Title: Story of A & C

Date: Dec 10, 2008 09:00 AM

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Abstract: In two dimensional CFTs the Zamolodchikov's c-theorem is fundamental in that it shows that the number of degrees of freedom decreases along the renormalization group flow. I will give a short history of and discuss recent developments in the quest to find its four-dimensional analogue using the central charges_x000D_ a & c.

Introduction to String Theory

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Institute for Advanced Study

Dec, 2008

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Contents

Real world consists of matter:

Electrons, quarks, ...

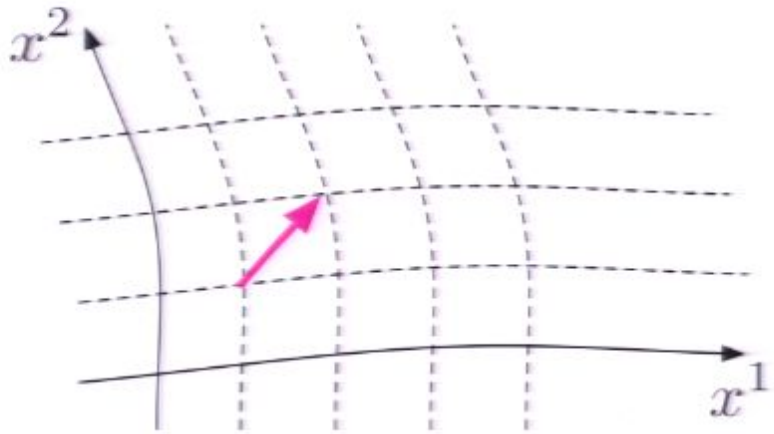
and forces:

gravity, photons, gluons...

Called the Standard model +gravity.

Describes the real world really, really well.

Gravity is described by metric $g_{\mu\nu}$



$$|\Delta s|^2 = g_{\mu\nu}(x)\Delta x^\mu \Delta x^\nu$$

Gauge fields A_μ

$$v'^a = (\delta^a_b + iA^a_{b\,\mu}\Delta x^\mu)v^b$$

Gauge group

$$G = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$$

Again, Standard Model + Gravity really works well.

We might become arrogant and ask

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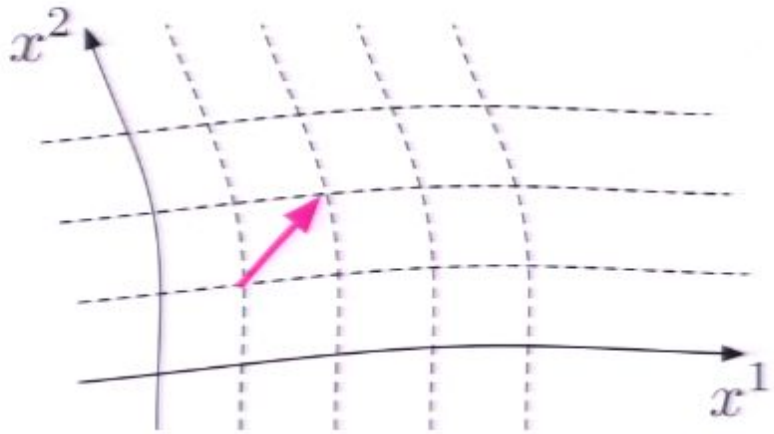

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- Why do we have gravity and gauge fields in the first place?
- Why are there 3+1 spacetime dimensions?

Fields come from the quantization of particles.

It was difficult to quantize particles relativistically. (established ~1950)

You can choose

- whatever matter contents (up to a very mild condition)
- whatever Lagrangian,
- whatever dimensions...

Stupid guys tried to quantize strings for strange reasons.
(definitely not to solve deep/arrogant questions.)

It was really difficult to quantize strings relativistically. (established 1984)

They found surprising things.

- Cannot choose arbitrary field content.
- Cannot choose arbitrary spacetime dimensions.

Consistency requires

- 9+1 dimensions
- Supersymmetry
- Particular gauge fields: either

None, **SO(32)**, $E_8 \times E_8$.

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If only we lived in 9+1 dimensions, supersymmetric, and with one of these gauge groups ...

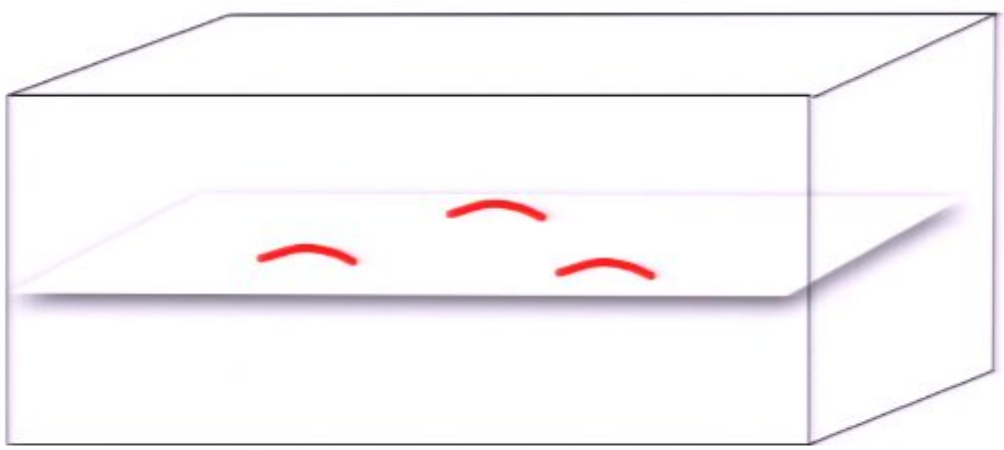
We would have been totally convinced by string theory. So rigid and satisfactory!

Of course we're not living in 10 dimensions, we don't see SUSY, etc.
String theory is totally wrong...

Of course we're not living in 10 dimensions, we don't see SUSY, etc.
String theory is totally wrong...but.

- Can "compactify" extra 6 dimensions

- Has "branes" on which gauge fields live



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- **SU(3)** $\times$ **SU(2)** $\times$ **U(1)** or its Grand Unified version
- chiral fermions

No definite phenomenological prediction ... yet.

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Is String Theory right ?

It's at least a consistent theory.

Influence to phenomenology/cosmology

- Supersymmetry
- Extra dimensions
- Brane models

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- 2d CFTs
- Non-perturbative studies of 4d QFTs
- Supergravities in various dimensions

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It's very RICH.

String theory has not given definite predictions to particle physics ...

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But did give surprising predictions to mathematics,
and they were 100% confirmed!

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  → Need to compactify on a small, 6-dimensional space.

- Need to solve Einstein equations in the internal dimensions

$$R_{ij} = 0$$

and also need to require supersymmetry → Calabi-Yau spaces.

Mirror Symmetry

People found

Type IIA String Theory on the CY M
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Type IIA String Theory on the CY M
is equivalent to
Type IIB String Theory on the CY W.

- Given M, W can be constructed
- and vice versa.
- Mirror symmetry in the physics sense.

It implies, as a small application, the following mathematical statement:

Let

$$P(q) = 1 + n_1 q + n_2 q^2 + \cdots$$

where n_d is the number of holomorphic S^2 of area d in the CY M.

Then it solves an ODE

$$D_W P(q) = 0$$

associated to the CY W.

N.B. this statement is too simplified and is wrong, but you get the flavor.

This is the Mirror Symmetry in the mathematical sense.

For the original example M: quintic, W: Mirror quintic,

- $n_1 = 2875$ and $n_2 = 609250$ was known.
- It gave an infinite number of predictions

$$n_3 = 317206375, n_4 = 242467530000, \ldots$$

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Mathematical Application of AdS/CFT duality

Quiver Q whose moduli space is a cone $C(X)$ over X_5

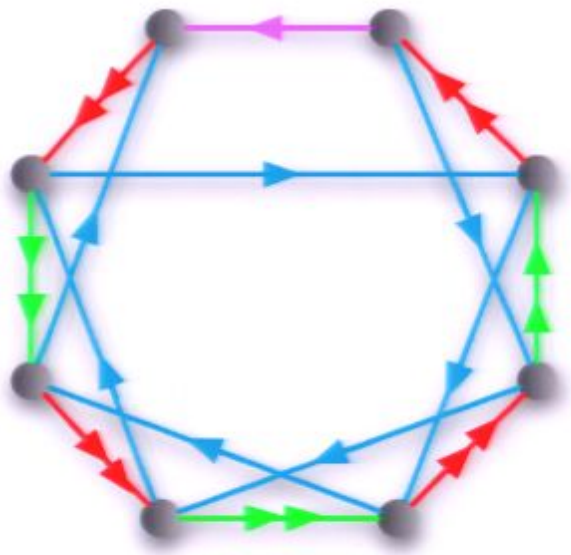
$$dr^2_{C(X)} = dr^2 + r^2 ds^2_X$$

is equivalent to Type IIB string on $AdS_5 \times X_5$.

This is the AdS/CFT correspondence. (I'll come back to its physics later.)

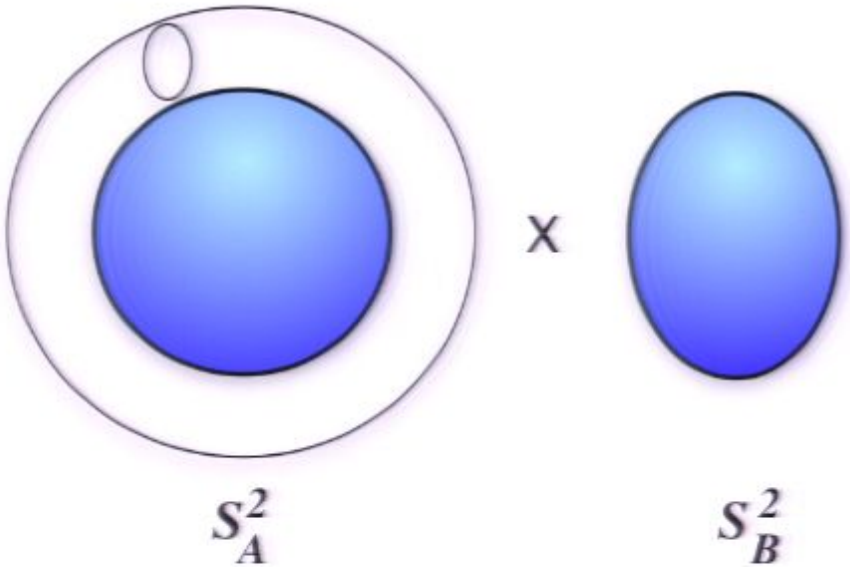
It also gives us a few mathematical surprises...

Quiver theory $\boldsymbol{Q}$ (e.g. $\boldsymbol{Y}^{\mathbf{4,3}}$)



- Each node stands for an $\mathbf{SU}(N)$ gauge group.
- Each arrow stands for a bifundamental matter fields.
- (Some of) the closed loops stand for the interaction terms.

Five-dimensional Space X. Example: $Y^{p,q}$



- S^1 fibration over $S_A^2 \times S_B^2$
- wraps p times on S_A^2 and q times on S_B^2

Given a quiver Q, there is a definite method to associate X topologically.

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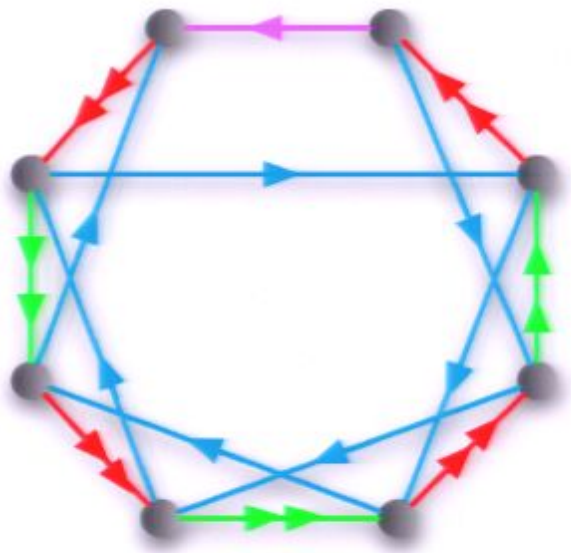

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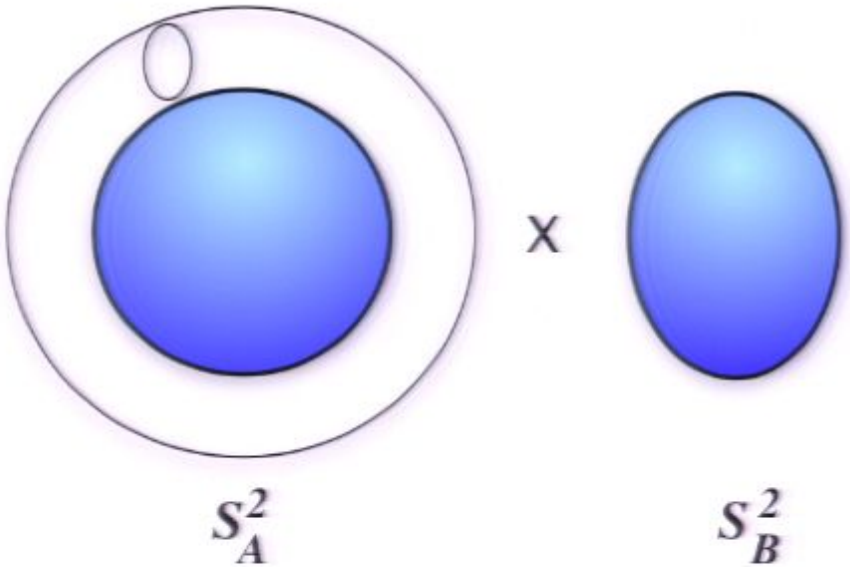

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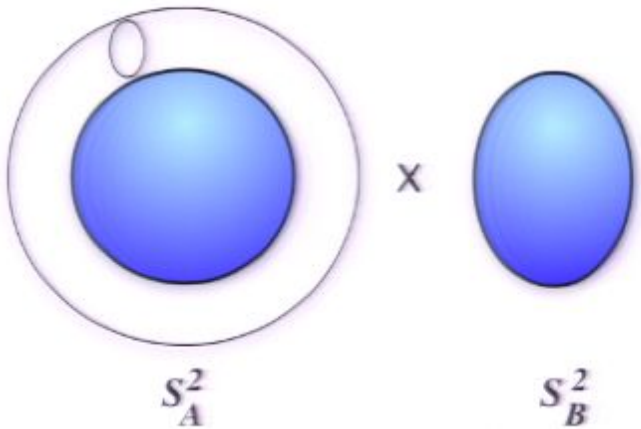
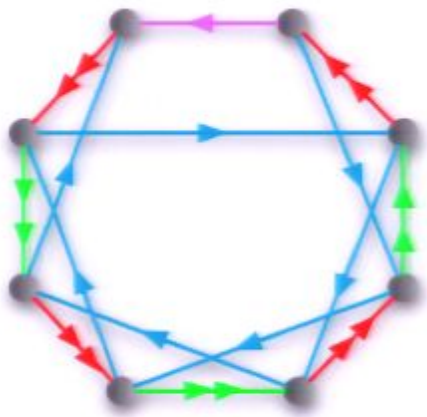


Find an Einstein metric on X normalized $R_{ij} = 4g_{ij}$.

For $Y^{p,q}$ explict metrics are known. We have

$$\mathrm{vol}(Y^{4,3}) = \frac{81\pi^3}{16(\sqrt{37}^3 - 188)}.$$

Given a quiver Q:



Assign numbers r_i to each arrow i, and consider the sum

$$a = \frac{9}{32}\left[\sum_{\text{nodes}} 1 + \sum_{\text{arrows } i} (r_i - 1)^3\right]$$

and maximize it under the condition

$$\sum_{\text{arrows } i \text{ in a loop}} r_i = 2,$$

$$\sum_{\text{arrows } i \text{ at a node}} \pm(r_i - 1) = 0,$$

For $Y^{4,3}$ we get

$$a(Y^{4,3}) = \frac{4}{81}(\sqrt{37}^3 - 188)$$

We find

$$a(Q) = \frac{\pi^3}{4\,\mathrm{vol}(X)}$$

in general.

Quiver Q	$\Leftrightarrow$	Space X
$\downarrow$		$\downarrow$
cubic function		solve Einstein eq.
$\downarrow$		$\downarrow$
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Why on earth does this combinatorial construction know the solution of the Einstein equation???

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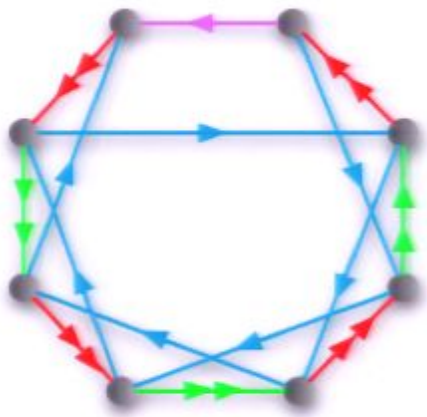

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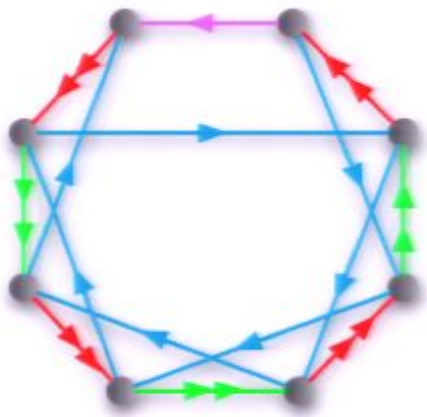

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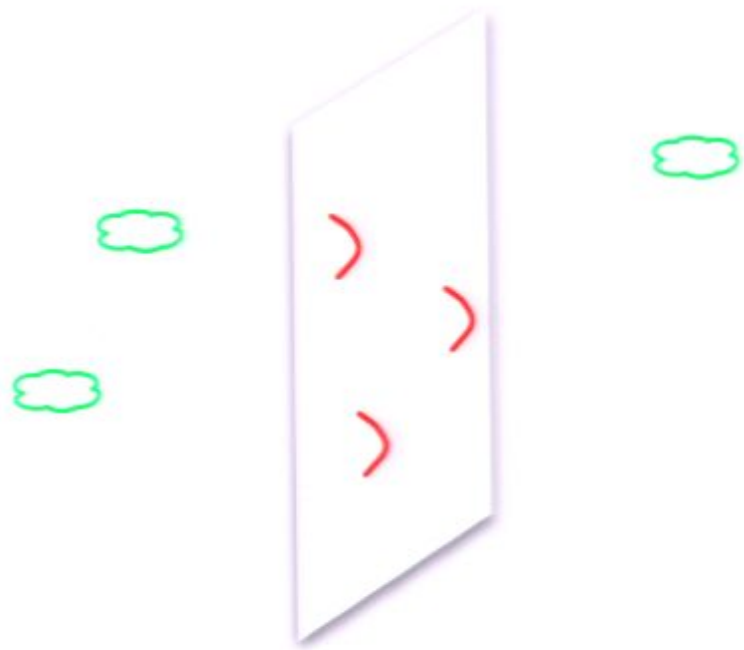
Contents

What is AdS/CFT ?

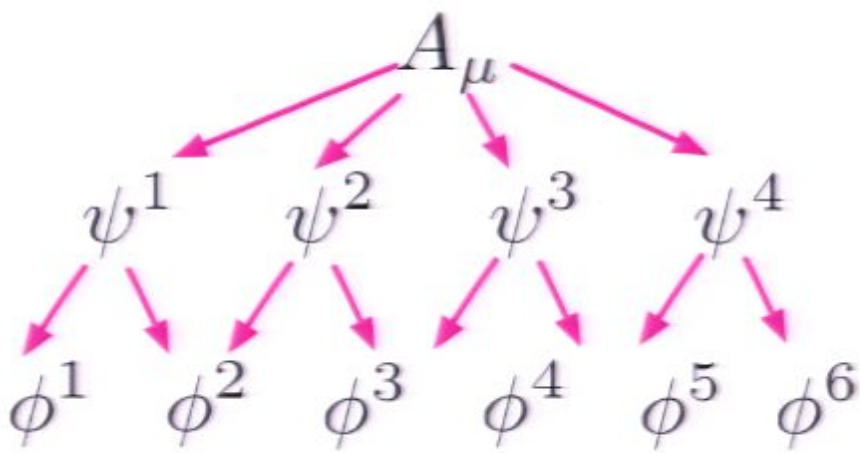
Consider N D3-branes in a flat 10d spacetime of Type IIB string theory

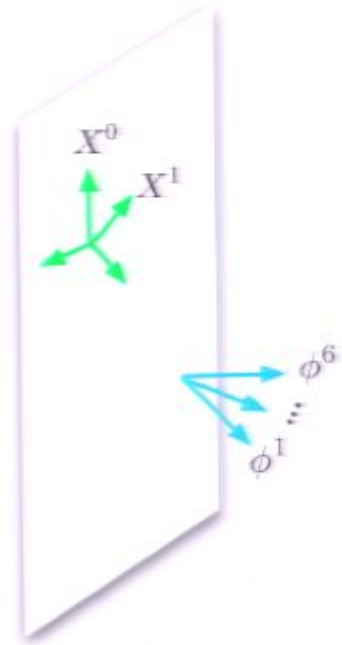


On D-branes strings can have ends.

On the stack of N D3-branes live

$\mathcal{N} = \mathbf{4}$ supersymmetric $\mathbf{U}(N)$ gauge theory.





We identify six scalars as transverse fluctuations.

4 (spacetime dimension of D3) + 6 (# of scalars)

$=$ 10 (natural dimensions for strings)

For N D3-branes, we need $\mathbf{U}(N)$ gauge theories.

$\phi^{1,\ldots,6}$: $N \times N$ matrices? Isn't it too many?

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$\phi^{1,\ldots,6}$: $N \times N$ matrices? Isn't it too many?

$$V = \sum_{1 \leq i,j \leq 6} |[\phi_i, \phi_j]|^2$$

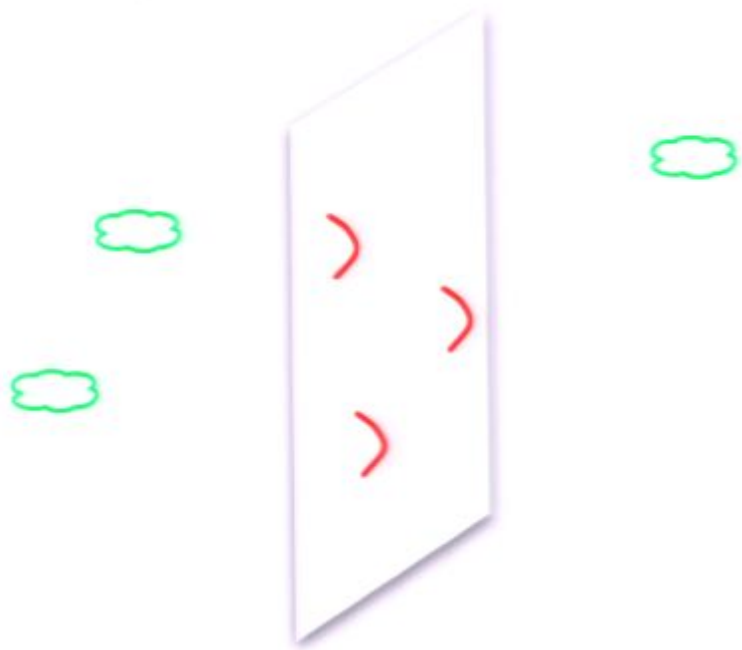
Zero energy configuration

→ ϕ's commute

→ simultaneously diagonalizable

$$\phi^1 = \mathbf{diag}(x_1, x_2, \ldots, x_N)$$
$$\phi^2 = \mathbf{diag}(y_1, y_2, \ldots, y_N)$$
$$\vdots$$

Now let's study the low energy limit of this system



$\mathcal{N} = 4$ U(N) gauge theory +Low energy gravitons in the bulk

They're decoupled.

Another way to take the low energy limit:

$N \gg 1$ → it's very heavy → Make the spacetime curved.

$$ds^2 = H(r)^{-1/2} dx_4^2 + H(r)^{1/2}(dr^2 + r^2 d\Omega_5^2)$$

where

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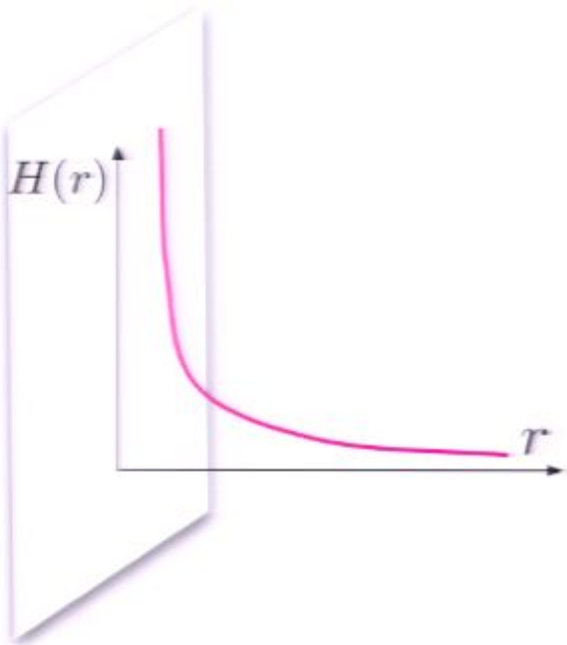
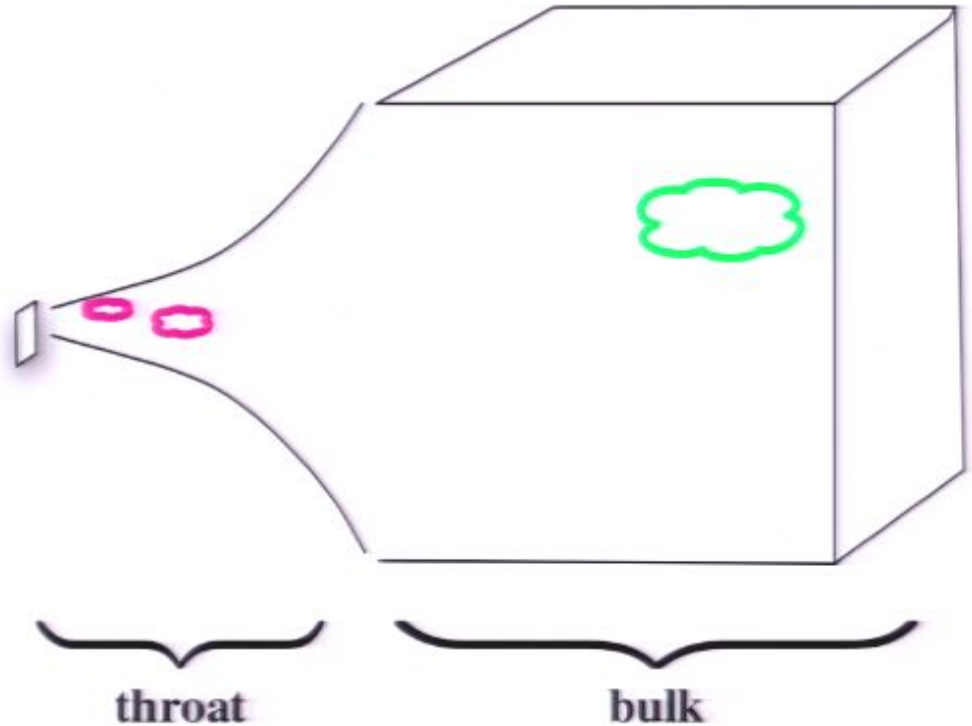
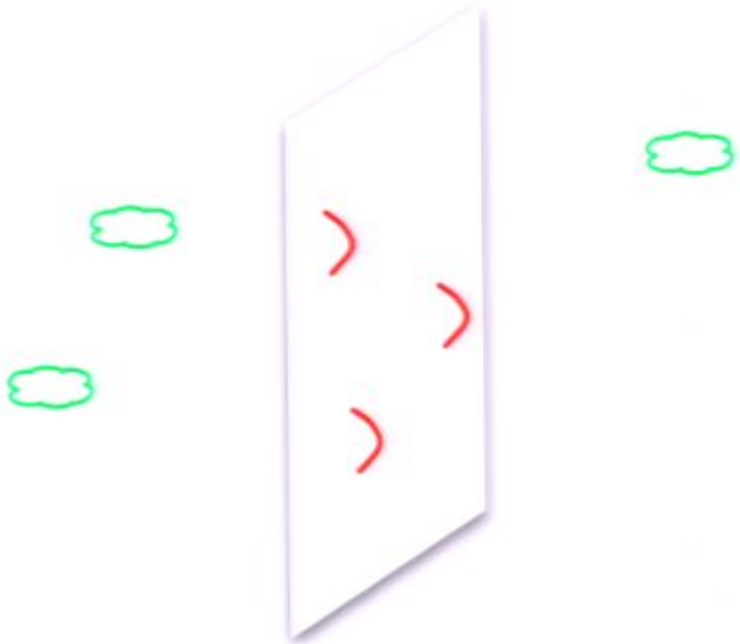


Important features at $r \to 0$:

- $g_{rr} \to \infty$
  → $r = 0$ becomes infinitely far away called the throat
- $g_{00} \to 0$
  → $r = 0$ becomes infinitely red-shifted.



Gravity trapped in the throat +Low energy gravitons in the bulk



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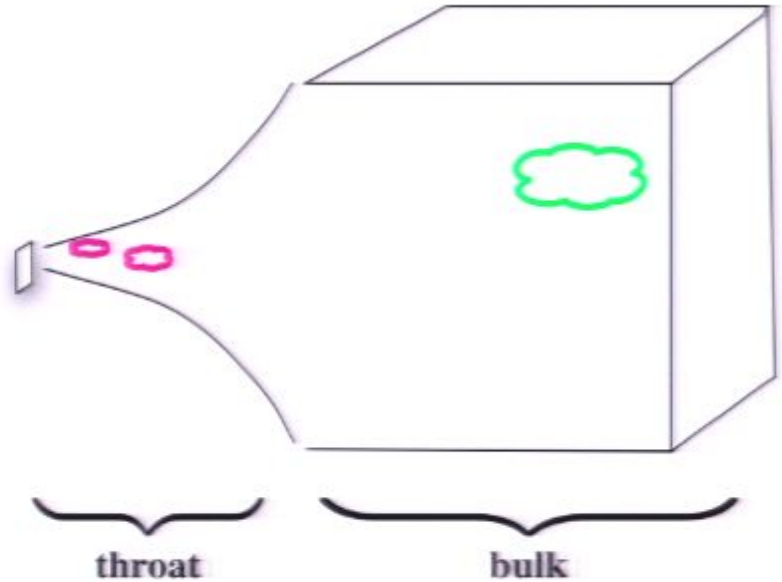

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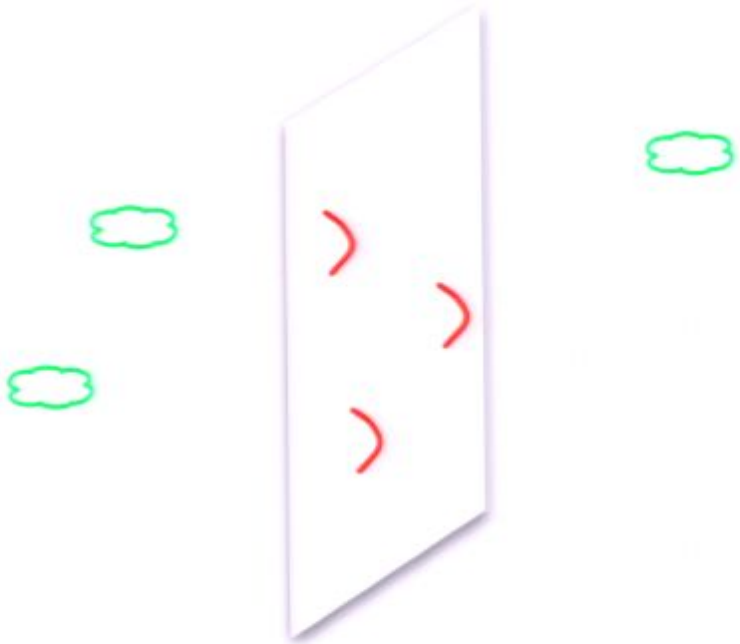

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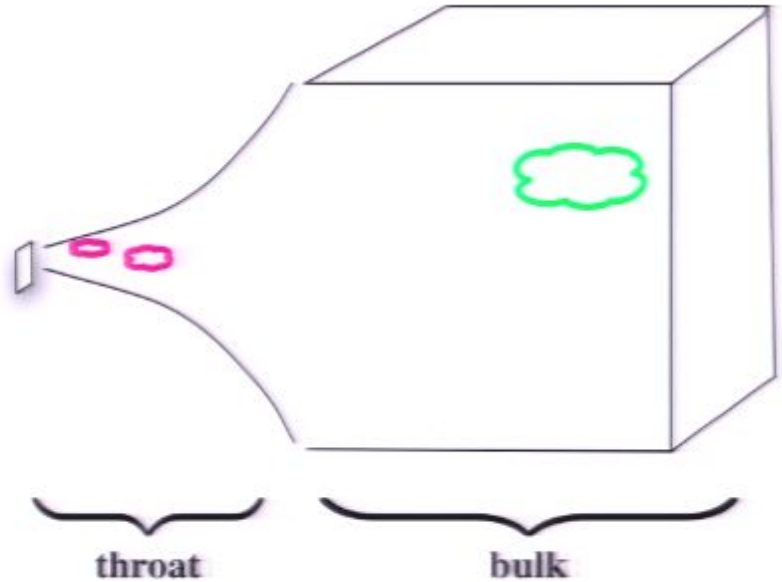

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$\mathcal{N} = 4$ U(N) gauge theory = Gravity trapped in the throat.

$$ds^2 = H(r)^{-1/2} dx_4^2 + H(r)^{1/2}(dr^2 + r^2 d\Omega_5^2)$$

where

$$H(r) = 1 + \frac{4\pi g_s N}{r^4}$$

To concentrate our attention to small r region,
take $r_{\text{new}} = c r_{\text{old}}$, $x^\mu_{\text{new}} = c x^\mu_{\text{old}}$ and let $c \to +\infty$:

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This is Anti-de Sitter$_5$ $\times$ Sphere^5.

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$$\frac{\text{Radius of } S^5}{\text{Typical length of strings}} = \lambda^{1/4}$$

where

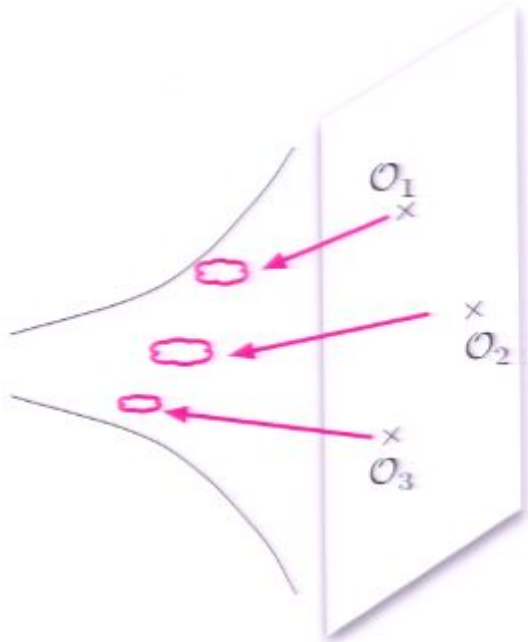
$$\lambda = g_{YM}^2 N$$

is the 't Hooft coupling.

Large coupling $\Leftrightarrow$ Weak curvature
Weak coupling $\Leftrightarrow$ Large curvature

Gauge invariant ops. $\mathcal{O}_i \Leftrightarrow$ bulk fields b_i

$$T^{\mu\nu} \Leftrightarrow g_{\mu\nu}$$



$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle =$$

Scattering amplitude of $b_{1,2,3}$ thrown in from $x_{1,2,3}$

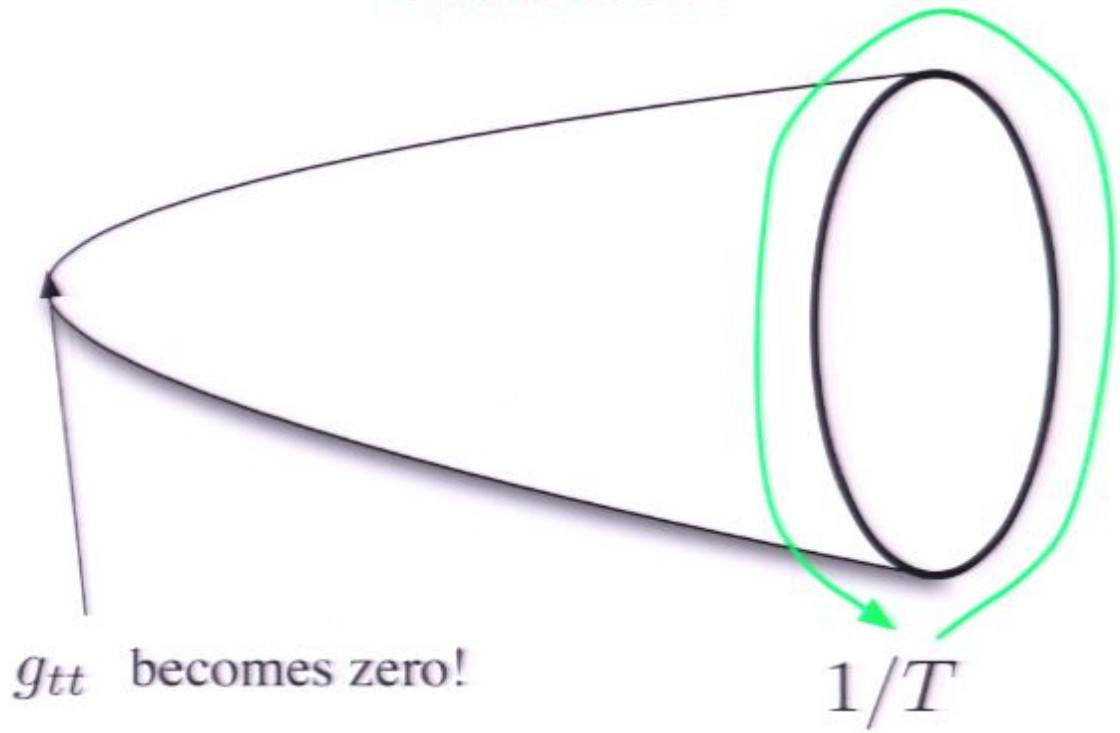
How can we go to finite temperature ?

Gauge theory side:

Take Euclidean time, and set $t \sim t + 1/T$.

Gravity side:

Black hole!



Viscosity/Entropy ratio

Viscosity η of the fluid

$\sim \langle TT \rangle$ via Kubo formula

$\sim$ Scattering amplitude of gravitons

$\propto$ Area of the black hole

Entropy s of the fluid

$\sim$ Bekenstein-Hawking entropy of the black hole

$\propto$ Area of the black hole

Get universal result:

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

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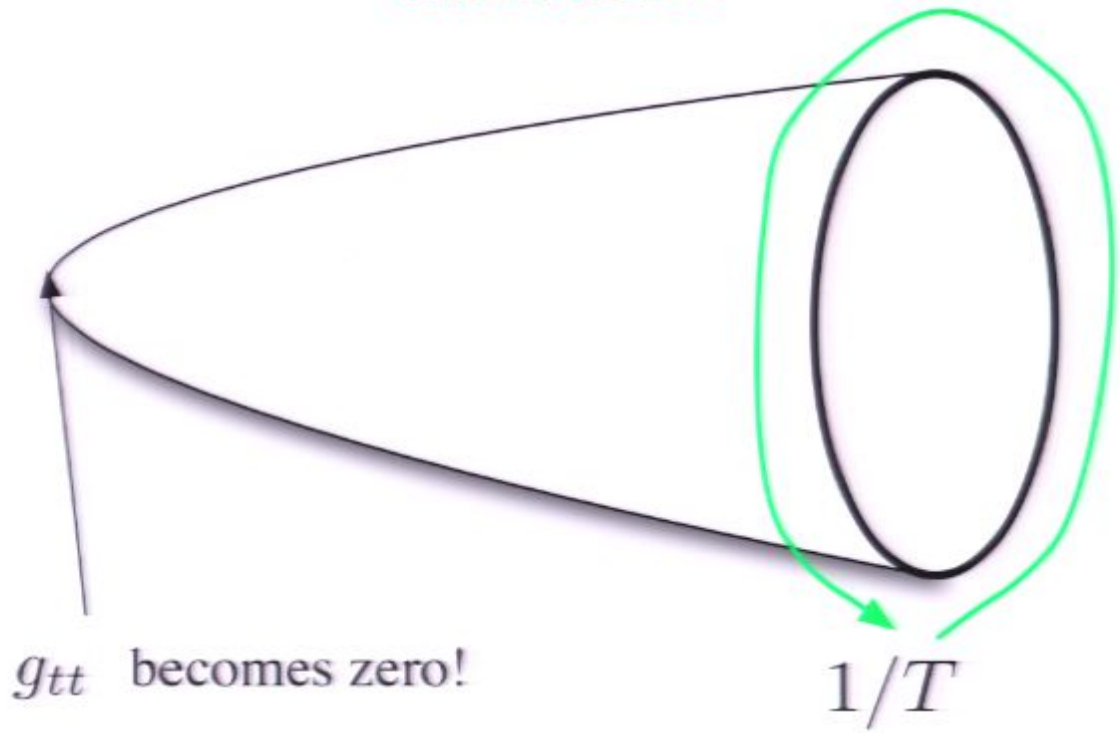

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This result is universal, once you assume

- there is a gravity dual
- which is weakly curved

E.g. Holds for any theories associated to quiver Q.

Corrections calculable.

Got quite a lot of attention,
because $\eta/s = 1/4\pi$ is quite low compared to other liquids.
Only comparable to that of QGP plasma.

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Summary

String theory

- has given a lot of ideas to particle physics/cosmology.
- has produced remarkable predictions to mathematics and all predictions have been confirmed so far.
- has given us the AdS/CFT duality, which can be used to study gauge theory.