Title: Interacting Anyonic Fermions in a Two-Body Color Code Model

Date: Jan 21, 2009 04:00 PM

URL: http://pirsa.org/09010008

Abstract: We introduce a two-body quantum Hamiltonian model of spin-1/2 on a 2D spatial lattice with exact topological degeneracy in all coupling regimes. There exists a gapped phase in which the low-energy sector reproduces an effective color code model. High energy excitations fall into three families of anyonic fermions that turn out to be strongly interacting. The model exhibits a Z_2xZ_2 gauge group symmetry and string-net integrals of motion, which are related to the existence of topological charges that are invisible to moving high-energy fermions.

Interacting Anyonic Fermions in a Two-Body Color Code Model

ArXiv:0811.0911

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Outline

- **Motivation**

  Anyons, stabilizer codes, 2-body models.

- **The model**

  Hamiltonian, string-net constants of motion.

- **Bosonic mapping**

  Emerging anyonic fermions, effective color code.

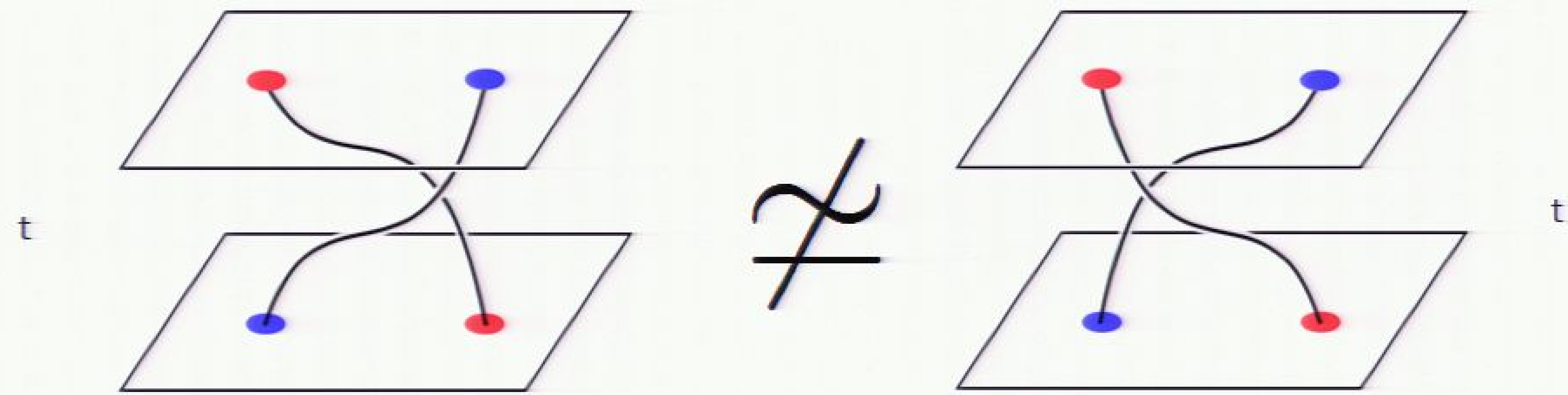
- **Future work and conclusions**

  Anyon condensation, color symmetry breaking

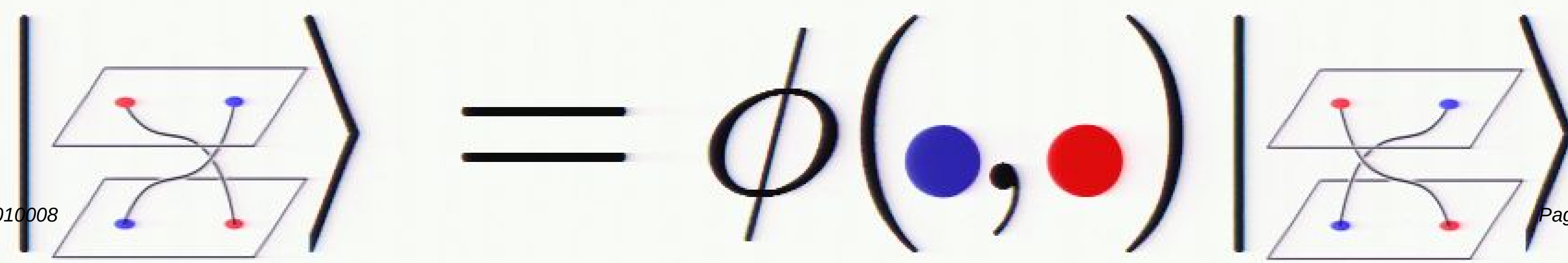
Motivation

Anyons

- In our everyday 3D world we only deal with **fermions** and **bosons**.
- Exchanging twice a pair of particles is a topologically trivial operation.
- In 2D this is no longer true and particles with other statistics are possible: **anyons.**



- When the difference is just a phase, the anyons are **abelian**:



Topological order

- Anyons are a signature of **topological order** (TO):
  - ✓ **Topological degeneracy** of the groud subspace (GS).
  - ✓ **Gapped** excited states: **localized quasiparticles**, **anyons**.
  - ✓ ...
- But where do we find TO?
- If we are lucky, on existing physical systems such as the quantum Hall effect.
- But we can also engineer suitable quantum Hamiltonian models. E. g., using polar molecules on optical lattices, as proposed by P. Zoller and collaborators.

Stabilizer code models

- Some of the simplest quantum Hamiltonian models with topological order can be obtained from local **stabilizer codes**.
- These are **spin-1/2 local models** of the form

$$H = -\sum_i S_i, \qquad S_i \in \mathcal{P} = \langle i, \sigma_1^x, \sigma_1^z, \dots \sigma_n^x, \sigma_n^z \rangle.$$

where the S_i generate an abelian group not containing -1.

- The **GS** is a stabilizer code...

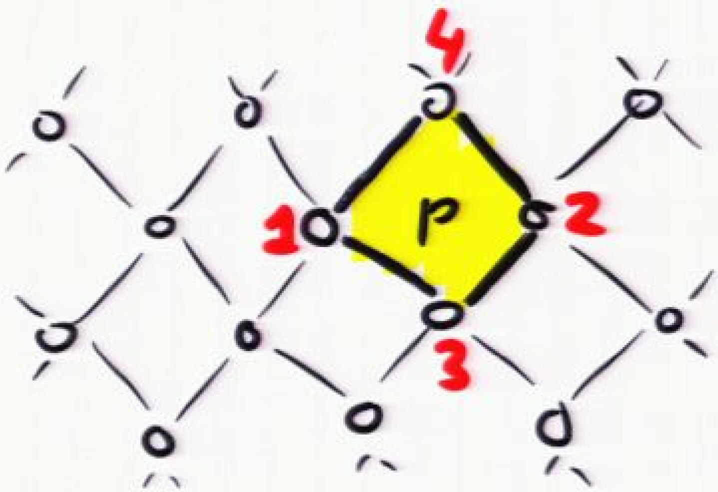
$$S_i \, |\mathrm{GS}\rangle = |\mathrm{GS}\rangle$$

...and excitations are **gapped** and correspond to error **syndromes**

$$S_i \, |\psi\rangle = -|\psi\rangle$$

Toric code

- The seminal example of topological stabilizer codes is the **toric code** (Kitaev, 97).
- In a **square lattice**, we place a **spin-1/2** system at each **vertex**.
- There is **one** stabilizer per **plaquette**:

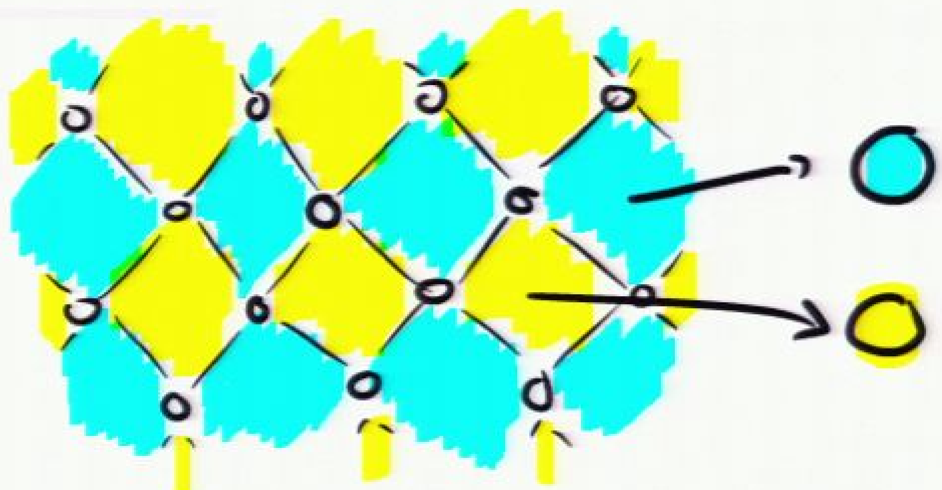


$$A_p := \sigma_1^x \sigma_2^x \sigma_3^z \sigma_4^z$$

$$H = -\sum_p A_p$$

Toric code

- There exist **two** kinds of basic **excitations**.
- To label them, we have to color the plaquettes as in a **chessboard**:



- There exist a total of **three nontrivial topological charges**:

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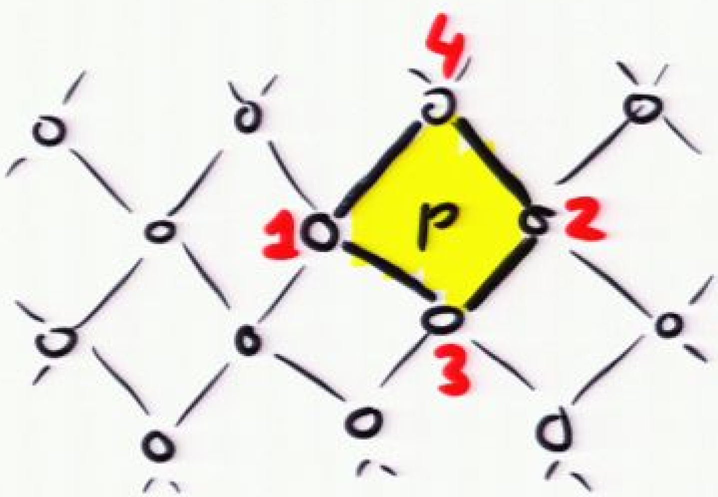

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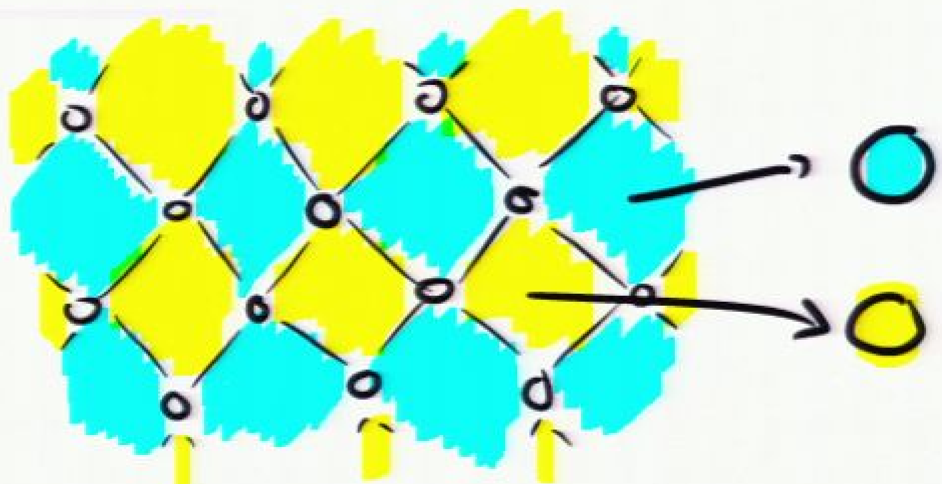

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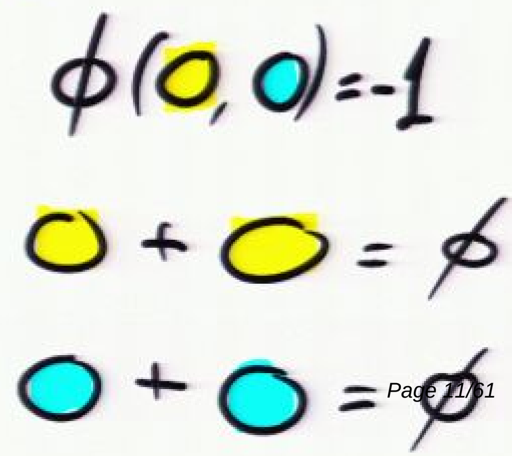

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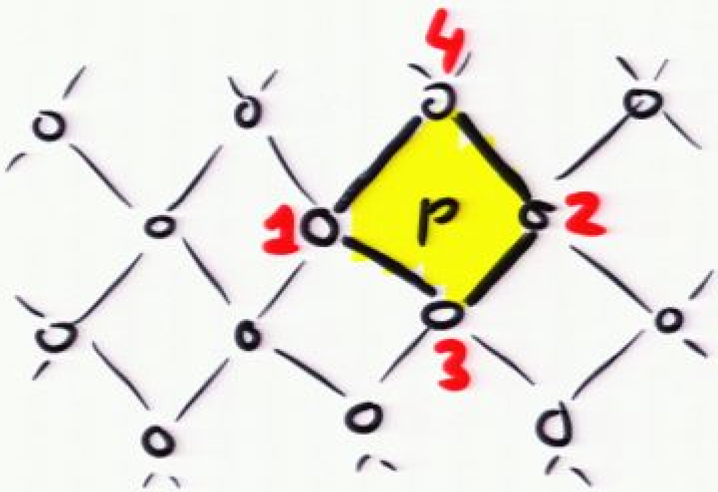

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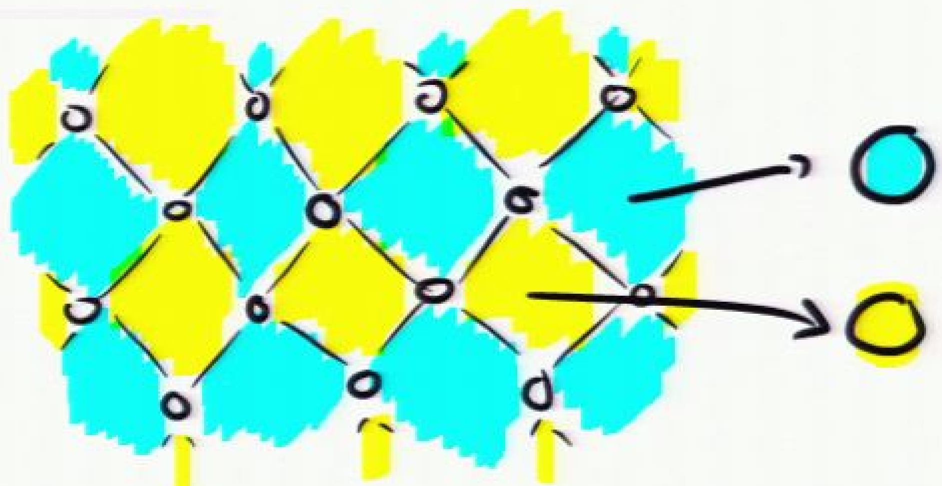

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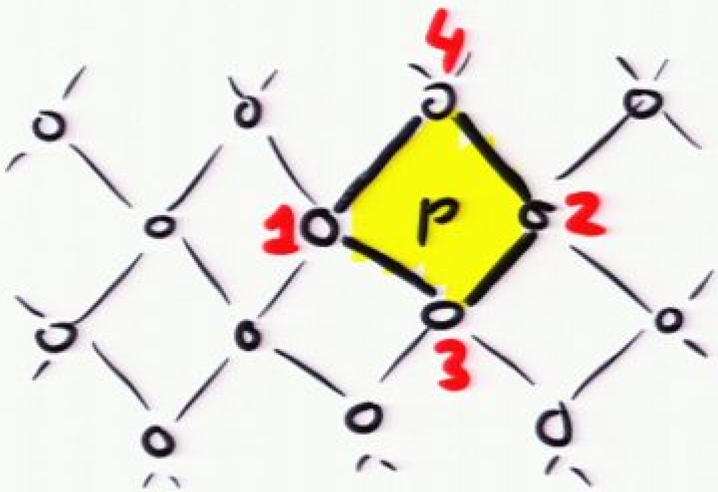

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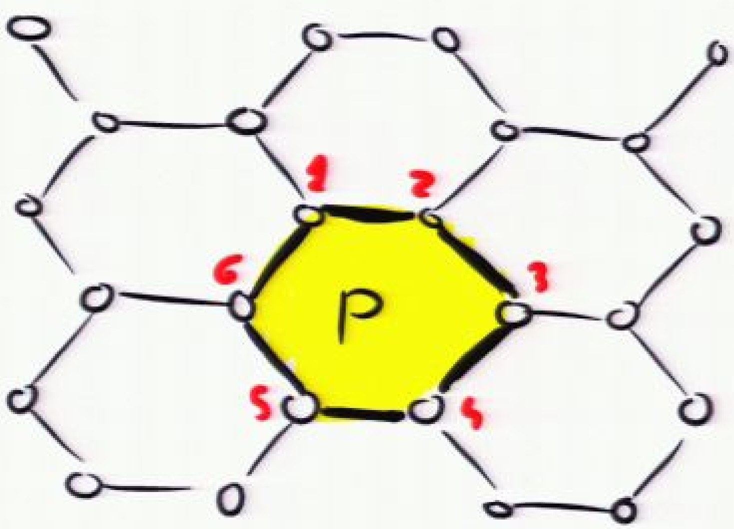
Color codes

- **Color codes** are another relevant example, with enhanced computational capabilities.
- In particular, they allow the **transversal** implementation of **Clifford** operations.
- In a **honeycomb lattice**, we place a **spin-1/2** system at each **vertex**.
- There are **two** stabilizers per **plaquette**:



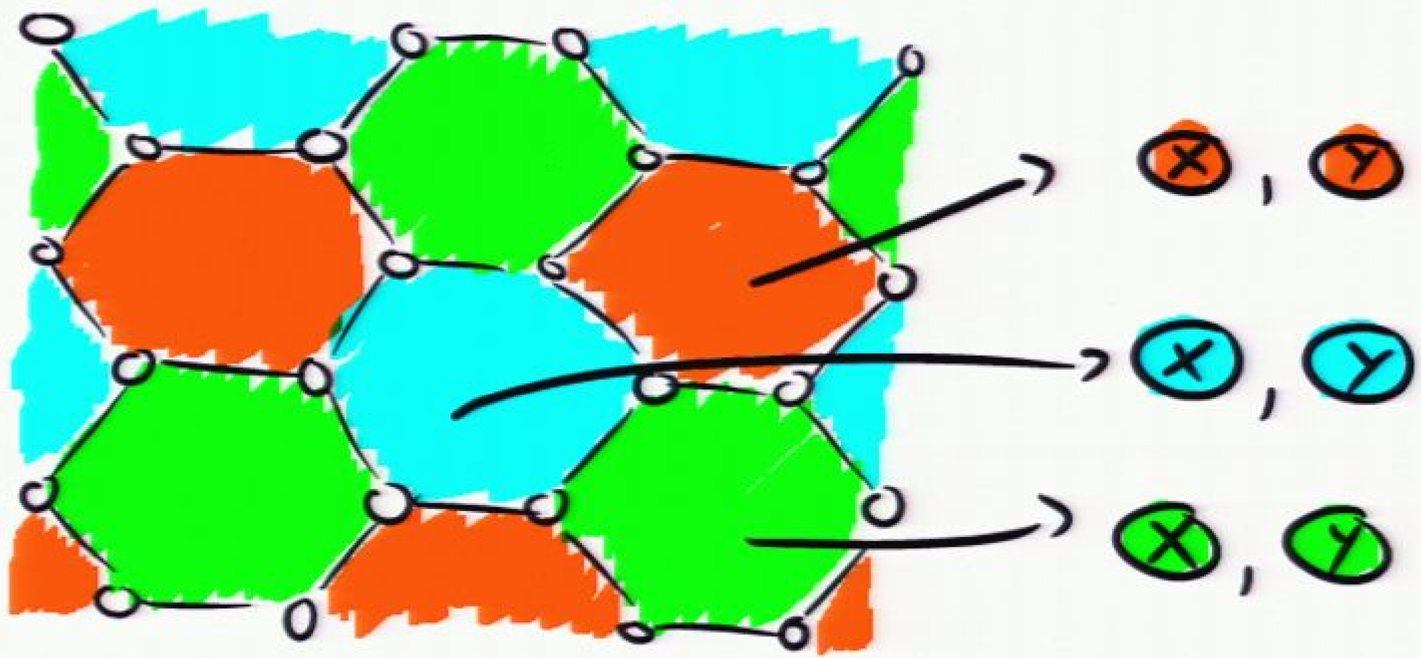
$$B_p^x := \sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x \sigma_5^x \sigma_6^x$$

$$B_p^y := \sigma_1^y \sigma_2^y \sigma_3^y \sigma_5^y \sigma_5^y \sigma_6^z$$

$$H = -\sum_p (B_p^x + B_p^y)$$

Color codes

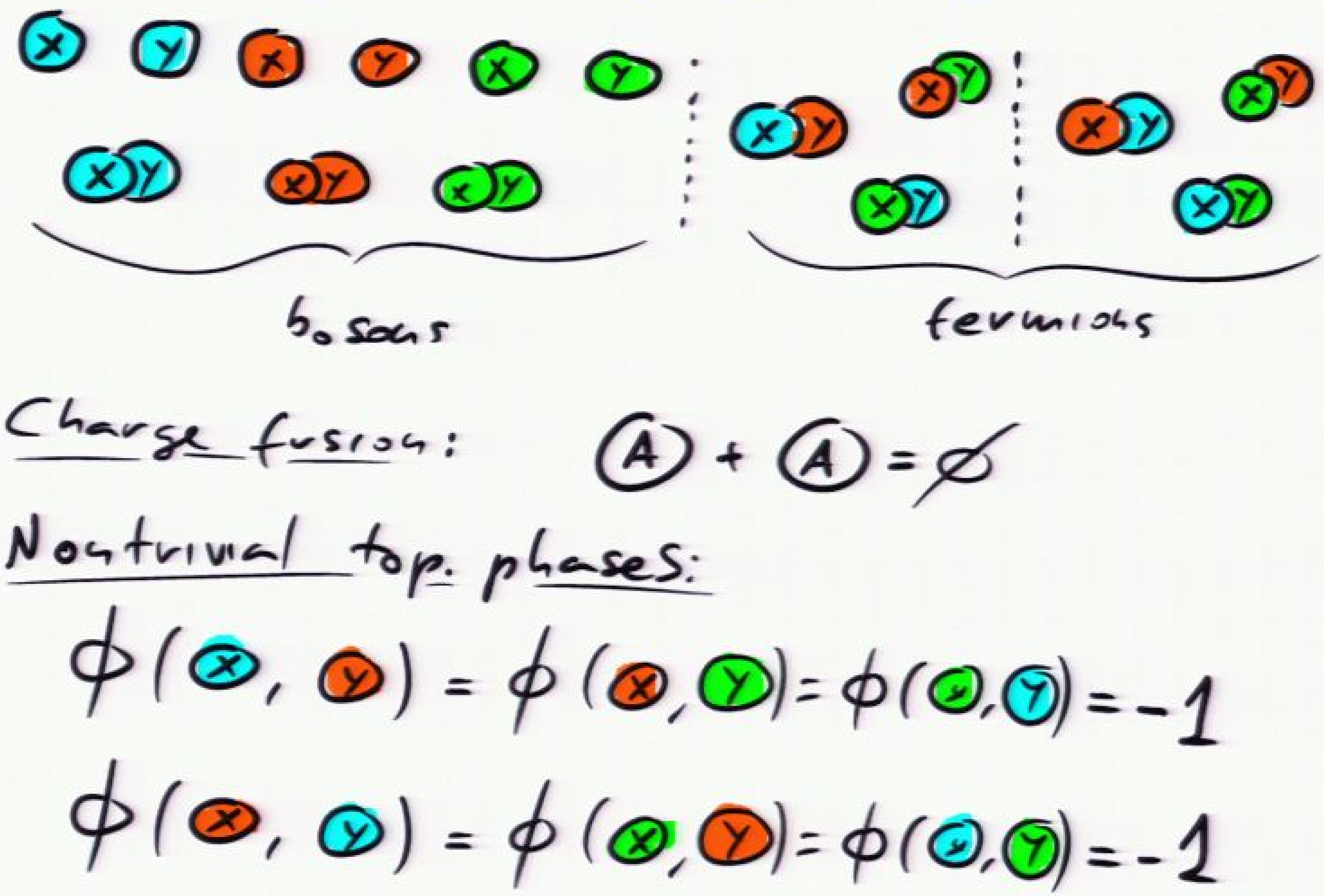
- There exist **six** kinds of basic **excitations**.
- To label them, we first label the plaquettes with **three colors**:



- Notice that the lattice is **3-valent** and has **3-colorable plaquettes**. We call such lattices **2-colexes**.
- One can define color codes in **any** 2-colex.

Color codes

- There exist a total of **15 nontrivial topological charges**:



- Each **family of fermions** is **closed under fusion**, and fermions from different families have **trivial mutual statistics**

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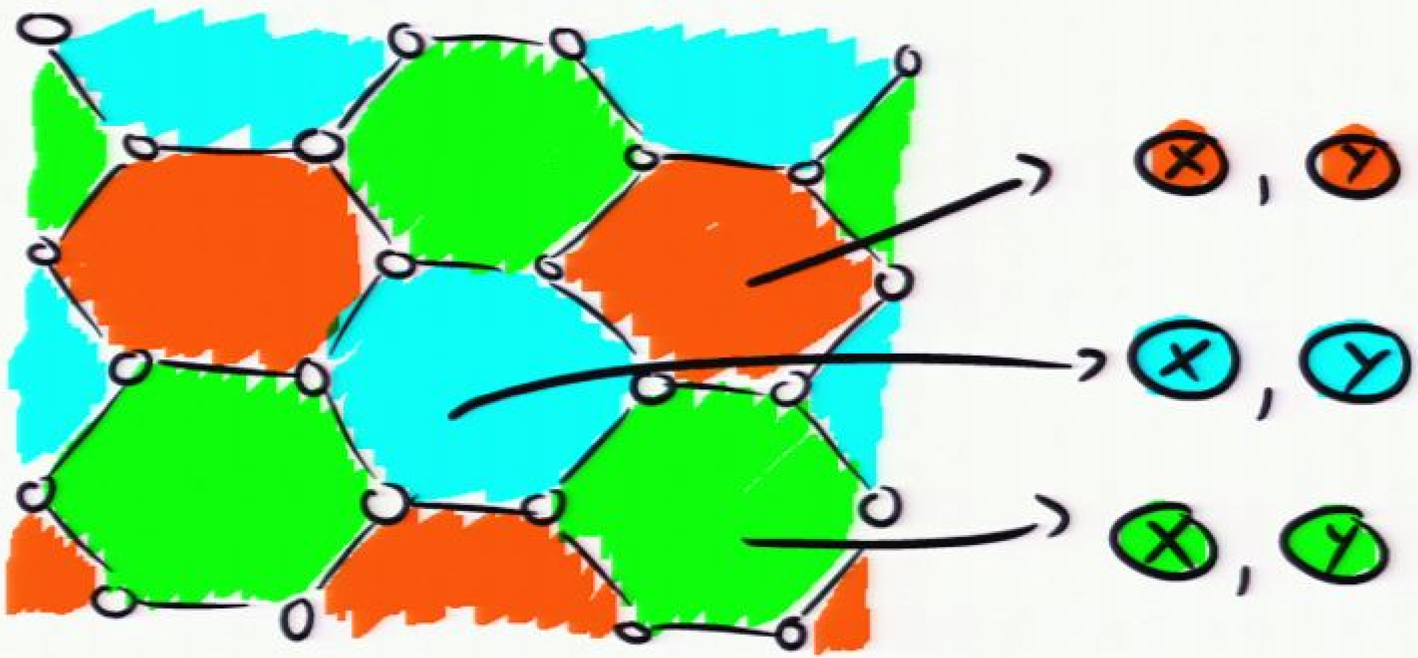

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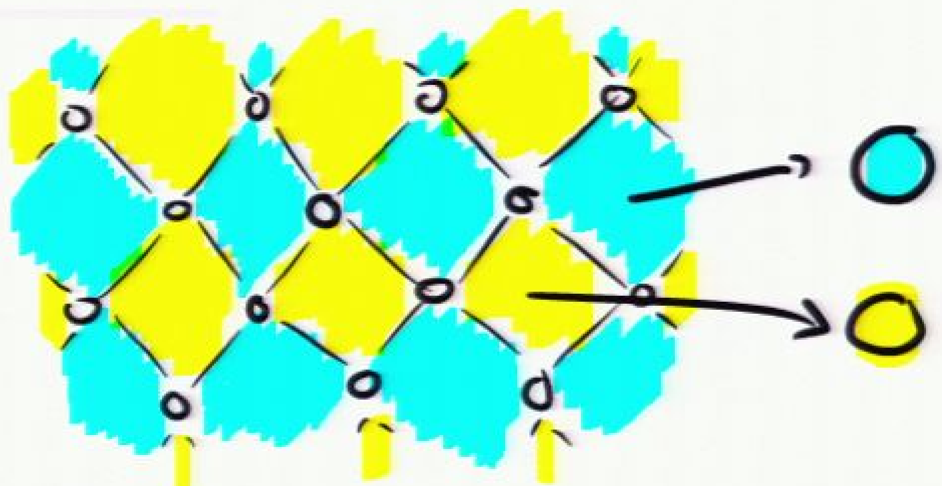

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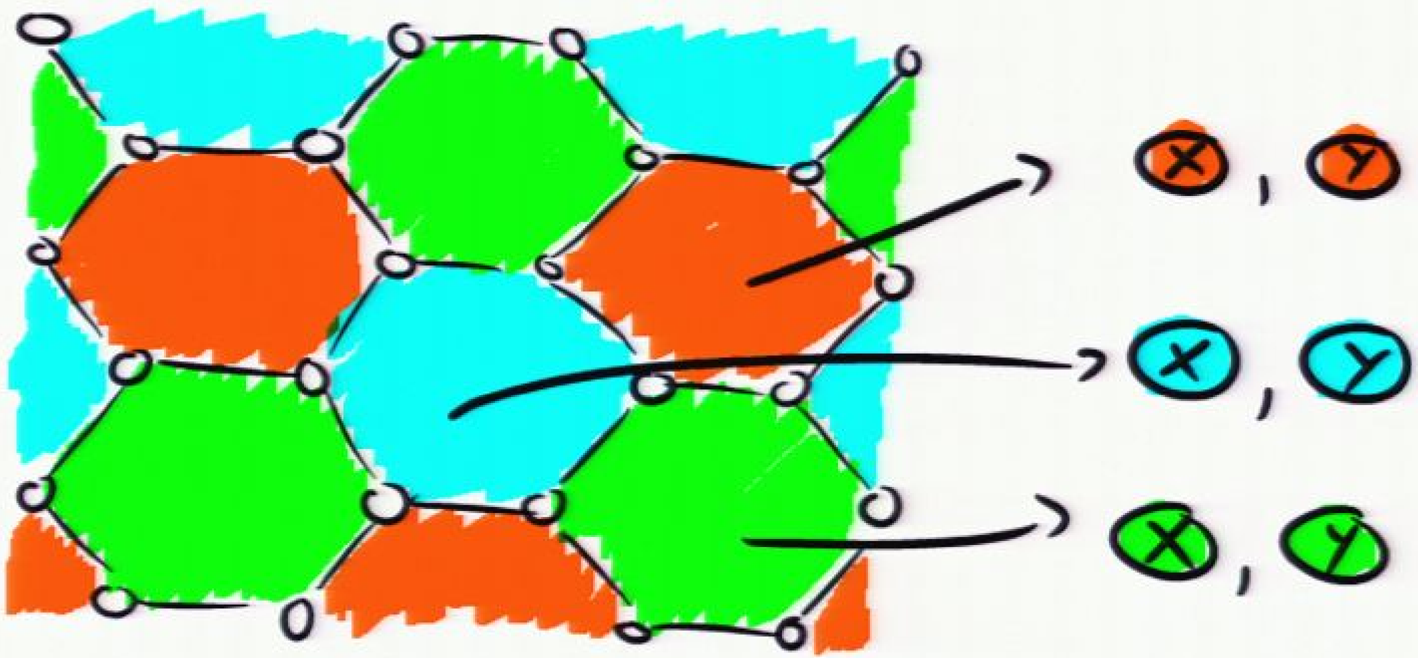

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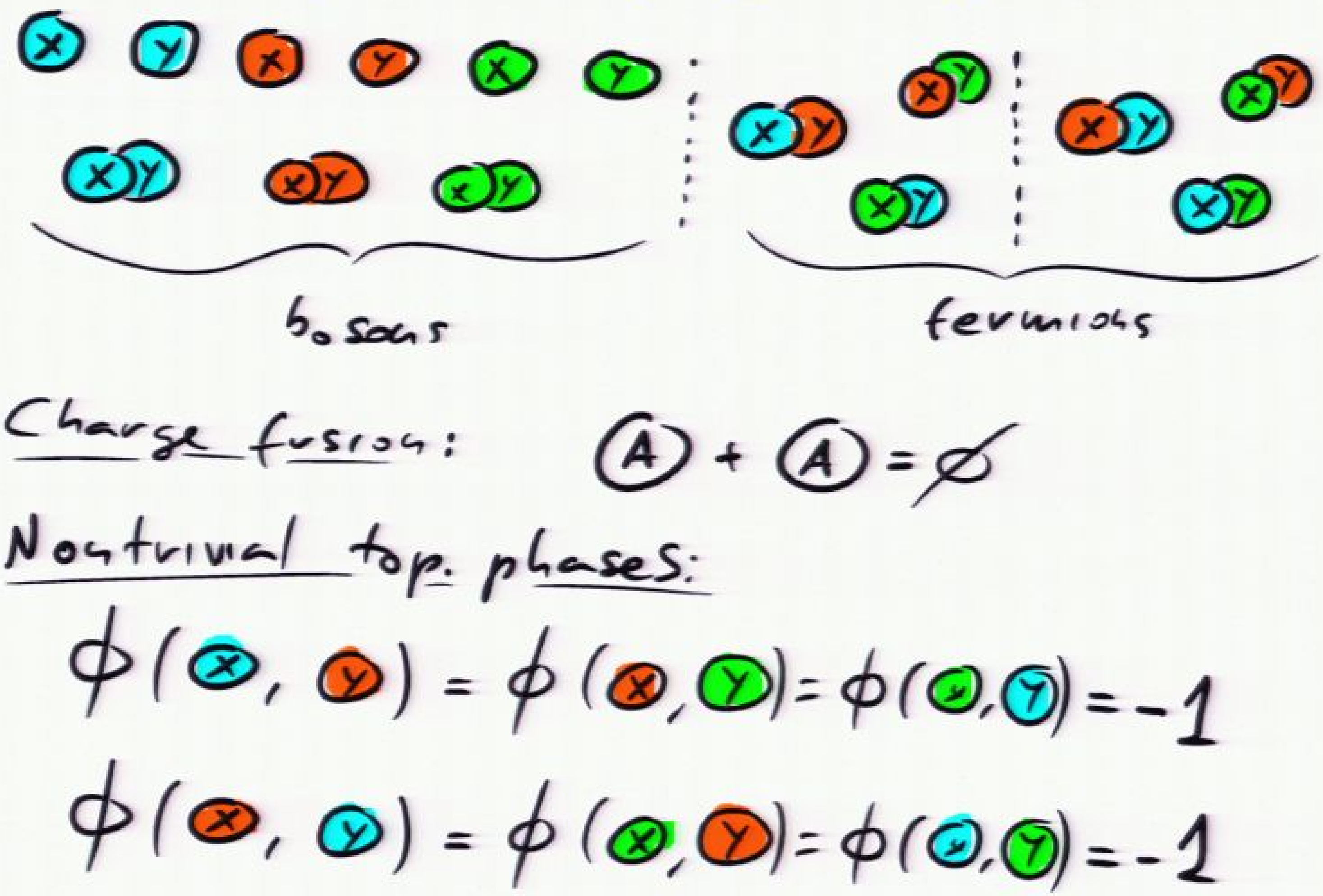


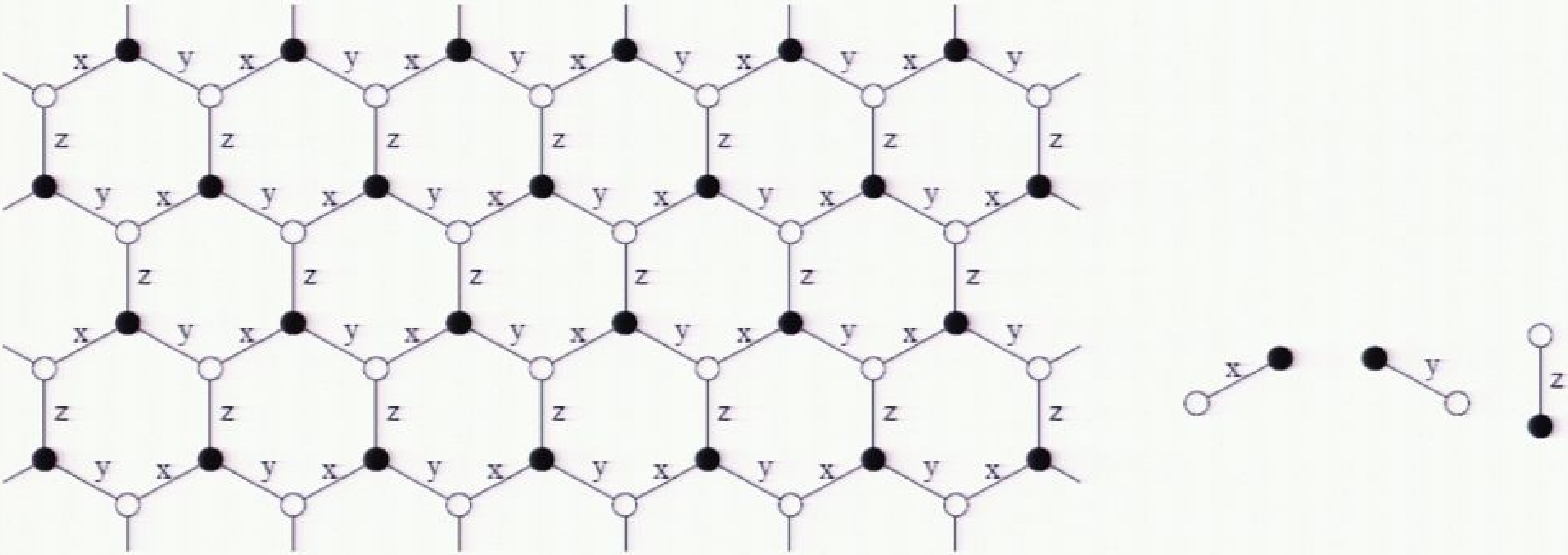
- Each **family of fermions** is **closed under fusion**, and fermions from different families have **trivial mutual statistics**

2-body models

- But both toric and color code models are **many-body.**
- This makes very **difficult** to **engineer** them directly.
- Kitaev introduced a **2-body** model in the **honeycomb** lattice that gives rise to an effective **toric code** model in one of its phases.

Kitaev's honeycomb model

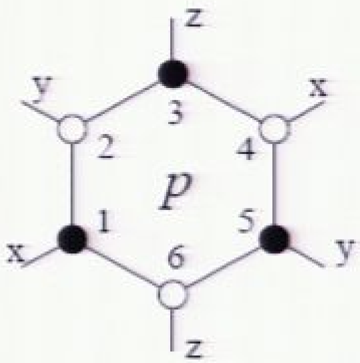
- A **2-body spin-1/2** model in a honeycomb lattice. 1 spin per vertex. 3-colored links.



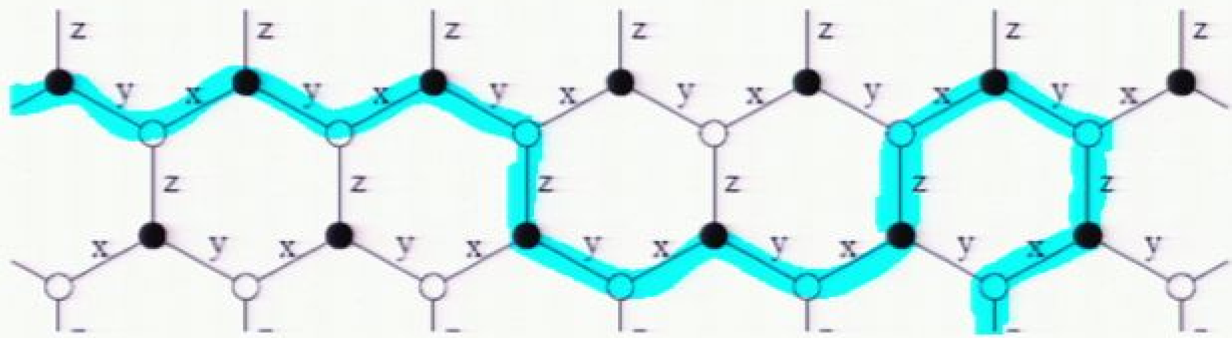
$$H = -J_x \sum_{x\text{-links}} \sigma_j^x \sigma_k^x - J_y \sum_{y\text{-links}} \sigma_j^y \sigma_k^y - J_z \sum_{z\text{-links}} \sigma_j^z \sigma_k^z,$$

Kitaev's honeycomb model

- The model features plaquette and strings **constants of motion**



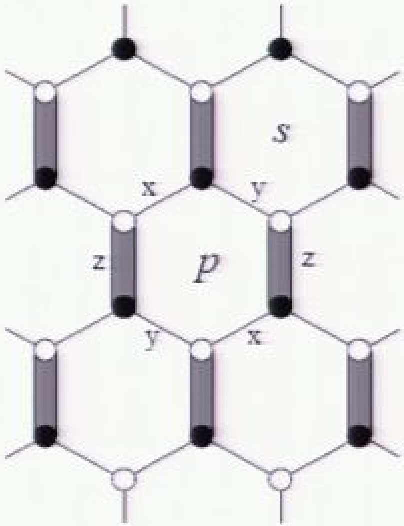
$$W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z$$

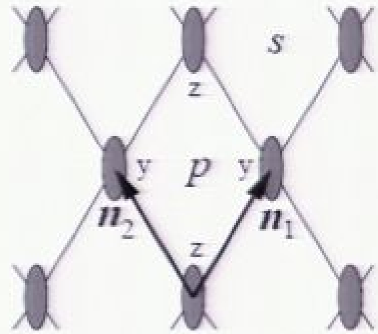


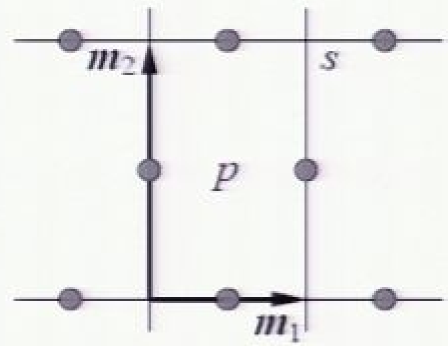
- **Exactly solvable** (3-valence!): emerging **free fermions** in the honeycomb lattice.
- If a magnetic field is added, it contains a non-abelian topological phase (although not enough for universal quantum computation).
- Good chances to be **implemented** (e.g., with polar molecules in optical lattices)

Effective toric codes

- A regime of the model gives rise to a **4-body** model, an effective toric code model:





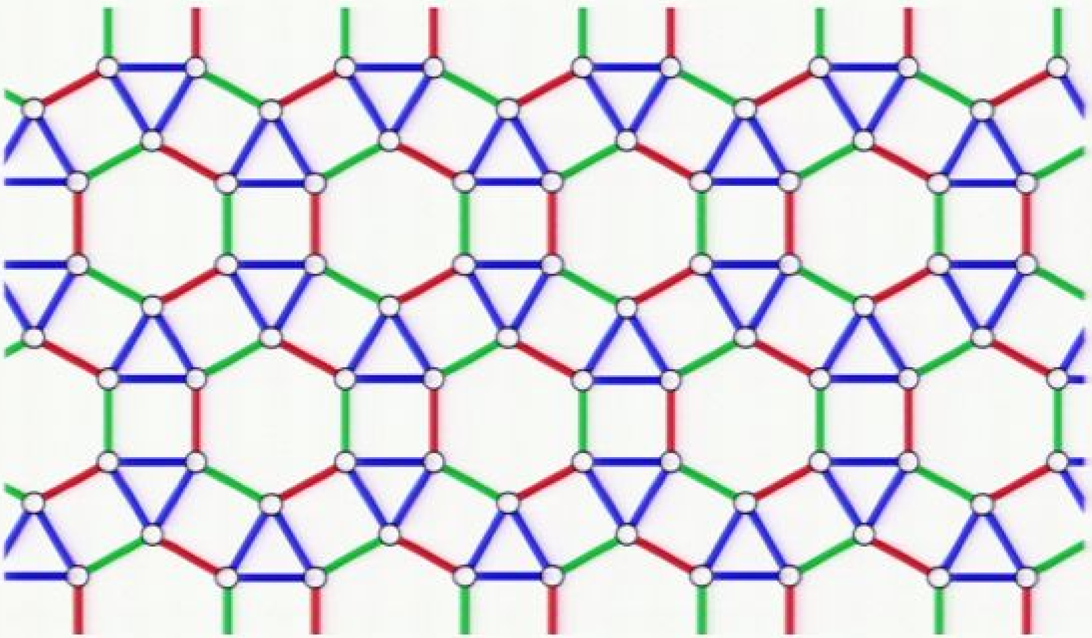


- **Can we get something similar for color codes?**

The model

Hamiltonian

- It is a **2-body spin-1/2** model in a 'ruby' lattice.
- We place one spin per **vertex**.
- **Links** come in 3 **colors**, each color representing a different interaction.



$$H = -\sum_{<i,j>} J_w \, \sigma_i^w \sigma_j^w$$

$$w = \begin{cases} x, & \text{red links} \\ y, & \text{green links} \\ z, & \text{blue links} \end{cases}$$

Hamiltonian

- For a suitable coupling regime, this model gives rise to an effective color code model.
- **New features**, not present in honeycomb-like models:
  - **Exact topological degeneracy** in all coupling regimes (4^g for genus g)
  - **String-net integrals of motion**.
  - Emergence of **3 families of strongly interacting fermions** with **semionic** mutual statistics => **Not (?) exactly solvable** (4-valence!).
  - $Z_2 \times Z_2$ **gauge symmetry**. Each family of fermions sees a different Z_2 gauge subgroup.

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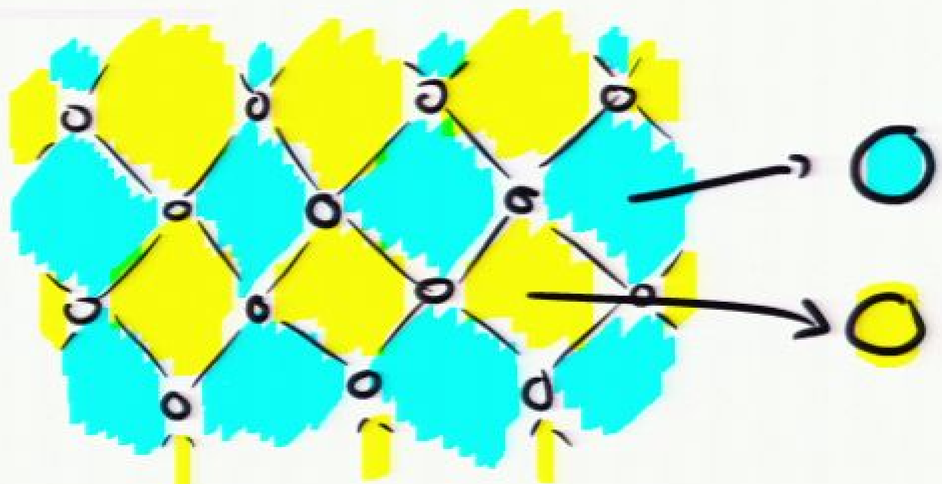

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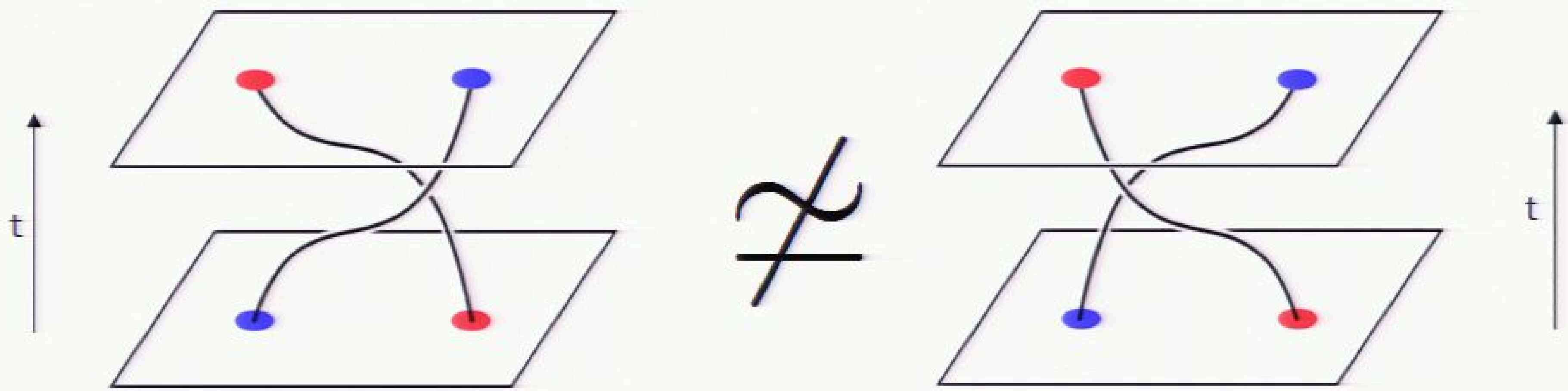

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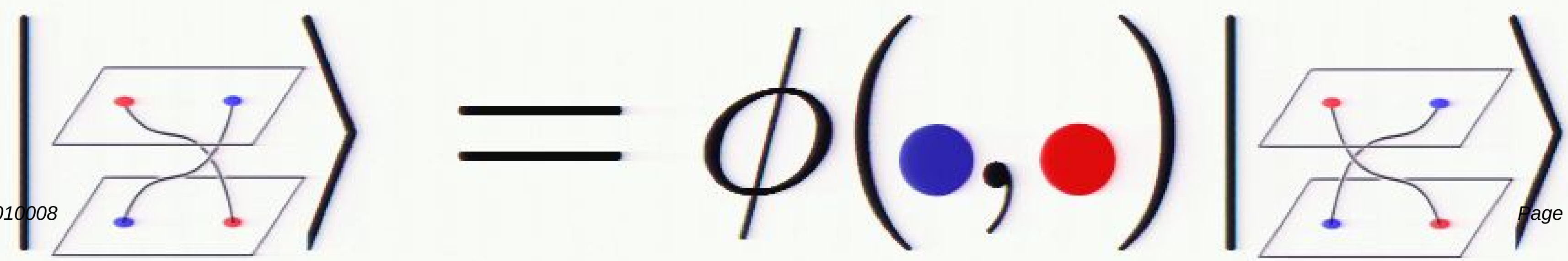

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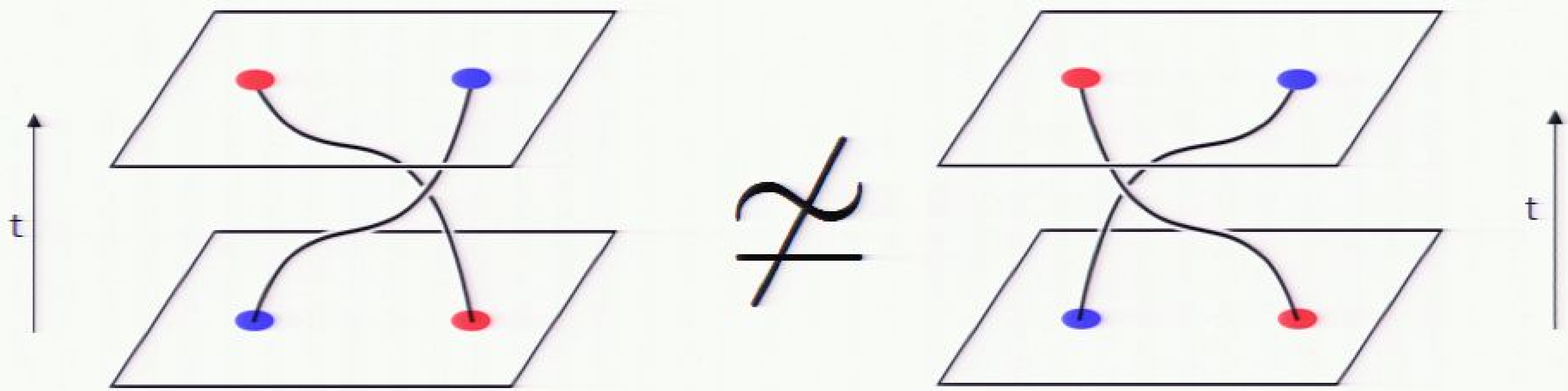

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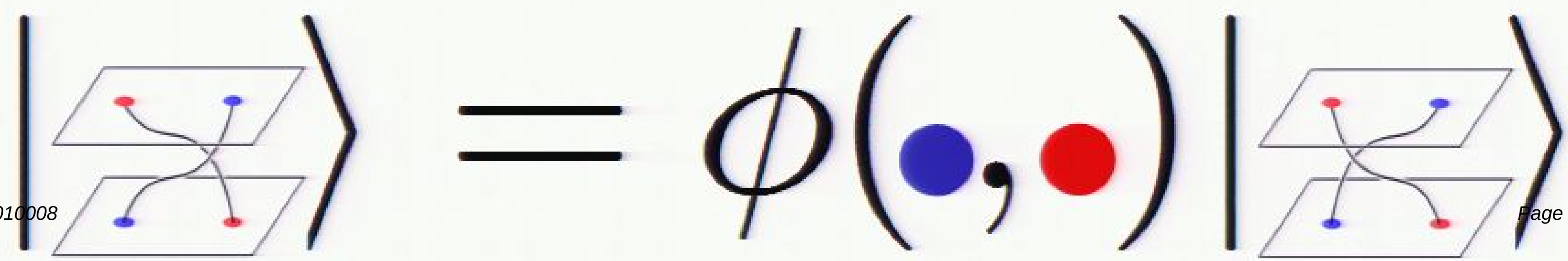

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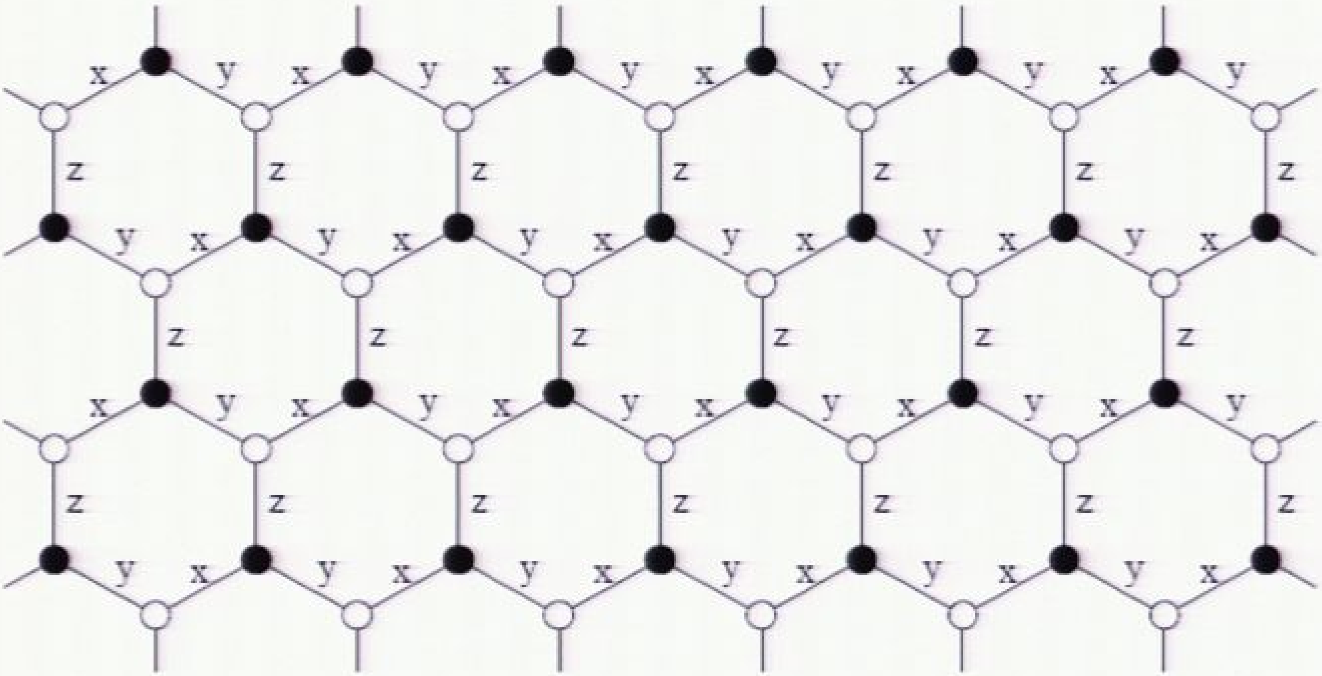

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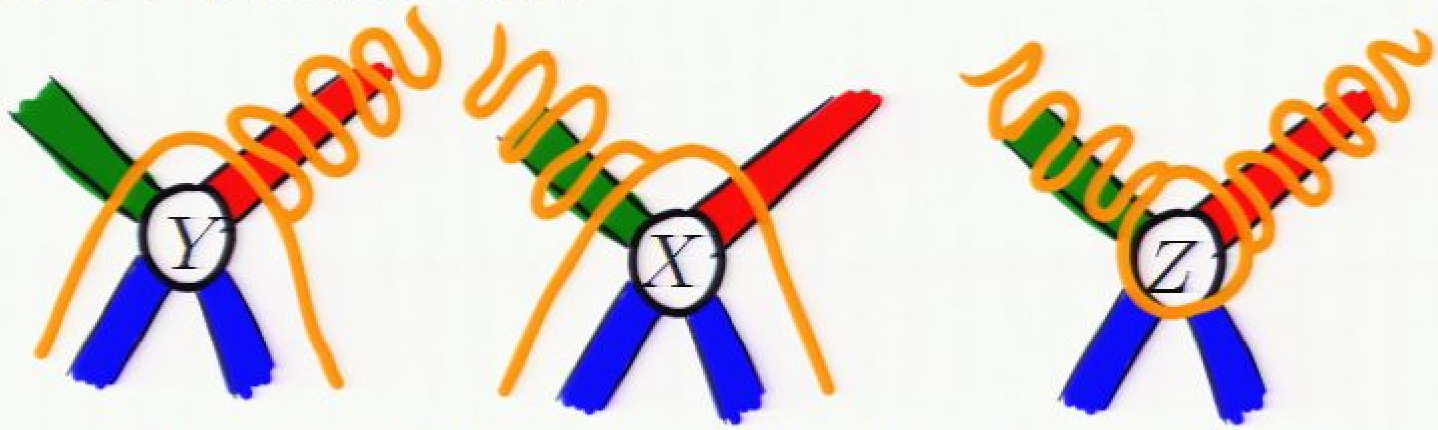
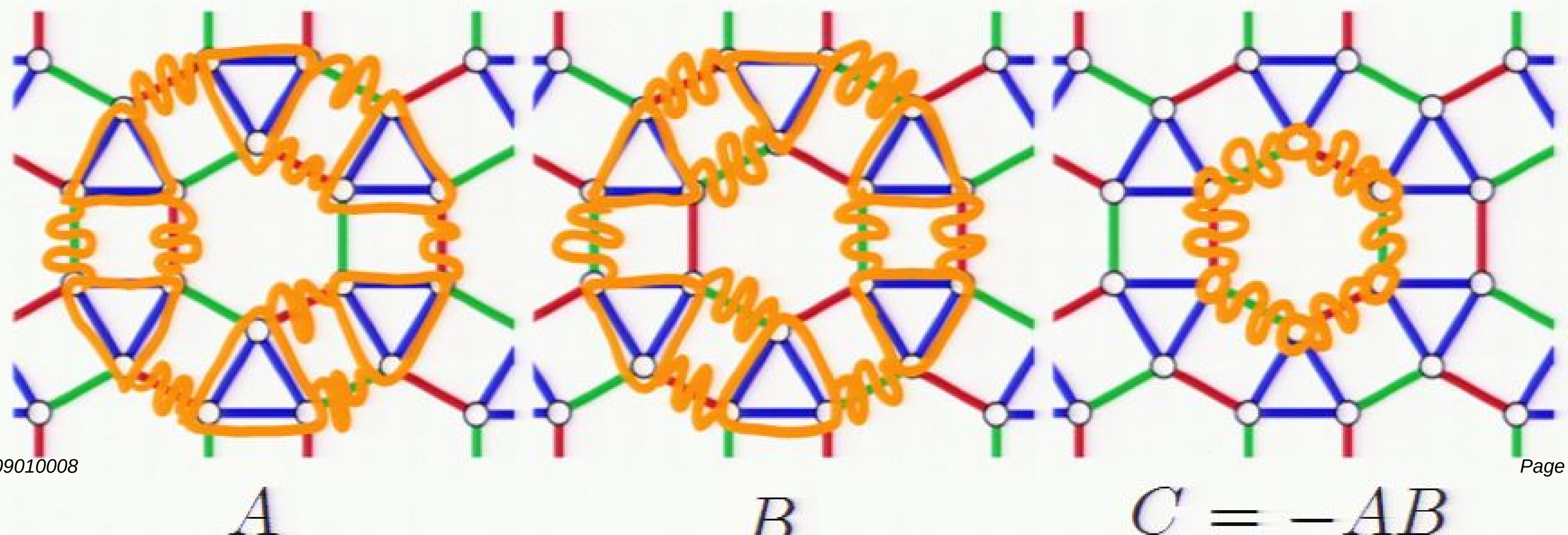
Integrals of motion

- We can construct **integrals of motion** (IOM) from drawings that translate into products of Pauli operators. Locally:

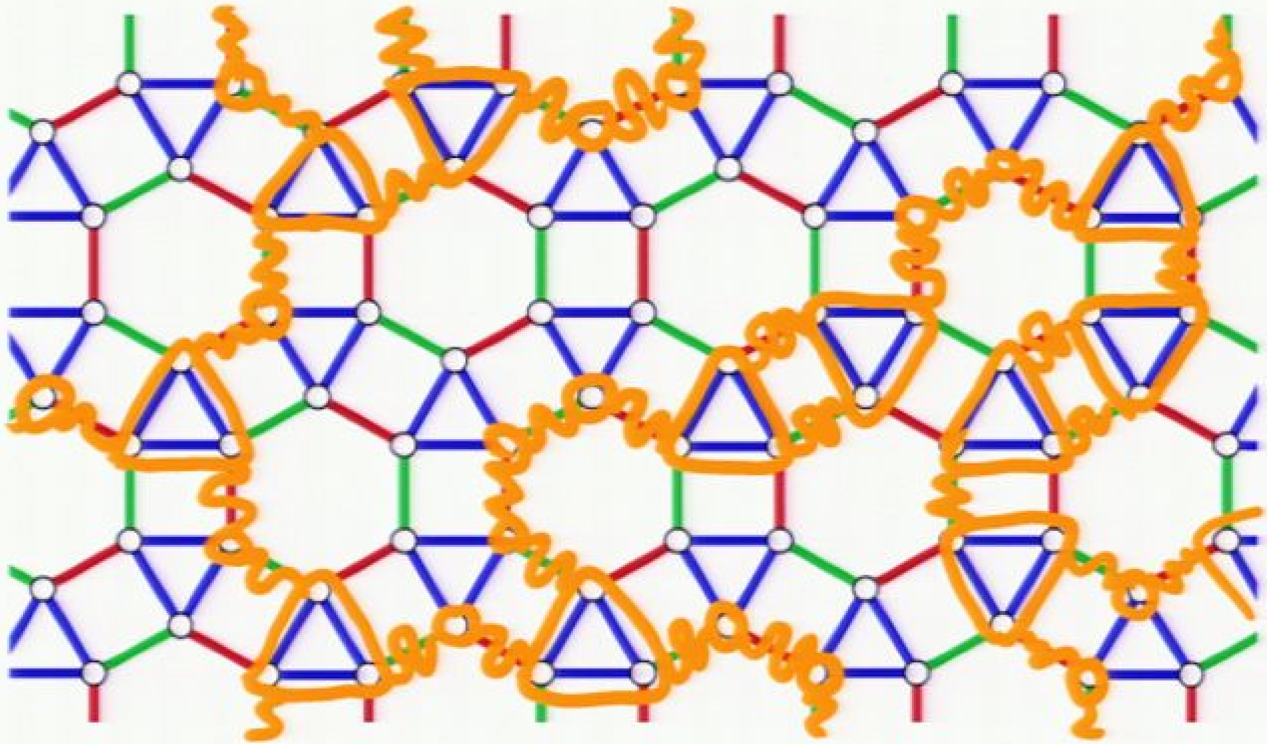


- There exist **2** independent IOM per **plaquette**: this is the $Z_2 x Z_2$ local symmetry.



String-net IOM

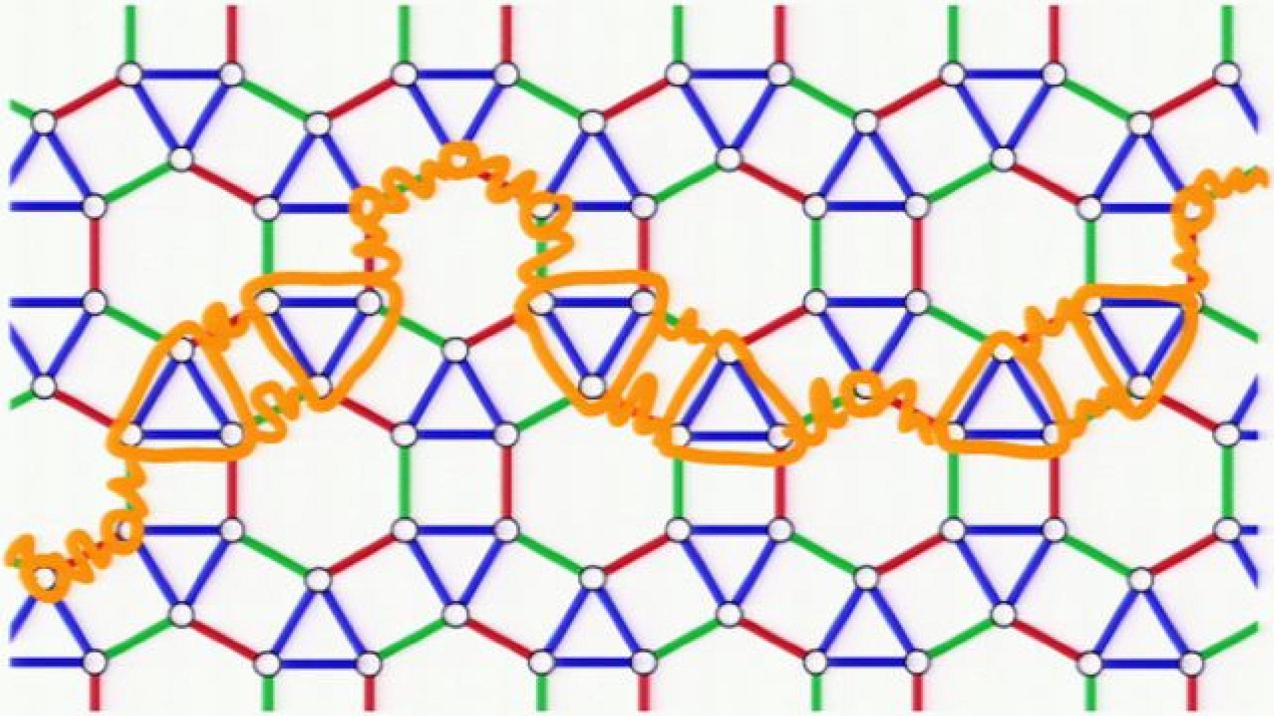
- More generally, we have **string-net** IOM.



- When they are defined on a **simply connected** piece of lattice they are products of plaquette operators.
- More generally, they can be **topologically non-trivial** and independent of plaquette operators.

String IOM

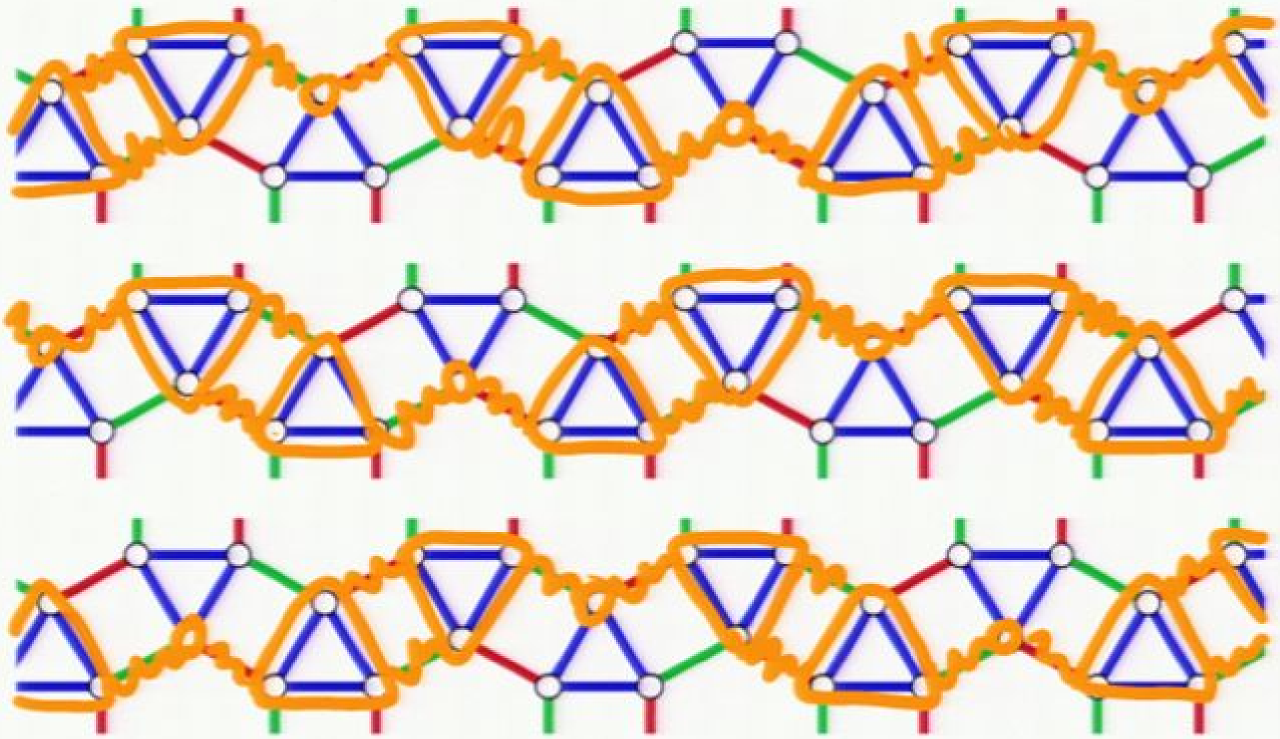
- As a special case we have **string** IOM:



- String IOM are **easier** to analyze.
- String-net IOM are **products** of string IOM.

String IOM

- For a given path, there exist **3 different** string IOM:



A

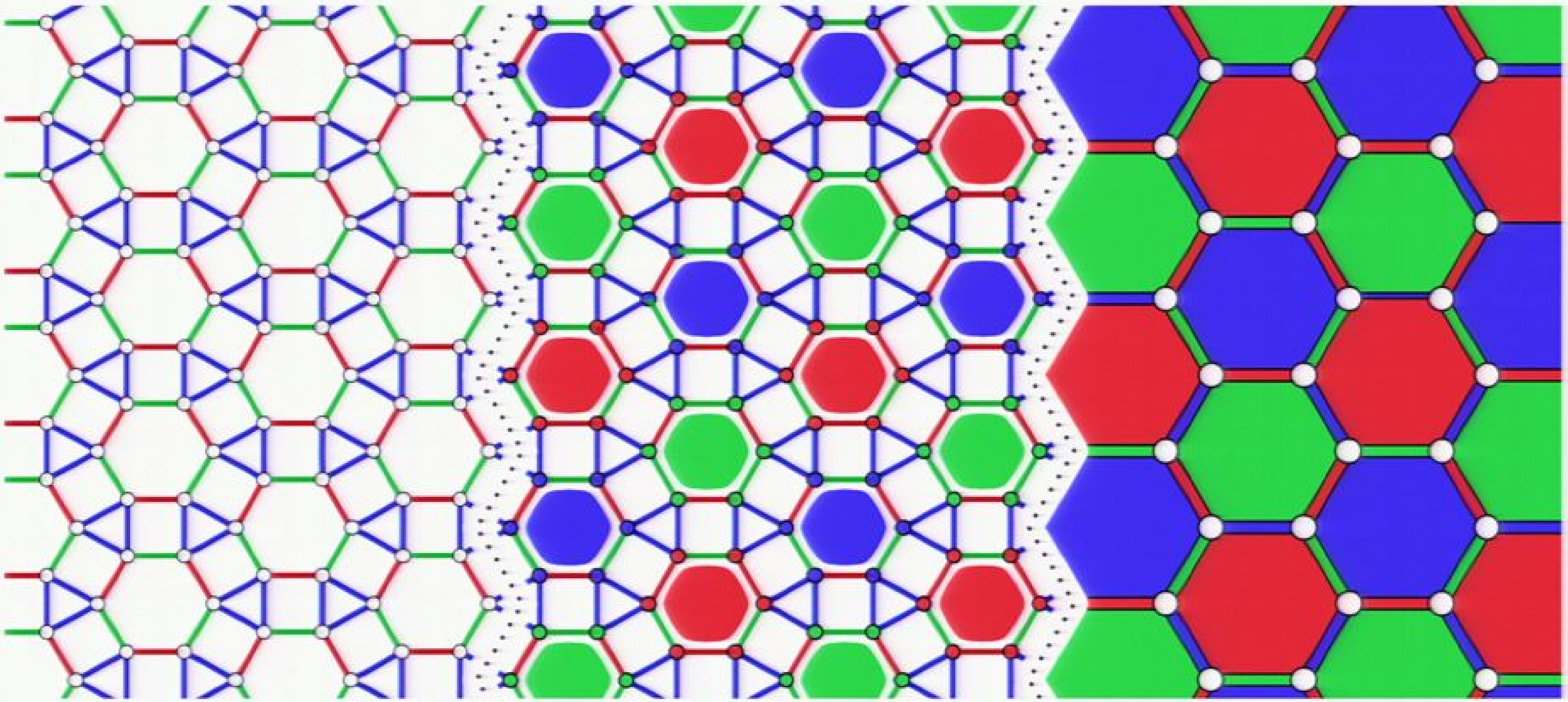
B

$C \propto AB$

- Only **two** of them are **independent**, as in plaquettes.
- To distinguish properly the three types we have to color the lattice.
- Strings are then **red, gree or blue**.

Underlying 2-colex

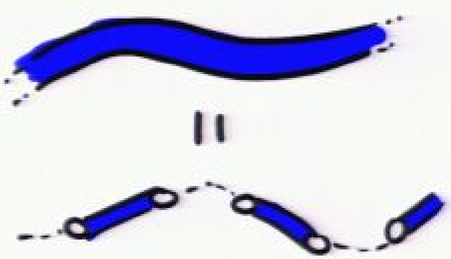
- The **hexagons and vertices** of the model are **3-colorable**:

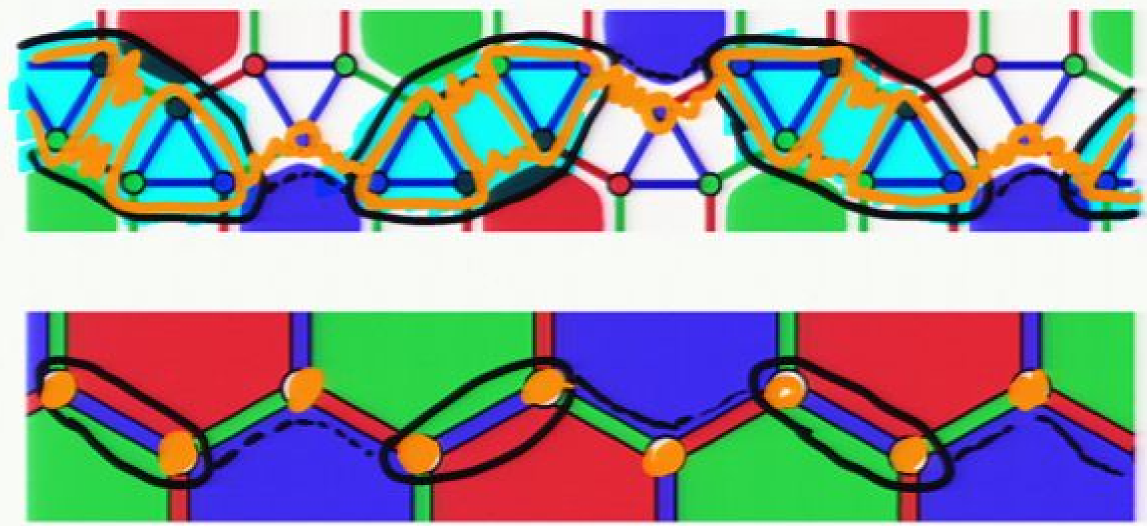


- If we regard **blue triangles** as the **sites** of a new lattice, we get a honeycomb.
- The model could be defined for **any** other **2-colex**.

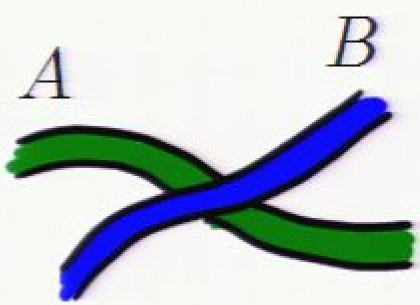
String IOM

- A **blue** string is composed of blue links:





- String IOM of different color that cross once **anticommute**:

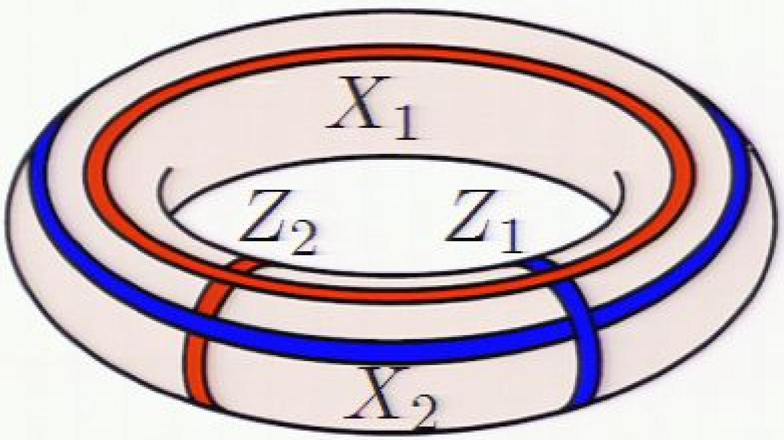


$$\{A, B\} = 0$$

- This feature is not available in honeycomb-like models.

String IOM

- In a torus, we can choose 4 independent string IOM that form the algebra of Pauli operators on 2 qubits.



$$[Z_1, Z_2] = 0 \qquad Z_1^2 = X_1^2 = 1$$
$$[X_1, X_2] = 0$$
$$[Z_1, X_2] = 0 \qquad \{Z_1, X_1\} = 0$$
$$[Z_2, X_1] = 0 \qquad \{Z_2, X_2\} = 0$$

- This implies an **exact 4−fold degeneracy.**
- More generally, in a surface of genus g we find a **4^g-fold topological degeneracy**.

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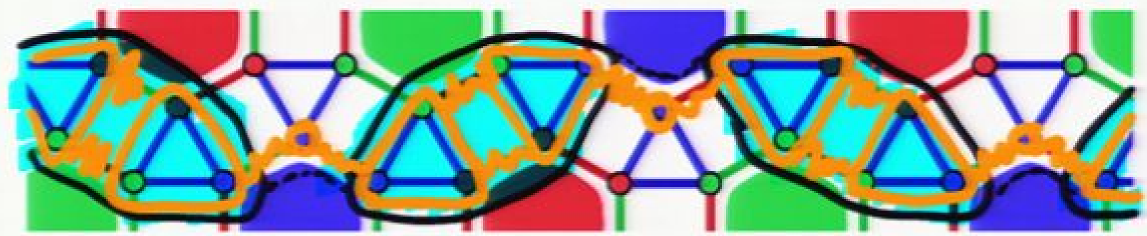
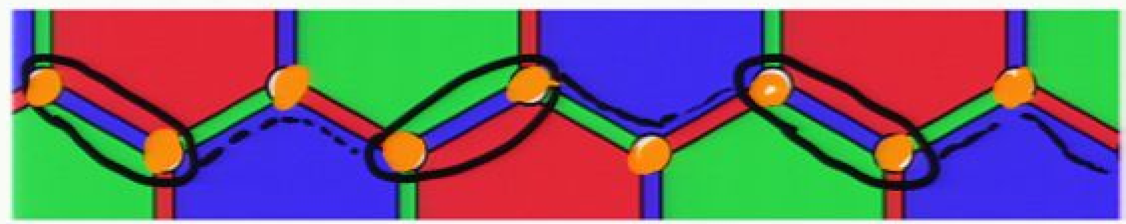
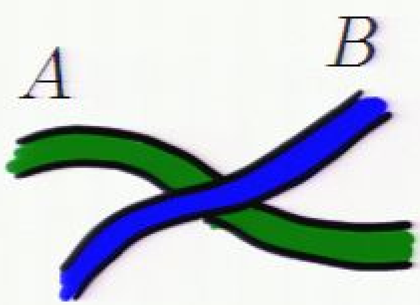
- A **blue** string is composed of blue links:





- String IOM of different color that cross once **anticommute**:

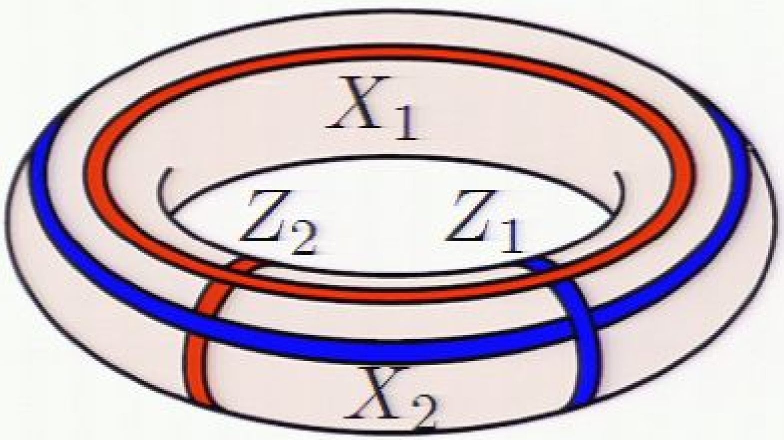


$$\{A, B\} = 0$$

- This feature is not available in honeycomb-like models.

String IOM

- In a torus, we can choose 4 independent string IOM that form the algebra of Pauli operators on 2 qubits.



$$[Z_1, Z_2] = 0 \qquad Z_1^2 = X_1^2 = 1$$
$$[X_1, X_2] = 0$$
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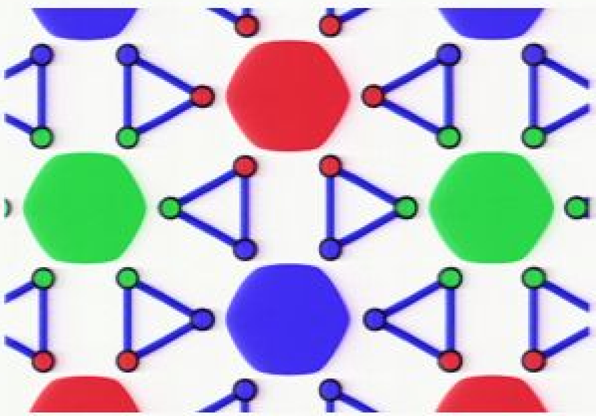
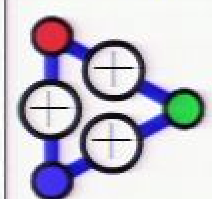
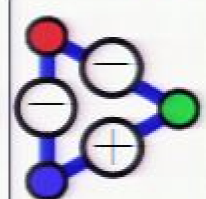
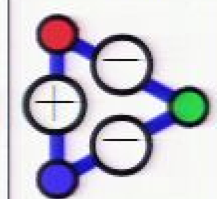
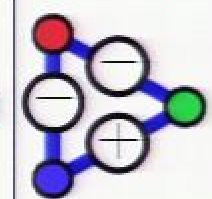
- This implies an **exact 4–fold degeneracy.**
- More generally, in a surface of genus g we find a **4^g-fold topological degeneracy**.

Bosonic mapping

Bosonic mapping

- We are specially interested in a particular regime: $J_x, J_y, J_z > 0, \quad J_x, J_y \ll J_z$
- We fix $J_z = 1/4$ and consider the extreme case $J_x = J_y = 0$



Energy	-3/4	+1/4	+1/4	+1/4
Configuration				
quasiparticle	0	$\color{red}\bullet$	$\color{green}\bullet$	$\color{blue}\bullet$

- This suggests an *exact* **mapping**: we substitute each triangle of spins by a **site**, formed by a **hardcore boson** and an **effective spin-1/2 system**.

$$|\uparrow, 0\rangle = |\uparrow\uparrow\uparrow\rangle,\ |\downarrow, 0\rangle = |\downarrow\downarrow\downarrow\rangle,$$
$$|\uparrow, {\color{red}\bullet}\rangle = |\uparrow\downarrow\downarrow\rangle,\ |\downarrow, {\color{red}\bullet}\rangle = |\downarrow\uparrow\uparrow\rangle,$$
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$$|\uparrow, {\color{blue}\bullet}\rangle = |\downarrow\downarrow\uparrow\rangle,\ |\downarrow, {\color{blue}\bullet}\rangle = |\uparrow\uparrow\downarrow\rangle.$$

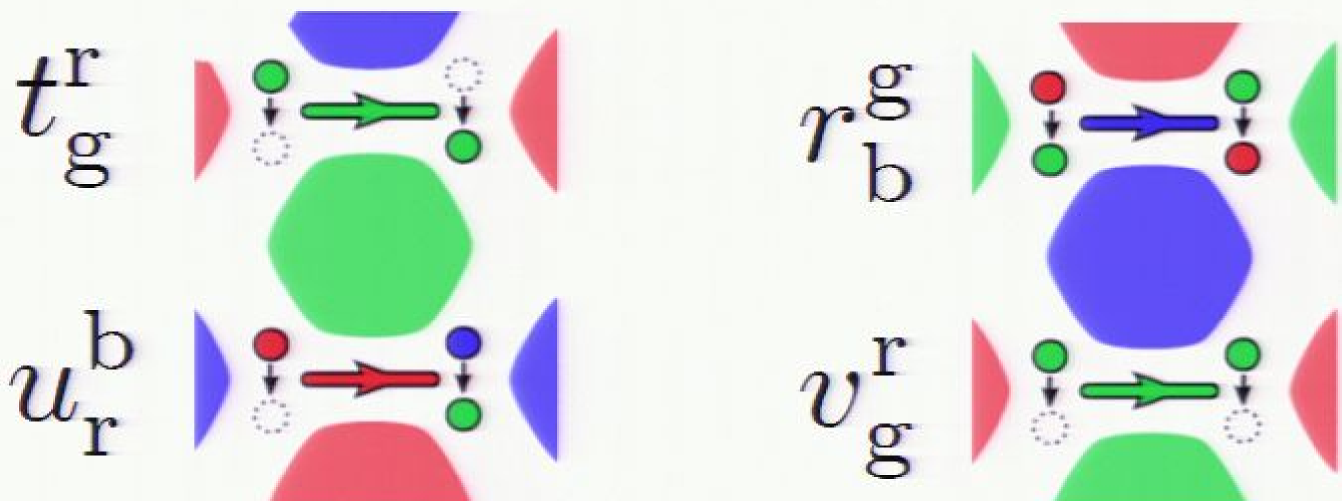
Bosonic mapping

- The resulting **Hamiltonian** takes the form:

$$H = -3N/4 + Q - \sum_{\Lambda} \sum_{c \neq c'} J_{c'|c} \left(t_c^{c'} + u_c^{c'} + \frac{r_c^{c'} + v_c^{c'} + \text{h.c.}}{2} \right),$$

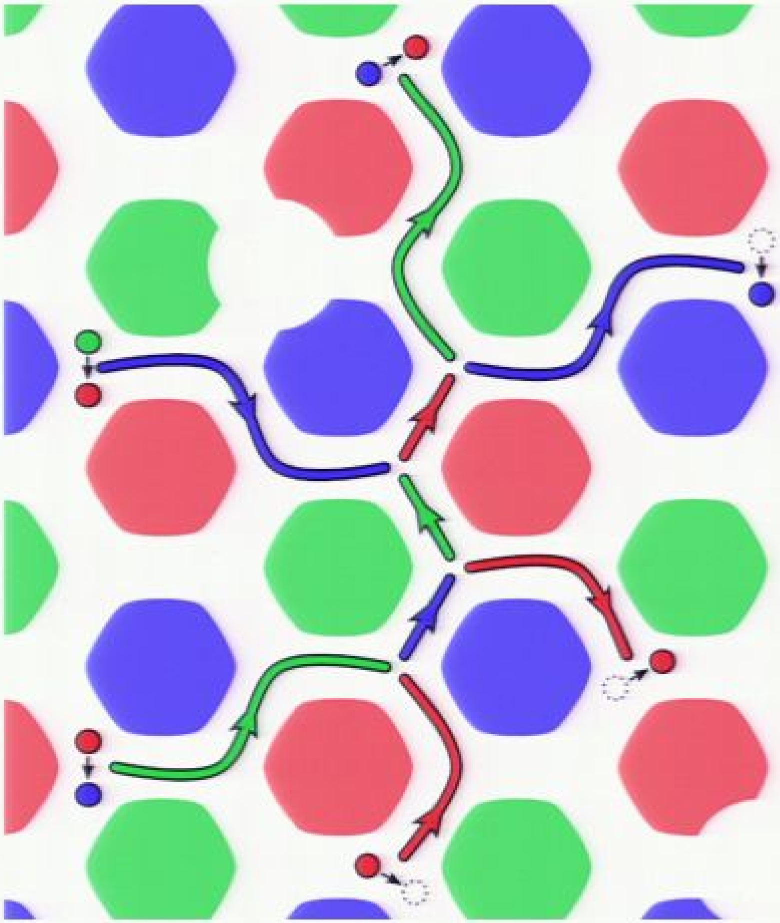
where N is the number of sites, Q counts the number of hard-core bosons, c and c' are colors and $\mathrm{r}|\mathrm{g} = \mathrm{g}|\mathrm{b} = \mathrm{b}|\mathrm{r} = y$, $\mathrm{r}|\mathrm{b} = \mathrm{g}|\mathrm{r} = \mathrm{b}|\mathrm{g} = x$.

- The t,u,r,v terms involve both effective spin and hard-core boson degrees of freedom.
- On the **hard-core boson** part, these are **hopping, fusion, color switching and pair annihilation terms**:



Interaction

- A typical **interaction process**, as viewed in the 2-colex lattice:



- The excitations interact through a 3-particle vertex.

- It is an **strong** interaction: the coupling for hopping and fusion terms is the same.

Color charge and vortices

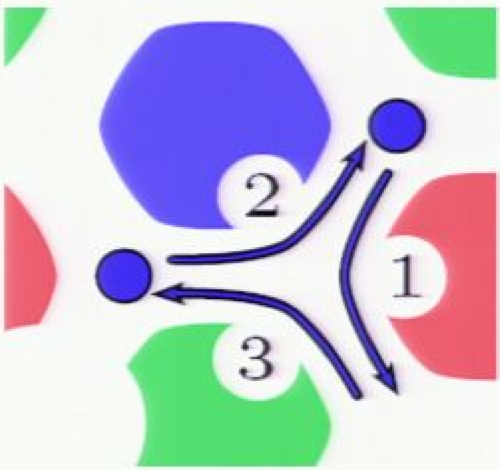
- The **color** of hardcore bosons can be regarded as a Z_2xZ_2 **charge** setting

$$\mathbf{Z}_2 \times \mathbf{Z}_2 = \{e, \mathrm{r}, \mathrm{g}, \mathrm{b}\}$$

- The Hamiltonian **preserves** the **total color charge** (e.g., fusion = Z_2xZ_2 product)
- The total color charge **constrains** the configurations of the plaquette IOM, which we will regard as **vortex degrees of freedom**.
- Once vortex degrees of freedom (and global fluxes) are **fixed**, we can forget effective spin degrees of freedom and label the states in each sector with **hardcore-boson occupancies**.
- **Which** are the quasiparticles **emerging** from hardcore-boson degrees of freedom**?**

Statistics

- Following Levin & Wen, we can recover their statistics from the **hopping terms**:

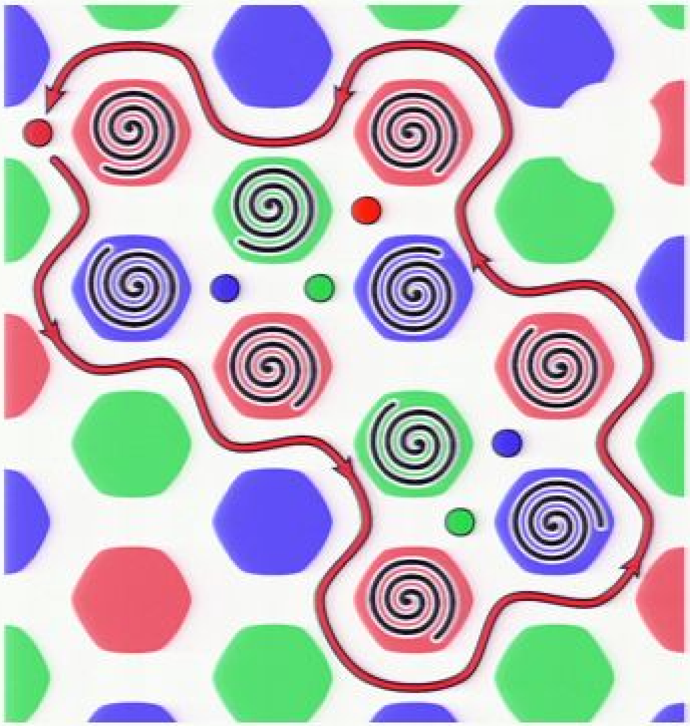


$$= -1$$

- It turns out that they are **fermions**. There are 3 familes, distinguished by color charge.
- Due to the existence of the 3-fermion vertex, they **must be anyons**.
- Also, they must be subject to a **non-trivial gauge field**.

Interacting anyonic fermions

- Following Levin & Wen again, we recover both aspects by **hopping fermions around a region**:



- The resulting phase **implies** two separate things:
  - Each family of fermions is subject to a different Z_2 **gauge field**, given by vortices.
  - Fermions with different color charge have **semionic mutual statistics**.

Effective color code

- We now focus on the regime:

$$J_z = \frac{1}{4}, \quad J_x, J_y > 0, \quad J_x, J_y \ll J_z$$

- In this regime, the anyonic fermions are **high-energy** excitations.
- Inspired by the work of J. Vidal *et al*. in the context of Kitaev's honeycomb model, we choose the **PCUTs** approach (Perturbative Continuous Unitary Transformations).
- At a given perturbative order, it produces an **effective Hamiltonian** such that

$$[H_{\rm eff}, Q] = 0$$

- We are specially interested in the **low-energy *Q*=0 sector**, where high-energy excitations are not present.
- In this sector, only **effective spin** degrees of freedom are relevant.

Effective color code

- Up to a constant, the **effective *Q*=0 Hamiltonian** at 9th perturbative order is:

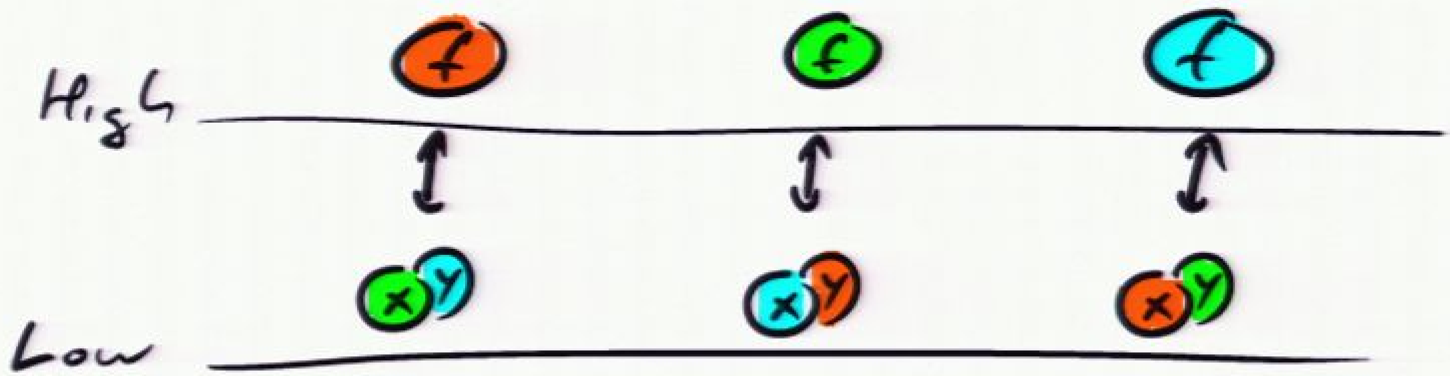
$$H_{\text{eff}} = -\sum_p \left(k_x B_p^x + k_y B_p^y + k_z B_p^x B_p^y \right)$$

$$k_z = \frac{3}{8} |J_x J_y|^3 + O(J^7), \quad \frac{k_x}{|J_y|^3} = \frac{k_y}{|J_x|^3} = \frac{55489}{13824} |J_x J_y|^3$$

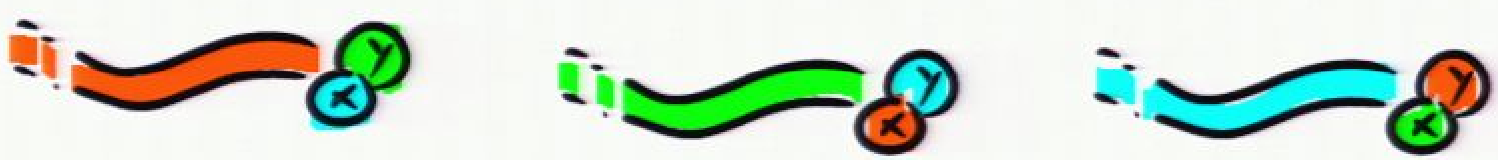
- This is a **color code** in the honeycomb lattice of effective spins!
- The **GS is the vortex free** sector.
- **Excitations are vortices**. They are **gapped** and **localized** at plaquettes.
- Higher order terms are products of vortex operators. This gives rise to short-range **vortex interactions**.

Topological charges

- We can **locally transform** high-energy excitations into low-energy ones.
- This **attaches a charge** from the low energy sector to high-energy fermions:



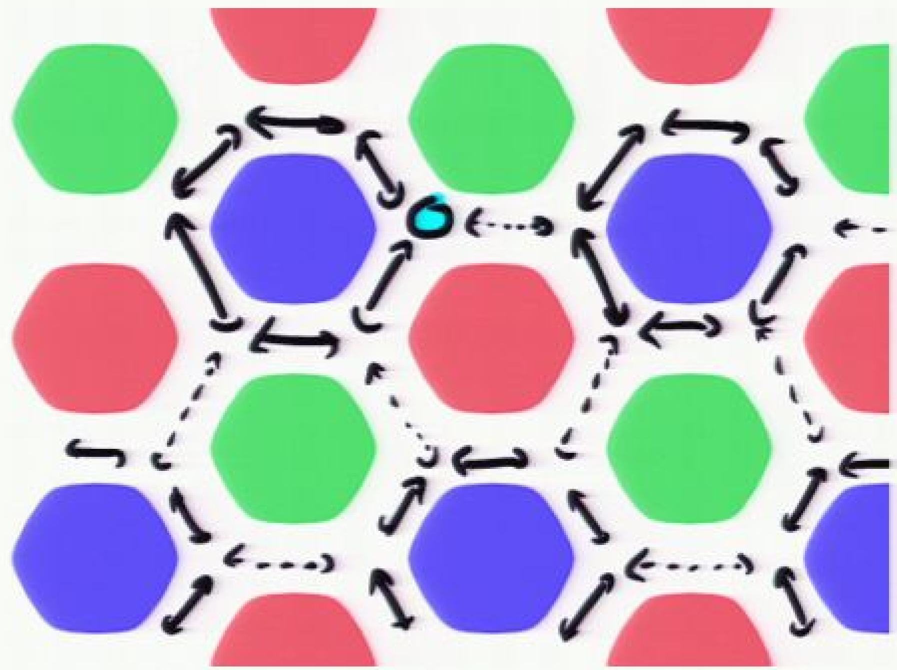
- Similarly, we may check which kind of **charges** are created at the **open ends** of the string-net IOM:



- They form the other family of fermions, so that they are **invisible** to moving high-energy fermions!

The 1-particle sector

- We can use the PCUTs method to study high-energy sectors.
- The **1-particle sector** is simple to analyze since there are no interactions:



- At **first** order, fermions only **jump around plaquettes** of their color.
- At **second** order they can **jump** from one orbit to another, in a **triangular lattice**:
- So the main contribution to the gap comes from the **gap** comes from the **orbital motion**.
- As the perturbative **couplings grow**, the **gap closes** till a phase transition occurs.
- This may be some form of **anyon condesation**.

Future work and conclusions

Future work

- We have only studied a **particular phase** of the system.
- The fact that all phases show a topological degeneracy anticipates a **rich phase diagram**.
- In this regard, one may **explicitly brake** the **color symmetry** that the model exhibits and still keep the features that we have discussed.
- It would be particularly interesting to check whether any of the phases displays **non-abelian anyons**.
- It seems that the model is **not solvable**, so that numerical calculations will be needed.

Conclusions

- We have introduced a **two-body spin-1/2 model** in a ruby lattice.
- The model exhibits an **exact topological degeneracy** in all coupling regimes.
- Using a bosonic mapping, we have discussed the **emergence of strongly interacting anyonic fermions**.
- A particular coupling regime gives rise to an **effective color code** model.

Bosonic mapping

- We are specially interested in a particular regime: $J_x, J_y, J_z > 0, \quad J_x, J_y \ll J_z$
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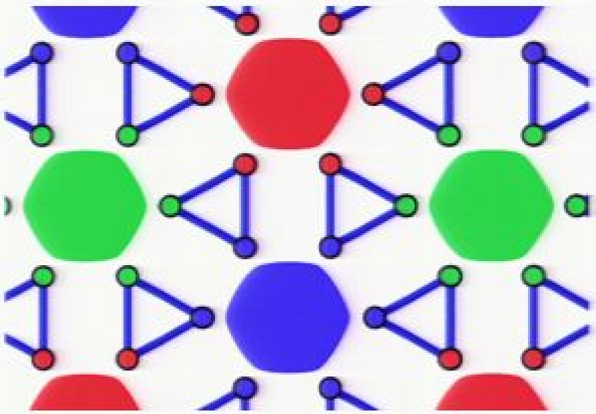

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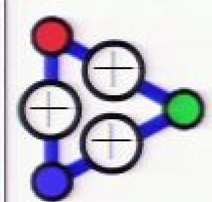
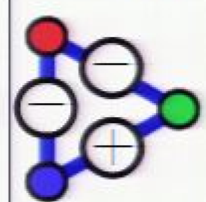
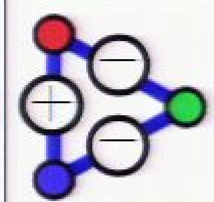
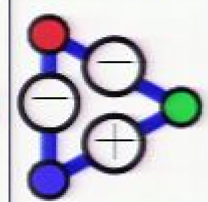
Configuration				
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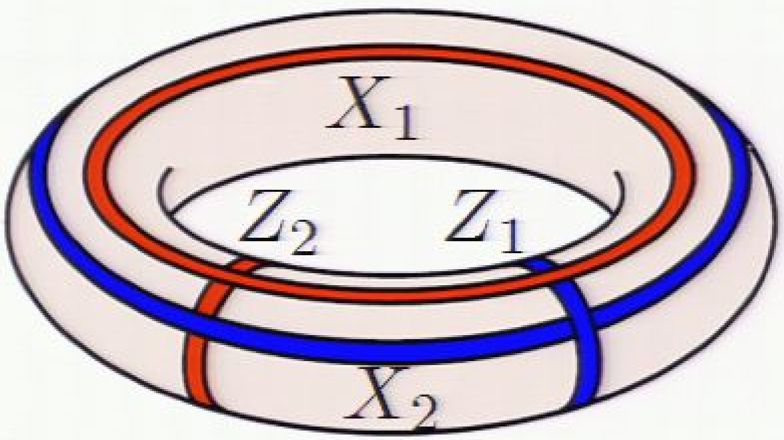

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