Title: Backreaction from Averaging in Cosmology

Date: Dec 11, 2008 10:40 AM

URL: http://pirsa.org/08120034

Abstract: There is an ongoing debate in the literature concerning the effects of averaging out inhomogeneities (``backreaction\'\') in cosmology. In particular, it has been suggested that the backreaction can play a significant role at late times, and that the standard perturbed FLRW framework is no longer a good approximation during structure formation, when the density contrast becomes nonlinear. After a brief introduction to the problem, I will show using Zalaletdinov\'s covariantaveraging scheme that as long as the metric of the universe can be described by the perturbed FLRW form, the corrections due to averaging remain negligibly small. Further, using a fully relativistic and reasonably generic model of pressureless spherical collapse, I will show that as long as matter velocities remain small (which is true in this model even at late times), the perturbed FLRW form of the metric can be explicitly recovered. Together with the observation that real peculiar velocities are in fact nonrelativistic, these results imply that the backreaction remains small even during nonlinear structure formation.

Backreaction from Averaging in Cosmology

Aseem Paranjape

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AP and T. P. Singh, PRL 101, 181101, 2008, arXiv:0806.3497;
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Young Researchers' Conference, Perimeter Institute, December 2008

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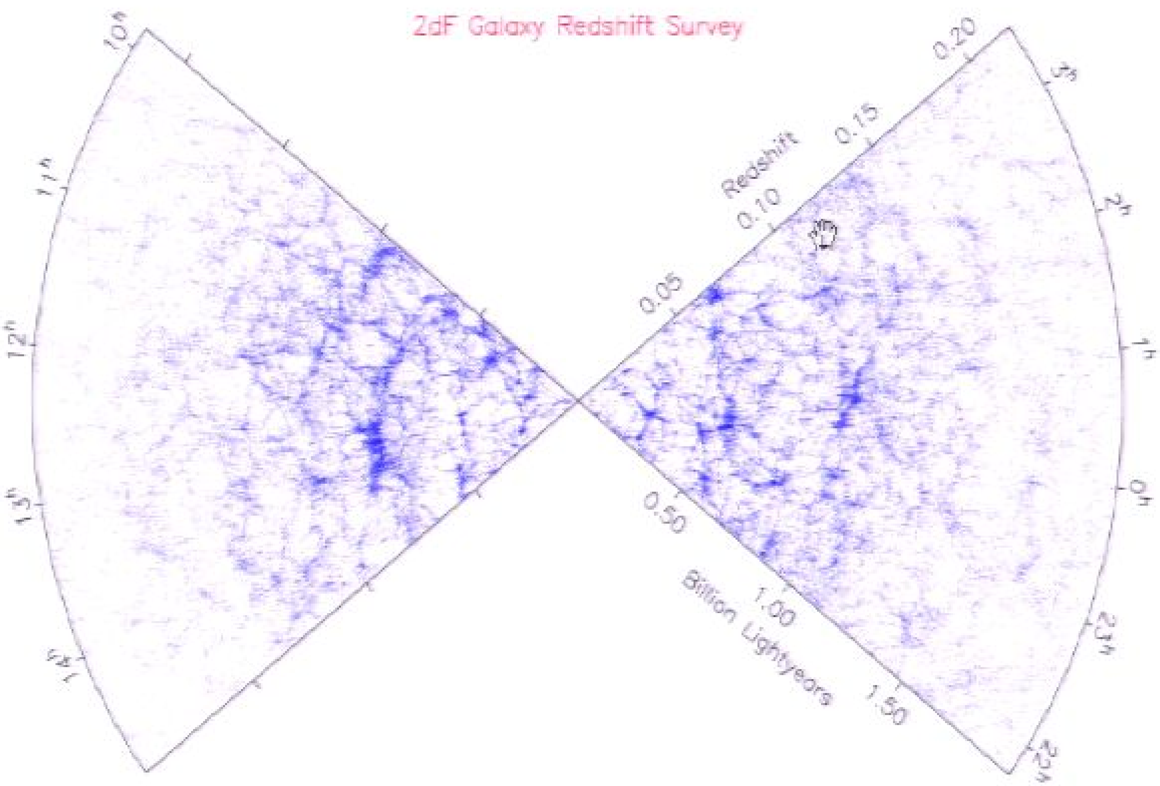
Introduction

We observe that the matter distribution in the Universe is approximately homogeneous on large scales.



Introduction

- Also, the CMBR is highly isotropic.
- The standard approach then follows :
  - Assume the Universe is almost homogeneous and isotropic.
  - Hence its metric must be FLRW, with small perturbations.
  - Solve Einstein's equations for the FLRW background sourced by a homogeneous and isotropic perfect fluid

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \; ; \; \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \; . \tag{1}$$

Introduction

However, homogeneity is not strictly valid on small scales.
Two issues are important :

- Nonlinearity of GR – Recall $\Gamma \sim \partial g$, $E[g] \sim \partial\Gamma + \Gamma^2$, and $E[g] = T$.
- In identifying FLRW as the background geometry, an implicit spatial averaging is (vaguely) implied in the standard approach.

Introduction

Two main approaches in understanding the effects of inhomogeneities on the cosmological expansion :

- Effect on average expansion via correlations in fluctuations or backreaction, arising from the fact that in general $E[\langle g \rangle] \neq \langle E[g] \rangle = \langle T \rangle$. (G. F. R. Ellis, *Gen. Rel. and Grav.*, 1984.)
- Effect on light propagation ("special observer" assumption).

Focus here is on the former.

Argument: Although technically possible, in the real world it is unlikely that backreaction significantly influences the average cosmological expansion.

Structure of Backreaction

Symbolically, the backreaction terms have the form

$$\mathcal{C} \sim \langle \widetilde{\Gamma}^2 \rangle - \langle \widetilde{\Gamma} \rangle^2$$

[Can be understood using $E[g] \sim \partial\Gamma + \Gamma^2$.]

Tilde represents any processing of the Christoffels required by the averaging operation.

Plan

1. Building the Argument
   - The Linear Regime
   - Lessons from linear theory

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Early times
1. First estimate

Assumption: Linear perturbation theory (in the metric *and* matter fluctuations) is a good approximation at early times (e.g. around recombination).

$$ds^2 = -(1+2\varphi)d\tau^2 + a(\tau)^2(1-2\varphi)d\vec{x}^2 .$$

Justified by amplitude of CMB anisotropies, coupled with Copernican belief.

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The dominant contribution to the backreaction $\mathcal{C}$ (e.g. in a perturbed Einstein-deSitter universe) is

$$\mathcal{C} \sim \frac{1}{a^2}\langle \nabla\varphi \cdot \nabla\varphi \rangle \sim \frac{1}{a^2} \cdot A \cdot (\frac{k_{eq}}{H_0})^2 \cdot H_0^2 \sim \frac{1}{a^2}(10^{-4})H_0^2 \,.$$

Broadly speaking, contribution of the backreaction at early times is expected to be small.

Early times
2. Self consistency

What we are after is a self consistent solution of the loop

$$\begin{tikzcd} a \arrow[dr] & & \mathcal{C} \arrow[ll] \\ & \varphi \arrow[ur] & \end{tikzcd}$$

Smallness of $\mathcal{C}$ at early times gives the hope that an iterative approach might work

$$a^{(0)} \longrightarrow \varphi^{(0)} \longrightarrow \mathcal{C}^{(0)} \longrightarrow a^{(1)} \longrightarrow \varphi^{(1)} \longrightarrow \dots$$

Early times

3. "Zeroth" iteration equations. [AP, arXiv:0806.2755, 2008.]

In the Zalaletdinov averaging framework [Zalaletdinov, GRG 24,1015,1992; GRG 25,673,1993] ▸ Details, assuming that perturbation theory holds for the *metric*,

$$ds^2 = -(1+2\varphi)d\tau^2 + a(\tau)^2(1-2\psi)d\vec{x}^2 \,.$$

one finds

$$\left(\frac{1}{a}\frac{da}{d\tau}\right)^2 = \frac{8\pi G_N}{3}\bar{\rho} - \frac{1}{6}\left[\mathcal{P}^{(1)} + \mathcal{S}^{(1)}\right] ,$$

$$\frac{1}{a}\frac{d^2a}{d\tau^2} = -\frac{4\pi G_N}{3}(\bar{\rho} + 3\bar{p}) + \frac{1}{3}\left[\mathcal{P}^{(1)} + \mathcal{P}^{(2)} + \mathcal{S}^{(2)}\right] ,$$

Early times

3. "Zeroth" iteration equations. [AP, arXiv:0806.2755, 2008.]

where (with $' \equiv \partial_\eta = a\partial_\tau$ and $\mathcal{H} \equiv a'/a$), in Fourier space, assuming $\varphi_{\vec{k}} = \psi_{\vec{k}} = \varphi_{i\vec{k}}\Phi_k(\eta)$ and replacing spatial averaging by ensemble averaging,

$$\mathcal{P}^{(1)} = -\frac{2}{a^2}\int \frac{dk}{2\pi^2} k^2 P_{\varphi i}(k)\left(\Phi_k'\right)^2,$$

$$\mathcal{S}^{(1)} = -\frac{2}{a^2}\int \frac{dk}{2\pi^2} k^2 P_{\varphi i}(k)\left(k^2\Phi_k^2\right),$$

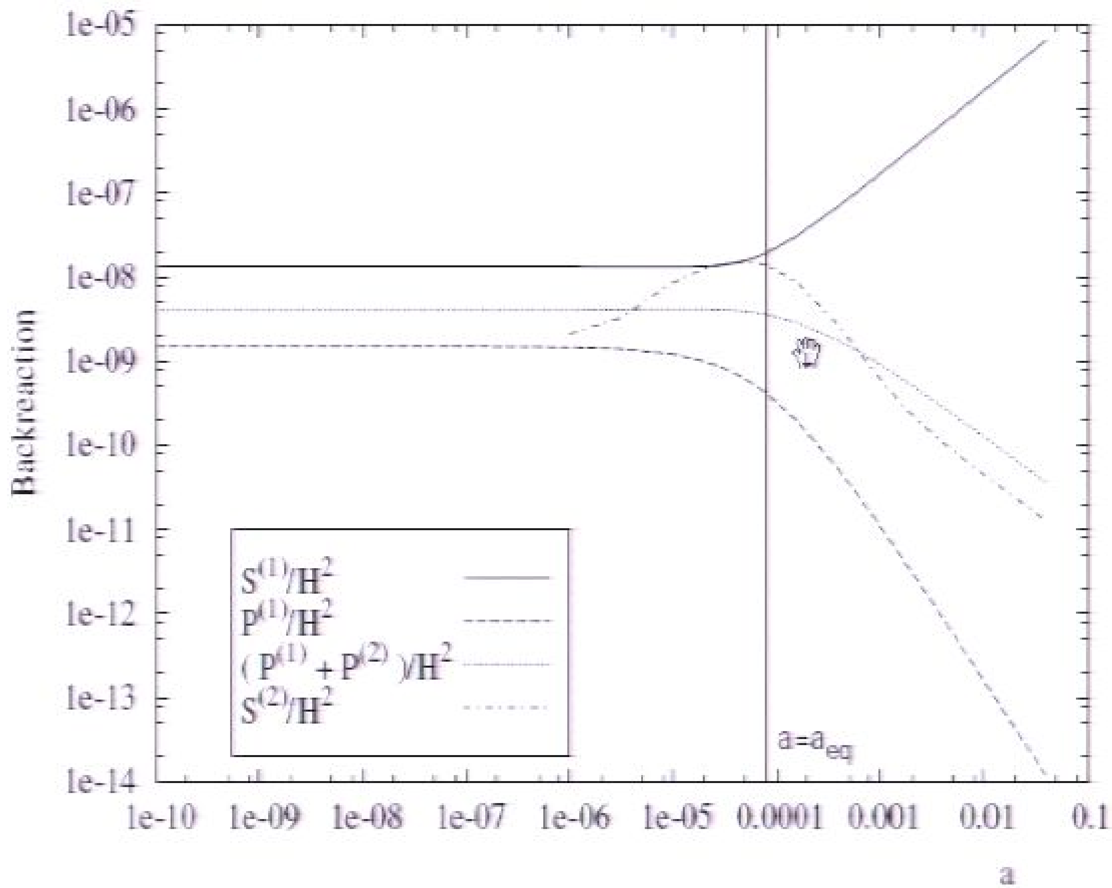
$$\mathcal{P}^{(1)} + \mathcal{P}^{(2)} = -\frac{8\mathcal{H}}{a^2}\int \frac{dk}{2\pi^2} k^2 P_{\varphi i}(k)\left(\Phi_k\Phi_k'\right),$$

$$\mathcal{S}^{(2)} = -\frac{2}{a^2}\int \frac{dk}{2\pi^2} k^2 P_{\varphi i}(k)\Phi_k''\left(\Phi_k - \frac{2\mathcal{H}}{k^2}\Phi_k'\right).$$

▸ Backreaction general expression

Early times

4. "Zeroth" iteration results. [AP, arXiv:0806.2755, 2008.]



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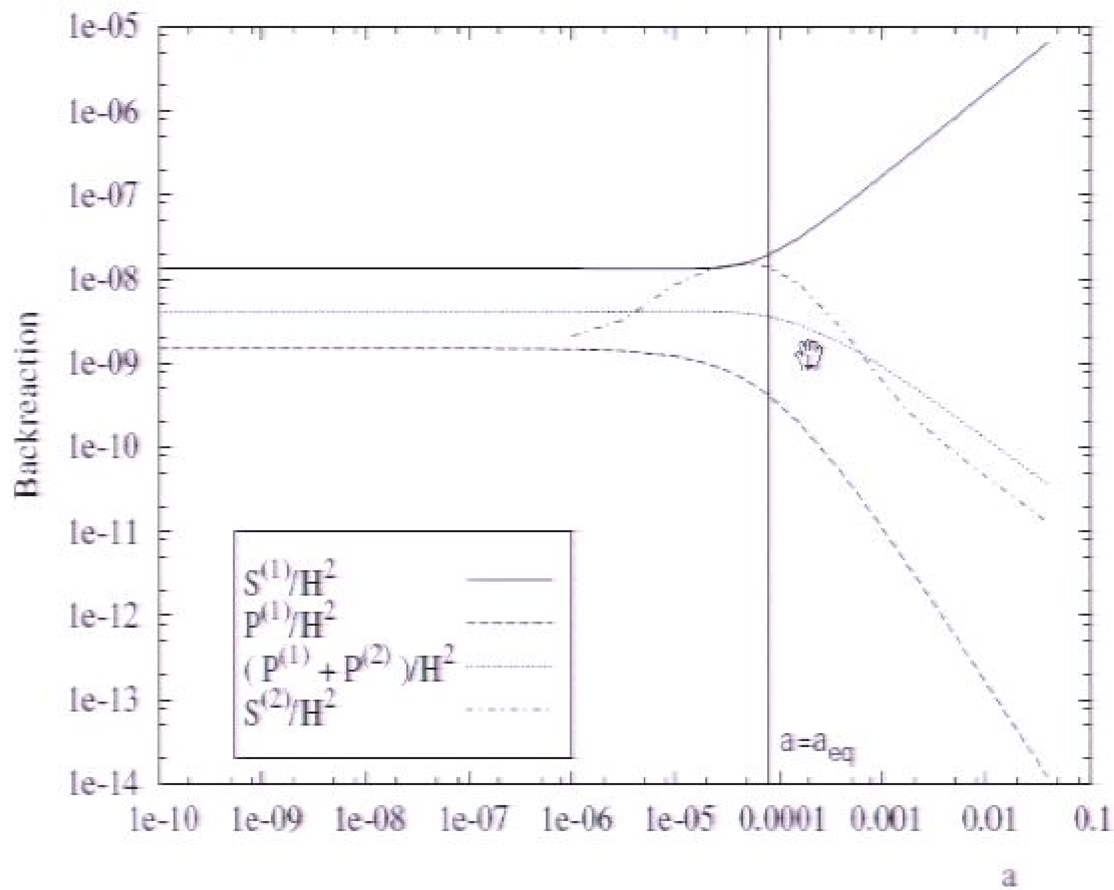

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Dynamical suppression of backreaction

Since $\mathcal{C}$ is small to begin with, the self consistency loop

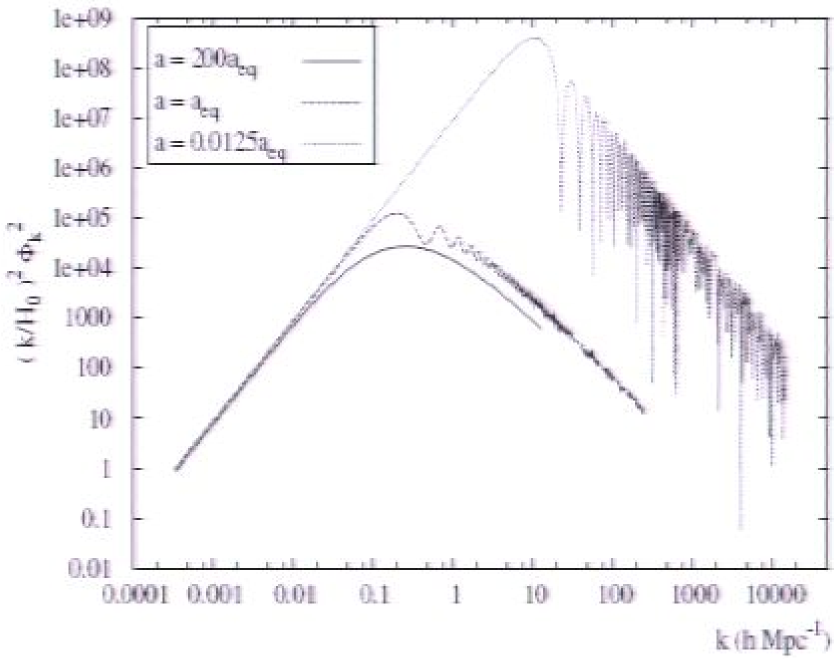
$$\begin{tikzcd} a \arrow[dr] & & \mathcal{C} \arrow[ll] \\ & \varphi \arrow[ur] & \end{tikzcd}$$

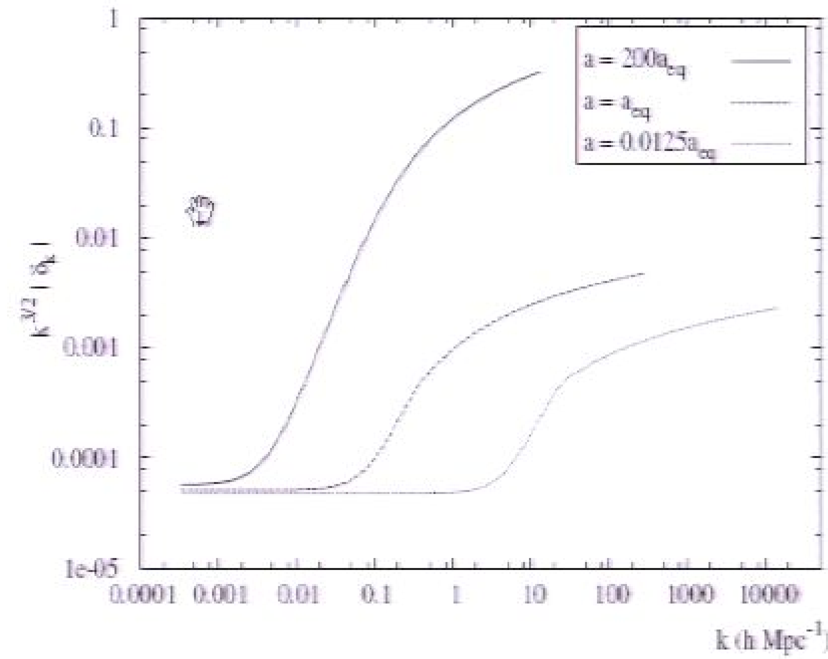
suppresses growth of $\mathcal{C}$ provided perturbation theory in the metric holds. [Large $\mathcal{C}$ needs nonstandard growth of φ, which needs nonstandard evolution of a, which needs large $\mathcal{C}$. But $\mathcal{C}$ is small to begin with.]

This argument is likely to carry forward to the nonlinear regime as well.

Small contribution from nonlinear scales

[AP, arXiv:0806.2755, 2008.]





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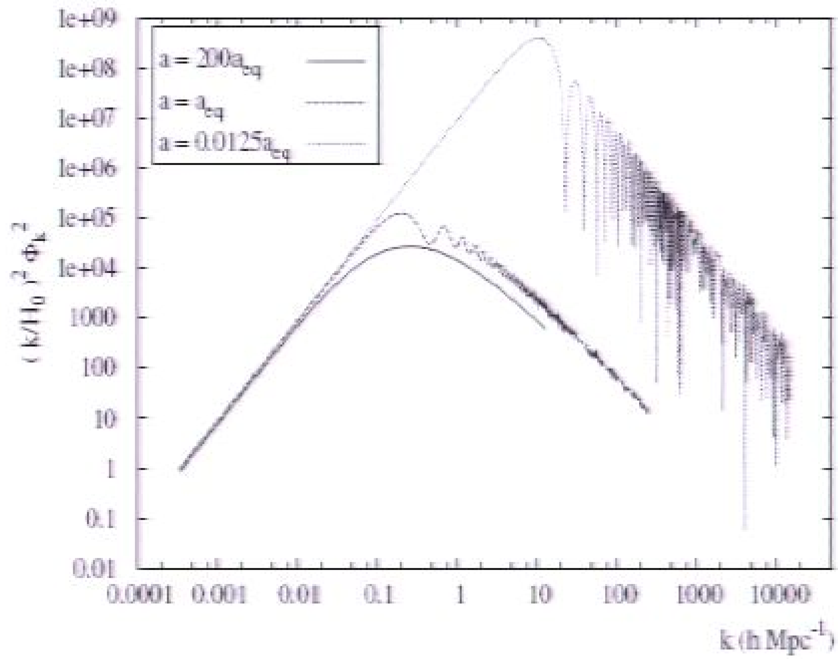

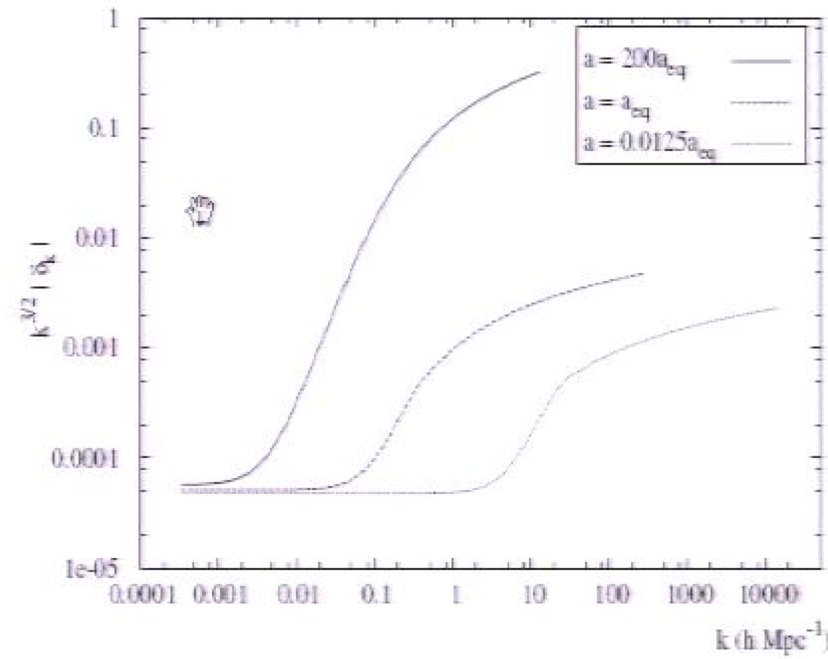

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Order of magnitude estimates

Suppose perturbation theory is valid for the metric, Then the relevant equation is

$$\frac{1}{a^2}\nabla^2\varphi = 4\pi G\bar{\rho}\delta \ ; \ \delta \equiv (\rho(t,\vec{x})/\bar{\rho}(t)) - 1 .$$

As before,

$$\mathcal{C} \sim a^{-2}\langle \nabla\varphi \cdot \nabla\varphi \rangle .$$

For an over/under-density of physical size R, treating $a^{-1}\nabla \sim R^{-1}$ and $G\bar{\rho} \sim H^2$, we have

$$|\varphi| \sim (HR)^2 \, |\delta| \ .$$

For voids, $\delta \sim -1$, and then $\mathcal{C} \sim H^2(HR)^2 \ll H^2$.

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In a typical spherical collapse situation,

$$R \sim (1 - \cos u)r \; ; \; H^{-1} \sim (G\bar{\rho})^{-1/2} \sim t \sim H_0^{-1}(u - \sin u)$$

$$G\rho \sim \frac{(H_0 r)^2}{R^2 R'} \sim \frac{H_0^2}{(1 - \cos u)^3} \; ; \; \delta \sim (\rho/\bar{\rho}) \sim \frac{(u - \sin u)^2}{(1 - \cos u)^3}$$

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$$|\varphi| \sim \frac{(H_0 r)^2}{(1 - \cos u)} \; ; \; \mathcal{C} \sim H^2 \left[(H_0 r)^2 \frac{(u - \sin u)^2}{(1 - \cos u)^4} \right] .$$

At late times we apparently have $\mathcal{C} \sim H^2$ implying large corrections, and later still, also $\varphi \sim 1$ implying a breakdown of perturbation theory.

Order of magnitude estimates

Crucial question is :
Is this situation actually realised, or are we taking these simple models too far?

Claim : Perturbation theory in the metric does not break down at late times, since observed peculiar velocities remain small. The spherical collapse model is not a good approximation when model peculiar velocities grow large.

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Spherical collapse

The LTB solution [AP and Singh, arXiv:0801.1546, 2008]

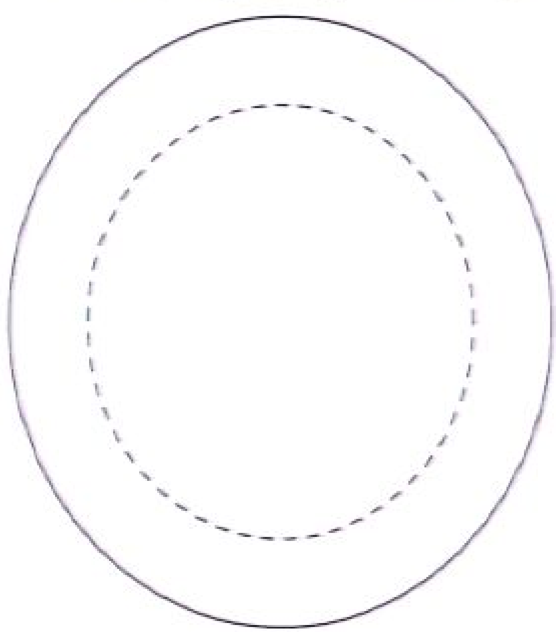
- The metric :

$$ds^2 = -dt^2 + \frac{R'^2 dr^2}{1 - k(r)r^2} + R^2 d\Omega^2 .$$

Details

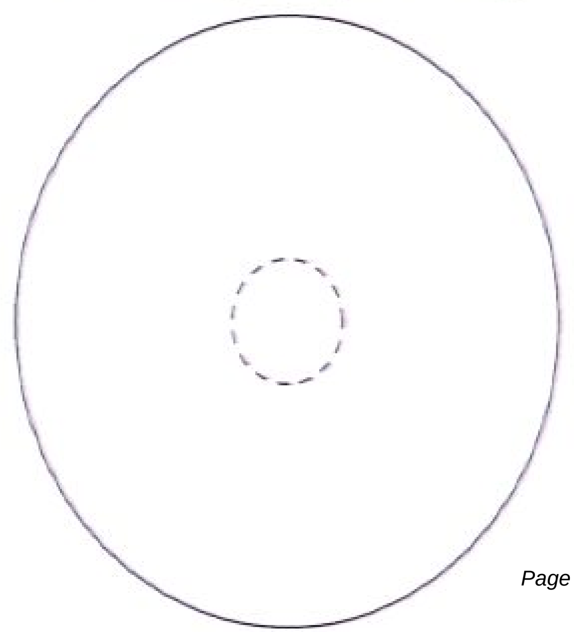
Inside out -- r*, rv

t = ti, R[ti,rv] = 23.5 kpc

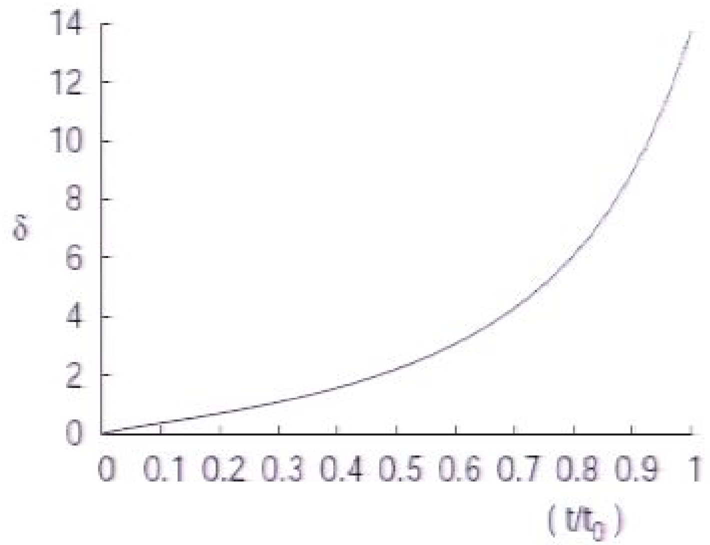


Inside out -- r*, rv

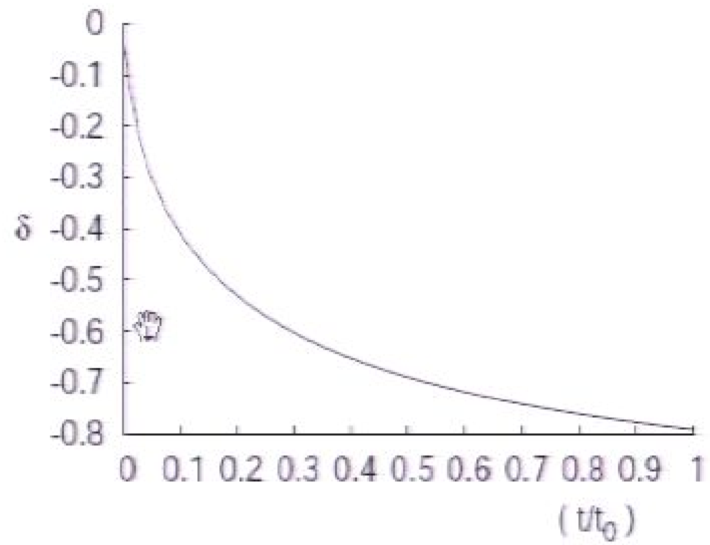
t = t0, R[t0,rv] = 33.3 Mpc



Spherical collapse
Behaviour of the model [AP and Singh, arXiv:0801.1546, 2008]



$r = 8.35\,\mathrm{Mpc}$



$r = 21.15\,\mathrm{Mpc}$

- Parameter values
- Aside : Acceleration from boundary conditions

Spherical collapse
Recovering perturbed FLRW [AP and Singh, arXiv:0801.1546, 2008]

Recall

$$ds^2 = -dt^2 + \frac{R'^2 dr^2}{1 - k(r)r^2} + R^2 d\Omega^2 .$$

We want

$$ds^2 = -(1+2\varphi)d\tau^2 + a^2(\tau)(1+2\psi)\left(d\tilde{r}^2 + \tilde{r}^2 d\Omega^2\right) ,$$

with at least $|\varphi|, |\psi| \ll 1$. Using

$$\tilde{r} = \frac{R(t,r)}{a(t)}(1+\xi(t,r)) \;\; ; \;\; \tau = t + \xi^0(t,r) \;\; ; \;\; |\,\xi\,|, |\,\xi^0 H\,| \ll 1 ,$$

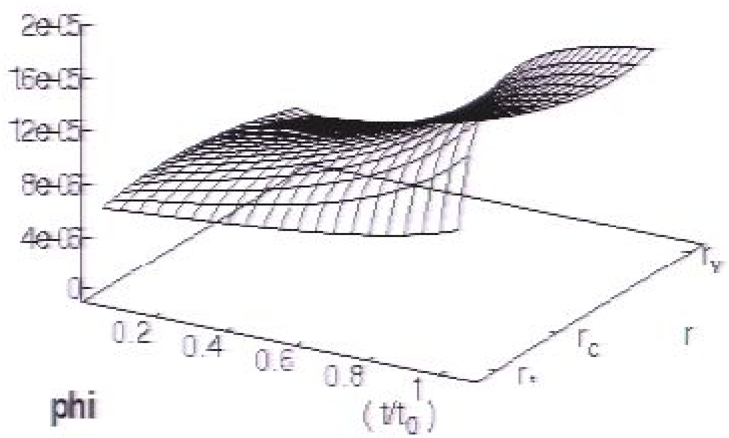
and given perturbed FLRW initial conditions, sole condition for a valid transformation is

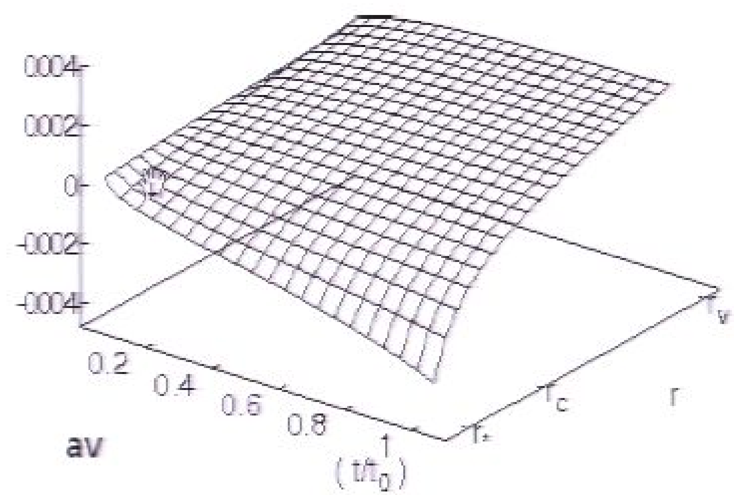
$$|av| \ll 1 \;\; ; \;\; v \equiv \frac{\partial \tilde{r}}{\partial t} \approx \partial_t\left(\frac{R}{a}\right) .$$

Physically v is the "comoving" peculiar velocity.

Spherical collapse

Results in $r_* < r < r_V$ [AP and Singh, arXiv:0801.1546, 2008]

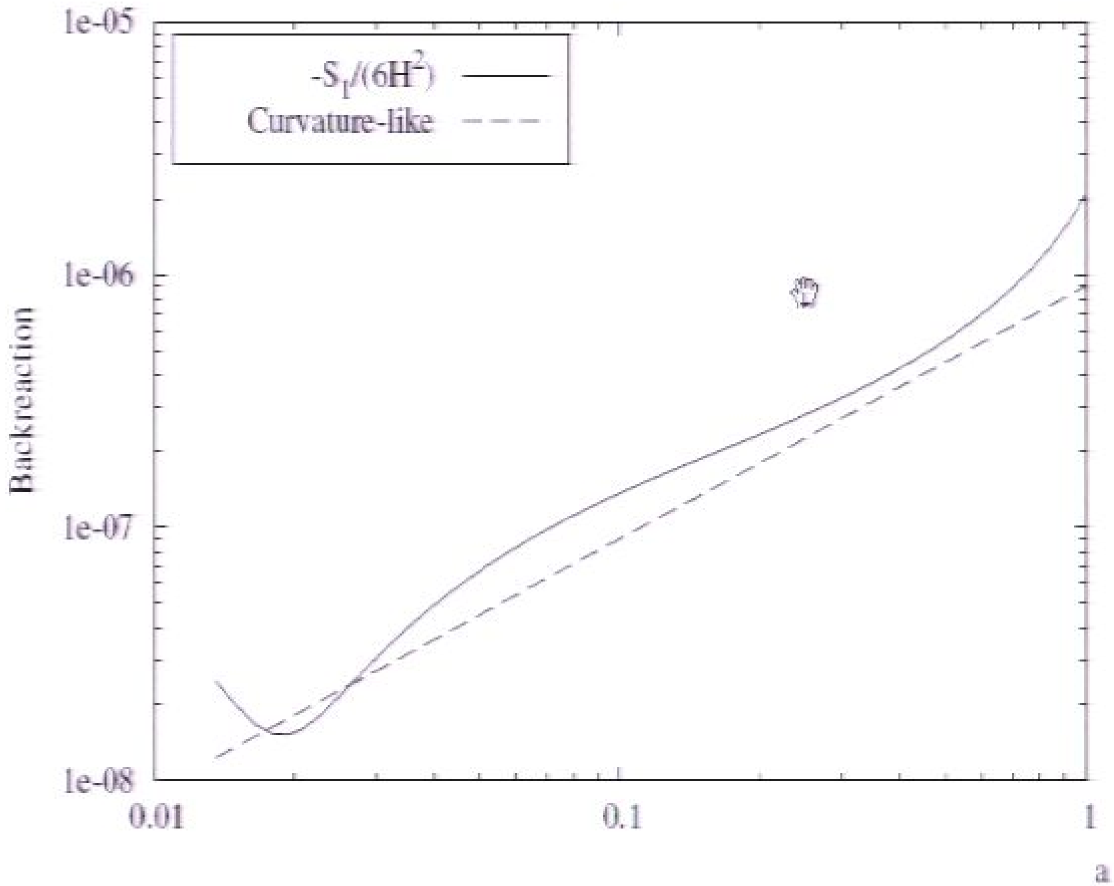




Backreaction in the spherical collapse model

[AP and Singh, arXiv:0806.3497, 2008]

The most dominant correction is $\mathcal{S}^{(1)}$ which appears on the right hand side of the Friedmann equation.



▸ Backreaction general expression

Summary

- Corrections to the cosmological equations due to averaging of inhomogeneities, are real and nontrivial, but appear to be negligible given realistic initial conditions and evolution models.
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  - They remain dynamically suppressed at later times provided perturbation theory in the metric is valid.
  - This appears to be the case provided peculiar velocities remain small.
  - Observed peculiar velocities are, in fact, nonrelativistic.
- Conclusions here were based on calculations in linear theory, and exactly solved toy models of structure formation.
- More accurate calculations may be possible using N-body simulations.

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Thank you.