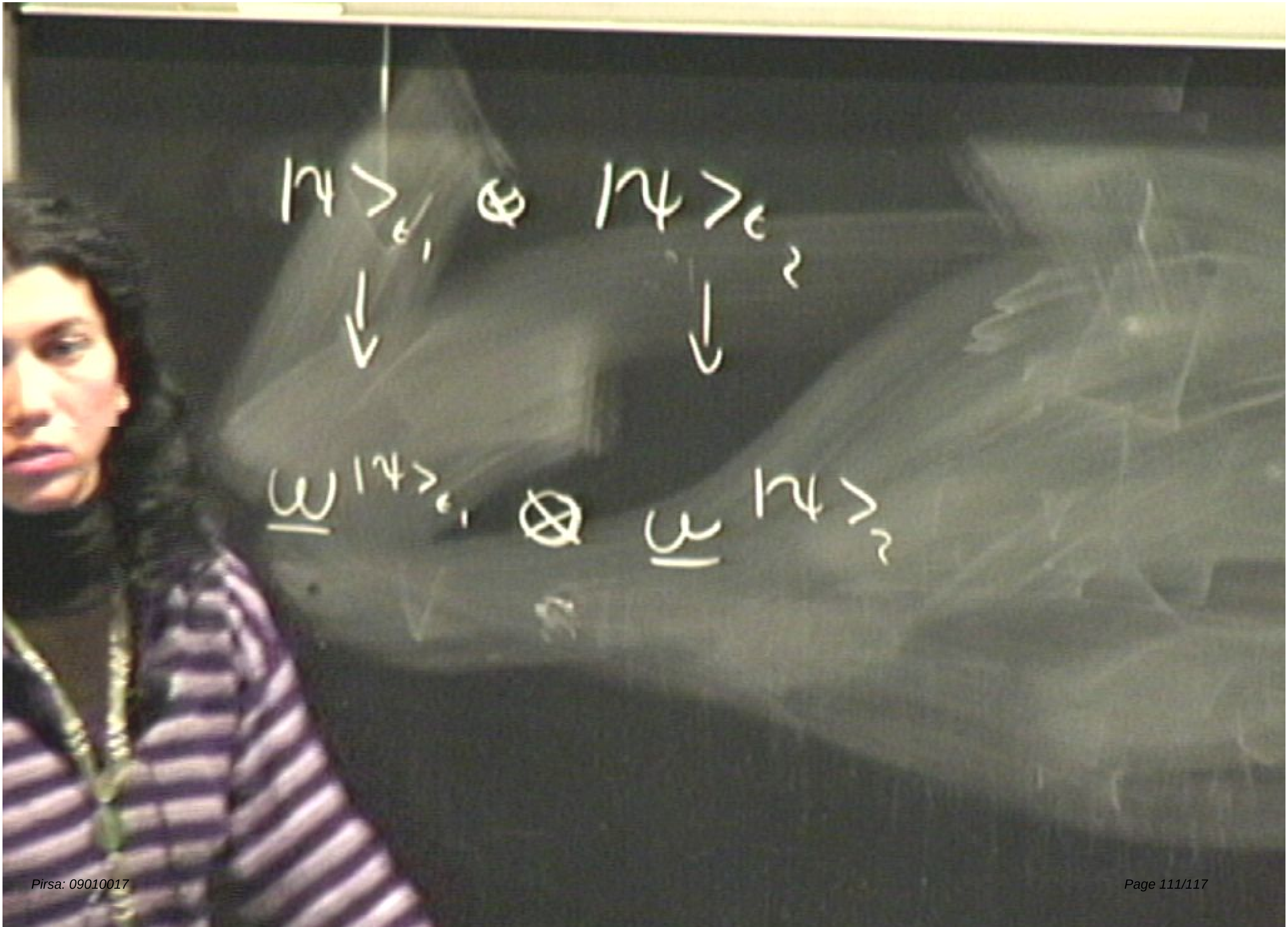
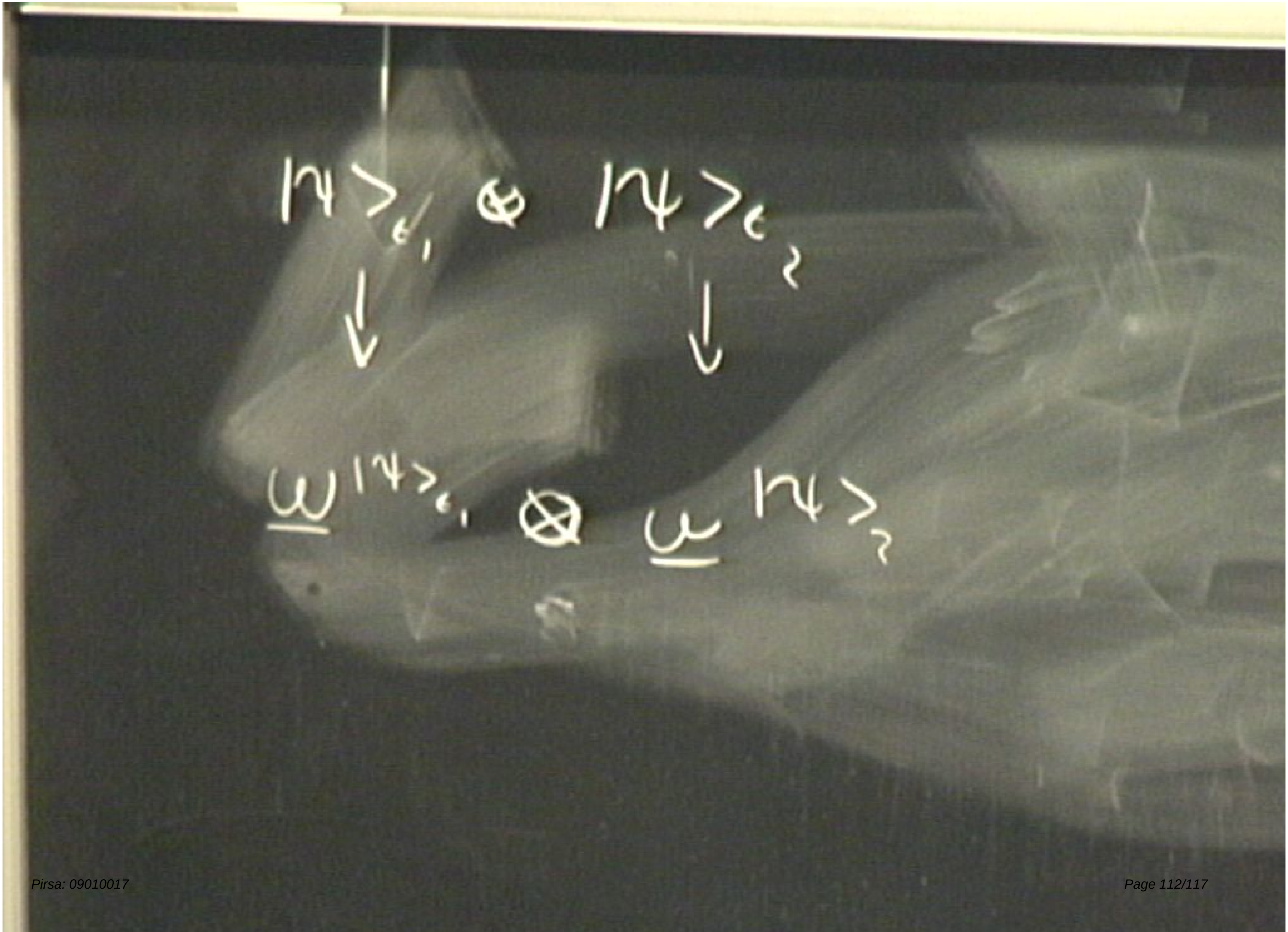



$|\psi\rangle_{t_1} \otimes |\psi\rangle_{t_2}$
$\downarrow \quad \downarrow$
$\underline{\omega}^{|\psi\rangle_{t_1}} \otimes \underline{\omega}^{|\psi\rangle_2}$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3.3 Daseinisation

- **'Daseinisation':**

$$\delta : P(\mathcal{H}) \to P(V)$$
$$P \mapsto \delta(\hat{P})_V$$

where $\delta(\hat{P})_V := \bigwedge\{\hat{\alpha} \in P(V) | \hat{\alpha} \geq \hat{P}\}$: the 'best' approximation of $\hat{P}$ (from above) by projectors in V.

- **Relation to** $Sub(\underline{\Sigma})$:

Any projector in V gives a subset of $\underline{\Sigma}_V$. Therefore get map, for each V, $\hat{P} \mapsto \underline{\delta(\hat{P})}_V$.

Can show that this corresponds to $\delta : P(\mathcal{H}) \to Sub(\underline{\Sigma})$!

Thus δ maps propositions about a quantum system to a *distributive* lattice in a *contextual* manner.

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4.2 Temporal Structure in a Heyting algebra

- **Aim**: Find a *topos* representation of the homogeneous history $\alpha = (A_1 \in \Delta_1)_{t_1} \sqcap (A_2 \in \Delta_2)_{t_2} \cdots \sqcap (A_n \in \Delta_n)_{t_n}$
  - Individual-time propositions are identified with sub-objects of the .spectral presheaf $\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_i})}$. Collection of all such sub-objects, $Sub(\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_{t_i})})$, forms a Heyting algebra.
  - Temporal structure of Heyting algebras requires tensor product of Heyting algebras? Yes!

4.4 Topos Formulation of HPO

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