Title: On quantum vs. classical probability

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Abstract: Both classical probability theory and quantum theory lend themselves to a Bayesian interpretation where probabilities represent degrees of belief, and where the various rules for combining and updating probabilities are but algorithms for plausible reasoning in the face of uncertainty. I elucidate the differences and commonalities of these two theories, and argue that they are in fact the only two algorithms to satisfy certain basic consistency requirements. In order to arrive at this result I develop an over-arching framework for plausible reasoning that incorporates both classical probability and quantum theory as special cases.

On quantum vs. classical probability

Jochen Rau
University of Frankfurt

Quantum Foundations Seminar
Perimeter Institute, 13 January 2009

Some fundamental laws of physics are inherently probabilistic

Probability in physics

Macroscopic domain

- Maximum entropy thermodynamics
- Second law

Some fundamental laws of physics are inherently probabilistic

Probability in physics

Macroscopic domain

- Maximum entropy thermodynamics
- Second law

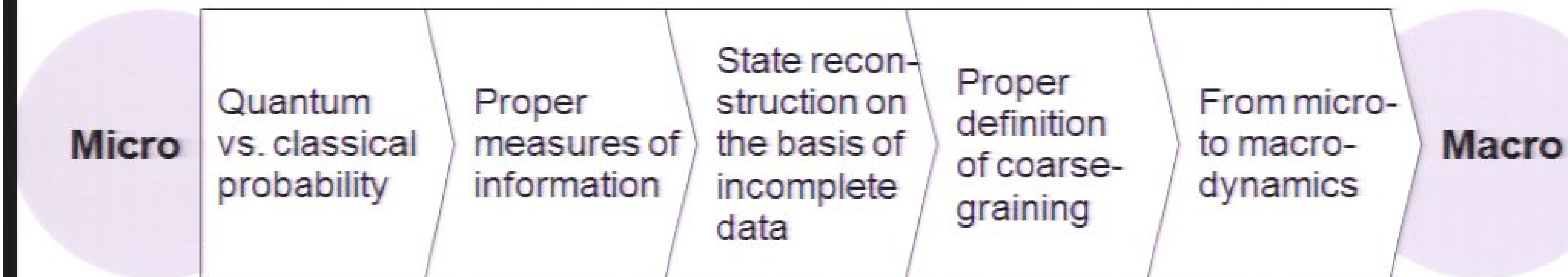
Microscopic domain

- Quantum theory

- Often difficult to grasp conceptually
- Discussions up to this day

Today's topic is the first part of a larger program to elucidate the role of probability in mathematical physics

Program overview



In the modern Bayesian view probability theory constitutes an extension of logic

Probability as extended logic

„Probability"

- embodies some agent's state of knowledge
- degree of belief rather than limit of relative frequency
- can be legitimately assigned not just to ensembles but also to individual systems

Cox 1946, Jaynes 2

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Consistency

Different ways of using the same information must lead to the same conclusions, irrespective of the particular path chosen

- Sum rule
- Bayes rule

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Framework for plausible reasoning in the absence of full information

Cox 1946, Jaynes 2

Laws of physics = laws of thought?

Physics as extended logic

"Physics is to be regarded not so much as the study of something a priori given, but rather as the development of methods for ordering and surveying human experience."

— Niels Bohr

For example, the second law reflects a basic constraint on any form of reasoning about the macroscopic world

Second law

Macroscopic process is reproducible

↓

A prediction never contains more information than the data on which it is based

↓

Second law

$$S_2 \geq S_1$$

Jaynes 1

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Second law

$$S_2 \geq S_1$$

Prerequisite for being able to subject a macroscopic process to scientific inquiry

Jaynes 1

Like classical probability theory, quantum theory deals with hypotheses and their probabilities

Quantum probability

Mathematical object	Interpretation
Subspace of Hilbert space or projector thereon	Hypothesis
Embedding into a larger subspace	Logical implication
Orthogonality	Logical contradiction
Density matrix, statistical operator	Probability distribution, knowledge

tr (ρP_x)	prob $(x\|\rho)$: probability that hypothesis x (represented by projector P_x) is true given ρ

Quantum mechanics as extended logic?

Fundamental issue

Traditional language:

Classical probability
theory constitutes
a framework for
plausible reasoning

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Classical probability theory constitutes a framework for plausible reasoning → Quantum mechanics is a peculiar variant of classical probability theory

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Modern language:

quantum theory → ? ← **information processing**

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Early attempt: „quantum logic“

Quantum logic

Idea

- Propositions form a lattice that is
  - complete
  - orthocomplemented
  - weakly modular
  - atomic
- Boolean operation ∩ („and“) is defined, albeit in a non-classical way

Birkhoff & v. Neumann 1934, Geneva School (Jauch, Piron et al) 19

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Only partially successful

- skew field unspecified – might also be **R** or **H**
- only for Hilbert space dimension ≥ 3

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More recent attempt: „Five reasonable axioms“

Hardy‘s approach

Quantum theory follows uniquely from five „reasonable axioms“:

1. **Probabilities:** are well defined as limits of relative frequencies
2. **Simplicity:** minimise the number of degrees of freedom
3. **Subspaces:** constrained big system = small system
4. **Composite systems:** dimension and number of degrees of freedom are multiplicative
5. **Continuity:** There exists a continuous reversible transformation between any two pure states

Hardy 2

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not in keeping with Bayesian approach

why?

why a special status for pure states?

Hardy 2

The past few years have seen the emergence of a Bayesian view on quantum theory

Quantum Bayesianism

„State"

- embodies some agent's knowledge about, rather than an objective property of, a physical system
- yields probabilities that reflect degrees of belief rather than limits of relative frequencies
- can be legitimately assigned to individual systems

Schack, Brun and Caves 2001, Caves, Fuchs and Schack 2

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Quantum Bayesianism

„State“	▪ embodies some agent‘s knowledge about, rather than an objective property of, a physical system ▪ yields probabilities that reflect degrees of belief rather than limits of relative frequencies ▪ can be legitimately assigned to individual systems
Quantum Bayes rule	▪ quantum analog of the classical Bayes rule ▪ ensures consistency of probabilistic reasoning ▪ allows agents to progress -via measurements on exchangeable sequences from a diverse array of subjective priors to a consensus posterior distributic (Such a consensus is implicit when one speaks of the state of a system as being the result of a well-defined, “objective” preparation procedure.)

Schack, Brun and Caves 2001, Caves, Fuchs and Schack 2

Today I will address three questions

Questions

- What are the essential differences between classical and quantum probability?
- What do they have in common?
- Is it conceivable that beyond these two theories there are still further frameworks for plausible reasoning?

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- What are the essential differences between classical and quantum probability?
- What do they have in common?
- Is it conceivable that beyond these two theories there are still further frameworks for plausible reasoning?

➡ **Conjecture: No**, not if they have to satisfy a minimal set of consistency requirements

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I shall assume that in both cases resources are finite

Model size

Size of probabilistic model

d:=
- classical: cardinality of hypothesis space
- quantum: Hilbert space dimension

Storage capacity

Maximum amount of information that can be extracted by way of measurement, or stored by way of preparation:

log d (both classical and quantum cases)

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Storage capacity

Maximum amount of information that can be extracted by way of measurement, or stored by way of preparation:

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- Resource available for information processing
- Assumption: **finite**

Quantum probability differs from classical probability in four important respects

Key differences

Classical	Quantum

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The irreducible probabilism of quantum mechanics is reflected in uncertainty relations

Irreducible probabilism

Accuracies satisfy uncertainty relations

⇕

Observables do not commute

⇕

Hypotheses are not jointly decidable

⇕

Boolean operation ∩ („and“) is not defined

} There are always hypotheses whose probabilities are neither 0 nor 1

The holism of quantum mechanics has its origin in the possibility of entanglement

Holism

A **Pure states:**

Composite system
is in a pure state

⇅ classical: ✓

States of individual
constituents are pure

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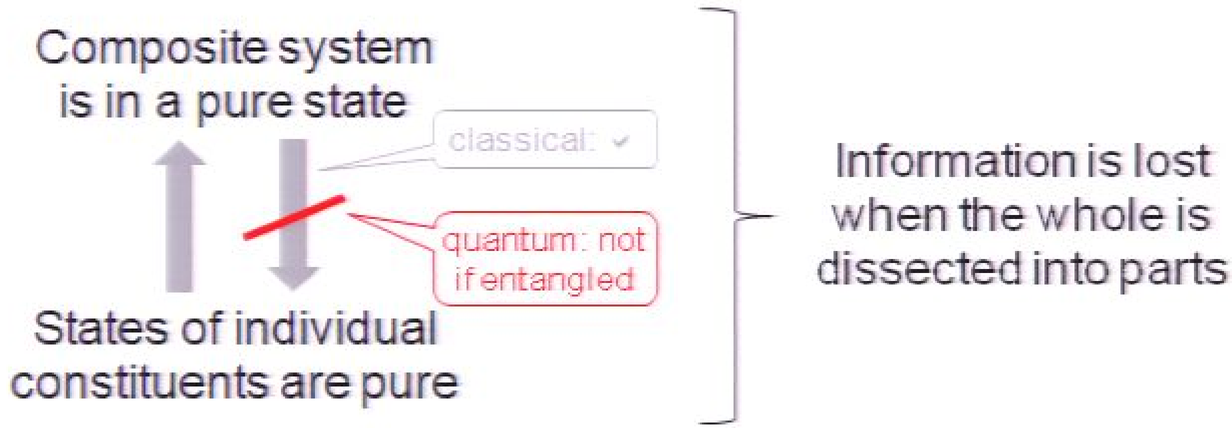

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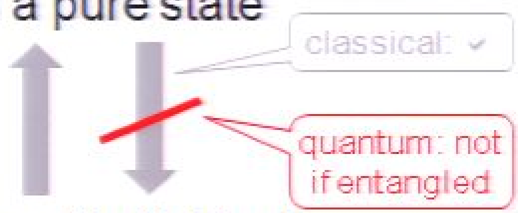

States of individual
constituents are pure

Information is lost when the whole is dissected into parts

B Mixed states:

There exist states that cannot be represented as mixtures of product states,

$$\rho_{AB} \neq \sum p_i \, \rho_A{}^i \times \rho_B{}^i$$

The innate observer-dependency of quantum mechanics manifests itself in multiple ways

Observer-dependency

1. **Measurement postulate:**
   - Measurement affects the state
   - The unknown prior state of an individual system cannot be learned by measurement

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1 Measurement postulate:

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2 Kochen-Specker and Bell's theorems:

It is impossible to assign to hypotheses truth values that are *preexisting* (i.e., merely revealed rather than influenced by the act of measurement) and at the same time...

a. ...noncontextual, i.e., independent of whichever group of mutually commuting observables one might choose to measure with it (Kochen-Specker theorem)

b. ...unaffected by any actions at a causally disconnected distance (Bell's theorem)

Quantum theory is „smoother“ than classical probability theory

Smoothness (1/2)

Given finit
resource:

	Classical
Hypothesis space, set of pure states	discrete set
Reversible Operations	symmetric group S_d (permutations)
Change of probability distribution under reversible operation	discontinuous

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Hypothesis space, set of pure states	discrete set	⇨	continuous manifold
Reversible Operations	symmetric group S_d (permutations)	⇨	Lie group U(d)
Change of probability distribution under reversible operation	discontinuous	⇨	continuous

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Change of probability distribution under reversible operation	discontinuous	⇨	continuous

Not to be confused with the „discontinuity“ of state change upon measurement:

- Reflects process of learning
- Occurs in classical probability, too (Bayes rule)

States change in a continuous fashion under reversible operations

Smoothness (2/2)

Probabilities that are initially greater than zero will not suddenly jump to zero upon an infinitesimal transformation:

$$\forall\, \epsilon(\rho) > 0\ \exists\, \delta > 0 :\ g(\rho)\,(x) > 0 \quad \forall \quad x \subseteq \mathrm{supp}(\rho)\,,\ x \neq \emptyset\,,\ g \in \mathcal{G}_\delta(d)\,.$$

smallest non-vanishing eigenvalue of ρ

$$\epsilon(\rho) := \min\{\rho(x)\mid x \subseteq \mathrm{supp}(\rho)\,,\ x \neq \emptyset\} > 0$$

$$x \perp \mathrm{supp}(\rho) :\Leftrightarrow\ \rho(x) = 0$$

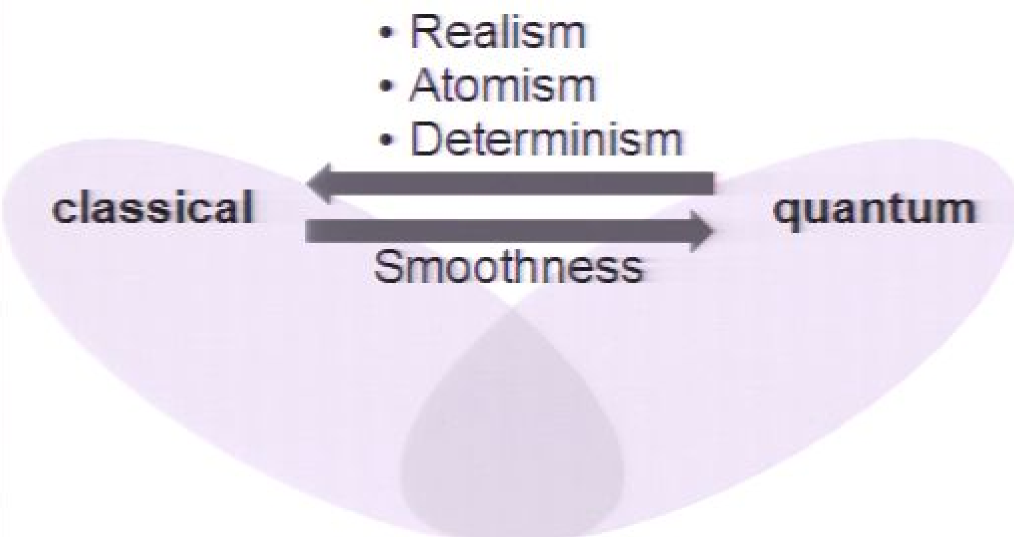
neighborhood of identity on Lie group G(d)≡U(d

$$\mathcal{G}_\delta(d) := \{g \in \mathcal{G}(d)\mid \mathrm{dist}(g, 1_{\mathcal{G}}) <$$

Real-, atom- and determinism are traded for the ability to reason about continua with only finite resources

Trade-off

Given finite resources



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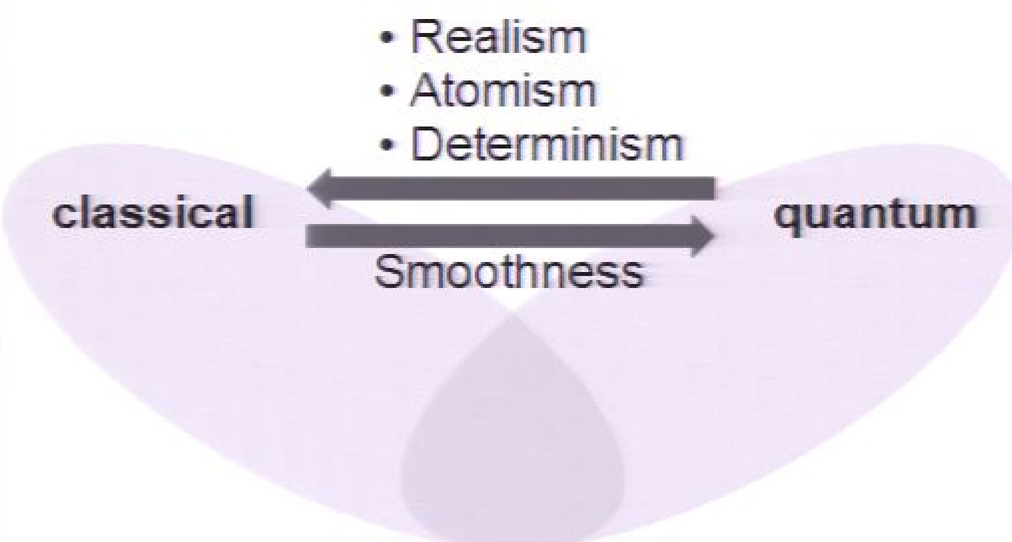

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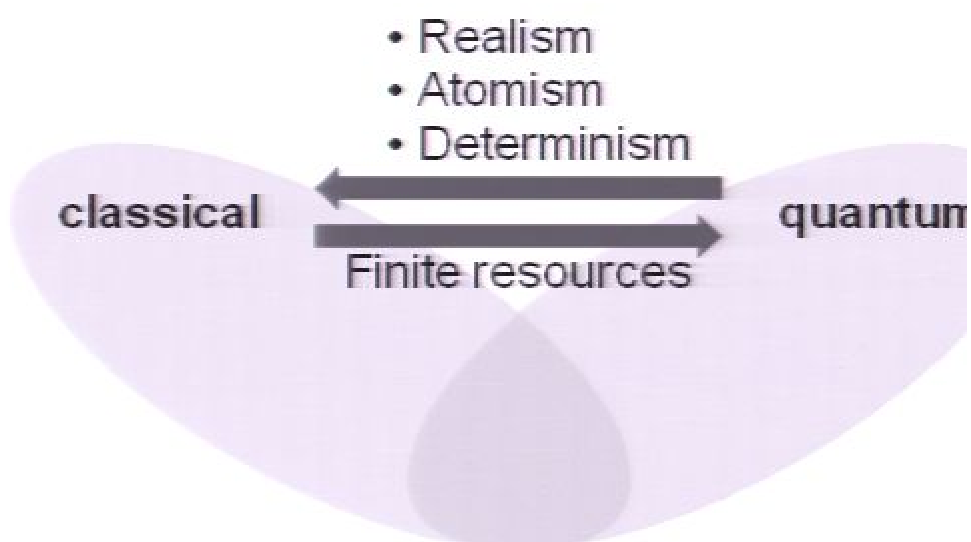

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Smoothness (1/2)

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Hypothesis space, set of pure states	discrete set
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Change of probability distribution under reversible operation	discontinuous

Why talk about commonalities?

Motivation

- Ever since the Einstein-Bohr debate the fundamental differences between quantum and classical probability have been scrutinised extensively
- Yet equally interesting, and less known, is the fact that both theories share some important commonalities
- These commonalities hint at the structure of a more general, over-arching framework for plausible reasoning that incorporates both classical and quantum probability as special cases

Classical and quantum probability are special cases of a more general framework for plausible reasoning

Generic notation (1/2)

		Mathematical manifestation	
Symbol	Meaning	Classical	Quantum

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$\{x_i\}_{i\in I}^{\perp} < \{y_k\}_{k\in K}^{\perp}$	Fine-graining	Cut into smaller subsets $y_k = \cup_{i\in I_k} x_i$, $I = \cup_{k\in K} I_k$	Orthogonal decomposition $P_{y_k} = \Sigma_{i\in I_k} P_{x_i}$, $I = \cup_{k\in K} I_k$

Classical and quantum probability are special cases of a more general framework for plausible reasoning

Generic notation (2/2)

Symbol	Meaning	Mathematical manifestation Classical	 Quantum

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p(x)≡prob(x|ρ)	Probability	$\Sigma_{e \subseteq x}\rho(e)$	$tr(\rho P_x)$

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p(x)≡prob(x|ρ)	Probability	$\Sigma_{e\subseteq x}\rho(e)$	$tr(\rho P_x)$
θ_b	Test	Map $\rho\rightarrow\rho\cdot\theta_b$ where $\theta_b(e):=1$ ($e\subseteq b$) or 0 (otherwise)	Map $\rho\rightarrow P_b\rho P_b$

Classical and quantum probability are special cases of a more general framework for plausible reasoning

Generic notation (2/2)

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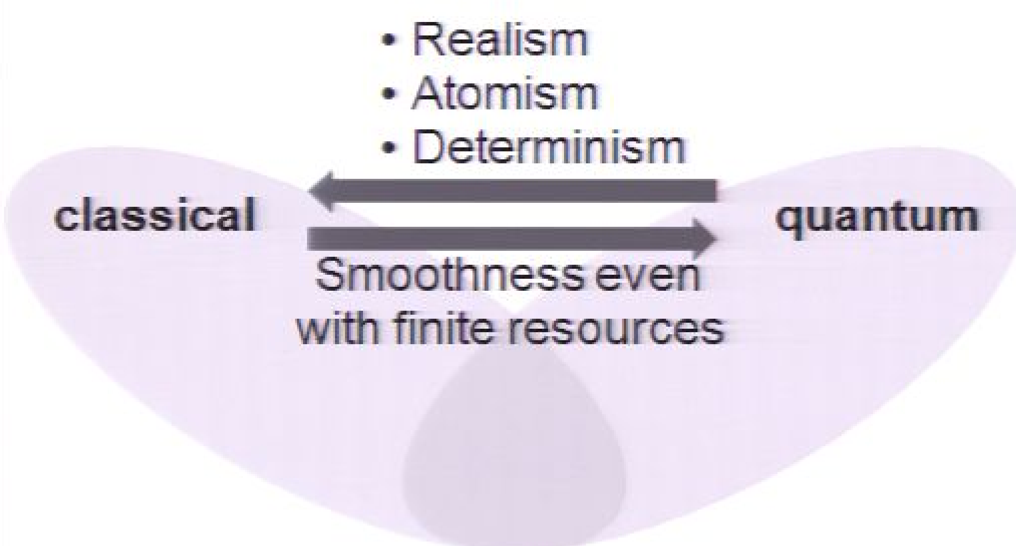
Not defined in general framework: ∩, ∪

Classical and quantum probability have considerable overlap

Principal commonalities



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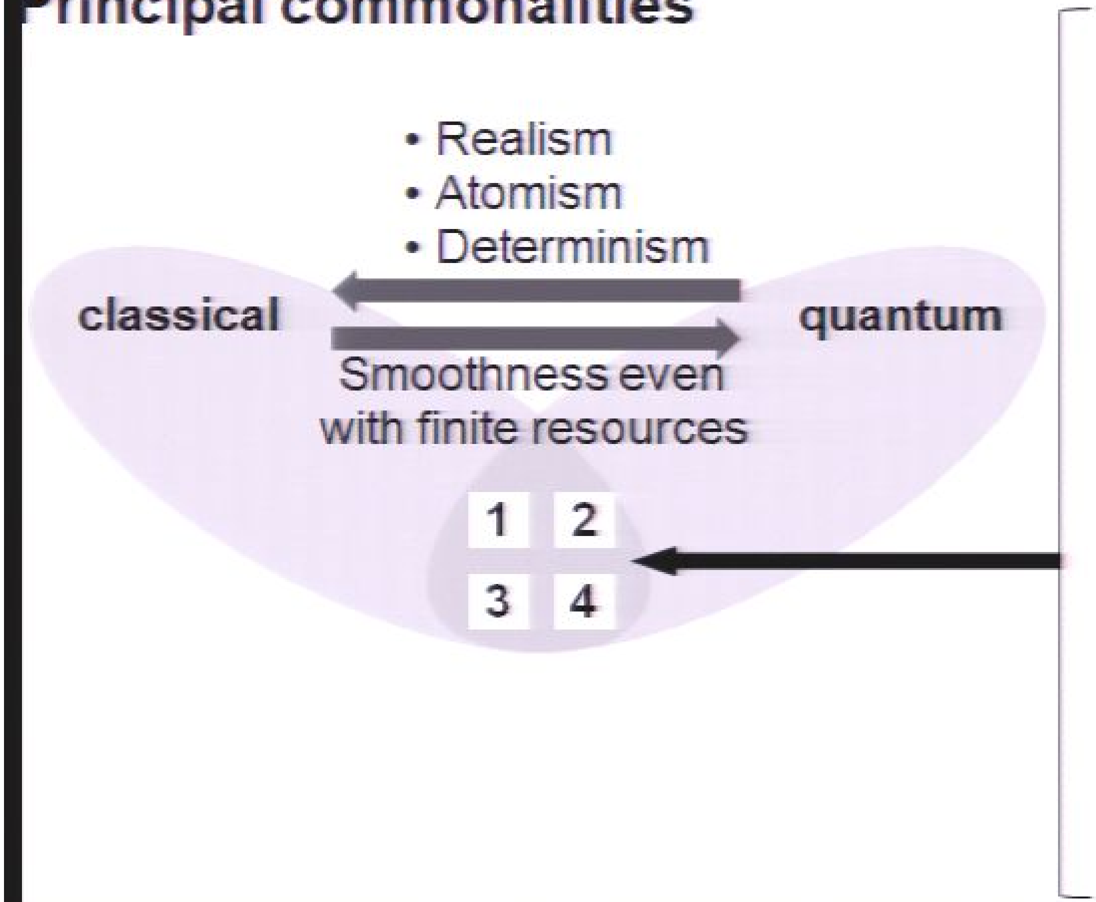

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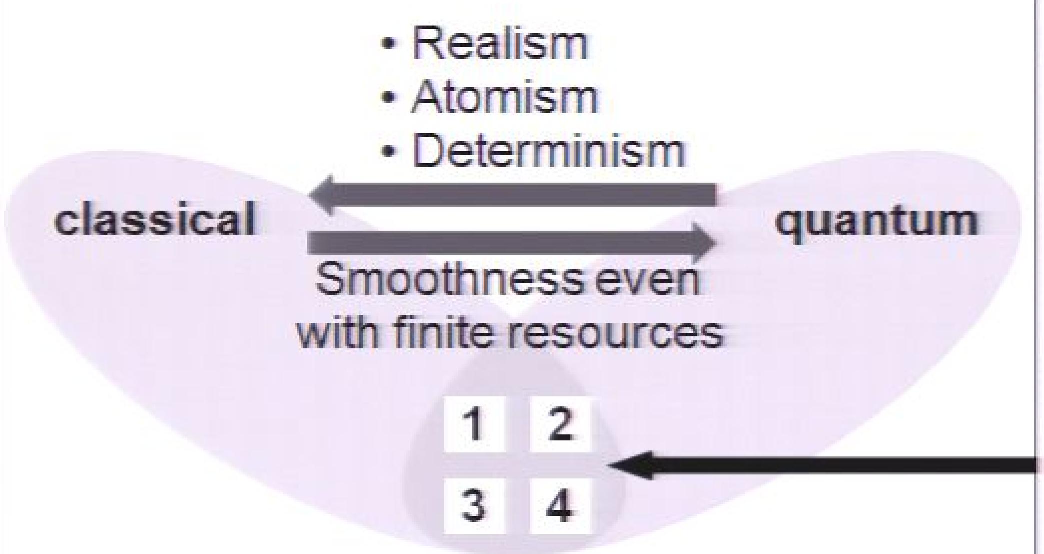

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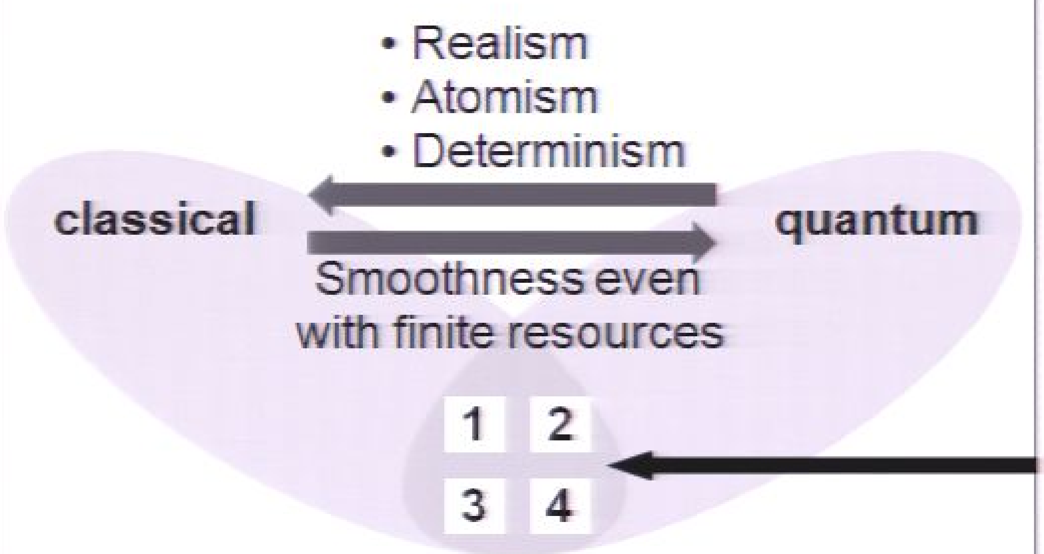

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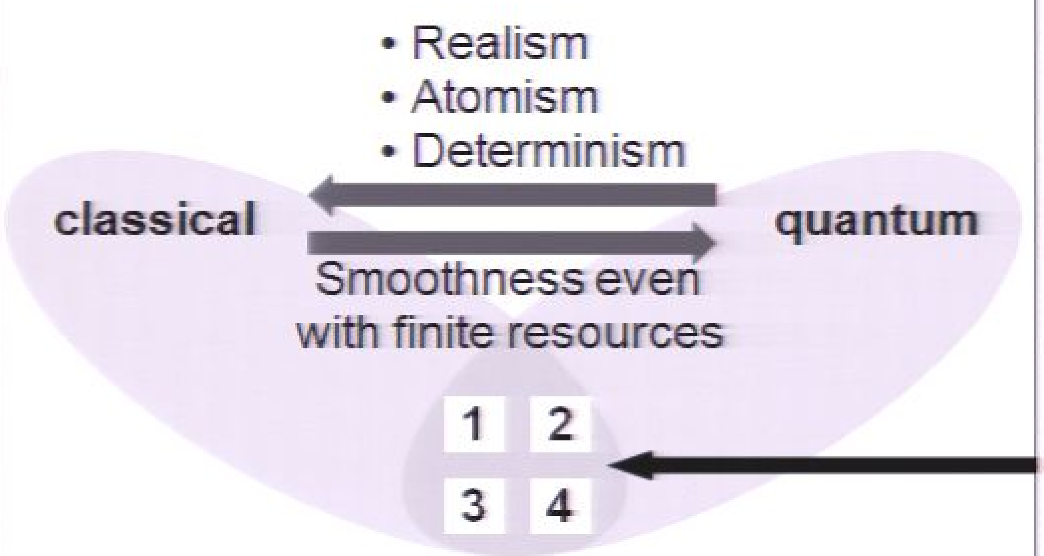

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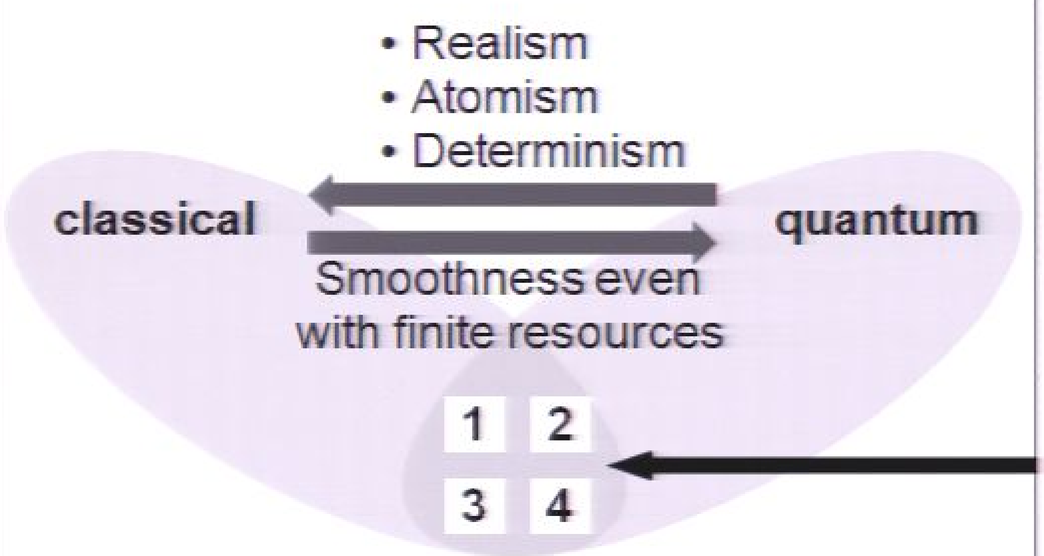

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4 **Learning:** It is possible to learn the single-constituent state by performing measurements on an exchangeable sequence

The minimal structure satisfies basic consistency requirements (1/5)

Logical relations

Logical implication ($\subseteq$) and fine-graining (<) constitute partial orders:

i. Reflexive

ii. Antisymmetric

iii. Transitive

The minimal structure satisfies basic consistency requirements (2/5)

Granularity

- $d(x)=0 \Leftrightarrow x=\emptyset$
- $x \subseteq y \Rightarrow d(x) \leq d(y)$
- Sum of granularities is invariant under fine-graining,
  $\{x_i\}^{\perp} < \{y_k\}^{\perp} \Rightarrow \Sigma_i d(x_i)=\Sigma_k d(y_k)$

The minimal structure satisfies basic consistency requirements (3/5)

Probability

- $\rho(\emptyset)=0$
- $x \subseteq y \Rightarrow \rho(x) \leq \rho(y)$
- Sum rule:
  $\{x_i\}^{\perp} < \{y_k\}^{\perp} \Rightarrow \Sigma_i \rho(x_i) = \Sigma_k \rho(y_k)$

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The minimal structure satisfies basic consistency requirements (4/5)

Test

Operational meaning	
Properties	

The minimal structure satisfies basic consistency requirements (4/5)

Test

Operational meaning

- Experiment: Test for b, $b \subseteq a$.
  - If b is found true: no further action
  - If b is found false: apparatus subsequently sets also a to „false“

  Example: Hypotheses a: „Photon exists“, b: „Photon has positive helicity“. Polarization filter lets photon pass only if helicity is positive.
- Prior knowledge pertaining to $x \subseteq a$: ρ
- This knowledge changes in two steps:
  (1) upon learning that test was performed, with outcome still unknown: $\rho \rightarrow \theta_b \rho$
  (2) upon learning the outcome: $\theta_b \rho \rightarrow \theta_b \rho / \mathrm{prob}(b|\rho)$ if b true, else $\theta_b \rho \rightarrow 0$.

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Properties

- $x \subseteq b \Rightarrow \theta_b \rho(x) = \rho(x)$
- $\mathrm{supp}(\theta_b \rho) \subseteq b$, with equality if and only if $\rho(e) > 0$ for all $e \subseteq b$
- Tests may narrow, but never broaden a distribution: $d(\mathrm{supp}(\theta_b \rho)) \leq d(\mathrm{supp}(\rho))$
  [**not:** $\mathrm{supp}(\theta_b \rho) \subseteq \mathrm{supp}(\rho)$]

The minimal structure satisfies basic consistency requirements (5/5)

Reversible operation

- Reversible operations constitute a group
- They act on states („Schrödinger picture") or hypotheses („Heisenberg picture"), respectively; pictures are related by $\mathrm{prob}(x|g(\rho))=\mathrm{prob}(g^{-1}(x)|\rho)$
- In Heisenberg picture reversible operations preserve
  - logical relations $\subseteq$, $\perp$, $<$
  - granularity
- $\mathrm{supp}(g(\rho))=g(\mathrm{supp}(\rho))$
- $g\circ\theta_b=\theta_{g(b)}\circ g$

Fundamental equivalence classes have as their sole parameter the model size

Single parameter

- Define
  - $\mathbf{L_a}:=\{x|x\subseteq a\}$
  - $\mathbf{M_a}(\{k_i\}):=\{\{x_i\}<a \mid d(x_i)=k_i\}$, $d(a)=\Sigma_i k_i$
  - $\mathbf{S_a}:=\{\rho|_a:L_a\to[0,1] \mid$ there exists a state ρ: $\rho|_a(x)=\rho(x)$ for all $x\subseteq a\}$ „constrained states", not necessarily normalised to $\rho|_a(a)=1$
  - $\mathbf{G_a}:=\{$reversible operations $g \mid g(a)=a, g(x)=x$ for all $x\perp a\}$ constitutes a group; acting on arbitrary hypotheses, not just on L_a
- Corresponding structures for $b\neq a$ are isomorphic to the above iff $d(b)=d(a)\equiv d$
- ⇨ Define equivalence classes $L(d)$, $S(d)$, $G(d)$, $M(\{k_i\})$ with $\Sigma_i k_i=d$

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Equivalence classes depend on granularity only, not on any specifics of the system under consideration

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Fine-grainings with identical granularities are connected by reversible operations

Connectedness

G(d) acts transitively on M({k_i}), hence isomorphism

$$\mathcal{M}(\{k_i\}) \sim \mathcal{G}(d) / \bigotimes_i \mathcal{G}(k_i) \ , \ d = \sum_i k_i$$

This implies

- Classical: cardinality

$$\#\mathcal{M}_{\rm cl}(\{k_i\}) = \frac{\#\mathcal{G}_{\rm cl}(d)}{\prod_i \#\mathcal{G}_{\rm cl}(k_i)} = \frac{d!}{\prod_i k_i!}$$

- Quantum: manifold dimension, with G(d)=U(d)

$$\dim \mathcal{M}_{\rm qu}(\{k_i\}) = \left(\sum_i k_i\right)^2 - \sum_i k_i^2$$

Parts can be freely combined into a whole (1/2)

Composite hypothesis space

- The whole encompasses the parts; it can never be less (but might be more) than the sum of its parts
- In particular, most accurate hypotheses pertaining to different constituents can be freely combined into most accurate hypotheses about the composite system:

$$M(\{1,d_A d_B-1\}) \supseteq M(\{1,d_A-1\}) \times M(\{1,d_B-1\})$$

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Classical	Quantum
$\#M(\{1,d_A d_B-1\}) = \#M(\{1,d_A-1\}) \cdot \#M(\{1,d_B-1\})$	$m(\{1,d_A d_B-1\}) > m(\{1,d_A-1\}) + m(\{1,d_B-1\})$
with $\#M(\{1,d-1\})=d$	with $m \equiv \dim M(\{1,d-1\})=2(d-1)$
	Not „=“, reflecting possibility of entanglemer

Parts can be freely combined into a whole (2/2)

Composite reversible operations

- Arbitrary concerted action of reversible operations on different constituents renders an allowed reversible operation on the composite system
- The Cartesian product of independent subsets of $G(d_A)$ and $G(d_B)$ must be isomorphic to an independent subset of $G(d_A d_B)$

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$\mu'(S_{d_A d_B}) \geq \mu'(S_{d_A}) \cdot \mu'(S_{d_B})$

with $G(d)=S_d$ symmetric group,
μ': size of largest independent subset,
$\mu'(S_d)=d-1$

Whiston 2000, Cameron and Cara 2

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Classical	Quantum
$\mu'(S_{d_A d_B}) \geq \mu'(S_{d_A}) \cdot \mu'(S_{d_B})$	$\dim U(d_A d_B) \geq \dim U(d_A) \cdot \dim U(d_B)$
with $G(d)=S_d$ symmetric group, μ': size of largest independent subset, $\mu'(S_d)=d-1$	[in fact: „="] with $G(d)=U(d)$ unitary group, $\dim U(d)=d^2$

Whiston 2000, Cameron and Cara 2

Further commonalities can be found by considering exchangeable sequences rather than individual systems

Exchangeable sequences

Idea

- Classical: Probability distribution of, say, a single die can be learnt by throwing the same die many times
- Quantum: Observer-dependency precludes determining the state of an individual system via repeated measurements on the same system
- Idea: Circumvent the latter limitation by performing instead measure-ments on many different members of an exchangeable sequence

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Exchangeable sequence

- Informally: finite subsequence of an infinite sequence of systems whose order is irrelevant
- Mathematically: symmetric and exchangeable,

$$\rho^{(N)}(x_{i_1}^{\pi(1)}, x_{i_2}^{\pi(2)}, \ldots, x_{i_N}^{\pi(N)}) = \rho^{(N)}(x_{i_1}^1, x_{i_2}^2, \ldots, x_{i_N}^N)$$

$$\rho^{(N)}(\cdot, \ldots, \cdot) = \frac{\rho^{(M)}(\overbrace{\cdot, \ldots, \cdot}^{N \text{ slots}}, \overbrace{I_d, \ldots, I_d}^{M-N \text{ slots}})}{\rho^{(M-N)}(I_d, \ldots, I_d)} \quad \forall M > N$$

- All constituents have the same reduced single-constituent state $\rho^{(1)}$
- Can learn $\rho^{(1)}$

When applied to exchangeable sequences, both classical and quantum probability permit learning

Learning

Rules for exchangeable sequences are always classical

- **Product rule:** $\mathrm{prob}(f^M, g^N) = \mathrm{prob}(f^M | g^N)\, \mathrm{prob}(g^N)$
  - f^M: measuring on M constituents the values {f} for observables {F}
  - g^N: same for g, but on N *different* constituents
  - Boolean „and" allowed because M,N pertain to different members of the sequence
- **Bayes rule:** $\mathrm{prob}(f^M | g^N) = \mathrm{prob}(g^N | f^M)\, \mathrm{prob}(f^M) / \mathrm{prob}(g^N)$
- **Marginalisation:** $\mathrm{prob}(g^N) = \Sigma_h\, \mathrm{prob}(g^N, h^K)$
  K pertains to yet another set of constituents

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Can learn single-constituent state

- {F},{G},{H}: each an info'ly complete set of single-constituent observables
- $M,K\to\infty$: $f^M\to\rho$, $h^K\to\sigma$ with $\langle F\rangle_\rho=f$, $\langle H\rangle_\sigma=h$; ρ is short for proposition: „single-constituent state is ρ"
- Bayes rule:
  $$prob(\rho|g^N)=prob(g^N|\rho)\ prob(\rho) / \int_{S(d)} d\sigma\ prob(g^N|\sigma)\ prob(\sigma)$$
  As $N\to\infty$, posterior converges to a sharply peaked distribution regardless of prior prob(ρ)

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The possibility of learning presupposes the existence of a de Finetti representation

de Finetti representation

Marginalisation and product rule imply

$\mathrm{prob}(g^N) = \int_{S(d)} d\rho \, \mathrm{prob}(g^N | \rho) \, \mathrm{prob}(\rho)$

- The single-constituent state ρ appears as a nuisance parameter
- True for arbitrary N-constituent observables g^N, hence

$\rho^{(N)} = \int_{S(d)} d\rho \, \mathrm{prob}(\rho) \, \rho^{\times N}$

(de Finetti representation)

de Finetti 1931, Caves, Fuchs and Schack 2

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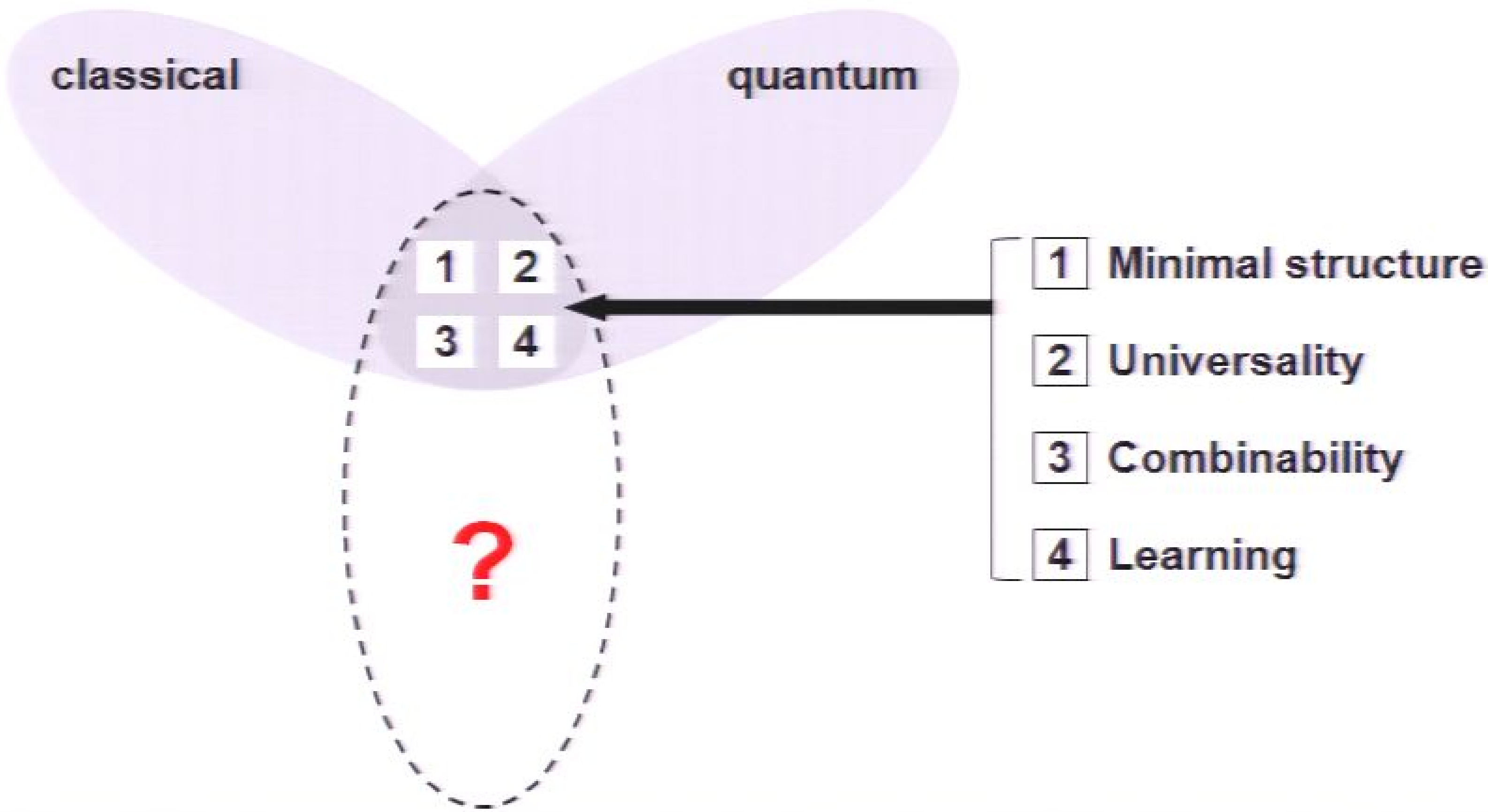
Contents

1. Introduction
2. How quantum probability differs from classical probability
3. What quantum and classical probability have in common
4. Tertium non datur

Are there any other probability theories that share the same commonalities?

Key question



I focus on the nontrivial situation where – as in the quantum case – hypotheses form a continuum

Overview of cases

Hypothesis space
M({1,d-1})

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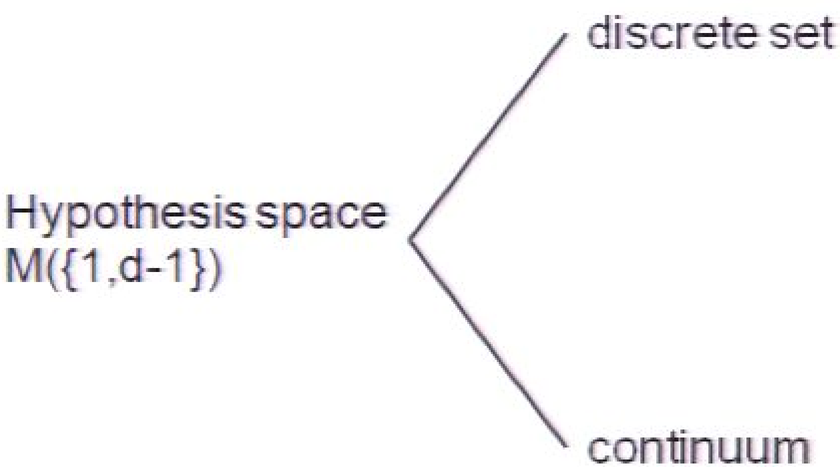

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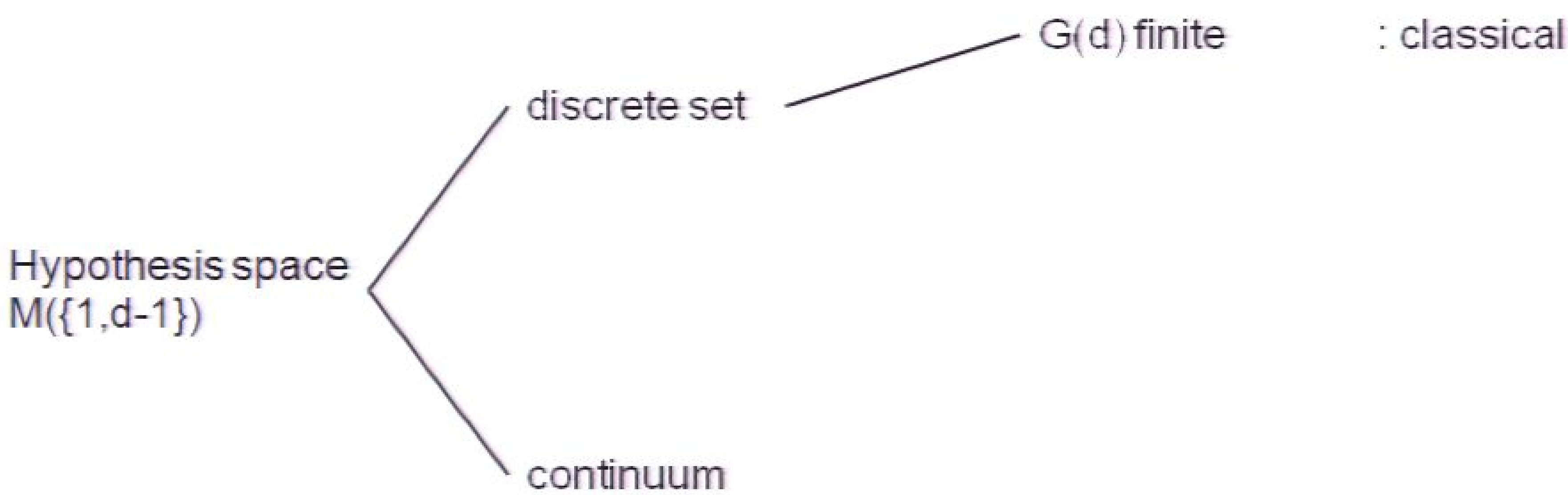

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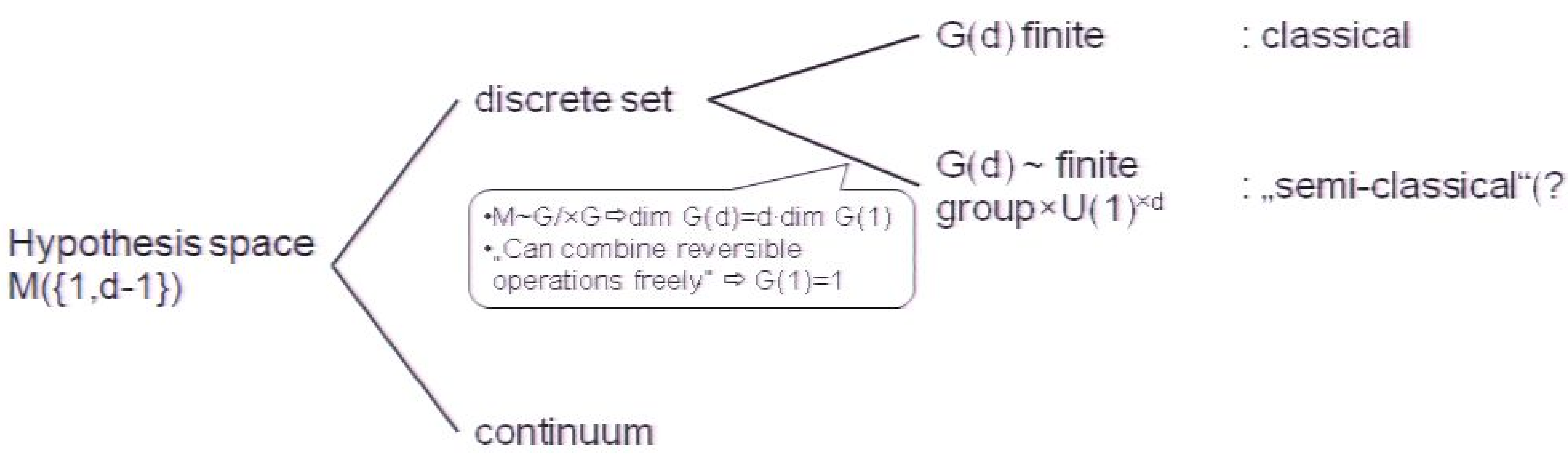

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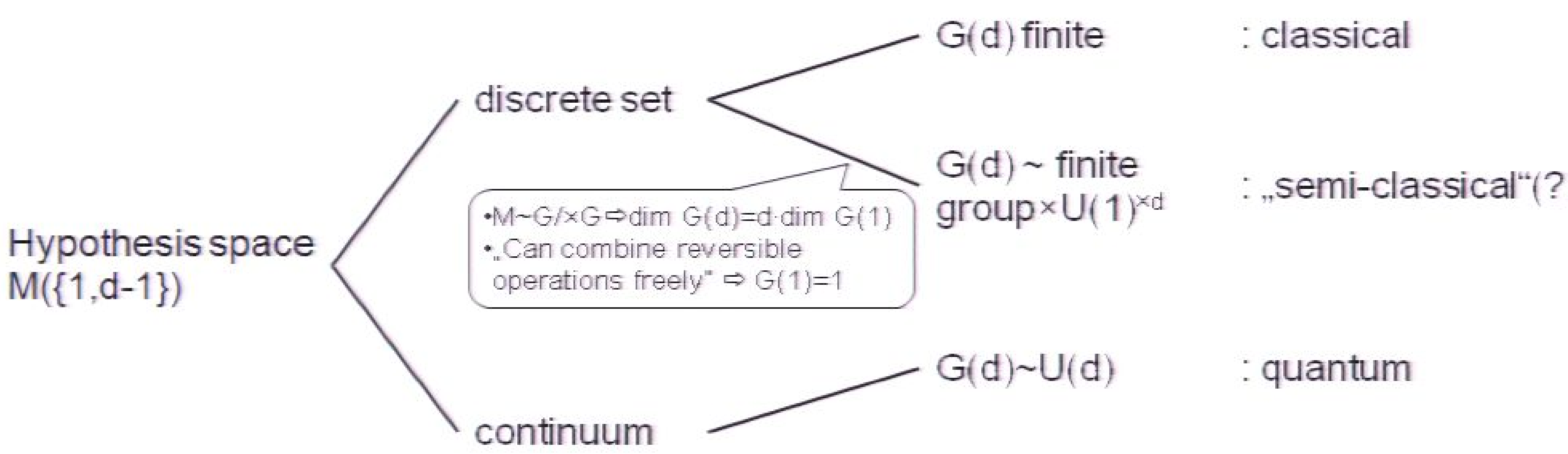

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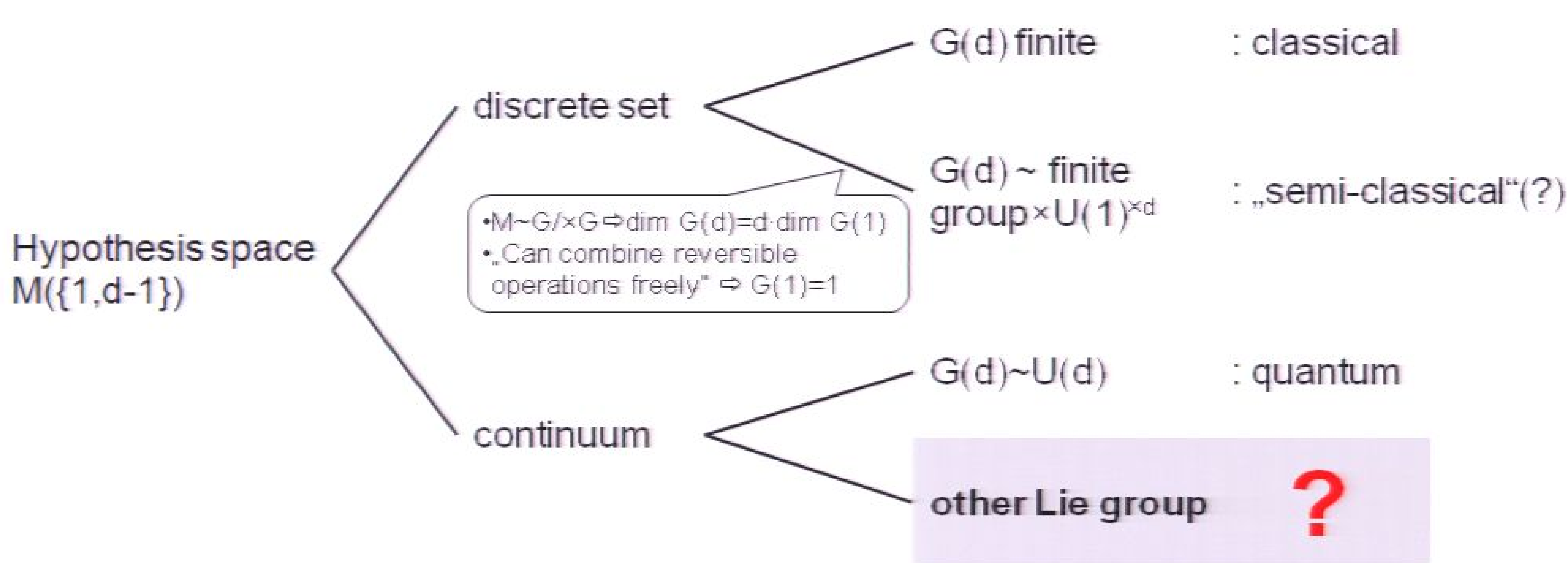

In the continuum case states shall change under reversible operations in a continuous fashion, as in quantum theory

Continuity

5 Probabilities that are initially greater than zero shall not suddenly jump to zero upon an infinitesimal transformation:

$$\forall\, \epsilon(\rho) > 0\ \exists\, \delta > 0 :\ g(\rho)\,(x) > 0 \quad \forall \quad x \subseteq \mathrm{supp}(\rho)\,,\ x \neq \emptyset\,, \\ g \in \mathcal{G}_\delta(d)\ .$$

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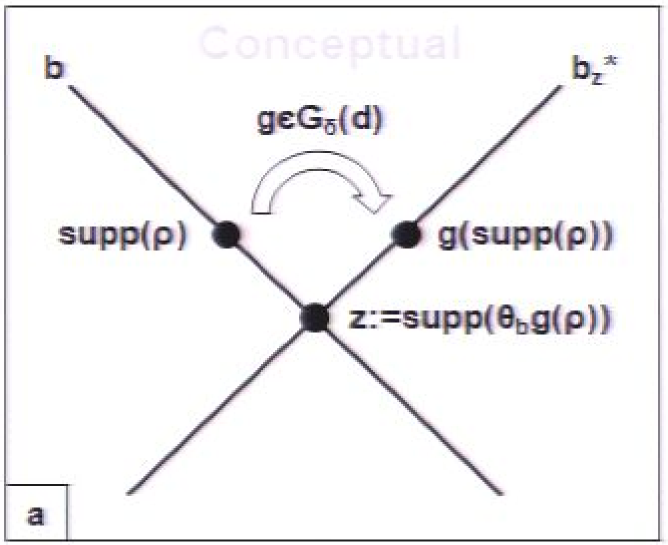
$$\mathrm{supp}\left[\theta_{\mathrm{supp}(\rho)}\, g(\rho)\right] = \mathrm{supp}(\rho)\ \forall\ g \in \mathcal{G}_\delta(d)$$

"Quantum Zeno"

Misra and Sudarshan 1

The continuity requirement imposes tight constraints on the group dimension

Dimensional analysis (1/2)



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⇨ d(z)=d(supp(ρ))

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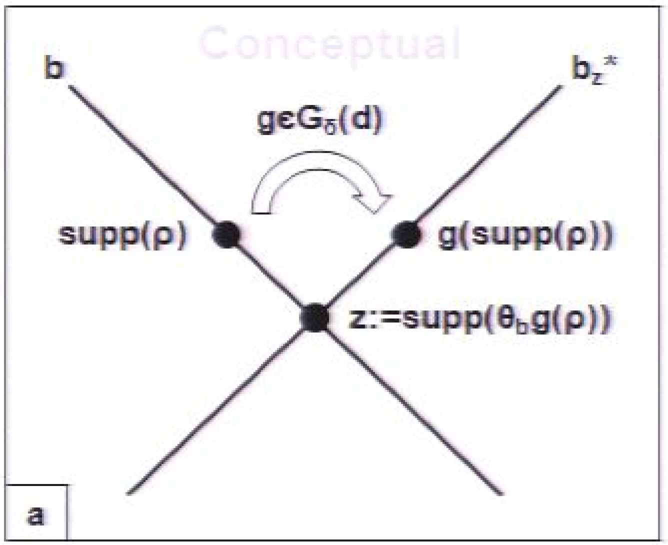

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- Given a, b, supp(ρ) there are two equivalent ways to specify g(supp(ρ)):
  - directly as a refinement of a
  - first z as a refinement of „base" b, then g(supp(ρ)) as a refinement of „fiber" b_z^*,

  yielding the sum rule

  dim M({k,d-k}) = dim M({k,l-k}) + dim M({k,d-l})

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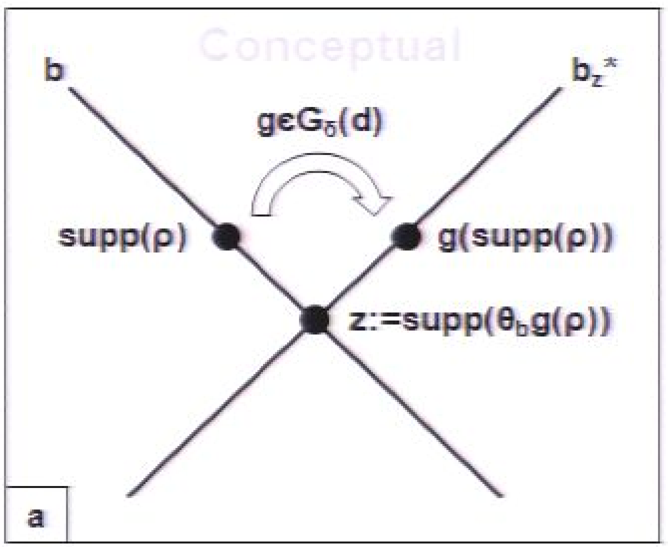

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- Together with transitivity

  $\dim M(\{k_i\}) = \dim G(\Sigma_i k_i) - \Sigma_i \dim G(k_i)$

  this constrains the group dimension to be of quadratic form

  $\dim G(d) = \dim M(\{1,1\})/2 \cdot d(d-1) + \dim G(1) \cdot d$

Combining constituent operations freely into composite operations is only possible with a Hilbert space structure

Dimensional analysis (2/2)

Constraints

- Quadratic form
- Continuum case: dim M({1,1})≥1
- Free combination of reversible operations:

  $\dim G(d_A d_B) \geq \dim G(d_A) \cdot \dim G(d_B)$

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Allowed dimensions and associated groups

dim M({1,1}) \ dim G(1)	0	1
1	**SO(d)** d(d-1)/2	**Sp(d)** d(d+1)/2
2	**SO(d)×SO(d)** d(d-1)	**U(d)** d^2

Combining constituent operations freely into composite operations is only possible with a Hilbert space structure

Dimensional analysis (2/2)

Constraints

- Quadratic form
- Continuum case: dim M({1,1})≥1
- Free combination of reversible operations:
  dim $G(d_A d_B) \geq \dim G(d_A) \cdot \dim G(d_B)$

Allowed dimensions and associated groups

dim M({1,1}) \ dim G(1)	0	1
1	**SO(d)** d(d-1)/2	**Sp(d)** d(d+1)/2
2	**SO(d)×SO(d)** d(d-1)	**U(d)** d^2

- 3 of the 4 allowed groups correspond to Hilbert space structure over some skew field (complex, real or quaternionic) ⇨ analogous to „quantum logic"
- The additional group SO(d)×SO(d) is somewhat elusive and may warrant further investigation. It yields the same manifold dimensions for M({k_i}) as (and hence might be locally equivalent to) quantum theory; yet global topology is different

In order to enable learning, Hilbert space must be over the complex numbers

Exclusion of other skew fields

In real and quaternionic Hilbert space there exist states of exchangeable sequences that do not have the de Finetti form

⇨ Some or all of the rules required for learning (product rule, Bayes rule, marginalisation) do not hold

⇨ The state of a system can never be learnt, not even by performing measurements on an exchangeable sequence of identical copies

⇨ ~~SO(d)~~, ~~Sp(d)~~

Caves, Fuchs and Schack 2

In sum, the only reasonable alternative to classical probability is quantum probability in complex Hilbert space

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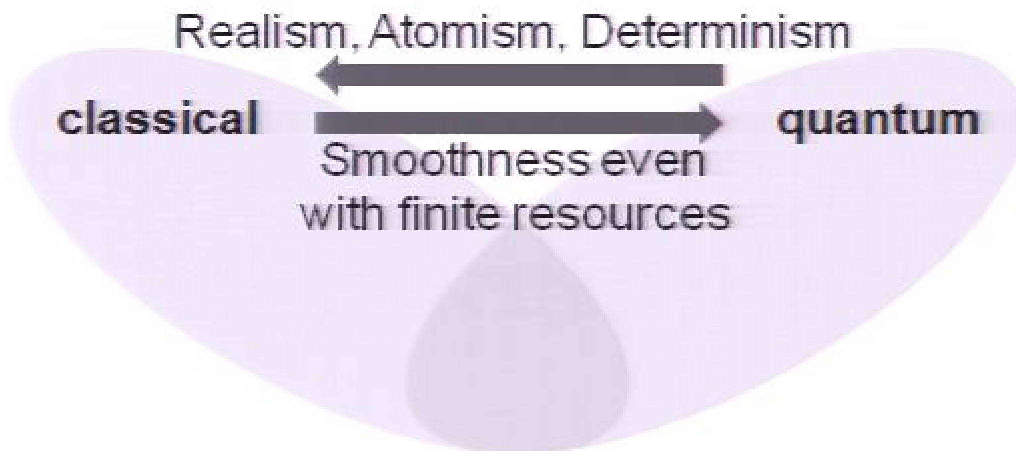

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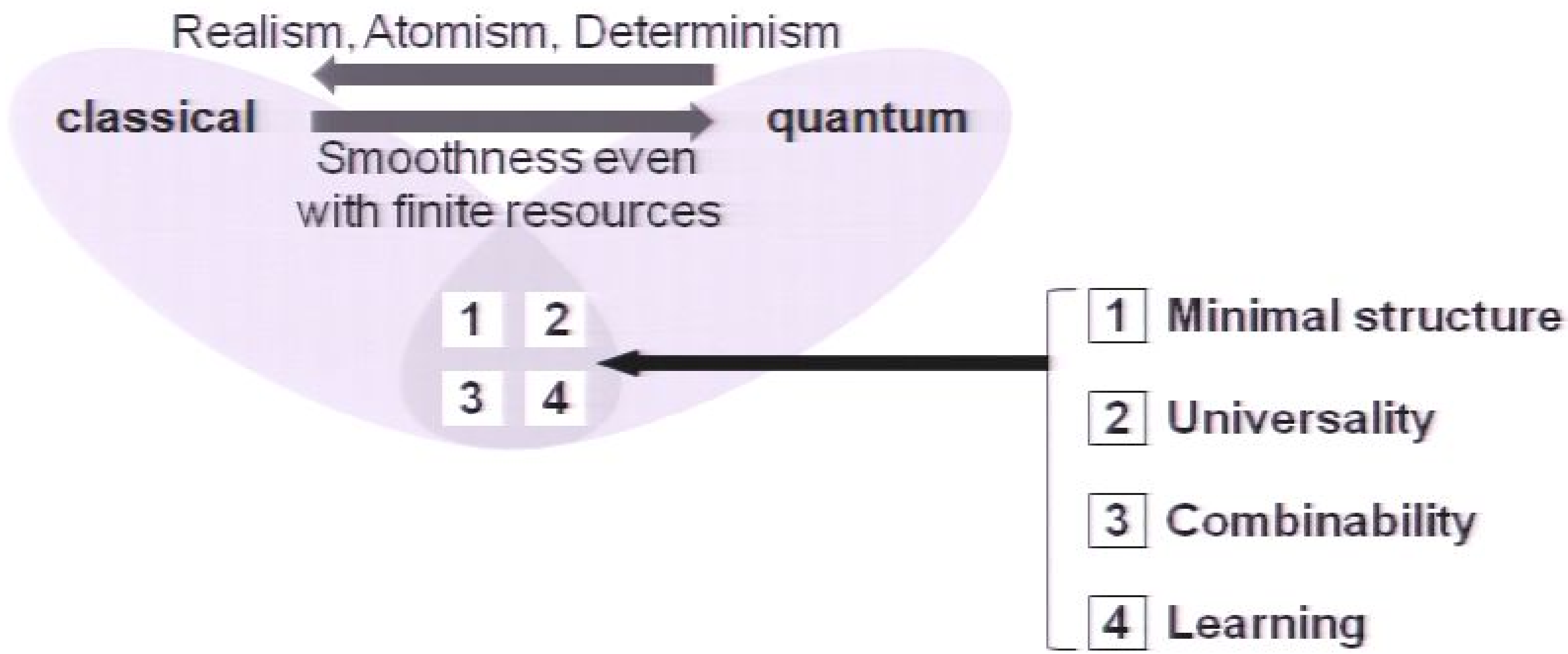
Summary



In sum, the only reasonable alternative to classical probability is quantum probability in complex Hilbert space

Summary