Title: On-shell methods in Quantum Field Theory

Date: Dec 10, 2008 11:50 AM

URL: http://pirsa.org/08120055

Abstract: The efficient computation of scattering amplitudes in quantum field theory has many important applications, ranging from the computation of QCD backgrounds at the LHC to the study of the perturbative finiteness of N=8 supergravity. \'On-shell methods\' are a crucial ingredient in the computation of gauge theory and gravity amplitudes because they are far more efficient than traditional Feynman diagram techniques. I give an introduction to the basic concepts used in this field. I explain one particularly elegant method, the MHV vertex expansion, and outline how we recently proved the validity of this expansion in N=4 Super Yang-Mills Theory.

On-shell methods in Quantum Field Theory

Michael Kiermaier

recommended reviews:

- arXiv:0704.2798 Zvi Bern, Lance Dixon, David Kosower
- hep-th/0504194 Freddy Cachazo, Peter Svrček
- hep-ph/9601359 Lance Dixon

Proof of MHV vertex expansion in $\mathcal{N} = 4$ SYM:

- arXiv:0811.3624 with Henriette Elvang, Daniel Z. Freedman
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Perimeter Institute, December 10, 2008

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1. Motivation
2. On-shell methods
3. MHV vertex expansion in $\mathcal{N} = 4$ SYM

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Textbook QFT

In textbooks, QFT is treated using

- a Lagrangian
- Feynman rules $\Rightarrow$ scattering amplitudes

In this formulation, the simplest fields are scalars.

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In this formulation, the simplest fields are scalars.

With gauge symmetries (vector fields, gravity)

- redundancy in the Lagrangian description
- "unphysical" degrees of freedom
- gauge-fixing required $\Rightarrow$ ghost fields, etc.

Textbook QFT

Scattering amplitudes

compute Feynman diagrams:

- many vertices, many diagrams
- each diagram gauge dependent $\Rightarrow$ unphysical
- final result for on-shell amplitudes often much simpler!

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Scattering amplitudes

compute Feynman diagrams:

- many vertices, many diagrams
- each diagram gauge dependent $\Rightarrow$ unphysical
- final result for on-shell amplitudes often much simpler!
- SUSY seems to make things worse (more fields, more vertices)
- $\mathcal{N}=8$ SUGRA is a nightmare
  ($\sim 10^{30}$ terms for a five-loop diagram)

Tree amplitudes in gauge theory

Let us study tree amplitudes in gauge theory
(QCD, $\mathcal{N} = 4$ Super Yang Mills,. . .)

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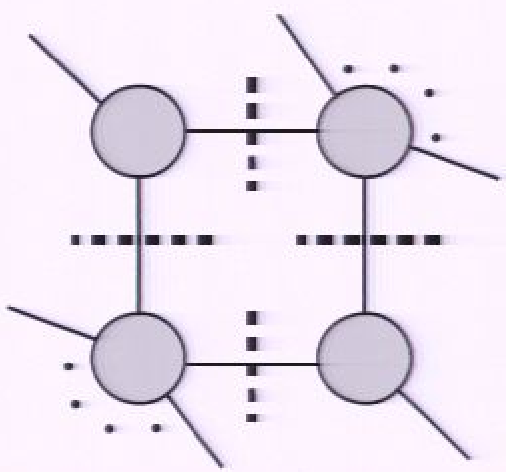
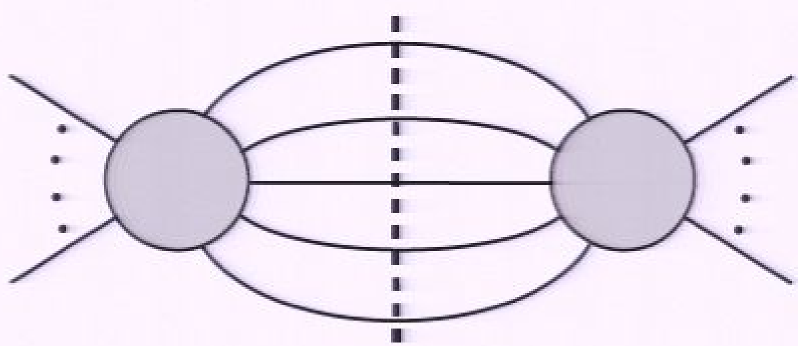
(QCD, $\mathcal{N} = 4$ Super Yang Mills,...)

Why tree amplitudes?

- building blocks for loop calculations
- (generalized) unitarity cuts of loop diagrams
  $\Rightarrow$ products of trees





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- QCD interesting for LHC applications (SM background)
- $\mathcal{N}=4$ SYM plays important role in AdS/CFT
- gravity amplitudes follow from gauge theory amplitudes via KLT relations

$$\text{gravity} \sim (\text{gauge theory})^2$$

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Technical preliminaries

Spinor-helicity formalism

spinors from null momenta

$$p_i^\mu \text{ with } p^2 = 0 \quad \Longleftrightarrow \quad \begin{aligned} u_+(i) &\leftrightarrow (\lambda_i)_\alpha \leftrightarrow |i\rangle\,, \\ u_-(i) &\leftrightarrow (\tilde{\lambda}_i)_{\dot{\alpha}} \leftrightarrow |i]\,. \end{aligned}$$

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momenta from spinors

$$p_i \equiv p_i^\mu (\sigma_\mu)_{\alpha\dot\alpha} = |i\rangle [i|\,, \qquad 2p_i \cdot p_j = \langle i\, j\rangle\, [j\, i]\,.$$

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Properties:

Dirac equation: $p_i|i] = 0\,, \qquad \langle i|p_i = 0\,.$

Antisymmetry: $\langle i\, j\rangle = -\langle j\, i\rangle \quad \Rightarrow \quad \langle i\, i\rangle = 0\,.$

Schouten identity: $\langle i\, j\rangle\langle k| + \text{cyclic} = 0\,.$

Technical preliminaries

Color-stripped amplitudes

defined by dropping the traces of group generators

$$A(1^{a_1}, \ldots, n^{a_n}) \;=\; \mathrm{Tr}\big(T^{a_1} \cdots T^{a_n}\big)\, \mathcal{A}(1, \ldots, n) \;+\; \text{perms}$$

$\Rightarrow$ color-stripped amplitudes depend on ordering of states

QCD amplitudes

Gluon amplitudes with zero or one negative-helicity gluon vanish:

$$\mathcal{A}(1^+, \ldots, n^+) = 0\,, \qquad \mathcal{A}(1^-, 2^+, \ldots, n^+) = 0\,.$$

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Maximally helicity violating (MHV) amplitudes

MHV amplitudes have exactly two negative-helicity gluons.
They take an extremely simple form [Parke, Taylor]:

$$\mathcal{A}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+) \;=\; \frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n-1,n\rangle\langle n\,1\rangle}\,.$$

Miraculous from Feynman diagrams, especially when $n>5$!

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Generalization

(Next-to)^{k}MHV amplitudes have $k+2$ negative helicity gluons

Complex shifts

Shift external momenta, with complex parameter z:

$$p_i \quad \rightarrow \quad \hat{p}_i = p_i + z\, q_i\,, \qquad (\hat{p}_i)^2 = 0\,, \quad \sum_i q_i = 0$$

The amplitude now depends on z:

$$\mathcal{A}(1,\ldots,n) \quad \rightarrow \quad \mathcal{A}(z) = \mathcal{A}(\hat{1},\ldots,\hat{n})\,.$$

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Assumption: $\mathcal{A}(z) \rightarrow 0$ as $z \rightarrow \infty\,.$

- non-trivial assumption
- naively worse for gauge/gravity than scalars

Recursion relations

Cauchy's theorem gives amplitude

$$0 = \oint \frac{dz}{2\pi i\, z} \mathcal{A}(z)$$

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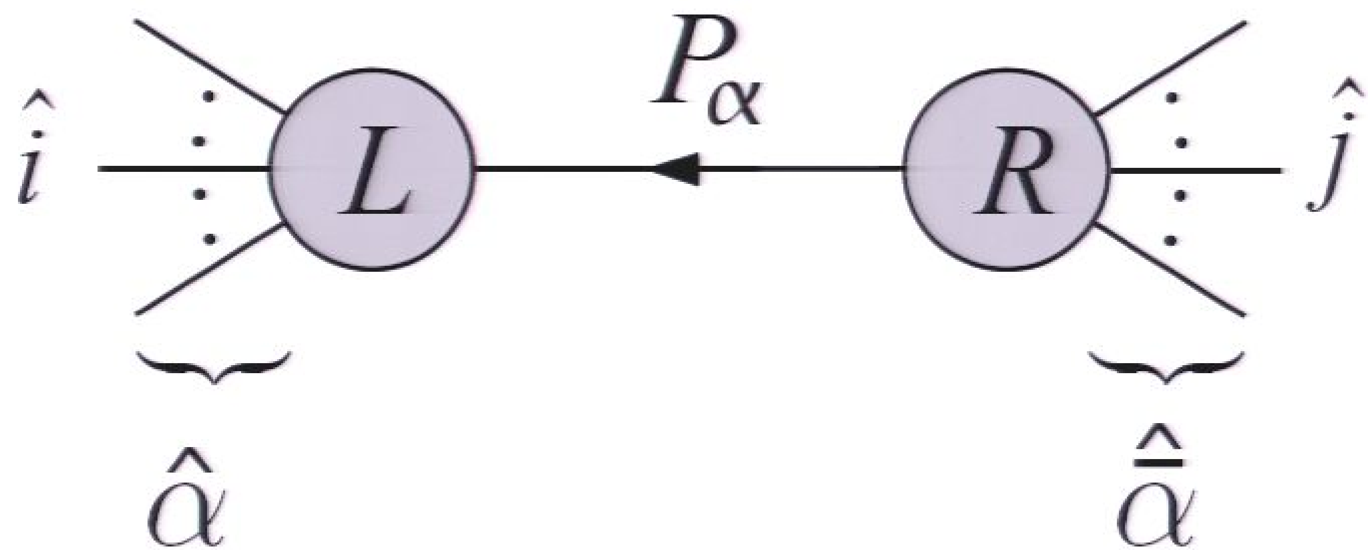

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Three-point subamplitudes in complex kinematics

$$
\mathcal{A}(1^-, 2^-, 3^+) = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}.
$$

BCFW [Britto, Cachazo, Feng, Witten]

Two-line shift $[i, j\rangle$

$$|i] \to |\hat{i}] = |i] + z|j] , \qquad |i\rangle \to |i\rangle$$
$$|j\rangle \to |\hat{j}\rangle = |j\rangle - z|i\rangle , \qquad |j] \to |j] .$$

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- $\hat{p}_i^\mu$, $\hat{p}_j^\mu$ still light-like:

$$\hat{p}_i \;=\; |i\rangle\Big([i| + z[j|\Big)\,, \qquad \hat{p}_j \;=\; \Big(|j\rangle - z|i\rangle\Big)[j|\,.$$

- momentum conserved

$$\delta P \;=\; |i\rangle\Big(z[j|\Big) \;-\; \Big(z|i\rangle\Big)[j| \;=\; 0$$

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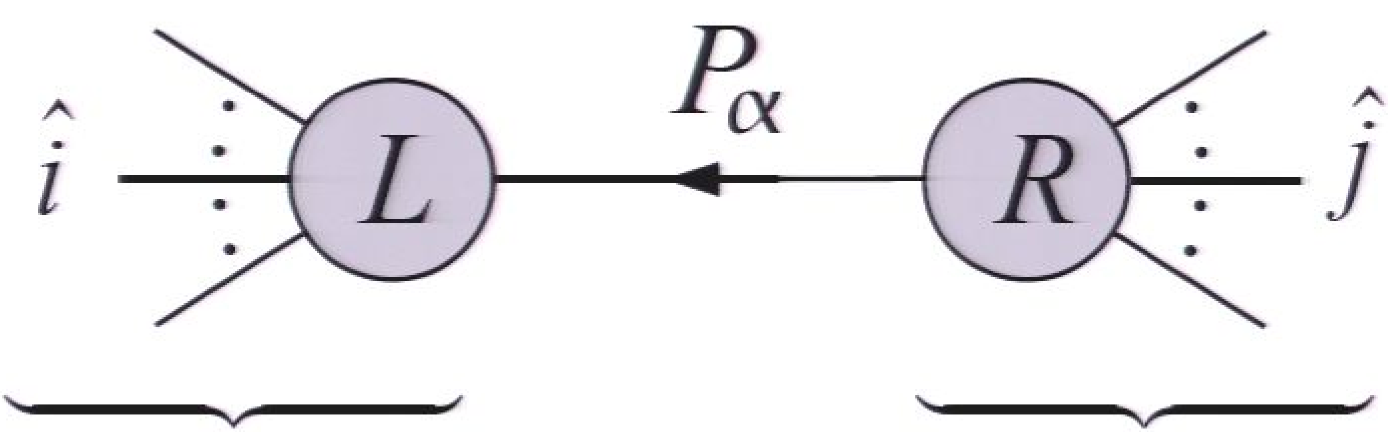
Some diagrams vanish by kinematics



vanishes if 3-point anti-MHV

vanishes if 3-point MHV

Square-spinor shifts

Three-line shift $|i,j,k]$ [Risager]

$$|i] \to |\hat{i}] = |i] + z\langle jk\rangle|X], \qquad |i\rangle \to |i\rangle,$$
$$|j] \to |\hat{j}] = |j] + z\langle ki\rangle|X], \qquad |j\rangle \to |j\rangle,$$
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- momentum conserved by Schouten identity

$$\delta P = z\left(|i\rangle\langle j\,k\rangle + |j\rangle\langle k\,i\rangle + |k\rangle\langle i\,j\rangle\right)[X| = 0\,.$$

- $|X]$ is an arbitrary reference spinor
- No anti-MHV three-point subamplitudes in expansion

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Particles

1 positive helicity gluon:	$A(i)$	
4 positive helicity gluinos:	$A^a(i)$,	$a = 1, \ldots, 4$
6 (self-dual) scalars:	$A^{ab}(i)$	
4 negative helicity gluinos:	$A^{abc}(i)$	
1 negative helicity gluon:	$A^{1234}(i)$.	

$\mathcal{N}=4$ SYM

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MHV amplitudes

$$\mathcal{A}(1^-,2^-,3^+,\ldots,n^+) = \langle A^{1234}(1)\,A^{1234}(2)\,A(3)\cdots A(n)\rangle$$
$$= \frac{\langle 1\,2\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\,.$$

and amplitudes related by SUSY.

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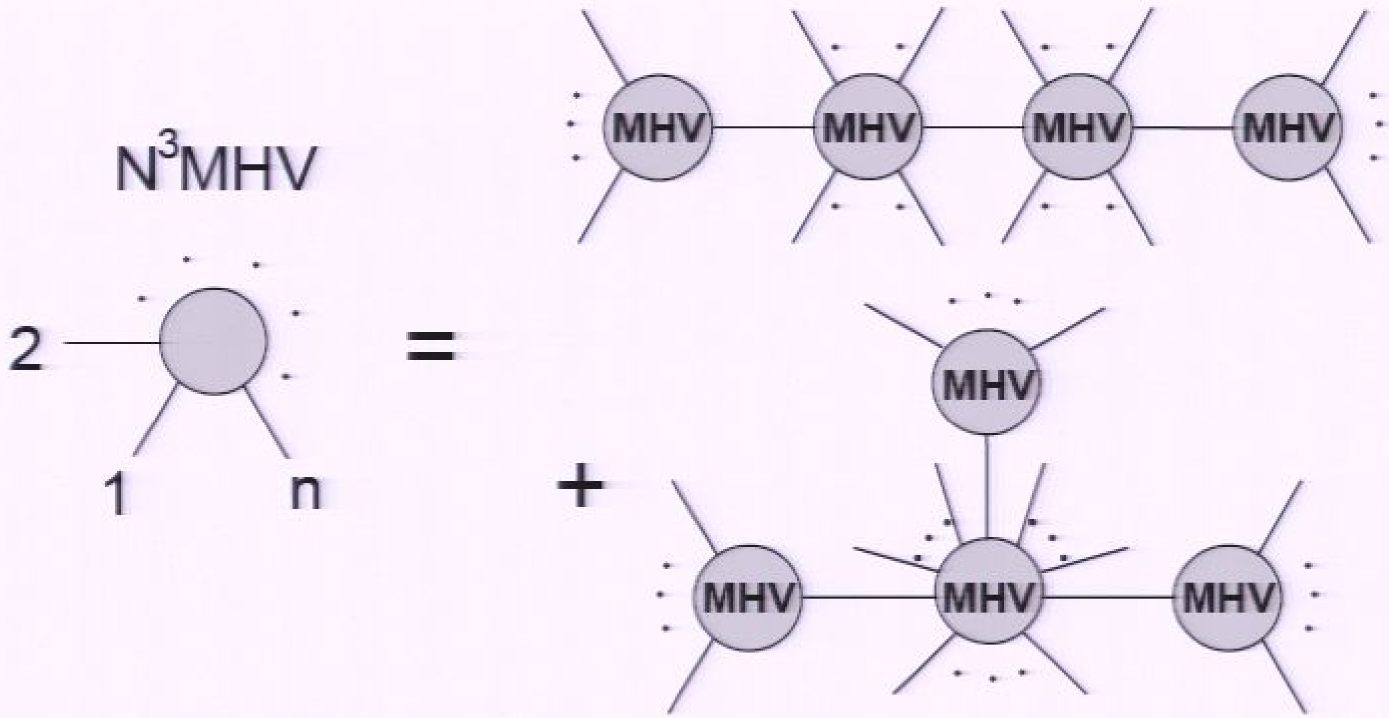
MHV amplitudes

$$\langle A^1(1)\,A^{23}(2)\,A(3)\,A^{234}(4)\,A^{14}(5)\rangle = \frac{\langle 15\rangle\langle 24\rangle^2\langle 45\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\,.$$

MHV vertex expansion

Prescription



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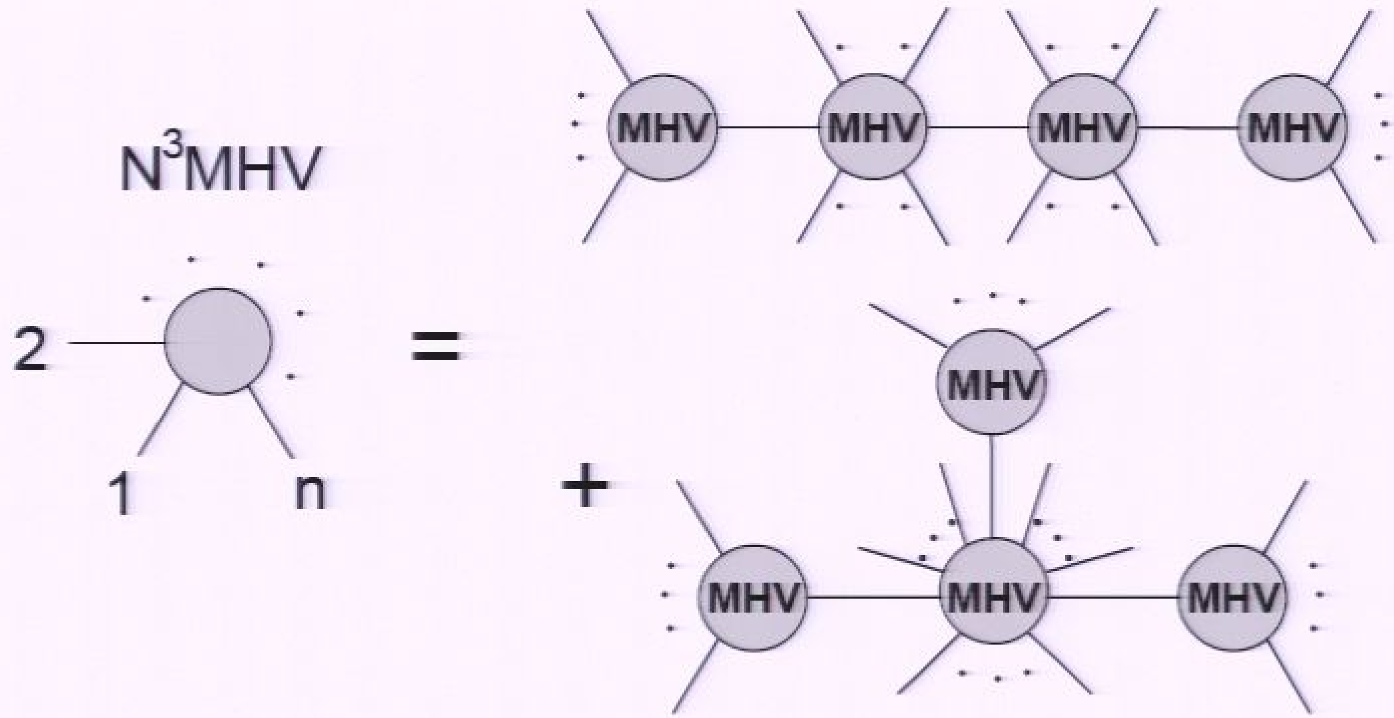

Supplement by CSW prescription [Cachazo, Svrček, Witten]:

$$|P_\alpha\rangle \equiv P_\alpha|X] \qquad (|X] \text{ is again an arbitrary reference spinor}).$$

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- proven for gauge theory [Risager]
- not valid for all gravity amplitudes ($|X]$ dependent!)

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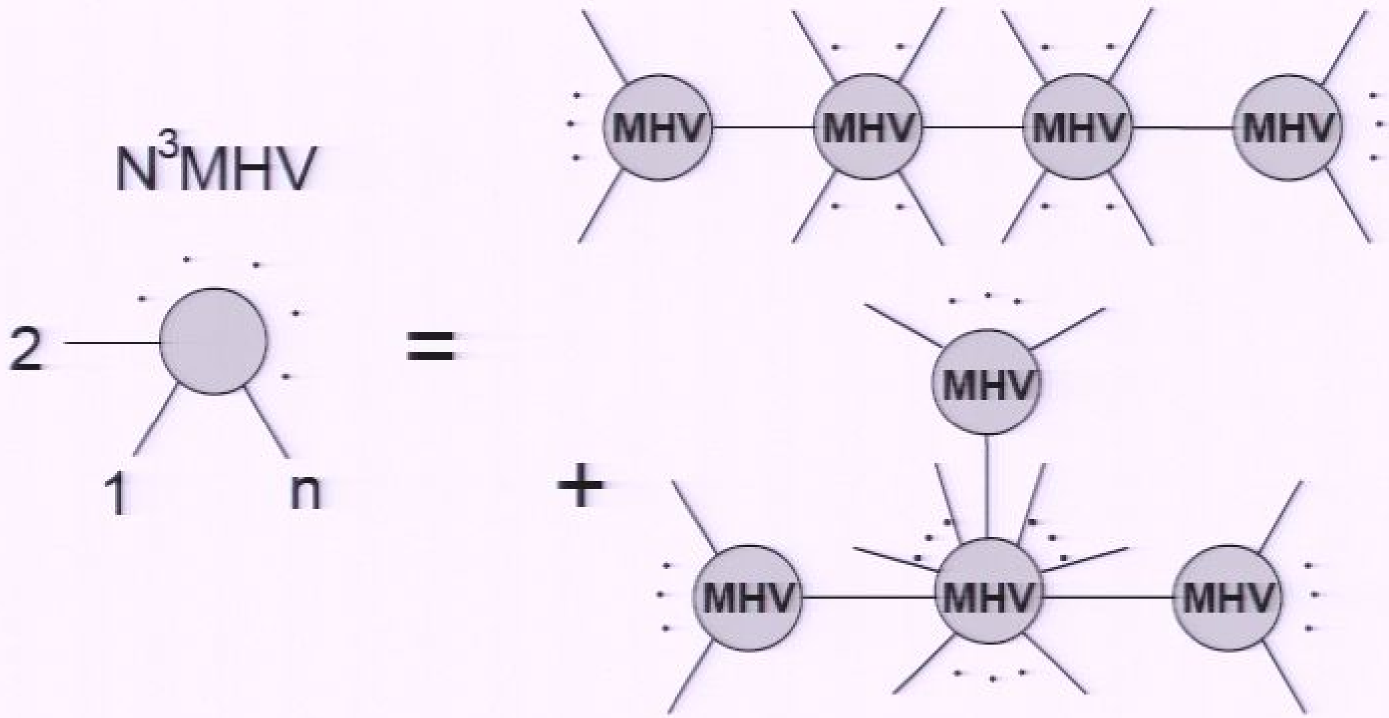

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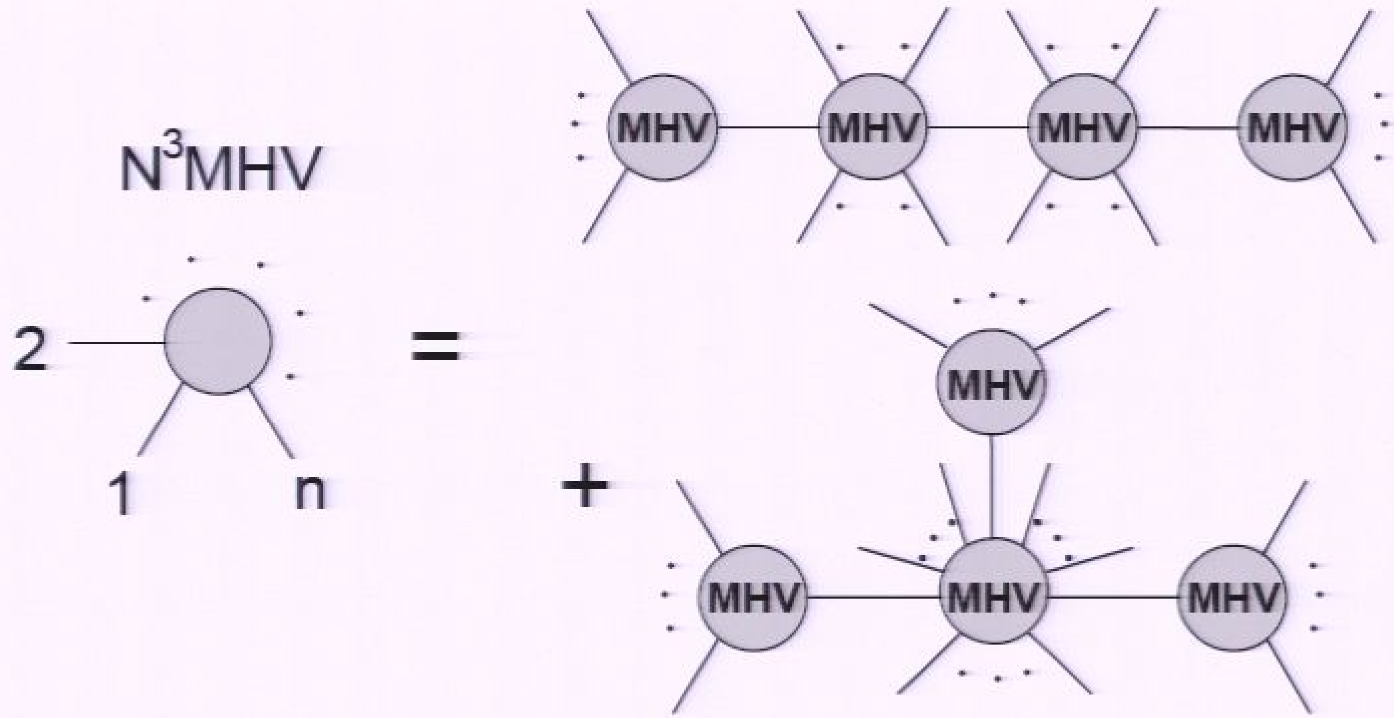

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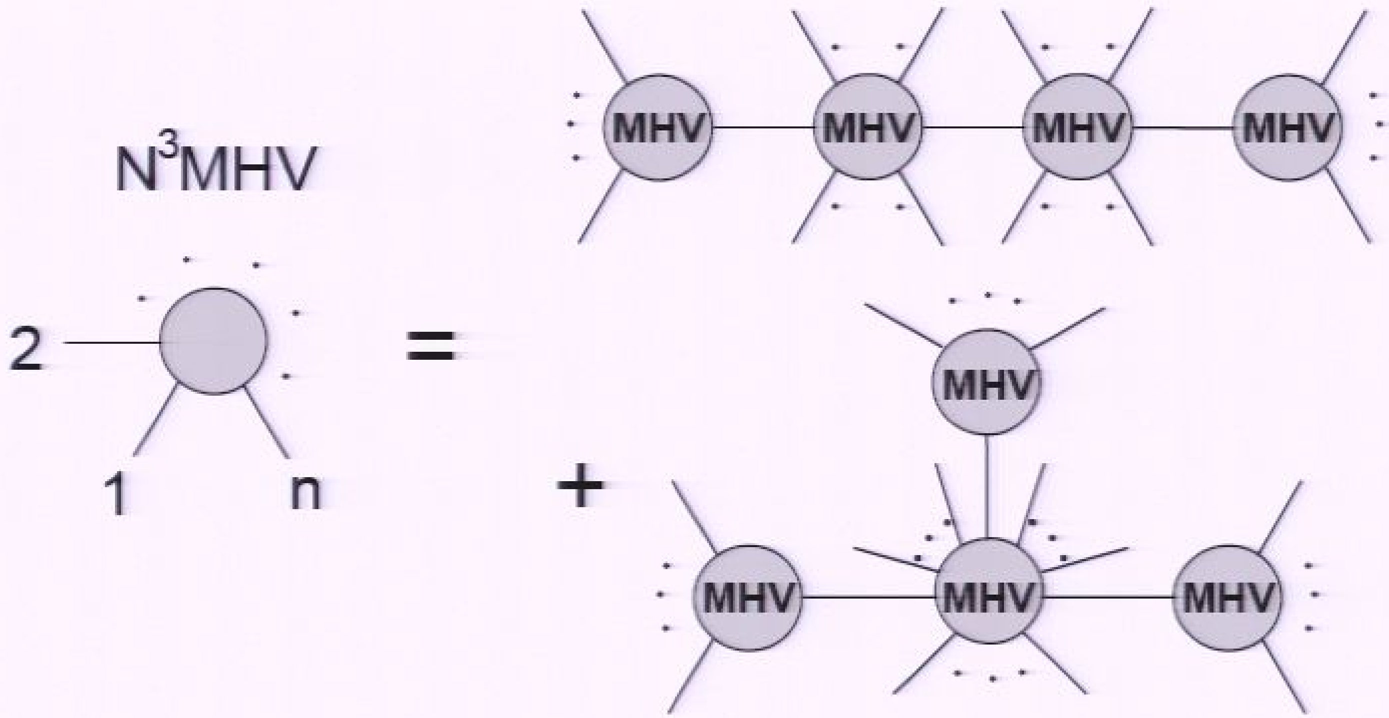

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MHV vertex expansion for $\mathcal{N} = 4$ SYM

- scalars, fermions $\Rightarrow$ non-trivial to prove
- but: supersymmetry helps
- for generic amplitudes:
  more powerful in $\mathcal{N} = 4$ SYM than in pure gauge theory

Proof of MHV vertex expansion in $\mathcal{N}=4$ SYM

Strategy

- Prove that all-line shifts

$$|i] \;\rightarrow\; |\hat{i}] = |i] + z c_i |X] \,, \qquad \sum_i c_i |i\rangle = 0$$

give at least $1/z^k$ falloff for $\mathrm{N}^k\mathrm{MHV}$ amplitudes

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Comment on efficiency

- rule of thumb:
  better large z-behavior $\Rightarrow$ less powerful recursion relation
- but: at level k, combine k diagrams to get MHV expansion!

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Claim

N^kMHV amplitudes go as $1/z^k$ under the shift

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MHV amplitudes ($k=0$)

only depend on angle brackets $\langle ab\rangle \;\Rightarrow\; \mathcal{O}(1)$ trivial:

$$\mathcal{A}(1,\ldots,n) \;=\; \frac{\langle ..\rangle\,\langle ..\rangle\,\langle ..\rangle\,\langle ..\rangle}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1,n\rangle\langle n\,1\rangle} \;\sim\; \mathcal{O}(1) .$$

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$$|i] \;\to\; |\hat{i}] = |i] + zc_i|X]\,, \qquad \sum_i c_i|i\rangle = 0\,.$$

anti-MHV amplitudes ($k = n-4$)

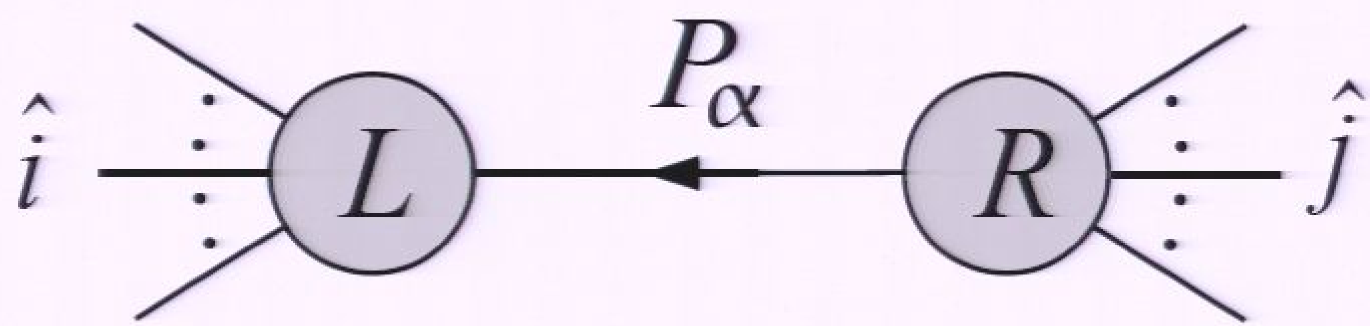
only depend on square brackets $[\hat{a}\,\hat{b}] \sim \mathcal{O}(z)$. Thus:

$$\mathcal{A}(1,\ldots,n) \;=\; \frac{\overbrace{[..]\,[..]\,[..]\,[..]}^{\text{4 square brackets}}}{\underbrace{[\hat{1}\,\hat{2}]\,[\hat{2}\,\hat{3}]\cdots[\hat{n}\,\hat{1}]}_{n\text{ square brackets}}} \;\sim\; 1/z^k\,.$$

Proof of MHV vertex expansion in $\mathcal{N} = 4$ SYM

All other amplitudes

- have a valid BCFW expansion from some shift $[i\, j\rangle$
  [Elvang, Freedman, MK; Cheung]



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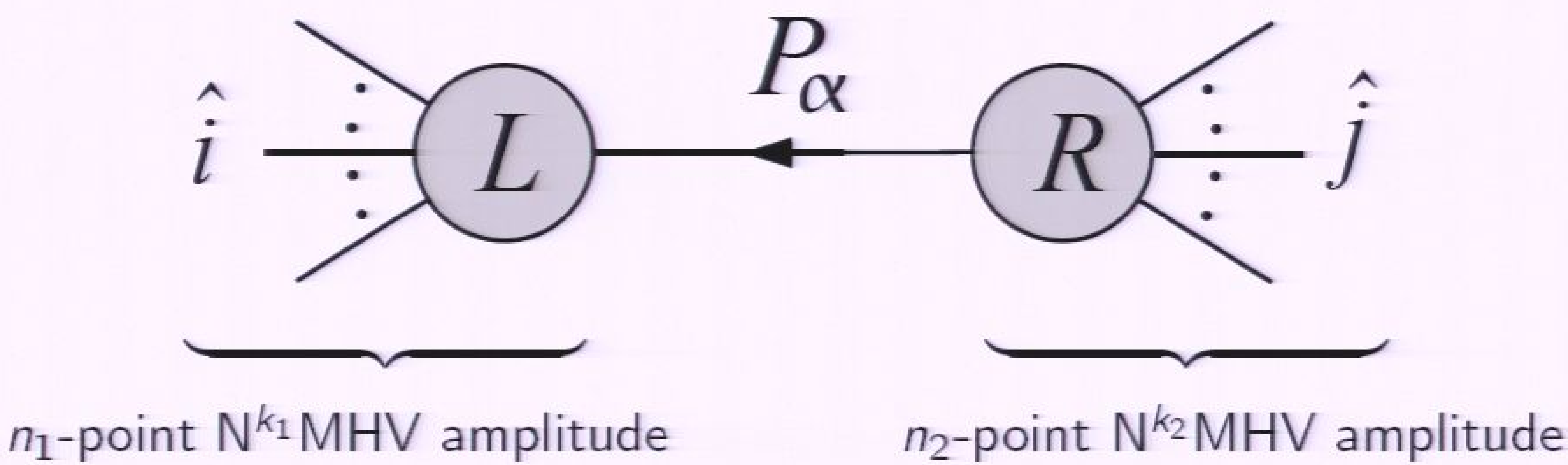

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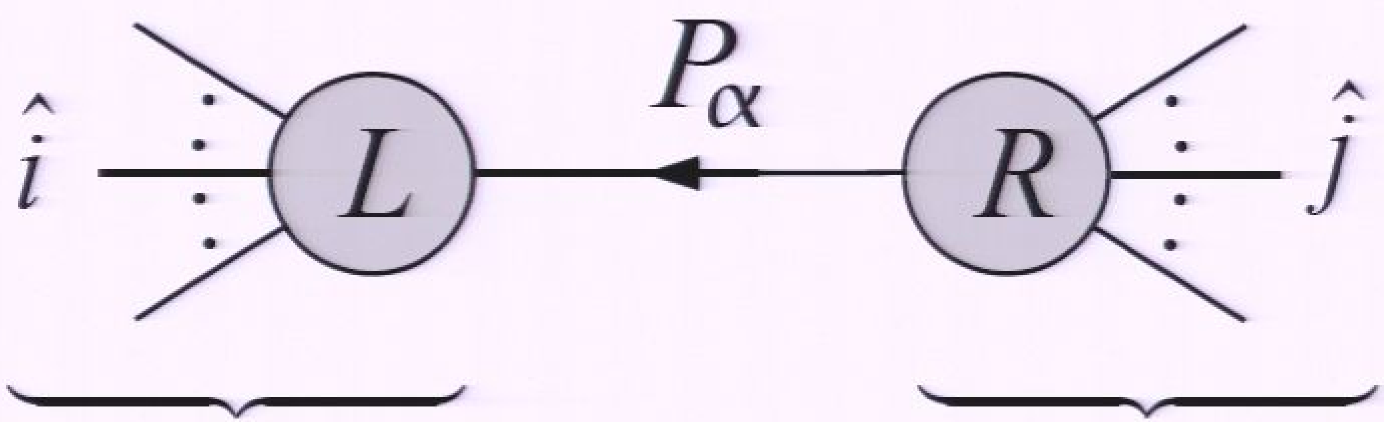


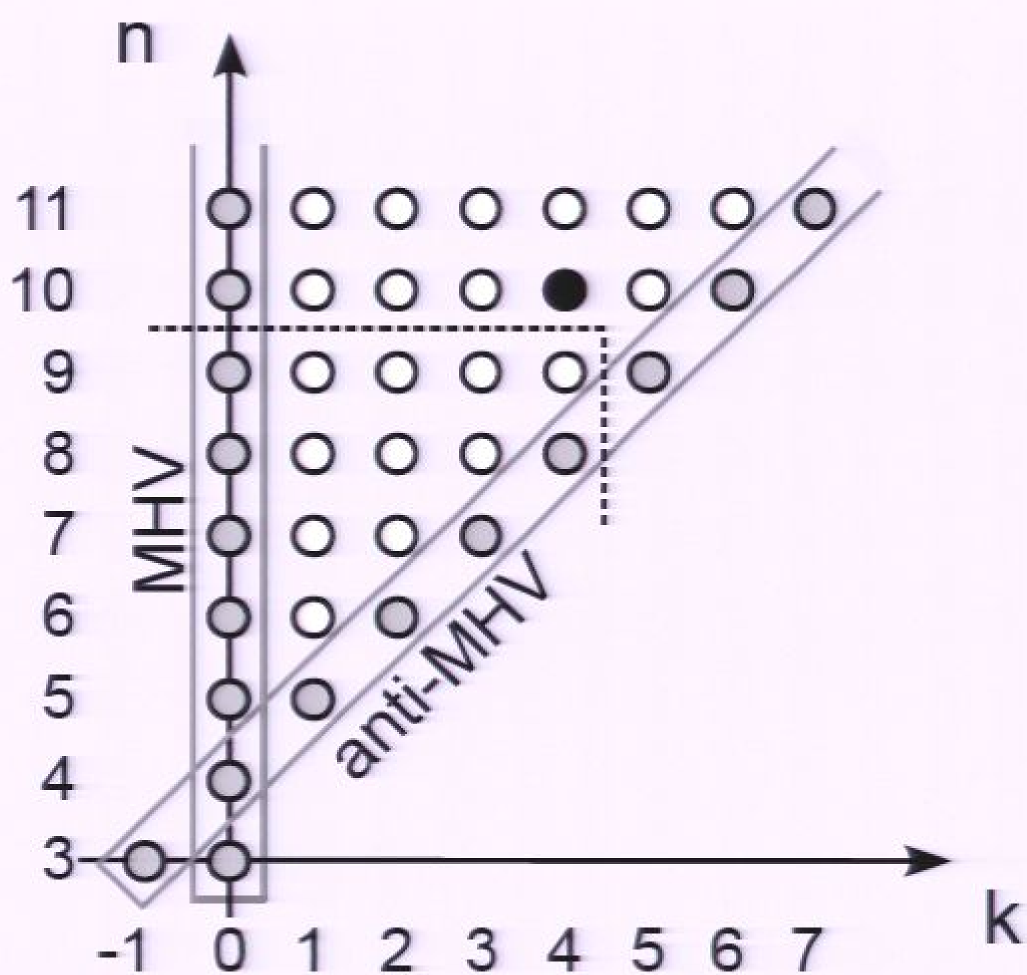
n_1-point N^{k_1}MHV amplitude $\qquad$ n_2-point N^{k_2}MHV amplitude

- All-line shift acts on subamplitudes L and R as all-line shift
- if L and R have $1/z^{k_i}$ falloff

$$\Rightarrow \mathcal{A}(z) \sim 1/z^{k_1} \times 1/z \times 1/z^{k_2} \sim 1/z^k .$$

Proof of MHV vertex expansion in $\mathcal{N} = 4$ SYM

Inductive proof of $1/z^k$ falloff



Consequences of the MHV vertex expansion

Efficient computation of amplitudes

Derive compact generating functions/superamplitudes

- functions of $4n$ Grassmann variables η_{ia}
- Compute any amplitude by acting with appropriate Grassmann differential operators

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Study behavior of $\mathcal{N} = 4$ SYM amplitudes under shifts

N^kMHV amplitudes go as $1/z^k$ under square spinor shifts

- of all lines
- of every other line
- of common-index lines (e.g. all $k+2$ lines carrying one specific SU(4) index a)

Conclusion and Outlook

On-shell methods

- crucial to compute tree and loop amplitudes
- important from LHC physics to supergravity

Tree amplitudes in $\mathcal{N} = 4$ SYM

- compact computation using the MHV vertex expansion
- alternative: superconformal invariants [Drummond,Henn]

Generalization to $\mathcal{N} = 8$ supergravity?

- can in principle use KLT
- recursion relations for $\mathcal{N} = 8$ supergravity
  - problematic, as large z falloff worse for large n
  - go to large k $\Rightarrow$ useful recursion relations?
  - generalize supershift to square spinor shift? [N A-H,Cachazo]