Title: Lattice chirality: a mission (im)possible?

Date: Jan 06, 2009 11:00 AM

URL: http://pirsa.org/09010016

Abstract: As LHC era is coming close, all sorts of ideas about physics beyond the standard model are being explored. It remains possible that a strong-coupling chiral theory could appear at TeV scale. When it comes to strongly coupled theories, lattice is still the most reliable and straightforward regularization method. But defining a chiral gauge theory on the lattice is formidable on its own. In this talk, I will present some most recent theoretical developments in attempt to tackle this problem, and explain some general theorems we proved for generic chiral gauge theories on lattice. These results should be useful in future studies in the field. I will also present some numerical results suggesting that the idea of constructing a chiral gauge theory by decoupling the mirror fermions using a high scale strong- coupling gauge symmetric phase suffers from severe constraints. I will end my talk by a brief outlook on how one may hope to reach a conclusive prove on the feasibility of this idea in general.

Might be possible on the lattice

Strong-coupling symmetric phase

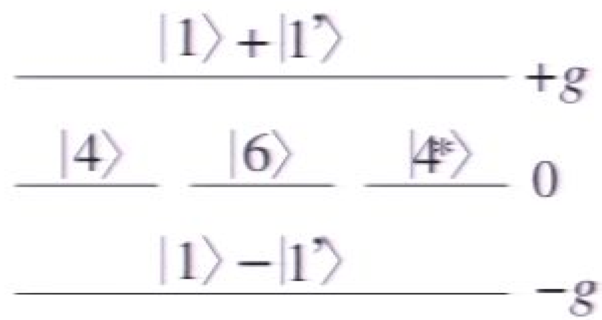
- Everybody knows that four-fermi interactions, if coupling taken strong enough, break chiral symmetries

$$\frac{g}{\Lambda^2}(\overline{\psi}\psi)(\overline{\psi}\psi), \quad gN > 8\pi^2$$

- However, if one takes coupling even stronger, the theory enters a "strong-coupling symmetric phase": with only massive excitations and unbroken chiral symmetry
- These phases are "lattice artifact" as the massive excitations are heavier than the UV cutoff
- Strong coupling expansion has a finite range of convergence and is applicable in any dimension.

"SU(4)" toy model continues …

- On each lattice site, there are 16 states$= \mathbf{1} + \mathbf{1}' + \mathbf{4} + \mathbf{4}^* + \mathbf{6}$
- $H_{4\psi}$ conserves F mod 4, and connects $\mathbf{1}$ (zero occupation) and $\mathbf{1}'$ (full occupation) only. The spectrum:

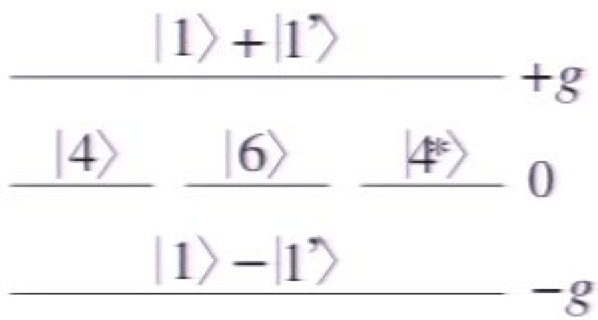


A unique ground state with a gap$= g$, singlet of $SU(4)$.

- At first order in $\frac{1}{g}$, hopping turns on, site-localized states form bands and propagate. The propagating degree of freedom is heavy, mass $\sim \frac{g}{a} \gg \frac{1}{a}$
- $1/g$ strong-coupling expansion has a finite radius of convergence. For sufficiently large g, the ground state analysis remains valid

"SU(4)" toy model continues ...

- On each lattice site, there are 16 states$= \mathbf{1} + \mathbf{1}' + \mathbf{4} + \mathbf{4}^* + \mathbf{6}$
- $H_{4\psi}$ conserves F mod 4, and connects $\mathbf{1}$ (zero occupation) and $\mathbf{1}'$ (full occupation) only. The spectrum:



A unique ground state with a gap$= g$, singlet of $SU(4)$.

- At first order in $\frac{1}{g}$, hopping turns on, site-localized states form bands and propagate. The propagating degree of freedom is heavy, mass $\sim \frac{g}{a} \gg \frac{1}{a}$
- $1/g$ strong-coupling expansion has a finite radius of convergence. For sufficiently large g, the ground state analysis remains valid

"SU(4)" toy model continues ...

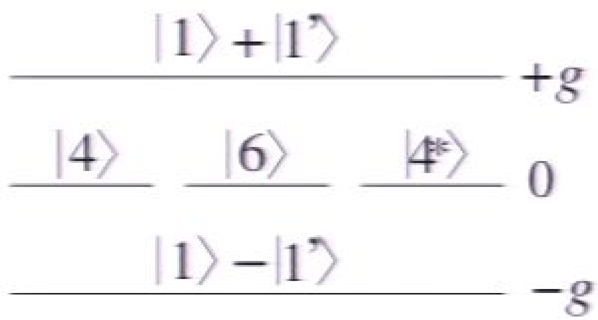
- If we turn on the gauge field, it appears only in hopping terms and the contributions to heavy sector should be $\sim \frac{1}{g}$.
- With a bit more group theory, same can be repeated for $SU(5)$ of E-P.
- Singlet needed:

$$g_1 : \mathbf{10}^* - \mathbf{5} - \mathbf{5} - \mathbf{1}$$
$$g_2 : \mathbf{10}^* - \mathbf{10}^* - \mathbf{10}^* - \mathbf{5}$$

  to break all mirror global symmetries, including anomalous ones to prevent extra zero modes (*more explanations come later*). At infinite g_1, g_2, $SU(5)$ ground state is unique and singlet.
- The idea is essentially to use this to decouple the mirror fermions

"SU(4)" toy model continues …

- On each lattice site, there are 16 states$= \mathbf{1} + \mathbf{1}' + \mathbf{4} + \mathbf{4}^* + \mathbf{6}$
- $H_{4\psi}$ conserves F mod 4, and connects $\mathbf{1}$ (zero occupation) and $\mathbf{1}'$ (full occupation) only. The spectrum:



A unique ground state with a gap$= g$, singlet of $SU(4)$.

- At first order in $\frac{1}{g}$, hopping turns on, site-localized states form bands and propagate. The propagating degree of freedom is heavy, mass $\sim \frac{g}{a} \gg \frac{1}{a}$
- $1/g$ strong-coupling expansion has a finite radius of convergence. For sufficiently large g, the ground state analysis remains valid

"SU(4)" toy model continues ...

- If we turn on the gauge field, it appears only in hopping terms and the contributions to heavy sector should be $\sim \frac{1}{g}$.
- With a bit more group theory, same can be repeated for $SU(5)$ of E-P.
- Singlet needed:

$$g_1 : \mathbf{10}^* - \mathbf{5} - \mathbf{5} - \mathbf{1}$$
$$g_2 : \mathbf{10}^* - \mathbf{10}^* - \mathbf{10}^* - \mathbf{5}$$

to break all mirror global symmetries, including anomalous ones to prevent extra zero modes (*more explanations come later*). At infinite g_1, g_2, $SU(5)$ ground state is unique and singlet.

- The idea is essentially to use this to decouple the mirror fermions

Gauged XY model

Unitary higgs field can induce a strong-coupling symmetric phase in lattice gauge theories.

- $$-S_\kappa = \sum_{\mathbf{x}} \left(\frac{\beta}{2} \prod_{\text{plaq}} U + \frac{\kappa}{2} \sum_{\hat{\mu}} \phi^*_{\mathbf{x}} U_{\mathbf{x},\mathbf{x}+\hat{\mu}} \phi_{\mathbf{x}+\hat{\mu}} \right) + \text{h.c.}$$

  where $\phi_{\mathbf{x}} = e^{i\eta_{\mathbf{x}}}$ is a unitary field.
- $\kappa < 1$, the theory is in a strong-coupling symmetric phase
- D. R. T. Jones, J. B. Kogut and D. K. Sinclair, Phys. Rev. D **19** (1979) 1882. ...

"'Light from chaos' in two dimensions,"

E. Poppitz and YS, arXiv:0801.0587 [hep-lat].

Gauged XY model

Unitary higgs field can induce a strong-coupling symmetric phase in lattice gauge theories.

- $$-S_{\kappa} = \sum_{\mathbf{x}} \left(\frac{\beta}{2} \prod_{\text{plaq}} U + \frac{\kappa}{2} \sum_{\hat{\mu}} \phi^*_{\mathbf{x}} U_{\mathbf{x},\mathbf{x}+\hat{\mu}} \phi_{\mathbf{x}+\hat{\mu}} \right) + \text{h.c.}$$

  where $\phi_{\mathbf{x}} = e^{i\eta_{\mathbf{x}}}$ is a unitary field.
- $\kappa < 1$, the theory is in a strong-coupling symmetric phase
- D. R. T. Jones, J. B. Kogut and D. K. Sinclair, Phys. Rev. D **19** (1979) 1882. …

"'Light from chaos' in two dimensions,"

E. Poppitz and YS, arXiv:0801.0587 [hep-lat].

"'Light from chaos' in two dimensions,"

E. Poppitz and YS, arXiv:0801.0587 [hep-lat].

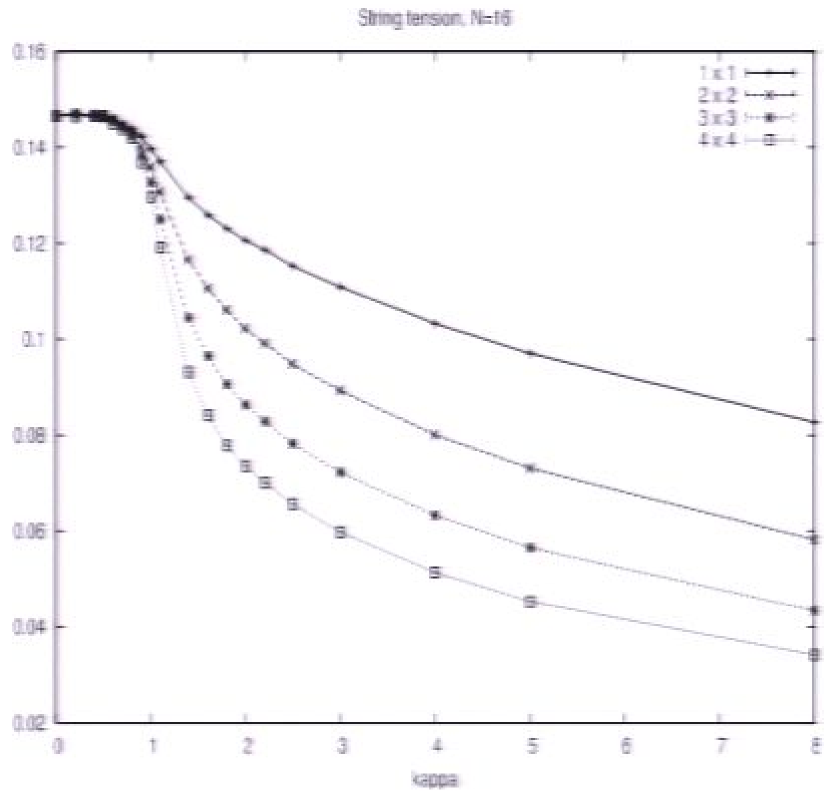


Figure: String tension vs κ for $N = 16$.

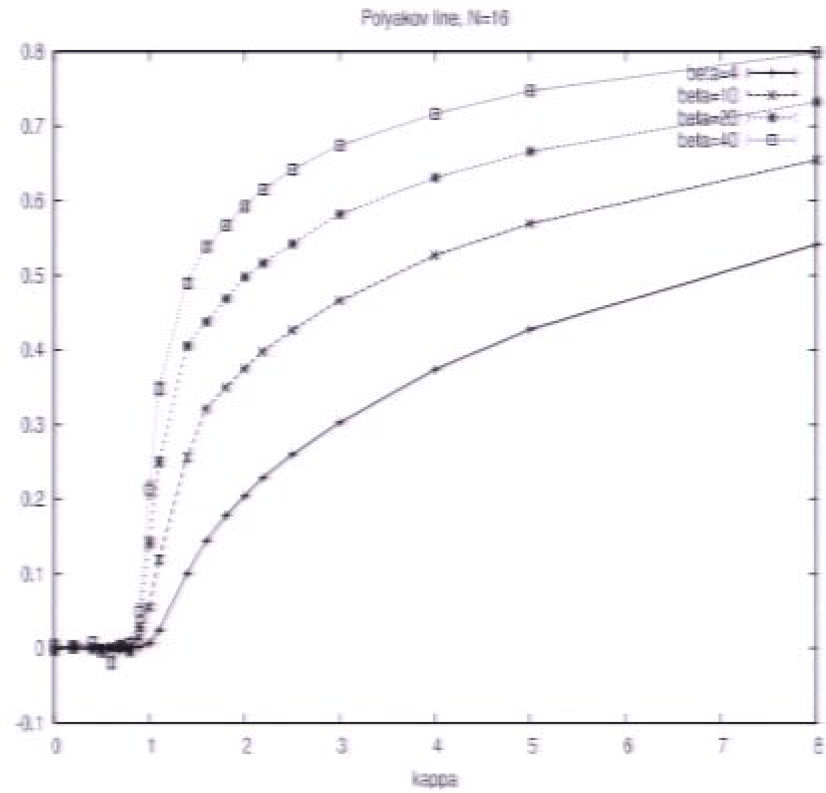


Figure: Polyakov line vs κ for $N = 16$.

Gauged XY model

Unitary higgs field can induce a strong-coupling symmetric phase in lattice gauge theories.

- $$-S_\kappa = \sum_{\mathbf{x}} \left(\frac{\beta}{2} \prod_{\text{plaq}} U + \frac{\kappa}{2} \sum_{\hat{\mu}} \phi^*_{\mathbf{x}} U_{\mathbf{x},\mathbf{x}+\hat{\mu}} \phi_{\mathbf{x}+\hat{\mu}} \right) + \text{h.c.}$$

  where $\phi_{\mathbf{x}} = e^{i\eta_{\mathbf{x}}}$ is a unitary field.
- $\kappa < 1$, the theory is in a strong-coupling symmetric phase
- D. R. T. Jones, J. B. Kogut and D. K. Sinclair, Phys. Rev. D **19** (1979) 1882. …

"'Light from chaos' in two dimensions,"

E. Poppitz and YS, arXiv:0801.0587 [hep-lat].

"'Light from chaos' in two dimensions,"

E. Poppitz and YS, arXiv:0801.0587 [hep-lat].

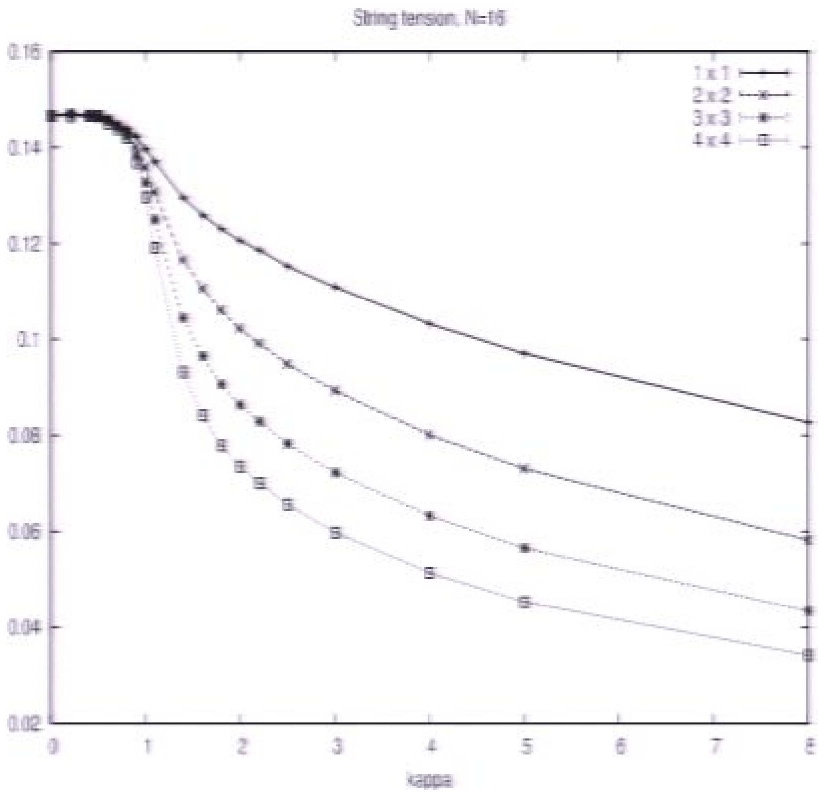


Figure: String tension vs κ for $N = 16$.

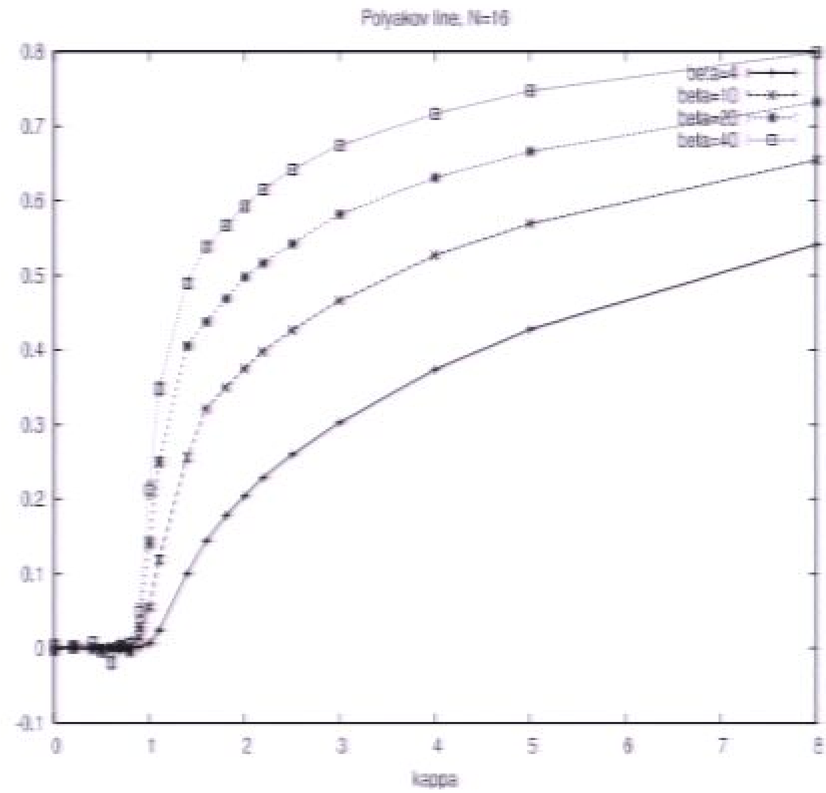


Figure: Polyakov line vs κ for $N = 16$.

So why wasn't this problem solved long ago?

- It was not known how to define an exact (non-gauged) chiral symmetry on the lattice.
- After a series of seminal papers in the 90's (Ginsparg/Wilson, Kaplan, Narayanan/Neuberger, Neuberger, P. Hasenfratz/Laliena/Niedermayer, Luescher, Neuberger), it was realized that an exact chiral symmetry can be defined on a lattice with finite spacing.
- Can defined L and R components of a Dirac spinor (Not the usual Weyl): overlap (domain wall) fermions.
- Lüscher proved the consistency of a chiral $U(1)$ gauge theory using the overlap fermions. (*More on this later.*)
- *An obvious question: can the "E-P" dream be resurrected now?* Some earlier numerical simulation suggested the answer might be yes

So why wasn't this problem solved long ago?

- It was not known how to define an exact (non-gauged) chiral symmetry on the lattice.
- After a series of seminal papers in the 90's (Ginsparg/Wilson, Kaplan, Narayanan/Neuberger, Neuberger, P. Hasenfratz/Laliena/Niedermayer, Luescher, Neuberger), it was realized that an exact chiral symmetry can be defined on a lattice with finite spacing.
- Can defined L and R components of a Dirac spinor (Not the usual Weyl): overlap (domain wall) fermions.
- Lüscher proved the consistency of a chiral $U(1)$ gauge theory using the overlap fermions. (*More on this later.*)
- *An obvious question: can the "E-P" dream be resurrected now?* Some earlier numerical simulation suggested the answer might be yes

So why wasn't this problem solved long ago?

- It was not known how to define an exact (non-gauged) chiral symmetry on the lattice.
- After a series of seminal papers in the 90's (Ginsparg/Wilson, Kaplan, Narayanan/Neuberger, Neuberger, P. Hasenfratz/Laliena/Niedermayer, Luescher, Neuberger), it was realized that an exact chiral symmetry can be defined on a lattice with finite spacing.
- Can defined L and R components of a Dirac spinor (Not the usual Weyl): overlap (domain wall) fermions.
- Lüscher proved the consistency of a chiral $U(1)$ gauge theory using the overlap fermions. (*More on this later.*)
- *An obvious question: can the "E-P" dream be resurrected now?* Some earlier numerical simulation suggested the answer might be yes

A toy model using overlap fermions: 0–1 model

J. Giedt and E. Poppitz, JHEP **0710**, 076 (2007) [arXiv:hep-lat/0701004].

- Using overlap fermions, studied a toy model:

$$
\begin{aligned}
S =& S_{\text{light}} + S_{\text{mirror}} \\
S_{\text{light}} =& (\overline{\psi}_+, \mathrm{D}_1 \psi_+) + (\overline{\chi}_-, \mathrm{D}_0 \chi_-) \\
S_{\text{mirror}} =& (\overline{\psi}_-, \mathrm{D}_1 \psi_-) + (\overline{\chi}_+, \mathrm{D}_0 \chi_+) \\
& + y\{(\overline{\psi}_-, \phi^* \chi_+) + (\overline{\chi}_+, \phi \psi_-) \\
& + h[(\psi_-^T, \phi \gamma_2 \chi_+) - (\overline{\chi}_+, \gamma_2 \phi^* \overline{\psi}_-^T)]\} \\
S_\kappa =& \frac{\kappa}{2} \sum_{\mathbf{x},\hat{\mu}} [2 - (\phi_{\mathbf{x}}^* U_{\mathbf{x},\mathbf{x}+\hat{\mu}} \phi_{\mathbf{x}+\hat{\mu}} + \text{h.c.})]
\end{aligned}
$$

$\phi_{\mathbf{x}} = e^{i\eta_{\mathbf{x}}}$ is a unitary higgs field and $(\psi, \chi) = \sum_{\mathbf{x}} \psi_{\mathbf{x}} \cdot \chi_{\mathbf{x}}$

- Zero gauge field background

Found evidence that while y large and $h > 1$, the charged mirror fermions and ϕ are heavy

Evidence: scalar is heavy

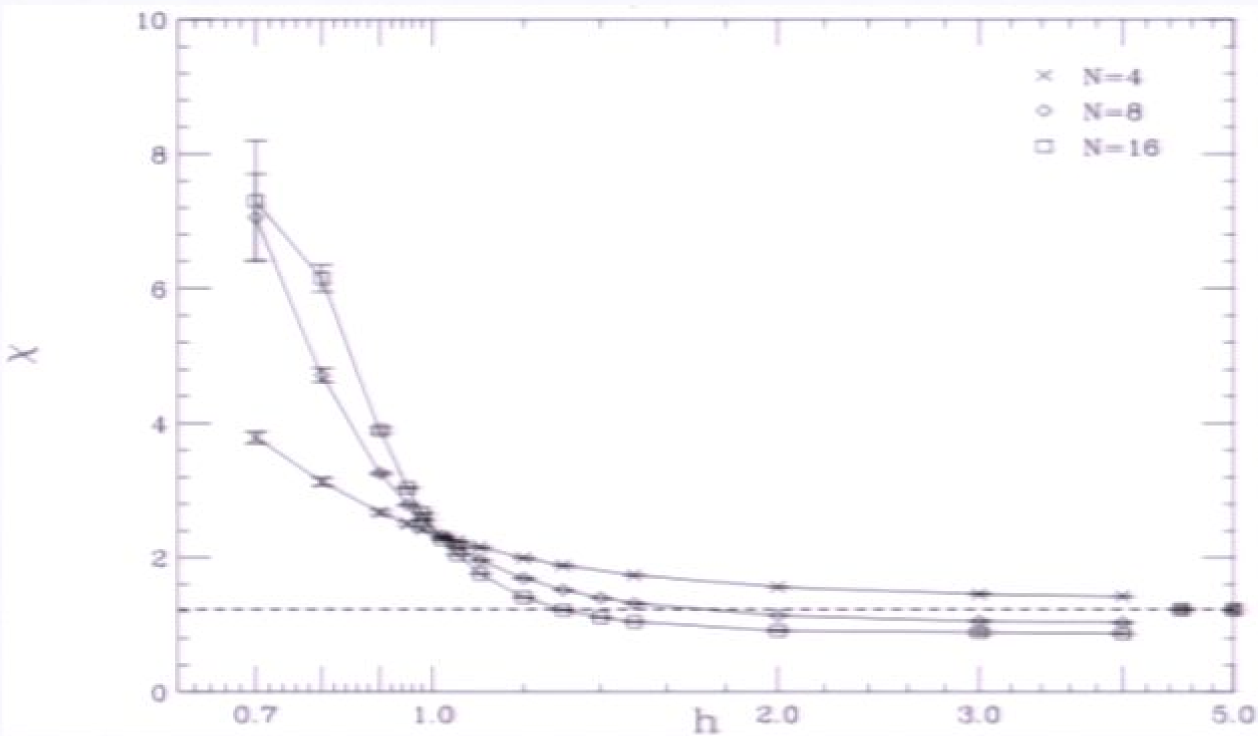


Figure: Susceptibilities of ϕ for $\kappa = 0.1$ and $N = 4, 8, 16$. Dash line indicates the susceptibility of ϕ in pure XY-model

A toy model using overlap fermions: 0–1 *model*

J. Giedt and E. Poppitz, JHEP **0710**, 076 (2007) [arXiv:hep-lat/0701004].

- Using overlap fermions, studied a toy model:

$$
\begin{aligned}
S =& S_{\text{light}} + S_{\text{mirror}} \\
S_{\text{light}} =& (\overline{\psi}_+, \mathrm{D}_1 \psi_+) + (\overline{\chi}_-, \mathrm{D}_0 \chi_-) \\
S_{\text{mirror}} =& (\overline{\psi}_-, \mathrm{D}_1 \psi_-) + (\overline{\chi}_+, \mathrm{D}_0 \chi_+) \\
&+ y\{(\overline{\psi}_-, \phi^* \chi_+) + (\overline{\chi}_+, \phi \psi_-) \\
&+ h[(\psi_-^T, \phi \gamma_2 \chi_+) - (\overline{\chi}_+, \gamma_2 \phi^* \overline{\psi}_-^T)]\} \\
S_\kappa =& \frac{\kappa}{2} \sum_{\mathbf{x},\hat{\mu}} [2 - (\phi_{\mathbf{x}}^* U_{\mathbf{x},\mathbf{x}+\hat{\mu}} \phi_{\mathbf{x}+\hat{\mu}} + \text{h.c.})]
\end{aligned}
$$

$\phi_{\mathbf{x}} = e^{i\eta_{\mathbf{x}}}$ is a unitary higgs field and $(\psi, \chi) = \sum_{\mathbf{x}} \psi_{\mathbf{x}} \cdot \chi_{\mathbf{x}}$

- Zero gauge field background

Found evidence that while y large and $h > 1$, the charged mirror fermions and ϕ are heavy

Evidence: scalar is heavy

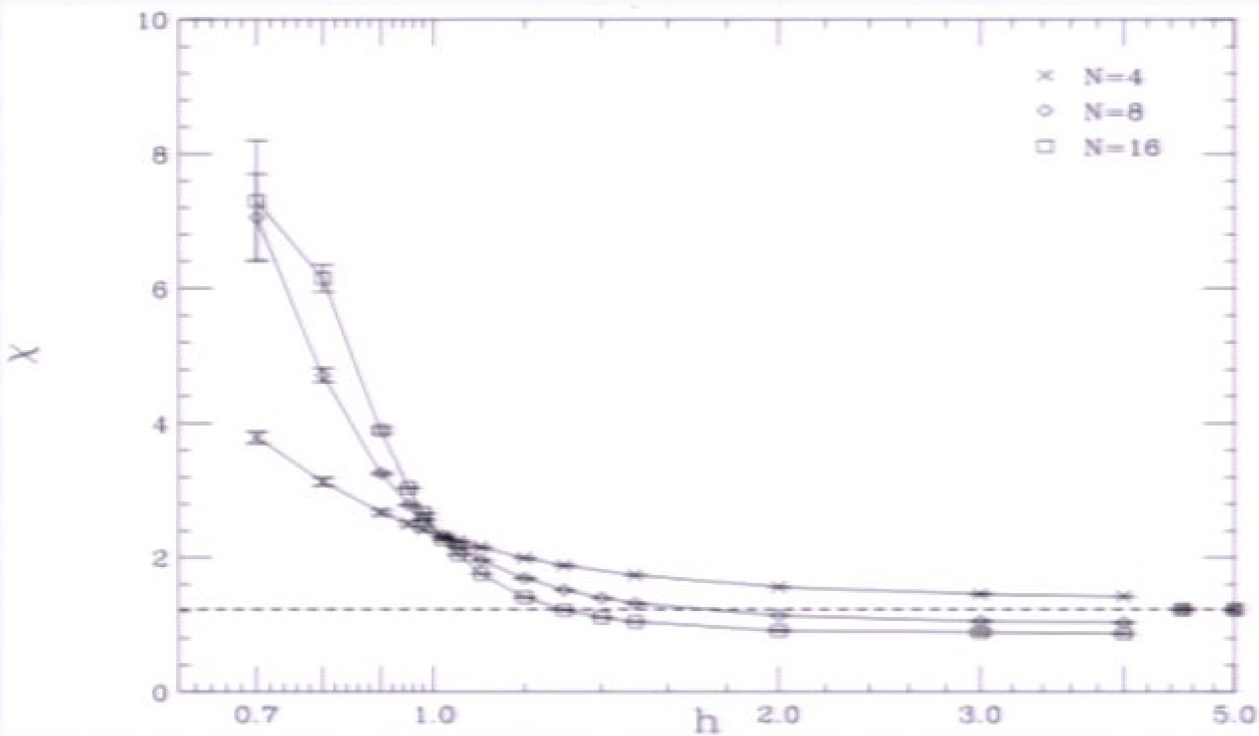


Figure: Susceptibilities of ϕ for $\kappa = 0.1$ and $N = 4, 8, 16$. Dash line indicates the susceptibility of ϕ in pure XY-model

Numerical evidence continues . . .

fermions are heavy

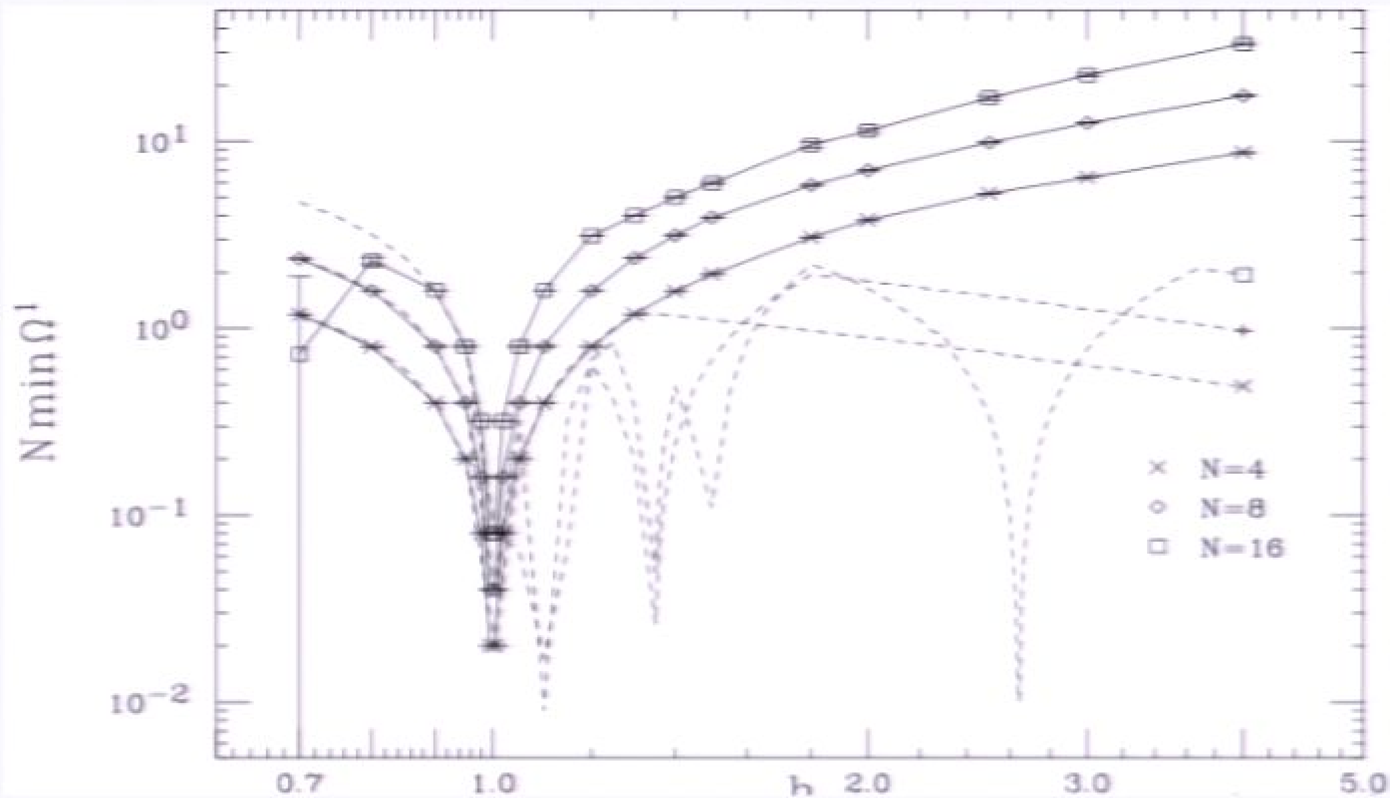


Figure: The lower bound on the charged mirror fermion mass for $\kappa = 0.1$

So did the dream come true?

- If the mirror parts are all heavy, at the low energy we get a chiral gauge theory on the lattice automatically, circumventing the difficulty of defining it explicitly. Great!
- Are we sure? Two big questions:
  - That entire mirror sector is heavy?
  - Is the continuum limit unitary?
- Why are we worried?
  The light content is anomalous (and so is the mirror sector).

$$S_{\text{light}} = (\overline{\psi}_-, D_1 \psi_-) + (\overline{\chi}_-, D_0 \chi_-)$$

The splitting between light and mirror, with both sectors anomalous but the mirror sector all heavy, in fact appears paradoxical.

So did the dream come true?

- If the mirror parts are all heavy, at the low energy we get a chiral gauge theory on the lattice automatically, circumventing the difficulty of defining it explicitly. Great!
- Are we sure? Two big questions:
  - That entire mirror sector is heavy?
  - Is the continuum limit unitary?
- Why are we worried?
  The light content is anomalous (and so is the mirror sector).

$$S_{\text{light}} = (\overline{\psi}_+, \mathrm{D}_1 \psi_+) + (\overline{\chi}_-, \mathrm{D}_0 \chi_-)$$

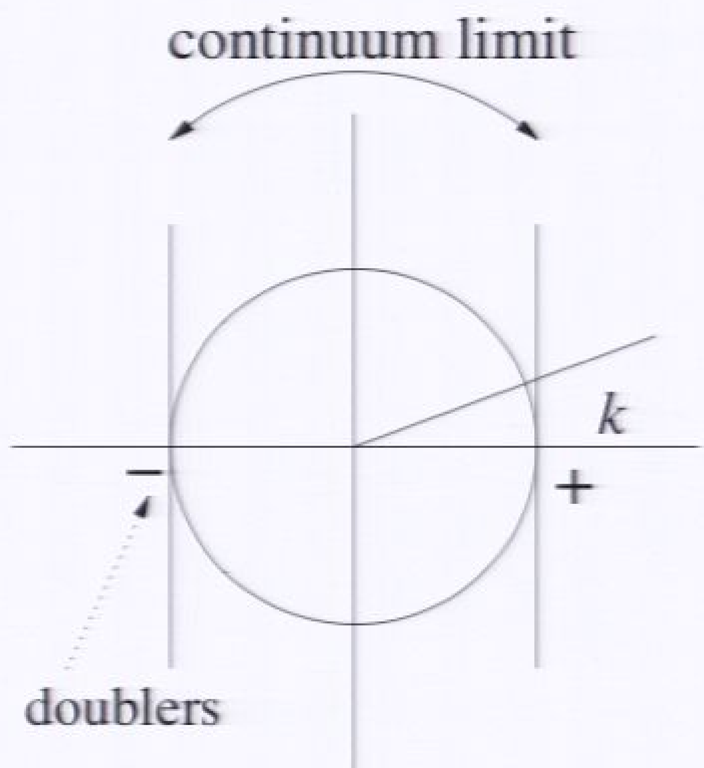
The splitting between light and mirror, with both sectors anomalous but the mirror sector all heavy, in fact appears paradoxical.

Outline

1. *Motivation and idea*
   - Why chiral gauge theory, why lattice
   - Why need the idea of "decoupling of mirror fermions"
   - Does it work: early numerical work suggest maybe
2. *A quick review of some most recent theoretical developments*
   - Exact lattice chiral symmetry
   - Put the formalism on a completely general ground
   - A powerful simple theorem
3. *A paradox in the* $0-1$ *model and the unitarity of the GW formalism*
4. *Toward a more analytical approach*

Fermion doubling problem

Naive discretization of Dirac operator causes fermion species doubling. On a finite lattice, the momentum is an angular variable that lives on a circle. The continuum limit is the small $\sin k$ limit, which is always paired.



Ginsparg-Wilson operator

Ginsparg-Wilson, 1982: "A remnant of chiral symmetry on the lattice"

$$\{\mathrm{D}, \gamma_5\} = a\mathrm{D}\gamma_5\mathrm{D}$$

$a = 1$ in our convention. D is γ_5 Hermitian:

$$(\gamma_5\mathrm{D})^\dagger = \gamma_5 D$$

As

$$\mathrm{D} \sim \mathbf{k}$$

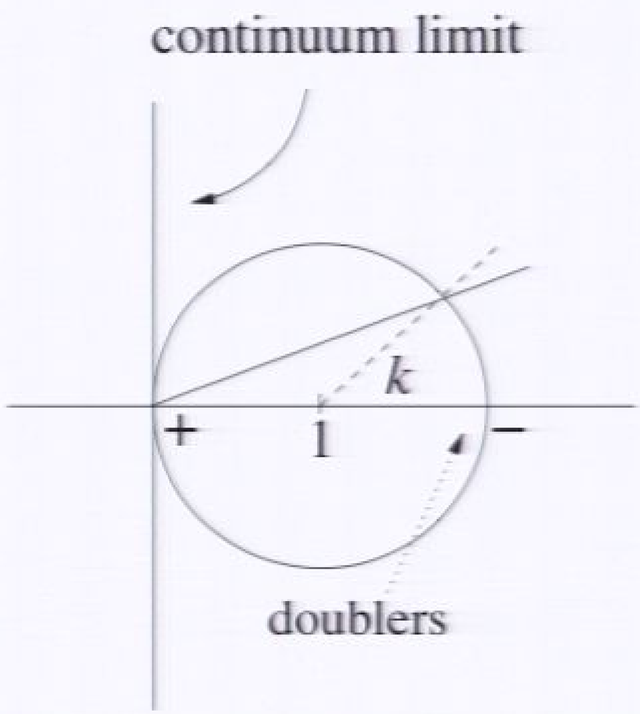
In the continuum limit: $\mathbf{k} \to 0$, the usual anti-commutative relationship between Dirac operator and γ_5 recovered

No fermion doubling problem

The GW properties imply that the eigenvalues of D, λ, satisfy

$$(\lambda^* - 1)(\lambda - 1) = 1$$

The “doublers” become heavy with mass equals 2 in lattice unit.



Ginsparg-Wilson operator

Ginsparg-Wilson, 1982: "A remnant of chiral symmetry on the lattice"

$$\{\mathrm{D}\,,\,\gamma_5\} = a\mathrm{D}\gamma_5\mathrm{D}$$

$a = 1$ in our convention. D is γ_5 Hermitian:

$$(\gamma_5\mathrm{D})^\dagger = \gamma_5 D$$

As

$$\mathrm{D} \sim \mathbf{k}$$

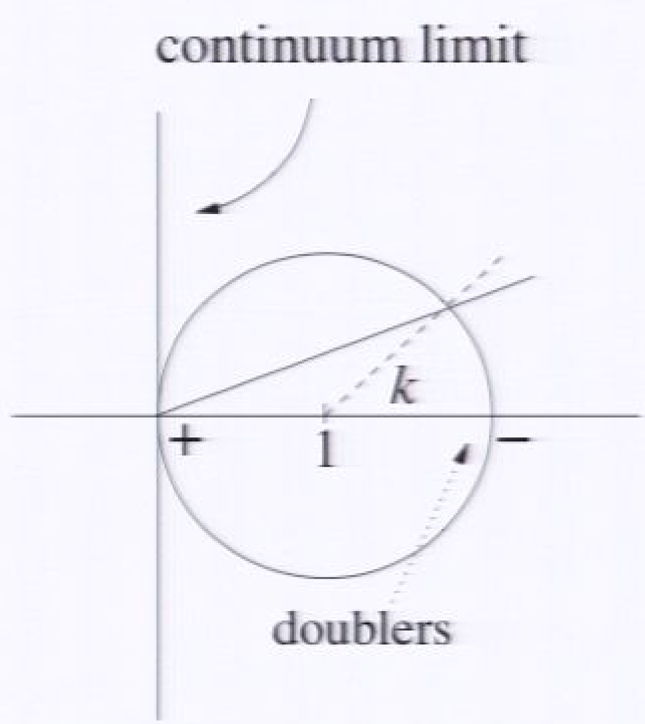
In the continuum limit: $\mathbf{k} \to 0$, the usual anti-commutative relationship between Dirac operator and γ_5 recovered

No fermion doubling problem

The GW properties imply that the eigenvalues of D, λ, satisfy

$$(\lambda^* - 1)(\lambda - 1) = 1$$

The "doublers" become heavy with mass equals 2 in lattice unit.



A new kind of "chiral symmetry" on the lattice

A new "γ_5"

If we define: $\hat{\gamma}_5 = (1 - \mathrm{D})\gamma_5$, GW implies

$$\hat{\gamma}_5^2 = 1 \quad \text{and} \quad \hat{\gamma}_5 \mathrm{D} = -\mathrm{D}\gamma_5$$

$\Rightarrow$ A new exact "chiral symmetry" on the lattice

The kinetic term in the action:

$$S = \sum_x \overline{\psi}_x \mathrm{D}_{xy} \psi_y$$

is invariant under the rotation:

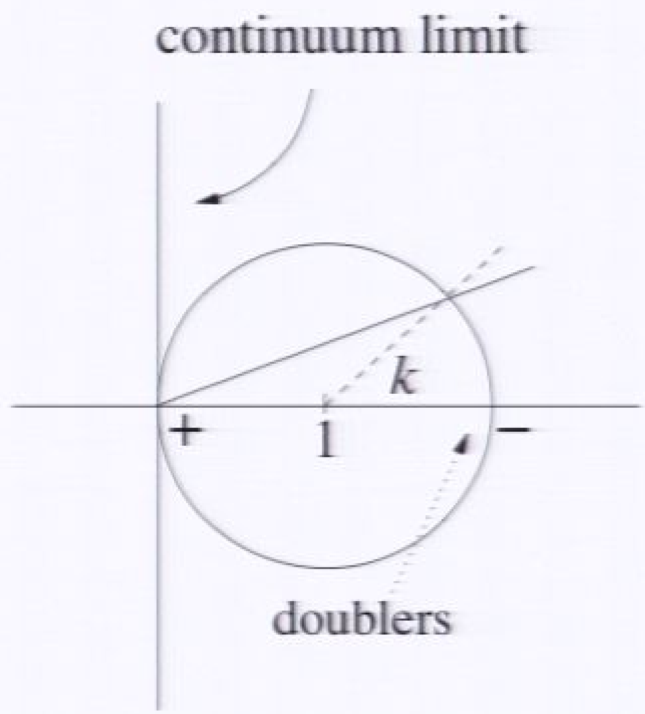
$$\psi \to e^{i\alpha\hat{\gamma}_5}\psi, \quad \overline{\psi} \to \overline{\psi} e^{i\alpha\gamma_5}$$

No fermion doubling problem

The GW properties imply that the eigenvalues of D, λ, satisfy

$$(\lambda^* - 1)(\lambda - 1) = 1$$

The "doublers" become heavy with mass equals 2 in lattice unit.



Ginsparg-Wilson operator

Ginsparg-Wilson, 1982: "A remnant of chiral symmetry on the lattice"

$$\{\mathrm{D}\,,\,\gamma_5\} = a\mathrm{D}\gamma_5\mathrm{D}$$

$a = 1$ in our convention. D is γ_5 Hermitian:

$$(\gamma_5\mathrm{D})^\dagger = \gamma_5 D$$

As

$$\mathrm{D} \sim \mathbf{k}$$

In the continuum limit: $\mathbf{k} \to 0$, the usual anti-commutative relationship between Dirac operator and γ_5 recovered

A new kind of "chiral symmetry" on the lattice

A new "γ_5"

If we define: $\hat{\gamma}_5 = (1 - D)\gamma_5$, GW implies

$$\hat{\gamma}_5^2 = 1 \quad \text{and} \quad \hat{\gamma}_5 D = -D\gamma_5$$

$\Rightarrow$ A new exact "chiral symmetry" on the lattice

The kinetic term in the action:

$$S = \sum_x \bar{\psi}_x D_{xy} \psi_y$$

is invariant under the rotation:

$$\psi \to e^{i\alpha\hat{\gamma}_5}\psi, \qquad \bar{\psi} \to \bar{\psi} e^{i\alpha\gamma_5}$$

Ginsparg-Wilson operator

Ginsparg-Wilson, 1982: "A remnant of chiral symmetry on the lattice"

$$\{ \mathrm{D}\,,\, \gamma_5 \} = a\mathrm{D}\gamma_5\mathrm{D}$$

$a = 1$ in our convention. D is γ_5 Hermitian:

$$(\gamma_5\mathrm{D})^\dagger = \gamma_5 D$$

As

$$\mathrm{D} \sim \mathbf{k}$$

In the continuum limit: $\mathbf{k} \to 0$, the usual anti-commutative relationship between Dirac operator and γ_5 recovered

A new kind of "chiral symmetry" on the lattice

A new "γ_5"

If we define: $\hat{\gamma}_5 = (1 - D)\gamma_5$, GW implies

$$\hat{\gamma}_5^2 = 1 \quad \text{and} \quad \hat{\gamma}_5 D = -D\gamma_5$$

$\Rightarrow$ A new exact "chiral symmetry" on the lattice

The kinetic term in the action:

$$S = \sum_x \bar{\psi}_x D_{xy} \psi_y$$

is invariant under the rotation:

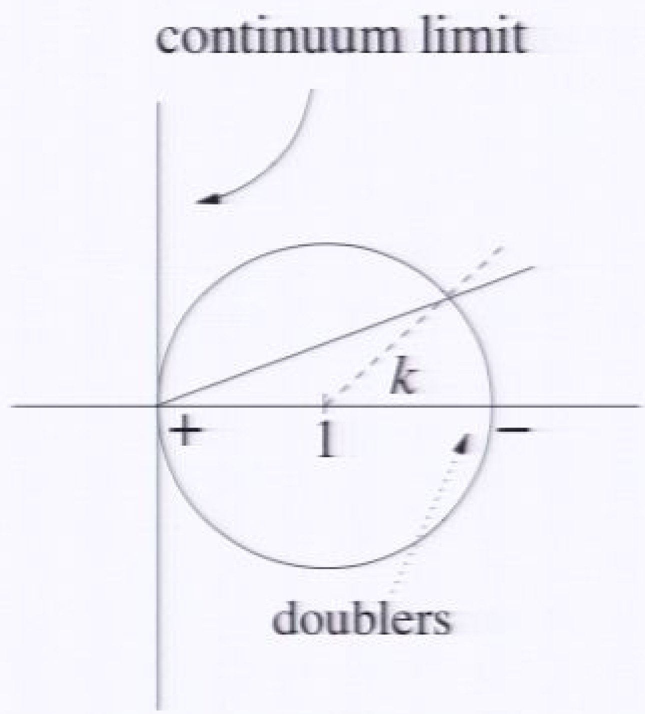
$$\psi \to e^{i\alpha\hat{\gamma}_5}\psi, \quad \bar{\psi} \to \bar{\psi} e^{i\alpha\gamma_5}$$

No fermion doubling problem

The GW properties imply that the eigenvalues of D, λ, satisfy

$$(\lambda^* - 1)(\lambda - 1) = 1$$

The "doublers" become heavy with mass equals 2 in lattice unit.



A new kind of "chiral symmetry" on the lattice

A new "γ_5"

If we define: $\hat{\gamma}_5 = (1 - \mathrm{D})\gamma_5$, GW implies

$$\hat{\gamma}_5^2 = 1 \quad \text{and} \quad \hat{\gamma}_5 \mathrm{D} = -\mathrm{D}\gamma_5$$

$\Rightarrow$ A new exact "chiral symmetry" on the lattice

The kinetic term in the action:

$$S = \sum_{\mathbf{x}} \overline{\psi}_{\mathbf{x}} \mathrm{D}_{\mathbf{xy}} \psi_{\mathbf{y}}$$

is invariant under the rotation:

$$\psi \to e^{i\alpha\gamma_5}\psi, \quad \overline{\psi} \to \overline{\psi} e^{i\alpha\hat{\gamma}_5}$$

The GW (overlap) chiral fermions

- Define chiral projection operator for ψ and $\overline{\psi}$ separately:

$$P_{\pm} = \frac{1 \pm \gamma_5}{2}, \quad \hat{P}_{\pm} = \frac{1 \mp \hat{\gamma}_5}{2}$$

- defined the chiral components of a Dirac spinor as

$$\psi_{\pm} = P_{\pm}\psi, \quad \overline{\psi}_{\pm} = \overline{\psi}\hat{P}_{\pm}$$

- then, build the "chiral theories" in terms of these new chiral spinors:

$$S = \sum_{\mathbf{x}} \overline{\psi}_{+} \mathrm{D} \psi_{+}$$

Fascinating theoretical achievement on lattice chiral gauge theory

Ginsparg, Wilson (1982); Callan, Harvey (1985); D.B. Kaplan (1992); Narayanan, Neuberger (1994); Neuberger (1997); P. Hasenfratz, Laliena, Niedermaier (1997); Luescher (1998); Neuberger (1998),

- No fermion doubling problem
- exact lattice chiral symmetry
- exact lattice gauge anomaly and lattice index theorem
- exact Ward identities, axial charge violation, . . .

But chiral gauge theory remains a hard problem

The fermion measure is ambiguous

- While gauging the chiral symmetry, D is convariantized. It depends on the gauge field background, so are $\hat{\gamma}_5$ and $\hat{P}$.
- Defining fermion measure in gauge theory becomes difficult

Only theories well studied before us were $U(1)$ gauged fermion bi-linear theory: $S = \overline{\psi}_+ \mathrm{D} \psi_+$, for which a non-ambiguous measure proven to exist by Lüscher

We need something more general to understand our “0-1” model

A new kind of "chiral symmetry" on the lattice

A new "γ_5"

If we define: $\hat{\gamma}_5 = (1 - \mathrm{D})\gamma_5$, GW implies

$$\hat{\gamma}_5^2 = 1 \quad \text{and} \quad \hat{\gamma}_5 \mathrm{D} = -\mathrm{D}\gamma_5$$

$\Rightarrow$ A new exact "chiral symmetry" on the lattice

The kinetic term in the action:

$$S = \sum_{\mathbf{x}} \overline{\psi}_{\mathbf{x}} \mathrm{D}_{\mathbf{xy}} \psi_{\mathbf{y}}$$

is invariant under the rotation:

$$\psi \to e^{i\alpha\gamma_5}\psi, \quad \overline{\psi} \to \overline{\psi} e^{i\alpha\hat{\gamma}_5}$$

The GW (overlap) chiral fermions

- Define chiral projection operator for ψ and $\overline{\psi}$ separately:

$$P_\pm = \frac{1 \pm \gamma_5}{2}, \quad \hat{P}_\pm = \frac{1 \mp \hat{\gamma}_5}{2}$$

- defined the chiral components of a Dirac spinor as

$$\psi_\pm = P_\pm \psi, \quad \overline{\psi}_\pm = \overline{\psi} \hat{P}_\pm$$

- then, build the “chiral theories” in terms of these new chiral spinors:

$$S = \sum_{\mathbf{x}} \overline{\psi}_+ \mathrm{D} \psi_+$$

But chiral gauge theory remains a hard problem

The fermion measure is ambiguous

- While gauging the chiral symmetry, D is convariantized. It depends on the gauge field background, so are $\hat{\gamma}_5$ and $\hat{P}$.
- Defining fermion measure in gauge theory becomes difficult

Only theories well studied before us were $U(1)$ gauged fermion bi-linear theory: $S = \overline{\psi}_+ D\psi_+$, for which a non-ambiguous measure proven to exist by Lüscher

We need something more general to understand our “0-1” model

Some general results in the GW formalism:

E. Poppitz and YS, “Lattice chirality and the decoupling of mirror fermions,” JHEP **0708**, 081 (2007) [arXiv:0706.1043 [hep-th]]

- Chiral action S, a functional of the spiniors that satisfies:

$$S[\psi, \overline{\psi}, O] = S[P\psi, \overline{\psi}, O] = S[\psi, \overline{\psi}\hat{P}, O]$$

  O denotes any local operators the theory may introduce. P and $\hat{P}$ are the projection operators.

- Choose particular sets of orthonormal basis $\{u_i, v_i\}$:

$$P u_i = u_i, \quad v_i^{\dagger}\hat{P} = v_i^{\dagger}$$

  and defined the partition function

$$Z = \int \prod_{i,j} dc_i d\overline{c}_j \, e^{S[\sum_i c_i u_i, \sum_j \overline{c}_j v_j^{\dagger}, O]}$$

Some general results in the GW formalism:

E. Poppitz and YS, "Lattice chirality and the decoupling of mirror fermions," JHEP **0708**, 081 (2007) [arXiv:0706.1043 [hep-th]]

- Chiral action S, a functional of the spiniors that satisfies:

$$S[\psi, \overline{\psi}, O] = S[P\psi, \overline{\psi}, O] = S[\psi, \overline{\psi}\hat{P}, O]$$

O denotes any local operators the theory may introduce. P and $\hat{P}$ are the projection operators.

- Choose particular sets of orthonormal basis $\{u_i, v_i\}$:

$$P u_i = u_i, \quad v_i^\dagger \hat{P} = v_i^\dagger$$

and defined the partition function

$$Z = \int \prod_{ij} \mathrm{d}c_i \mathrm{d}\bar{c}_j \, e^{S[\sum c_i u_i, \sum_j \bar{c}_j v_j^\dagger, O]}$$

Some general results in the GW formalism:

E. Poppitz and YS, "Lattice chirality and the decoupling of mirror fermions," JHEP **0708**, 081 (2007) [arXiv:0706.1043 [hep-th]]

- Chiral action S, a functional of the spinors that satisfies:

$$S[\psi, \overline{\psi}, O] = S[P\psi, \overline{\psi}, O] = S[\psi, \overline{\psi}\hat{P}, O]$$

  O denotes any local operators the theory may introduce. P and $\hat{P}$ are the projection operators.
- Choose particular sets of orthonormal basis $\{u_i,\ v_i\}$:

$$P\,u_i = u_i, \quad v_i^\dagger \hat{P} = v_i^\dagger$$

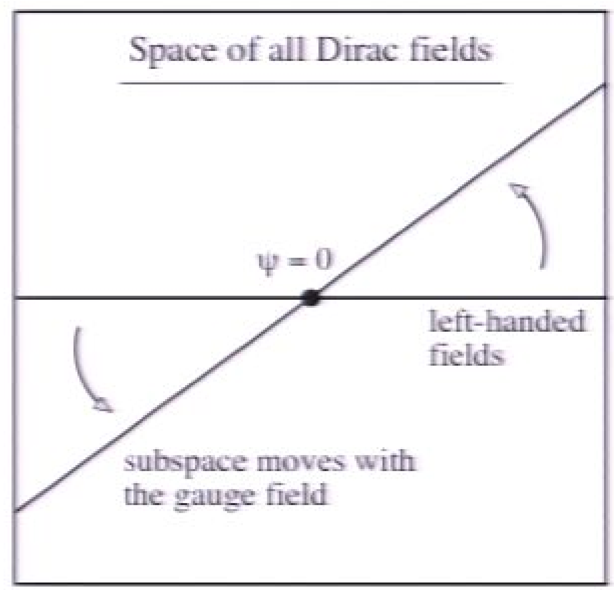
  and defined the partition function

$$Z = \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\overline{c}_j \, e^{S\left[\sum_i c_i u_i, \sum_j \overline{c}_j v_j^\dagger, O\right]}$$

Chiral partition function is ambiguous

- Suppose we choose a different set of basis $v'_i = \mathcal{U}_{ij} v_j$, $\mathcal{U}$ unitary matrix, then $Z \to \det \mathcal{U} \cdot Z$, the ambiguity is always a pure phase
- Usually not a problem because this phase is just an unphysical constant
- A serious problem in GW-formalism:

P depends on the gauge backgroud. $\Rightarrow$ chiral spinors live in different subspace when the gauge field varies. It appears that the effective action of the gauge field U contains a completely arbitrary phase.



Some general results in the GW formalism:

E. Poppitz and YS, "Lattice chirality and the decoupling of mirror fermions," JHEP **0708**, 081 (2007) [arXiv:0706.1043 [hep-th]]

- Chiral action S, a functional of the spiniors that satisfies:

$$S[\psi,\overline{\psi},O]=S[P\psi,\overline{\psi},O]=S[\psi,\overline{\psi}\hat{P},O]$$

  O denotes any local operators the theory may introduce. P and $\hat{P}$ are the projection operators.
- Choose particular sets of orthonormal basis $\{u_i,\ v_i\}$:

$$P\,u_i=u_i,\quad v_i^\dagger\hat{P}=v_i^\dagger$$

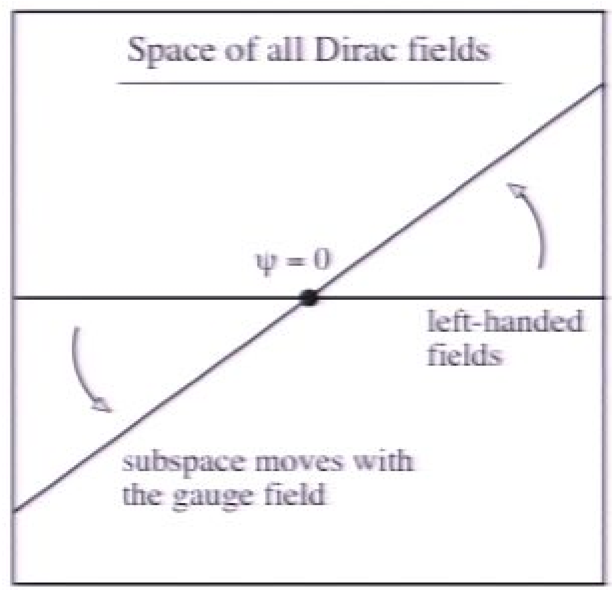
  and defined the partition function

$$Z=\int\prod_{i,j}\mathrm{d}c_i\mathrm{d}\overline{c}_j\,e^{S\left[\sum_i c_iu_i,\sum_j\overline{c}_jv_j^\dagger,O\right]}$$

Chiral partition function is ambiguous

- Suppose we choose a different set of basis $v'_i = \mathcal{U}_{ij} v_j$, $\mathcal{U}$ unitary matrix, then $Z \to \det \mathcal{U} \cdot Z$, the ambiguity is always a pure phase
- Usually not a problem because this phase is just an unphysical constant
- A serious problem in GW-formalism:

P depends on the gauge backgroud. $\Rightarrow$ chiral spinors live in different subspace when the gauge field varies. It appears that the effective action of the gauge field U contains a completely arbitrary phase.



Some general results in the GW formalism:

E. Poppitz and YS, "Lattice chirality and the decoupling of mirror fermions," JHEP **0708**, 081 (2007) [arXiv:0706.1043 [hep-th]]

- Chiral action S, a functional of the spiniors that satisfies:

$$S[\psi, \overline{\psi}, O] = S[P\psi, \overline{\psi}, O] = S[\psi, \overline{\psi}\hat{P}, O]$$

  O denotes any local operators the theory may introduce. P and $\hat{P}$ are the projection operators.
- Choose particular sets of orthonormal basis $\{u_i,\ v_i\}$:

$$P\, u_i = u_i, \quad v_i^\dagger \hat{P} = v_i^\dagger$$

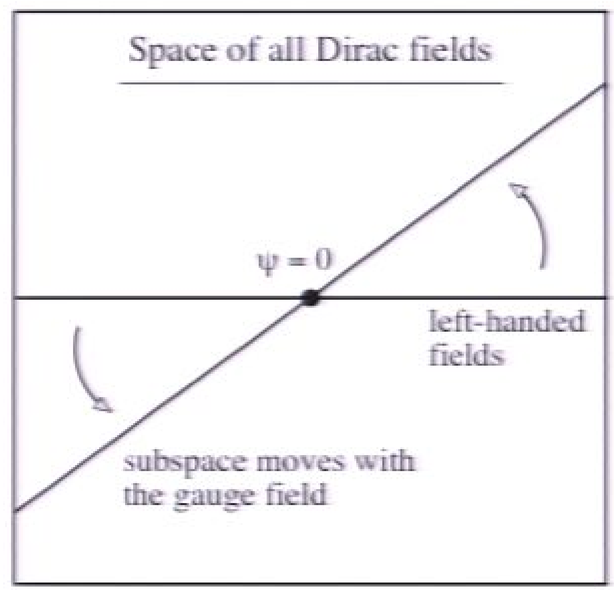
  and defined the partition function

$$Z = \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\overline{c}_j \, e^{S\left[\sum_i c_i u_i, \sum_j \overline{c}_j v_j^\dagger, O\right]}$$

Chiral partition function is ambiguous

- Suppose we choose a different set of basis $v'_i = \mathcal{U}_{ij} v_j$, $\mathcal{U}$ unitary matrix, then $Z \to \det \mathcal{U} \cdot Z$, the ambiguity is always a pure phase
- Usually not a problem because this phase is just an unphysical constant
- A serious problem in GW-formalism:

P depends on the gauge backgroud. $\Rightarrow$ chiral spinors live in different subspace when the gauge field varies. It appears that the effective action of the gauge field U contains a completely arbitrary phase.



Some general results in the GW formalism:

E. Poppitz and YS, "Lattice chirality and the decoupling of mirror fermions," JHEP **0708**, 081 (2007) [arXiv:0706.1043 [hep-th]]

- Chiral action S, a functional of the spiniors that satisfies:

$$S[\psi, \overline{\psi}, O] = S[P\psi, \overline{\psi}, O] = S[\psi, \overline{\psi}\hat{P}, O]$$

  O denotes any local operators the theory may introduce. P and $\hat{P}$ are the projection operators.

- Choose particular sets of orthonormal basis $\{u_i, v_i\}$:

$$P\, u_i = u_i, \quad v_i^\dagger \hat{P} = v_i^\dagger$$

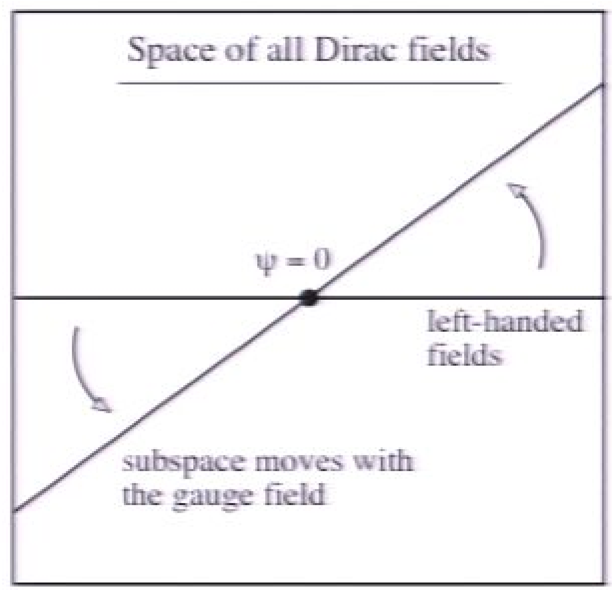
  and defined the partition function

$$Z = \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\overline{c}_j \, e^{S\left[\sum_i c_i u_i, \sum_j \overline{c}_j v_j^\dagger, O\right]}$$

Chiral partition function is ambiguous

- Suppose we choose a different set of basis $v_i' = \mathcal{U}_{ij} v_j$, $\mathcal{U}$ unitary matrix, then $Z \to \det \mathcal{U} \cdot Z$, the ambiguity is always a pure phase
- Usually not a problem because this phase is just an unphysical constant
- A serious problem in GW-formalism:

P depends on the gauge backgroud. $\Rightarrow$ chiral spinors live in different subspace when the gauge field varies. It appears that the effective action of the gauge field U contains a completely arbitrary phase.



Some general results in the GW formalism:

E. Poppitz and YS, "Lattice chirality and the decoupling of mirror fermions," JHEP **0708**, 081 (2007) [arXiv:0706.1043 [hep-th]]

- Chiral action S, a functional of the spinors that satisfies:

$$S[\psi, \overline{\psi}, O] = S[P\psi, \overline{\psi}, O] = S[\psi, \overline{\psi}\hat{P}, O]$$

  O denotes any local operators the theory may introduce. P and $\hat{P}$ are the projection operators.
- Choose particular sets of orthonormal basis $\{u_i,\ v_i\}$:

$$P\,u_i = u_i, \quad v_i^{\dagger}\hat{P} = v_i^{\dagger}$$

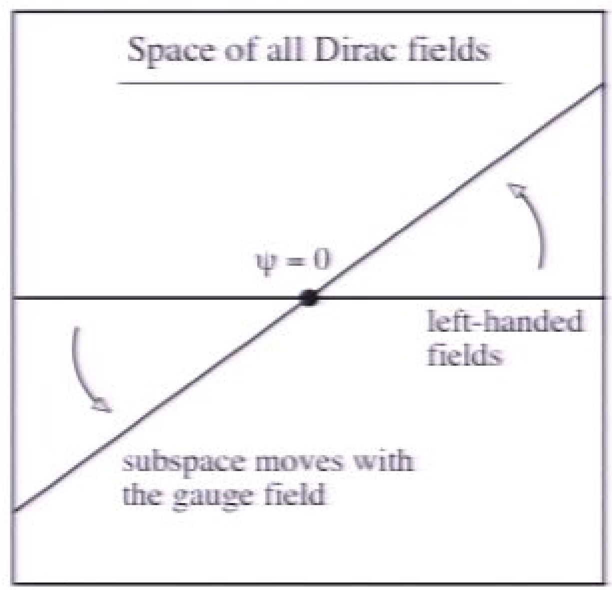
  and defined the partition function

$$Z = \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\overline{c}_j\, e^{S\left[\sum_i c_i u_i, \sum_j \overline{c}_j v_j^{\dagger}, O\right]}$$

Chiral partition function is ambiguous

- Suppose we choose a different set of basis $v'_i = \mathcal{U}_{ij} v_j$, $\mathcal{U}$ unitary matrix, then $Z \to \det \mathcal{U} \cdot Z$, the ambiguity is always a pure phase
- Usually not a problem because this phase is just an unphysical constant
- A serious problem in GW-formalism:

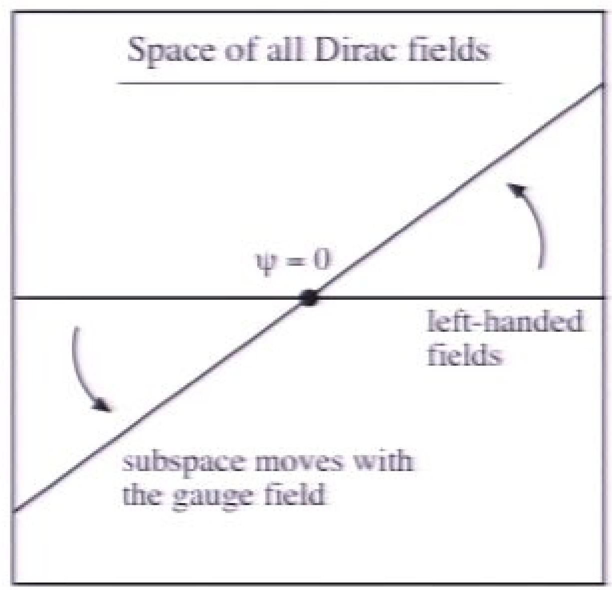
P depends on the gauge backgroud. $\Rightarrow$ chiral spinors live in different subspace when the gauge field varies. It appears that the effective action of the gauge field U contains a completely arbitrary phase.



Chiral partition function is ambiguous

- Suppose we choose a different set of basis $v_i' = \mathcal{U}_{ij} v_j$, $\mathcal{U}$ unitary matrix, then $Z \to \det\mathcal{U} \cdot Z$, the ambiguity is always a pure phase
- Usually not a problem because this phase is just an unphysical constant
- A serious problem in GW-formalism:

$\hat{P}$ depends on the gauge backgroud. $\Rightarrow$ chiral spinors live in different subspace when the gauge field varies. It appears that the effective action of the gauge field U contains a completely arbitrary phase.



Some general results in the GW formalism:

E. Poppitz and YS, "Lattice chirality and the decoupling of mirror fermions," JHEP **0708**, 081 (2007) [arXiv:0706.1043 [hep-th]]

- Chiral action S, a functional of the spinors that satisfies:

$$S[\psi, \overline{\psi}, O] = S[P\psi, \overline{\psi}, O] = S[\psi, \overline{\psi}\hat{P}, O]$$

  O denotes any local operators the theory may introduce. P and $\hat{P}$ are the projection operators.
- Choose particular sets of orthonormal basis $\{u_i,\ v_i\}$:

$$P\, u_i = u_i, \quad v_i^\dagger \hat{P} = v_i^\dagger$$

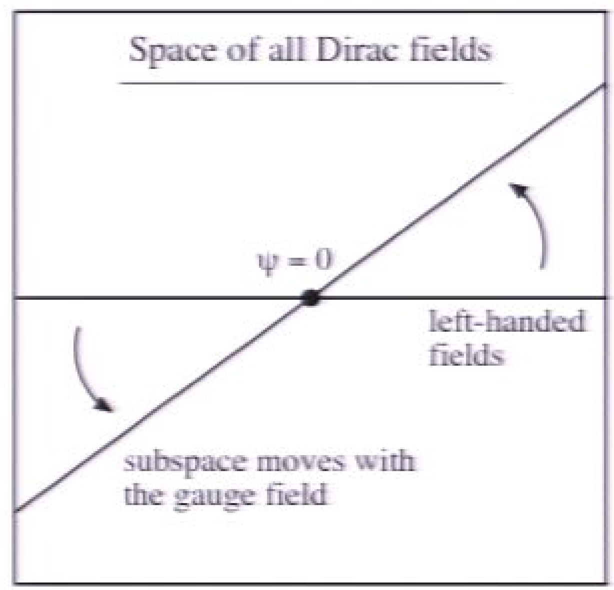
  and defined the partition function

$$Z = \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\overline{c}_j \, e^{S\left[\sum_i c_i u_i, \sum_j \overline{c}_j v_j^\dagger, O\right]}$$

Chiral partition function is ambiguous

- Suppose we choose a different set of basis $v'_i = \mathcal{U}_{ij} v_j$, $\mathcal{U}$ unitary matrix, then $Z \to \det \mathcal{U} \cdot Z$, the ambiguity is always a pure phase
- Usually not a problem because this phase is just an unphysical constant
- A serious problem in GW-formalism:

$\hat{P}$ depends on the gauge backgroud. $\Rightarrow$ chiral spinors live in different subspace when the gauge field varies. It appears that the effective action of the gauge field U contains a completely arbitrary phase.



A splitting theorem for general chiral partition functions

(E. Popptiz and YS, JHEP **0708**, 081 (2007) [arXiv:0706.1043])

- For any general chiral action that satisfies

$$S[v, \overline{v}, O] = S[Pv, \overline{v}, O] = S[v, \overline{v}P, O]$$

Ignoring the ambiguity, the partition function is given by

$$Z = \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\overline{c}_j \, e^{S\left[\sum_i c_i u_i, \sum_j \bar{c}_j v_j^\dagger, O\right]}$$

- Let us study its variations. Under any variation of both the basis vectors and the operators:

$$u_i \to u_i + \delta u_i, \; v_j = v_j + \delta v_j, O \to \delta O$$

we proved that

A splitting theorem for general chiral partition functions

(E. Popptiz and YS, JHEP **0708**, 081 (2007) [arXiv:0706.1043])

- For any general chiral action that satisfies

$$S[\psi, \overline{\psi}, O] = S[P\psi, \overline{\psi}, O] = S[\psi, \overline{\psi}\hat{P}, O]$$

  Ignoring the ambiguity, the partition function is given by

$$Z = \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\overline{c}_j \, e^{S\left[\sum_i c_i u_i, \sum_j \bar{c}_j v_j^\dagger, O\right]}$$

- Let us study its variations. Under any variation of both the basis vectors and the operators:

$$u_i \to u_i + \delta u_i, \; v_i = v_i + \delta v_j, O \to \delta O$$

  we proved that

A splitting theorem for general chiral partition functions

(E. Popptiz and YS, JHEP **0708**, 081 (2007) [arXiv:0706.1043])

- For any general chiral action that satisfies

$$S[\psi, \overline{\psi}, O] = S[P\psi, \overline{\psi}, O] = S[\psi, \overline{\psi}\hat{P}, O]$$

Ignoring the ambiguity, the partition function is given by

$$Z = \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\overline{c}_j \, e^{S\left[\sum_i c_i u_i, \sum_j \overline{c}_j v_j^\dagger, O\right]}$$

- Let us study its variations. Under any variation of both the basis vectors and the operators:

$$u_i \rightarrow u_i + \delta u_i, \; v_i = v_i + \delta v_i, O \rightarrow \delta O$$

we proved that

A splitting theorem for general chiral partition functions

(E. Popptiz and YS, JHEP **0708**, 081 (2007) [arXiv:0706.1043])

- For any general chiral action that satisfies

$$S[\psi, \overline{\psi}, O] = S[P\psi, \overline{\psi}, O] = S[\psi, \overline{\psi}\hat{P}, O]$$

Ignoring the ambiguity, the partition function is given by

$$Z = \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\overline{c}_j \, e^{S\left[\sum_i c_i u_i, \sum_j \overline{c}_j v_j^\dagger, O\right]}$$

- Let us study its variations. Under any variation of both the basis vectors and the operators:

$$u_i \to u_i + \delta u_i, \;\; v_i = v_i + \delta v_i, O \to \delta O$$

we proved that

$$\delta \log Z = \sum_i (u_i^\dagger \cdot \delta u_i) + \sum_i (\delta v_i^\dagger \cdot v_i) + \left\langle \frac{\delta S}{\delta O} \delta O \right\rangle$$

A splitting theorem for general chiral partition functions

(E. Popptiz and YS, JHEP **0708**, 081 (2007) [arXiv:0706.1043])

- For any general chiral action that satisfies

$$S[\psi, \overline{\psi}, O] = S[P\psi, \overline{\psi}, O] = S[\psi, \overline{\psi}\hat{P}, O]$$

Ignoring the ambiguity, the partition function is given by

$$Z = \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\overline{c}_j \, e^{S\left[\sum_i c_i u_i, \sum_j \overline{c}_j v_j^\dagger, O\right]}$$

- Let us study its variations. Under any variation of both the basis vectors and the operators:

$$u_i \to u_i + \delta u_i, \;\; v_i = v_i + \delta v_i, O \to \delta O$$

we proved that

$$\delta \log Z = 0 + \sum_i (\delta v_i^\dagger \cdot v_i) + \left\langle \frac{\delta S}{\delta O} \delta O \right\rangle$$

A splitting theorem for general chiral partition functions

(E. Popptiz and YS, JHEP **0708**, 081 (2007) [arXiv:0706.1043])

- For any general chiral action that satisfies

$$S[\psi, \overline{\psi}, O] = S[P\psi, \overline{\psi}, O] = S[\psi, \overline{\psi}\hat{P}, O]$$

Ignoring the ambiguity, the partition function is given by

$$Z = \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\overline{c}_j \, e^{S\left[\sum_i c_i u_i, \sum_j \overline{c}_j v_j^\dagger, O\right]}$$

- Let us study its variations. Under any variation of both the basis vectors and the operators:

$$u_i \to u_i + \delta u_i, \;\; v_i = v_i + \delta v_i, O \to \delta O$$

we proved that

$$\delta \log Z = \mathcal{J} + \left\langle \frac{\delta S}{\delta O} \delta O \right\rangle$$

The topological obstruction

$\mathcal{J}$ is the "measure" current and it captures all

The measure current defined as

$$\mathcal{J}_\mu(x) \equiv \sum_i \left(\frac{\delta v_i^\dagger}{\delta A_\mu(x)} \cdot v_i \right)$$

plays an essential role in this study. $\mathcal{J}_\mu$ is v_i-choice dependent. It captures all the ambiguity of the gauge field dependent phase of the chiral partition function Z explained above.

The topological obstruction

A v_i-choice independent quantity

While the measure current $\mathcal{J}_\mu(x)$ depends on the choice of the eigenvectors v_i, it may appears surprising that the "curvature" defined as

$$\mathcal{F}_{\mu\nu} = \delta_\mu \mathcal{J}_\nu - \delta_\nu \mathcal{J}_\mu = \mathrm{Tr}\left(\hat{P}[\,\delta_\mu \hat{P}\,,\,\delta_\nu \hat{P}\,]\right)$$

is basis choice independent and well-defined on the entire gauge field configurations space.

Gauge anomaly back in the picture

Gauge anomaly is related to the topological property of the measure current

The integration of $\mathcal{F}_{\mu\nu}$ over any non-trivial cycles in the gauge field configuration space is quantized:

$$\propto \sum_i q_{i+}^2 - \sum_j q_{j-}^2, \quad \text{in 2-D, or}$$

$$\propto \sum_i q_{i+}^3 - \sum_j q_{j-}^3 \quad \text{in 4-D.}$$

It is non-vanishing if the gauge anomaly cancellation is not satisfied, (Neuberger, Lüscher), making a smooth definition of $\mathcal{J}_\mu$ impossible. It's just like the magnetic monopole and Dirac string.

The topological obstruction

No “magnetic monopoles” iff anomaly cancellation conditions are satisfied.

Lüscher proved that $\mathcal{J}_\mu$ can be chosen uniquely as a smooth current of the $U(1)$ gauge field, and vanishing along the directions of gauge transformations, if and only if anomaly cancellation condition is satisfied (1999-2000).

Remark

despite the fact that $\mathcal{J}_\mu$ maybe be smooth, some of the vectors v_i are necessarily singular somewhere in the gauge field configuration space, gauge anomaly cancellation conditions satisfied or not.

For non-Abelian groups, the problem still remains open.

The topological obstruction

No "magnetic monopoles" iff anomaly cancellation conditions are satisfied.

Lüscher proved that $\mathcal{J}_\mu$ can be chosen uniquely as a smooth current of the $U(1)$ gauge field, and vanishing along the directions of gauge transformations, if and only if anomaly cancellation condition is satisfied (1999-2000).

Remark

despite the fact that $\mathcal{J}_\mu$ maybe be smooth, some of the vectors v_i are necessarily singular somewhere in the gauge field configuration space, gauge anomaly cancellation conditions satisfied or not.

For non-Abelian groups, the problem still remains open.

Gauge invariance

- If under the gauge variation:

$$\delta_\omega X = i\omega X,\ \delta_\omega Y = i\omega Y,\ \delta_\omega O = i[\,\omega\,,\,O\,]$$

the chiral action $S[X, Y^\dagger, O]$ is invariant:

$$0 = \delta_\omega S = \frac{\delta S}{\delta X}\delta_\omega X + \delta_\omega Y^\dagger \frac{\delta S}{\delta Y^\dagger} + \frac{\delta S}{\delta O}\delta_\omega O$$

- then by the "splitting theorem", for any chiral partition function:

$$\delta_\omega \log Z = \mathcal{J}_\omega + \frac{i}{2}\mathrm{Tr}\,\omega\gamma_5$$

- Anomaly free: $\mathrm{Tr}\,\omega\gamma_5 = 0$, and $\mathcal{J}_\omega = 0$,
therefore: $\delta_\omega \log Z = 0$ completely general

The topological obstruction

No "magnetic monopoles" iff anomaly cancellation conditions are satisfied.

Lüscher proved that $\mathcal{J}_\mu$ can be chosen uniquely as a smooth current of the $U(1)$ gauge field, and vanishing along the directions of gauge transformations, if and only if anomaly cancellation condition is satisfied (1999-2000).

Remark

despite the fact that $\mathcal{J}_\mu$ maybe be smooth, some of the vectors v_i are necessarily singular somewhere in the gauge field configuration space, gauge anomaly cancellation conditions satisfied or not.

For non-Abelian groups, the problem still remains open.

A splitting theorem for general chiral partition functions

(E. Popptiz and YS, JHEP **0708**, 081 (2007) [arXiv:0706.1043])

- For any general chiral action that satisfies

$$S[\psi, \overline{\psi}, O] = S[P\psi, \overline{\psi}, O] = S[\psi, \overline{\psi}\hat{P}, O]$$

Ignoring the ambiguity, the partition function is given by

$$Z = \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\overline{c}_j \, e^{S\left[\sum_i c_i u_i, \sum_j \overline{c}_j v_j^\dagger, O\right]}$$

- Let us study its variations. Under any variation of both the basis vectors and the operators:

$$u_i \to u_i + \delta u_i, \;\; v_i = v_i + \delta v_i, O \to \delta O$$

we proved that

$$\delta \log Z = \mathcal{J} + \left\langle \frac{\delta S}{\delta O} \delta O \right\rangle$$

A splitting theorem for general chiral partition functions

(E. Popptiz and YS, JHEP **0708**, 081 (2007) [arXiv:0706.1043])

- For any general chiral action that satisfies

$$S[\psi,\overline{\psi},O]=S[P\psi,\overline{\psi},O]=S[\psi,\overline{\psi}\hat{P},O]$$

  Ignoring the ambiguity, the partition function is given by

$$Z=\int\prod_{i,j}\mathrm{d}c_i\mathrm{d}\overline{c}_j\,e^{S\left[\sum_i c_i u_i,\sum_j \overline{c}_j v_j^\dagger,O\right]}$$

- Let us study its variations. Under any variation of both the basis vectors and the operators:

$$u_i\rightarrow u_i+\delta u_i,\ v_i=v_i+\delta v_i,O\rightarrow\delta O$$

  we proved that

$$\delta\log Z=0+\sum_i(\delta v_i^\dagger\cdot v_i)+\left\langle\frac{\delta S}{\delta O}\delta O\right\rangle$$

Some general results in the GW formalism:

E. Poppitz and YS, "Lattice chirality and the decoupling of mirror fermions," JHEP **0708**, 081 (2007) [arXiv:0706.1043 [hep-th]]

- Chiral action S, a functional of the spiniors that satisfies:

$$S[\,\psi, \overline{\psi}, O\,] = S[\,P\psi, \overline{\psi}, O\,] = S[\,\psi, \overline{\psi}\hat{P}, O\,]$$

  O denotes any local operators the theory may introduce. P and $\hat{P}$ are the projection operators.
- Choose particular sets of orthonormal basis $\{u_i,\ v_i\}$:

$$P\,u_i = u_i, \quad v_i^\dagger \hat{P} = v_i^\dagger$$

  and defined the partition function

$$Z = \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\overline{c}_j \, e^{S\left[\sum_i c_i u_i, \sum_j \overline{c}_j v_j^\dagger, O\right]}$$

A splitting theorem for general chiral partition functions

(E. Popptiz and YS, JHEP **0708**, 081 (2007) [arXiv:0706.1043])

- For any general chiral action that satisfies

$$S[\psi,\overline{\psi},O] = S[P\psi,\overline{\psi},O] = S[\psi,\overline{\psi}\hat{P},O]$$

Ignoring the ambiguity, the partition function is given by

$$Z = \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\overline{c}_j \, e^{S\left[\sum_i c_i u_i, \sum_j \overline{c}_j v_j^\dagger, O\right]}$$

- Let us study its variations. Under any variation of both the basis vectors and the operators:

$$u_i \to u_i + \delta u_i, \;\; v_i = v_i + \delta v_i, O \to \delta O$$

we proved that

$$\delta \log Z = \sum_i (u_i^\dagger \cdot \delta u_i) + \sum_i (\delta v_i^\dagger \cdot v_i) + \left\langle \frac{\delta S}{\delta O} \delta O \right\rangle$$

A splitting theorem for general chiral partition functions

(E. Popptiz and YS, JHEP **0708**, 081 (2007) [arXiv:0706.1043])

- For any general chiral action that satisfies

$$S[\psi, \overline{\psi}, O] = S[P\psi, \overline{\psi}, O] = S[\psi, \overline{\psi}\hat{P}, O]$$

Ignoring the ambiguity, the partition function is given by

$$Z = \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\overline{c}_j \, e^{S\left[\sum_i c_i u_i, \sum_j \overline{c}_j v_j^\dagger, O\right]}$$

- Let us study its variations. Under any variation of both the basis vectors and the operators:

$$u_i \to u_i + \delta u_i, \;\; v_i = v_i + \delta v_i, O \to \delta O$$

we proved that

$$\delta \log Z = \mathcal{J} + \left\langle \frac{\delta S}{\delta O} \delta O \right\rangle$$

The topological obstruction

A v_i-choice independent quantity

While the measure current $\mathcal{J}_\mu(x)$ depends on the choice of the eigenvectors v_i, it may appears surprising that the "curvature" defined as

$$\mathcal{F}_{\mu\nu} = \delta_\mu \mathcal{J}_\nu - \delta_\nu \mathcal{J}_\mu = \mathrm{Tr}\left(\hat{P}[\,\delta_\mu \hat{P}\,,\,\delta_\nu \hat{P}\,]\right)$$

is basis choice independent and well-defined on the entire gauge field configurations space.

The topological obstruction

No "magnetic monopoles" iff anomaly cancellation conditions are satisfied.

Lüscher proved that $\mathcal{J}_\mu$ can be chosen uniquely as a smooth current of the $U(1)$ gauge field, and vanishing along the directions of gauge transformations, if and only if anomaly cancellation condition is satisfied (1999-2000).

Remark

despite the fact that $\mathcal{J}_\mu$ maybe be smooth, some of the vectors v_i are necessarily singular somewhere in the gauge field configuration space, gauge anomaly cancellation conditions satisfied or not.

For non-Abelian groups, the problem still remains open.

Gauge invariance

- If under the gauge variation:

$$\delta_\omega X = i\omega X,\ \delta_\omega Y = i\omega Y,\ \delta_\omega O = i[\,\omega\,,\,O\,]$$

  the chiral action $S[X, Y^\dagger, O]$ is invariant:

$$0 = \delta_\omega S = \frac{\delta S}{\delta X}\delta_\omega X + \delta_\omega Y^\dagger \frac{\delta S}{\delta Y^\dagger} + \frac{\delta S}{\delta O}\delta_\omega O$$

- then by the "splitting theorem", for any chiral partition function:

$$\delta_\omega \log Z = \mathcal{J}_\omega + \frac{i}{2}\mathrm{Tr}\,\omega\gamma_5$$

- Anomaly free: $\mathrm{Tr}\,\omega\gamma_5 = 0$, and $\mathcal{J}_\omega = 0$.
  therefore: $\delta_\omega \log Z = 0$ completely general

Smoothness

For a general chiral action, must apply our "splitting theorem" recursively:

- Assuming that action $S[X, Y, O]$ has no poles. Therefore $\langle \frac{\delta S}{\delta O} \delta O \rangle < \infty$
- Proved that $\langle \frac{\delta S}{\delta O} \delta O \rangle$ can be viewed as the partition function of a new "chiral action" $S^{(1)}$
- Apply the "splitting" to $S^{(1)}$ while taking further derivatives
- Since $\delta^n \log Z$ is finite for any n, we proved that $\log Z$ is smooth as long as $\mathcal{J}$ is.
- Remarks:
  - although $\mathcal{J} = \sum_i \delta v_i^\dagger \cdot v_i$ is smooth, always some of the v_i is singular
  - "splitting theorem" also useful in deriving correlation functions

Gauge invariance

- If under the gauge variation:

$$\delta_\omega X = i\omega X,\ \delta_\omega Y = i\omega Y,\ \delta_\omega O = i[\,\omega\,,\,O\,]$$

the chiral action $S[X, Y^\dagger, O]$ is invariant:

$$0 = \delta_\omega S = \frac{\delta S}{\delta X}\delta_\omega X + \delta_\omega Y^\dagger \frac{\delta S}{\delta Y^\dagger} + \frac{\delta S}{\delta O}\delta_\omega O$$

- then by the “splitting theorem”, for any chiral partition function:

$$\delta_\omega \log Z = \mathcal{J}_\omega + \frac{i}{2}\mathrm{Tr}\omega\hat{\gamma}_5$$

- Anomaly free: $\mathrm{Tr}\omega\hat{\gamma}_5 = 0$, and $\mathcal{J}_\omega = 0$, therefore: $\delta_\omega \log Z = 0$ completely general

The topological obstruction

No "magnetic monopoles" iff anomaly cancellation conditions are satisfied.

Lüscher proved that $\mathcal{J}_\mu$ can be chosen uniquely as a smooth current of the $U(1)$ gauge field, and vanishing along the directions of gauge transformations, if and only if anomaly cancellation condition is satisfied (1999-2000).

Remark

despite the fact that $\mathcal{J}_\mu$ maybe be smooth, some of the vectors v_i are necessarily singular somewhere in the gauge field configuration space, gauge anomaly cancellation conditions satisfied or not.

For non-Abelian groups, the problem still remains open.

Gauge invariance

- If under the gauge variation:

$$\delta_\omega X = i\omega X,\ \delta_\omega Y = i\omega Y,\ \delta_\omega O = i[\,\omega\,,\,O\,]$$

the chiral action $S[X, Y^\dagger, O]$ is invariant:

$$0 = \delta_\omega S = \frac{\delta S}{\delta X}\delta_\omega X + \delta_\omega Y^\dagger \frac{\delta S}{\delta Y^\dagger} + \frac{\delta S}{\delta O}\delta_\omega O$$

- then by the "splitting theorem", for any chiral partition function:

$$\delta_\omega \log Z = \mathcal{J}_\omega + \frac{i}{2}\mathrm{Tr}\omega\hat{\gamma}_5$$

- Anomaly free: $\mathrm{Tr}\omega\hat{\gamma}_5 = 0$, and $\mathcal{J}_\omega = 0$,
therefore: $\delta_\omega \log Z = 0$ completely general

Smoothness

For a general chiral action, must apply our "splitting theorem" recursively:

- Assuming that action $S[X, Y, O]$ has no poles. Therefore $\langle \frac{\delta S}{\delta O} \delta O \rangle < \infty$
- Proved that $\langle \frac{\delta S}{\delta O} \delta O \rangle$ can be viewed as the partition function of a new "chiral action" $S^{(1)}$
- Apply the "splitting" to $S^{(1)}$ while taking further derivatives
- Since $\delta^n \log Z$ is finite for any n, we proved that $\log Z$ is smooth as long as $\mathcal{J}$ is.
- Remarks:
  - although $\mathcal{J} = \sum_i \delta v_i^\dagger \cdot v_i$ is smooth, always some of the v_i is singular
  - "splitting theorem" also useful in deriving correlation functions

Gauge invariance

- If under the gauge variation:

$$\delta_\omega X = i\omega X,\ \delta_\omega Y = i\omega Y,\ \delta_\omega O = i[\,\omega\,,\,O\,]$$

the chiral action $S[X, Y^\dagger, O]$ is invariant:

$$0 = \delta_\omega S = \frac{\delta S}{\delta X}\delta_\omega X + \delta_\omega Y^\dagger \frac{\delta S}{\delta Y^\dagger} + \frac{\delta S}{\delta O}\delta_\omega O$$

- then by the "splitting theorem", for any chiral partition function:

$$\delta_\omega \log Z = \mathcal{J}_\omega + \frac{i}{2}\mathrm{Tr}\,\omega\hat{\gamma}_5$$

- Anomaly free: $\mathrm{Tr}\,\omega\hat{\gamma}_5 = 0$, and $\mathcal{J}_\omega = 0$, therefore: $\delta_\omega \log Z = 0$ completely general

Smoothness

For a general chiral action, must apply our "splitting theorem" recursively:

- Assuming that action $S[X, Y, O]$ has no poles. Therefore $\langle \frac{\delta S}{\delta O} \delta O \rangle < \infty$
- Proved that $\langle \frac{\delta S}{\delta O} \delta O \rangle$ can be viewed as the partition function of a new "chiral action" $S^{(1)}$
- Apply the "splitting" to $S^{(1)}$ while taking further derivatives
- Since $\delta^n \log Z$ is finite for any n, we proved that $\log Z$ is smooth as long as $\mathcal{J}$ is.
- Remarks:
  - although $\mathcal{J} = \sum_i \delta v_i^\dagger \cdot v_i$ is smooth, always some of the v_i is singular
  - "splitting theorem" also useful in deriving correlation functions

Smoothness

For a general chiral action, must apply our "splitting theorem" recursively:

- Assuming that action $S[X, Y, O]$ has no poles. Therefore $\langle \frac{\delta S}{\delta O} \delta O \rangle < \infty$
- Proved that $\langle \frac{\delta S}{\delta O} \delta O \rangle$ can be viewed as the partition function of a new "chiral action" $S^{(1)}$
- Apply the "splitting" to $S^{(1)}$ while taking further derivatives
- Since $\delta^n \log Z$ is finite for any n, we proved that $\log Z$ is smooth as long as $\mathcal{J}$ is.
- Remarks:
  - although $\mathcal{J} = \sum_i \delta v_i^\dagger \cdot v_i$ is smooth, always some of the v_i is singular
  - "splitting theorem" also useful in deriving correlation functions

A brief recap

- Combining Lüscher's proof and our "splitting theorem" $\Rightarrow$ splitting of $\log Z$ of a vector-like theory into chicral sectors: $\log Z = \log Z_{\text{light}} + \log Z_{\text{mirror}}$ is smooth iff both sectors are anomaly free.
- With dynamical gauge field, the splitting of the spectrum of the $0-1$ model into the anomalous light and mirror sector doesn't make sense.
- But in anomalous cases, the obstacle is topological ("magnetic monopoles" and "Dirac string"). Can always be circumvented locally (in gauge field configuraiton space) by tuning the boundary conditions. $\Rightarrow$ GP's study on the $0-1$ model at vanishing gauge field background is still correct.
- Lüscher's proof of the smoothness of $\mathcal{J}$ is "constructive" but computationally unuseful, because The proof is inductive on the dimension of the gauge field configuration space (N^2+2 in 2-d).

A brief recap

- Combining Lüscher's proof and our "splitting theorem" $\Rightarrow$ splitting of $\log Z$ of a vector-like theory into chicral sectors: $\log Z = \log Z_{\text{light}} + \log Z_{\text{mirror}}$ is smooth iff both sectors are anomaly free.
- With dynamical gauge field, the splitting of the spectrum of the $0-1$ model into the anomalous light and mirror sector doesn't make sense.
- But in anomalous cases, the obstacle is topological ("magnetic monopoles" and "Dirac string"). Can always be circumvented locally (in gauge field configuraiton space) by tuning the boundary conditions. $\Rightarrow$ GP's study on the $0-1$ model at vanishing gauge field background is still correct.
- Lüscher's proof of the smoothness of $\mathcal{J}$ is "constructive" but computationally unuseful, because The proof is inductive on the dimension of the gauge field configuration space (N^2+2 in 2-d).

A brief recap

- Combining Lüscher's proof and our "splitting theorem" $\Rightarrow$ splitting of $\log Z$ of a vector-like theory into chicral sectors: $\log Z = \log Z_{\text{light}} + \log Z_{\text{mirror}}$ is smooth iff both sectors are anomaly free.
- With dynamical gauge field, the splitting of the spectrum of the $0-1$ model into the anomalous light and mirror sector doesn't make sense.
- But in anomalous cases, the obstacle is topological ("magnetic monopoles" and "Dirac string"). Can always be circumvented locally (in gauge field configuraiton space) by tuning the boundary conditions. $\Rightarrow$ GP's study on the $0-1$ model at vanishing gauge field background is still correct.
- Lüscher's proof of the smoothness of $\mathcal{J}$ is "constructive" but computationally unuseful. because The proof is inductive on the dimension of the gauge field configuration space (N^2+2 in 2-d)

A brief recap

- Combining Lüscher's proof and our "splitting theorem" $\Rightarrow$ splitting of $\log Z$ of a vector-like theory into chicral sectors: $\log Z = \log Z_{\text{light}} + \log Z_{\text{mirror}}$ is smooth iff both sectors are anomaly free.
- With dynamical gauge field, the splitting of the spectrum of the $0-1$ model into the anomalous light and mirror sector doesn't make sense.
- But in anomalous cases, the obstacle is topological ("magnetic monopoles" and "Dirac string"). Can always be circumvented locally (in gauge field configuraiton space) by tuning the boundary conditions. $\Rightarrow$ GP's study on the $0-1$ model at vanishing gauge field background is still correct.
- Lüscher's proof of the smoothness of $\mathcal{J}$ is "constructive" but computationally unuseful, because The proof is inductive on the dimension of the gauge field configuration space (N^2+2 in 2-d).

A brief recap

- Combining Lüscher's proof and our "splitting theorem" $\Rightarrow$ splitting of $\log Z$ of a vector-like theory into chicral sectors:
  $\log Z = \log Z_{\text{light}} + \log Z_{\text{mirror}}$ is smooth iff both sectors are anomaly free.
- With dynamical gauge field, the splitting of the spectrum of the $0-1$ model into the anomalous light and mirror sector doesn't make sense.
- But in anomalous cases, the obstacle is topological ("magnetic monopoles" and "Dirac string"). Can always be circumvented locally (in gauge field configuraiton space) by tuning the boundary conditions. $\Rightarrow$ GP's study on the $0-1$ model at vanishing gauge field background is still correct.
- Lüscher's proof of the smoothness of $\mathcal{J}$ is "constructive" but computationally unuseful. because The proof is inductive on the dimension of the gauge field configuration space ($N^2 + 2$ in 2-d)

A brief recap

- Combining Lüscher's proof and our "splitting theorem" $\Rightarrow$ splitting of $\log Z$ of a vector-like theory into chicral sectors: $\log Z = \log Z_{\text{light}} + \log Z_{\text{mirror}}$ is smooth iff both sectors are anomaly free.
- With dynamical gauge field, the splitting of the spectrum of the $0-1$ model into the anomalous light and mirror sector doesn't make sense.
- But in anomalous cases, the obstacle is topological ("magnetic monopoles" and "Dirac string"). Can always be circumvented locally (in gauge field configuraiton space) by tuning the boundary conditions. $\Rightarrow$ GP's study on the $0-1$ model at vanishing gauge field background is still correct.
- Lüscher's proof of the smoothness of $\mathcal{J}$ is "constructive" but computationally unuseful, because The proof is inductive on the dimension of the gauge field configuration space (N^2+2 in 2-d).

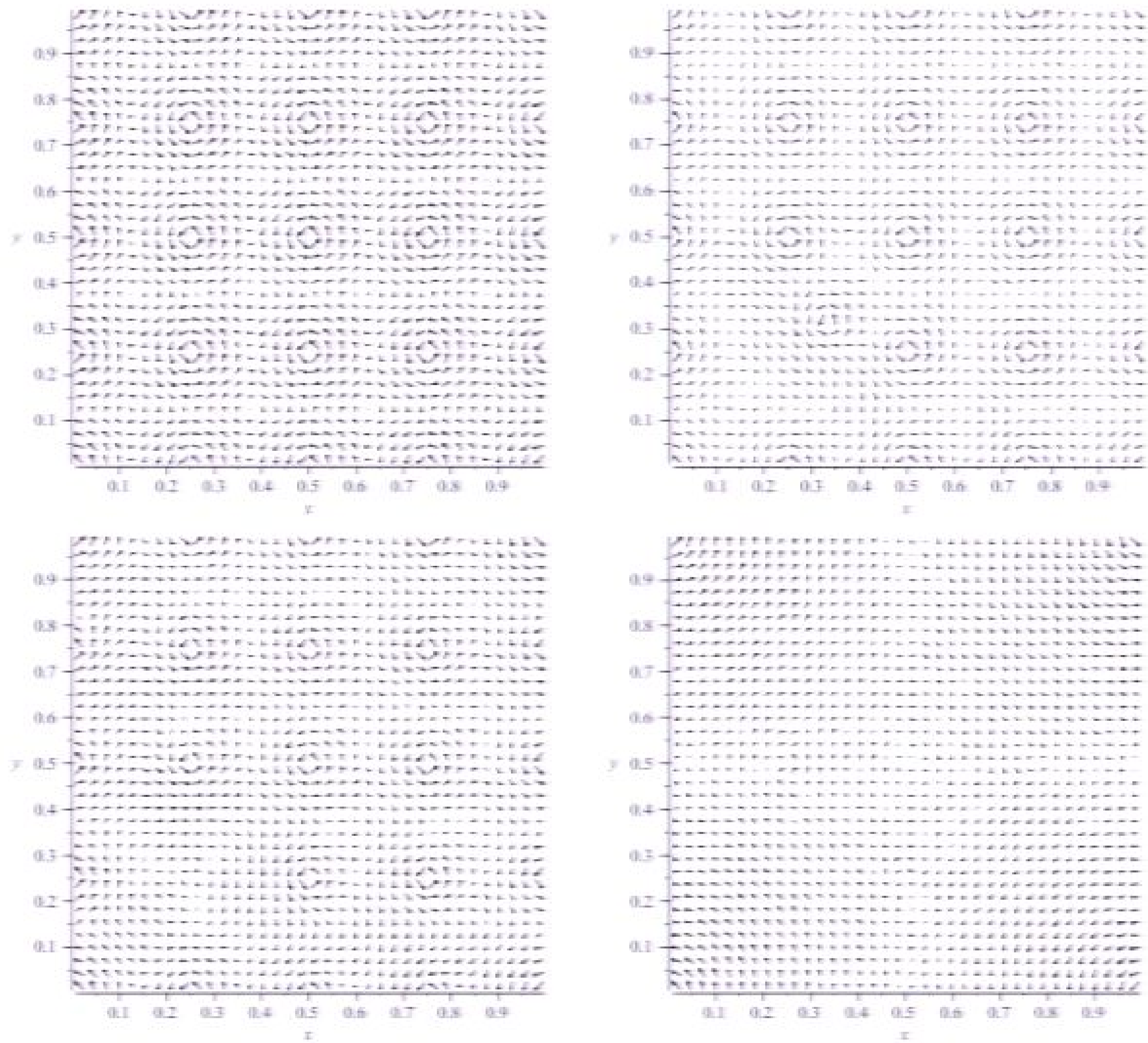


Figure: 1st panel: the 16 singularities of $\mathcal{J}_{\mu}^{4}$, 2nd: one singular vortex slightly shifted; 3rd: one vortex moved to $\mathbf{h} = (0, 0)$ so that two singularities coincide; 4th: all 16 vortices shifted to the corner.

Outline

1. *Motivation and idea*
   - Why chiral gauge theory, why lattice
   - Why need the idea of "decoupling of mirror fermions"
   - Does it work: early numerical work suggest maybe
2. *A quick review of some most recent theoretical developments*
   - Exact lattice chiral symmetry
   - Put the formalism on a completely general ground
   - A powerful simple theorem
3. *A paradox in the* $0-1$ *model and the unitarity of the GW formalism*
4. *Toward a more analytical approach*

A paradox of the $0-1$ toy model

So we still have a "paradox" in the 0-1 model at the zero gauge field background (Erich Poppitz and YS, soon to appear on arXiv.)

- If we study the photon polarization operator, focusing only on the mirror sector, at vanishing gauge field background, we find

$$\Pi_{\mu\nu}(x,y) \equiv \frac{\delta^2 \ln Z[A]}{\delta A_\mu(x)\delta A_\nu(y)} = \frac{\delta \mathcal{J}_\mu(x)}{\delta A_\nu(y)} + \Pi'_{\mu\nu}(x,y)$$

$$\Pi'_{\mu\nu} = \frac{\delta}{\delta A_\nu(x)} \left\langle \frac{\delta O}{\delta A_\mu(x)} \right\rangle$$

- O represents a complicated operator in terms of the original mirror fermions and ϕ.
- Using the splitting theorem, one can verify that $\left\langle \frac{\delta O}{\delta A_\mu(x)} \right\rangle$ is basis choice independent and gauge invariant.

Photon polarization operator

- Notice while $\Pi_{\mu\nu}$ is symmetrical, $\Pi'_{\mu\nu}$ is not. Amusingly, its anti-symmetric part $\Pi'^{A}_{\mu\nu}(x,y) = \frac{1}{2}\mathcal{F}_{\mu\nu}$ is theory & basis independent and known!
- The divergence of $\mathcal{F}_{\mu\nu}$ can be computed exactly:

$$\begin{aligned}\nabla^*_\mu \mathcal{F}_{\mu\nu} =& i\frac{\delta}{\delta\omega}\mathrm{Tr}[\delta_\nu \hat{P}\hat{P}(\omega\hat{P} - \hat{P}\omega) - \hat{P}\delta_\nu\hat{P}(\omega\hat{P} - \hat{P}\omega)] \\ =& i\frac{\delta}{\delta\omega}\mathrm{Tr}\delta_\nu\hat{P}\omega = \frac{i}{2}\delta_\nu \mathrm{tr}\hat{\gamma}^5_{xx}\end{aligned}$$

- A very useful identity is used:

$$\nabla^*_\mu \delta_\mu O = \frac{\delta}{\delta\omega(x)}\delta_\omega O,$$

where δ_ω denote the gauge variation of any operator and ω is the gauge parameter: $A_\mu \rightarrow A_\mu - \nabla_\mu\omega$.

The anomaly equation on lattice and the paradox

The symmetric part of $\Pi'_{\mu\nu}$, $\Pi'^S_{\mu\nu}$ satisfies:

$$\nabla^*_\mu \Pi'^S_{\mu\nu} = \frac{i}{4}\delta_\nu \mathrm{tr}\hat{\gamma}^5_{xx}$$

Precisely the anomalous divergence of $\Pi_{\mu\nu}$. In the continuum limit:

$$\delta_\nu \mathrm{tr}\hat{\gamma}_5 \sim \epsilon_{\nu\rho}k_\rho, \quad \text{when } k \to 0$$

The only solution to the anomaly equation at $k \to 0$ limit is

$$\tilde{\Pi}'^S_{\mu\nu}(k) \sim i\frac{\epsilon^{\mu\rho}k_\rho k_\nu + \epsilon^{\nu\rho}k_\rho k_\mu}{k^2} + \text{any divergence free piece}$$

It indicats a light degree of freedom in the continuum limit, contradicting GP's finding.

The anomaly equation on lattice and the paradox

The symmetric part of $\Pi'_{\mu\nu}$, $\Pi'^{S}_{\mu\nu}$ satisfies:

$$\nabla^*_\mu \Pi'^{S}_{\mu\nu} = \frac{i}{4}\delta_\nu \mathrm{tr}\hat{\gamma}^5_{xx}$$

Precisely the anomalous divergence of $\Pi_{\mu\nu}$. In the continuum limit:

$$\delta_\nu \mathrm{tr}\hat{\gamma}_5 \sim \epsilon_{\nu\rho}k_\rho, \quad \text{when } k \to 0$$

The only solution to the anomaly equation at $k \to 0$ limit is

$$\tilde{\Pi}'^{S}_{\mu\nu}(k) \sim i\frac{\epsilon^{\mu\rho}k_\rho k_\nu + \epsilon^{\nu\rho}k_\rho k_\mu}{k^2} + \text{any divergence free piece}$$

It indicats a light degree of freedom in the continuum limit, contradicting GP's finding.

Constraints by unitarity

Is the "massless pole" a sufficient evidence for a massless fermion? After all, the anomaly is cancelled in the full theory.

- In continuum limit, if the mirror is unitary, apart from a contact term, the photon two point polarization operator is given by

$$\Pi_{\mu\nu}(x,y) \sim \langle j_\mu(x) j_\nu(y) \rangle$$

where $j_\mu(x)$ is just the current that couples to the photon, and in 2-D, it can always be decomposed into

$$j_\mu(x) = \partial_\mu n(x) + i\epsilon_{\mu\nu}\partial_\nu m(x)$$

Here, $n(x)$ and $m(x)$ are real. We therefore have

$$\Pi_{\mu\nu}(x) \sim \partial_\mu\partial_\nu \langle n(x)n(y) + m(x)m(y) \rangle - i\epsilon_{\mu\rho}\partial_\rho \langle n(x)m(y) \rangle + \mu - \nu, x - y$$

The anomaly equation on lattice and the paradox

The symmetric part of $\Pi'_{\mu\nu}$, $\Pi'^{S}_{\mu\nu}$ satisfies:

$$\nabla^*_\mu \Pi'^{S}_{\mu\nu} = \frac{i}{4}\delta_\nu \mathrm{tr}\hat{\gamma}^5_{xx}$$

Precisely the anomalous divergence of $\Pi_{\mu\nu}$. In the continuum limit:

$$\delta_\nu \mathrm{tr}\hat{\gamma}_5 \sim \epsilon_{\nu\rho} k_\rho, \quad \text{when } k \to 0$$

The only solution to the anomaly equation at $k \to 0$ limit is

$$\tilde{\Pi}'^{S}_{\mu\nu}(k) \sim i\frac{\epsilon^{\mu\rho} k_\rho k_\nu + \epsilon^{\nu\rho} k_\rho k_\mu}{k^2} + \text{any divergence free piece}$$

It indicats a light degree of freedom in the continuum limit, contradicting GP's finding.

Constraints by unitarity

Is the “massless pole” a sufficient evidence for a massless fermion? After all, the anomaly is cancelled in the full theory.

- In continuum limit, if the mirror is unitary, apart from a contact term, the photon two point polarization operator is given by

$$\Pi_{\mu\nu}(x,y) \sim \langle j_\mu(x) j_\nu(y) \rangle$$

where $j_\mu(x)$ is just the current that couples to the photon, and in 2-D, it can always be decomposed into

$$j_\mu(x) = \partial_\mu n(x) + i\epsilon_{\mu\nu}\partial_\nu m(x)$$

Here, $n(x)$ and $m(x)$ are real. We therefore have

$$\Pi_{\mu\nu}(x) \sim \partial_\mu\partial_\nu \langle n(x)n(y) + m(x)m(y) \rangle$$
$$+ i\epsilon_{\mu\rho}\partial_\rho \langle n(x)m(y) \rangle + \mu - \nu, x - y$$

Constraints by unitarity

Is the "massless pole" a sufficient evidence for a massless fermion? After all, the anomaly is cancelled in the full theory.

- In continuum limit, if the mirror is unitary, apart from a contact term, the photon two point polarization operator is given by

$$\Pi_{\mu\nu}(x,y) \sim \langle j_\mu(x) j_\nu(y) \rangle$$

where $j_\mu(x)$ is just the current that couples to the photon, and in 2-D, it can always be decomposed into

$$j_\mu(x) = \partial_\mu n(x) + i\epsilon_{\mu\nu}\partial_\nu m(x)$$

Here, $n(x)$ and $m(x)$ are real. We therefore have

$$\Pi_{\mu\nu}(x) \sim \partial_\mu\partial_\nu \langle n(x)n(y) + m(x)m(y) \rangle + i\epsilon_{\mu\rho}\partial_\rho \langle n(x)m(y) \rangle + \mu \leftrightarrow \nu, x \leftrightarrow y$$

- Unitarity demands that $\langle (n(x) - m(x))(n(y) - m(y)) \rangle \geq 0$, therefore

$$\langle n(x)n(y) + m(x)m(y) \rangle \geq 2 \langle n(x)m(y) \rangle$$

We conclude that

$$\tilde{\Pi}_{\mu\nu}(k) \sim A \frac{k_\mu k_\nu - g_{\mu\nu} k^2}{k^2} - i B \frac{\epsilon_{\mu\rho} k_\rho k_\nu + \epsilon_{\nu\rho} k_\rho k_\mu}{k^2}$$

and

$$A \geq 2|B|$$

if the theory is unitary in the continuum limit.

- A free fermion saturates the bound precisely.

Possible resolution to the paradox

Two possibilities to resolve the paradox

- The theory does not have a unitary continuum limit. Very bad.
- The theory does contain a light degree of freedom, but it's a non-local composite state in terms of the original fields and eluded the examinations carried out by GP.

How can we tell which is the answer?

- We had to resort to numerical tests as we don't have the analytic power to compute everything at this moment.
- But the theoretical develoments, in particular the "splitting-theorem", are vitally important even for numerical simulations to become possible.

Possible resolution to the paradox

Two possibilities to resolve the paradox

- The theory does not have a unitary continuum limit. Very bad.
- The theory does contain a light degree of freedom, but it's a non-local composite state in terms of the original fields and eluded the examinations carried out by GP.

How can we tell which is the answer?

- We had to resort to numerical tests as we don't have the analytic power to compute everything at this moment.
- But the theoretical develoments, in particular the "splitting-theorem", are vitally important even for numerical simulations to become possible.

A bit more analytic work

- Again, using the splitting theorem many times, we can derive in $y \to \infty$ limit:

$$\Pi'_{\mu\nu} = \Pi^{\kappa}_{\mu\nu} + \Pi^{y\kappa}_{\mu\nu} + \Pi^{y}_{\mu\nu}$$

where

$$\Pi^{\kappa}_{\mu\nu} = \frac{\kappa}{2} \langle (\phi^* \cdot \delta_\nu \delta_\mu U \cdot \phi) + \text{h.c.} \rangle + \frac{\kappa^2}{4} \langle [(\phi^* \cdot \delta_\mu U \cdot \phi) + \text{h.c.}] \, [(\phi^* \cdot \delta_\nu U \cdot \phi) + \text{h.c.}] \rangle^C$$

$$\Pi^{y\kappa}_{\mu\nu} = -\frac{y\kappa}{2} \Big\{ \Big\langle [(\phi^* \cdot \delta_\mu U \cdot \phi) + \text{h.c.}] \Big[\overline{\alpha}^i_- \beta^j_+ (w_i^\dagger \cdot \delta_\nu \hat{P}_+ \cdot \phi^* v_j) + h \overline{\alpha}^i_- \overline{\beta}^j_+ (u_j^\dagger \gamma_2 \cdot \phi^* \cdot \delta_\nu \hat{P}^T_+ \cdot w_i^*) \Big] \Big\rangle^C + (\mu \leftrightarrow \nu) \Big\}$$

- $$\begin{aligned}\Pi^{y}_{\mu\nu} = & - y \, \langle \overline{\alpha}^i_- \beta^j_+ \, (w_i^\dagger \cdot \delta_\nu (\hat{P}_+ \delta_\mu \hat{P}_+) \cdot \phi^* v_j) \rangle \\ & - yh \, \langle \overline{\alpha}^i_- \overline{\beta}^j_+ \, (u_j^\dagger \gamma_2 \cdot \phi^* \cdot \delta_\nu (\delta_\mu \hat{P}^T_+ \cdot \hat{P}^T_+) \cdot w_i^*) \rangle \\ & + y^2 \Big\langle \Big(\overline{\alpha}^i_- \beta^j_+ (w_i^\dagger \cdot \delta_\mu \hat{P}_+ \cdot \phi^* v_j) + h \overline{\alpha}^i_- \overline{\beta}^j_+ (u_j^\dagger \gamma_2 \cdot \phi^* \cdot \delta_\mu \hat{P}^T_+ \cdot w_i^*) \Big) \\ & \times \Big(\overline{\alpha}^k_- \beta^l_+ (w_k^\dagger \cdot \delta_\nu \hat{P}_+ \cdot \phi^* v_l) + h \overline{\alpha}^k_- \overline{\beta}^l_+ (u_l^\dagger \gamma_2 \cdot \phi^* \cdot \delta_\nu \hat{P}^T_+ \cdot w_k^*) \Big) \Big\rangle^C \end{aligned}$$
- Expand the Ginsparg-wilson operators to the second order in terms of the gauge field, all the expressions inside $\langle \cdot \rangle$ can be computed exactly.
- Put on to a cluster to run simulations. A huge number of momenta sum requires computing power of a super computer. We ran it on 400 CPU's on CITA's Sunnyvale and it takes 5 hours for each run.

A numerical trick

But, how do we distinguish the possible scenarios on a 8×8 lattice?

Angular singularity:

- If there's a massless mode, $Re(\tilde{\Pi}_{00}(k)) \sim \frac{k_0^2 - k^2}{k^2} \sim -\sin^2\theta$, where $(k_0, k_1) \equiv (k\cos\theta, k\sin\theta)$.
- If all excitations are massive, $\tilde{\Pi}_{\mu\nu}(k) \to 0$ when $k \to 0$.

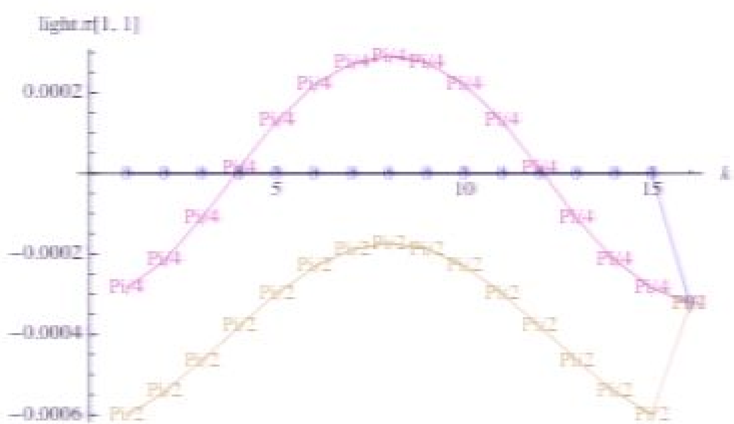


Figure: $\tilde{\Pi}_{\mu\nu}(k)$ of a massless fermion

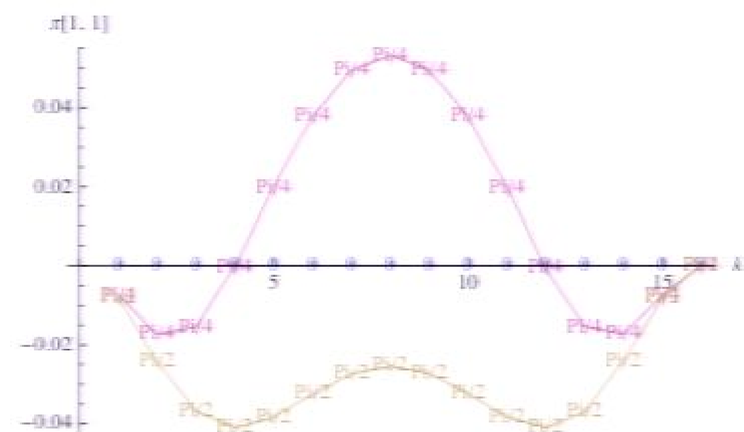


Figure: $\tilde{\Pi}_{\mu\nu}(k)$ of a massive fermion

- Unitarity demands that $\langle (n(x) - m(x))(n(y) - m(y)) \rangle \geq 0$, therefore

$$\langle n(x)n(y) + m(x)m(y) \rangle \geq 2 \langle n(x)m(y) \rangle$$

We conclude that

$$\tilde{\Pi}_{\mu\nu}(k) \sim A \frac{k_\mu k_\nu - g_{\mu\nu} k^2}{k^2} - i B \frac{\epsilon_{\mu\rho} k_\rho k_\nu + \epsilon_{\nu\rho} k_\rho k_\mu}{k^2}$$

and

$$A \geq 2|B|$$

if the theory is unitary in the continuum limit.

- A free fermion saturates the bound precisely.

The numerical results

- After a series non-trivial checks (divergences and so on)

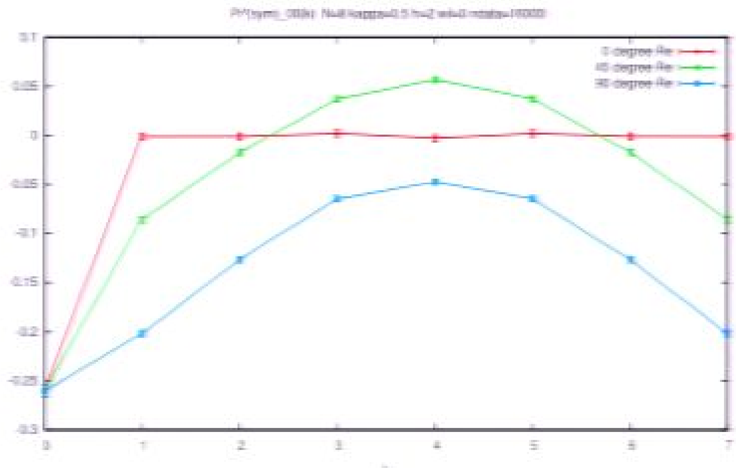


Figure: $\tilde{\Pi}'^{S}_{\mu\nu}(k)$ of the mirror sector at $y \to \infty$

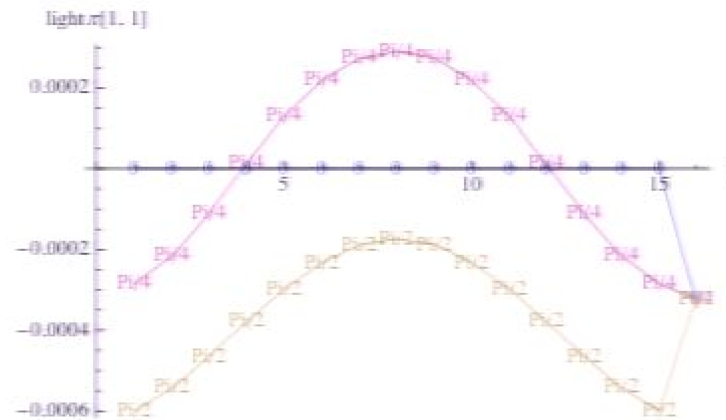


Figure: $\tilde{\Pi}_{\mu\nu}(k)$ of a massless fermion

Indeed a fermionic light degree of freedom in the mirror sector.

- The imaginary part is also checked to give rise the precise angular singularity of $\frac{1}{4\pi}(\epsilon_{\mu\rho}k_\rho k_\mu + \epsilon_{\nu\rho}k_\rho k_\nu)/k^2$, and the unitary bound is satisfied.
- Evidence that the theory is indeed unitary in the continuum limit

A numerical trick

But, how do we distinguish the possible scenarios on a 8×8 lattice?

Angular singularity:

- If there's a massless mode, $Re(\tilde{\Pi}_{00}(k)) \sim \frac{k_0^2 - k^2}{k^2} \sim -\sin^2\theta$, where $(k_0, k_1) \equiv (k\cos\theta, k\sin\theta)$.
- If all excitations are massive, $\tilde{\Pi}_{\mu\nu}(k) \to 0$ when $k \to 0$.

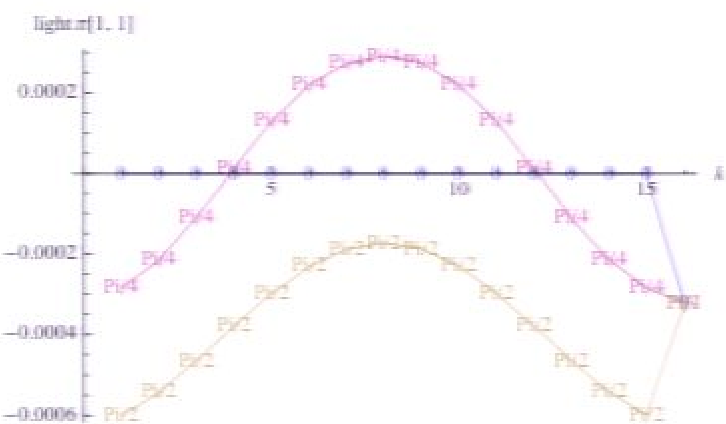


Figure: $\tilde{\Pi}_{\mu\nu}(k)$ of a massless fermion

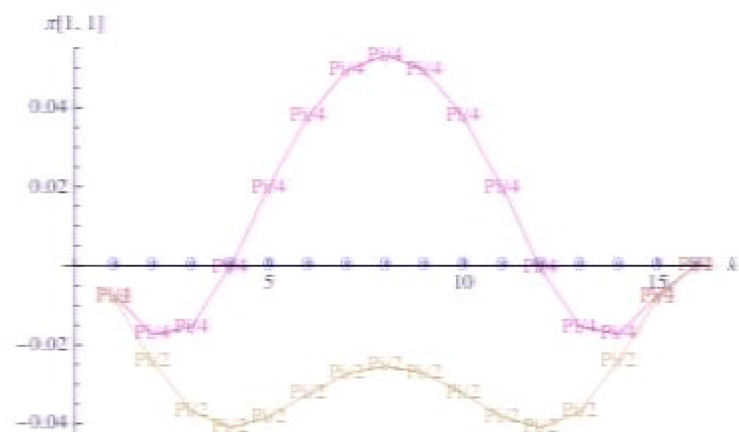


Figure: $\tilde{\Pi}_{\mu\nu}(k)$ of a massive fermion

The numerical results

- After a series non-trivial checks (divergences and so on)

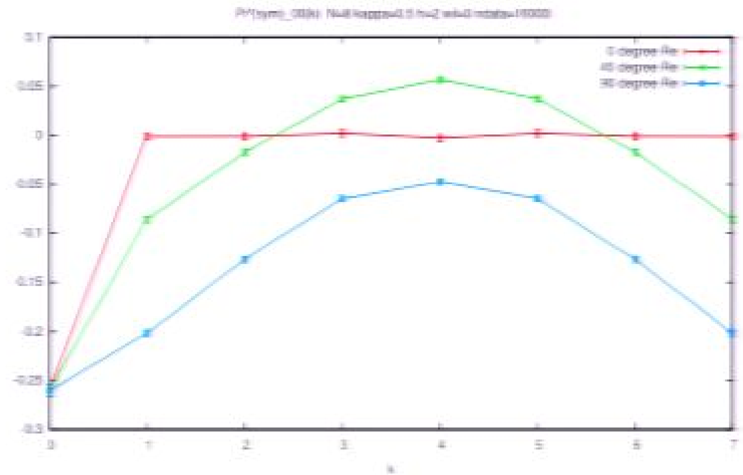


Figure: $\tilde{\Pi}'^{S}_{\mu\nu}(k)$ of the mirror sector at $y \to \infty$

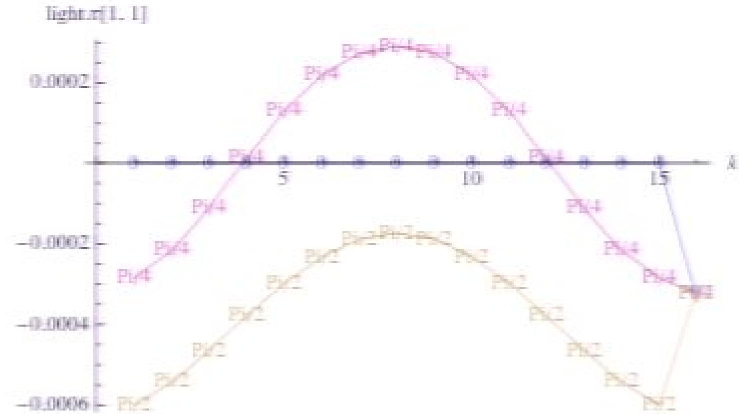


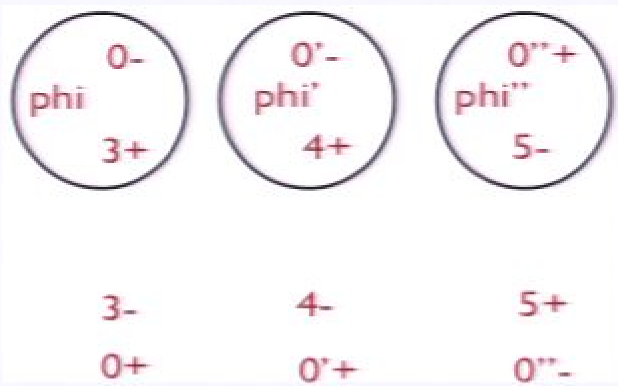
Figure: $\tilde{\Pi}_{\mu\nu}(k)$ of a massless fermion

Indeed a fermionic light degree of freedom in the mirror sector.

- The imaginary part is also checked to give rise the precise angular singularity of $\frac{1}{4\pi}(\epsilon_{\mu\rho}k_\rho k_\mu + \epsilon_{\nu\rho}k_\rho k_\nu)/k^2$, and the unitary bound is satisfied.
- Evidence that the theory is indeed unitary in the continuum limit

So, what about anomaly free theories?

Trivial cancellation of gauge anomaly will not help



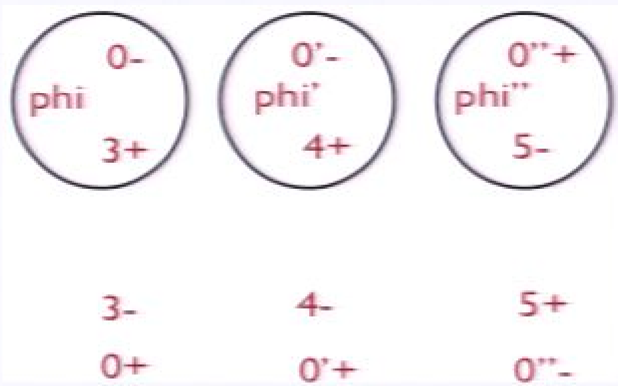
This model is bound to have light fermionic degrees of freedom

Most generally, if there exists a charge re-assignment that allows the theory to become anomalous, it must contain mode.

In other words, if there exists any anomalous global symmetry, there exists light modes (thinking of gauging this symmetry to make an anomalous theory, t' Hooft anomaly matching.).

So, what about anomaly free theories?

Trivial cancellation of gauge anomaly will not help



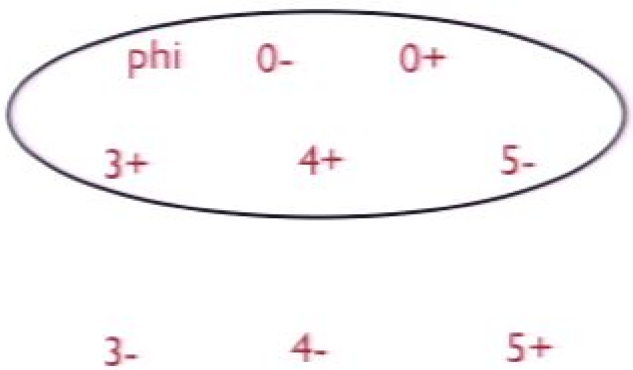
This model is bound to have light fermionic degrees of freedom

Most generally, if there exists a charge re-assignment that allows the theory to become anomalous, it must contain mode.

In other words, if there exists any anomalous global symmetry, there exists light modes (thinking of gauging this symmetry to make an anomalous theory, t' Hooft anomaly matching.).

Anomaly free theories ...

- More involved models may still survive



break all global symmetries that involve mirror fields, so that at $g = 0$, only global symmetries are the trhee light chiral $U(1)$'s.

- We have no convincing evidence that such theories won't work.
- However, some analytic considerations might lead to an argument disproving such possibilities.
- Numerical experiment is doable in 2-d as most of the groundwork is done. 345 or 11112 are ready to be tested.

Outline

1. *Motivation and idea*
   - Why chiral gauge theory, why lattice
   - Why need the idea of "decoupling of mirror fermions"
   - Does it work: early numerical work suggest maybe
2. *A quick review of some most recent theoretical developments*
   - Exact lattice chiral symmetry
   - Put the formalism on a completely general ground
   - A powerful simple theorem
3. *A paradox in the* $0-1$ *model and the unitarity of the GW formalism*
4. *Toward a more analytical approach*

A different viewpoint

- The mirror sector at $y \to \infty$ limit:

$$S_{\text{kinetic}} = -(\overline{\psi}_+ \mathrm{D}_1 \psi_+) - (\overline{\chi}_- \mathrm{D}_0 \chi_-)$$

$$S_{\text{Yukawa}} = y\left\{(\overline{\psi}_+ \phi^* \chi_+) + (\overline{\chi}_- \phi \psi_-) + h[(\psi_-^T \phi \gamma_2 \chi_+) - (\overline{\chi}_- \gamma_2 \phi^* \overline{\psi}_+^T)]\right\}$$

- Make a variable transformation:

$$\overline{\psi} \to \overline{\psi}\frac{1}{2 - \mathrm{D}_1}, \quad \overline{\chi} \to \overline{\chi}\frac{1}{2 - \mathrm{D}_0}$$

A different way of understanding

- Now we find

$$S'_{\text{kinetic}} = -\left(\overline{\psi}_+ \frac{\mathrm{D}_1}{2-\mathrm{D}_1}\psi_+\right) - \left(\overline{\chi}_- \frac{\mathrm{D}_0}{2-\mathrm{D}_0}\chi_-\right)$$

$$S'_{\text{Yukawa}} = \frac{y}{2}\left[(\overline{\psi}_+\phi^*\chi_+) + (\overline{\chi}_-\phi\psi_-)\right]$$
$$+ yh\Bigg[(\psi_-^T\phi\gamma_2\chi_+) - \frac{1}{4}(\overline{\chi}_-\gamma_2\phi^*\overline{\psi}_+^T)$$
$$- \frac{1}{4}\left(\overline{\chi}_- \frac{\mathrm{D}_0}{2-\mathrm{D}_0}\gamma_2\phi^*\left(\frac{\mathrm{D}_1}{2-\mathrm{D}_1}\right)^T\overline{\psi}_+^T\right)\Bigg]$$

A very interesting observation

- The key point is

$$\frac{1}{2-\mathrm{D}}\hat{\gamma}_5 = \gamma_5 \frac{1}{2-\mathrm{D}}, \qquad \left\{ \frac{\mathrm{D}}{2-\mathrm{D}}, \gamma_5 \right\} = 0$$

chiral fermions $\overline{\psi}_+$ and $\overline{\chi}_-$ are the usual Weyl fermions defined with γ_5 now.

$$Z = \det{}_-(2-D_0)\det{}_-(2-\mathrm{D}_1)\int \mathrm{d}\psi_-\mathrm{d}\overline{\psi}_-\mathrm{d}\chi_+\mathrm{d}\overline{\chi}_+\mathrm{d}\phi e^{-S'_{\mathrm{Yukawa}}}$$

Chiral anomalies are captured by the variation of the $\det{}_-(2-\mathrm{D}_1)$.

- $\det{}_-(2-\mathrm{D}_1)$ contains 3 light fermions, the three doublers at $\mathrm{D} \sim 2$.
- when $h > 1$, the Majorana coupling terms evidently contains two "ghost" fermions (the kinetic term has a pole near $\mathrm{D}_1 \sim 2$).

A conjecture

- Most generally, any lattice chiral gauge theory defined with overlap fermions

$$S[X, Y^\dagger, O] = S[PX, Y^\dagger, O] = S[X, Y^\dagger \hat{P}, O]$$

with the orthonormal basis $\{u_i,\ v_i\}$, $P u_i = u_i, v_i^\dagger \hat{P} = v_i^\dagger$, the partition function is given by:

$$Z = \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\bar{c}_j \, e^{S\left[\sum_i c_i u_i, \sum_j \bar{c}_j v_j^\dagger, O\right]}$$

- After the field redefinition

$$Z = \det{}_{-}(2-\mathrm{D}_0)\det{}_{-}(2-\mathrm{D}_1) \int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\bar{c}_j \, e^{S\left[\sum_i c_i u_i', \sum_j \bar{c}_j v_j'^\dagger, O, \mathrm{D}_1(2-\mathrm{D}_1)^{-1}\right]}$$

where u_i and v_i are usual Weyl fermions.

A conjecture

Any chiral lattice gauge theory with overlap fermions are equivalent to

$$Z = \det{}_{-}(2 - \mathrm{D}_0)\det{}_{-}(2 - \mathrm{D}_1)\int \prod_{i,j} \mathrm{d}c_i \mathrm{d}\bar{c}_j \, e^{S\left[\sum_i c_i u'_i, \sum_j \bar{c}_j v'^{\dagger}_j, O, \mathrm{D}_1(2-\mathrm{D}_1)^{-1}\right]}$$

Conjecture

Theories that do not have chiral anomalies necessarily contain fermion doublers $\Rightarrow$ always contain even number of light fermions

But the det's always contain odd number of light fermionic modes $\Rightarrow$ the final theory should always contain light fermions?

Need to make sense of the "ghost" fermions in the continuum limit (*in progress*).

Summary

- GW formalism is theoretically elegant but not so practically useful
- Combining GW formalism and the idea of decoupling of the mirror fermions appeared to make chiral lattice gauge theories possible
- Some earlier numerical work was encouraging but also paradoxical
- Our "splitting theorem" is a general and powerful result, which often leads to surprisingly strong conclusions.
- Examining the photon polarization operator "proved" the existence of light fermions in the mirror sector, and excluded a large class of candidate models even if they are anomaly free.

- Open questions
  - Is this the end or would more involved models survive?
  - A general proof might be more easily achieved after a field redefinition (*in progress*).
  - what about non-Abeliean gauge theories?

Summary

- GW formalism is theoretically elegant but not so practically useful
- Combining GW formalism and the idea of decoupling of the mirror fermions appeared to make chiral lattice gauge theories possible
- Some earlier numerical work was encouraging but also paradoxical
- Our "splitting theorem" is a general and powerful result, which often leads to surprisingly strong conclusions.
- Examining the photon polarization operator "proved" the existence of light fermions in the mirror sector, and excluded a large class of candidate models even if they are anomaly free.
- Open questions
  - Is this the end or would more involved models survive?
  - A general proof might be more easily achieved after a field redefinition (*in progress*).
  - what about non-Abeliean gauge theories?

Summary

- GW formalism is theoretically elegant but not so practically useful
- Combining GW formalism and the idea of decoupling of the mirror fermions appeared to make chiral lattice gauge theories possible
- Some earlier numerical work was encouraging but also paradoxical
- Our "splitting theorem" is a general and powerful result, which often leads to surprisingly strong conclusions.
- Examining the photon polarization operator "proved" the existence of light fermions in the mirror sector, and excluded a large class of candidate models even if they are anomaly free.

- Open questions
  - Is this the end or would more involved models survive?
  - A general proof might be more easily achieved after a field redefinition (*in progress*).
  - what about non-Abeliean gauge theories?

Summary

- GW formalism is theoretically elegant but not so practically useful
- Combining GW formalism and the idea of decoupling of the mirror fermions appeared to make chiral lattice gauge theories possible
- Some earlier numerical work was encouraging but also paradoxical
- Our "splitting theorem" is a general and powerful result, which often leads to surprisingly strong conclusions.
- Examining the photon polarization operator "proved" the existence of light fermions in the mirror sector, and excluded a large class of candidate models even if they are anomaly free.
- Open questions
  - Is this the end or would more involved models survive?
  - A general proof might be more easily achieved after a field redefinition (*in progress*).
  - what about non-Abeliean gauge theories?

Summary

- GW formalism is theoretically elegant but not so practically useful
- Combining GW formalism and the idea of decoupling of the mirror fermions appeared to make chiral lattice gauge theories possible
- Some earlier numerical work was encouraging but also paradoxical
- Our "splitting theorem" is a general and powerful result, which often leads to surprisingly strong conclusions.
- Examining the photon polarization operator "proved" the existence of light fermions in the mirror sector, and excluded a large class of candidate models even if they are anomaly free.

- Open questions
  - Is this the end or would more involved models survive?
  - A general proof might be more easily achieved after a field redefinition (*in progress*).
  - what about non-Abeliean gauge theories?

Summary

- GW formalism is theoretically elegant but not so practically useful
- Combining GW formalism and the idea of decoupling of the mirror fermions appeared to make chiral lattice gauge theories possible
- Some earlier numerical work was encouraging but also paradoxical
- Our "splitting theorem" is a general and powerful result, which often leads to surprisingly strong conclusions.
- Examining the photon polarization operator "proved" the existence of light fermions in the mirror sector, and excluded a large class of candidate models even if they are anomaly free.
- Open questions
    - Is this the end or would more involved models survive?
    - A general proof might be more easily achieved after a field redefinition (*in progress*).
    - what about non-Abeliean gauge theories?

THANK YOU

A numerical trick

But, how do we distinguish the possible scenarios on a 8×8 lattice?

Angular singularity:

- If there's a massless mode, $Re(\tilde{\Pi}_{00}(k)) \sim \frac{k_0^2 - k^2}{k^2} \sim -\sin^2\theta$, where $(k_0, k_1) \equiv (k\cos\theta, k\sin\theta)$.
- If all excitations are massive, $\tilde{\Pi}_{\mu\nu}(k) \to 0$ when $k \to 0$.

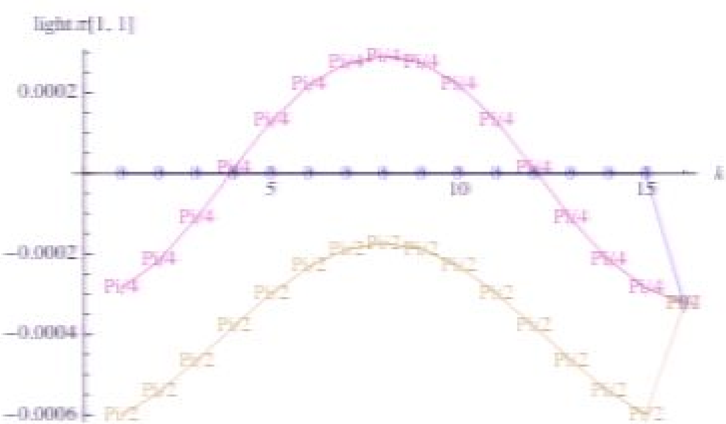


Figure: $\tilde{\Pi}_{\mu\nu}(k)$ of a massless fermion

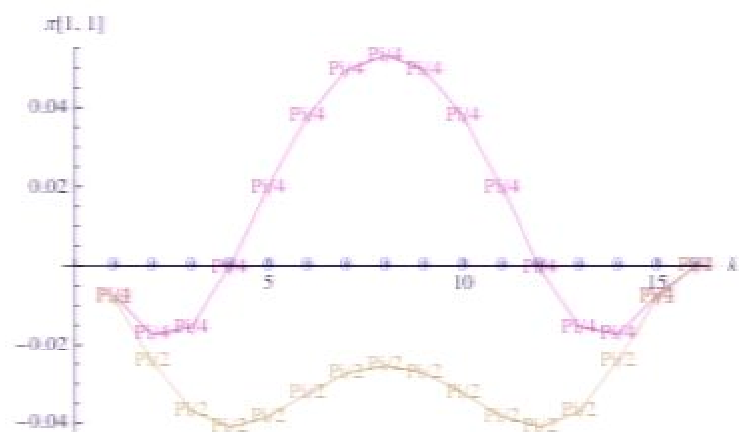


Figure: $\tilde{\Pi}_{\mu\nu}(k)$ of a massive fermion

- Unitarity demands that $\langle (n(x) - m(x))(n(y) - m(y)) \rangle \geq 0$, therefore

$$\langle n(x)n(y) + m(x)m(y) \rangle \geq 2 \langle n(x)m(y) \rangle$$

We conclude that

$$\tilde{\Pi}_{\mu\nu}(k) \sim A \frac{k_\mu k_\nu - g_{\mu\nu} k^2}{k^2} - iB \frac{\epsilon_{\mu\rho} k_\rho k_\nu + \epsilon_{\nu\rho} k_\rho k_\mu}{k^2}$$

and

$$A \geq 2|B|$$

if the theory is unitary in the continuum limit.

- A free fermion saturates the bound precisely.