Title: Admissible transformations of quantum networks and their applications in quantum information processing

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Abstract: Quantum operations are known to be the most general state transformations that can be applied to parts of compound systems compatibly with the probabilistic structure of quantum mechanics. What about the most general transformations of quantum operations? It turns out that any such general transformation can be realized by a quantum network with an open slot in which the input operation can be inserted, thus programming the resulting circuit. Moreover, one can recursively iterate this construction, generating an infinite hierarchy of admissible transformations and proving their realization within the circuit model of quantum mechanics. These results provide the basis of a new method to optimize quantum networks for information processing tasks, including e.g. gate estimation, discrimination, programming, and cloning. As examples of application, I will present here the optimal quantum networks for estimation of group transformations, for the alignment of reference frames with multiple communication rounds, and for universal cloning of unitary transformations.

# *ADMISSIBLE TRANSFORMATIONS OF QUANTUM NETWORKS*

***Giulio Chiribella***

A rhapsody on joint themes with G M D'Ariano and P Perinotti
Quantum Information Theory Group
Pavia University

Young Researchers Conference
Perimeter Institute
Waterloo, 8-12 December 2008

# OUTLINE

- Part I: Admissible quantum transformations:
  - -abstract definition
  - -circuital realization
- Part II: Optimization of quantum networks
- Part III: Applications:
  - -optimal networks for estimation
  - -multi-round alignment of reference frames
  - -universal cloning of unitary gates

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# QUANTUM OPERATIONS (QO'S)

Most general transformations a quantum state can undergo: linear, completely positive, trace non-increasing maps

$$\rho \in \mathcal{S}(\mathcal{H}_{in}) \longmapsto \mathcal{E}(\rho) \in \mathcal{S}(\mathcal{H}_{out})$$

Linear: mixture of input states is mapped into mixture of output states

$$\mathcal{E}\left(\sum_i p_i \rho_i\right) = \sum_i p_i \mathcal{E}(\rho_i)$$

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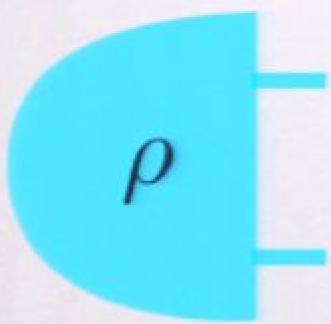

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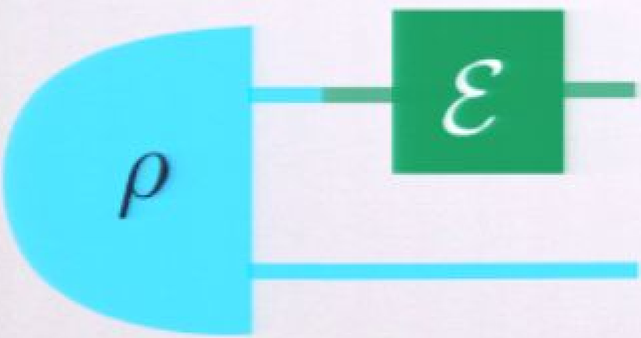

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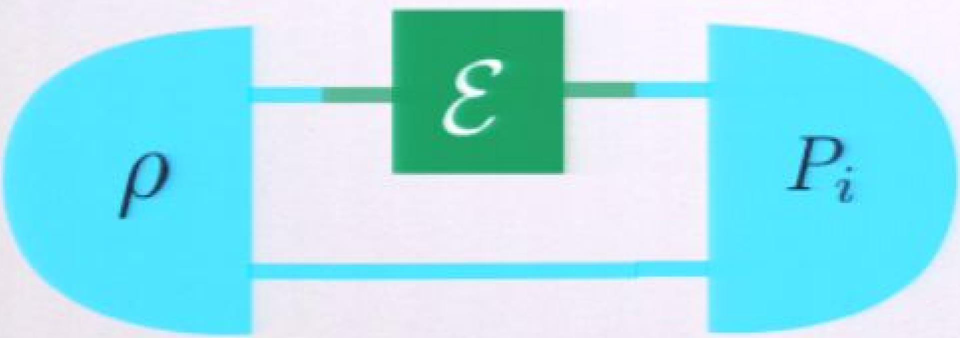

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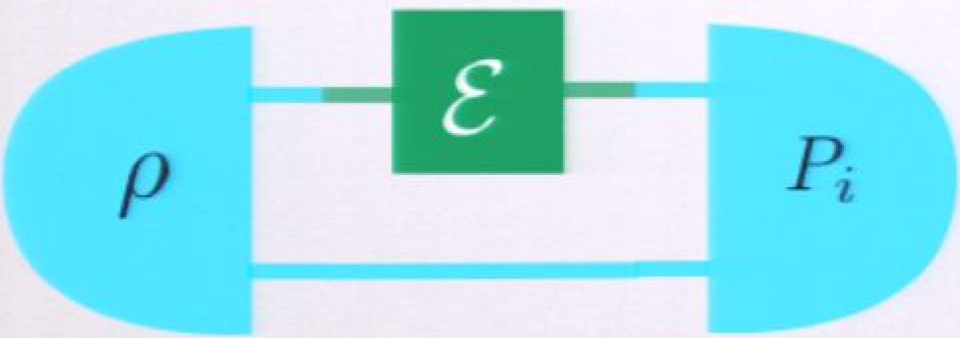


$$p(i|\rho, \mathcal{E}) \geq 0$$

Trace non-increasing: probabilities must be upper bounded by 1

Trace-preserving maps = deterministic QOs = quantum channels

# REALIZATION: OPEN SYSTEM EVOLUTIONS

QO's can be interpreted as evolutions of open systems:

$$\mathcal{E}(\rho) = \mathrm{Tr}_{\mathrm{env}}[U(\rho \otimes \sigma_{env})U^{\dagger}(I_{out} \otimes P_{env})]$$

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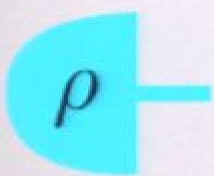

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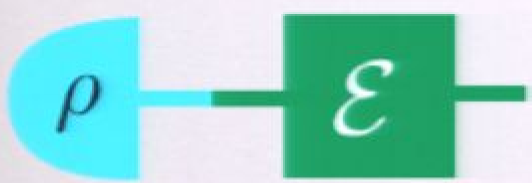

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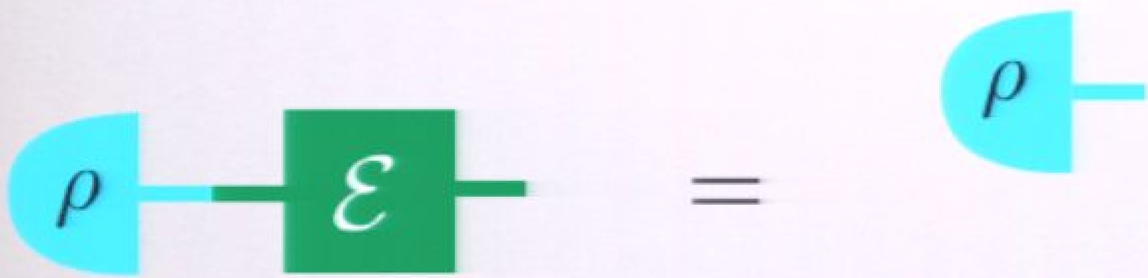

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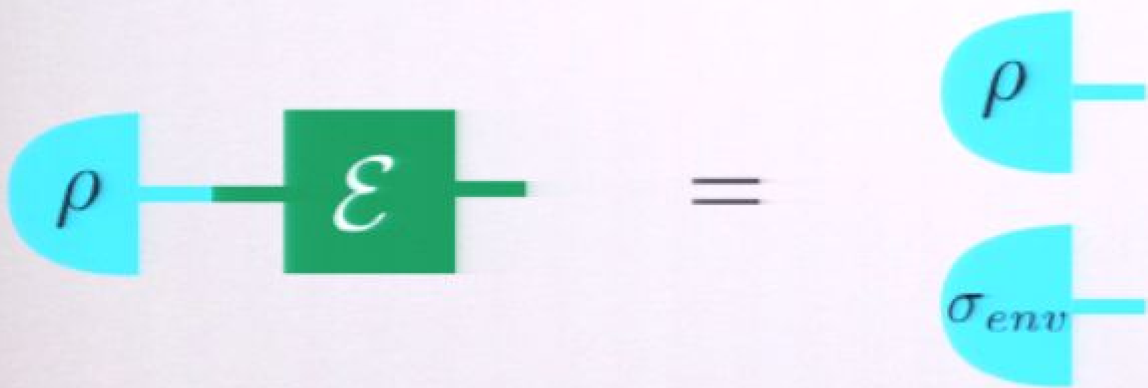

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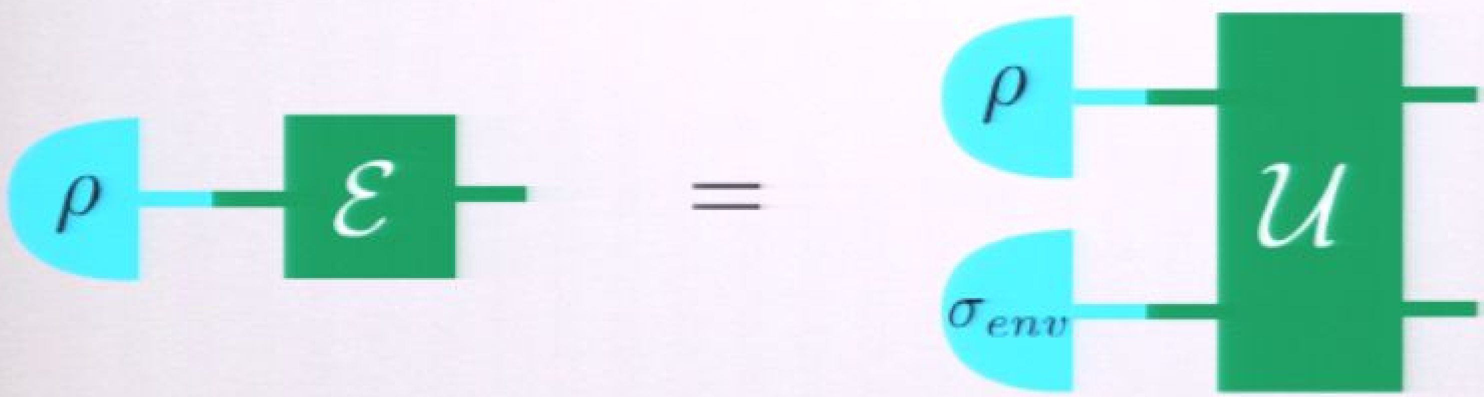

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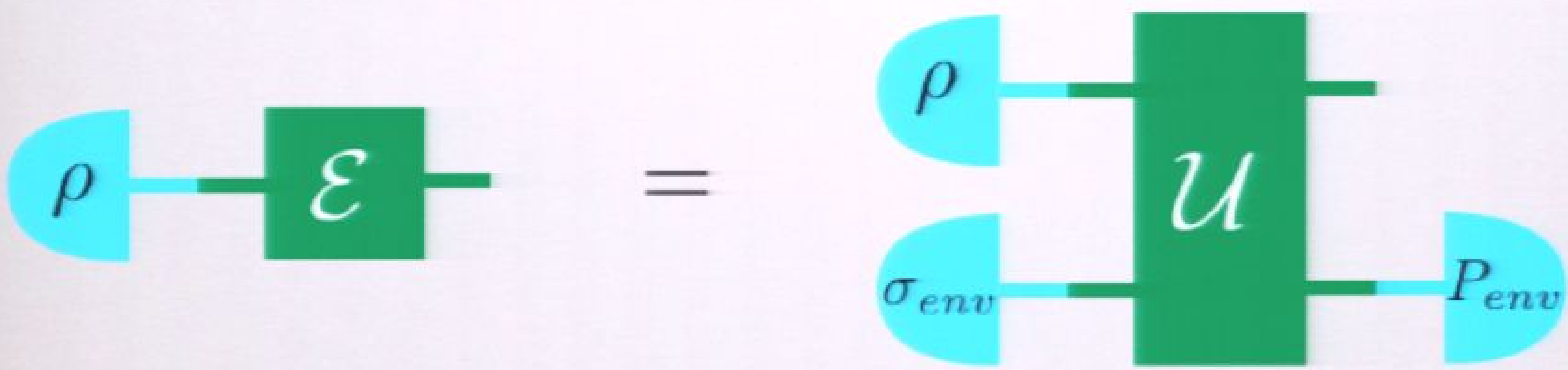

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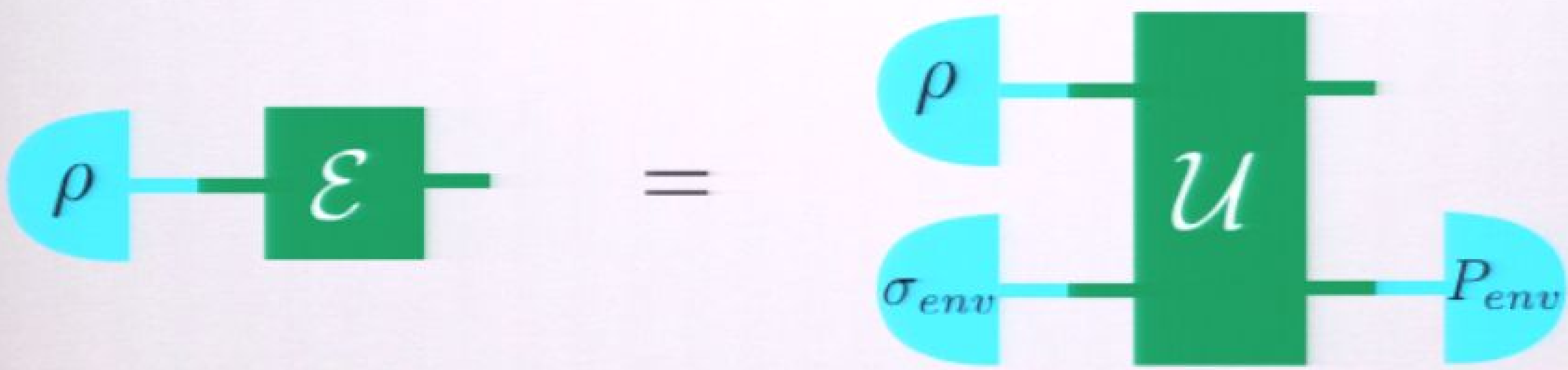


Trace decreasing: corresponds to a particular outcome of the measurement on the environment

Trace preserving: sum over all outcomes
the environment is discarded
$P_{env} = I$

# TRANSFORMATIONS OF QO'S

Two questions:

- QOs are the most general state transformations, which are the most general transformations of QOs?
- QOs can be realized as open system evolutions, what about their transformations?

A transformation of QOs must be a linear supermap

$$\mathcal{E} \in QO(\mathcal{H}_{in}, \mathcal{H}_{out}) \longmapsto \mathcal{S}(\mathcal{E}) \in QO(\mathcal{H}'_{in}, \mathcal{H}'_{out})$$

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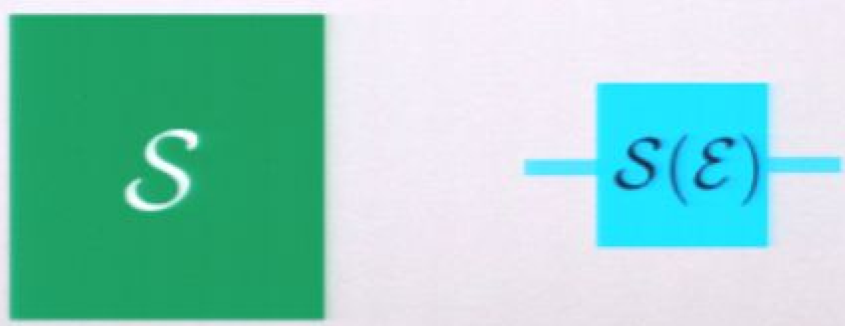

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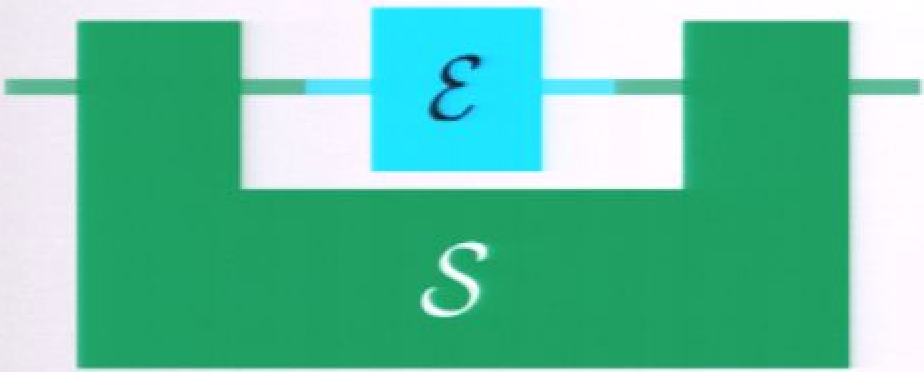

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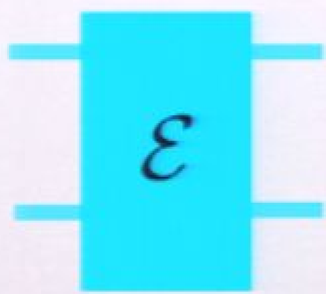

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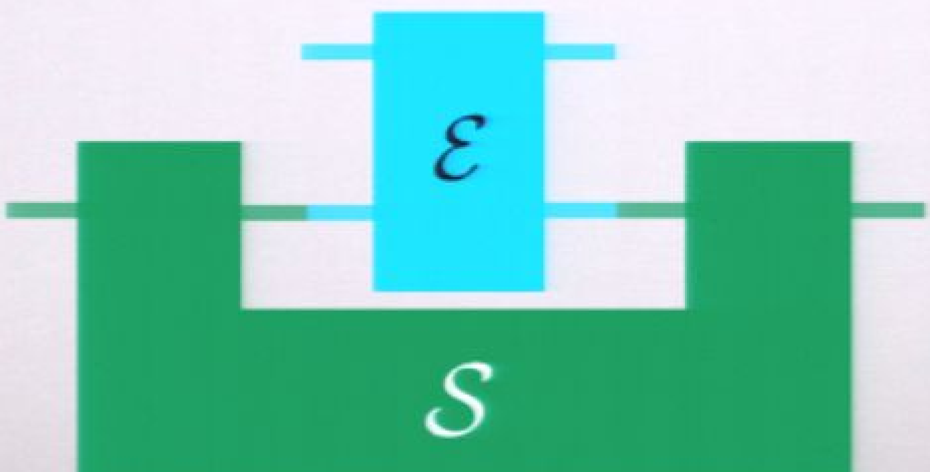

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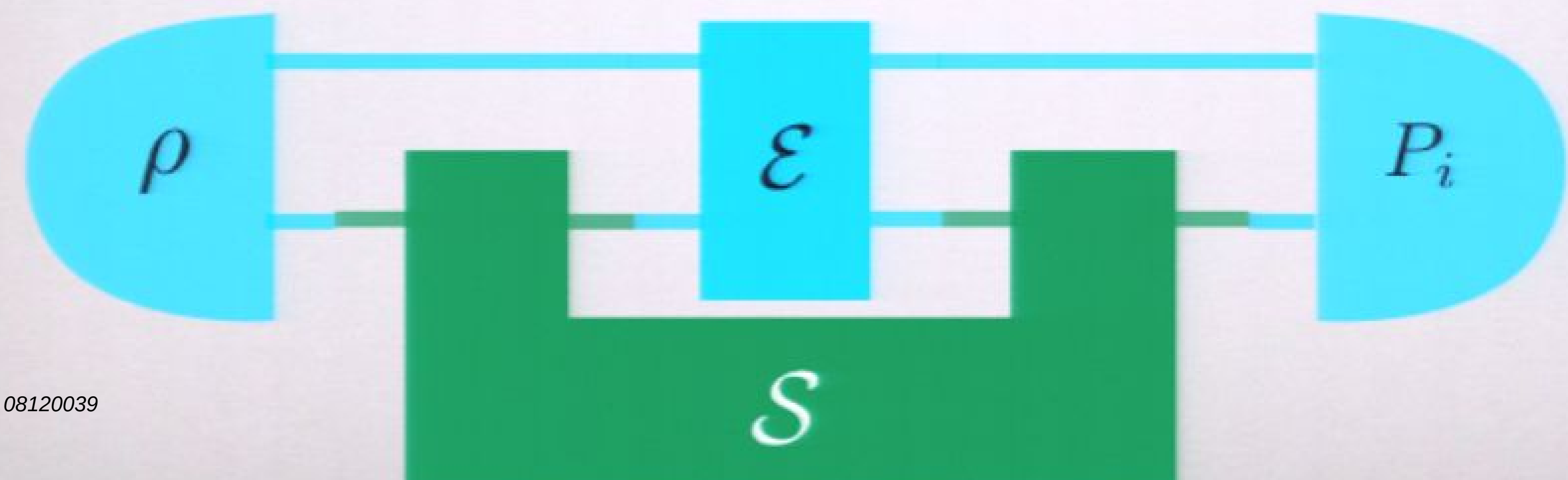

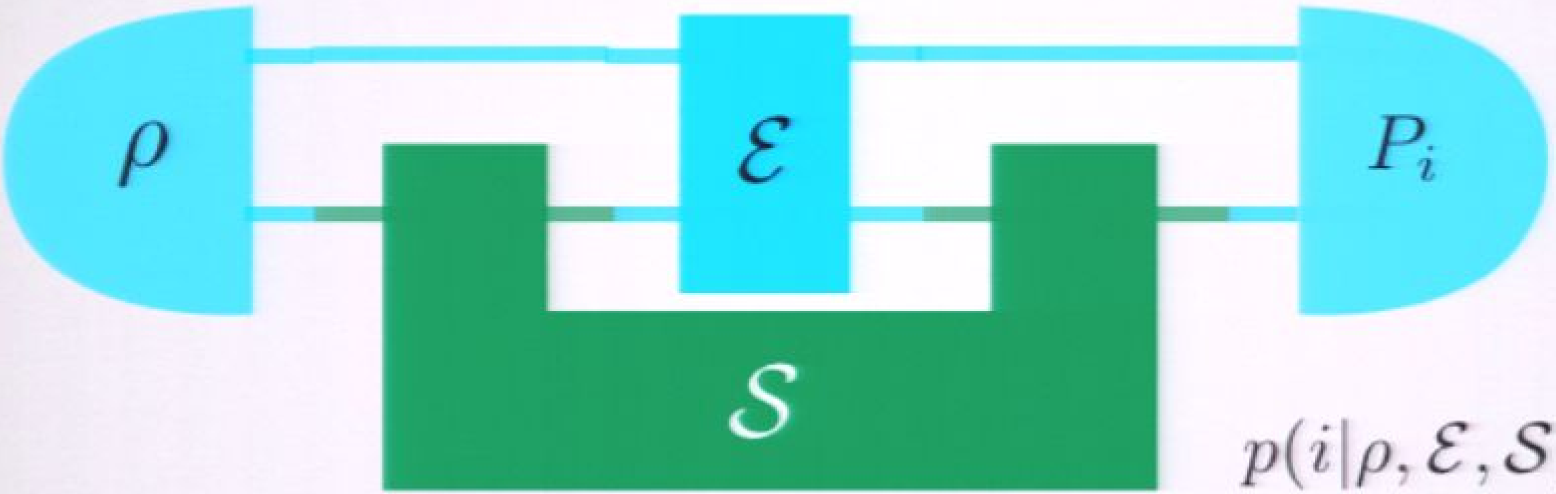


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Deterministic transformation: all channels are mapped into channels

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Theorem:
any admissible transformation can be realized by a quantum circuit consisting in

- a pre-processing channel [from the new input to the old input + ancilla]
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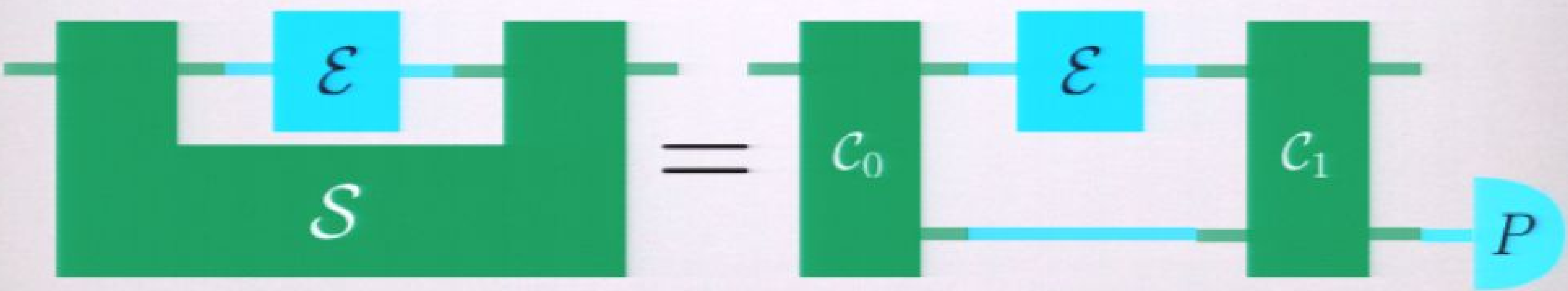





GC, G M D'Ariano, and P Perinotti, Europhys. Lett. 83, 30004 (2008)

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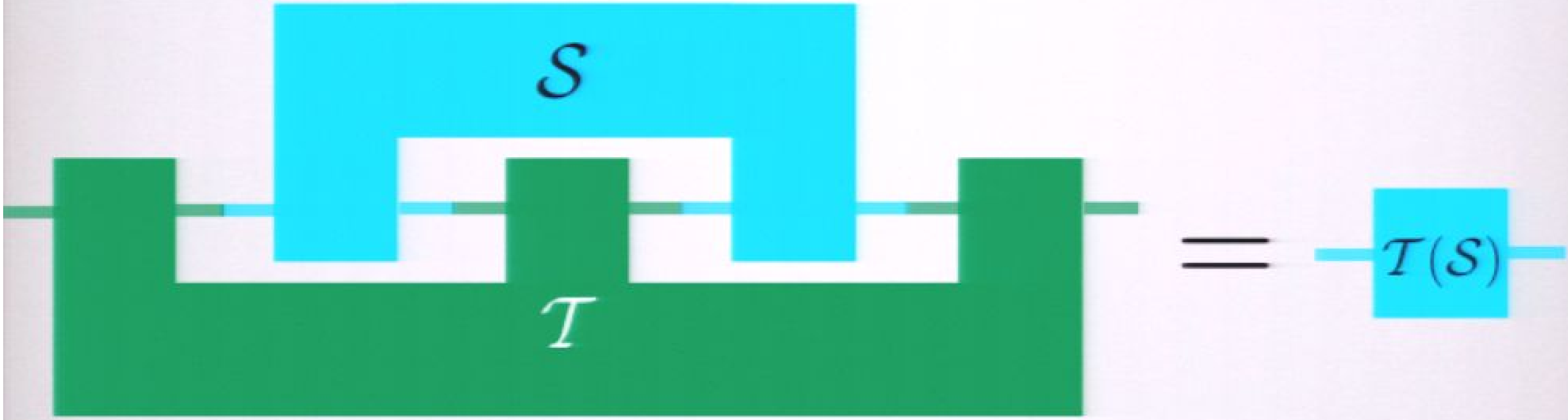


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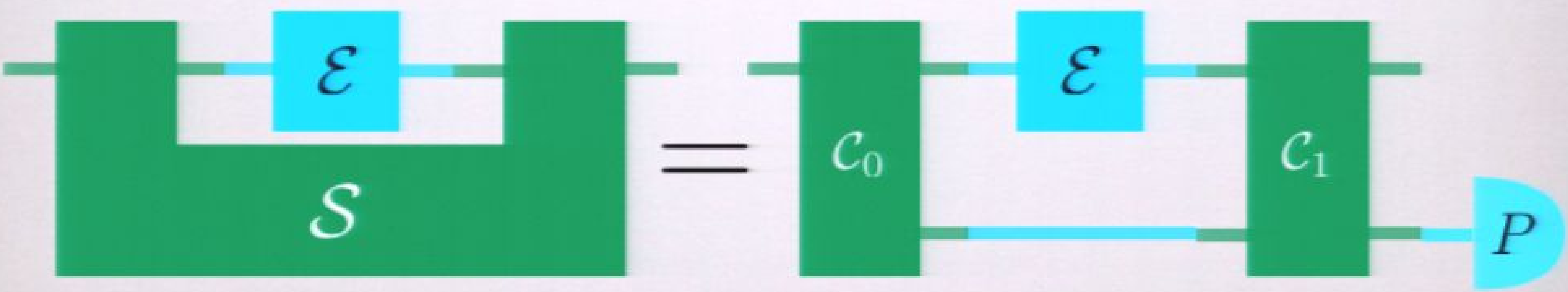





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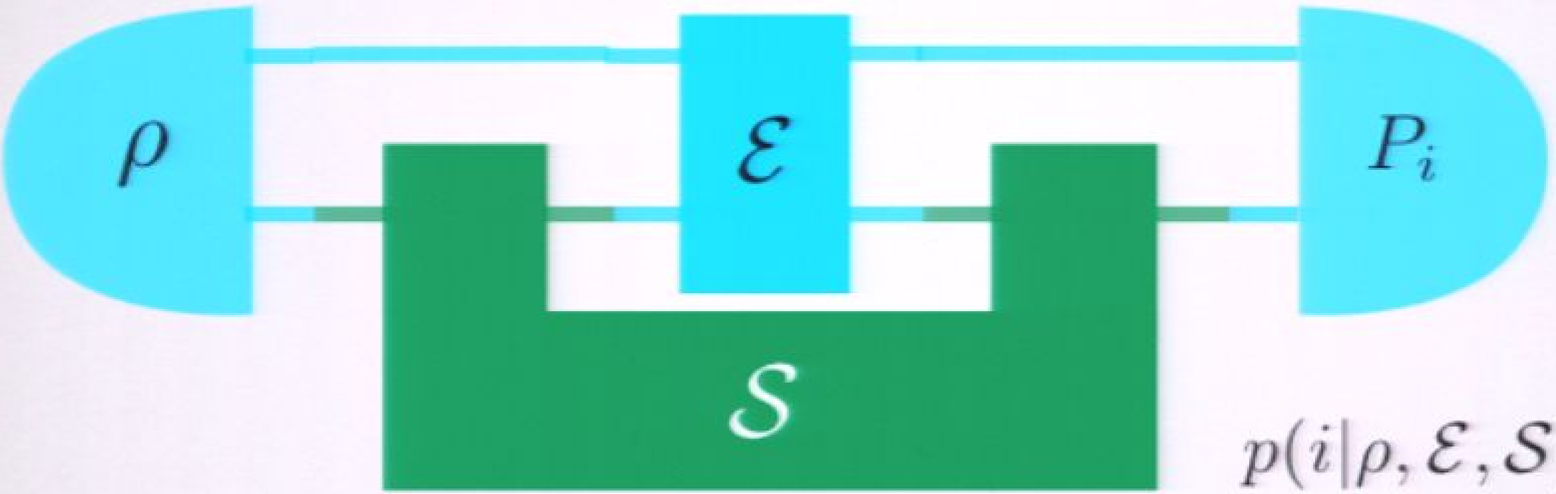


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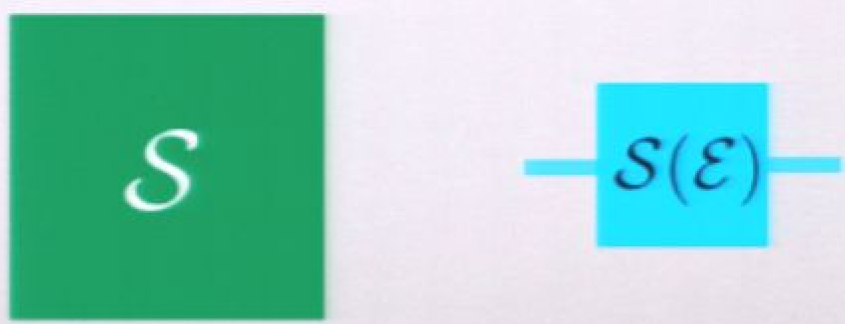

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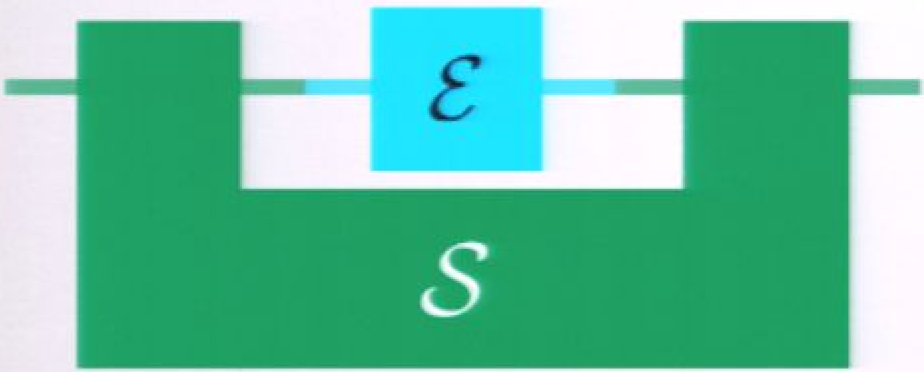


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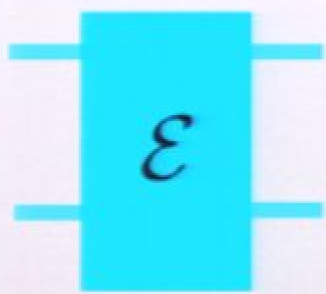

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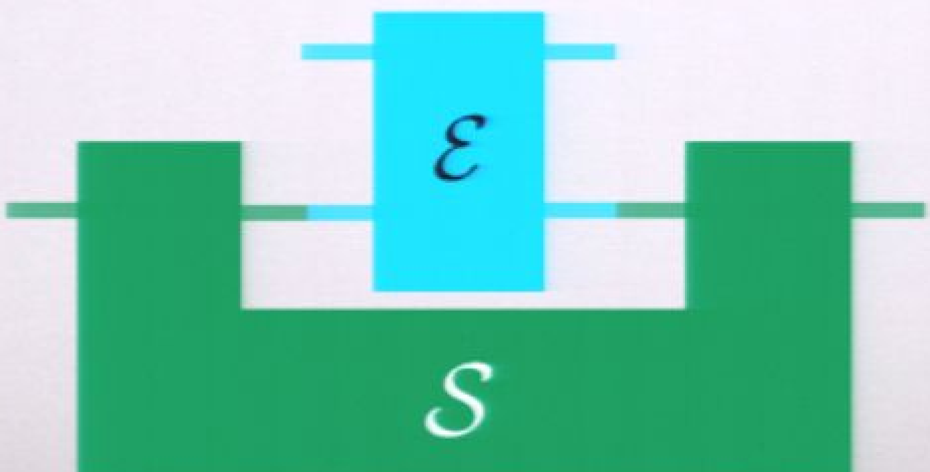

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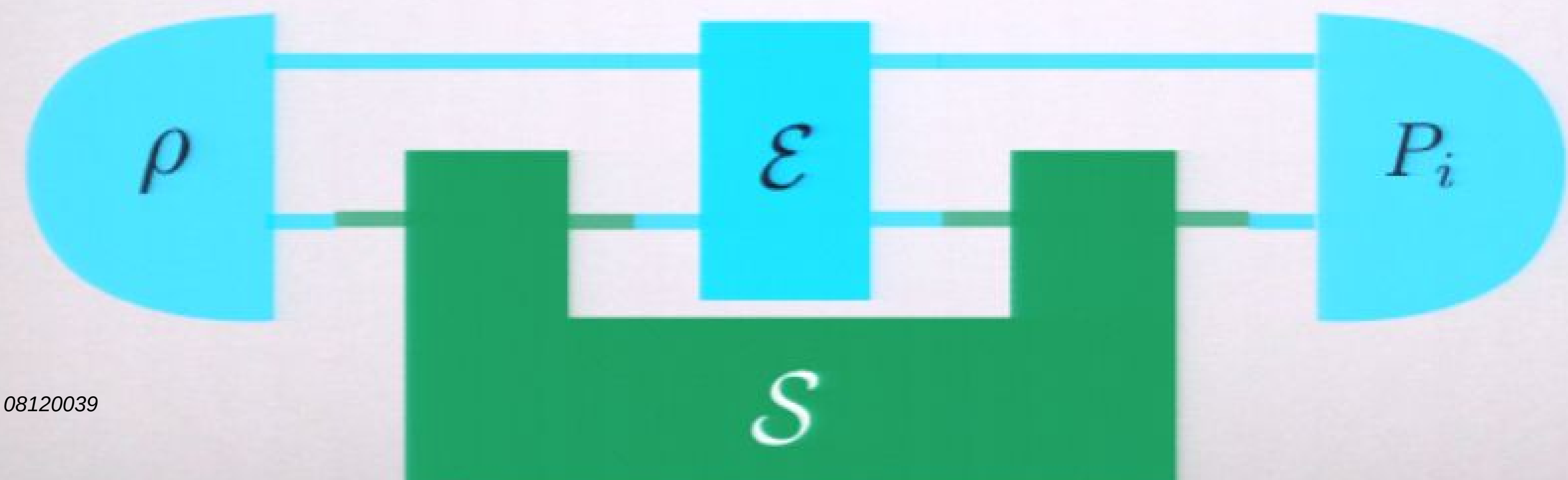

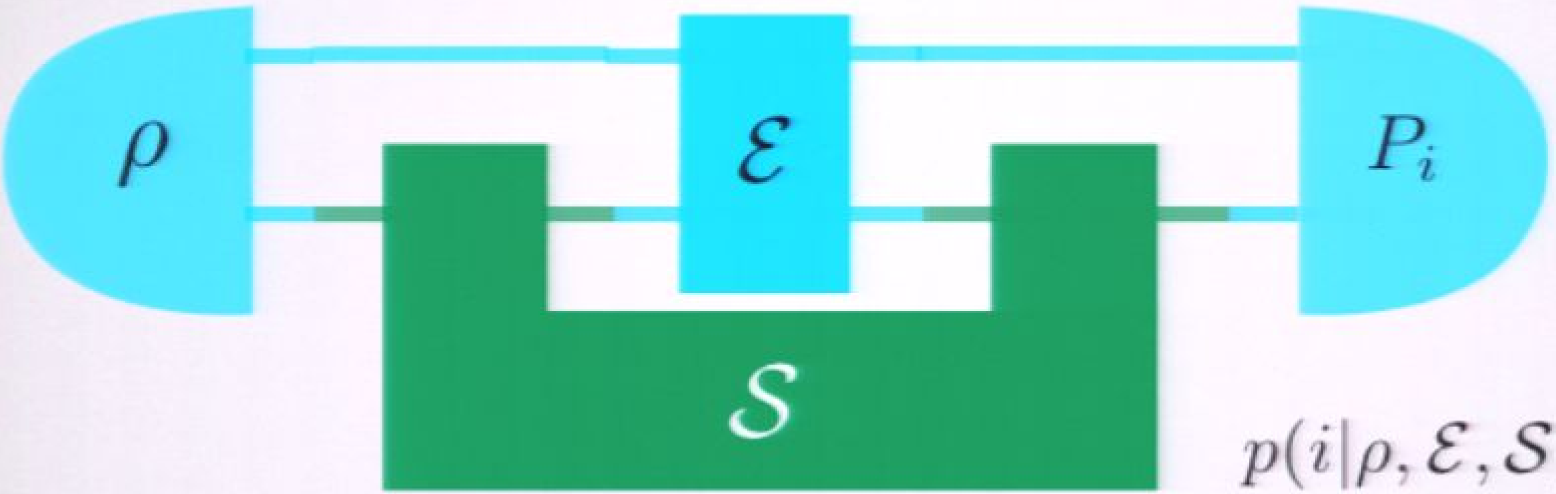


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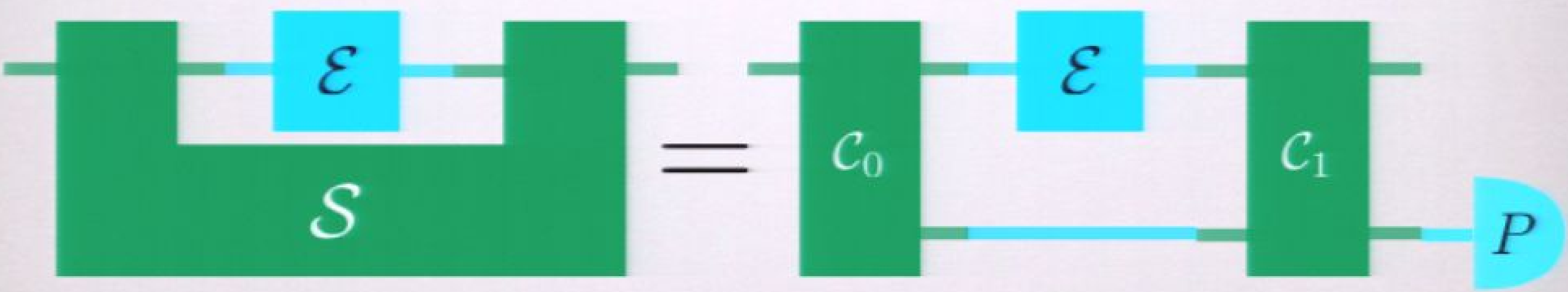





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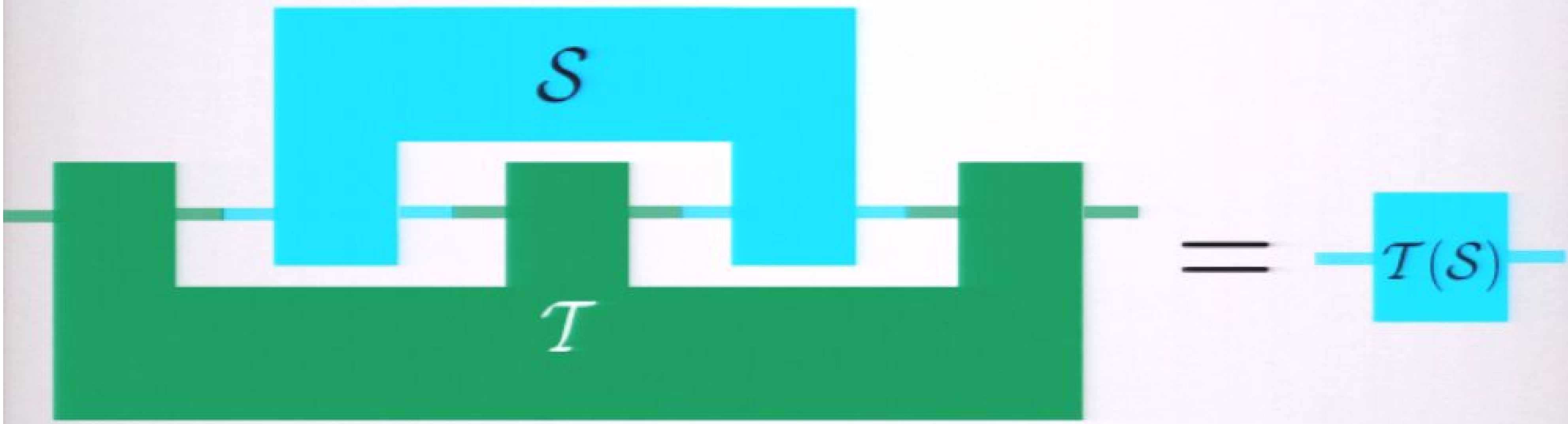


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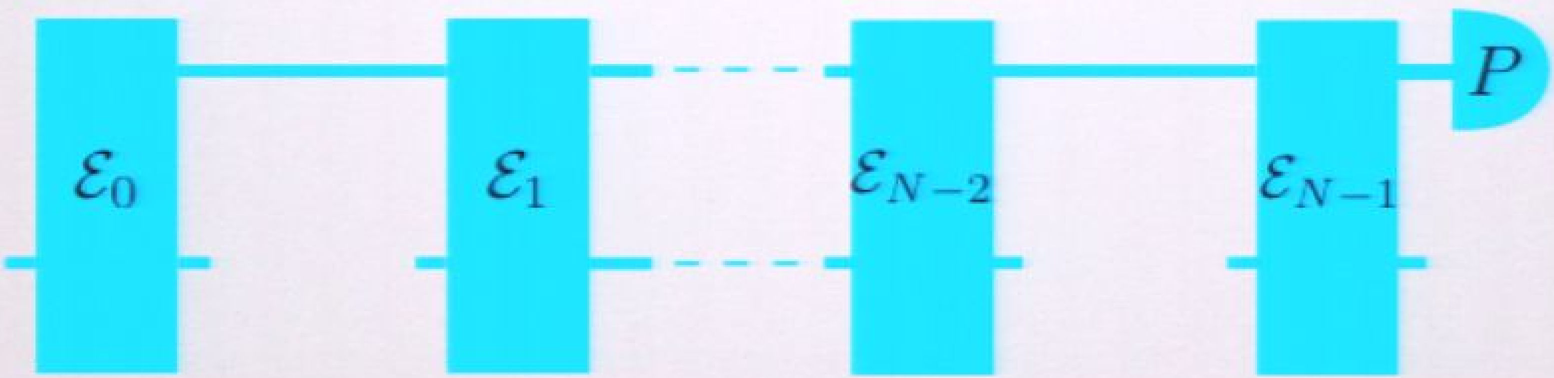

# CIRCUITAL REALIZATION OF ADMISSIBLE N-MAPS

Theorem:
any admissible N-map can be realized by a sequential network of quantum channels with memory, followed by a measurement on an ancilla.

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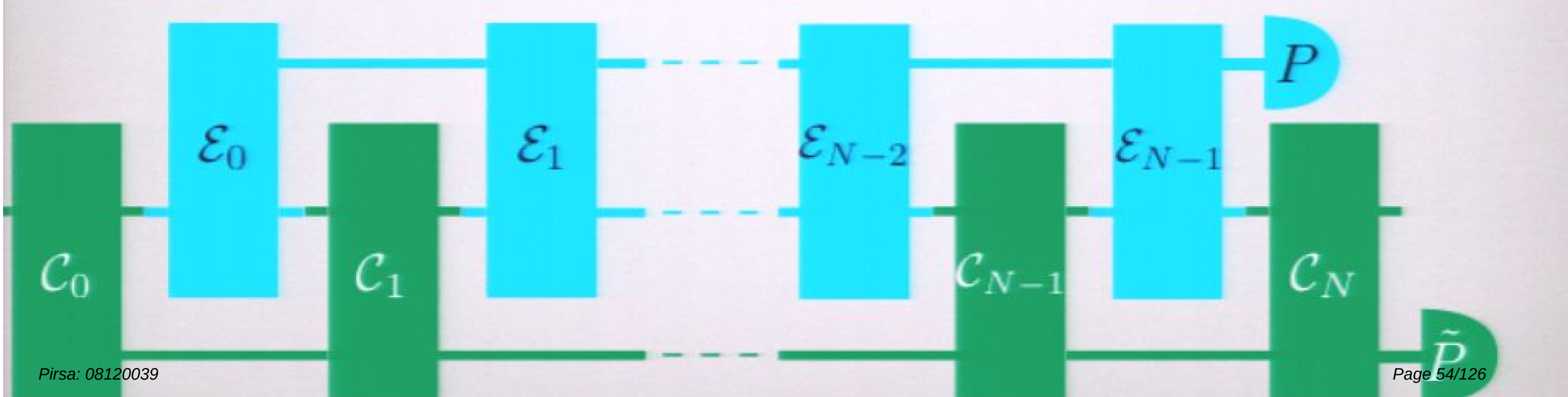


# QUANTUM TESTERS

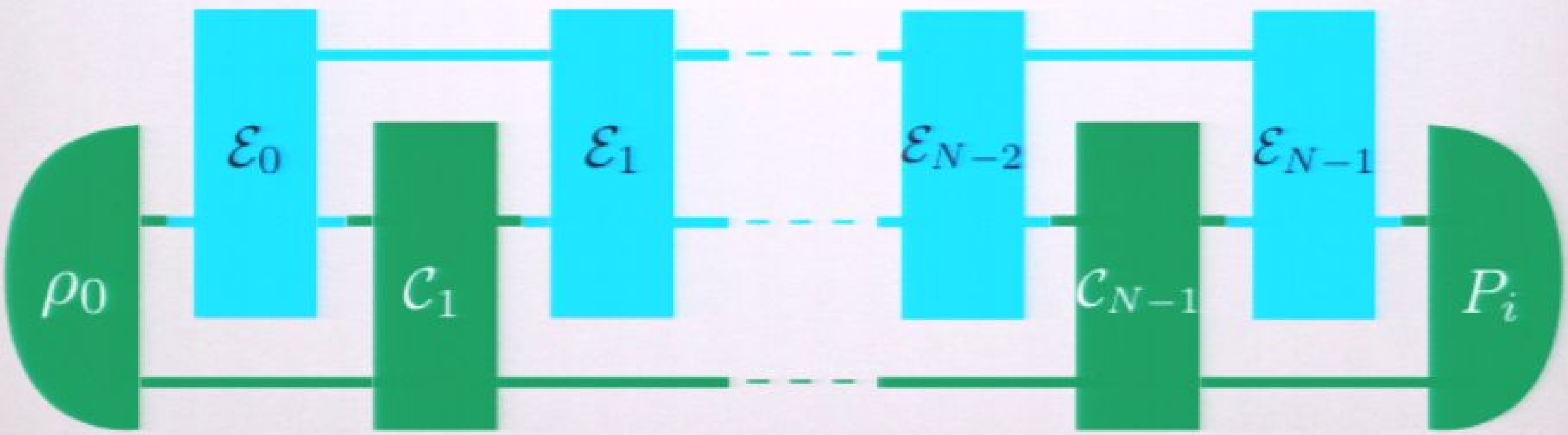
Interesting case: collections of N-maps that transform (N-1)-maps into probabilities:

$$p_i = \mathcal{T}_i^{(N)}\left(\mathcal{S}^{(N-1)}\right), \qquad \sum_i p_i = 1$$

A collection of this kind must satisfy

$$\sum_i \mathcal{T}_i^{(N)} = \mathcal{T}^{(N)} \quad \mathcal{T}^{(N)} \text{ deterministic N} - \text{map}$$

Realization theorem for testers:



# SUMMARY OF PART I

- In Quantum Mechanics the only admissible N-maps are the obvious ones: sequential networks of QOs
  [open question: is this property generic for any probabilistic theory?]

- For quantum N-maps the transformation and the transformed object are of the same kind.

- All that matters is the interlinking of quantum (sequential) networks

Aim of the next part: providing an efficient method for treating quantum networks and their interlinking

# CHOI REPRESENTATION

Convenient representation of linear maps: Choi-Jamiolokwski operator (in infinite dimensions Belavkin-Staszewski)

$$E = (\mathcal{E} \otimes I)(\Omega) \qquad |\Omega\rangle\rangle = \sum_{n=1}^{d} |n\rangle|n\rangle$$

$$\mathcal{E} \in Lin(Lin(\mathcal{H}_{in}), Lin(\mathcal{H}_{out})) \iff E \in Lin(\mathcal{H}_{out} \otimes \mathcal{H}_{in})$$

Completely positive map $\iff$ positive Choi operator

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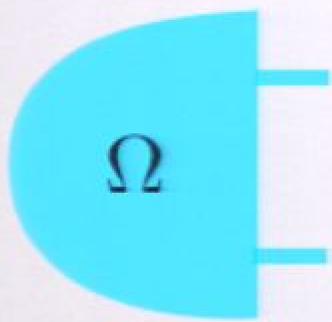


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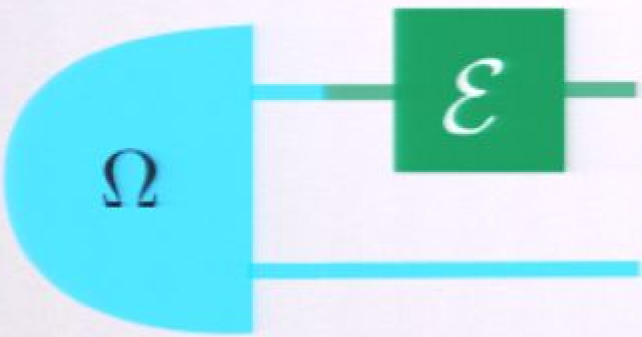


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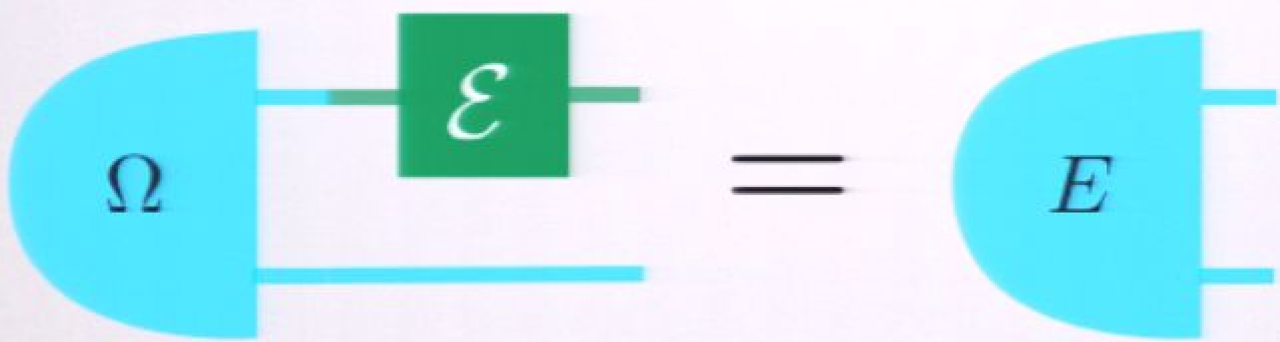


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# LINK PRODUCT

Convenient representation of composition of linear maps: link product

$$\mathcal{F} \circ \mathcal{E} \Longleftrightarrow F_{cb} * E_{ba} := \mathrm{Tr}_b[(F_{cb} \otimes I_a)(I_c \otimes E_{ba}^{\tau_b})]$$
$$= \mathrm{Tr}_{b'b}[(F_{cb'} \otimes E_{ba})(I_c \otimes \Omega_{b'b} \otimes I_a)]$$

$F_{cb} * E_{ba} = E_{ba} * F_{cb}$ up to permutation of Hilbert spaces



GC, G M D'Ariano, and P Perinotti, Phys. Rev. Lett. 101, 060401 (2008)

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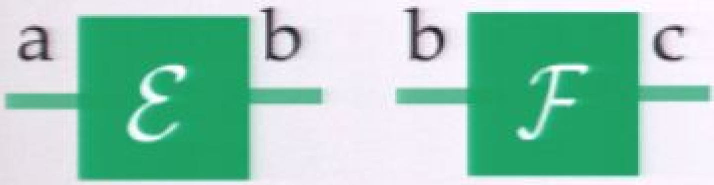


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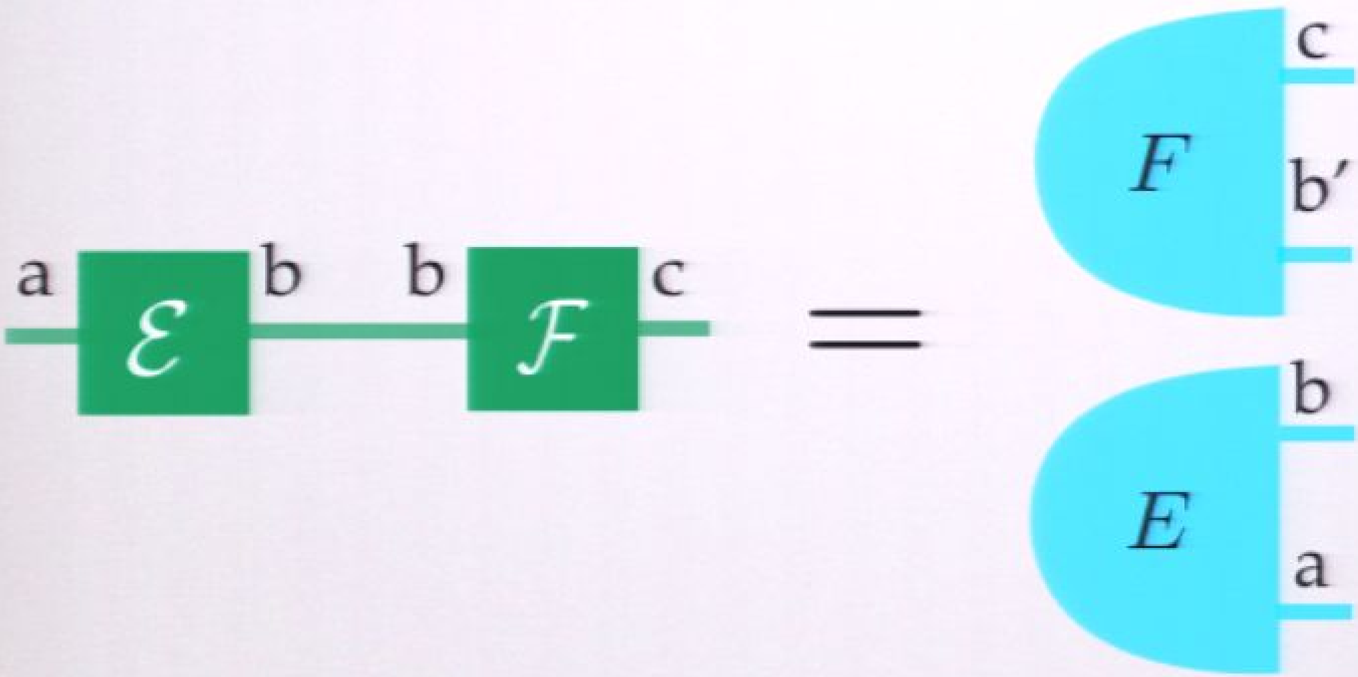


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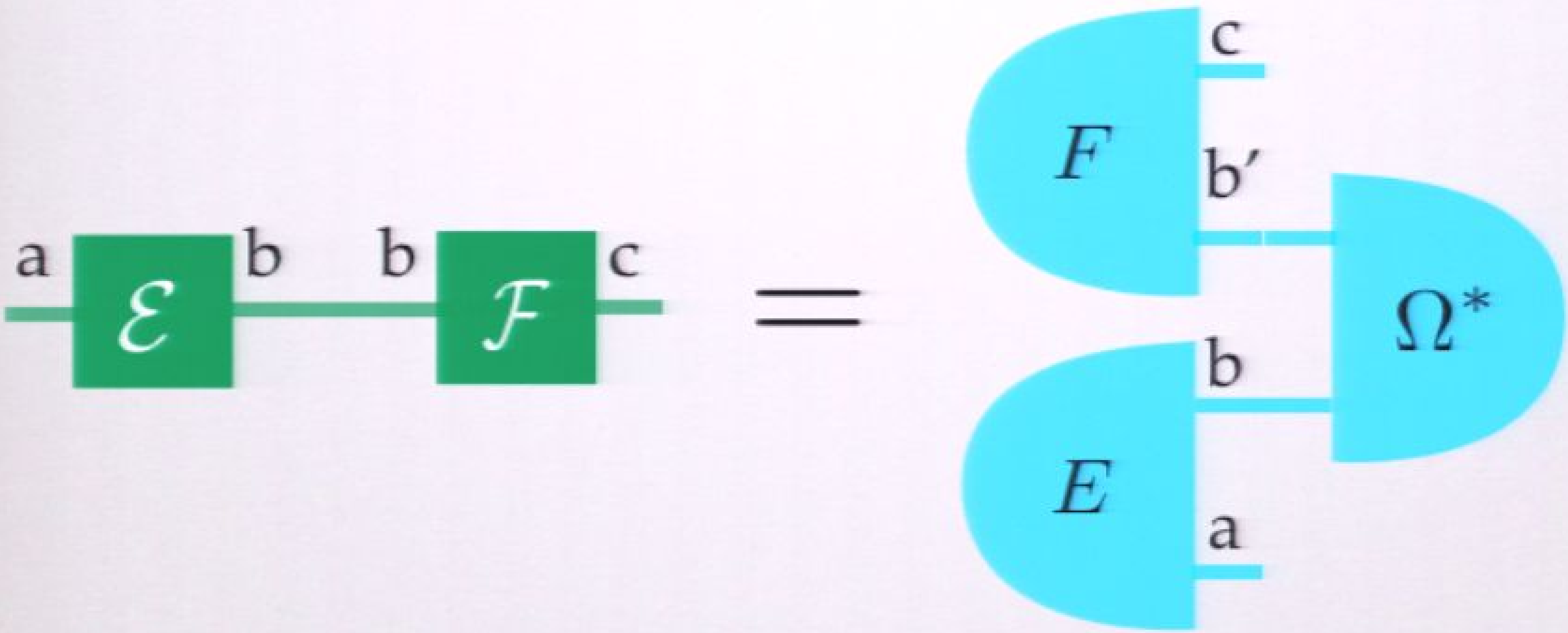


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# KNOWN FORMULAS IN TERMS OF LINK PRODUCT

- Tensor product of states:

$$\rho_a \otimes \sigma_b = \rho_a * \sigma_b$$

- Born statistical formula:

$$\mathrm{Tr}[\rho P] = \rho_a * P_a^\tau$$

- Transformation of states:

$$\mathcal{E}(\rho) = E_{out,in} * \rho_{in}$$

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Is this a state
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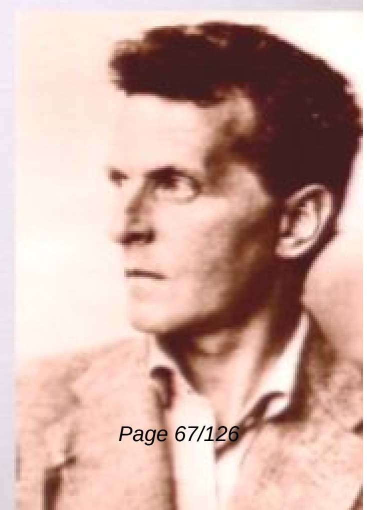
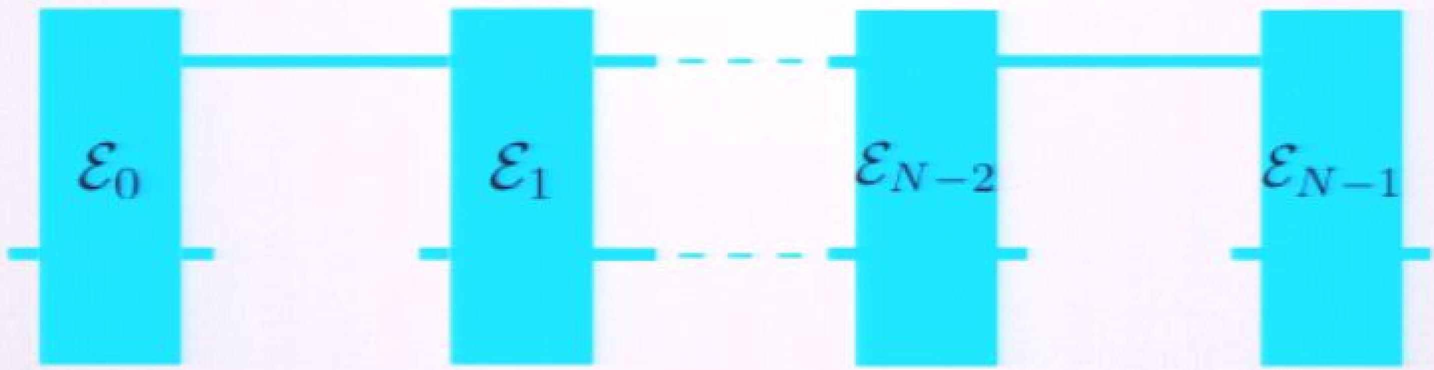


# CHOI OPERATOR OF A QUANTUM NETWORK

We are interested in sequential networks of quantum operations:

$$S^{(N)} = E_0 * E_1 * \cdots * E_{N-2} * E_{N-1}$$



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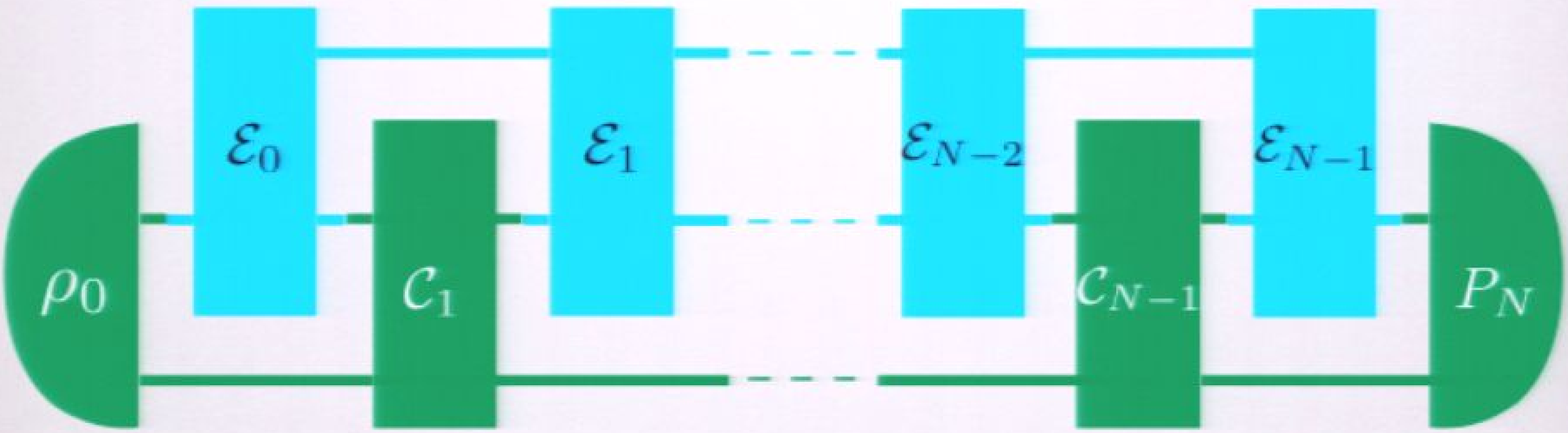


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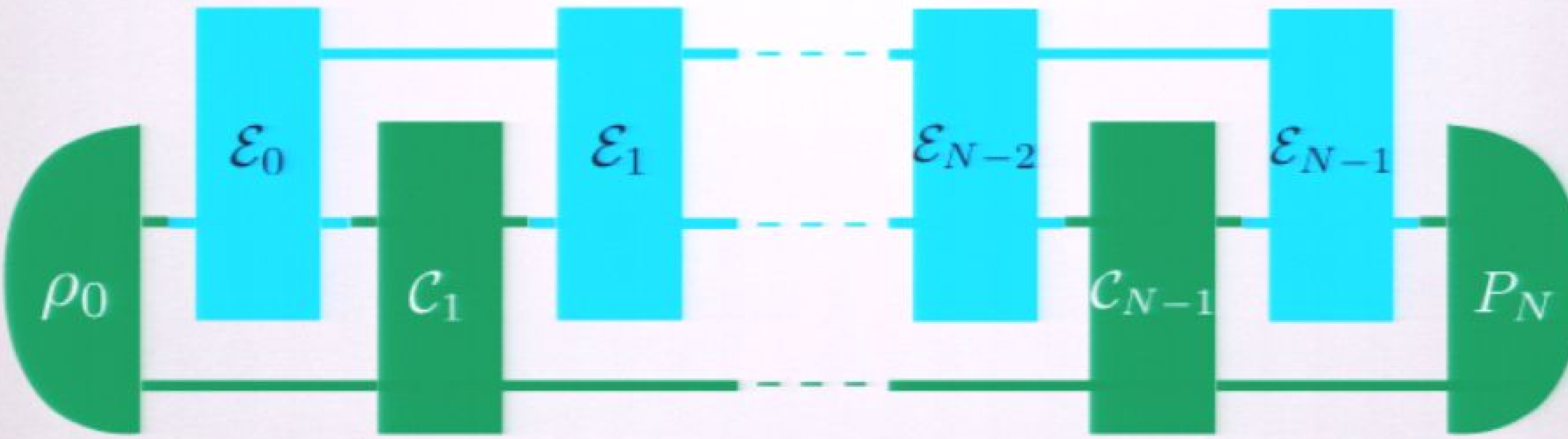



$$T^{(N+1)} = \rho_0 * C_1 * \cdots * C_{N-1} * P_N$$

Born rule for probabilities: $p = S^{(N)} * T^{(N+1)}$

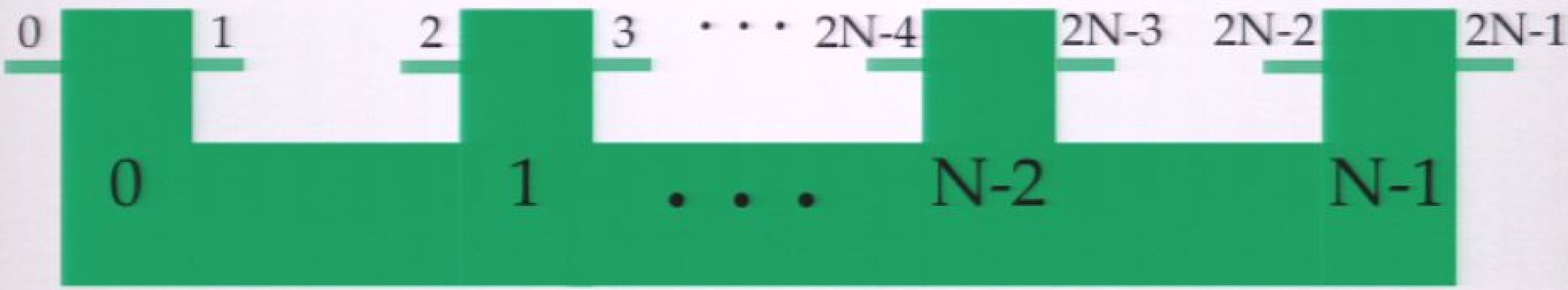
In any possible experiment, the probabilities depend only on the Choi operator, and not on the internal structure of the network.

# QUANTUM COMBS

For many purposes, the complete specification of all QOs in a network is a superfluous information: it is sufficient to give the Choi operator.

Quantum N-comb = equivalence class of networks of N QOs that have the same external systems and the same Choi operator

Diagrammatic representation of N-combs:




Choi operator:

$$S^{(N)} \in Lin\left(\bigotimes_{j=0}^{2N-1} \mathcal{H}_j\right) \qquad S^{(N)} \geq 0$$

# DETERMINISTIC AND PROBABILISTIC COMBS

- Deterministic N-combs = networks of N channels with memory = deterministic N-maps

Recursive normalization of deterministic combs:

$$\mathrm{Tr}_{2N-1}[S^{(N)}] = I_{2N-2} \otimes S^{(N-1)}$$

or else,

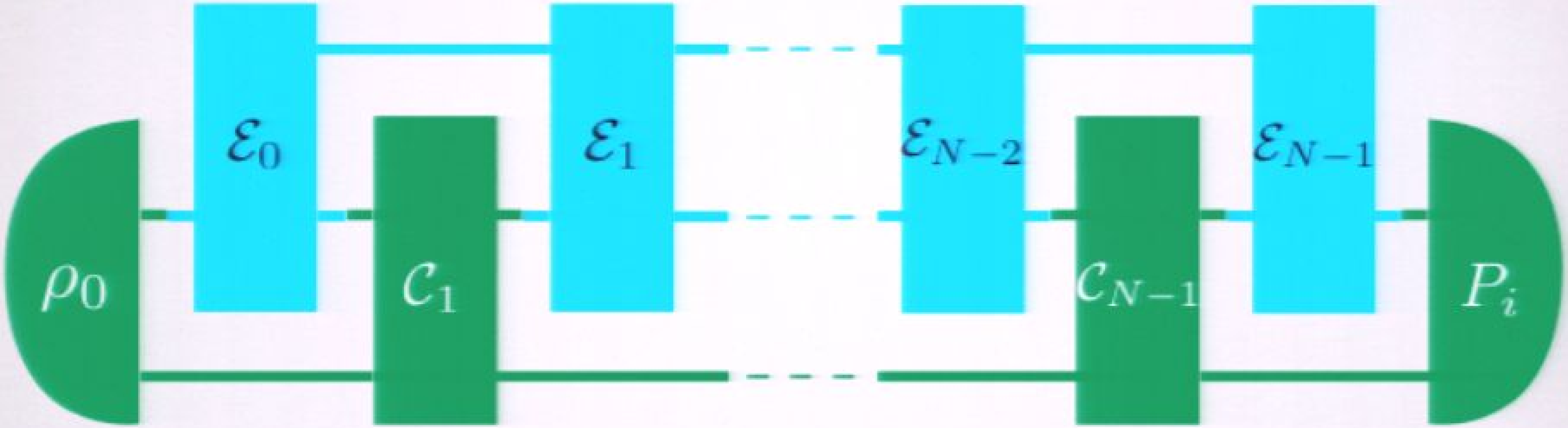
$$I_{2N-1} * S^{(N)} = I_{2N-2} * S^{(N-1)}$$

- Probabilistic N-combs = networks of N QOs with memory = probabilistic N-maps

An operator $S^{(N)}$ is a probabilistic N-comb is there exists a deterministic N-comb $T^{(N)}$ such that $S^{(N)} \leq T^{(N)}$

G Gutoski and J Watrous, STOC 2007, 565
GC, G M D'Ariano, and P Perinotti, Phys. Rev. Lett. 101, 060401 (2008)

# QUANTUM TESTERS




Quantum tester = quantum network beginning with a state preparation and ending with a measurement = collection of positive operators with suitable normalization.

$$\{T_i^{(N)}\} \qquad T_i^{(N)} \geq 0 \quad \sum_i T_i = \langle T^{(N)} \rangle$$

Born rule for quantum networks: $p_i = \mathrm{Tr}[S^{(N)} T_i^{(N)}]$

# DECOMPOSITION OF QUANTUM TESTERS

## Theorem

Any tester can be split into two parts

- a deterministic map transforming quantum networks into states
- a quantum measurement

in the following way:

$$p_i = \mathrm{Tr}[T_i S] = \mathrm{Tr}\left[\mathcal{T}(S)\ P_i\right]$$

$$\mathcal{T}(S) = \langle T \rangle^{\frac{1}{2}} S\ \langle T \rangle^{\frac{1}{2}} \in \mathcal{S}\left(\bigotimes_{j=0}^{2N-1} \mathcal{H}_j\right)$$

$$\{P_i\} = \text{quantum measurement (for states)}$$

## Operational distance between two quantum networks:

$$d_{op}(S_0, S_1) = \sup_{\langle T \rangle} \| \sqrt{\langle T \rangle}(S_0 - S_1)\sqrt{\langle T \rangle} \|$$




GC, G M D'Ariano, and P Perinotti, Phys. Rev. Lett. 101, 180501 (2008)

# APPLICATION I: OPTIMAL GATE ESTIMATION

Problem: a black box performs a transformation belonging to a given symmetry group.
Suppose we have N uses of it at disposal:

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Problem: a black box performs a transformation belonging to a given symmetry group.
Suppose we have N uses of it at disposal:


Which is the best way to estimate g?
that is, Which is the best way to connect the boxes?
and Which is the ultimate precision we can reach?

Examples: quantum interferometry [U(1)],
estimation of rotations [SO(3)]
full gate estimation [SU(d)]

## Parallel architectures:

In this case the optimal strategy (optimal input state + optimal measurement) is known:

Phase estimation: Buzek, Derka, Massar, Phys. Rev. Lett. 82, 2207 (1999)

Estimation of rotations: GC, G M D'Ariano, P Perinotti, and M F Sacchi, Phys. Rev. Lett 93, 180503 (2004)

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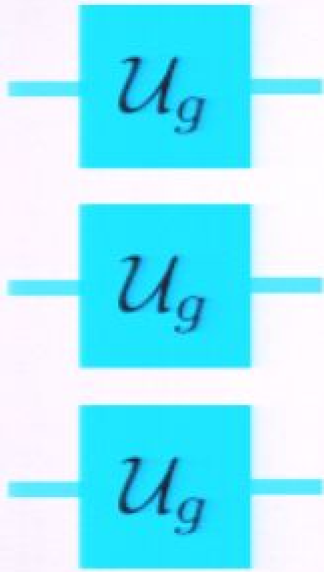


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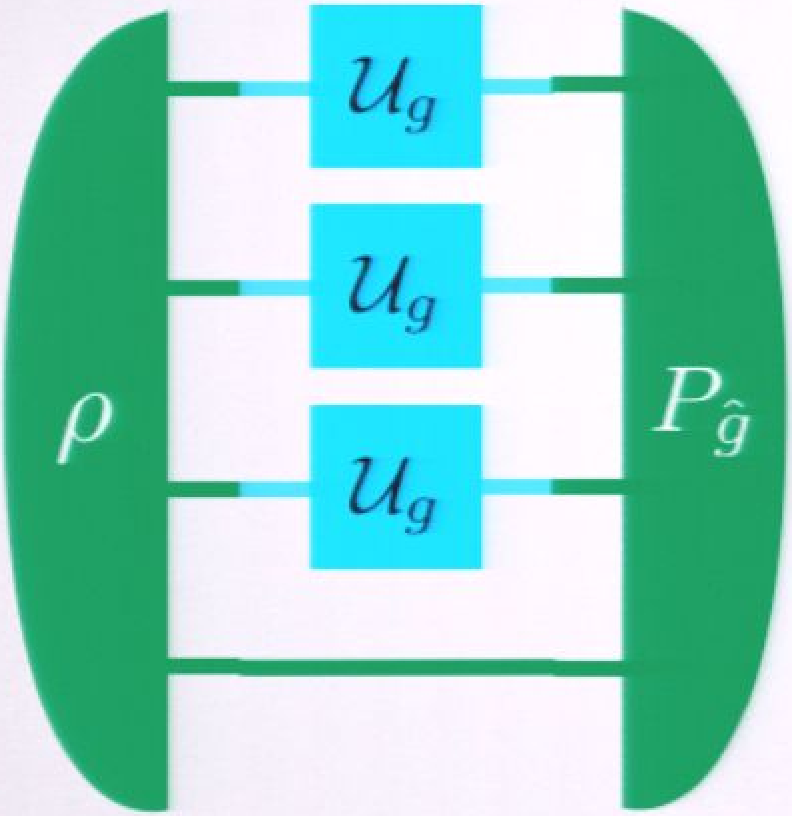


$$\hat{g}, \qquad p(\hat{g}|g) = \mathrm{Tr}[P_{\hat{g}}\rho_g]$$

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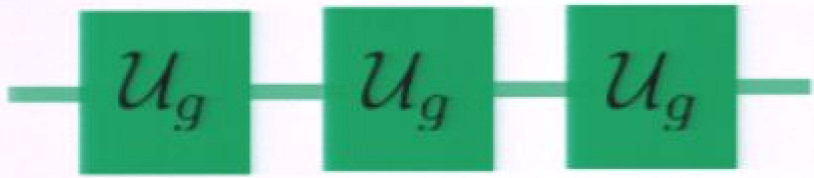

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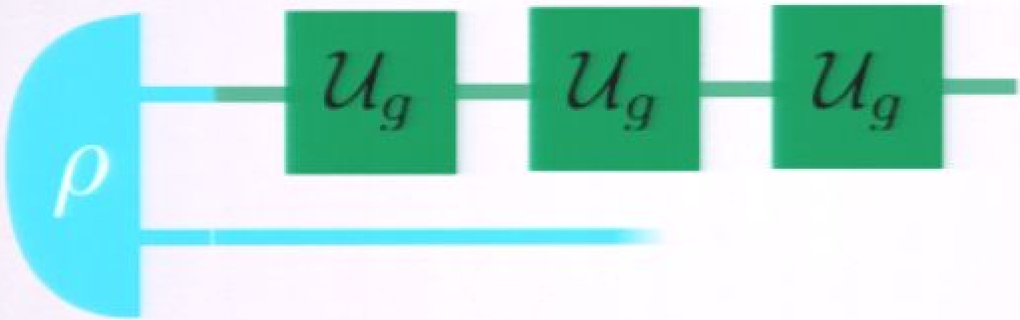

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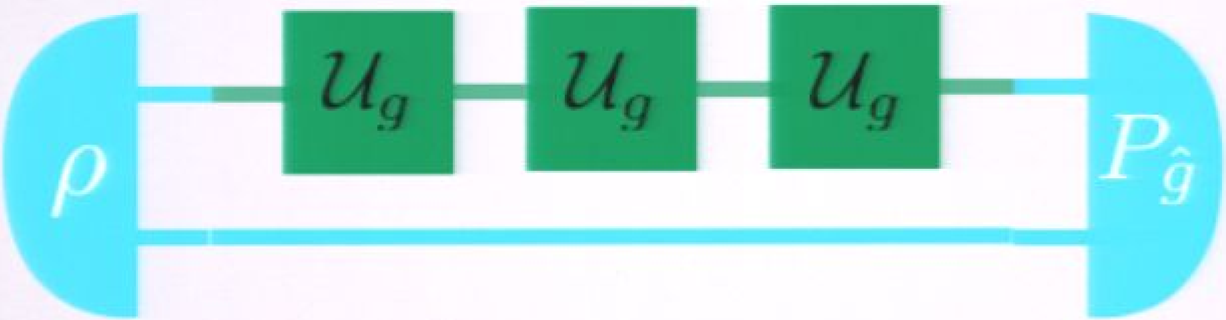


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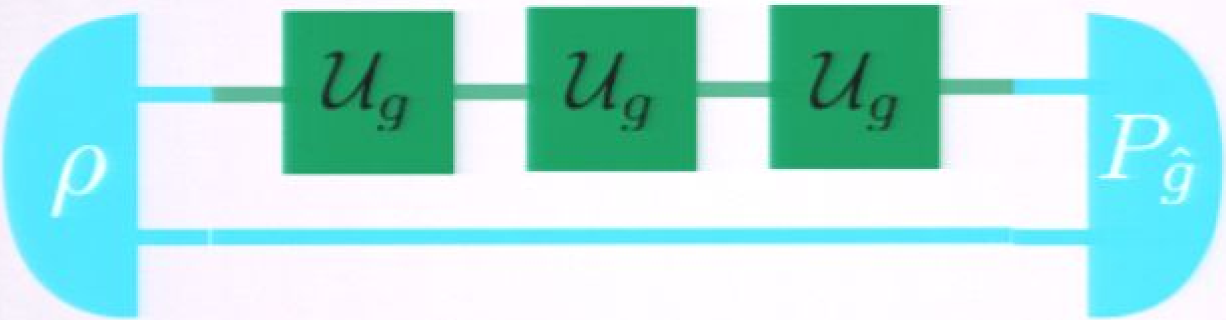


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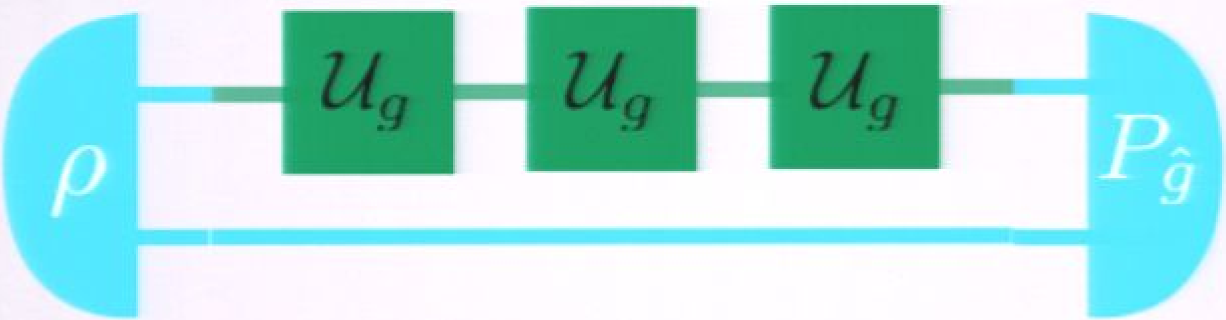


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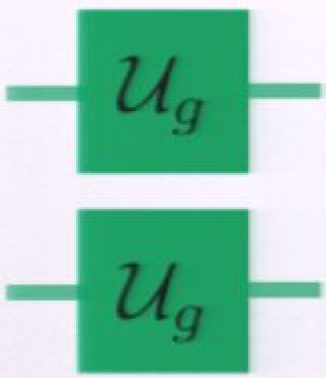

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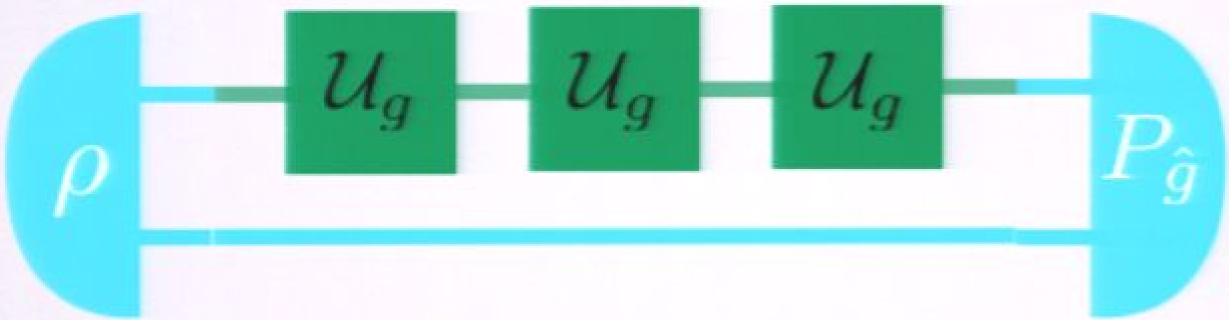


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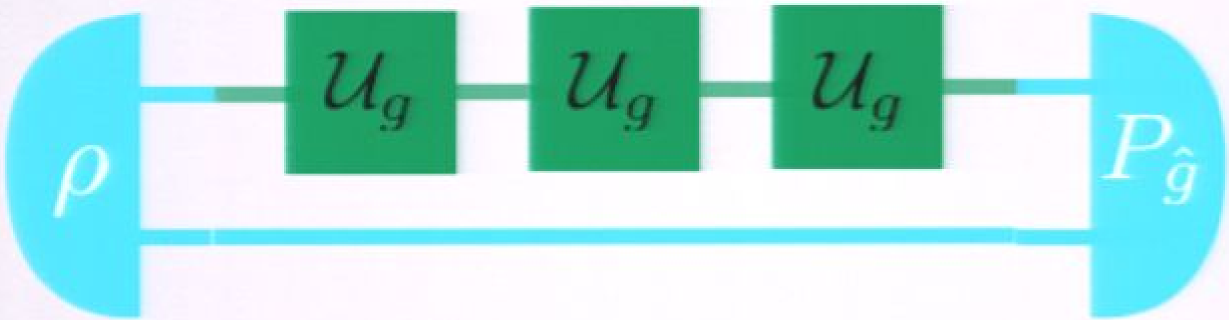


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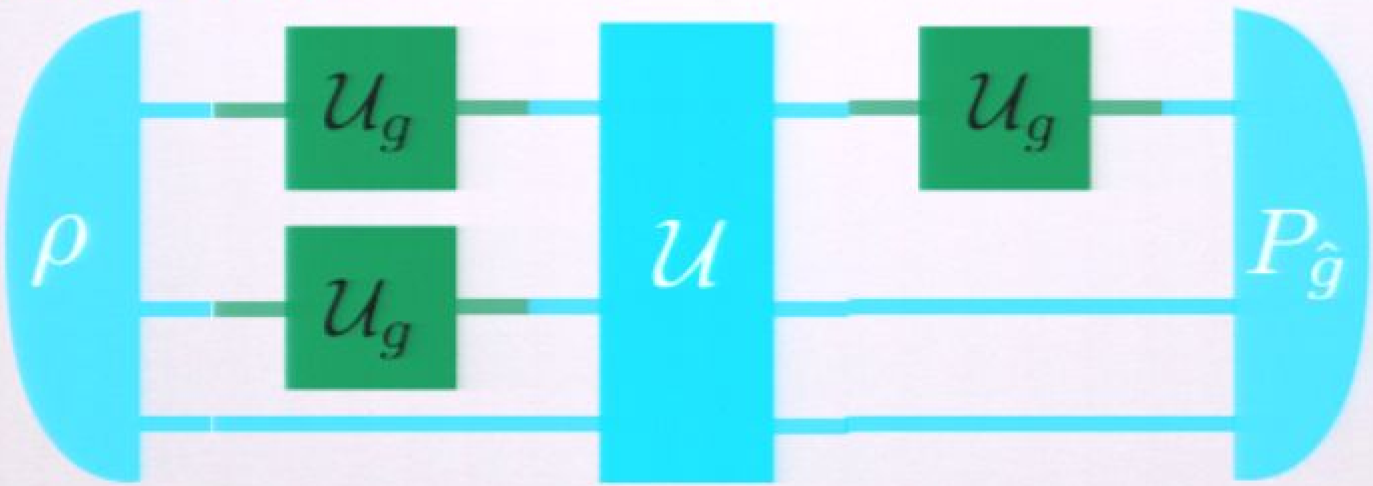

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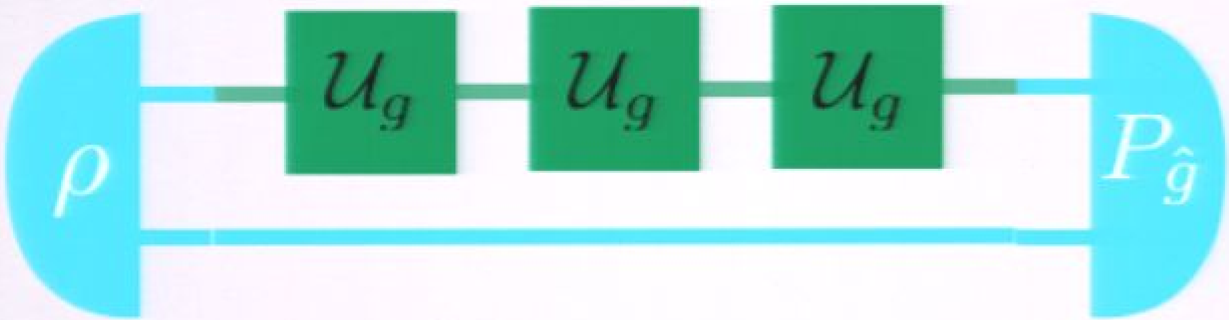


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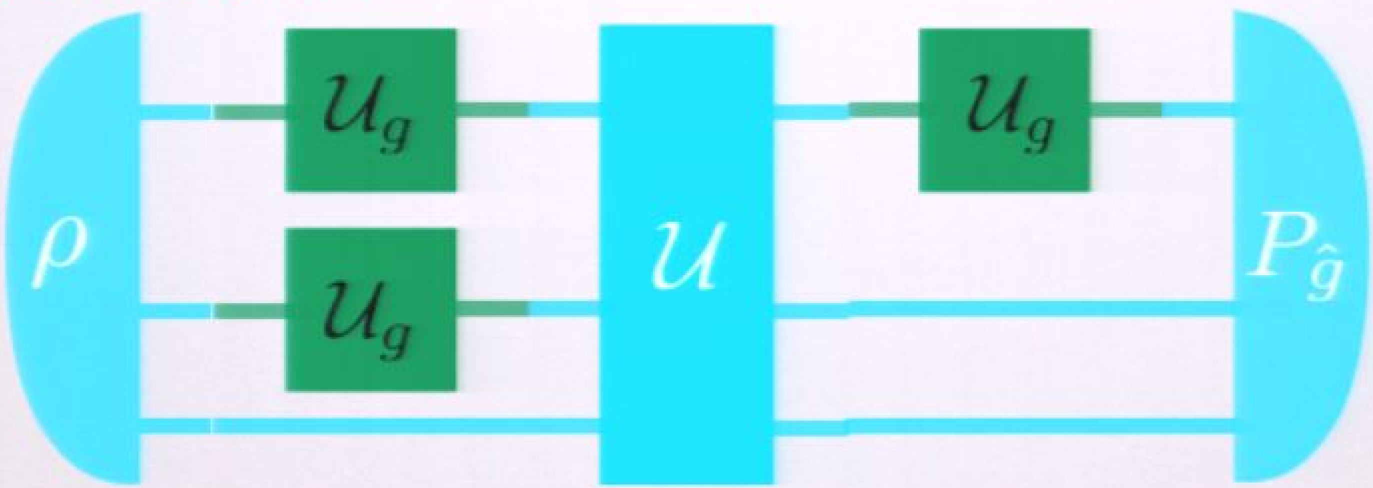


Example of optimization over all architectures:

optimal network for phase estimation [van Dam, D'Ariano, Ekert, Macchiavello, Mosca, Phys. Rev. Lett. 98, 090501 (2007)]

# OPTIMAL TESTERS

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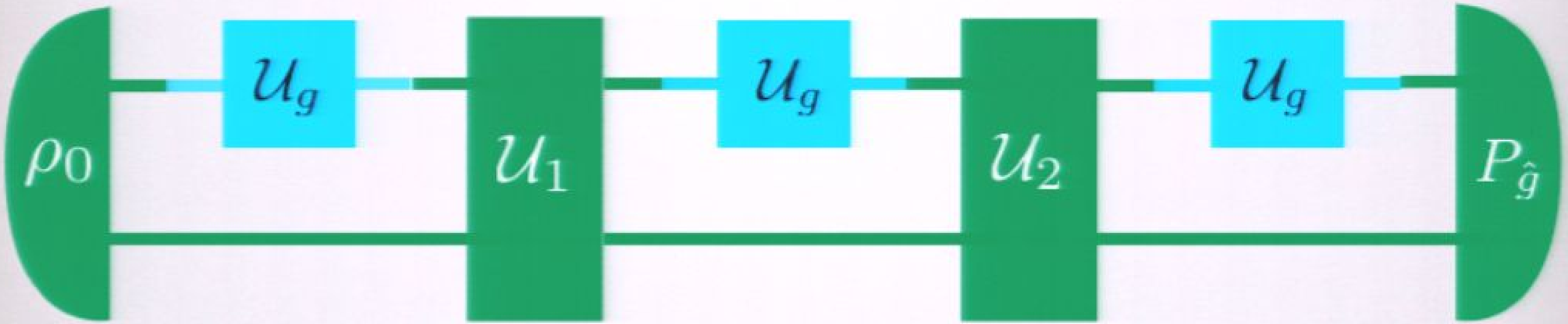

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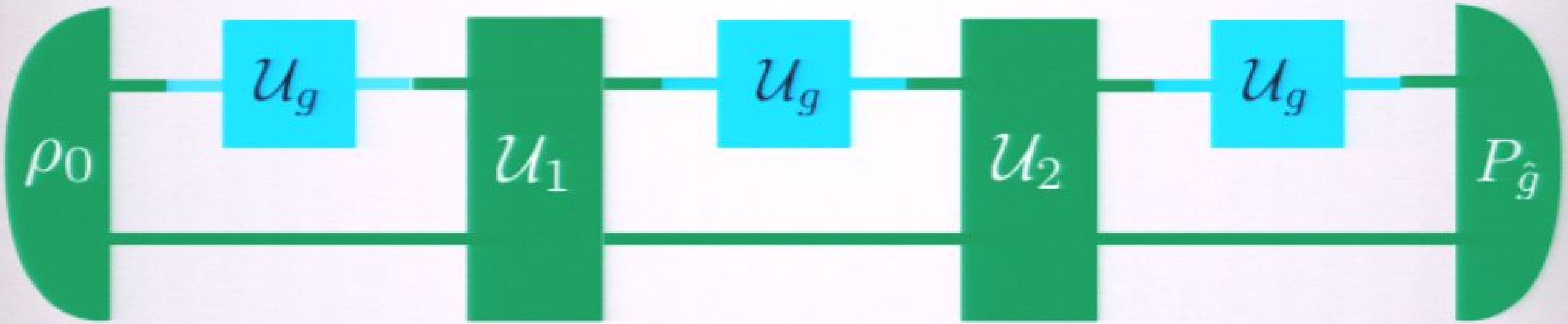



Choi operator of the measured network: $S_g = |U_g\rangle\rangle\langle\langle U_g|^{\otimes N}$

Tester of the measuring network: $T_{\hat{g}} = (U_{\hat{g}} \otimes I)^{\otimes N} T_0 (U_{\hat{g}} \otimes I)^{\dagger \otimes N}$

Normalization: $\langle T \rangle = \int \mathrm{d}\hat{g}\, T_{\hat{g}}$ $\qquad [\langle T \rangle, (U_g \otimes I)^{\otimes N}] = 0$

# OPTIMALITY PROOF FOR PARALLEL STRATEGIES

Decomposition of the tester: measurement on the quantum state

$$\mathcal{T}(S_g) = \langle T \rangle^{\frac{1}{2}} S_g \; \langle T \rangle^{\frac{1}{2}}$$

Since $\quad [\langle T \rangle, (U_g \otimes I)^{\otimes N}] = 0$

the state is of the form $\quad \rho_g = (U_g \otimes I)^{\otimes N} \; \rho_0 \; (U_g \otimes I)^{\dagger \otimes N}$

But this is the form the output states in a parallel architecture.

Conclusion: for any group G, the parallel architectures achieve the optimum among all possible architectures

# QUANTUM GYROSCOPES

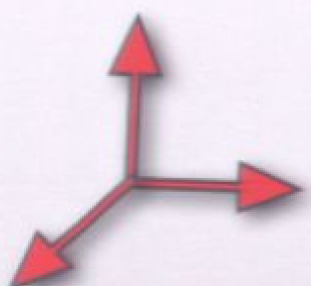
Spin $\frac{1}{2}$ particle, rotation $g \in \mathbb{SO}(3)$ $g = (\mathbf{n}, \varphi)$

State change: $U_g = e^{i\varphi \mathbf{n}\cdot\sigma} = \cos(\varphi/2) + i\sin(\varphi/2)\mathbf{n}\cdot\sigma$

encodes a spatial direction:

N qubits: $|A\rangle \in \mathcal{H}^{\otimes N}$ $|A_g\rangle = U_g^{\otimes N}|A\rangle$

encode a Cartesian frame:



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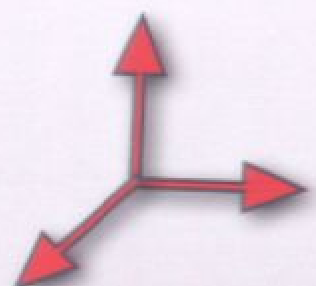

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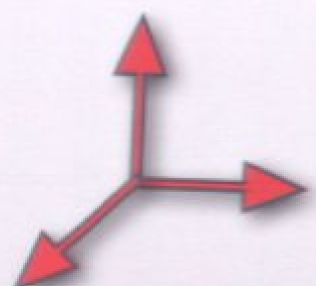


# QUANTUM GYROSCOPES

Spin $\frac{1}{2}$ particle, rotation $g \in \mathbb{SO}(3)$ $g = (\mathbf{n}, \varphi)$

State change: $U_g = e^{i\varphi \mathbf{n}\cdot\sigma} = \cos(\varphi/2) + i\sin(\varphi/2)\mathbf{n}\cdot\sigma$

encodes a spatial direction:

N qubits: $|A\rangle \in \mathcal{H}^{\otimes N}$ $|A_g\rangle = U_g^{\otimes N}|A\rangle$

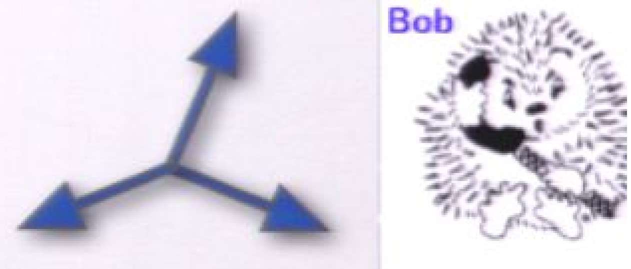
encode a Cartesian frame:

# ALIGNING AXES WITH QUANTUM GYROSCOPES

Suppose Alice and Bob have different Cartesian frames (different axes): a state that is $|A\rangle$ for Alice is $U_g|A\rangle$ for Bob.
However, using quantum communication they can try to establish a shared reference frame:

Alice


Bob




Problem: find the optimal quantum state and the optimal estimation strategy for aligning Cartesian frames

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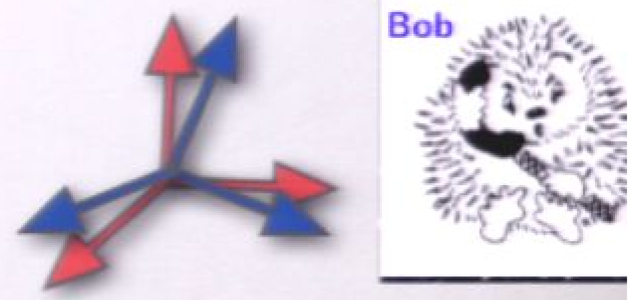


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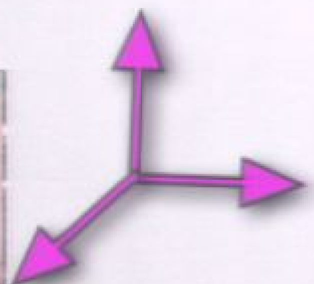


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# ULTIMATE PRECISION LIMITS FOR N PARTICLES

- For a quantum gyroscope made of N identical spin 1/2 particles:

$$\langle c \rangle \approx \sum_{i=x,y,z} \Delta\theta_i^2 = 3\Delta\theta_x^2 \approx \frac{2\pi^2}{N^2}$$

GC, G M D'Ariano, P Perinotti, and M F Sacchi, Phys. Rev. Lett 93, 180503 (2004)

However, this result is the optimal one
if we assume that Alice sends all particles in a single shot.

In other words, this result is about protocols with a single-round of forward quantum communication.

What about multi-round protocols?

# MULTI-ROUND ALIGNMENT PROTOCOLS

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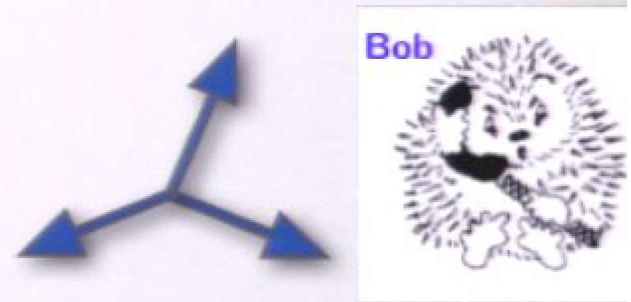


Allow
- unlimited amount of classical communication
- k rounds of quantum communication, in which batches of spin 1/2 particles are sent.

Then find the best way of estimating the mismatch of alignment.

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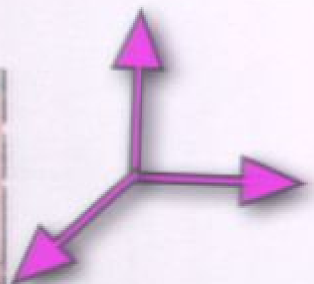

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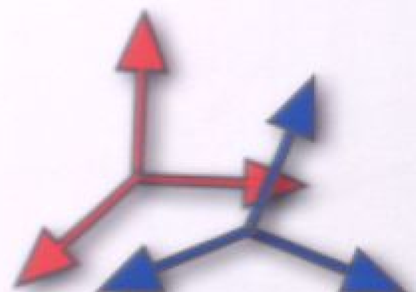


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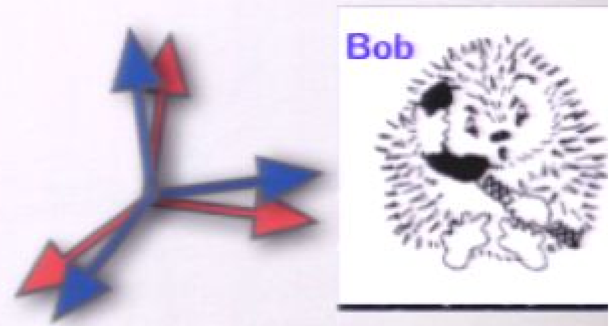


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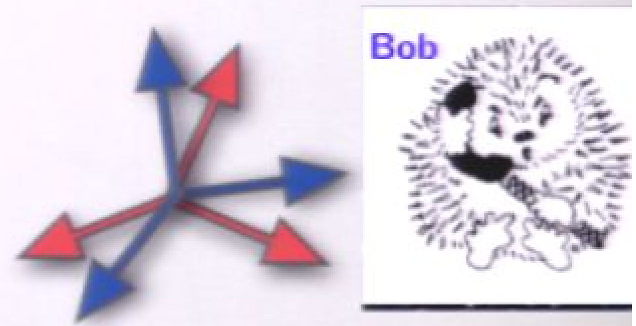


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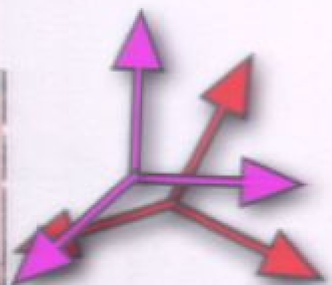

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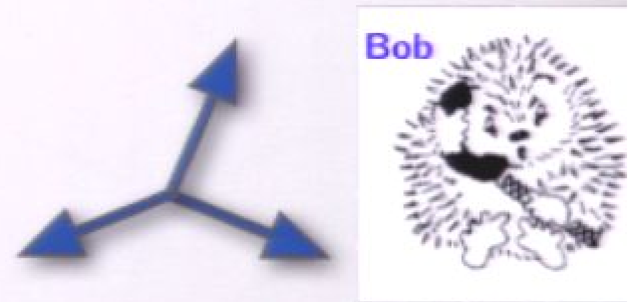


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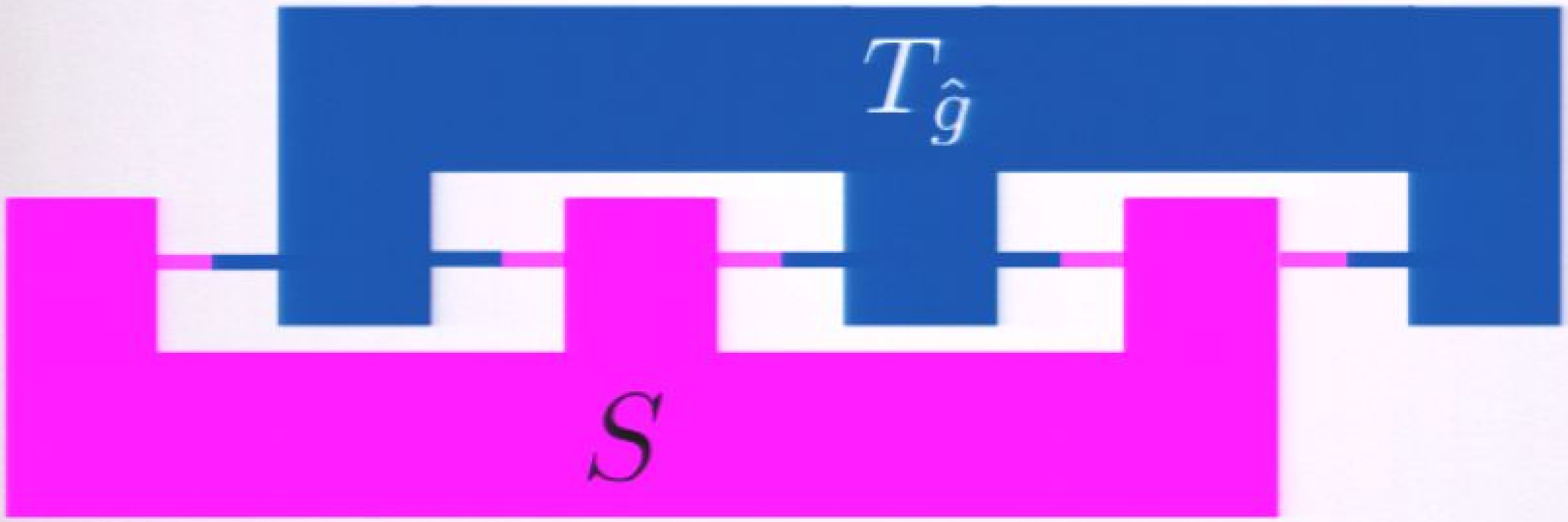

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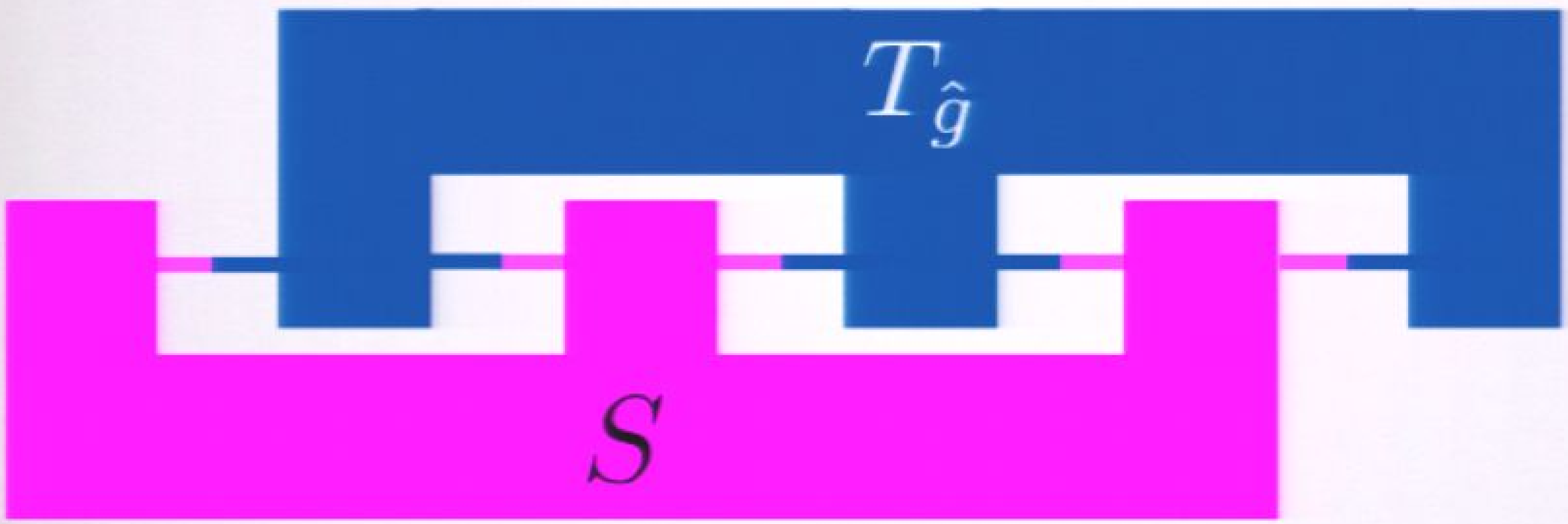


Alice's moves, in her description, are given by comb S
In Bob's description:

$$S_g = (U_g^{\otimes N_{A\to B}} \otimes U_g^{*\otimes N_{B\to A}} \otimes I_C) S (U_g^{\dagger \otimes N_{A\to B}} \otimes U_g^{\tau * \otimes N_{B\to A}} \otimes I_C)$$

Bob's estimation strategy: tester

$$T_{\hat{g}} = (U_{\hat{g}}^{\otimes N_{A\to B}} \otimes U_{\hat{g}}^{*\otimes N_{B\to A}} \otimes I_C)\ T_0\ (U_{\hat{g}}^{\otimes N_{A\to B}} \otimes U_{\hat{g}}^{*\otimes N_{B\to A}} \otimes I_C$$

# OPTIMALITY PROOF FOR ONE-WAY STRATEGIES

Decomposition of the tester: measurement on the quantum state

$$\mathcal{T}(S_g) = \langle T \rangle^{\frac{1}{2}} S_g \; \langle T \rangle^{\frac{1}{2}}$$

Since $\quad [\langle T \rangle, U_g^{\otimes N_{A\to B}} \otimes U_g^{*\otimes N_{B\to A}} \otimes I_C] = 0$

the state is of the form

$$\rho_g = (U_g^{\otimes N_{A\to B}} \otimes U_g^{*\otimes N_{B\to A}} \otimes I_C) \; \rho_0 \; (U_g^{\otimes N_{A\to B}} \otimes U_g^{*\otimes N_{B\to A}} \otimes I_C)^\dagger$$

Conclusions:

- a single round with $N_{tot} = N_{A\to B} + N_{B\to A}$ transmitted particles is enough.
- classical communication is useless
- obvious generalization to arbitrary groups

# CLONING OF QUANTUM TRANSFORMATIONS

What does it mean to clone a transformation?

Use the corresponding black box only once,
to simulate two independent uses of it on a bipartite system.

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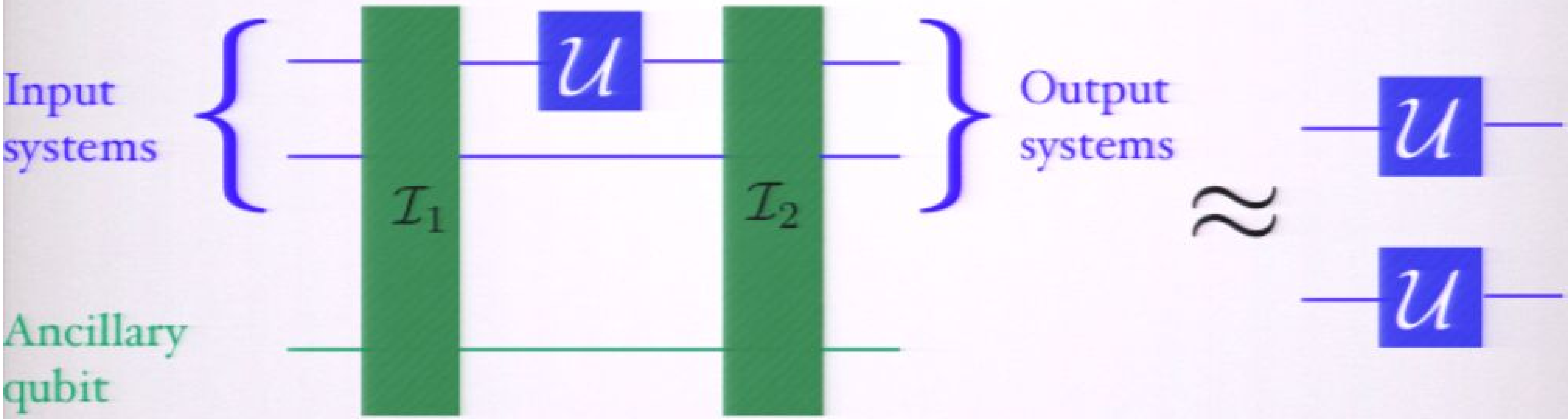
# OPTIMAL UNIVERSAL GATE CLONING




Pre-processing interaction: controlled swap

Post-processing interaction: extension of pure state cloning

$$F_{clon}(1 \to 2) = \frac{d + \sqrt{d^2 - 1}}{d^3} \qquad F_{est}(1 \to 2) = \frac{6}{d^4}$$

CC, G M D'Ariano, and P Perinotti, Phys. Rev. Lett. 101, 180504 (2008)

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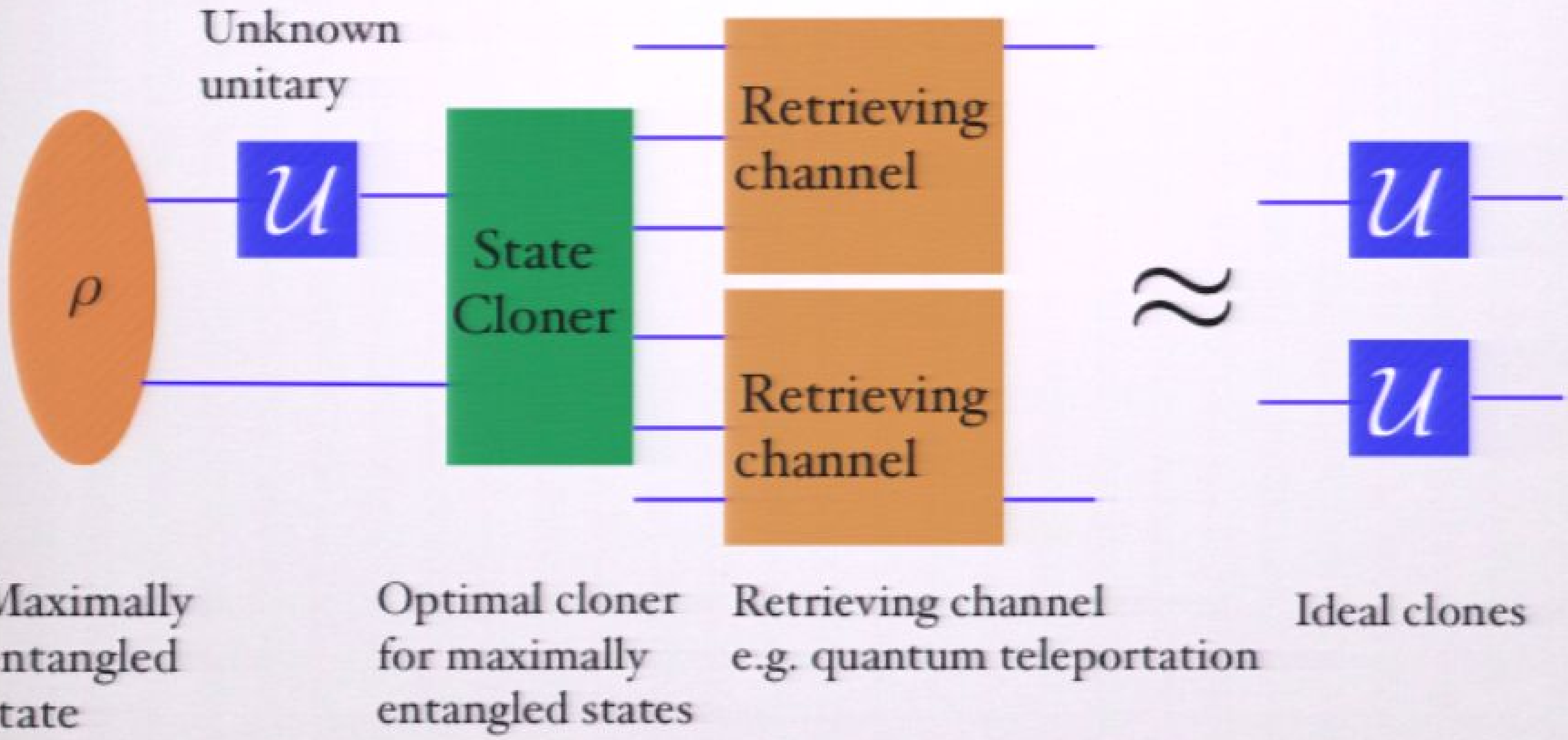
**Natural question:** is it possible to achieve the optimal cloning of a unitary via the optimal cloning of a maximally entangled state? i.e. by cloning the Choi state?



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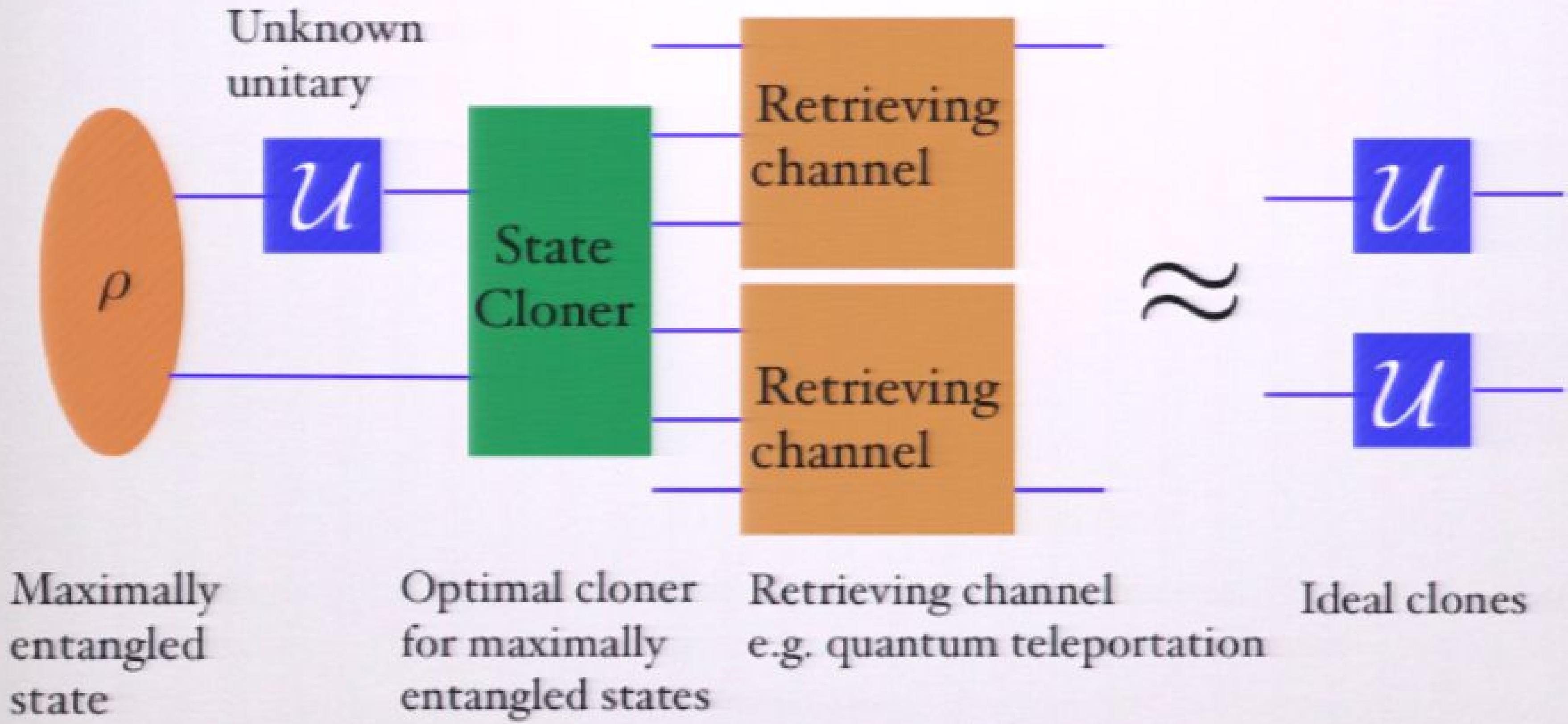


**Answer:** No, this is a strictly suboptimal strategy.

# OTHER APPLICATIONS IN QIP

- Optimal storing/retrieving of quantum gates
- Optimal programming of quantum games
- Analysis of multi-round quantum games/cryptographic protocols
  cf G Gutoski and J Watrous, STOC 2007, 565
- Information-disturbance trade-off for quantum transformations

*Thank you for your attention.*